Unit - 4

Topics under this chapter

* Double Interals
* Change of Order And Change of variables
* Triple Integeration
* Gamma, Béta Function.

Double Integral

We know that

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \partial x\to 0}} \left[f(x_1)\partial(x_1) + f(x_2)\partial(x_2) + \cdots f(x_n)\partial(x_n) \right]$$

In the case of two variables x & y it becomes

$$\iint_R f(x,y)\,dx\,dy = \lim_{\substack{n\to\infty \\ \partial A\to 0}} \left[f(x_1,y_1)\partial A_1 + f(x_2,y_2)\partial A_2 + \cdots f(x_n,y_n)\partial A_n \right]$$

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⇒ Evaluation of Double Integeration :

$$\iint_A f(x,y)\,dA = \int_a^b \left[\int_{y_1}^{y_2} f(x,y)\,dx \right] dy$$

where A is described as :

$y_1 \le y \le y_2$ Or $f_1(x) \le y \le f_2(x)$

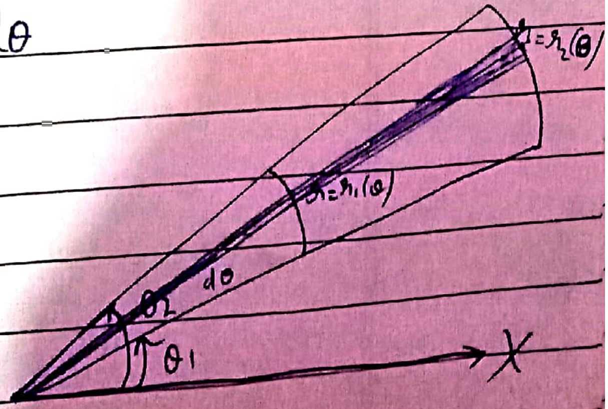
$a \le x \le b$

⇐ y

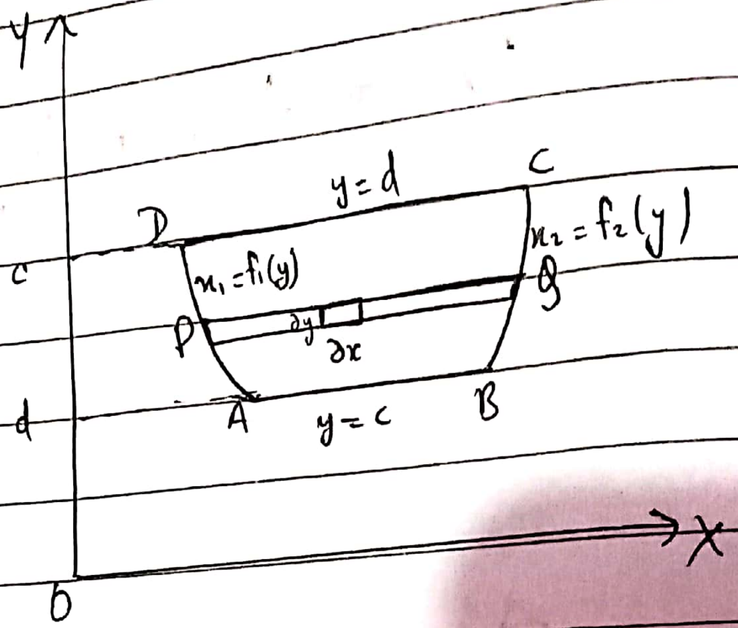
$$\iint_A f(x,y)\,dA = \int_a^b \left[\int_{x_1}^{x_2} f(x,y)\,dy \right] dx$$

2) Evaluation of Double Integrals in Polar Co-ordinates

$$\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\,dr\,d\theta$$



* To Calculate the integral over a given region



In this we take a small area $dx\,dy$. The integeration w.r.t x between the limits x_1 & x_2 will give the area ~~lst~~ of strip PQ the integerating w.r.t 'y' between limits $y = c$ & $y = d$ will g like extend the strip PQ from AB to CD and in this manner it will cover the area of given region

Change of Order And Change of Variable :

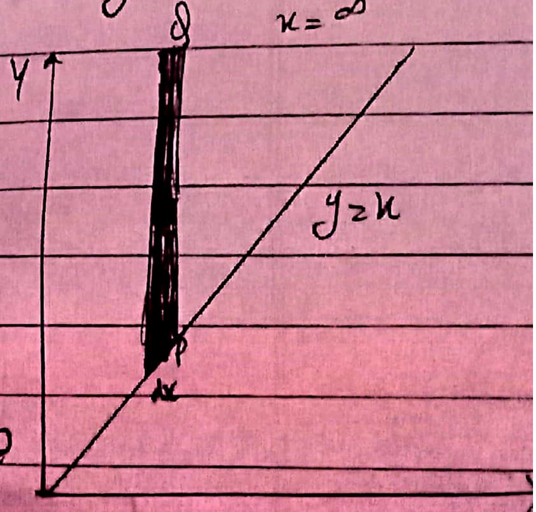
On ~~ord~~ Changing the order, the limit of integration change.

We ~~we~~ change the order to simply the integertion which may be hard when we are integerating them in other limit.

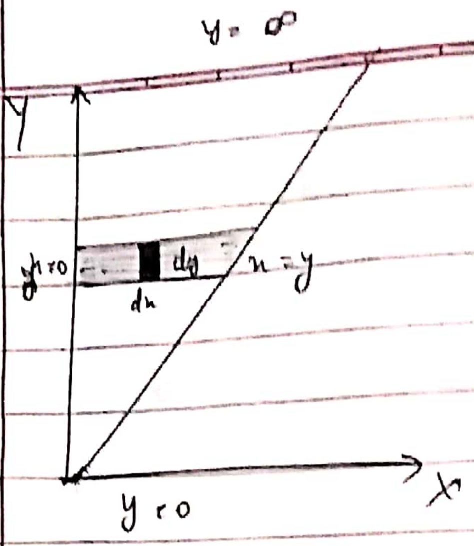
For finding the new limit, we first have to draw a rough digram of region of integration.

Example : $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$

Solution : $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$



Here x varies from
$x = 0$ to $x = \infty$
y varies from
$y = x$ to $y = \infty$



Now; x varies from 0 to y

y varies from 0 to ∞

Now, integerating along these limits

$$= \int_0^{\infty}\left[\frac{e^{-y}}{y}dy\right]_0^{y} dx$$

$$= \int_0^{\infty}\frac{e^{-y}}{y}dy\,[x]_0^{y}$$

$$= \int_0^{\infty}\frac{e^{-y}}{y}\times y\,dy$$

$$= \int_0^{\infty} e^{-y}\,dy$$

$$= \left[-e^{-y}\right]_0^{\infty}$$

$$= [0-(-1)] = \underline{\underline{1}}$$

Triple Integeration:

The expression of triple integertion

$$\iiint_S f(x,y,z)\,dx\,dy\,dz$$

$$\int_{x=a}^{x=b} \phi(x)\,dx \int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} \phi(x,y)\,dy \int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz$$

In this we integeration according to limit

means,

If first limit is combination of x,y then we first integerate w.r.t "Z" and second limit is function of x then now we integerate w.r.t "y", third limit is constant we integerate w.r.t "x" (what is left now from above Eg. it is x).

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Gamma Beta Function :

Gamma Function:

Gamma function is denoted by Γn (it is called Gamma n).

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$$

Above is the expression of Gamma.

* Some Formulaes

(i) $\Gamma n+1 = n \Gamma n$

(ii) $\Gamma \frac{1}{2} = \sqrt{\pi}$

(iii) $\int_0^\infty e^{-ky} y^{n-1} \, dy = \frac{\Gamma n}{k^n}$

Beta Function :

Beta function is denoted by $\beta(l,m)$ (it is called Beta of l, m)

$$\beta(l,m) = \int_0^1 x^{l-1} (1-x)^{m-1} \, dx$$

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$$\beta(l,m) = \frac{(l-1)!}{m(m+1)\cdots(m+l-1)}$$

Property of Beta Function:

$$\beta(l,m) = \beta(m,l)$$

Relation between Beta And Gamma:

Statement: $\beta(l,m) = \dfrac{\Gamma l \, \Gamma m}{\Gamma l+m}$

Proof:-

We know, $\Gamma l = \int_0^\infty e^{-x} x^{l-1} dx$

Put $zx = y$

$$\frac{\Gamma l}{z^l} = \int_0^\infty e^{-zx} x^{l-1} dx$$

$$\Gamma l = \int_0^\infty z^l e^{-zx} x^{l-1} dx$$

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Multiplying both side by $e^{-z} z^{m-1}$

$$\Gamma l \cdot e^{-z} z^{m-1} = \int_0^\infty e^{-z} z^{m-1} \cdot z^{l} e^{-zx} x^{l-1} dx$$

$$\Gamma l \cdot e^{-z} z^{m-1} = \int_0^\infty z^{m-1+l} x^{l-1} e^{-z(x+1)} dx$$

Integerating both side w.r.t "z".

$$\int_0^\infty \Gamma l \cdot e^{-z} z^{m-1} dz = \int_0^\infty \int_0^\infty z^{m+l-1} x^{l-1} e^{-z(x+1)} dx\, dz$$

$$\Gamma l \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty x^{l-1} dx \int_0^\infty z^{m+l-1} e^{-z(x+1)} dz$$

$$\Gamma l\, \Gamma m = \int_0^\infty x^{l-1} dx \frac{\Gamma m+l}{(1+x)^{l+m}}$$

$$\left\{ \because \beta(l,m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{m+l}} \right\}$$

$$\Gamma l\, \Gamma m = \Gamma l+m \int_0^\infty \frac{x^{l-1} dx}{(1+x)^{l+m}}$$

$$\Gamma l\, \Gamma m = \Gamma l+m \;\; \beta(l,m)$$

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$$\boxed{\beta(l, m) = \frac{\Gamma l \, \Gamma m}{\Gamma l+m}}$$

Dirichelet's Integral

$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma l+m+n+1}$$

If x, y, z in the triple integral have some power then from those power we can easily find l, m, n & then can solve triple integration by above formula.