PARTIAL DERIVATIVE

DEF. The usual derivative of a function of several variables with respect to one of the independent variables keeping all other independent variables as constant is called the partial derivative of the function with respect to that variable.

Let $z = f(x,y)$; $(x,y) \in \mathbb{R}^2$, $z \in \mathbb{R}$

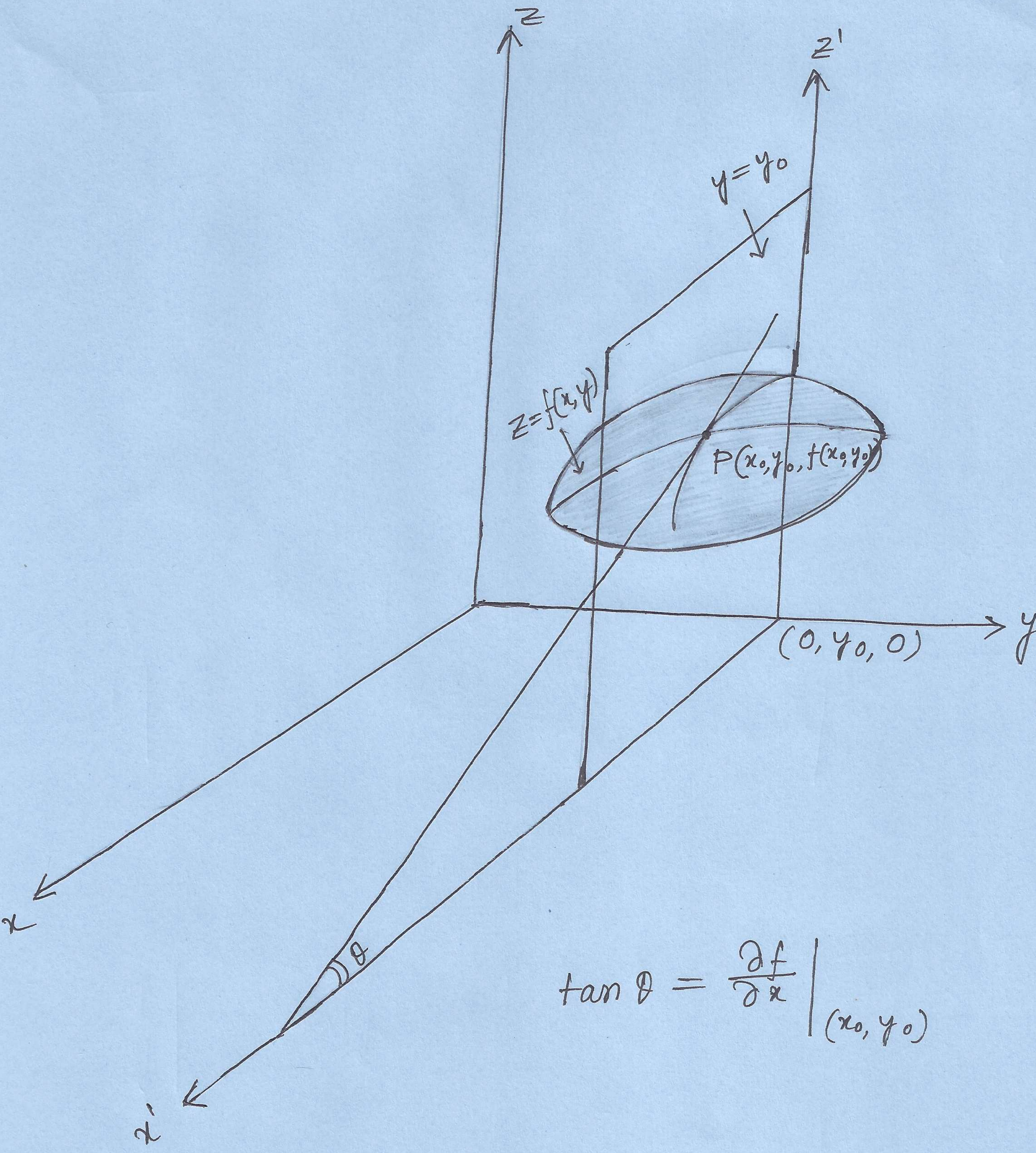
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0,y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$$

$$= \left.\frac{d}{dx} f(x,y_0)\right|_{x=x_0}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0,y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0,y_0)}{\Delta y}$$

$$= \left.\frac{d}{dy} f(x_0,y)\right|_{y=y_0}$$

GEOMETRIC INTERPRETATION:



$$\tan\theta = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}$$

Ex. Find the value of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (x, y) of the following function

(i) $f(x,y) = y e^{-x}$ (ii) $f(x,y) = \sin(2x+3$

from the first principles.

SOL: (i)

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{y e^{-(x+\Delta x)} - y e^{-x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{y e^{-x}\{e^{-\Delta x} - 1\}}{\Delta x}$$

$$= y e^{-x} \lim_{\Delta x \to 0} \frac{\{1 - \Delta x - \frac{\Delta x^2}{2!} - \dots - 1\}}{\Delta x}$$

$$= -y e^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(y+\Delta y)\, e^{-x} - y e^{-x}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} e^{-x}$$

$$= e^{-x}$$

ii) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\sin\{2(x+\Delta x)+3y\} - \sin(2x+3y)}{\Delta x}$$

$$\left[\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$= \lim_{\Delta x \to 0} \frac{2\cdot\cos(2x+3y+\Delta x)\cdot\sin\Delta x}{\Delta x}$$

$$= 2\cos(2x+3y)$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{\sin(2x + 3(y + \Delta y)) - \sin(2x+3y}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2 \cdot \cos\left(2x + 3y + \frac{3\Delta y}{2}\right) \cdot \sin\left(\frac{3}{2} \cdot \right.}{\frac{2}{3} \cdot \left(\frac{3}{2}\Delta y\right)}$$

$$= 3\cos(2x+3y) \cdot \lim_{\Delta y \to 0} \frac{\sin\left(\frac{3}{2}\Delta y\right)}{\frac{3}{2}\Delta y}$$

$$= 3\cos(2x+3y)$$

RELATIONSHIP BETWEEN CONTINUITY AND THE EXISTENCE OF PARTIAL DERIVATIVES

A function can have partial derivatives with respect to both x and y at a point without being continous there. On the other hand a continuous function may not have partial derivatives.

EXAMPLE: Show that the function

$$f(x,y) = \begin{cases} (x+y)\,\text{Sin}\left(\frac{1}{x+y}\right), & (x,y) \neq (0, \\ 0 & , (x,y) = (0, \end{cases}$$

is continuous at $(0,0)$ but its partial derivatives do not exist at $(0,0)$.

SOLUTION:

$$
\begin{aligned}
|f(x,y) - f(0,0)| \\
&= \left| (x+y) \sin\left(\frac{1}{x+y}\right) \right| \\
&\leq |x+y| \\
&\left.\begin{aligned}&\leq |x| + |y| \\ &\leq \sqrt{2} \cdot \sqrt{x^2+y^2}\end{aligned}\right\}^{*} \\
&< \epsilon
\end{aligned}
$$

Choose $\delta < \dfrac{\epsilon}{\sqrt{2}}$, then

$|f(x,y) - f(0,0)| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} <$

$$
\Rightarrow \boxed{\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)}
$$

Hence the function is continuous at $(0,0)$

(*)

$$(|x| - |y|)^2 \geqslant 0$$

$$\Rightarrow \quad x^2 + y^2 \geqslant 2\,|x|\,|y|$$

$$\Rightarrow \quad 2\,(x^2+y^2) \geqslant x^2 + y^2 + 2\,|x|\,|y|$$

$$\Rightarrow \quad 2\,(x^2+y^2) \geqslant (|x| + |y|)^2$$

$$\Rightarrow \quad (|x| + |y|) \leq \sqrt{2}\sqrt{(x^2+y^2)}$$

(III) ALTERNATIVE : 1

$$\lim_{(x,y)\to(0,0)} (x+y)\,\operatorname{Sin}\left(\frac{1}{x+y}\right)$$

put $x + y = t$

Then $\displaystyle\lim_{t\to 0} t\,\operatorname{Sin}\left(\frac{1}{t}\right)$

$= 0$

Now consider

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \ \sin\left(\frac{1}{\Delta x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \sin\left(\frac{1}{\Delta x}\right)$$

$\Rightarrow$ $f_x(0, 0)$ does not exist.

Similarly $f_y(0, 0)$ does not exist.

EXAMPLE:

Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exist at $(0,0)$.

SOLUTION:

Choose the path $y = mx$

The limit

$$\lim_{x \to 0} \frac{x \cdot mx}{x^2 + 2m^2x^2}$$

$$= \frac{m}{1+m^2}$$

depends on the path.

Hence the function is not continuou at $(0,0)$.

Now consider

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x}$$

$$= 0 = f_x(0,0)$$

$$\lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y}$$

$$= 0 = f_y(0,0)$$

$\Rightarrow$ The partial derivatives f_x and f_y exist at $(0,0)$

THEOREM: [SUFFICIENT CONDITION FOR CONTINUITY AT (0, 0)]

One of the first order partial derivative exist and is bounded in the neighbourhood of (x_0, y_0) and the other exist at (x_0, y_0).

PARTIAL DERIVATIVES OF HIGHER ORD

$$\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)}_{f_{xx}}, \quad \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}_{f_{yx}}, \quad \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)}_{f_{xy}}$$

are called second order partial derivatives of f.

The derivatives f_{xy} and f_{yx} are called mixed derivatives.

If the mixed derivatives f_{yx} and f_{xy} are continuous in an open domain D, then at any point $(x,y) \in D$

$$f_{xy} = f_{yx}.$$

EXAMPLE: Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

SOLUTION:

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y}$$

$$= 0$$

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \cdot y^3}{\Delta x + y^2} \cdot \frac{1}{\Delta x}$$

$$= y$$

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{x \cdot \Delta y^{\cancel{3}\,2}}{x + \Delta y^2} \cdot \frac{1}{\cancel{\Delta y}}$$

$$= 0$$

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$$

$$= 0$$

$$\therefore \boxed{f_{xy}(0,0) = 0}$$

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(\Delta y - 0)}{\Delta y}$$

$$= 1$$

$$\therefore \boxed{f_{yx}(0,0) = 1}$$

Since $f_{xy}(0,0) \neq f_{yx}(0,0)$, f_{xy} and f_{yx} are not continuous at $(0,0)$.

III <u>CHECK</u>: For $(x,y) \neq (0,0)$

$$f_{yx}(x,y) = \frac{y^6 + 5xy^4}{(x+y^2)^3} = f_{xy}(x,y)$$

Along the path $x = my^2$ the limit $\lim_{(x,y)\to(0,0)} f_{yx}(x,y)$ depends on the path.

This implies, f_{yx} is not continuous at $(0,0)$.