Probablity and Statistics

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Unit 1: PROBABLITY

Random experiment: A random experiment is an experiment in which.

1. the set of all possible outcomes of the experiment is known in advance.
2. The outcome of a particular trial cannot be predicted in advance.
3. and the experiment can be repeated ~~in identical~~ under identical conditions.

S — the set of all possible outcomes of the experiment
↓
sample space

$E \subseteq S$. (subset E (event) of S).
↓
an event.

An outcome ω is favourable to E if $\omega \in E$

→ $E \cup F$ occured ⇒ either E occurred or F occurred (outcome belongs either to E or F)

→ $E \cap F$ occured ⇒ (both events occured ⇒ outcome belongs to both E and F)

1. finite
2. countable set → either finite or onto fun^n from $\mathbb{N}$ to that set
   - countably infinite
   - countable finite

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...bability that seats A, C, D, H, I an

countable
↓
even if uncountable then also one-one and onto mapping map to nat no.

$$S = \bigcup_{n \in \mathbb{N}} E_n$$

$$E_n = \left\{ \frac{m}{n} \,\middle|\, m \in \mathbb{Z} \right\}$$

E and F are said to be mutually exclusive if $E \cap F = \phi$
(we cannot get simultaneously both events)

Classical Method: if outcomes are equally likely and finite.

Relative frequency method.

Classical Relative Frequency (Emprical).

Consider a random experiment with sample space S. For each event E, we assume that a real number P(E) is defined ~~assign~~, we have following properties.
Axiomatic defn of probablity (3):

(i) $0 \leq P(E) \leq 1$.

(ii) $P(S) = 1$.

as mapping to nat. no.s

(iii) If $E_1, E_2, \ldots, E_n$ is a countably infinite collection of mutually exclusive events i.e. $E_i \cap E_j = \phi$, if $i \neq j$, then

$$\underbrace{P\left(\bigcup_{i=1}^{\infty} E_i\right)}_{\text{Real number}} = \underbrace{\sum_{i=1}^{\infty} P(E_i)}_{} \rightarrow \text{Infinite series}$$

↳ sequence from $\mathbb{N}$ to $\mathbb{R}$
so series convergent to

$$P\left(\bigcup_{i=1}^{\infty} E_i\right)$$

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Thus the infinite series is convergent to a real number.

then $P(E)$ is called probablity of event E.

$S = \{ w_1, w_2 \dots \}$

countable set + bijective map from $\mathbb{N}$ to this set.

$P(w_i) = p_i$

$P(S) = 1$

every outcome is made a event (singleton) prob. ke sum = 1

$$\sum_{i=1}^{\infty} P(w_i) = 1.$$

$$P(A) = \sum_{w_i \in A} P(w_i)$$

↳ then $P: P(S) \to [0,1]$ (power sets) s.t it satisfies 3 prop.

$E_1 = \{w_1\}$
$E_2 = \{w_2\}$ singleton set. $P(E_i) = p_i$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

$$P(S) = \sum_{i=1}^{\infty} P(E_i) = 1$$

continuum, if $f: E \to \mathbb{R}$.

* A class f of subset of S (i.e. subset of power set of S.) is called sigma-field if

prob is def for an event subset of sample space

prob. fun^ → domain is power set

(i) $S \in f$.

(ii) $A \in f \Rightarrow A^c \in f$.

(iii) $A_1, \dots A_n, \dots \in f$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in f$$

A probality is a function from sigma-field to [0,1].

$$P: f \to [0,1]$$

Remark:

P is a function from power set to [0,1]

↳ $P: \mathcal{P}(S) \to [0,1]$.

power set → samp sp ke har subset ko lega but all are not events

∴ sigma field → cont. ~~event~~ subsets which are events

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event → subset of sample space ... that

triples is called.

$(S, \mathcal{P}(S), P) \rightarrow$ Probability space.
↳ power set of S

In general,

$(S, \mathcal{F}, P) \rightarrow$ probability space.
↳ sigma

Theorem: Assume probability P of each event is defined, then.

(i) $P(\phi) = 0$ (empty set)

(ii) For mutually exclusive events $E_1, E_2, \ldots, E_n$ we have.

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

(iii) $P(E^c) = 1 - P(E)$

(iv) $E_1 \subseteq E_2 \Rightarrow P(E_2 - E_1) = P(E_2) - P(E_1)$.

and $P(E_1) \leq P(E_2)$

(v) $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Proof: (i) $E_1 = S$, $E_2 = \phi = E_3 = \cdots E_n = \cdots$

$$E_1 = \bigcup_{i=1}^{\infty} E_i$$

$$1 = P(E_1) = \sum_{i=1}^{\infty} P(E_i) = 1 + \sum_{i=2}^{\infty} P(\phi)$$
(by 2nd prop)

$$\Rightarrow \sum_{i=2}^{\infty} P(\phi) = 0$$

$$\Rightarrow P(\phi) = 0$$ (constant series is convergent if and only if ~~terms~~ constant term is 0).

(ii) $E_{n+1} = E_{n+2} = E_{n+3} = \cdots = \phi$

$$P\left(\bigcup_{i=1}^{n} E_i\right) = P\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$= \sum_{i=1}^{n} P(E_i)$$

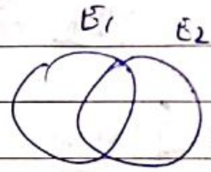
$i=1$ to n .. $n+1 \cdots \infty$

$\hookrightarrow 0$.

(iii) Trivial.

(iv) Again from (ii)

(v)



$$E_1 \cup E_2 = E_1 \cup (E_2 - E_1)$$

$$\Rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2 - E_1) \rightarrow (1)$$

$$E_2 = (E_2 - E_1) \cup (E_1 \cap E_2)$$

by (ii) $P(E_2) - P(E_1 \cap E_2) = P(E_2 - E_1)$. — (2)

using (2) in (1).

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

$$P(E_1 \cup E_2 \cup E_3) = P(E_1 \cup E_2) + P(E_3) - P((E_1 \cup E_2) \cap E_3)$$

$$= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P\left((E_1 \cap E_3) \cup (E_2 \cap E_3)\right)$$

$$= \sum_{i=1}^{3} P(E_i) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3).$$

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* $P(E_1 \cup E_2 \cup \cdots E_n)$.

$$= \sum_{i=1}^{n} P(E_i) - \sum_{i<j} (E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k)$$

$$+ \cdots + (-1)^{n+1} P(E_1 \cap E_2 \cap \cdots \cap E_n).$$

This is called inclusion - exclusion identity.

Ex: Toss two faire coins.

$E \rightarrow$ the event that upper face of first coin is head.
$F \rightarrow$ the event " " " " second " " head.

$S = \{(H,H), (H,T), (T,H), (T,T)\}$.

$P(E) = 1/2$, $P(F) = 1/2$

Conditional Probablity

Ex: Roll two faire dice.

cardinality of S is 36.

$S = \{(i,j) ; 1 \leq i, j \leq 6\}$

E - the event that sum of dice is six.
(card = 5)

F - the event that the first dice is four.

$P(E) = \frac{5}{36}$ $P(F) = \left(\frac{6}{36}\right) = 1/6$

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Prob of outcomes that belongs to F

$$P((i,j)/F) = \begin{cases} 1/6 & \text{if } (i,j) \in F \\ 0 & \text{if } (i,j) \notin F \end{cases}$$

P(E) if outcomes belongs to F.

$P(E/F) = 1/6$

$\hookrightarrow = \frac{P(E \cap F)}{P(F)} = 1/6$

$(S, \mathcal{P}(S), P(\cdot/F)) \rightarrow$ probablity space.

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Lec 3 (7/8/19)

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Conditional Probability

1

Consider an experiment with sample space S such that the probablity of each event is defined.

let F be an event s.t $P(F) > 0$. Then the probablity of an event E given that F has occurred is called the conditional probability of E given that outcomes is in F. It is denoted by $P(E/F)$

$$\boxed{P(E/F) = \frac{P(E \cap F)}{P(F)}}$$

Ex: Suppose cards numbered one to ten are placed in a hat, mixed up, and then one of the card is drawn. If we are told that the number on the drawn card is atleast five, then what is the conditional probablity that it is ten.

Sol:

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{1/10}{6/10} = 1/6$$

Ex: Suppose that each of three ~~a~~ men at a party throw his hat into the centre of the room. The hats are first mixed up and then each man randomly selects a hat. What is probablity that none of the three men select his own hat?

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$\begin{pmatrix}1\\2\\3\end{pmatrix} \rightarrow I$; $\begin{matrix}1 \rightarrow 2\\ 2 \rightarrow 1\\ 3 \rightarrow 3\end{matrix}$, $\boxed{(1,3)\ (2 \rightarrow 2)}$

$((2,3),\ 1 \rightarrow 1)$ $(1\ 2\ 3)$ $(1,3,2)$

$2/6 = \underline{1/3}$

Or. E_i : event that i^{th} man gets his own hat $1 \le i \le 3$.

$\underline{P(E_1 \cup E_2 \cup E_3)^c} = 1 - P(E_1 \cup E_2 \cup E_3)$.

$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2)$
$- P(E_2 \cap E_3) - P(E_1 \cap E_3)$
$+ P(E_1 \cap E_2 \cap E_3)$

$= \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}}$

$P(E_i) = 1/3 \quad \forall \ 1 \le i \le 3$

$P(E_i \cap E_j) = P(E_i / E_j) \times P(E_j)$
$= \frac{1}{2} \times \frac{1}{3} = 1/6 \quad \forall \ \underline{i \ne j}$

$P(E_1 \cap E_2 \cap E_3) = \underbrace{P(E_3 / E_1 \cap E_2)}_{1} . \underbrace{P(E_1 \cap E_3)}_{1/6}$
$= 1/6$.

$= (1/3) \times 3 - \left(\frac{1}{6}\right) \times 3 + \frac{1}{6}$

$= 1 + \frac{1}{6} - \frac{1}{2} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6}\ \underline{\underline{2/3}}$

$P(E_1 \cup E_2 \cup E_3)^c = 1 - 2/3 = \underline{\underline{1/3}}$

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Ex: Suppose an urn contains seven black balls and five white balls. We draw two balls from the urn without replacement. Assuming that each ball in the urn is equally likely to be drawn, what is the probability that both drawn balls are black.

Solⁿ

7B
5W.

$$\frac{7}{12} \times \frac{6}{11} = \frac{7}{22}$$

Or E :- first ball is black.
F : second drawn ball is black.

$P(E \cap F) = ?$

$P(E \cap F) = \cancel{P(E) \times P} \; P(E) \times P(F/E).$

$$= \frac{7}{12} \times \frac{6}{11}$$

index. (finite or countable)

Defⁿ: A countable collection $\{E_i : i \in \Lambda\}$ of event is said to be exhaustive if

$$P\left(\bigcup_{i \in \Lambda} E_i\right) = 1, \text{ where } \Lambda \text{ is an index set.}$$

→ if $A = B$, then $P(A) = P(B)$, but converse not true.

→ if $P(A) = 1$, ~~then~~ $\not\Rightarrow A = S$

Ex: $S = (0, \infty)$ $\quad P(A) = \int_A e^{-t} dt$. } verify it this prob on this set.

$A = (0, \infty) - \{a\}$
↳ singleton.

Independent events

Two events A and B are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$.

Remark: (i) $P(B) = 0$, take any other event

every event is independent with B if $P(B)=0$

$\Rightarrow P(A \cap B) = 0 = P(A) \cdot P(B)$

i.e. every event is independent on B.

(ii) $P(B) > 0$, $P(A/B) = P(A)$.

$\frac{P(A \cap B)}{P(B)}$

(if A and B are independent)

$P(A) \cdot P(B)$

in the presence of B, there is no change in prob. of A (means independent)

Def^n: (i) Events $E_1, E_2, ---, E_n$ are said to be independent if for any subcollection $\{E_{i_1}, E_{i_2}, ----- E_{i_k}\}$ of $\{E_1, E_2, --- E_n\}$. we have,

$$P\left(\bigcap_{j=1}^{k} E_{i_j}\right) = \prod_{j=1}^{k} P(E_{i_j})$$

to verify that coll. cont. n events → independent.

$2^n - n - 1$.

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A independent $\Rightarrow$ pairwise independent
$\nLeftarrow$ (converse not true)

Ex $S = \{1, 2, 3, 4\}$.

$A = \{1, 4\}$, $B = \{2, 4\}$, $C = \{3, 4\}$

$P(A \cap B \cap C) = 1/4$.
$\neq 1/8 = P(A) \cdot P(B) \cdot P(C)$

A, B, C are not independent but are pairwise independent

Ex: Toss two fair dice.
E_1 – the event that sum of the dice is six.
E_2 – " " " " " " " " seven

F – the event that first die is equal to four.

$P(E_2) = 6/36$ $P(E_1) = 5/36$

$P(E_1 \cap F) = 1/36 \neq P(E_1) \cdot P(F)$ $\Rightarrow$ E_1 is not
$= 5/36 \cdot 1/6$. ind. on F.

$P(E_2 \cap F) = 1/36 = P(E_2) \cdot P(F) = 1/6 \times 1/6 = 1/36$

$\Rightarrow$ E_2 is independent on F.

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Baye's Formula.

$$E = (E\cap F) \cup (E \cap F^c)$$

$$P(E) = P(E\cap F) + P(E\cap F^c)$$

$$= P(E/F).P(F) + P(E/F^c).P(F^c)$$

(if $P(F)>0$)

Lec-4 (9/8/19) $\quad P(F\cup F^c) = 1. \qquad P(F\cup F^c) = 1.$

F and F^c are exhaustive events.

$F_1, F_2, F_3, \ldots F_n$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$

$$E = E\cap S = E\cap \left(\bigcup_{i=1}^{n} F_i\right)$$

$$= \bigcup_{i=1}^{n} (E\cap F_i)$$

$$\Rightarrow P(E) = \sum_{i=1}^{n} P(E\cap F_i)$$

$$\boxed{P(E) = \sum_{i=1}^{n} P(E/F_i).P(F_i).}$$ Total Probability theorem.

Remark:

$F_1, F_2, \ldots F_n$ are mutually exclusive events s.t.

$$P\left(\bigcup_{i=1}^{n} F_i\right) = 1$$

$$F = \bigcup_{i=1}^{n} F_i$$

$$P(F) = 1.$$

$$\Rightarrow P(F^c) = 0$$

$$\boxed{P(E) = P(E\cap F) + \underbrace{P(E\cap F^c)}_{=0}}$$

(d) What is the probability that seats A, C, D, H, I and J

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$$P(F_j/E) = \frac{P(E \cap F_j)}{P(E)}$$

$$= \frac{P(E/F_j) \cdot P(F_j)}{P(E)}$$

$$\boxed{P(F_j/E) = \frac{P(E/F_j) \cdot P(F_j)}{\sum_{i=1}^{n} P(E/F_i) \cdot P(F_i)}} \quad \forall \ 1 \leq j \leq n.$$

↳ Bayes formula
(if we have mutually exclusive and exhaustive events).

Ex: Consider two urns - the first contains two white and seven black balls, and second contains five white and six black balls. We flip a fair coin and then draw ball from the first or the second urn depending on whether the outcome was head or tail. What is conditional probablity that the outcome of the toss was head given that a white ball was selected.

2W 7B   5W 6B

H → the event that outcome of the toss (upper face of coin) is head.

W → the event that white ball is drawn.

$$P(H/W) = \frac{P(W/H) \cdot P(H)}{P(W/H) \cdot P(H) + P(W/H^c) \cdot P(H^c)}$$

$$= \frac{\frac{2}{9} \times \frac{1}{2}}{\frac{2}{9} \times \frac{1}{2} + \frac{5}{11} \times \frac{1}{2}} = \frac{\frac{1}{9}}{\frac{1}{9} + \frac{1}{11}} = \frac{\frac{1}{9}}{\frac{11+9}{9\cdot}}$$

Ex: Urn
Urn
A fair
upper
urn U
the
(i) Gi
(ii) Gi
con

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Ex: Urn U_1 contains of 4 white and 6 black balls

Urn U_2 contains of 6 white and 4 black balls.

A fair die is cast and urn U_1 is selected if the upper face of die shows five or six dots. Otherwise urn U_2 is selected. A ball is drawn at random from the selected urn.

(i) Given that the drawn ball is white, find the conditional prob. that it comes from urn U_1.

(ii) Given that the drawn ball is white, find the conditional prob. that it comes from urn U_2. $3/4$

$3/4$

$W \to$ drawn ball is white

U_1: 4, 6 — $\left(\frac{4}{10}\right)$ $\left(\frac{6}{10}\right)$

U_2: 6, 4 — $\left(\frac{6}{10}\right)$ $\left(\frac{4}{10}\right)$

die: $\frac{4}{6}$, $(5, 6)$ $\frac{2}{6}$

$P(U_1/W)$ and $P(U_2/W)$

2 events are mutually exclusive and exhaustive.

$P(U_1 \cup U_2 / W) = 1$ $\Rightarrow \frac{P(U_1 \cup U_2 \cap W)}{P(W)} = 1.$

relation b/w $P(U_1/W)$ and $P(U_1)$
$(<)$.

$P(U_2/W) > P(U_2)$

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(d) What is the probability that seats A, C, ... H, I and J

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Lec-5
(14/8/19)

Ex: Urn 1 contains one white and two black marbles, urn 2 contains one black and two white marbles, and urn 3 contains three black and three white marbles. A die is rolled.
If a 1, 2 or 3 shown up, urn 1 is selected.
If a 4 is shown up, urn 2 is selected.
If 5, 6, is shown up, urn 3 is selected.

1 marble is drawn at random from the selected urn.
A - the event that marble drawn is white.
U, V & W → denotes the event the urn 1, 2 or 3 got selected respectively.

Then what is the probability that urn 2 is selected given that marble drawn is white.

1W 2B (1).   1B 2W (2).   3B 3W (3).

die → 1,2,3 ; 4 ; 5,6.

3/6.   1/6   2/6

$P(U) = 1/2 \quad P(V) = 1/6 \quad P(W) = 1/3$

$$P(U/W) = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + \frac{1}{\cancel{6}} \times \frac{\cancel{2}}{3} + \frac{1}{3} \times \frac{3}{6}}$$

$$= \frac{1/6}{1/6 + \frac{1}{9} + 1/6} = \frac{1/6}{1/3 + 1/9}$$

$$= \frac{1/6}{\frac{4}{9}} = \frac{\cancel{9}\,3}{4 \times \cancel{6}\,2} = 3/8$$