Instructor's Solutions Manual
Part I

to accompany

Thomas' Calculus

Tenth Editon

Instructor's Solutions Manual
Part I

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to accompany

Thomas' Calculus

Tenth Edition

Based on the original work by
George B. Thomas, Jr.
Massachusetts Institute of Technology

As revised by
Ross L. Finney,
Maurice D. Weir,
and Frank R. Giordano

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PREFACE TO THE INSTRUCTOR

This Instructor's Solutions Manual contains the solutions to every exercise in the 10th Edition of Thomas' CALCULUS as revised by Ross L. Finney, Maurice D. Weir and Frank R. Giordano. The corresponding Student's Solutions Manual omits the solutions to the even-numbered exercises as well as the solutions to the CAS exercises (because the CAS command templates would give them all away).

In addition to including the solutions to all of the new exercises in this edition of Thomas' CALCULUS, we have carefully reviewed every solution which appeared in previous solutions manuals to ensure that each solution

- conforms exactly to the methods, procedures and steps presented in the text
- is mathematically correct
- includes all of the steps necessary so a typical calculus student can follow the logical argument and algebra
- includes a graph or figure whenever called for by the exercise or, if needed, to help with the explanation
- is formatted in an appropriate style to aid in its understanding

Every CAS exercise is solved in both the MAPLE and MATHEMATICA computer algebra systems. A template showing an example of the CAS commands needed to execute the solution is provided for each exercise type. Similar exercises within the text grouping require a change only in the input function or other numerical input parameters associated with the problem (such as the interval endpoints or the number of iterations).

Acknowledgments

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TABLE OF CONTENTS

PRELIMINARY CHAPTER

P.1 LINES

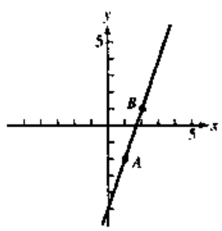
1. (a) $\Delta x = -1 - 1 = -2$
$\Delta y = -1 - 2 = -3$

(b) $\Delta x = -1 - (-3) = 2$
$\Delta y = -2 - 2 = -4$

2. (a) $\Delta x = -8 - (-3) = -5$
$\Delta y = 1 - 1 = 0$

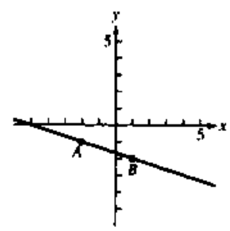
(b) $\Delta x = 0 - 0 = 0$
$\Delta y = -2 - 4 = -6$

3. (a)



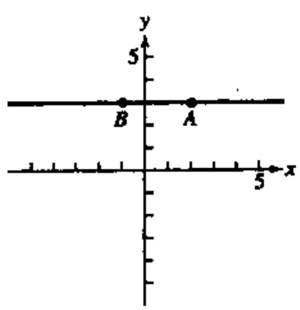
$$m = \frac{1-(-2)}{2-1} = \frac{3}{1} = 3$$

(b)



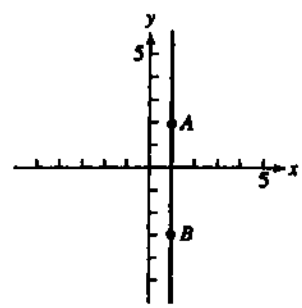
$$m = \frac{-2-(-1)}{1-(-2)} = \frac{-1}{3} = -\frac{1}{3}$$

4. (a)



$$m = \frac{3-3}{-1-2} = \frac{0}{-3} = 0$$

(b)



$$m = \frac{-3-2}{1-1} = \frac{-5}{0} \text{ (undefined)}$$

5. (a) $x = 2,\ y = 3$

(b) $x = -1,\ y = \frac{4}{3}$

6. (a) $x = 0,\ y = -\sqrt{2}$

(b) $x = -\pi,\ y = 0$

7. (a) $y = 1(x-1) + 1$

(b) $y = -1[x - (-1)] + 1 = -1(x+1) + 1$

8. (a) $y = 2(x-0) + 3$

(b) $y = -2[x - (-4)] + 0 = -2(x+4) + 0$

9. (a) $m = \frac{3-0}{2-0} = \frac{3}{2}$

$y = \frac{3}{2}(x-0) + 0$

$y = \frac{3}{2}x$

(b) $m = \frac{1-1}{2-1} = \frac{0}{1} = 0$

$y = 0(x-1) + 1$

$y = 1$

$2y = 3x$

$3x - 2y = 0$

10. (a) $m = \frac{-2-0}{-2-(-2)} = \frac{-2}{0}$ (undefined)

Vertical line: $x = -2$

(b) $m = \frac{-2-1}{2-(-2)} = \frac{-3}{4} = -\frac{3}{4}$

$y = -\frac{3}{4}[x-(-2)] + 1$

$4y = -3(x+2) + 4$

$4y = -3x - 2$

$3x + 4y = -2$

11. (a) $y = 3x - 2$

(b) $y = -1x + 2$ or $y = -x + 2$

12. (a) $y = -\frac{1}{2}x - 3$

(b) $y = \frac{1}{3}x - 1$

13. The line contains $(0,0)$ and $(10,25)$.

$m = \frac{25-0}{10-0} = \frac{25}{10} = \frac{5}{2}$

$y = \frac{5}{2}x$

14. The line contains $(0,0)$ and $(5,2)$.

$m = \frac{2-0}{5-0} = \frac{2}{5}$

$y = \frac{2}{5}x$

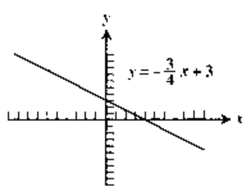
15. (a) $3x + 4y = 12$

$4y = -3x + 12$

$y = -\frac{3}{4}x + 3$

i) Slope: $-\frac{3}{4}$

ii) y-intercept: 3

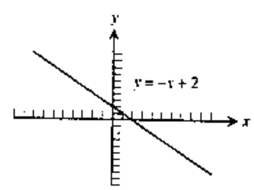


(b) $x + y = 2$

$y = -x + 2$

i) Slope: -1

ii) y-intercept: 2



16. (a) $\frac{x}{3} + \frac{y}{4} = 1$

$\frac{y}{4} = -\frac{x}{3} + 1$

$y = -\frac{4}{3}x + 4$

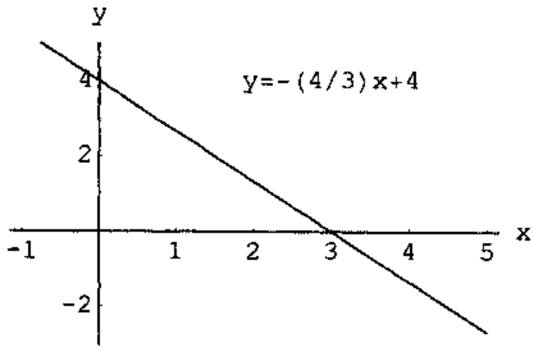
i) Slope: $-\frac{4}{3}$

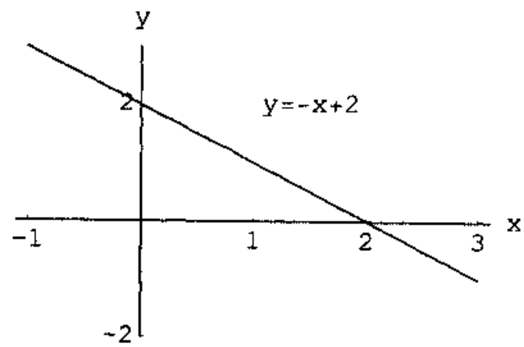
ii) y-intercept: 4

(b) $y = 2x + 4$

i) Slope: 2

ii) y-intercept: 4





17. (a) i) The desired line has slope -1 and passes through $(0,0)$: $y=-1(x-0)+0$ or $y=-x$.

ii) The desired line has slope $\frac{-1}{-1}=1$ and passes through $(0,0)$: $y=1(x-0)+0$ or $y=x$.

(b) i) The given equation is equivalent to $y=-2x+4$. The desired line has slope -2 and passes through $(-2,2)$: $y=-2(x+2)+2$ or $y=-2x-2$.

ii) The desired line has slope $\frac{-1}{-2}=\frac{1}{2}$ and passes through $(-2,2)$: $y=\frac{1}{2}(x+2)+2$ or $y=\frac{1}{2}x+3$.

18. (a) i) The given line is vertical, so we seek a vertical line through $(-2,4)$: $x=-2$.

ii) We seek a horizontal line through $(-2,4)$: $y=4$.

(b) i) The given line is horizontal, so we seek a horizontal line through $\left(-1,\frac{1}{2}\right)$: $y=\frac{1}{2}$.

ii) We seek a vertical line through $\left(-1,\frac{1}{2}\right)$: $x=-1$.

19. $m=\frac{9-2}{3-1}=\frac{7}{2}$

$f(x)=\frac{7}{2}(x-1)+2=\frac{7}{2}x-\frac{3}{2}$

Check: $f(5)=\frac{7}{2}(5)-\frac{3}{2}=16$, as expected.

Since $f(x)=\frac{7}{2}x-\frac{3}{2}$, we have $m=\frac{7}{2}$ and $b=-\frac{3}{2}$.

20. $m=\frac{-4-(-1)}{4-2}=\frac{-3}{2}=-\frac{3}{2}$

$f(x)=-\frac{3}{2}(x-2)+(-1)=-\frac{3}{2}x+2$

Check: $f(6)=-\frac{3}{2}(6)+2=-7$, as expected.

Since $f(x)=-\frac{3}{2}x+2$, we have $m=-\frac{3}{2}$ and $b=2$.

21.
$$\begin{aligned}-\frac{2}{3}&=\frac{y-3}{4-(-2)}\\ -\frac{2}{3}(6)&=y-3\\ -4&=y-3\\ -1&=y\end{aligned}$$

22.
$$\begin{aligned}2&=\frac{2-(-2)}{x-(-8)}\\ 2(x+8)&=4\\ x+8&=2\\ x&=-6\end{aligned}$$

23. $y=1\cdot(x-3)+4$
$y=x-3+4$
$y=x+1$
This is the same as the equation obtained in Example 5.

24. (a) When $y=0$, we have $\frac{x}{c}=1$, so $x=c$.

When $x=0$, we have $\frac{y}{d}=1$, so $y=d$.

(b) When $y=0$, we have $\frac{x}{c}=2$, so $x=2c$.

When $x=0$, we have $\frac{y}{d}=2$, so $y=2d$.

The x-intercept is $2c$ and the y-intercept is $2d$.

25. (a) The given equations are equivalent to $y = -\frac{2}{k}x + \frac{3}{k}$ and $y = -x + 1$, respectively, so the slopes are $-\frac{2}{k}$ and -1. The lines are parallel when $-\frac{2}{k} = -1$, so $k = 2$.

(b) The lines are perpendicular when $-\frac{2}{k} = \frac{-1}{-1}$, so $k = -2$.

26. (a) $m \approx \frac{68 - 69.5}{0.4 - 0} = \frac{-1.5}{0.4} = -3.75$ degrees/inch

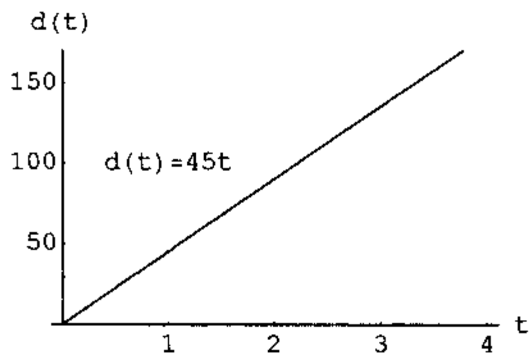
(b) $m \approx \frac{10 - 68}{4 - 0.4} = \frac{-58}{3.6} \approx -16.1$ degrees/inch

(c) $m \approx \frac{5 - 10}{4.7 - 4} = \frac{-5}{0.7} = -7.1$ degrees/inch

(d) Best insulator: Fiberglass insulation
Poorest insulator: Gypsum wallboard
The best insulator will have the largest temperature change per inch, because that will allow larger temperature differences on opposite sides of thinner layers.

27. Slope: $k = \frac{\Delta p}{\Delta d} = \frac{10.94 - 1}{100 - 0} = \frac{9.94}{100} = 0.0994$ atmospheres per meter
At 50 meters, the pressure is $p = 0.0994(50) + 1 = 5.97$ atmospheres.

28. (a) $d(t) = 45t$
(b)



(c) The slope is 45, which is the speed in miles per hour.
(d) Suppose the car has been traveling 45 mph for several hours when it is first observed at point P at time $t = 0$.
(e) The car starts at time $t = 0$ at a point 30 miles past P.

29. (a) Suppose x°F is the same as x°C.

$x = \frac{9}{5}x + 32$

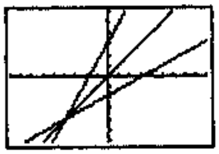
$\left(1 - \frac{9}{5}\right)x = 32$

$-\frac{4}{5}x = 32$

$x = -40$

Yes, −40°F is the same as −40°C.

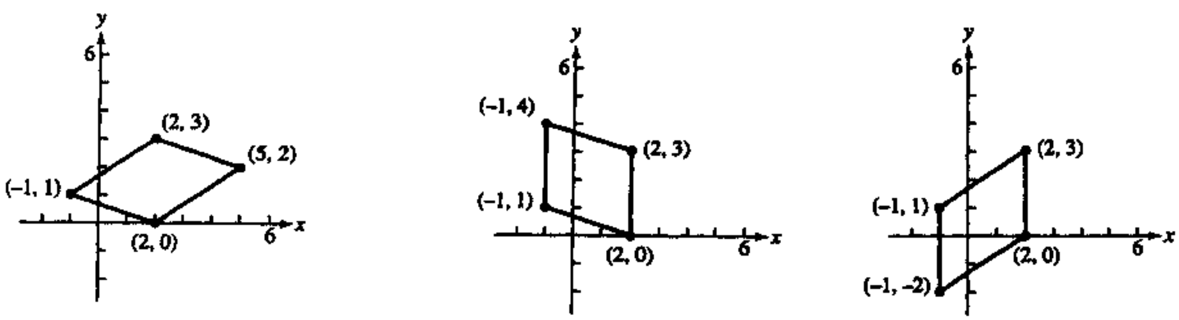
(b)



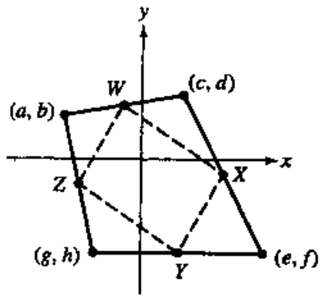
$[-90, 90]$ by $[-60, 60]$

It is related because all three lines pass through the point $(-40, -40)$ where the Fahrenheit and Celsius temperatures are the same.

30. The coordinates of the three missing vertices are $(5, 2)$, $(-1, 4)$ and $(-1, -2)$, as shown below.



31.



Suppose that the vertices of the given quadrilateral are (a, b), (c, d), (e, f), and (g, h). Then the midpoints of the consecutive sides are $W\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$, $X\left(\frac{c+e}{2}, \frac{d+f}{2}\right)$, $Y\left(\frac{e+g}{2}, \frac{f+h}{2}\right)$, and $Z\left(\frac{g+a}{2}, \frac{h+b}{2}\right)$. When these four points are connected, the slopes of the sides of the resulting figure are:

WX: $\dfrac{\frac{d+f}{2}-\frac{b+d}{2}}{\frac{c+e}{2}-\frac{a+c}{2}} = \frac{f-b}{e-a}$

XY: $\dfrac{\frac{f+h}{2}-\frac{d+f}{2}}{\frac{e+g}{2}-\frac{c+e}{2}} = \frac{h-d}{g-c}$

ZY: $\dfrac{\frac{f+h}{2}-\frac{h+b}{2}}{\frac{e+g}{2}-\frac{g+a}{2}} = \frac{f-b}{e-a}$

WZ: $\dfrac{\frac{h+b}{2}-\frac{b+d}{2}}{\frac{g+a}{2}-\frac{a+c}{2}} = \frac{h-d}{g-c}$

Opposite sides have the same slope and are parallel.

32. The radius through (3, 4) has slope $\frac{4-0}{3-0} = \frac{4}{3}$.

The tangent line is perpendicular to this radius, so its slope is $\frac{-1}{4/3} = -\frac{3}{4}$. We seek the line of slope $-\frac{3}{4}$ that passes through (3, 4).

$y = -\frac{3}{4}(x-3)+4$

$y = -\frac{3}{4}x+\frac{9}{4}+4$

$y = -\frac{3}{4}x+\frac{25}{4}$

33. (a) The equation for line L can be written as

$y = -\frac{A}{B}x+\frac{C}{B}$, so its slope is $-\frac{A}{B}$. The perpendicular line has slope $\frac{-1}{-A/B} = \frac{B}{A}$ and passes through (a, b), so its equation is $y = \frac{B}{A}(x-a)+b$.

(b) Substituting $\frac{B}{A}(x-a)+b$ for y in the equation for line L gives:

$Ax + B\left[\frac{B}{A}(x-a)+b\right] = C$

$$A^2x + B^2(x-a) + ABb = AC$$
$$\left(A^2+B^2\right)x = B^2a + AC - ABb$$
$$x = \frac{B^2a + AC - ABb}{A^2+B^2}$$

Substituting the expression for x in the equation for line L gives:

$$A\left(\frac{B^2a + AC - ABb}{A^2+B^2}\right) + By = C$$

$$By = \frac{-A\left(B^2a + AC - ABb\right)}{A^2+B^2} + \frac{C\left(A^2+B^2\right)}{A^2+B^2}$$

$$By = \frac{-AB^2a - A^2C + A^2Bb + A^2C + B^2C}{A^2+B^2}$$

$$By = \frac{A^2Bb + B^2C - AB^2a}{A^2+B^2}$$

$$y = \frac{A^2b + BC - ABa}{A^2+B^2}$$

The coordinates of Q are $\left(\dfrac{B^2a + AC - ABb}{A^2+B^2}, \dfrac{A^2b + BC - ABa}{A^2+B^2}\right)$.

(c) Distance $= \sqrt{(x-a)^2 + (y-b)^2}$

$$= \sqrt{\left(\frac{B^2a + AC - ABb}{A^2+B^2} - a\right)^2 + \left(\frac{A^2b + BC - ABa}{A^2+B^2} - b\right)^2}$$

$$= \sqrt{\left(\frac{B^2a + AC - ABb - a\left(A^2+B^2\right)}{A^2+B^2}\right)^2 + \left(\frac{A^2b + BC - ABa - b\left(A^2+B^2\right)}{A^2+B^2}\right)^2}$$

$$= \sqrt{\left(\frac{AC - ABb - A^2a}{A^2+B^2}\right)^2 + \left(\frac{BC - ABa - B^2b}{A^2+B^2}\right)^2}$$

$$= \sqrt{\left(\frac{A(C - Bb - Aa)}{A^2+B^2}\right)^2 + \left(\frac{B(C - Aa - Bb)}{A^2+B^2}\right)^2}$$

$$= \sqrt{\frac{A^2(C - Aa - Bb)^2}{\left(A^2+B^2\right)^2} + \frac{B^2(C - Aa - Bb)^2}{\left(A^2+B^2\right)^2}}$$

$$= \sqrt{\frac{\left(A^2+B^2\right)(C - Aa - Bb)^2}{\left(A^2+B^2\right)^2}}$$

$$= \sqrt{\frac{(C - Aa - Bb)^2}{A^2+B^2}}$$

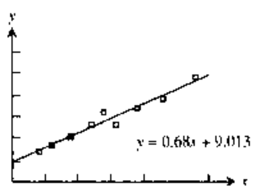
$$= \frac{|C - Aa - Bb|}{\sqrt{A^2 + B^2}}$$
$$= \frac{|Aa + Bb - C|}{\sqrt{A^2 + B^2}}$$

34. The line of incidence passes through $(0,1)$ and $(1,0) \Rightarrow$ The line of reflection passes through $(1,0)$ and $(2,1)$ $\Rightarrow m = \frac{1-0}{2-1} = 1 \Rightarrow y - 0 = 1(x-1) \Rightarrow y = x - 1$ is the line of reflection.

35. $m = \frac{37.1}{100} = \frac{14}{\Delta x} \Rightarrow \Delta x = \frac{14}{.371}$. Therefore, distance between first and last rows is $\sqrt{(14)^2 + \left(\frac{14}{.371}\right)^2} \approx 40.25$ ft.

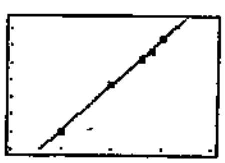
36. (a) $(-1,4)$ (b) $(3,-2)$ (c) $(5,2)$ (d) $(0,x)$
(e) $(-y,0)$ (f) $(-y,x)$ (g) $(3,-10)$

37. (a) $y = 0.680x + 9.013$
(b) The slope is 0.68. It represents the approximate average weight gain in pounds per month.
(c)



(d) When $x = 30$, $y \approx 0.680(30) + 9.013 = 29.413$.
She weighs about 29 pounds.

38. (a) $y = 1060.4233x - 2{,}077{,}548.669$
(b) The slope is 1060.4233. It represents the approximate rate of increase in earnings in dollars per year.
(c)



[1975, 1995] by [20,000, 35,000]

(d) When $x = 2000$, $y \approx 1060.4233(2000) - 2{,}077{,}548.669 \approx 43{,}298$.
In 2000, the construction workers' average annual compensation will be about \$43,298.

39. (a) $y = 5632x - 11{,}080{,}280$
(b) The rate at which the median price is increasing in dollars per year
(c) $y = 2732x - 5{,}362{,}360$
(d) The median price is increasing at a rate of about \$5632 per year in the Northeast, and about \$2732 per year in the Midwest. It is increasing more rapidly in the Northeast.

P.2 FUNCTIONS AND GRAPHS

1. base $= x$; $(\text{height})^2 + \left(\frac{x}{2}\right)^2 = x^2 \Rightarrow$ height $= \frac{\sqrt{3}}{2}x$; area is $a(x) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(x)\left(\frac{\sqrt{3}x}{2}\right) = \frac{\sqrt{3}}{4}x^2$; perimeter is $p(x) = x + x + x = 3x$.

2. $s =$ side length $\Rightarrow s^2 + s^2 = d^2 \Rightarrow s = \frac{d}{\sqrt{2}}$; and area is $a = s^2 \Rightarrow a = \frac{1}{2}d^2$

3. Let $D =$ diagonal of a face of the cube and $\ell =$ the length of an edge. Then $\ell^2 + D^2 = d^2$ and (by Exercise 2) $D^2 = 2\ell^2 \Rightarrow 3\ell^2 = d^2 \Rightarrow \ell = \frac{d}{\sqrt{3}}$. The surface area is $6\ell^2 = \frac{6d^2}{3} = 2d^2$ and the volume is $\ell^3 = \left(\frac{d}{\sqrt{3}}\right)^{3/2}$ $= \frac{d^3}{3\sqrt{3}}$.

4. The coordinates of P are $\left(x, \sqrt{x}\right)$ so the slope of the line joining P to the origin is $m = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$ $(x > 0)$. Thus $\sqrt{x} = \frac{1}{m}$ and the x-coordinate of P is $x = \frac{1}{m^2}$; the y-coordinate of P is $y = \frac{1}{m}$.

5. (a) Not the graph of a function of x since it fails the vertical line test.
   (b) Is the graph of a function of x since any vertical line intersects the graph at most once.

6. (a) Not the graph of a function of x since it fails the vertical line test.
   (b) Not the graph of a function of x since it fails the vertical line test.

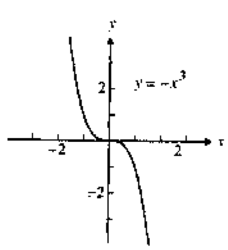
7. (a) domain $= (-\infty, \infty)$; range $= [1, \infty)$ (b) domain $= [0, \infty)$; range $= (-\infty, 1]$

8. (a) domain $= (0, \infty)$; y in range $\Rightarrow y = \frac{1}{\sqrt{t}}$, $t > 0 \Rightarrow y^2 = \frac{1}{t}$ and $y > 0 \Rightarrow$ y can be any positive real number $\Rightarrow$ range $= (0, \infty)$.
   (b) domain $= [0, \infty)$; y in range $\Rightarrow y = \frac{1}{1 + \sqrt{t}}$, $t > 0$. If $t = 0$, then $y = 1$ and as t increases, y becomes a smaller and smaller positive real number $\Rightarrow$ range $= (0, 1]$.

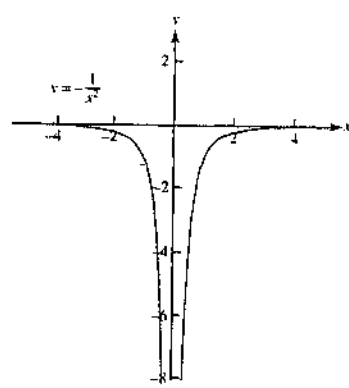
9. $4 - z^2 = (2 - z)(2 + z) \geq 0 \Leftrightarrow z \in [-2, 2] =$ domain. Largest value is $g(0) = \sqrt{4} = 2$ and smallest value is $g(-2) = g(2) = \sqrt{0} = 0 \Rightarrow$ range $= [0, 2]$.

10. domain $= (-\infty, \infty)$; range $= (-\infty, \infty)$

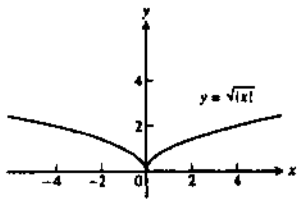
11. (a) Symmetric about the origin



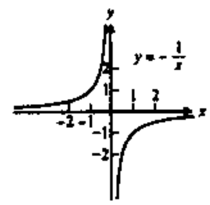
(b) Symmetric about the y-axis



12. (a) Symmetric about the y-axis

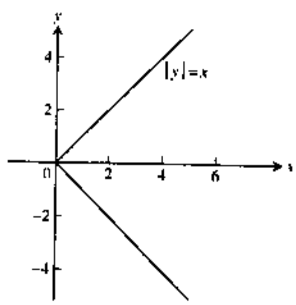


(b) Symmetric about the origin

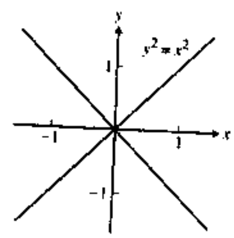


13. Neither graph passes the vertical line test

(a)

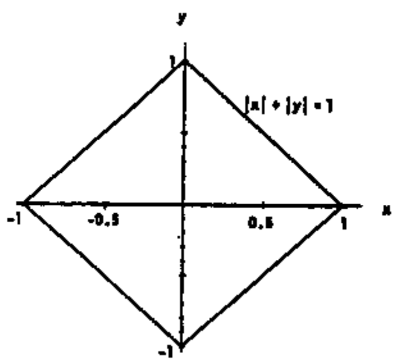


(b)

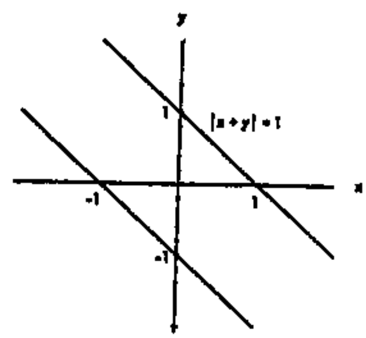


14. Neither graph passes the vertical line test

(a)



(b)



$$|x+y| = 1 \Leftrightarrow \left\{\begin{matrix} x+y=1 \\ \text{or} \\ x+y=-1 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} y=1-x \\ \text{or} \\ y=-1-x \end{matrix}\right\}$$

15. (a) even
    (b) odd

16. (a) even
    (b) neither

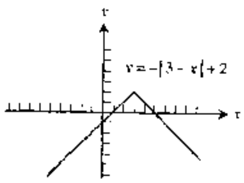
17. (a) odd
    (b) even

18. (a) even
    (b) odd

19. (a) neither
    (b) even

20. (a) even
    (b) even

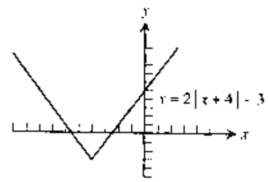
21. (a)



Note that $f(x) = -|x-3| + 2$, so its graph is the graph of the absolute value function reflected across the x-axis and then shifted 3 units right and 2 units upward.

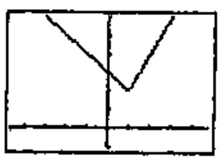
$(-\infty, \infty)$
$(-\infty, 2]$

(b) The graph of f(x) is the graph of the absolute value function stretched vertically by a factor of 2 and then shifted 4 units to the left and 3 units downward



$(-\infty, \infty)$ or all real numbers
$[-3, \infty)$

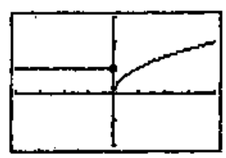
22. (a)



[−4.7, 4.7] by [−1, 6]

$(-\infty, \infty)$ or all real numbers
$[2, \infty)$

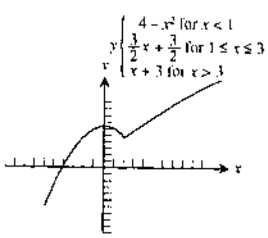
(b)



[−4, 4] by [−2, 3]

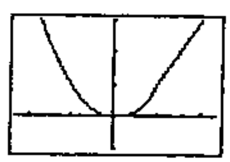
$(-\infty, \infty)$ or all real numbers
$[0, \infty)$

23. (a)



(b) $(-\infty, \infty)$ or all real numbers
(c) $(-\infty, \infty)$ or all real numbers

24. (a)



[−2.35, 2.35] by [−1, 3]

(b) $(-\infty, \infty)$ or all real numbers
(c) $[0, \infty)$

25. Because if the vertical line test holds, then for each x-coordinate, there is at most one y-coordinate giving a point on the curve. This y-coordinate corresponds to the value assigned to the x-coordinate. Since there is only one y-coordinate, the assignment is unique.

26. If the curve is not $y = 0$, there must be a point (x, y) on the curve where $y \neq 0$. That would mean that (x, y) and $(x, -y)$ are two different points on the curve and it is not the graph of a function, since it fails the vertical line test.

27. (a) Line through $(0,0)$ and $(1,1)$: $y = x$
Line through $(1,1)$ and $(2,0)$: $y = -x + 2$

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ -x+2, & 1 < x \le 2 \end{cases}$$

(b) $$f(x) = \begin{cases} 2, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 0, & 3 \le x \le 4 \end{cases}$$

(c) Line through $(0,2)$ and $(2,0)$: $y = -x + 2$

Line through $(2,1)$ and $(5,0)$: $m = \frac{0-1}{5-2} = \frac{-1}{3} = -\frac{1}{3}$, so $y = -\frac{1}{3}(x-2) + 1 = -\frac{1}{3}x + \frac{5}{3}$

$$f(x) = \begin{cases} -x+2, & 0 < x \le 2 \\ -\frac{1}{3}x + \frac{5}{3}, & 2 < x \le 5 \end{cases}$$

(d) Line through $(-1,0)$ and $(0,-3)$: $m = \frac{-3-0}{0-(-1)} = -3$, so $y = -3x - 3$

Line through $(0,3)$ and $(2,-1)$: $m = \frac{-1-3}{2-0} = \frac{-4}{2} = -2$, so $y = -2x + 3$

$$f(x) = \begin{cases} -3x-3, & -1 < x \le 0 \\ -2x+3, & 0 < x \le 2 \end{cases}$$

28. (a) Line through $(-1,1)$ and $(0,0)$: $y = -x$

Line through $(0,1)$ and $(1,1)$: $y = 1$

Line through $(1,1)$ and $(3,0)$: $m = \frac{0-1}{3-1} = \frac{-1}{2} = -\frac{1}{2}$, so $y = -\frac{1}{2}(x-1) + 1 = -\frac{1}{2}x + \frac{3}{2}$

$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 1, & 0 < x \le 1 \\ -\frac{1}{2}x + \frac{3}{2}, & 1 < x < 3 \end{cases}$$

(b) Line through $(-2,-1)$ and $(0,0)$: $y = \frac{1}{2}x$

Line through $(0,2)$ and $(1,0)$: $y = -2x + 2$

Line through $(1,-1)$ and $(3,-1)$: $y = -1$

$$f(x) = \begin{cases} \frac{1}{2}x, & -2 \le x \le 0 \\ -2x+2, & 0 < x \le 1 \\ -1, & 1 < x \le 3 \end{cases}$$

(c) Line through $\left(\frac{T}{2}, 0\right)$ and $(T, 1)$: $m = \frac{1-0}{T-(T/2)} = \frac{2}{T}$, so $y = \frac{2}{T}\left(x - \frac{T}{2}\right) + 0 = \frac{2}{T}x - 1$

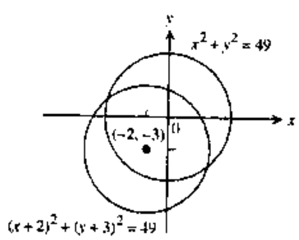
$$f(x) = \begin{cases} 0, & 0 \le x \le \frac{T}{2} \\ \frac{2}{T}x - 1, & \frac{T}{2} < x \le T \end{cases}$$

(d) $$f(x) = \begin{cases} A, & 0 \le x < \frac{T}{2} \\ -A, & \frac{T}{2} \le x < T \\ A, & T \le x < \frac{3T}{2} \\ -A, & \frac{3T}{2} \le x \le 2T \end{cases}$$

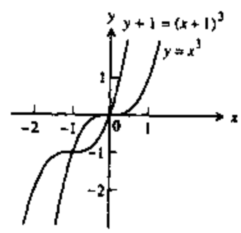
29. (a) Position 4 (b) Position 1 (c) Position 2 (d) Position 3

30. (a) $y = -(x-1)^2 + 4$ (b) $y = -(x+2)^2 + 3$ (c) $y = -(x+4)^2 - 1$ (d) $y = -(x-2)^2$

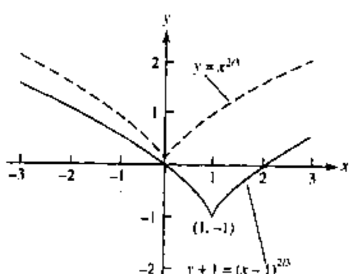
31.



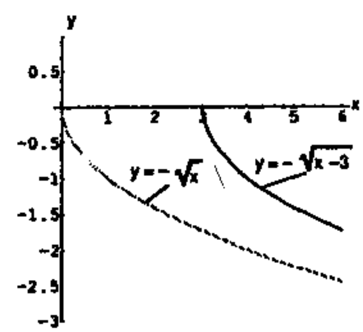
32.



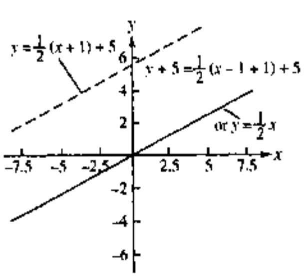
33.



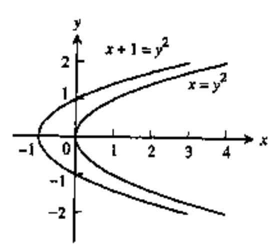
34.



35.



36.



37. (a) $f(g(0)) = f(-3) = 2$

(b) $g(f(0)) = g(5) = 22$

(c) $f(g(x)) = f(x^2 - 3) = x^2 - 3 + 5 = x^2 + 2$

(d) $g(f(x)) = g(x + 5) = (x + 5)^2 - 3 = x^2 + 10x + 22$

(e) $f(f(-5)) = f(0) = 5$

(f) $g(g(2)) = g(1) = -2$

(g) $f(f(x)) = f(x + 5) = (x + 5) + 5 = x + 10$

(h) $g(g(x)) = g(x^2 - 3) = (x^2 - 3)^2 - 3 = x^4 - 6x^2 + 6$

38. (a) $f\left(g\left(\frac{1}{2}\right)\right) = f\left(\frac{2}{3}\right) = -\frac{1}{3}$

(b) $g\left(f\left(\frac{1}{2}\right)\right) = g\left(-\frac{1}{2}\right) = 2$

(c) $f(g(x)) = f\left(\frac{1}{x+1}\right) = \frac{1}{x+1} - 1 = \frac{-x}{x+1}$

(d) $g(f(x)) = g(x - 1) = \frac{1}{(x-1)+1} = \frac{1}{x}$

(e) $f(f(2)) = f(1) = 0$

(f) $g(g(2)) = g\left(\frac{1}{3}\right) = \frac{1}{\frac{4}{3}} = \frac{3}{4}$

(g) $f(f(x)) = f(x - 1) = (x - 1) - 1 = x - 2$

(h) $g(g(x)) = g\left(\frac{1}{x+1}\right) = \frac{1}{\frac{1}{x+1} + 1} = \frac{x+1}{x+2}$ $(x \neq -1 \text{ and } x \neq -2)$

39. (a) $u(v(f(x))) = u\left(v\left(\frac{1}{x}\right)\right) = u\left(\frac{1}{x^2}\right) = 4\left(\frac{1}{x}\right)^2 - 5 = \frac{4}{x^2} - 5$

(b) $u(f(v(x))) = u\left(f\left(x^2\right)\right) = u\left(\frac{1}{x^2}\right) = 4\left(\frac{1}{x^2}\right) - 5 = \frac{4}{x^2} - 5$

(c) $v(u(f(x))) = v\left(u\left(\frac{1}{x}\right)\right) = v\left(4\left(\frac{1}{x}\right) - 5\right) = \left(\frac{4}{x} - 5\right)^2$

(d) $v(f(u(x))) = v(f(4x - 5)) = v\left(\frac{1}{4x-5}\right) = \left(\frac{1}{4x-5}\right)^2$

(e) $f(u(v(x))) = f\left(u\left(x^2\right)\right) = f\left(4\left(x^2\right) - 5\right) = \frac{1}{4x^2 - 5}$

(f) $f(v(u(x))) = f(v(4x - 5)) = f\left((4x - 5)^2\right) = \frac{1}{(4x-5)^2}$

40. (a) $h(g(f(x))) = h\left(g\left(\sqrt{x}\right)\right) = h\left(\frac{\sqrt{x}}{4}\right) = 4\left(\frac{\sqrt{x}}{4}\right) - 8 = \sqrt{x} - 8$

(b) $h(f(g(x))) = h\left(f\left(\frac{x}{4}\right)\right) = h\left(\sqrt{\frac{x}{4}}\right) = 4\sqrt{\frac{x}{4}} - 8 = 2\sqrt{x} - 8$

(c) $g(h(f(x))) = g\left(h\left(\sqrt{x}\right)\right) = g\left(4\sqrt{x} - 8\right) = \frac{4\sqrt{x} - 8}{4} = \sqrt{x} - 2$

(d) $g(f(h(x))) = g(f(4x - 8)) = g\left(\sqrt{4x - 8}\right) = \frac{\sqrt{4x - 8}}{4} = \frac{\sqrt{x - 2}}{2}$

(e) $f(g(h(x))) = f(g(4x - 8)) = f\left(\frac{4x - 8}{4}\right) = f(x - 2) = \sqrt{x - 2}$

(f) $f(h(g(x))) = f\left(h\left(\frac{x}{4}\right)\right) = f\left(4\left(\frac{x}{4}\right) - 8\right) = f(x - 8) = \sqrt{x - 8}$

41. (a) $y = g(f(x))$
(b) $y = j(g(x))$
(c) $y = g(g(x))$
(d) $y = j(j(x))$
(e) $y = g(h(f(x)))$
(f) $y = h(j(f(x)))$

42. (a) $y = f(j(x))$
(b) $y = h(g(x)) = g(h(x))$
(c) $y = h(h(x))$
(d) $y = f(f(x))$
(e) $y = j(g(f(x)))$
(f) $y = g(f(h(x)))$

43. (a) Since $(f \circ g)(x) = \sqrt{g(x) - 5} = \sqrt{x^2 - 5}$, $g(x) = x^2$.

(b) Since $(f \circ g)(x) = 1 + \frac{1}{g(x)} = x$, we know that $\frac{1}{g(x)} = x - 1$, so $g(x) = \frac{1}{x - 1}$.

(c) Since $(f \circ g)(x) = f\left(\frac{1}{x}\right) = x$, $f(x) = \frac{1}{x}$.

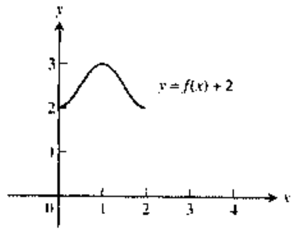
(d) Since $(f \circ g)(x) = f\left(\sqrt{x}\right) = |x|$, $f(x) = x^2$.
The completed table is shown. Note that the absolute value sign in part (d) is optional.

$g(x)$	$f(x)$	$(f \circ g)(x)$
x^2	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
$\frac{1}{x - 1}$	$1 + \frac{1}{x}$	$x, x \neq -1$
$\frac{1}{x}$	$\frac{1}{x}$	$x, x \neq 0$
$\sqrt{x}$	x^2	$\|x\|, x \geq 0$

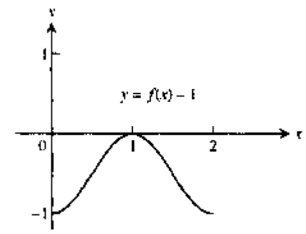
44.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
(a)	$x - 7$	$\sqrt{x}$	$\sqrt{x - 7}$
(b)	$x + 2$	$3x$	$3(x + 2) = 3x + 6$
(c)	x^2	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
(d)	$\frac{x}{x - 1}$	$\frac{x}{x - 1}$	$\frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{x}{x - (x - 1)} = x$
(e)	$\frac{1}{x - 1}$	$1 + \frac{1}{x}$	$1 + \frac{1}{\frac{1}{x-1}} = 1 + (x - 1) = x$
(f)	$\frac{1}{x}$	$\frac{1}{x}$	$\frac{1}{\frac{1}{x}} = x$

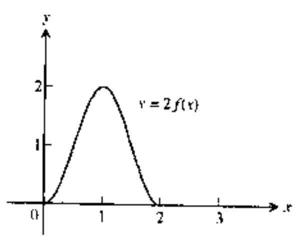
45. (a) domain: $[0,2]$; range: $[2,3]$



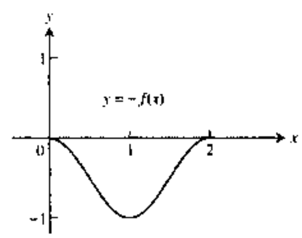
(b) domain: $[0,2]$; range: $[-1,0]$



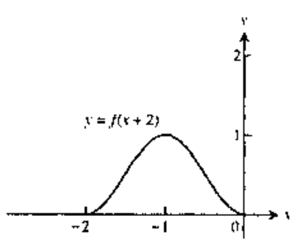
(c) domain: $[0,2]$; range: $[0,2]$



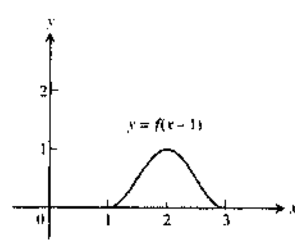
(d) domain: $[0,2]$; range: $[-1,0]$



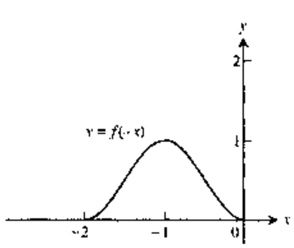
(e) domain: $[-2,0]$; range: $[0,1]$



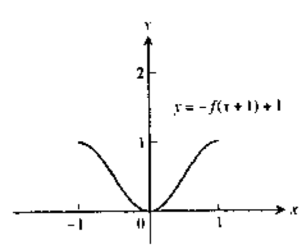
(f) domain: $[1,3]$; range: $[0,1]$



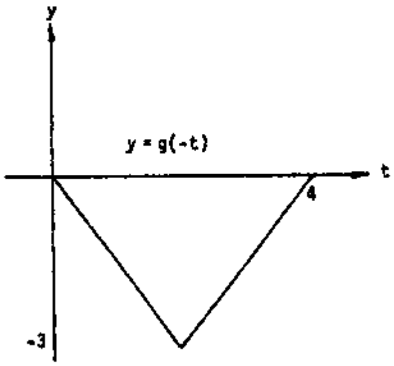
(g) domain: $[-2,0]$; range: $[0,1]$



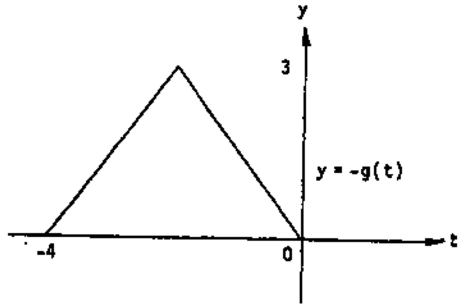
(h) domain: $[-1,1]$; range: $[0,1]$



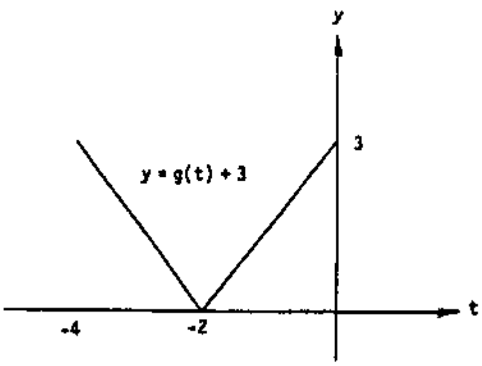
46. (a) domain: $[0,4]$; range: $[-3,0]$



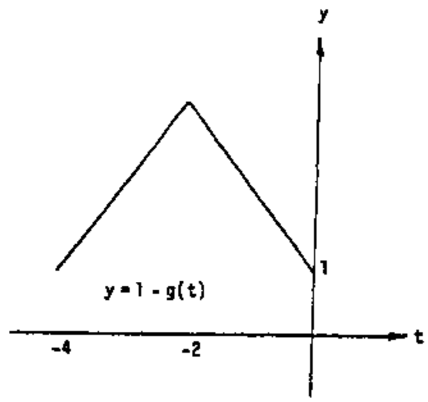
(b) domain: $[-4,0]$; range: $[0,3]$



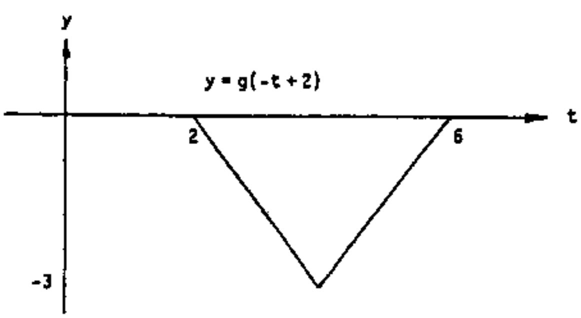
(c) domain: $[-4,0]$; range: $[0,3]$



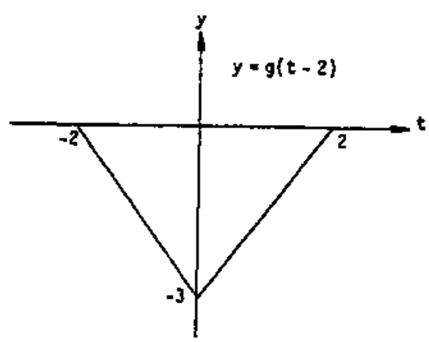
(d) domain: $[-4,0]$; range: $[1,4]$



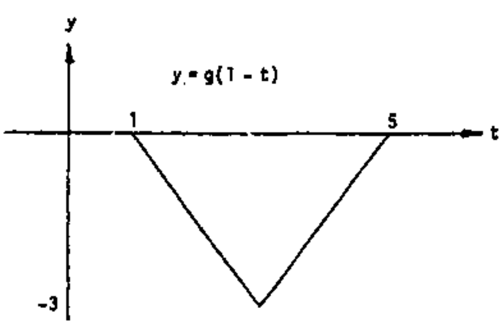
(e) domain: $[2,6]$; range: $[-3,0]$



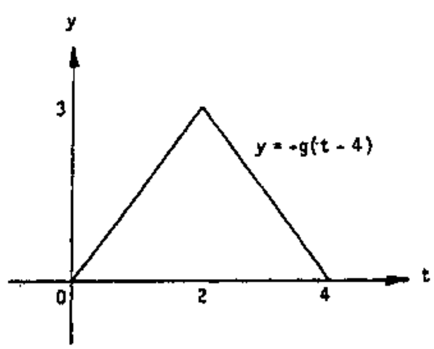
(f) domain: $[-2,2]$; range: $[-3,0]$



(g) domain: $[1,5]$; range: $[-3,0]$



(h) domain: $[0,4]$; range: $[0,3]$



47. (a) Because the circumference of the original circle was 8π and a piece of length x was removed.

(b) $r = \frac{8\pi - x}{2\pi} = 4 - \frac{x}{2\pi}$

(c) $h = \sqrt{16 - r^2} = \sqrt{16 - \left(4 - \frac{x}{2\pi}\right)} = \sqrt{16 - \left(16 - \frac{4x}{\pi} + \frac{x^2}{4\pi^2}\right)} = \sqrt{\frac{4x}{\pi} - \frac{x^2}{4\pi^2}} = \sqrt{\frac{16\pi x}{4\pi^2} - \frac{x^2}{4\pi^2}} = \frac{\sqrt{16\pi x - x^2}}{2\pi}$

(d) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{8\pi - x}{2\pi}\right)^2 \cdot \frac{\sqrt{16\pi x - x^2}}{2\pi} = \frac{(8\pi - x)^2\sqrt{16\pi x - x^2}}{24\pi^2}$

48. (a) Note that 2 mi = 10,560 ft, so there are $\sqrt{800^2 + x^2}$ feet of river cable at \$180 per foot and (10,560 − x) feet of land cable at \$100 per foot. The cost is $C(x) = 180\sqrt{800^2 + x^2} + 100(10{,}560 - x)$

(b)
$C(0) = \$1{,}200{,}000$
$C(500) \approx \$1{,}175{,}812$
$C(1000) \approx \$1{,}186{,}512$
$C(1500) = \$1{,}212{,}000$
$C(2000) \approx \$1{,}243{,}732$
$C(2500) \approx \$1{,}278{,}479$
$C(3000) \approx \$1{,}314{,}870$

Values beyond this are all larger. It would appear that the least expensive location is less than 2000 ft from point P.

49. (a) Yes. Since $(f \cdot g)(-x) = f(-x) \cdot g(-x) = f(x) \cdot g(x) = (f \cdot g)(x)$, the function $(f \cdot g)(x)$ will also be even.

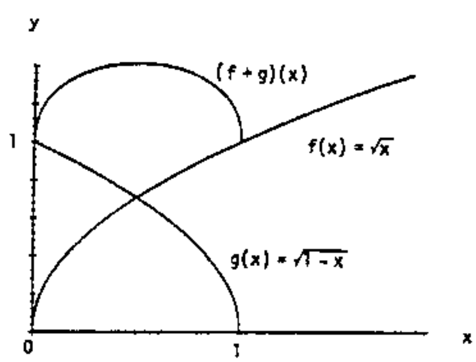
(b) The product will be even, since
$$\begin{aligned}(f \cdot g)(-x) &= f(-x) \cdot g(-x) \\ &= (-f(x)) \cdot (-g(x)) \\ &= f(x) \cdot g(x) \\ &= (f \cdot g)(x).\end{aligned}$$

(c) Yes, $f(x) = 0$ is both even and odd since $f(-x) = -f(x) = f(x)$

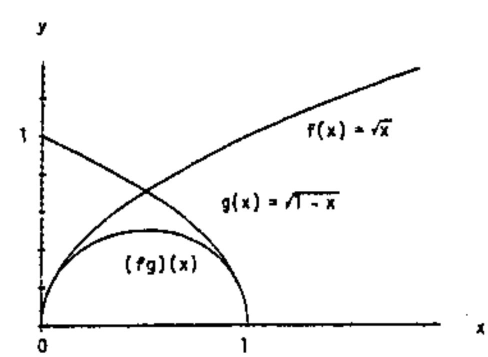
50. (a) Pick 11, for example: $11 + 5 = 16 \to 2 \cdot 16 = 32 \to 32 - 6 = 26 \to 26/2 = 13 \to 13 - 2 = 11$, the original number.

(b) $f(x) = \frac{2(x+5) - 6}{2} - 2 = x$, the number you started with.

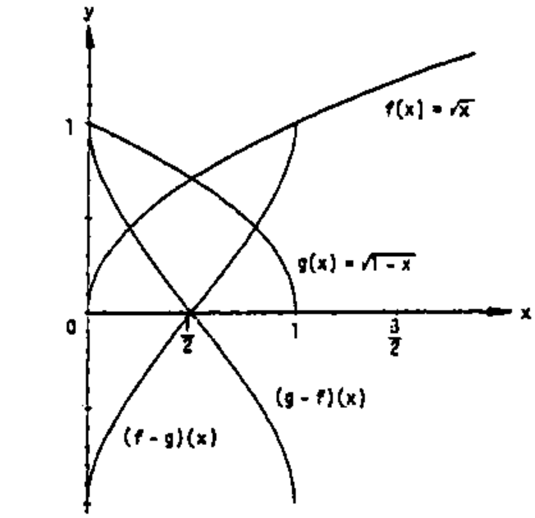
51. (a)



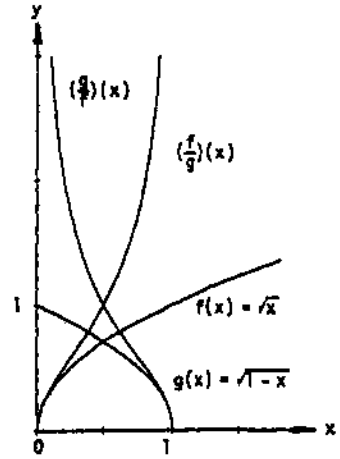
(b)



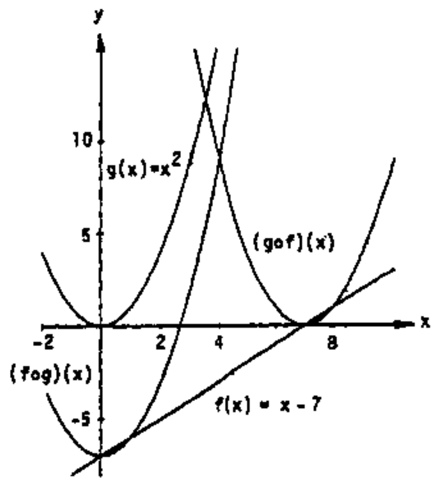
(c)



(d)

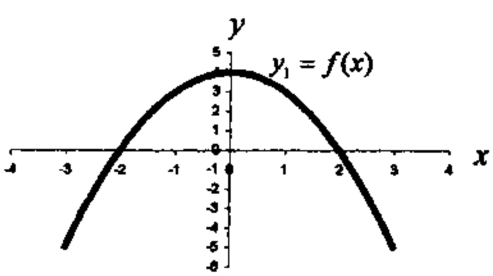


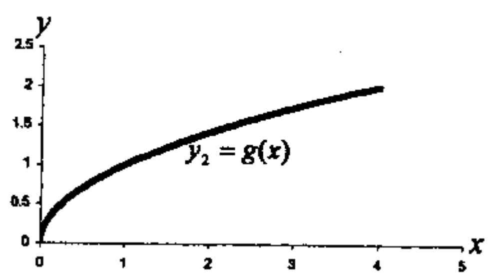
52.

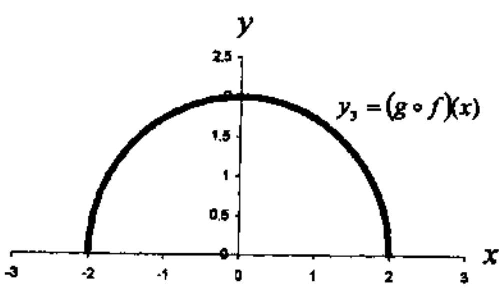


53. (a) $y_4 = (f \circ g)(x)$; $y_3 = (g \circ f)(x)$

(b)

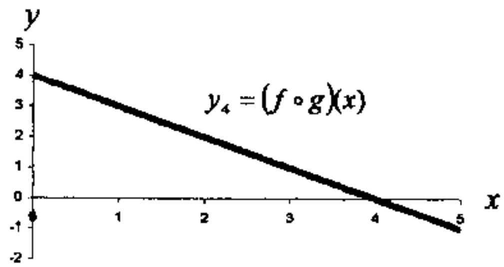






$D(g \circ f) = [-2, 2]$; The domain of $g \circ f$ is the set of all values of x in the domain of f for which the values $y_1 = f(x)$ are in the domain of g.
$R(g \circ f) = [0, 2]$; The range of $g \circ f$ is the subset of the range of g that includes all the values of g(x) evaluated at the values from the range of f where g(x) is defined.

(c) The graphs of $y_1 = f(x)$ and $y_2 = g(x)$ are shown in part (a).



$D(f \circ g) = [0, \infty)$; The domain of $f \circ g$ is the set of all values of x in the domain of g for which the values $y_2 = g(x)$ are in the domain of f.
$R(f \circ g) = (-\infty, 4]$; The range of $f \circ g$ is the subset of the range of f that includes all the values of f(x) evaluated at the values from the range of g where f(x) is defined.

(d) $(g \circ f)(x) = \sqrt{4 - x^2}$; $D(g \circ f) = [-2, 2]$; $R(g \circ f) = [0, 2]$
$(f \circ g)(x) = 4 - (\sqrt{x})^2 = 4 - x$ for $x \geq 0$; $D(f \circ g) = [0, \infty)$; $R(f \circ g) = (-\infty, 4]$

54. (a)

$[-3, 3]$ by $[-1, 3]$

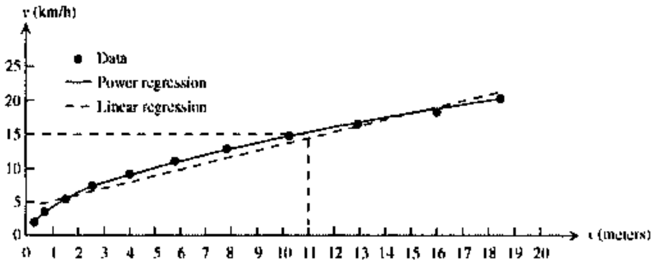
(b) Domain of y_1: $[0, \infty)$
Domain of y_2: $(-\infty, 1]$
Domain of y_3: $[0, 1]$

(c) The functions $y_1 - y_2$, $y_2 - y_1$, and $y_1 \cdot y_2$ all have domain $[0, 1]$, the same as the domain of $y_1 + y_2$ found in part (b).
Domain of $\frac{y_1}{y_2}$: $[0, 1)$
Domain of $\frac{y_2}{y_1}$: $(0, 1]$

(d) The domain of a sum, difference, or product of two functions is the intersection of their domains.
The domain of a quotient of two functions is the intersection of their domains with any zeros of the denominator removed.

55. (a) The power regression function on the TI-92 Plus calculator gives $y = 4.44647x^{0.511414}$

(b)

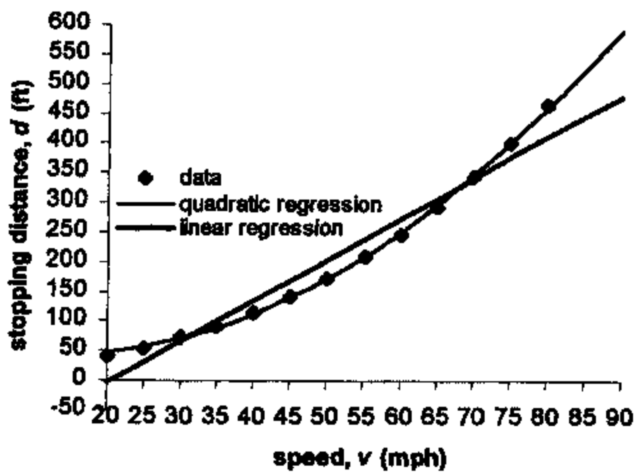


(c) 15.2 km/h

(d) The linear regression function on the TI-92 Plus calculator gives $y = 0.913675x + 4.189976$ and it is shown on the graph in part (b). The linear regression function gives a speed of 14.2 km/h when $y = 11$m. The power regression curve in part (a) better fits the data.

56. (a) Let v represent the speed in miles per hour and d the stopping distance in feet. The quadratic regression function on the TI-92 Plus calculator gives $d = 0.0886v^2 - 1.97v + 50.1$.

(b)



(c) From the graph in part (b), the stopping distance is about 370 feet when the vehicle speed is 72 mph and it is about 525 feet when the speed is 85 mph.
Algebraically: $d_{quadratic}(72) = 0.0886(72)^2 - 1.97(72) + 50.1 = 367.6$ ft.

$$d_{quadratic}(85) = 0.0886(85)^2 - 1.97(85) + 50.1 = 522.8 \text{ ft.}$$

(d) The linear regression function on the TI-92 Plus calculator gives $d = 6.89v - 140.4 \Rightarrow d_{linear}(72)$ $= 6.89(72) - 140.4 = 355.7$ ft and $d_{linear}(85) = 6.89(85) - 140.4 = 445.2$ ft. The linear regression line is shown on the graph in part (b). The quadratic regression curve clearly gives the better fit.

P.3 EXPONENTIAL FUNCTIONS

1. The graph of $y = 2^x$ is increasing from left to right and has the negative x-axis as an asymptote. (a)

2. The graph of $y = 3^{-x}$ or, equivalently, $y = \left(\frac{1}{3}\right)^x$, is decreasing from left to right and has the positive x-axis as an asymptote. (d)

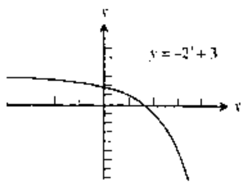
3. The graph of $y = -3^{-x}$ is the reflection about the x-axis of the graph in Exercise 2. (e)

4. The graph of $y = -0.5^{-x}$ or, equivalently, $y = -2^{x}$, is the reflection about the x-axis of the graph in Exercise 1. (c)

5. The graph of $y = 2^{-x} - 2$ is decreasing from left to right and has the line $y = -2$ as an asymptote. (b)

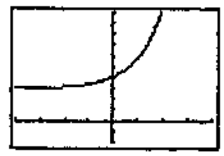
6. The graph of $y = 1.5^{x} - 2$ is increasing from left to right and has the line $y = -2$ as an asymptote. (f)

7.



Domain: $(-\infty, \infty)$
Range: $(-\infty, 3)$
x-intercept: ≈ 1.585
y-intercept: 2

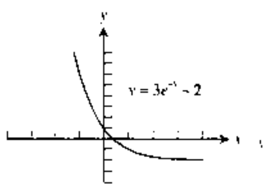
8.



[−4, 4] by [−2, 10]

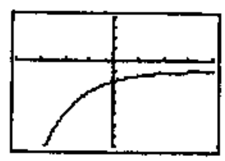
Domain: $(-\infty, \infty)$
Range: $(3, \infty)$
x-intercept: None
y-intercept: 4

9.



Domain: $(-\infty, \infty)$
Range: $(-2, \infty)$
x-intercept: ≈ 0.405
y-intercept: 1

10.



[−4, 4] by [−8, 4]

Domain: $(-\infty, \infty)$
Range: $(-\infty, -1)$
x-intercept: None
y-intercept: -2

11. $9^{2x} = \left(3^2\right)^{2x} = 3^{4x}$

12. $16^{3x} = \left(2^4\right)^{3x} = 2^{12x}$

13. $\left(\frac{1}{8}\right)^{2x} = \left(2^{-3}\right)^{2x} = 2^{-6x}$

14. $\left(\frac{1}{27}\right)^{x} = \left(3^{-3}\right)^{x} = 3^{-3x}$

15.

x	y	Δy
1	−1	
		2
2	1	
		2
3	3	
		2
4	5	

16.

x	y	Δy
1	1	
		−3
2	−2	
		−3
3	−5	
		−3
4	−8	

17.

x	y	Δy
1	1	
		3
2	4	
		5
3	9	
		7
4	16	

18.

x	y	ratio
1	8.155	
		2.718
2	22.167	
		2.718
3	60.257	
		2.718
4	163.79	

19. The slope of a straight line is $m = \frac{\Delta y}{\Delta x} \to \Delta y = m(\Delta x)$. In Exercise 15, each $\Delta x = 1$ and $m = 2 \to$ each $\Delta y = 2$, and in problem 16, each $\Delta x = 1$ and $m = -3 \to$ each $\Delta y = -3$. If the changes in x are constant for a linear function, say $\Delta x = c$, then the changes in y are also constant, specifically, $\Delta y = mc$.

20. From the table in Exercise 17, it can be seen that $\Delta y = 2x + 1$. Some examples are: $\Delta y = 9 - 4 = 5 = 2(2) + 1 = 2x + 1$ and $\Delta y = 16 - 9 = 7 = 2(3) + 1 = 2x + 1$. As x changes from $x = 1000$ to $x = 1001$, the change in y is $\Delta y = 2(1000) + 1 = 2001$. As x changes from n to $n + 1$, where n is an arbitrary positive integer, the change in y is $\Delta y = 2n + 1$.

21. Since $f(1) = 4.5$ we have $ka = 4.5$, and since $f(-1) = 0.5$ we have $ka^{-1} = 0.5$.
Dividing, we have

$$\frac{ka}{ka^{-1}} = \frac{4.5}{0.5}$$

$$a^2 = 9$$

$$a = \pm 3$$

Since $f(x) = k \cdot a^x$ is an exponential function, we require $a > 0$, so $a = 3$. Then $ka = 4.5$ gives $3k = 4.5$, so $k = 1.5$. The values are $a = 3$ and $k = 1.5$.

22. Since $f(1) = 1.5$ we have $ka = 1.5$, and since $f(-1) = 6$ we have $ka^{-1} = 6$.
Dividing, we have

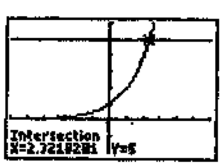
$$\frac{ka}{ka^{-1}} = \frac{1.5}{6}$$

$$a^2 = 0.25$$

$$a = \pm 0.5$$

Since $f(x) = k \cdot a^x$ is an exponential function, we require $a > 0$, so $a = 0.5$. Then $ka = 1.5$ gives $0.5k = 1.5$, so $k = 3$. The values are $a = 0.5$ and $k = 3$.

23.



[−6, 6] by [−2, 6]

$x \approx 2.3219$

24.

[−6, 6] by [−2, 6]

$x \approx 1.3863$

25.

[−6, 6] by [−3, 5]

$x \approx -0.6309$

26.

[−6, 6] by [−3, 5]

$x \approx -1.5850$

27. $5422(1.018)^{19} \approx 7609.7$ million

28. (a) When $t = 0$, $B = 100e^0 = 100$. There were 100 bacteria present initially.
    (b) When $t = 6$, $B = 100e^{0.693(6)} \approx 6394.351$. After 6 hours, there are about 6394 bacteria.
    (c) Solving $100e^{0.693t} = 200$ graphically, we find that $t \approx 1.000$. The population will be 200 after about 1 hour. Since the population doubles (from 100 to 200) in about 1 hour, the doubling time is about 1 hour.

29. Let t be the number of years. Solving $500{,}000(1.0375)^t = 1{,}000{,}000$ graphically, we find that $t \approx 18.828$. The population will reach 1 million in about 19 years.

30. (a) The population is given by $P(t) = 6250(1.0275)^t$, where t is the number of years after 1890.
    Population in 1915: $P(25) \approx 12{,}315$
    Population in 1940: $P(50) \approx 24{,}265$
    (b) Solving $P(t) = 50{,}000$ graphically, we find that $t \approx 76.651$. The population reached 50,000 about 77 years after 1890, in 1967.

31. (a) $A(t) = 6.6\left(\frac{1}{2}\right)^{t/14}$
    (b) Solving $A(t) = 1$ graphically, we find that $t \approx 38$. There will be 1 gram remaining after about 38.1145 days.

32. Let t be the number of years. Solving $2300(1.06)^t = 4150$ graphically, we find that $t \approx 10.129$. It will take about 10.129 years. (If the interest is not credited to the account until the end of each year, it will take 11 years.)

33. Let A be the amount of the initial investment, and let t be the number of years. We wish to solve $A(1.0625)^t = 2A$, which is equivalent to $1.0625^t = 2$. Solving graphically, we find that $t \approx 11.433$. It will take about 11.433 years. (If the interest is credited at the end of each year, it will take 12 years.)

34. Let A be the amount of the initial investment, and let t be the number of years. We wish to solve $A\left(1 + \frac{0.0625}{12}\right)^{12t} = 2A$, which is equivalent to $\left(1 + \frac{0.0625}{12}\right)^{12t} = 2$. Solving graphically, we find that $t \approx 11.119$. It will take about 11.119 years. (If the interest is credited at the end of each month, it will take 11 years 2 months.)

35. Let A be the amount of the initial investment, and let t be the number of years. We wish to solve $Ae^{0.0625t} = 2A$, which is equivalent to $e^{0.0625t} = 2$. Solving graphically, we find that $t \approx 11.090$. It will take about 11.090 years.

36. Let A be the amount of the initial investment, and let t be the number of years. We wish to solve $A(1.0575)^t = 3A$, which is equivalent to $1.0575^t = 3$. Solving graphically, we find that $t \approx 19.650$. It will take about 19.650 years. (If the interest is credited at the end of each year, it will take 20 years.)

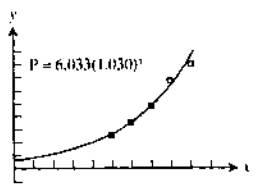
37. Let A be the amount of the initial investment, and let t be the number of years. We wish to solve $A\left(1+\frac{0.0575}{365}\right)^{365t} = 3A$, which is equivalent to $\left(1+\frac{0.0575}{365}\right)^{365t} = 3$. Solving graphically, we find that $t \approx 19.108$. It will take about 19.108 years.

38. Let A be the amount of the initial investment, and let t be the number of years. We wish to solve $Ae^{0.0575t} = 3A$, which is equivalent to $e^{0.0575t} = 3$. Solving graphically, we find that $t \approx 19.106$. It will take about 19.106 years.

39. After t hours, the population is $P(t) = 2^{t/0.5}$ or, equivalently, $P(t) = 2^{2t}$. After 24 hours, the population is $P(24) = 2^{48} \approx 2.815 \times 10^{14}$ bacteria.

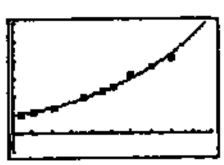
40. (a) Each year, the number of cases is $100\% - 20\% = 80\%$ of the previous year's number of cases. After t years, the number of cases will be $C(t) = 10{,}000(0.8)^t$. Solving $C(t) = 1000$ graphically, we find that $t \approx 10.319$. It will take about 10.319 years.
(b) Solving $C(t) = 1$ graphically, we find that $t \approx 41.275$. It will take about 41.275 years.

41. (a) Let $x = 0$ represent 1900, $x = 1$ represent 1901, and so on. The regression equation is $P(x) = 6.033(1.030)^x$.



(b) The regression equation gives an estimate of $P(0) \approx 6.03$ million, which is not very close to the actual population.
(c) Since the equation is of the form $P(x) = P(0) \cdot 1.030^x$, the annual rate of growth is about 3%.

42. (a) The regression equation is $P(x) = 4.831(1.019)^x$.



[0, 100] by [−5, 30]

(b) $P(90) \approx 26.3$ million
(c) Since the equation is of the form $P(x) = P(0) \cdot 1.019^x$, the annual rate of growth is approximately 1.9%.

P.4 FUNCTIONS AND LOGARITHMS

1. Yes one-to-one, the graph passes the horizontal test.

2. Not one-to-one, the graph fails the horizontal test.

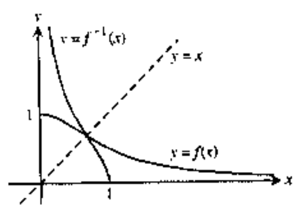
3. Not one-to-one since (for example) the horizontal line $y = 2$ intersects the graph twice.

4. Not one-to-one, the graph fails the horizontal test

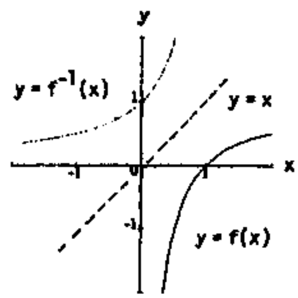
5. Yes one-to-one, the graph passes the horizontal test

6. Yes one-to-one, the graph passes the horizontal test

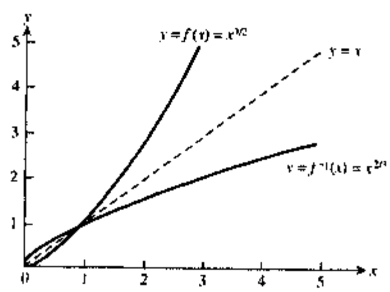
7. Domain: $0 < x \leq 1$, Range: $y \geq 0$



8. Domain: $x < 1$, Range: $y > 0$

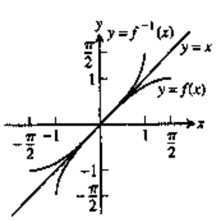


9.



Domain: $x \geq 0$, Range: $y \geq 0$

10. Domain: $-1 \leq x \leq 1$, Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



11. Step 1: $y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y-1}$
Step 2: $y = \sqrt{x-1} = f^{-1}(x)$

12. Step 1: $y = x^2 \Rightarrow x = -\sqrt{y}$
Step 2: $y = -\sqrt{x} = f^{-1}(x)$

13. Step 1: $y = x^3 - 1 \Rightarrow x^3 = y + 1 \Rightarrow x = (y+1)^{1/3}$
Step 2: $y = \sqrt[3]{x+1} = f^{-1}(x)$

14. Step 1: $y = x^2 - 2x + 1 \Rightarrow y = (x-1)^2 \Rightarrow \sqrt{y} = x - 1 \Rightarrow x = \sqrt{y} + 1$
Step 2: $y = 1 + \sqrt{x} = f^{-1}(x)$

15. Step 1: $y = (x+1)^2 \Rightarrow \sqrt{y} = x + 1 \Rightarrow x = \sqrt{y} - 1$
Step 2: $y = \sqrt{x} - 1 = f^{-1}(x)$

16. Step 1: $y = x^{2/3} \Rightarrow x = y^{3/2}$
Step 2: $y = x^{3/2} = f^{-1}(x)$

17. $y = 2x + 3 \to y - 3 = 2x \to \frac{y-3}{2} = x$. Interchange x and y: $\frac{x-3}{2} = y \to f^{-1}(x) = \frac{x-3}{2}$
Verify.
$(f \circ f^{-1})(x) = f\left(\frac{x-3}{2}\right) = 2\left(\frac{x-3}{2}\right) + 3 = (x-3) + 3 = x$
$(f^{-1} \circ f)(x) = f^{-1}(2x+3) = \frac{(2x+3)-3}{2} = \frac{2x}{2} = x$

18. $y = 5 - 4x \to 4x = 5 - y \to x = \frac{5-y}{4}$. Interchange x and y: $y = \frac{5-x}{4} \to f^{-1}(x) = \frac{5-x}{4}$
Verify.
$(f \circ f^{-1})(x) = f\left(\frac{5-x}{4}\right) = 5 - 4\left(\frac{5-x}{4}\right) = 5 - (5-x) = x$
$(f^{-1} \circ f)(x) = f^{-1}(5-4x) = \frac{5-(5-4x)}{4} = \frac{4x}{4} = x$

19. $y = x^3 - 1 \to y + 1 = x^3 \to (y+1)^{1/3} = x$. Interchange x and y: $(x+1)^{1/3} = y$
$\to f^{-1}(x) = (x+1)^{1/3}$ or $\sqrt[3]{x+1}$
Verify.
$(f \circ f^{-1})(x) = f\left(\sqrt[3]{x+1}\right) = \left(\sqrt[3]{x+1}\right)^3 - 1 = (x+1) - 1 = x$
$(f^{-1} \circ f)(x) = f^{-1}(x^3 - 1) = \sqrt[3]{(x^3-1)+1} = \sqrt[3]{x^3} = x$

20. $y = x^2 + 1,\ x \geq 0 \to y - 1 = x^2,\ x \geq 0 \to \sqrt{y-1} = x$.
Interchange x and y: $\sqrt{x-1} = y \to f^{-1}(x) = \sqrt{x-1}$ or $(x-1)^{1/2}$
Verify. For $x \geq 1$ (the domain of f^{-1}),
$(f \circ f^{-1})(x) = f\left(\sqrt{x-1}\right) = \left(\sqrt{x-1}\right)^2 + 1 = (x-1) + 1 = x$
For $x > 0$ (the domain of f),
$(f^{-1} \circ f)(x) = f^{-1}(x^2+1) = \sqrt{(x^2+1)-1} = \sqrt{x^2} = |x| = x$

21. $y = x^2,\ x \le 0 \to x = -\sqrt{y}$. Interchange x and y: $y = -\sqrt{x} \to f^{-1}(x) = -\sqrt{x}$ or $-x^{1/2}$

Verify.

For $x \ge 0$ (the domain of f^{-1}), $(f \circ f^{-1})(x) = f(-\sqrt{x}) = (-\sqrt{x})^2 = x$

For $x \le 0$ (the domain of f), $(f^{-1} \circ f)(x) = f^{-1}(x^2) = -\sqrt{x^2} = -|x| = x$

22. $y = x^{2/3},\ x \ge 0 \to y^{3/2} = (x^{2/3})^{3/2},\ x \ge 0 \to y^{3/2} = x$

Interchange x and y: $x^{3/2} = y \to f^{-1}(x) = x^{3/2}$

Verify.

For $x \ge 0$ (the domain of f^{-1}), $(f \circ f^{-1})(x) = f(x^{3/2}) = (x^{3/2})^{2/3} = x$

For $x \ge 0$ (the domain of f), $(f^{-1} \circ f)(x) = f^{-1}(x^{2/3}) = (x^{2/3})^{3/2} = |x| = x$

23. $y = -(x-2)^2,\ x \le 2 \to (x-2)^2 = -y,\ x \le 2 \to x - 2 = -\sqrt{-y} \to x = 2 - \sqrt{-y}$.

Interchange x and y: $y = 2 - \sqrt{-x} \to f^{-1}(x) = 2 - \sqrt{-x}$ or $2 - (-x)^{1/2}$

Verify.

For $x \le 0$ (the domain of f^{-1})

$(f \circ f^{-1})(x) = f(2 - \sqrt{-x}) = -[(2 - \sqrt{-x}) - 2]^2 = -(-\sqrt{-x})^2 = -|x| = x$

For $x \le 2$ (the domain of f),

$(f^{-1} \circ f)(x) = f^{-1}(-(x-2)^2) = 2 - \sqrt{(x-2)^2} = 2 - |x-2| = 2 + (x-2) = x$

24. $y = (x^2 + 2x + 1),\ x \ge -1 \to y = (x+1)^2,\ x \ge -1 \to \sqrt{y} = x + 1 \to \sqrt{y} - 1 = x$.

Interchange x and y: $\sqrt{x} - 1 = y \to f^{-1}(x) = \sqrt{x} - 1$ or $x^{1/2} - 1$

Verify.

For $x \ge 0$ (the domain of f^{-1}),

$(f \circ f^{-1})(x) = f(\sqrt{x} - 1) = \left[(\sqrt{x} - 1)^2 + 2(\sqrt{x} - 1) + 1\right] = (\sqrt{x})^2 - 2\sqrt{x} + 1 + 2\sqrt{x} - 2 + 1 = (\sqrt{x})^2 = x$

For $x \ge -1$ (the domain of f),

$(f^{-1} \circ f)(x) = f^{-1}(x^2 + 2x + 1) = \sqrt{x^2 + 2x + 1} - 1 = \sqrt{(x+1)^2} - 1 = |x+1| - 1 = (x+1) - 1 = x$

25. $y = \frac{1}{x^2},\ x > 0 \to x^2 = \frac{1}{y},\ x > 0 \to x = \sqrt{\frac{1}{y}} = \frac{1}{\sqrt{y}}$.

Interchange x and y: $y = \frac{1}{\sqrt{x}} \to f^{-1}(x) = \frac{1}{\sqrt{x}}$ or $\frac{1}{x^{1/2}}$

Verify.

For $x > 0$ (the domain of f^{-1}), $(f \circ f^{-1})(x) = f\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{(1/\sqrt{x})^2} = x$

For $x > 0$ (the domain of f), $(f^{-1} \circ f)(x) = f^{-1}\left(\frac{1}{x^2}\right) = \frac{1}{\sqrt{1/x^2}} = \sqrt{x^2} = |x| = x$

26. $y = \frac{1}{x^3} \to x^3 = \frac{1}{y} \to x = \sqrt[3]{\frac{1}{y}} = \frac{1}{\sqrt[3]{y}}$.

Interchange x and y: $y = \frac{1}{\sqrt[3]{x}} \to f^{-1}(x) = \frac{1}{\sqrt[3]{x}}$ or $\frac{1}{x^{1/3}}$

Verify.

$$(f \circ f^{-1})(x) = f\left(\frac{1}{\sqrt[3]{x}}\right) = \frac{1}{\left(1/\sqrt[3]{x}\right)^3} = x$$

$$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{1}{x^3}\right) = \frac{1}{\sqrt[3]{1/x^3}} = x$$

27. $y = \frac{2x+1}{x+3} \to xy + 3y = 2x + 1 \to xy - 2x = 1 - 3y \to (y-2)x = 1 - 3y \to x = \frac{1-3y}{y-2}$.

Interchange x and y: $y = \frac{1-3x}{x-2} \to f^{-1}(x) = \frac{1-3x}{x-2}$

Verify.

$$(f \circ f^{-1})(x) = f\left(\frac{1-3x}{x-2}\right) = \frac{2\left(\frac{1-3x}{x-2}\right)+1}{\frac{1-3x}{x-2}+3} = \frac{2(1-3x)+(x-2)}{(1-3x)+3(x-2)} = \frac{-5x}{-5} = x$$

$$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{2x+1}{x+3}\right) = \frac{1-3\left(\frac{2x+1}{x+3}\right)}{\frac{2x+1}{x+3}-2} = \frac{(x+3)-3(2x+1)}{(2x+1)-2(x+3)} = \frac{-5x}{-5} = x$$

28. $y = \frac{x+3}{x-2} \to xy - 2y = x + 3 \to xy - x = 2y + 3 \to x(y-1) = 2y + 3 \to x = \frac{2y+3}{y-1}$.

Interchange x and y: $y = \frac{2x+3}{x-1} \to f^{-1}(x) = \frac{2x+3}{x-1}$

Verify.

$$(f \circ f^{-1})(x) = f\left(\frac{2x+3}{x-1}\right) = \frac{\frac{2x+3}{x-1}+3}{\frac{2x+3}{x-1}-2} = \frac{(2x+3)+3(x-1)}{(2x+3)-2(x-1)} = \frac{5x}{5} = x$$

$$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{x+3}{x-2}\right) = \frac{2\left(\frac{x+3}{x-2}\right)+3}{\frac{x+3}{x-2}-1} = \frac{2(x+3)+3(x-2)}{(x+3)-(x-2)} = \frac{5x}{5} = x$$

29. $y = \left(e^a\right)^x - 1 \to e^a = 3 \to a = \ln 3 \to y = e^{x \ln 3} - 1$

(a) $D = (-\infty, \infty)$ (b) $R = (-1, \infty)$

30. $y = \left(e^a\right)^{x+1} \to e^a = 4 \to a = \ln 4 \to y = e^{(x+1)\ln 4} = e^{x \ln 4}e^{\ln 4} = 4e^{x \ln 4}$

(a) $D = (-\infty, \infty)$ (b) $R = (0, \infty)$

31. $y = 1 - (\ln 3) \log_3 x = 1 - (\ln 3) \frac{\ln x}{\ln 3} = 1 - \ln x$

(a) $D = (0, \infty)$

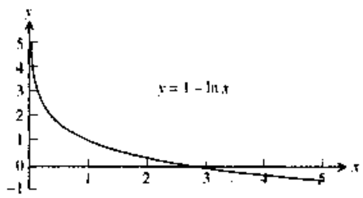
(b) $R = (-\infty, \infty)$

32. $y = (\ln 10) \log(x+2) = (\ln 10)\frac{\ln(x+2)}{\ln 10} = \ln(x+2)$

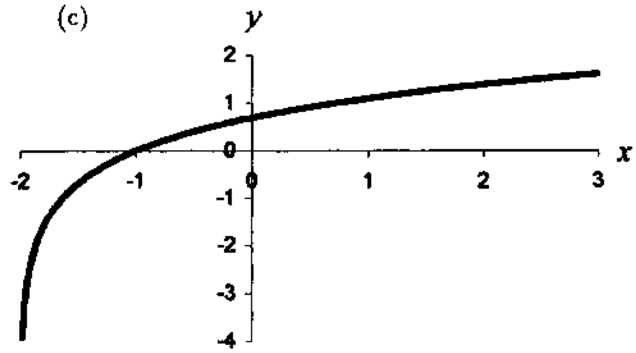
(a) $D = (-2, \infty)$

(b) $R = (-\infty, \infty)$

(c)



(c)



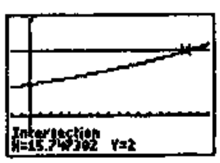
33. $(1.045)^t = 2$

$\ln (1.045)^t = \ln 2$

$t \ln 1.04 = \ln 2$

$t = \dfrac{\ln 2}{\ln 1.045} \approx 15.75$

Graphical support:



$[-2, 18]$ by $[-1, 3]$

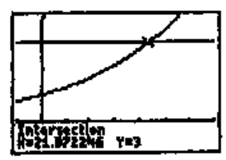
34. $e^{0.05t} = 3$

$\ln e^{0.05t} = \ln 3$

$0.05t = \ln 3$

$t = \dfrac{\ln 3}{0.05} = 20 \ln 3 \approx 21.97$

Graphical support:



$[-5, 35]$ by $[-1, 4]$

35. $e^x + e^{-x} = 3$

$e^x - 3 + e^{-x} = 0$

$e^x\left(e^x - 3 + e^{-x}\right) = e^x(0)$

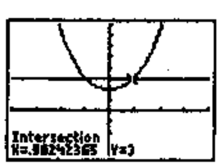
$\left(e^x\right)^2 - 3e^x + 1 = 0$

$$e^x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)}$$

$$e^x = \frac{3 \pm \sqrt{5}}{2}$$

$$x = \ln\left(\frac{3 \pm \sqrt{5}}{2}\right) \approx -0.96 \text{ or } 0.96$$

Graphical support:



$[-4, 4]$ by $[-4, 8]$

36. $2^x + 2^{-x} = 5$

$2^x - 5 + 2^{-x} = 0$

$2^x\left(2^x - 5 + 2^{-x}\right) = 2^x(0)$

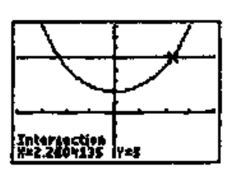
$\left(2^x\right)^2 - 5(2^x) + 1 = 0$

$$2^x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(1)}}{2(1)}$$

$$2^x = \frac{5 \pm \sqrt{21}}{2}$$

$$x = \log_2\left(\frac{5 \pm \sqrt{21}}{2}\right) \approx -2.26 \text{ or } 2.26$$

Graphical support:



$[-4, 4]$ by $[-4, 8]$

37. $\ln y = 2t + 4 \to e^{\ln y} = e^{2t+4} \to y = e^{2t+4}$

38. $\ln(y-1) - \ln 2 = x + \ln x \to \ln(y-1) = x + \ln x + \ln 2 \to e^{\ln(y-1)} = e^{x + \ln x + \ln 2} \to y - 1 = e^x(x)(2)$
$\to y = 2xe^x + 1$

39. (a) $y = \frac{100}{1+2^{-x}} \to 1 + 2^{-x} = \frac{100}{y} \to 2^{-x} = \frac{100}{y} - 1 \to \log_2(2^{-x}) = \log_2\left(\frac{100}{y} - 1\right) \to -x = \log_2\left(\frac{100}{y} - 1\right)$

$\to x = -\log_2\left(\frac{100}{y} - 1\right) = -\log_2\left(\frac{100-y}{y}\right) = \log_2\left(\frac{y}{100-y}\right).$

Interchange x and y: $y = \log_2\left(\frac{x}{100-x}\right) \to f^{-1}(x) = \log_2\left(\frac{x}{100-x}\right)$

Verify.

$$(f \circ f^{-1})(x) = f\left(\log_2\left(\frac{x}{100-x}\right)\right) = \frac{100}{1 + 2^{-\log_2\left(\frac{x}{100-x}\right)}} = \frac{100}{1 + 2^{\log_2\left(\frac{100-x}{x}\right)}} = \frac{100}{1 + \frac{100-x}{x}}$$

$$= \frac{100x}{x + (100-x)} = \frac{100x}{100} = x$$

$$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{100}{1+2^{-x}}\right) = \log_2\left(\frac{\frac{100}{1+2^{-x}}}{100 - \frac{100}{1+2^{-x}}}\right) = \log_2\left(\frac{100}{100(1+2^{-x}) - 100}\right)$$

$$= \log_2\left(\frac{1}{2^{-x}}\right) = \log_2(2^x) = x$$

(b) $y = \frac{50}{1+1.1^{-x}} \to 1 + 1.1^{-x} = \frac{50}{y} \to 1.1^{-x} = \frac{50}{y} - 1 \to \log_{1.1}(1.1^{-x}) = \log_{1.1}\left(\frac{50}{y} - 1\right) \to -x = \log_{1.1}\left(\frac{50}{y} - 1\right)$

$\to x = -\log_{1.1}\left(\frac{50}{y} - 1\right) = -\log_{1.1}\left(\frac{50-y}{y}\right) = \log_{1.1}\left(\frac{y}{50-y}\right).$

Interchange x and y: $y = \log_{1.1}\left(\frac{x}{50-x}\right) \to f^{-1}(x) = \log_{1.1}\left(\frac{x}{50-x}\right)$

Verify.

$$(f \circ f^{-1})(x) = f\left(\log_{1.1}\left(\frac{x}{50-x}\right)\right) = \frac{50}{1 + 1.1^{-\log_{1.1}\left(\frac{x}{50-x}\right)}} = \frac{50}{1 + 1.1^{\log_{1.1}\left(\frac{50-x}{x}\right)}} = \frac{50}{1 + \frac{50-x}{x}}$$

$$= \frac{50x}{x + (50-x)} = \frac{50x}{50} = x$$

$$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{50}{1+1.1^{-x}}\right) = \log_{1.1}\left(\frac{\frac{50}{1+1.1^{-x}}}{50 - \frac{50}{1+1.1^{-x}}}\right) = \log_{1.1}\left(\frac{50}{50(1+1.1^{-x}) - 50}\right)$$

$$= \log_{1.1}\left(\frac{1}{1.1^{-x}}\right) = \log_{1.1}(1.1^x) = x$$

40. (a) Suppose that $f(x_1) = f(x_2)$. Then $mx_1 + b = mx_2 + b$ so $mx_1 = mx_2$. Since $m \neq 0$, this gives $x_1 = x_2$.

(b) $y = mx + b \to y - b = mx \to \frac{y-b}{m} = x.$

Interchange x and y: $\frac{x-b}{m} = y \rightarrow f^{-1}(x) = \frac{x-b}{m}$

The slopes are reciprocals.

(c) If the original functions both have slope m, each of the inverse functions will have slope $\frac{1}{m}$. The graphs of the inverses will be parallel lines with nonzero slope.

(d) If the original functions have slopes m and $-\frac{1}{m}$, respectively, then the inverse functions will have slopes $\frac{1}{m}$ and $-m$, respectively. Since each of $\frac{1}{m}$ and $-m$ is the negative reciprocal of the other, the graphs of the inverses will be perpendicular lines with nonzero slopes.

41. (a) Amount $= 8\left(\frac{1}{2}\right)^{t/12}$

(b) $8\left(\frac{1}{2}\right)^{t/12} = 1 \rightarrow \left(\frac{1}{2}\right)^{t/12} = \frac{1}{8} \rightarrow \left(\frac{1}{2}\right)^{t/12} = \left(\frac{1}{2}\right)^{3} \rightarrow \frac{t}{12} = 3 \rightarrow t = 36$

There will be 1 gram remaining after 36 hours.

42. $500(1.0475)^t = 1000 \rightarrow 1.0475^t = 2 \rightarrow \ln(1.0475^t) = \ln 2 \rightarrow t \ln 1.0475 = \ln 2 \rightarrow t = \frac{\ln 2}{\ln 1.0475} \approx 14.936$

It will take about 14.936 years. (If the interest is paid at the end of each year, it will take 15 years.)

43. $375{,}000(1.0225)^t = 1{,}000{,}000 \rightarrow 1.0225^t = \frac{8}{3} \rightarrow \ln(1.0225^t) = \ln\left(\frac{8}{3}\right) \rightarrow t \ln 1.0225 = \ln\left(\frac{8}{3}\right)$

$\rightarrow t = \frac{\ln(8/3)}{\ln 1.0225} \approx 44.081$

It will take about 44.081 years.

44. Let O = original sound level $= 10 \log_{10}(I \times 10^{12})$ db from Equation (1) in the text. Solving

$O + 10 = 10 \log_{10}(kI \times 10^{12})$ for k $\Rightarrow 10 \log_{10}(I \times 10^{12}) + 10 = 10 \log_{10}(kI \times 10^{12})$

$\Rightarrow \log_{10}(I \times 10^{12}) + 1 = \log_{10}(kI \times 10^{12}) \Rightarrow \log_{10}(I \times 10^{12}) + 1 = \log_{10} k + \log_{10}(I \times 10^{12})$

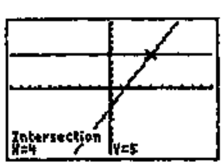
$\Rightarrow 1 = \log_{10} k \Rightarrow 1 = \frac{\ln k}{\ln 10} \Rightarrow \ln k = \ln 10 \Rightarrow k = 10$

45. Sound level with intensity = 10I is $10 \log_{10}(10I \times 10^{12}) = 10\left[\log_{10} 10 + \log_{10}(I \times 10^{12})\right]$

$= 10 + 10 \log_{10}(I \times 10^{12}) =$ original sound level $+ 10 \Rightarrow$ an increase of 10 db

46. $y = y_0 e^{-0.18t}$ represents the decay equation; solving $(0.9)y_0 = y_0 e^{-0.18t} \Rightarrow t = \frac{\ln(0.9)}{-0.18} \approx 0.585$ days

47.



[−10, 10] by [−10, 10]

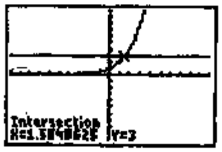
$(4, 5)$

48.

[−10, 10] by [−10, 10]

$\left(\frac{8}{3}, -3\right) \approx (2.67, -3)$

49. (a)



[−10, 10] by [−10, 10]

(1.58, 3)

(b) No points of intersection, since $2^x > 0$ for all values of x.

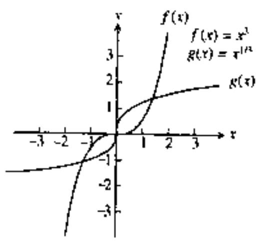
50. (a)

[−10, 10] by [−10, 10]

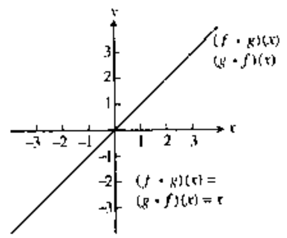
(−1.39, 4)

(b) No points of intersection, since $e^{-x} > 0$ for all values of x.

51. (a)

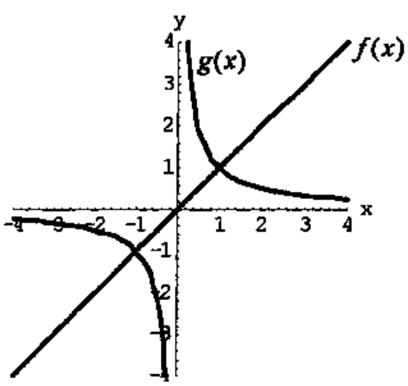


(b) and (c)

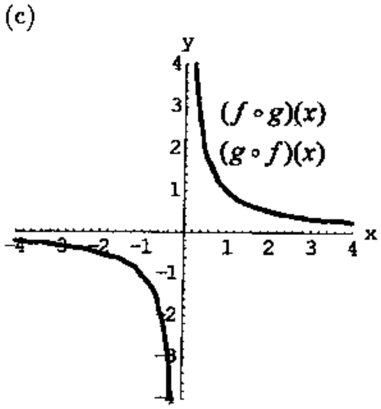


We conclude that f and g are inverses of each other because $(f \circ g)(x) = (g \circ f)(x) = x$, the identity function.

52. (a)

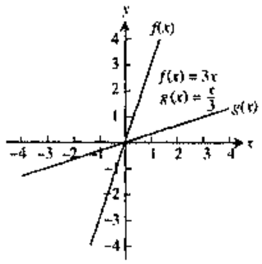


(b) and (c)

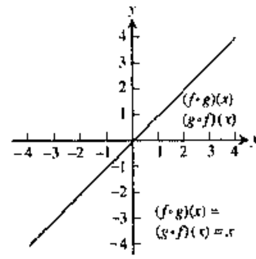


We conclude that f is the identity function because $(f \circ g)(x) = (g \circ f)(x) = \frac{1}{x} = g(x)$

53. (a)

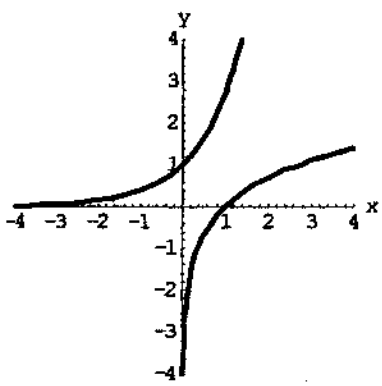


(b) and (c)

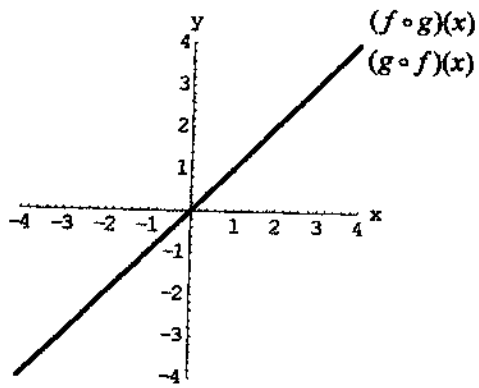


We conclude that f and g are inverses of each other because $(f \circ g)(x) = (g \circ f)(x) = x$, the identity function.

54. (a)

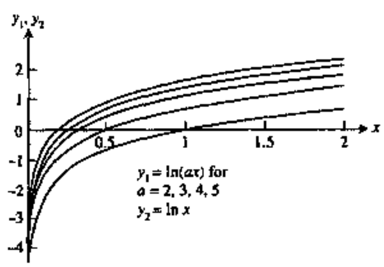


(b) and (c)

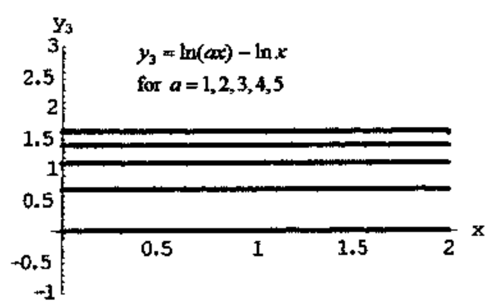


We conclude that f and g are inverses of each other because $(f \circ g)(x) = (g \circ f)(x) = x$, the identity function.

55. (a)



(b)



The graphs of y_1 appear to be vertical translates of y_2

The graphs of $y_1 - y_2$ support the finding in part (a).

(c) $y_3 = y_1 - y_2 = \ln ax - \ln x = (\ln a + \ln x) - \ln x = \ln a$, a constant.

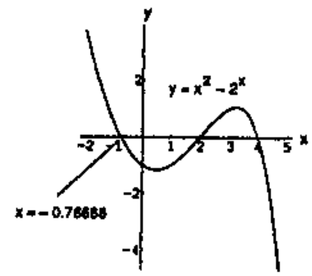
56. (a) y_2 is a vertical shift (upward) of y_1, although it's difficult to see that near the vertical asymptote at $x = 0$. One might use "trace" or "table" to verify this.

(b) Each graph of y_3 is a horizontal line.

(c) The graphs of y_4 and $y = a$ are the same.

(d) $e^{y_2 - y_1} = a$, $\ln\left(e^{y_2 - y_1}\right) = \ln a$, $y_2 - y_1 = \ln a$, $y_1 = y_2 - \ln a = \ln x - \ln a$

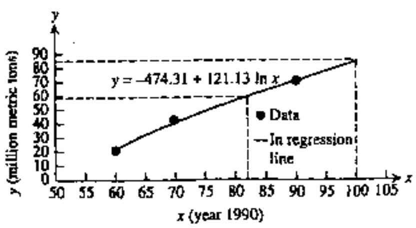
57. From zooming in on the graph at the right, we estimate the third root to be $x \approx -0.76666$



58. The functions $f(x) = x^{\ln 2}$ and $g(x) = 2^{\ln x}$ appear to have identical graphs for $x > 0$. This is no accident, because $x^{\ln 2} = e^{\ln 2 \cdot \ln x} = \left(e^{\ln 2}\right)^{\ln x} = 2^{\ln x}$.

59. (a) The LnReg command on the TI-92 Plus calculator gives $y(x) = -474.31 + 121.13 \ln x$ $\Rightarrow y(82) = -474.31 + 121.13 \ln(82) = 59.48$ million metric tons produced in 1982 and $y(100) = -474.31 + 121.13 \ln(100) = 83.51$ million metric tons produced in 2000.

(b)



(c) From the graph in part (b), $y(82) \approx 59$ and $y(100) \approx 84$.

60. (a) $y = -2539.852 + 636.896 \ln x$

(b) When $x = 75$, $y \approx 209.94$. About 209.94 million metric tons were produced.

(c) $-2539.852 + 636.896 \ln x = 400$

$$636.896 \ln x = 2939.852$$

$$\ln x = \frac{2939.852}{636.896}$$

$$x = e^{\frac{2939.852}{636.896}} \approx 101.08$$

According to the regression equation, Saudi Arabian oil production will reach 400 million metric tons when $x \approx 101.08$, in about 2001.

P.5 TRIGONOMETRIC FUNCTIONS AND THEIR INVERSES

1. (a) $s = r\theta = (10)\left(\frac{4\pi}{5}\right) = 8\pi$ m (b) $s = r\theta = (10)(110^\circ)\left(\frac{\pi}{180^\circ}\right) = \frac{110\pi}{18} = \frac{55\pi}{9}$ m

2. $\theta = \frac{s}{r} = \frac{10\pi}{8} = \frac{5\pi}{4}$ radians and $\frac{5\pi}{4}\left(\frac{180^\circ}{\pi}\right) = 225^\circ$

3.

θ	$-\pi$	$-\frac{2\pi}{3}$	0	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
$\sin\theta$	0	$-\frac{\sqrt{3}}{2}$	0	1	$\frac{1}{\sqrt{2}}$
$\cos\theta$	-1	$-\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{2}}$
$\tan\theta$	0	$\sqrt{3}$	0	und.	-1
$\cot\theta$	und.	$\frac{1}{\sqrt{3}}$	und.	0	-1
$\sec\theta$	-1	-2	1	und.	$-\sqrt{2}$
$\csc\theta$	und.	$-\frac{2}{\sqrt{3}}$	und.	1	$\sqrt{2}$

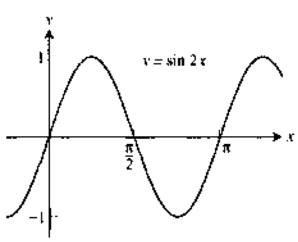
4.

θ	$-\frac{3\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{5\pi}{6}$
$\sin\theta$	1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\cos\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$
$\tan\theta$	und.	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	1	$-\frac{1}{\sqrt{3}}$
$\cot\theta$	0	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	1	$-\sqrt{3}$
$\sec\theta$	und.	2	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	$-\frac{2}{\sqrt{3}}$
$\csc\theta$	1	$-\frac{2}{\sqrt{3}}$	-2	$\sqrt{2}$	2

5. (a) $\cos x = -\frac{4}{5}$, $\tan x = -\frac{3}{4}$ (b) $\sin x = -\frac{2\sqrt{2}}{3}$, $\tan x = -2\sqrt{2}$

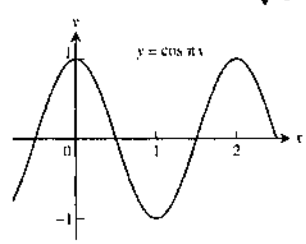
6. (a) $\sin x = -\frac{1}{\sqrt{5}}$, $\cos x = -\frac{2}{\sqrt{5}}$ (b) $\cos x = -\frac{\sqrt{3}}{2}$, $\tan x = \frac{1}{\sqrt{3}}$

7. (a)



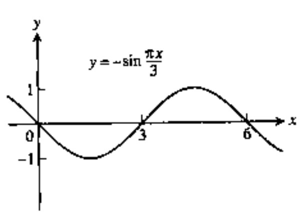
period $= \pi$

(b)



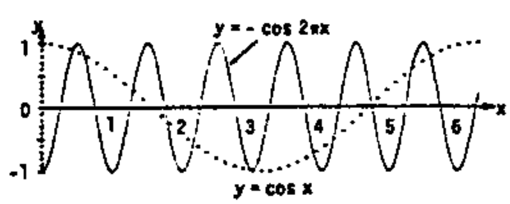
period = 2

8. (a)



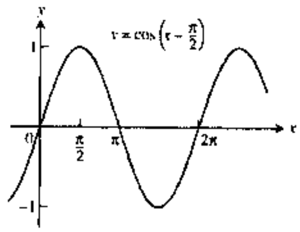
period = 6

(b)



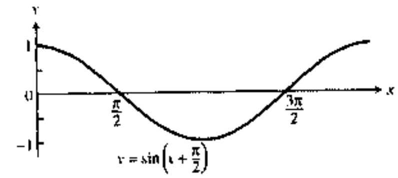
period = 1

9. (a)



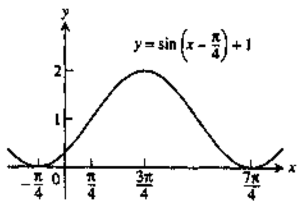
period $= 2\pi$

(b)



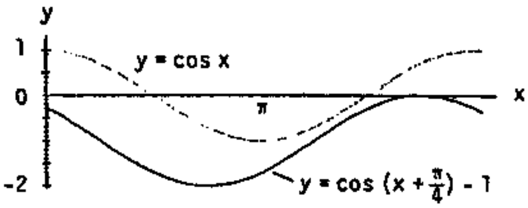
period $= 2\pi$

10. (a)



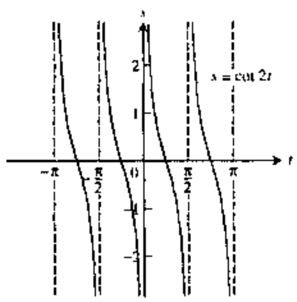
period $= 2\pi$

(b)

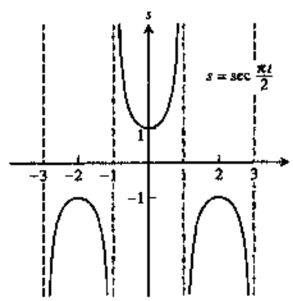


period $= 2\pi$

11. period $= \frac{\pi}{2}$, symmetric about the origin



12. period $= 4$, symmetric about the y-axis



13. (a) $\cos(\pi + x) = \cos\pi\cos x - \sin\pi\sin x = (-1)(\cos x) - (0)(\sin x) = -\cos x$
(b) $\sin(2\pi - x) = \sin 2\pi\cos(-x) + \cos(2\pi)\sin(-x) = (0)(\cos(-x)) + (1)(\sin(-x)) = -\sin x$

14. (a) $\sin\left(\frac{3\pi}{2} - x\right) = \sin\left(\frac{3\pi}{2}\right)\cos(-x) + \cos\left(\frac{3\pi}{2}\right)\sin(-x) = (-1)(\cos x) + (0)(\sin(-x)) = -\cos x$

(b) $\cos\left(\frac{3\pi}{2} + x\right) = \cos\left(\frac{3\pi}{2}\right)\cos x - \sin\left(\frac{3\pi}{2}\right)\sin x = (0)(\cos x) - (-1)(\sin x) = \sin x$

15. (a) $\cos\left(x - \frac{\pi}{2}\right) = \cos x\cos\left(-\frac{\pi}{2}\right) - \sin x\sin\left(-\frac{\pi}{2}\right) = (\cos x)(0) - (\sin x)(-1) = \sin x$

$$\cos(A - B) = \cos(A + (-B)) = \cos A\cos(-B) - \sin A\sin(-B) = \cos A\cos B - \sin A(-\sin B)$$
$$= \cos A\cos B + \sin A\sin B$$

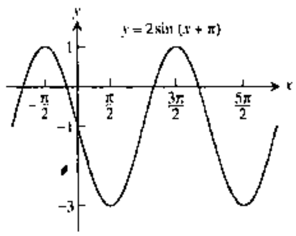
16. (a) $\sin\left(x + \frac{\pi}{2}\right) = \sin x\cos\left(\frac{\pi}{2}\right) + \cos x\sin\left(\frac{\pi}{2}\right) = (\sin x)(0) + (\cos x)(1) = \cos x$

(b) $\sin(A-B) = \sin(A+(-B)) = \sin A \cos(-B) + \cos A \sin(-B) = \sin A \cos B + \cos A(-\sin B)$
$= \sin A \cos B - \cos A \sin B$

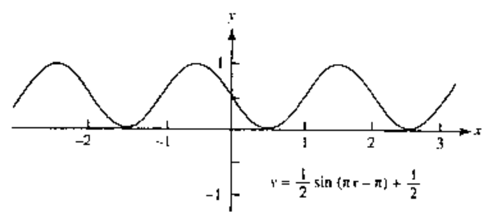
17. If $B = A$, $A - B = 0 \Rightarrow \cos(A-B) = \cos 0 = 1$. Also $\cos(A-B) = \cos(A-A) = \cos A \cos A + \sin A \sin A$ $= \cos^2 A + \sin^2 A$. Therefore, $\cos^2 A + \sin^2 A = 1$.

18. If $B = 2\pi$, then $\cos(A+2\pi) = \cos A \cos 2\pi - \sin A \sin 2\pi = (\cos A)(1) - (\sin A)(0) = \cos A$ and $\sin(A+2\pi) = \sin A \cos 2\pi + \cos A \sin 2\pi = (\sin A)(1) + (\cos A)(0) = \sin A$. The result agrees with the fact that the cosine and sine functions have period 2π.

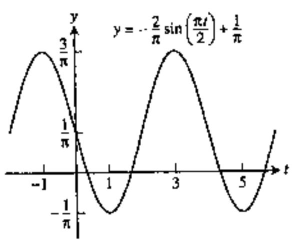
19. (a) $A = 2$, $B = 2\pi$, $C = -\pi$, $D = -1$



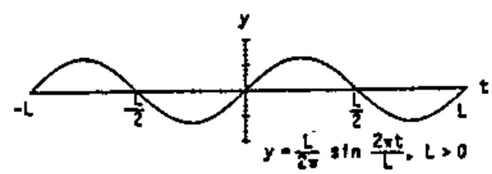
(b) $A = \frac{1}{2}$, $B = 2$, $C = 1$, $D = \frac{1}{2}$



20. (a) $A = -\frac{2}{\pi}$, $B = 4$, $C = 0$, $D = \frac{1}{\pi}$



(b) $A = \frac{L}{2\pi}$, $B = L$, $C = 0$, $D = 0$



21. (a) amplitude $= |A| = 37$
(b) period $= |B| = 365$
(c) right horizontal shift $= C = 101$
(d) upward vertical shift $= D = 25$

22. (a) It is highest when the value of the sine is 1 at $f(101) = 37\sin(0) + 25 = 62°F$.
The lowest mean daily temp is $37(-1) + 25 = -12°F$.
(b) The average of the highest and lowest mean daily temperatures $= \frac{62° + (-12)°}{2} = 25°F$.
The average of the sine function is its horizontal axis, $y = 25$.

23. (a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$

24. (a) $-\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{3}$

25. (a) $\frac{\pi}{3}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{6}$

26. (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$

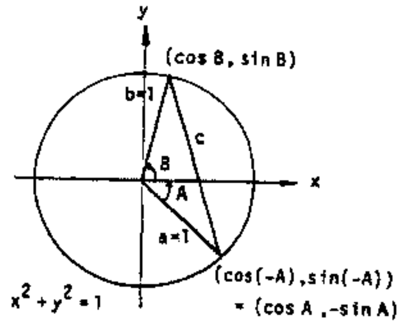
27. The angle α is the large angle between the wall and the right end of the blackboard minus the small angle between the left end of the blackboard and the wall $\Rightarrow \alpha = \cot^{-1}\left(\frac{x}{15}\right) - \cot^{-1}\left(\frac{x}{3}\right)$.

28. $65° + (90° - \beta) + (90° - \alpha) = 180° \Rightarrow \alpha = 65° - \beta = 65° - \tan^{-1}\left(\frac{21}{50}\right) \approx 65° - 22.78° \approx 42.22°$

29. According to the figure in the text, we have the following: By the law of cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$ $= 1^2 + 1^2 - 2\cos(A-B) = 2 - 2\cos(A-B)$. By distance formula, $c^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$

$= \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B = 2 - 2(\cos A\cos B + \sin A\sin B)$. Thus $c^2 = 2 - 2\cos(A - B) = 2 - 2(\cos A\cos B + \sin A\sin B) \Rightarrow \cos(A - B) = \cos A\cos B + \sin A\sin B$.

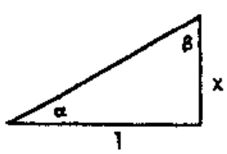
30. Consider the figure where $\theta = A + B$ is the sum of two angles. By the law of cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$ $= 1^2 + 1^2 - 2\cos(A + B) = 2 - 2\cos(A + B)$.
Also, by the distance formula,
$c^2 = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$
$= \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A + 2\sin A\sin B + \sin^2 B$
$= 2 - 2(\cos A\cos B - \sin A\sin B)$. Thus,
$2 - 2\cos(A + B) = 2 - 2(\cos A\cos B - \sin A\sin B)$
$\Rightarrow \cos(A + B) = \cos A\cos B - \sin A\sin B$.



31. Take each square as a unit square. From the diagram we have the following: the smallest angle α has a tangent of $1 \Rightarrow \alpha = \tan^{-1} 1$; the middle angle β has a tangent of $2 \Rightarrow \beta = \tan^{-1} 2$; and the largest angle γ has a tangent of $3 \Rightarrow \gamma = \tan^{-1} 3$. The sum of these three angles is $\pi \Rightarrow \alpha + \beta + \gamma = \pi$
$\Rightarrow \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$.

32. (a) From the symmetry of the diagram, we see that $\pi - \sec^{-1} x$ is the vertical distance from the graph of $y = \sec^{-1} x$ to the line $y = \pi$ and this distance is the same as the height of $y = \sec^{-1} x$ above the x-axis at $-x$; i.e., $\pi - \sec^{-1} x = \sec^{-1}(-x)$.
(b) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, where $-1 \le x \le 1 \Rightarrow \cos^{-1}\left(-\frac{1}{x}\right) = \pi - \cos^{-1}\left(\frac{1}{x}\right)$, where $x \ge 1$ or $x \le -1$
$\Rightarrow \sec^{-1}(-x) = \pi - \sec^{-1} x$

33. $\sin^{-1}(1) + \cos^{-1}(1) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$; $\sin^{-1}(0) + \cos^{-1}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2}$; and $\sin^{-1}(-1) + \cos^{-1}(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$.
If $x \in (-1, 0)$ and $x = -a$, then $\sin^{-1}(x) + \cos^{-1}(x) = \sin^{-1}(-a) + \cos^{-1}(-a) = -\sin^{-1} a + \left(\pi - \cos^{-1} a\right)$
$= \pi - \left(\sin^{-1} a + \cos^{-1} a\right) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$ from Equations (7) and (9) in the text.

34. 

$\Rightarrow \tan\alpha = x$ and $\tan\beta = \frac{1}{x} \Rightarrow \frac{\pi}{2} = \alpha + \beta = \tan^{-1} x + \tan^{-1}\frac{1}{x}$.

35. From the figures in the text, we see that $\sin B = \frac{h}{c}$. If C is an acute angle, then $\sin C = \frac{h}{b}$. On the other hand, if C is obtuse (as in the figure on the right), then $\sin C = \sin(\pi - C) = \frac{h}{b}$. Thus, in either case, $h = b\sin C = c\sin B \Rightarrow ah = ab\sin C = ac\sin B$.

By the law of cosines, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ and $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$. Moreover, since the sum of the interior angles of a triangle is π, we have $\sin A = \sin(\pi - (B + C)) = \sin(B + C) = \sin B\cos C + \cos B\sin C$
$= \left(\frac{h}{c}\right)\left[\frac{a^2 + b^2 - c^2}{2ab}\right] + \left[\frac{a^2 + c^2 - b^2}{2ac}\right]\left(\frac{h}{b}\right) = \left(\frac{h}{2abc}\right)\left(2a^2 + b^2 - c^2 + c^2 - b^2\right) = \frac{ah}{bc} \Rightarrow ah = bc\sin A$.

Combining our results we have ah = ab sin C, ah = ac sin B, and ah = bc sin A. Dividing by abc gives $\frac{h}{bc} = \underbrace{\frac{\sin A}{a} = \frac{\sin C}{c} = \frac{\sin B}{b}}_{\text{law of sines}}$.

36. $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \cos B}{\cos A \cos B - \sin A \sin B} = \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

37. (a) $c^2 = a^2 + b^2 - 2ab \cos C = 2^2 + 3^2 - 2(2)(3)\cos(60°) = 4 + 9 - 12 \cos(60°) = 13 - 12\left(\frac{1}{2}\right) = 7$.
Thus, $c = \sqrt{7} \approx 2.65$.

(b) $c^2 = a^2 + b^2 - 2ab \cos C = 2^2 + 3^2 - 2(2)(3)\cos(40°) = 13 - 12 \cos(40°)$. Thus,
$c = \sqrt{13 - 12 \cos 40°} \approx 1.951$.

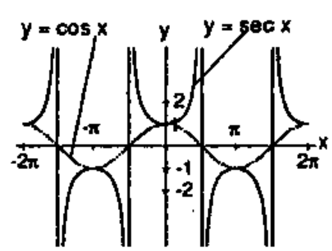
38. (a) By the law of sines, $\frac{\sin A}{2} = \frac{\sin B}{3} = \frac{\sqrt{3}/2}{c}$. By Exercise 55 we know that $c = \sqrt{7}$.
Thus $\sin B = \frac{3\sqrt{3}}{2\sqrt{7}} \simeq 0.982$.

(a) From the figure at the right and the law of cosines,
$b^2 = a^2 + 2^2 - 2(2a) \cos B = a^2 + 4 - 4a\left(\frac{1}{2}\right) = a^2 - 2a + 4$.

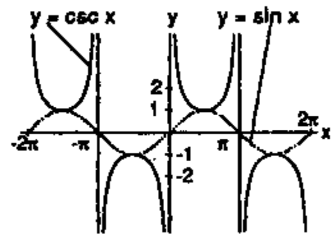
Applying the law of sines to the figure, $\frac{\sin A}{a} = \frac{\sin B}{b}$
$\Rightarrow \frac{\sqrt{2}/2}{a} = \frac{\sqrt{3}/2}{b} \Rightarrow b = \sqrt{\frac{3}{2}}\,a$. Thus, combining results,
$a^2 - 2a + 4 = b^2 = \frac{3}{2}a^2 \Rightarrow 0 = \frac{1}{2}a^2 + 2a - 4$
$\Rightarrow 0 = a^2 + 4a - 8$. From the quadratic formula and the
fact that $a > 0$, we have $a = \frac{-4 + \sqrt{4^2 - 4(1)(-8)}}{2} = \frac{4\sqrt{3} - 4}{2} \simeq 1.464$.

39. (a) The graphs of y = sin x and y = x nearly coincide when x is near the origin (when the calculator is in radians mode).
(b) In degree mode, when x is near zero degrees the sine of x is much closer to zero than x itself. The curves look like intersecting straight lines near the origin when the calculator is in degree mode.

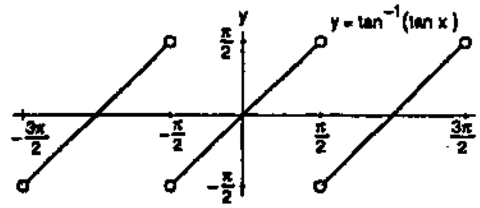
40. (a) Cos x and sec x are positive in QI and QIV and negative in QII and QIII. Sec x is undefined when cos x is 0. The range of sec x is $(-\infty, -1] \cup [1, \infty)$; the range of cos x is $[-1, 1]$.



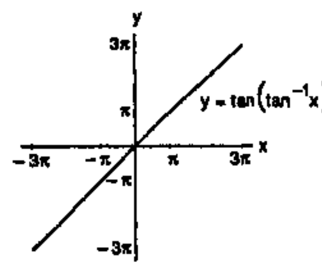
(b) Sin x and csc x are positive in QII and negative in QIII and QIV. Csc x is undefined when sin x is 0. The range of csc x is $(-\infty,-1]\cup[1,\infty)$; the range of sin x is $[-1,1]$.



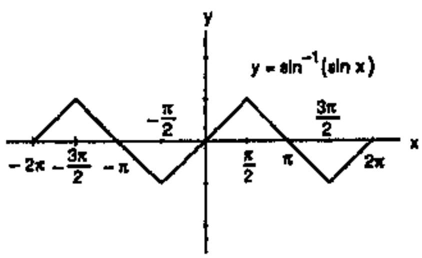
41. (a) Domain: all real numbers except those having the form $\frac{\pi}{2}+k\pi$ where k is an integer.
Range: $-\frac{\pi}{2}<y<\frac{\pi}{2}$



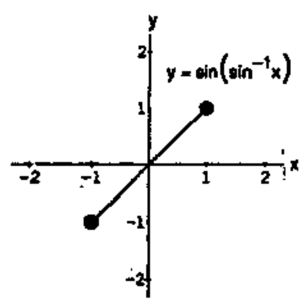
(b) Domain: $-\infty<x<\infty$; Range: $-\infty<y<\infty$
The graph of $y=\tan^{-1}(\tan x)$ is periodic, the graph of $y=\tan\left(\tan^{-1}x\right)=x$ for $-\infty\le x<\infty$.



42. (a) Domain: $-\infty<x<\infty$; Range: $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$



(b) Domain: $-1\le x\le 1$; Range: $-1\le y\le 1$
The graph of $y=\sin^{-1}(\sin x)$ is periodic; the graph of $y=\sin\left(\sin^{-1}x\right)=x$ for $-1\le x\le 1$.



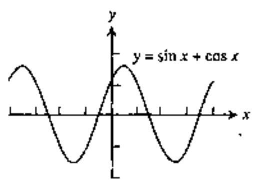
43. The angle $\tan^{-1}(2.5)\approx 1.190$ is the solution to this equation in the interval $-\frac{\pi}{2}<x<\frac{\pi}{2}$. Another solution in $0\le x<2\pi$ is $\tan^{-1}(2.5)+\pi\approx 4.332$. The solutions are $x\approx 1.190$ and $x\approx 4.332$.

44. The angle $\cos^{-1}(-0.7) \approx 2.346$ is the solution to this equation in the interval $0 \le x \le \pi$. Since the cosine function is even, the value $-\cos^{-1}(-0.7) \approx -2.346$ is also a solution, so any value of the form $\pm\cos^{-1}(-0.7) + 2k\pi$ is a solution, where k is an integer. In $2\pi \le x < 4\pi$ the solutions are $x = \cos^{-1}(-0.7) + 2\pi \approx 8.629$ and $x = -\cos^{-1}(-0.7) + 4\pi \approx 10.220$.

45. This equation is equivalent to $\cos x = -\frac{1}{3}$, so the solution in the interval $0 \le x \le \pi$ is $y = \cos^{-1}\left(-\frac{1}{3}\right) \approx 1.911$. Since the cosine function is even, the solutions in the interval $-\pi \le x < \pi$ are $x \approx -1.911$ and $x \approx 1.911$.

46. The solutions in the interval $0 \le x < 2\pi$ are $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$. Since $y = \sin x$ has period 2π, the solutions are all of the form $x = \frac{7\pi}{6} + 2k\pi$ or $x = \frac{11\pi}{6} + 2k\pi$, where k is any integer.

47. (a)



The graph is a sine/cosine type graph, but it is shifted and has an amplitude greater than 1.

(b) Amplitude ≈ 1.414 (that is, $\sqrt{2}$)

Period $= 2\pi$

Horizontal shift ≈ -0.785 $\left(\text{that is, } -\frac{\pi}{4}\right)$ or 5.498 $\left(\text{that is, } \frac{7\pi}{4}\right)$ relative to sin x.

Vertical shift: 0

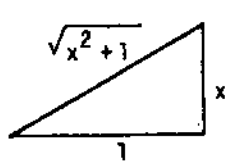
(c)
$$\begin{aligned}\sin\left(x + \frac{\pi}{4}\right) &= (\sin x)\left(\cos \frac{\pi}{4}\right) + (\cos x)\left(\sin \frac{\pi}{4}\right)\\ &= (\sin x)\left(\frac{1}{\sqrt{2}}\right) + (\cos x)\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{1}{\sqrt{2}}(\sin x + \cos x)\end{aligned}$$

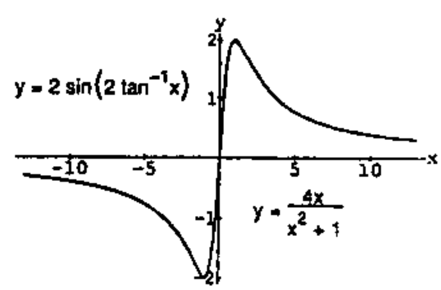
Therefore, $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

48. The graphs are identical for $y = 2\sin(2\tan^{-1}x)$

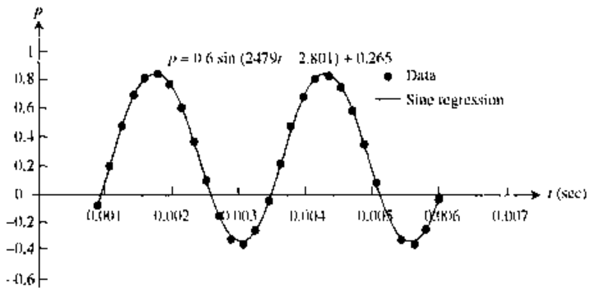
$= 4[\sin(\tan^{-1}x)][\cos(\tan^{-1}x)] = 4\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right)$

$= \frac{4x}{x^2+1}$ from the triangle





49. (a) The sinusoidal regression on the TI-92 Plus calculator gives $p = 0.599 \sin(2479t - 2.801) + 0.265$



(b) The period is approximately $\frac{2\pi}{2479}$ seconds, so the frequency is approximately $\frac{2479}{2\pi} \approx 395$ Hz

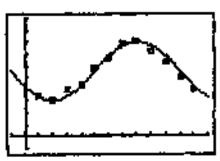
50. (a) $b = \frac{2\pi}{12} = \frac{\pi}{6}$

(b) It's half of the difference, so $a = \frac{80 - 30}{2} = 25$.

(c) $k = \frac{80 + 30}{2} = 55$

(d) The function should have its minimum at $t = 2$ (when the temperature is 30°F) and its maximum at $t = 8$ (when the temperature is 80°F). The value of h is $\frac{2+8}{2} = 5$. Equation: $y = 25 \sin\left[\frac{\pi}{6}(x - 5)\right] + 55$

(e)

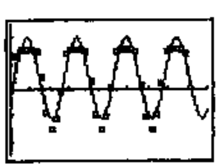


[−1, 13] by [−10, 100]

51. (a) Using a graphing calculator with the sinusoidal regression feature, the equation is $y = 3.0014 \sin(0.9996x + 2.0012) + 2.9999$.

(b) $y = 3 \sin(x + 2) + 3$

52. (a) Using a graphing calculator with the sinusoidal regression feature, the equation is $y = 1.543 \sin(2468.635x - 0.494) + 0.438$.

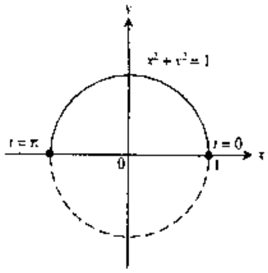


[0, 0.01] by [−2.5, 2.5]

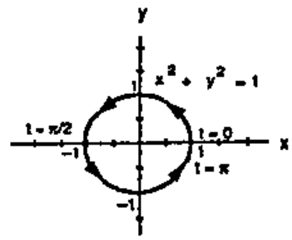
(b) The frequency is 2468.635 radians per second, which is equivalent to $\frac{2468.635}{2\pi} \approx 392.9$ cycles per second (Hz). The note is a "G."

P.6 PARAMETRIC EQUATIONS

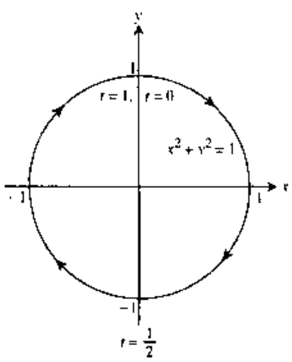
1. $x = \cos t,\ y = \sin t,\ 0 \le t \le \pi$
$\Rightarrow \cos^2 t + \sin^2 t = 1 \Rightarrow x^2 + y^2 = 1$



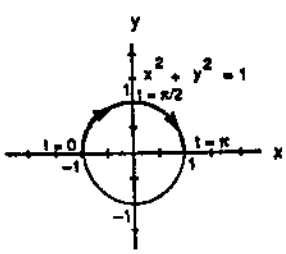
2. $x = \cos 2t,\ y = \sin 2t,\ 0 \le t \le \pi$
$\Rightarrow \cos^2 2t + \sin^2 2t = 1 \Rightarrow x^2 + y^2 = 1$



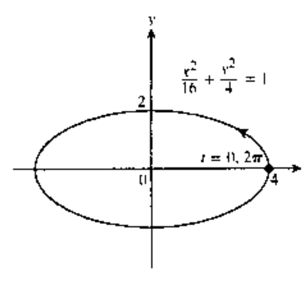
3. $x = \sin(2\pi t),\ y = \cos(2\pi t),\ 0 \le t \le 1$
$\sin^2(2\pi t) + \cos^2(2\pi t) = 1 \Rightarrow x^2 + y^2 = 1$



4. $x = \cos(\pi - t),\ y = \sin(\pi - t),\ 0 \le t \le \pi$
$\Rightarrow \cos^2(\pi - t) + \sin^2(\pi - t) = 1$
$\Rightarrow x^2 + y^2 = 1$

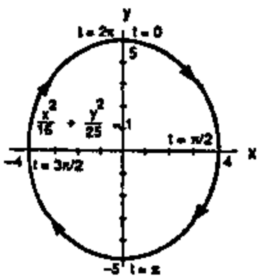


5. $x = 4\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$
$\Rightarrow \frac{16\cos^2 t}{16} + \frac{4\sin^2 t}{4} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$

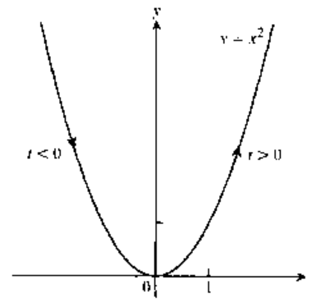


6. $x = 4 \sin t,\ y = 5 \cos t,\ 0 \le t \le 2\pi$

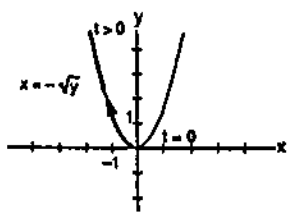
$\Rightarrow \dfrac{16 \sin^2 t}{16} + \dfrac{25 \cos^2 t}{25} = 1 \Rightarrow \dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$



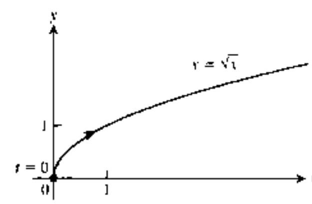
7. $x = 3t,\ y = 9t^2,\ -\infty < t < \infty \Rightarrow y = x^2$



8. $x = -\sqrt{t},\ y = t,\ t \ge 0 \Rightarrow x = -\sqrt{y}$
or $y = x^2,\ x \le 0$

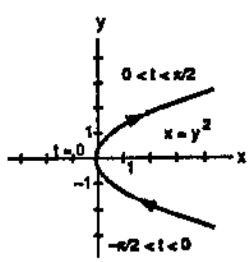


9. $x = t,\ y = \sqrt{t},\ t \ge 0 \Rightarrow y = \sqrt{x}$



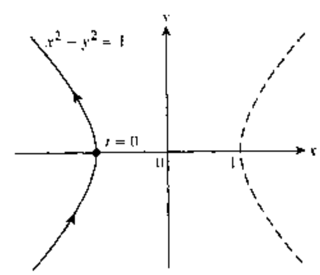
10. $x = \sec^2 t - 1,\ y = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$

$\Rightarrow \sec^2 t - 1 = \tan^2 t \Rightarrow x = y^2$



11. $x = -\sec t,\ y = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$

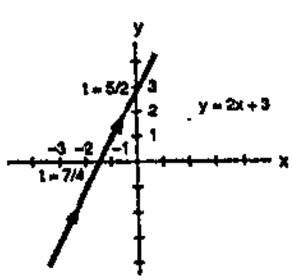
$\Rightarrow \sec^2 t - \tan^2 t = 1 \Rightarrow x^2 - y^2 = 1$



12. $x = 2t - 5,\ y = 4t - 7,\ -\infty < t < \infty$

$\Rightarrow x + 5 = 2t \Rightarrow 2(x + 5) = 4t$

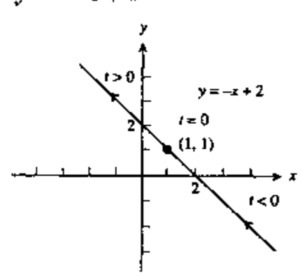
$\Rightarrow y = 2(x + 5) - 7 \Rightarrow y = 2x + 3$



13. $x = 1 - t,\ y = 1 + t,\ -\infty < t < \infty$

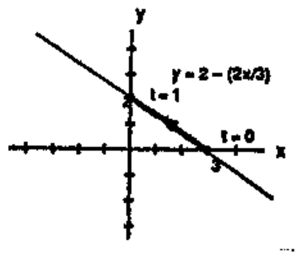
$\Rightarrow 1 - x = t \Rightarrow y = 1 + (1 - x)$

$\Rightarrow y = -x + 2$



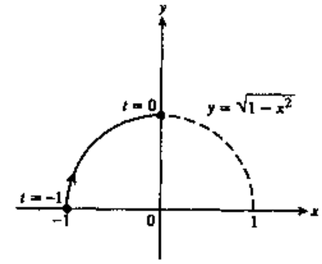
14. $x = 3 - 3t,\ y = 2t,\ 0 \le t \le 1 \Rightarrow \frac{y}{2} = t$

$\Rightarrow x = 3 - 3\left(\frac{y}{2}\right) \Rightarrow 2x = 6 - 3y \Rightarrow y = 2 - \frac{2}{3}x$



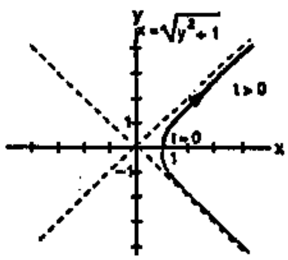
15. $x = t,\ y = \sqrt{1 - t^2},\ -1 \le t \le 0$

$\Rightarrow y = \sqrt{1 - x^2}$



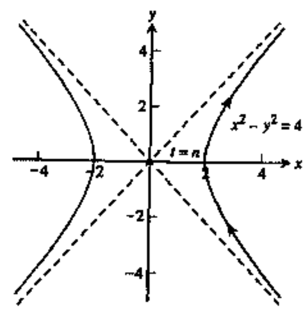
16. $x = \sqrt{t + 1},\ y = \sqrt{t},\ t \ge 0$

$\Rightarrow y^2 = t \Rightarrow x = \sqrt{y^2 + 1},\ y \ge 0$



17. $x = e^t + e^{-t},\ y = e^t - e^{-t},\ -\infty < t < \infty$

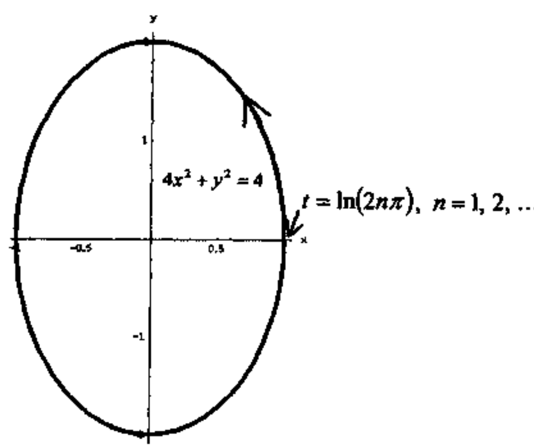
$\left(e^t + e^{-t}\right)^2 - \left(e^t - e^{-t}\right)^2 = \left(e^{2t} + 2 + e^{-2t}\right)$
$- \left(e^{2t} - 2 + e^{-2t}\right) = 4 \Rightarrow x^2 - y^2 = 4$



18. $x = \cos\left(e^t\right),\ y = 2\sin\left(e^t\right),\ -\infty < t < \infty$

$\cos^2\left(e^t\right) + \sin^2\left(e^t\right) = 1 \Rightarrow x^2 + (y/2)^2 = 1$

$\Rightarrow 4x^2 + y^2 = 4$



19. (a) $x = a\cos t,\ y = -a\sin t,\ 0 \le t \le 2\pi$

(b) $x = a\cos t,\ y = a\sin t,\ 0 \le t \le 2\pi$

(c) $x = a\cos t,\ y = -a\sin t,\ 0 \le t \le 4\pi$

(d) $x = a\cos t,\ y = a\sin t,\ 0 \le t \le 4\pi$

20. (a) $x = a\sin t,\ y = b\cos t,\ \frac{\pi}{2} \le t \le \frac{5\pi}{2}$

(b) $x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi$

(c) $x = a\sin t,\ y = b\cos t,\ \frac{\pi}{2} \le t \le \frac{9\pi}{2}$

(d) $x = a\cos t,\ y = b\sin t,\ 0 \le t \le 4\pi$

21. Using $(-1, -3)$ we create the parametric equations $x = -1 + at$ and $y = -3 + bt$, representing a line which goes through $(-1, -3)$ at $t = 0$. We determine a and b so that the line goes through $(4, 1)$ when $t = 1$.
Since $4 = -1 + a$, $a = 5$.

Since $1 = -3 + b$, $b = 4$.
Therefore, one possible parametrization is $x = -1 + 5t$, $y = -3 + 4t$, $0 \le t \le 1$.

22. Using $(-1, -3)$ we create the parametric equations $x = -1 + at$ and $y = 3 + bt$, representing a line which goes through $(-1, 3)$ at $t = 0$. We determine a and b so that the line goes through $(3, -2)$ $t = 1$.
Since $3 = -1 + a$, $a = 4$.
Since $-2 = 3 + b$, $b = -5$.
Therefore, one possible parametrization is $x = -1 + 4t$, $y = 3 - 5t$, $0 \le t \le 1$.

23. The lower half of the parabola is given by $x = y^2 + 1$ for $y \le 0$. Substituting t for y, we obtain one possible parametrization $x = t^2 + 1$, $y = t$, $t \le 0$.

24. The vertex of the parabola is at $(-1, -1)$, so the left half of the parabola is given by $y = x^2 + 2x$ for $x \le -1$. Substituting t for x, we obtain one possible parametrization: $x = t$, $y = t^2 + 2t$, $t \le -1$.

25. For simplicity, we assume that x and y are linear functions of t and that the point (x, y) starts at $(2, 3)$ for $t = 0$ and passes through $(-1, -1)$ at $t = 1$. Then $x = f(t)$, where $f(0) = 2$ and $f(1) = -1$.
Since slope $= \dfrac{\Delta x}{\Delta t} = \dfrac{-1-2}{1-0} = -3$, $x = f(t) = -3t + 2 = 2 - 3t$. Also, $y = g(t)$, where $g(t) = 3$ and $g(1) = -1$.
Since slope $= \dfrac{\Delta y}{\Delta t} = \dfrac{-1-3}{1-0} = -4$, $y = g(t) = -4t + 3 = 3 - 4t$.
One possible parametrization is: $x = 2 - 3t$, $y = 3 - 4t$, $t \ge 0$.

26. For simplicity, we assume that x and y are linear functions of t and that the point (x, y) starts at $(-1, 2)$ for $t = 0$ and passes through $(0, 0)$ at $t = 1$. Then $x = f(t)$, where $f(0) = -1$ and $f(1) = 0$.
Since slope $= \dfrac{\Delta x}{\Delta t} = \dfrac{0-(-1)}{1-0} = 1$, $x = f(t) = 1t + (-1) = -1 + t$. Also, $y = g(t)$, where $g(0) = 2$ and $g(1) = 0$.
Since slope $= \dfrac{\Delta y}{\Delta t} = \dfrac{0-2}{1-0} = -2$, $y = g(t) = -2t + 2 = 2 - 2t$.
One possible parametrization is: $x = -1 + t$, $y = 2 - 2t$, $t \ge 0$.

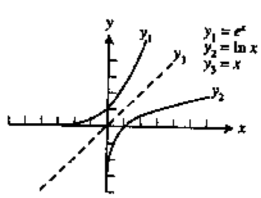
27. Graph (c). Window: $[-4, 4]$ by $[-3, 3]$, $0 \le t \le 2\pi$

28. Graph (a). Window: $[-2, 2]$ by $[-2, 2]$, $0 \le t \le 2\pi$

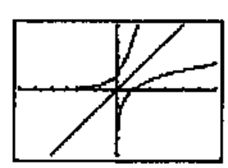
29. Graph (d). Window: $[-10, 10]$ by $[-10, 10]$, $0 \le t \le 2\pi$

30. Graph (b). Window: $[-15, 15]$ by $[-15, 15]$, $0 \le t \le 2\pi$

31. Graph of f: $x_1 = t$, $y_1 = e^t$
Graph of f^{-1}: $x_2 = e^t$, $y_2 = t$
Graph of $y = x$: $x_3 = t$, $y_3 = t$

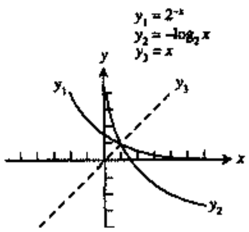


32. Graph of f: $x_1 = t$, $y_1 = 3^t$
Graph of f^{-1}: $x_2 = 3^t$, $y_2 = t$
Graph of $y = x$: $x_3 = t$, $y_3 = t$

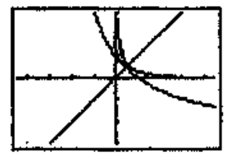


$[-6, 6]$ by $[-4, 4]$

33. Graph of f: $x_1 = t$, $y_1 = 2^{-t}$
Graph of f^{-1}: $x_2 = 2^{-t}$, $y_2 = t$
Graph of $y = x$: $x_3 = t$, $y_3 = t$

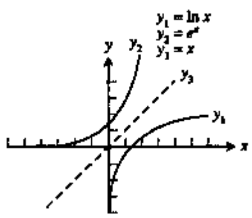


34. Graph of f: $x_1 = t$, $y_1 = 3^{-t}$
Graph of f^{-1}: $x_2 = 3^{-t}$, $y_2 = t$
Graph of $y = x$: $x_3 = t$, $y_3 = t$

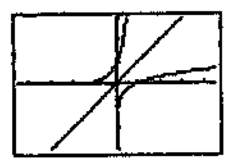


[−4.5, 4.5] by [−3, 3]

35. Graph of f: $x_1 = t$, $y_1 = \ln t$
Graph of f^{-1}: $x_2 = \ln t$, $y_2 = t$
Graph of $y = x$: $x_3 = t$, $y_3 = t$

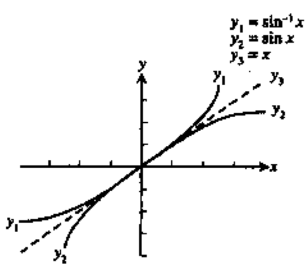


36. Graph of f: $x_1 = t$, $y_1 = \log t$
Graph of f^{-1}: $x_2 = \log t$, $y_2 = t$
Graph of $y = x$: $x_3 = t$, $y_3 = t$

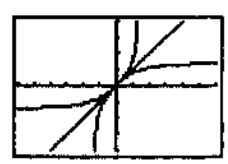


[−4.5, 4.5] by [−3, 3]

37. Graph of f: $x_1 = t$, $y_1 = \sin^{-1} t$
Graph of f^{-1}: $x_2 = \sin^{-1} t$, $y_2 = t$
Graph of $y = x$: $x_3 = t$, $y_3 = t$



38. Graph of f: $x_1 = t$, $y_1 = \tan^{-1} t$
Graph of f^{-1}: $x_2 = \tan^{-1} t$, $y_2 = t$
Graph of $y = x$: $x_3 = t$, $y_3 = t$



[−6, 6] by [−4, 4]

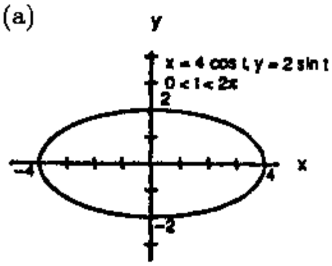
39. The graph is in Quadrant I when $0 < y < 2$, which corresponds to $1 < t < 3$. To confirm, note that $x(1) = 2$ and $x(3) = 0$.

40. The graph is in Quadrant II when $2 < y \le 4$, which corresponds to $3 < t \le 5$. To confirm, note that $x(3) = 0$ and $x(5) = -2$.

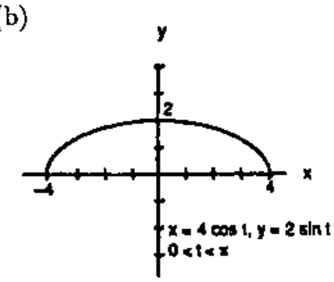
41. The graph is in Quadrant III when $-6 \le y < -4$, which corresponds to $-5 \le t < -3$. To confirm, note that $x(-5) = -2$ and $x(-3) = 0$.

42. The graph is in Quadrant IV when $-4 < y < 0$, which corresponds to $-3 < t < 1$. To confirm, note that $x(-3) = 0$ and $x(1) = 2$.

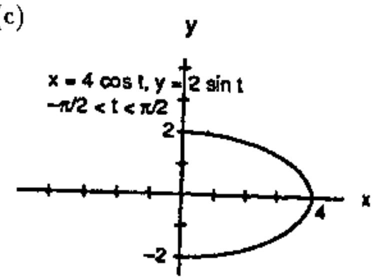
43. (a)



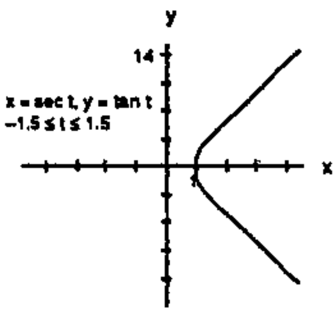
(b)



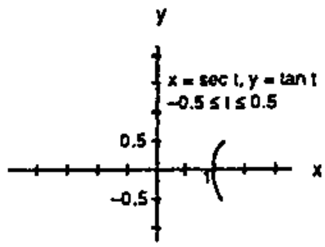
(c)



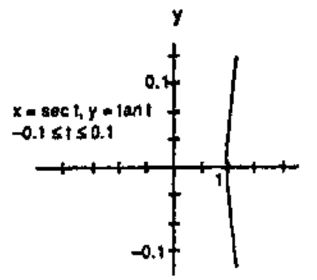
44. (a)



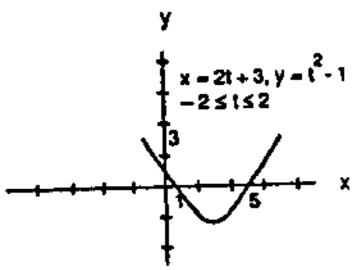
(b)



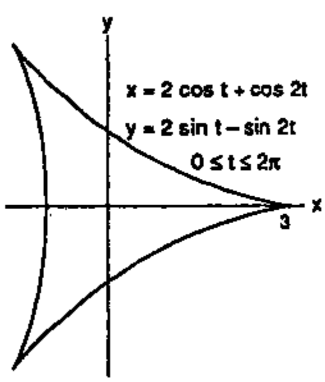
(c)



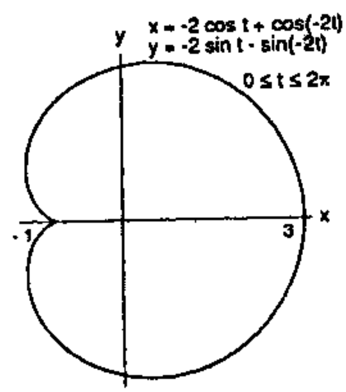
45.



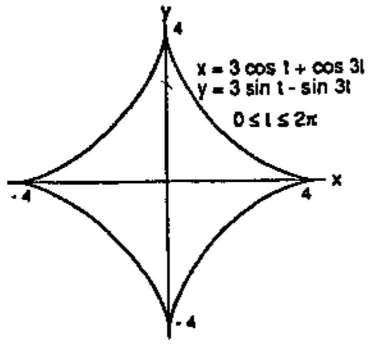
46. (a)



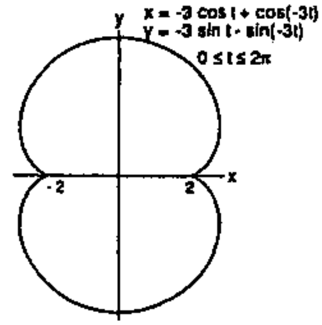
(b)



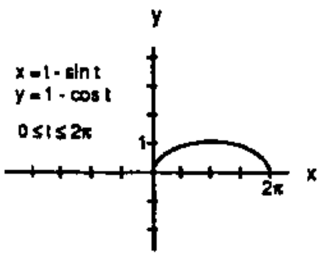
47. (a)



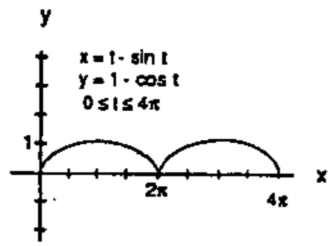
(b)



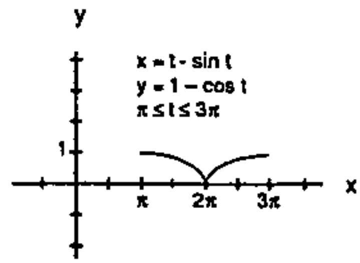
48. (a)



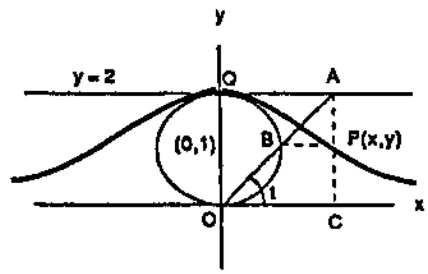
(b)



(c)



49. Extend the vertical line through A to the x-axis and let C be the point of intersection. Then $OC = AQ = x$ and $\tan t = \frac{2}{OC} = \frac{2}{x} \Rightarrow x = \frac{2}{\tan t} = 2 \cot t$; $\sin t = \frac{2}{OA}$ $\Rightarrow OA = \frac{2}{\sin t}$; and $(AB)(OA) = (AQ)^2 \Rightarrow AB\left(\frac{2}{\sin t}\right) = x^2$ $\Rightarrow AB\left(\frac{2}{\sin t}\right) = \left(\frac{2}{\tan t}\right)^2 \Rightarrow AB = \frac{2 \sin t}{\tan^2 t}$. Next $y = 2 - AB \sin t \Rightarrow y = 2 - \left(\frac{2 \sin t}{\tan^2 t}\right) \sin t =$



$2 - \frac{2 \sin^2 t}{\tan^2 t} = 2 - 2\cos^2 t = 2 \sin^2 t$. Therefore let $x = 2 \cot t$ and $y = 2 \sin^2 t$, $0 < t < \pi$.

50. (a) $x = x_0 + (x_1 - x_0)t$ and $y = y_0 + (y_1 - y_0)t \Rightarrow t = \frac{x - x_0}{x_1 - x_0} \Rightarrow y = y_0 + (y_1 - y_0)\left(\frac{x - x_0}{x_1 - x_0}\right)$

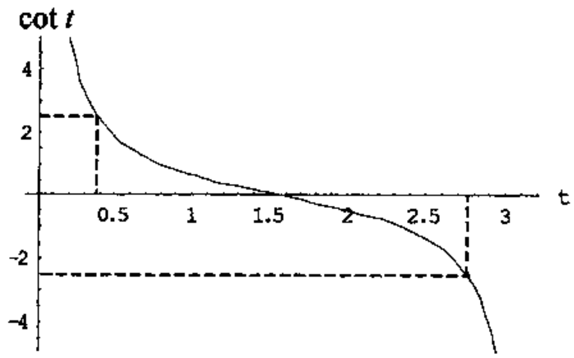
$\Rightarrow y - y_0 = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0)$ which is an equation of the line through the points (x_0, y_0) and (x_1, y_1)

(b) Let $x_0 = y_0 = 0$ in (a) $\Rightarrow x = x_1 t$, $y = y_1 t$ (the answer is not unique)

(c) Let $(x_0, y_0) = (-1, 0)$ and $(x_1, y_1) = (0, 1)$ or let $(x_0, y_0) = (0, 1)$ and $(x_1, y_1) = (-1, 0)$ in part (a) $\Rightarrow x = -1 + t$, $y = t$ or $x = -t$, $y = 1 - t$ (the answer is not unique)

51. (a) $-5 \le x \le 5 \Rightarrow -5 \le 2 \cot t \le 5 \Rightarrow -\frac{5}{2} \le \cot t \le \frac{5}{2}$

The graph of cot t shows where to look for the limits on t.



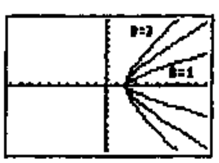
$\tan^{-1}\left(\frac{2}{5}\right) \le t \le \pi + \tan^{-1}\left(-\frac{2}{5}\right) \Rightarrow 0.381 \le t \le 2.761$

The curve is traced from right to left and extends infinitely in both directions from the origin.

(b) For $-\frac{\pi}{2} < t < \frac{\pi}{2}$, the curve is the same as that which is given. It first traces from the vertex at $(0, 2)$ to the left extreme point in the window, and then from the right extreme point in the window to the vertex point. For $0 < t < \frac{\pi}{2}$, only the right half of the curve appears, and it traces from the right extreme of the window to the vertex at $(0, 2)$ and terminates there. For $\frac{\pi}{2} < t < \pi$, only the left half of the curve appears, and it traces from the vertex to the left extreme of the window.

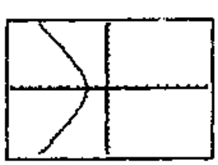
(c) For $x = -2 \cot t$, the curve traces from left to right rather than from right to left. For $x = 2 \cot(\pi - t)$, the curve traces from right to left, as it does with the original parametrization.

52\. (a) The resulting graph appears to be the right half of a hyperbola in the first and fourth quadrants. The parameter a determines the x-intercept. The parameter b determines the shape of the hyperbola. If b is smaller, the graph has less steep slopes and appears "sharper." If b is larger, the slopes are steeper and the graph appears more "blunt." The graphs for a = 2 and b = 1, 2, and 3 are shown.



$[-10, 10]$ by $[-10, 10]$

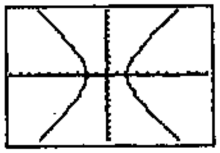
(b)



$[-10, 10]$ by $[-10, 10]$

This appears to be the left half of the same hyperbola.

(c)



$[-10, 10]$ by $[-10, 10]$

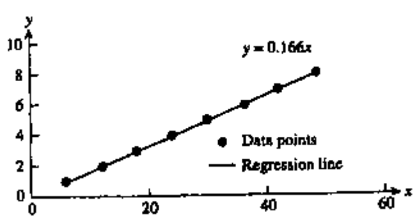
One must be careful because both sec t and tan t are discontinuous at these points. This might cause the grapher to include extraneous lines (the asymptotes of the hyperbola) in its graph. The extraneous lines can be avoided by using the grapher's dot mode instead of connected mode.

(d) Note that $\sec^2 t - \tan^2 t = 1$ by a standard trigonometric identity. Substituting $\frac{x}{a}$ for sec t and $\frac{y}{b}$ for tan t gives $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$.

(e) This changes the orientation of the hyperbola. In this case, b determines the y-intercept of the hyperbola, and a determines the shape. The parameter interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ gives the upper half of the hyperbola. The parameter interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ gives the lower half. The same values of t cause discontinuities and may add extraneous lines to the graph. Substituting $\frac{y}{b}$ for sec t and $\frac{x}{a}$ for tan t in the identity $\sec^2 t - \tan^2 t = 1$ gives $\left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1$.

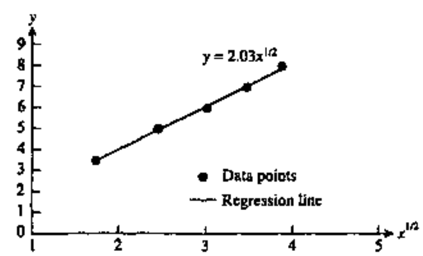
P.7 MODELING CHANGE

1. (a)



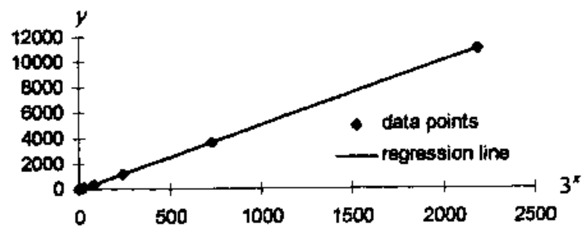
The graph supports the assumption that y is proportional to x. The constant of proportionality is estimated from the slope of the regression line, which is 0.166.

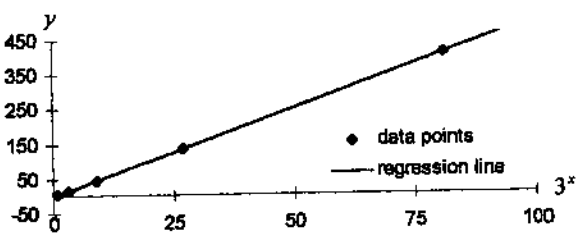
(b)



The graph supports the assumption that y is proportional to $x^{1/2}$. The constant of proportionality is estimated from the slope of the regression line, which is 2.03.

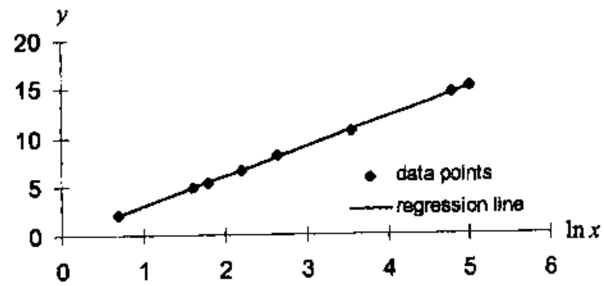
(c) Because of the wide range of values of the data, two graphs are needed to observe all of the points in relation to the regression line.





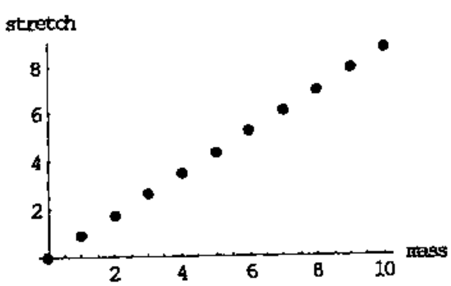
The graphs support the assumption that y is proportional to 3^x. The constant of proportionality is estimated from the slope of the regression line, which is 5.00.

(d)

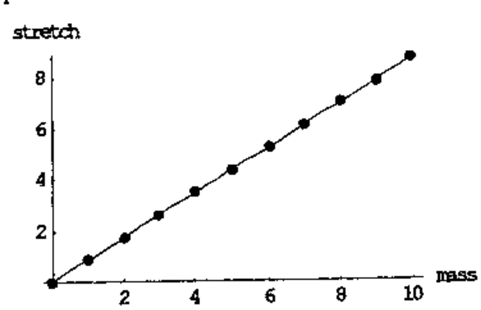


The graph supports the assumption that y is proportional to ln x. The constant of proportionality is estimated from the slope of the regression line, which is 2.99.

2. (a) Plot the data to see if there is a recognizable pattern.

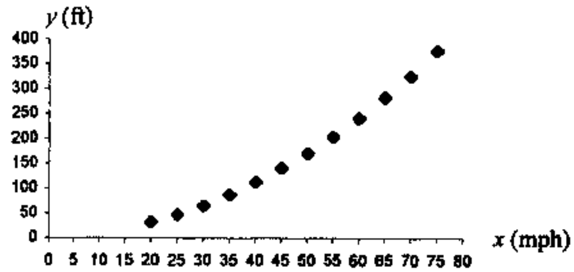


The data clearly suggests a linear relationship. The line of best fit, or the regression line, is s = 0.8742m where s is the stretch in the spring and m is the mass. Now we superimpose the regression line on the graph of the data.

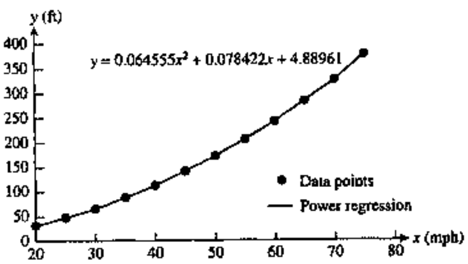


(b) The model fits the data very well.
(c) When $m = 13$, the model gives a stretch of $s = 0.8742(13) = 11.365$. Since this data point is outside the range of the data that the model is based upon, one should feel uncomfortable with this prediction of the stretch without further experimental verification.

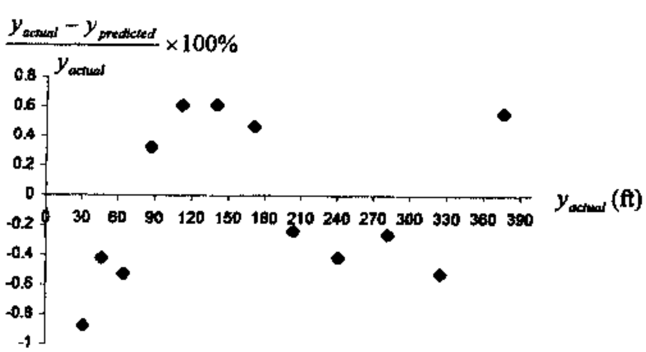
3. First, plot the braking distance versus the speed.



The shape of the graph suggests either a power function or an exponential function to describe the relationship. First, try to fit a quadratic function. Using quadratic regression on the TI-92 Plus calculator gives
$y = 0.064555x^2 + 0.078422x + 4.88961$.



The quadratic regression fits the data well as seen by the following plot of the relative errors versus the actual stopping distance.

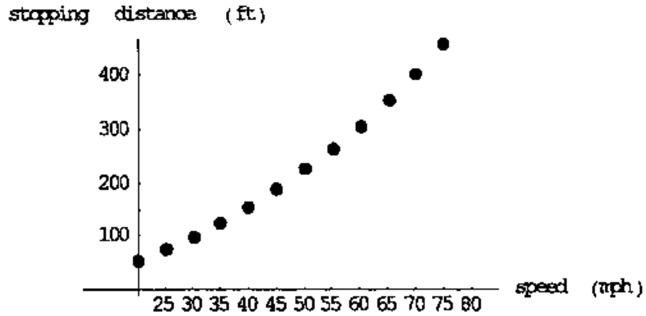


The largest relative error is less than 1%.

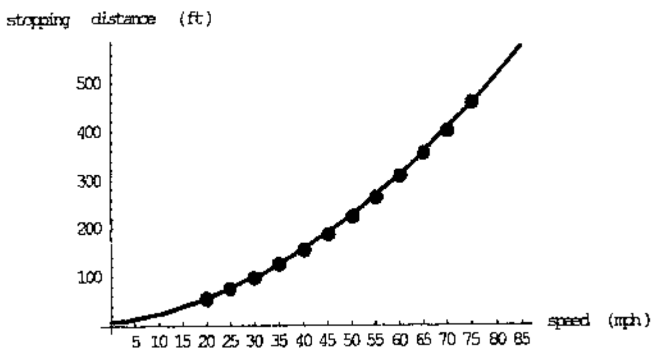
4. The following table gives the total stopping distance (reaction distance + braking distance) for automobile speeds ranging from 20 to 75 miles per hour.

speed	20	25	30	35	40	45	50	55	60	65	70	75
stopping distance	54	75	98	126	156	190	226	265	307	354	402	459

Plot the total stopping distance versus speed.



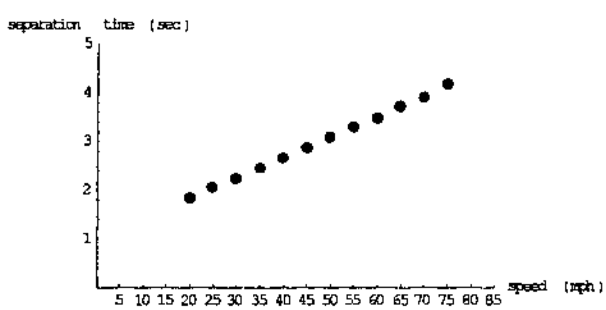
The graph suggests a possible quadratic relationship. Quadratic regression on the data gives $d = 0.0646v^2 + 1.181v + 5.040$ where d is the total stopping distance in feet and v is the travel speed in miles per hour. Now superimpose the quadratic regression on the graph of the data.



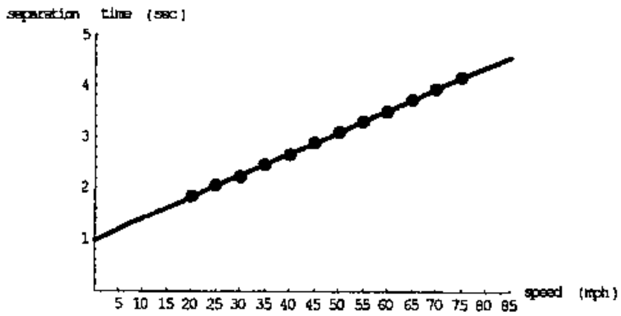
The quadratic regression fits the data very well. To test the 2-second "rule of thumb," calculate the time the vehicle will travel the distance d when it is traveling at speed v. (Don't forget to convert mph into ft/sec using 60 mph = 88 ft/sec.) The following table gives separation times versus travel speed.

v (mph)	20	25	30	35	40	45	50	55	60	65	70	75
t (sec)	1.84	2.05	2.23	2.45	2.66	2.88	3.08	3.29	3.49	3.71	3.92	4.17

Plot the data.

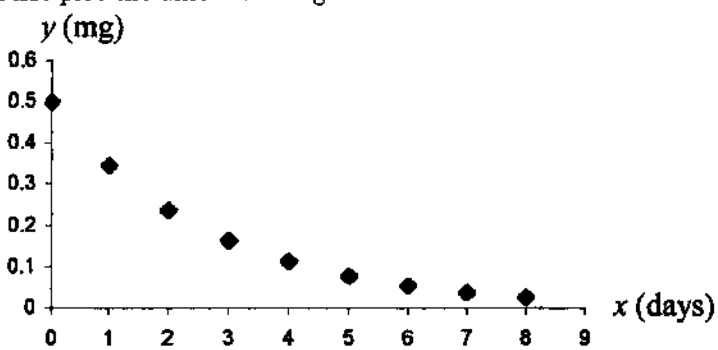


The graph suggests a linear relationship and a linear regression gives $t = 0.042v + 0.983$. Now superimpose the linear regression function on the graph of the data.

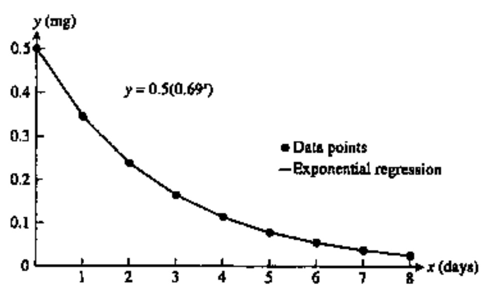


Based on the preceding analysis, a better rule of thumb would be to keep a minimum separation time of 2 seconds and add 1 sec for every 20 mph increment of speed above 20 mph. So, for example, if you are traveling at 40 mph your separation should be $2 + 1(1) = 3$ seconds, at 60 mph your separation should be $2 + 2(1) = 4$ seconds, at 80 mph it should be $2 + 3(1) = 5$ seconds, and so on.

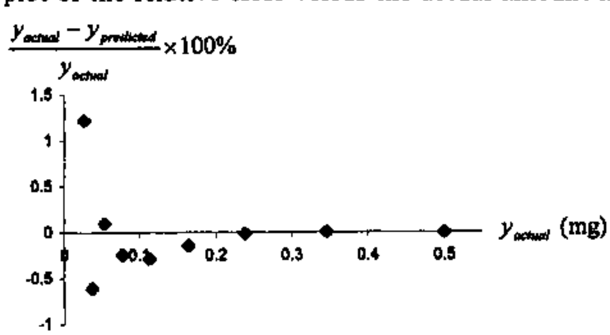
5. (a) First plot the amount of digoxin in the blood versus time.



The graph suggests that the amount decays exponentially with time. The exponential regression on the TI-92 Plus calculator gives $y = 0.5(0.69^x) = 0.5e^{-0.371x}$.

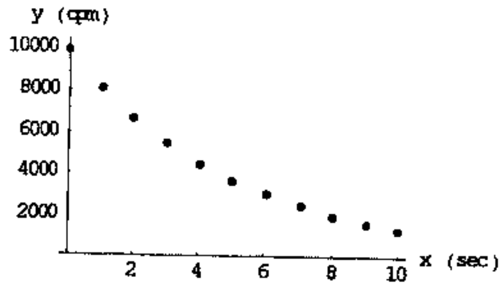


(b) The exponential function fits the data very well as demonstrated by the graph above and the following is a plot of the relative error versus the actual amount in the blood.

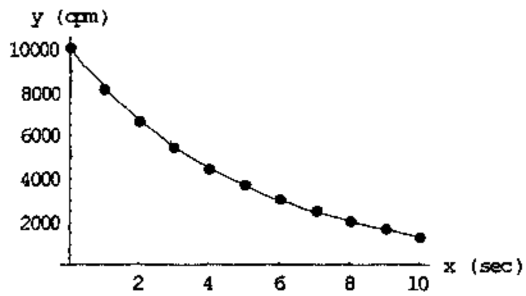


(c) $y(12) = 0.5e^{-0.371(12)} = 0.00583$, therefore, the model predicts that after 12 hours, the amount of digoxin in the blood will be less than 0.006 mg.

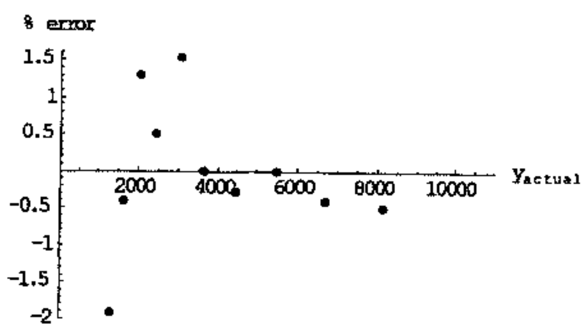
6. (a) Plot the data.



An exponential regression on the TI-92 Plus calculator gives $y = 10{,}037e^{-0.2005x}$. Superimpose the regression function on the graph of the data.



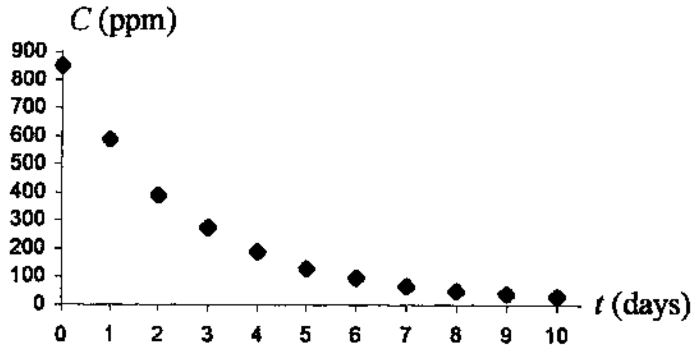
(b) The exponential function fits the data very well as indicated by the graph above. The following is a graph of the relative error, $\frac{y_{\text{predicted}} - y_{\text{actual}}}{y_{\text{actual}}} \times 100\%$, versus y_{actual}.



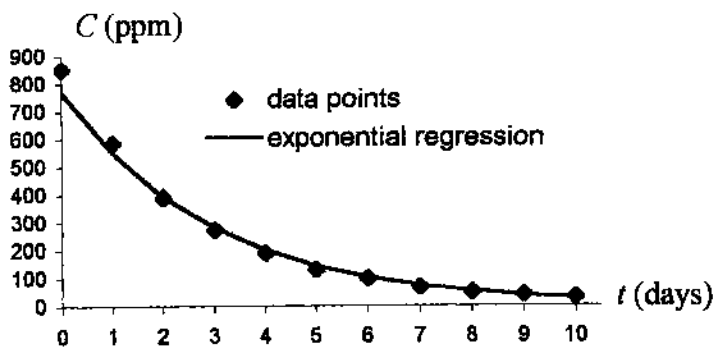
The largest relative error is less than 2% in magnitude.

(c) $500 = 10{,}037e^{-0.2005x} \Rightarrow -0.2005x = \ln\left(\frac{500}{10{,}037}\right) \Rightarrow x = 15.0$ minutes.

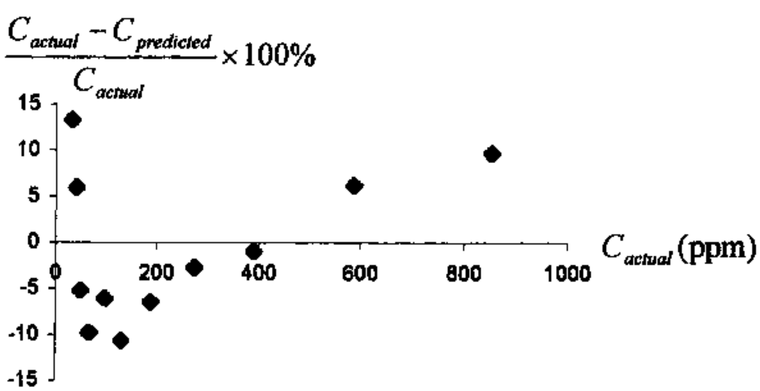
7. (a) First, plot a graph of the blood concentration versus time. Let t represent the elapsed time in days and C the blood concentration in parts per million.



The graph suggests that the amount decays exponentially with time. The exponential regression function on the TI-92 Plus calculator gives $C = 770(0.7146^t) = 770e^{-0.336t}$.



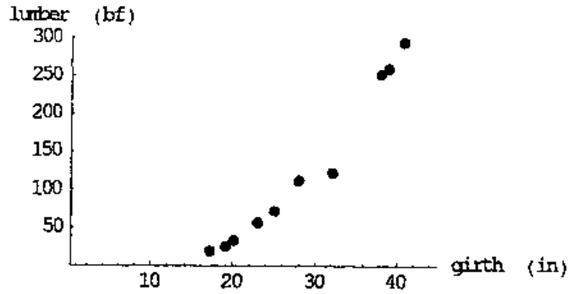
(b) The exponential function appears to capture a trend for this data. The following graph shows the relative errors in the model estimates.



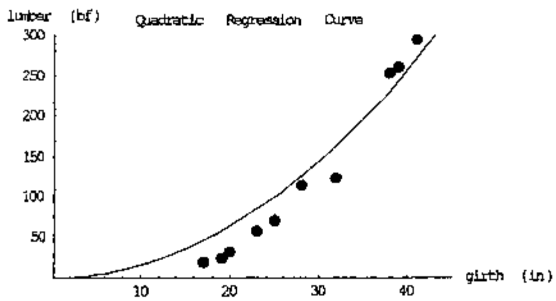
The relative errors in the predicted values are as large as 13.2% and the errors are large for small as well as large blood concentrations. The pattern of the residual errors does not suggest an obvious improvement of the model.

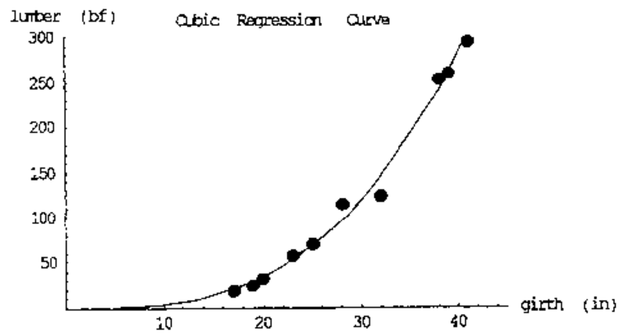
(c) $10 = 770e^{-0.336t} \Rightarrow t = -\frac{1}{0.336}\ln\left(\frac{10}{770}\right) = 12.93$ days. Therefore, the model predicts that the blood concentration will fall below 10 ppm after 12 days and 22 hours.

8. Plot the data.



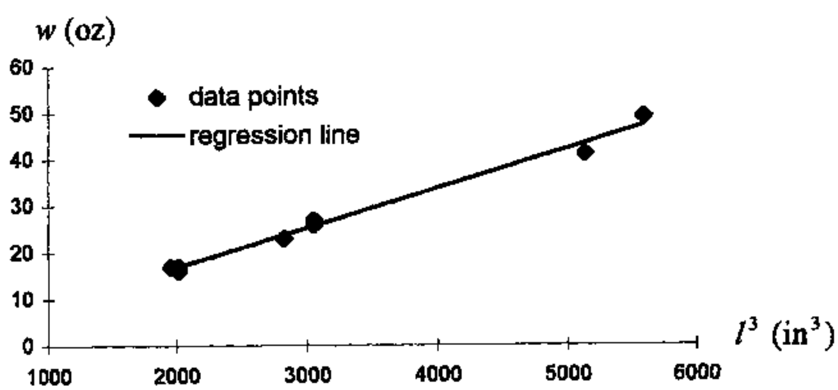
(a) and (b) The quadratic regression function is $y = 0.1579x^2$ where x represents the girth in inches and y the amount of usable lumber in board feet. The cubic regression function is $y = 0.00436x^3$. Superimpose the two regression functions on the graph of the data.



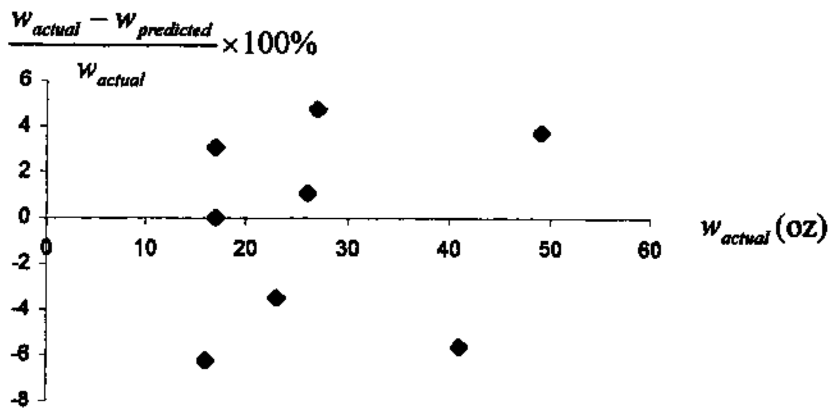


The graphs show that the cubic relationship provides the better model.
Explanation of the model: The unit of board feet is a measure of the volume and, if a tree is modeled as a right circular cone, its volume would be $y = \frac{1}{3}\pi r^2 h$. The girth is the circumference of the tree near the base so that $x = 2\pi r \Rightarrow r = \frac{x}{2\pi}$. If, in addition, we assume that as a tree grows the proportion $\frac{h}{r} = k$, a constant, then we have that $y = \frac{1}{3}\pi\left(\frac{x}{2\pi}\right)^2(kr) = \frac{1}{3}\pi\left(\frac{x}{2\pi}\right)^2\left(\frac{kx}{2\pi}\right) = \frac{k}{24\pi}x^3$, which shows that $y = 0.00436x^3$ is a rational model.

9.

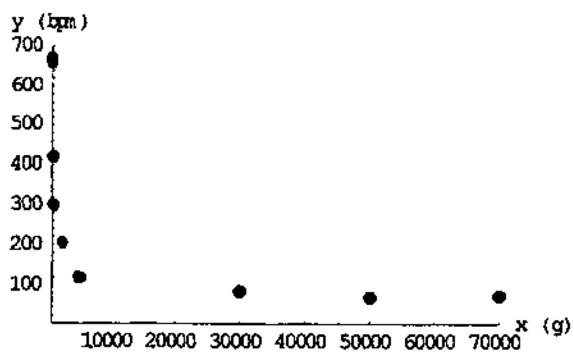


The slope of the regression line is 0.008435, so the model that estimates the weight as a function of L is $w = 0.008435L^3$. The model fits the data reasonably well as demonstrated by the following plot of the relative errors in the weight estimates by the model.

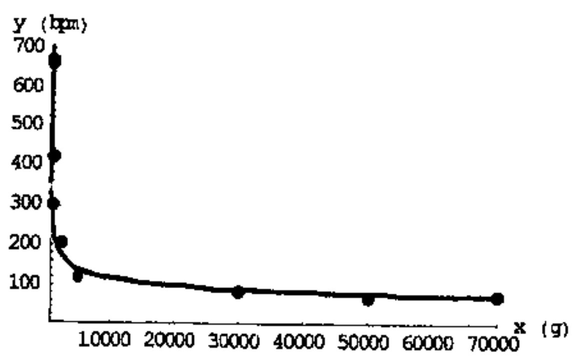


The relative error in the estimated values is always less than 7%.

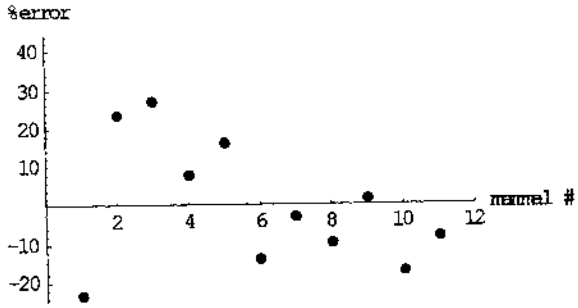
10. The following plot of the data does not include the ox and elephant. However, the data for these mammals are used in the analysis that follows.



There does appear to be a trend. After trying regressions with n = 1, 2, 3, 4, 5, the best fit was found with n = 4. The following graph superimposes the regression function $y = 1150x^{-1/4}$ on the data points.



To test the model, calculate the relative errors (i.e., $\frac{y_{\text{predicted}} - y_{\text{actual}}}{y_{\text{actual}}} \times 100\%$) for all of the mammals in the sample set. These are shown in the following graph.



The errors appear to be random and the largest relative errors are for the two larger animals (i.e., the ox and the elephant) with magnitudes of 92% and 43%, respectively. The model appears to capture a trend in the data, which could be useful in understanding the relationship between mammal size and heart rate; however, it probably would not be useful as a predictive tool.

11. Graph (c). For some drugs, the rate of elimination is proportional to the concentration of the drug in the blood-stream. Graph (c) matches this behavior because the graph falls faster at higher concentrations.

12. Graph (d) or graph (f). Often times, when we begin to learn a new subject, we master the basics quickly at first, but then as the subject becomes more intricate our proficiency increases more slowly. This learning behavior would be described by graph (d). Some subjects have high overhead in terms of learning basic skills and so our proficiency increases slowly at first but, as we acquire the basic skills, our proficiency increases more rapidly. Then, as we reach our intellectual capacity or as our interest wanes, our proficiency will increase more slowly. Graph (f) would match this learning pattern.

13. Graph (c). The rate of decay of radioactive Carbon-14 is proportional to the amount of Carbon-14 present in the artwork. Graph (c) matches this behavior because the graph falls faster at higher amounts.

14. (a) Graph (e). At first, the water velocity is high but as the tank drains the velocity will decrease. When the water level in the tank is high the discharge velocity will decrease slowly, but as time progresses and the water level drops, the discharge velocity will decrease more rapidly.
    (b) Graph (c). Assuming the tank is an upright circular cylinder, the rate at which the water level in the tank falls will be proportional to the rate at which the volume of water in the tank decreases. Also, the rate at which the volume decreases will be proportional to the discharge velocity. Therefore, when the discharge velocity is high at the start, the rate at which the volume decreases will be high and so will the rate of decrease in the water depth. As the discharge velocity decreases, the rate at which the water depth drops will also decrease. This behavior is depicted by graph (c).

15. (a) One possibility: If an item sells for \$p and x is the number of items sold, then the revenue from sales will be $y = px$, and the graph of the revenue function looks like graph (a).
    (b) One possibility: If y is the number of deer in a very large game reserve with unlimited resources to support the deer and x represents the number of years elapsed, then the population would exhibit unconstrained growth over time. In this situation, the population can be modeled by an exponential growth function like $y = y_0 e^{kx}$, where y_0 is the initial deer population, k is a constant, and the growth of the function looks like graph (b).
    (c) One possibility: If y represents the selling price per unit that can be realized for a certain commodity, say grape jelly for example, and x represents the availability of the commodity, then the unit selling price for the commodity is often times inversely proportional to its availability. This relationship can be modeled with a function of the form $y = \frac{y_0}{(x+1)^\alpha}$, where y_0 is the unit selling price when no product is available, α is a positive constant, and the graph of the function looks like graph (c).

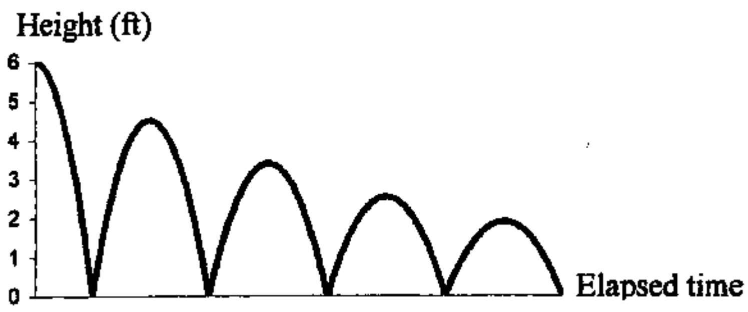
(d) One possibility: Let y represent the speed of your car and x represent the amount of time after you punch the accelerator. At first you will rapidly accelerate but, as the car picks up speed, the rate of acceleration (i.e., the rate at which the car speeds up) decreases. This can be modeled by a function like $y = y_{new} + (y_0 - y_{new})e^{-kx}$, where y_0 is the speed you were traveling when you stepped on it, y_{new} is the new speed you achieve when you are done accelerating, and k is a positive constant (determined in part by the size of your engine and how good your traction is). The graph of this function looks like graph(d).

(e) One possibility: Let y represent the amount you owe on your credit card and x represent the number of monthly payments you have made. At first the amount you owe decreases slowly because most of your payment goes toward paying the monthly interest charge. But, as the amount you owe decreases, the interest charge decreases and your payment makes a bigger difference toward reducing the debt. This can be modeled with a function like the one represented by graph (c).

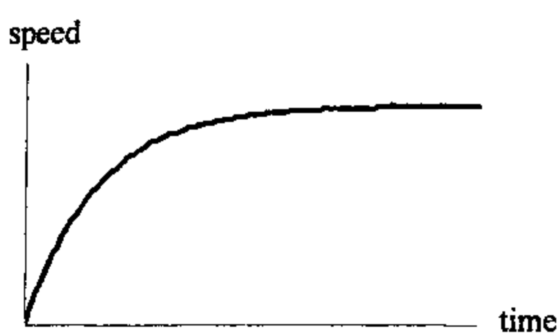
(f) One possibility: Let y represent the number of people in your school who have the flu and let x represent the number of days that have elapsed after the first person gets sick. At first the flu doesn't spread very quickly because there are only a few sick people to pass it on. But, as more people get sick the disease spreads more rapidly. The most volatile mixture is when half the people are sick, because then there are a lot of sick people to spread the disease and a lot of uninfected people who can still catch it. As time continues and more people get sick, there are fewer and fewer people available to catch the flu and the spread of the disease begins to slow down. This behavior can be modeled with a function like the one represented by graph (f).

16. The intensity of light will probably decrease linearly as the number of layers of plastic increases. If I_0 is the intensity with no layers of plastic, then the relationship would be $I = I_0 - kn$, where n is the number of layers and k is a constant.

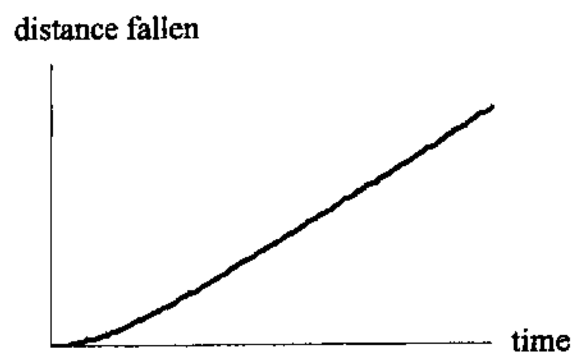
17.



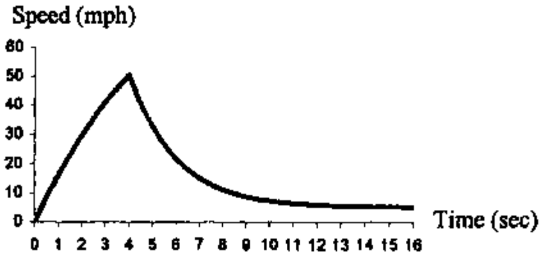
18. (a)



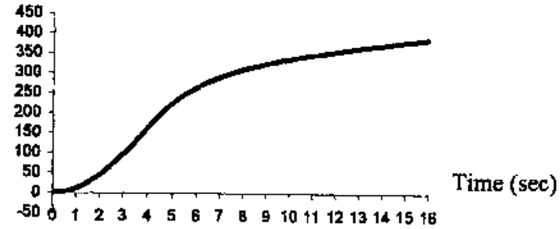
(b)



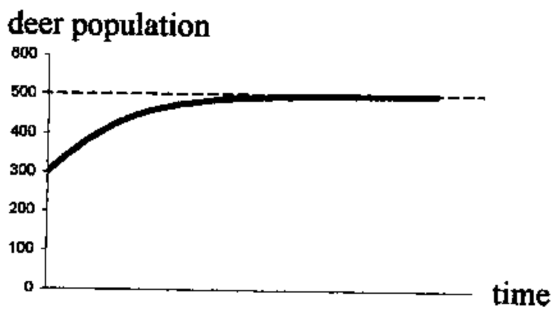
19. (a)



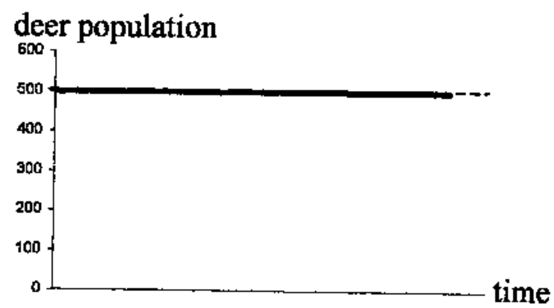
(b)



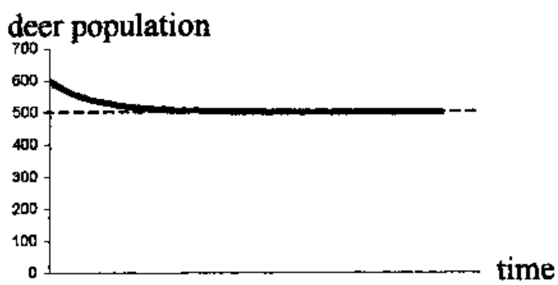
20. (a)



(b)



(c)



21. (a) The graph could represent the angle that a pendulum makes with the vertical as it swings back and forth. The variable y represents the angle and x represents time. Because of friction, the amplitude of the oscillation decays, as depicted by the graph. When y is positive, the pendulum is on one side of the vertical and when y is negative, the pendulum is on the other side.

(b) The graph could represent the angle the playground swing makes with the vertical as a child "pumps" on the swing to get it going. The variable y represents the angle and x represents time. Because the child puts mechanical energy into the system (swing + child), the amplitude of the oscillation grows with time, as depicted by the graph. When y is positive, the swing is on one side of the vertical and when y is negative, the swing is on the other side.

22. Answers will vary. An example follows.

(f) I would like to study the effect that the geometric configuration of a group of four light poles and fixtures would have on the illumination intensity on the ground. More specifically, if four light poles are arranged in a square, how is the light intensity on the ground at the center of the square (where the light intensity is assumed to be minimum) affected by the spacing of the poles? To determine the effect of the pole spacing, I will need to design an experiment to measure the intensity of light at the center of the square pattern as the pole spacing is varied. After collecting the data I would then try to find a mathematical model to fit the data. The parking lot designer could use this model to determine the maximum pole spacing, given the minimum required light intensity on the ground. In addition to the spacing of the poles, some other variables that affect the illumination are the type, size and brightness of the light sources, shadowing by other objects, and the height of the poles.

PRELIMINARY CHAPTER PRACTICE EXERCISES

1. $y = 3(x-1) + (-6)$
   $y = 3x - 9$

2. $y = -\frac{1}{2}(x+1) + 2$

   $y = -\frac{1}{2}x + \frac{3}{2}$

3. $x = 0$

4. $m = \frac{-2-6}{1-(-3)} = \frac{-8}{4} = -2$

   $y = -2(x+3) + 6$

   $y = -2x$

5. $y = 2$

6. $m = \frac{5-3}{-2-3} = \frac{2}{-5} = -\frac{2}{5}$

   $y = -\frac{2}{5}(x-3) + 3$

   $y = -\frac{2}{5}x + \frac{21}{5}$

7. $y = -3x + 3$

8. Since $2x - y = -2$ is equivalent to $y = 2x + 2$, the slope of the given line (and hence the slope of the desired line) is 2.
   $y = 2(x-3) + 1$
   $y = 2x - 5$

9. Since $4x + 3y = 12$ is equivalent to $y = -\frac{4}{3}x + 4$, the slope of the given line (and hence the slope of the desired line) is $-\frac{4}{3}$.

   $y = -\frac{4}{3}(x-4) - 12$

   $y = -\frac{4}{3}x - \frac{20}{3}$

10. Since $3x - 5y = 1$ is equivalent to $y = \frac{3}{5}x - \frac{1}{5}$, the slope of the given line is $\frac{3}{5}$ and the slope of the perpendicular line is $-\frac{5}{3}$.

$y = -\frac{5}{3}(x+2) - 3$

$y = -\frac{5}{3}x - \frac{19}{3}$

11. Since $\frac{1}{2}x + \frac{1}{3}y = 1$ is equivalent to $y = -\frac{3}{2}x + 3$, the slope of the given line is $-\frac{3}{2}$ and the slope of the perpendicular line is $\frac{2}{3}$.

$y = \frac{2}{3}(x+1) + 2$

$y = \frac{2}{3}x + \frac{8}{3}$

12. The line passes through $(0, -5)$ and $(3, 0)$.

$m = \frac{0-(-5)}{3-0} = \frac{5}{3}$

$y = \frac{5}{3}x - 5$

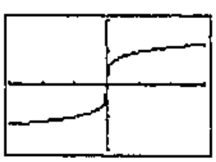
13. The area is $A = \pi r^2$ and the circumference is $C = 2\pi r$. Thus, $r = \frac{C}{2\pi} \Rightarrow A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{4\pi}$.

14. The surface area is $S = 4\pi r^2 \Rightarrow r = \left(\frac{S}{4\pi}\right)^{1/2}$ The volume is $V = \frac{4}{3}\pi r^3 \Rightarrow r = \sqrt[3]{\frac{3V}{4\pi}}$. Substitution into the formula for surface area gives $S = 4\pi r^2 = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$

15. The coordinates of a point on the parabola are (x, x^2). The angle of inclination θ joining this point to the origin satisfies the equation $\tan\theta = \frac{x^2}{x} = x$. Thus the point has coordinates $(x, x^2) = (\tan\theta, \tan^2\theta)$.

16. $\tan\theta = \frac{\text{rise}}{\text{run}} = \frac{h}{500} \Rightarrow h = 500 \tan\theta$ ft.

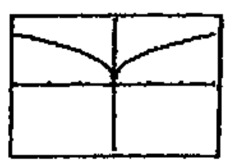
17.



$[-3, 3]$ by $[-2, 2]$

Symmetric about the origin.

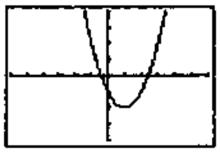
18.



$[-3, 3]$ by $[-2, 2]$

Symmetric about the y-axis.

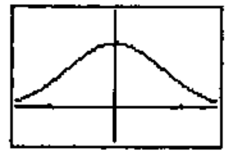
19.



[−6, 6] by [−4, 4]

Neither

20.



[−1.5, 1.5] by [−0.5, 1.5]

Symmetric about the y-axis.

21. $y(-x) = (-x)^2 + 1 = x^2 + 1 = y(x)$
Even

22. $y(-x) = (-x)^5 - (-x) = -x^5 + x^3 + x = -y(x)$
Odd

23. $y(-x) = 1 - \cos(-x) = 1 - \cos x = y(x)$
Even

24. $y(-x) = \sec(-x)\tan(-x)$
$$= \frac{\sin(-x)}{\cos^2(-x)} = \frac{-\sin x}{\cos^2 x}$$
$$= -\sec x \tan x = -y(x)$$
Odd

25. $y(-x) = \dfrac{(-x)^4 + 1}{(-x)^3 - 2(-x)} = \dfrac{x^4+1}{-x^3+2x} = -\dfrac{x^4+1}{x^3-2x} = -y(x)$

Odd

26. $y(-x) = 1 - \sin(-x) = 1 + \sin x$
Neither even nor odd

27. $y(-x) = -x + \cos(-x) = -x + \cos x$
Neither even nor odd

28. $y(-x) = \sqrt{(-x)^4 - 1} = \sqrt{x^4 - 1} = y(x)$
Even

29. (a) The function is defined for all values of x, so the domain is $(-\infty, \infty)$.
(b) Since $|x|$ attains all nonnegative values, the range is $[-2, \infty)$.

30. (a) Since the square root requires $1 - x \geq 0$, the domain is $(-\infty, 1]$.
(b) Since $\sqrt{1-x}$ attains all nonnegative values, the range is $[-2, \infty)$.

31. (a) Since the square root requires $16 - x^2 \geq 0$, the domain is $[-4, 4]$
(b) For values of x in the domain, $0 \leq 16 - x^2 \leq 16$, so $0 \leq \sqrt{16 - x^2} \leq 4$. The range is $[0, 4]$.

32. (a) The function is defined for all value os x, so the domain is $(-\infty, \infty)$.
(b) Since 3^{2-x} attains all possible values, the range is $(1, \infty)$.

33. (a) The function is defined for all values of x, so the domain is $(-\infty, \infty)$.
(b) Since $2e^{-x}$ attains all positive values, the range is $(-3, \infty)$.

34. (a) The function is equivalent to $y = \tan 2x$, so we require $2x \neq \frac{k\pi}{2}$ for odd integers k. The domain is given by $x \neq \frac{k\pi}{4}$ for odd integers k.
(b) Since the tangent function attains all values, the range is $(-\infty, \infty)$.

35. (a) The function is defined for all values of x, so the domain is $(-\infty, \infty)$.
(b) The sine function attains values from -1 to 1, so $-2 \leq 2\sin(3x + \pi) \leq 2$, and hence $-3 \leq 2\sin(3x + \pi) - 1 \leq 1$. The range is $[-3, 1]$.

36. (a) The function is defined for all values of x, so the domain is $(-\infty, \infty)$.
(b) The function is equivalent to $y = \sqrt[5]{x^2}$, which attains all nonnegative values. The range is $[0, \infty)$.

37. (a) The logarithm requires $x - 3 > 0$, so the domain is $(3, \infty)$.
(b) The logarithm attains all real values, so the range is $(-\infty, \infty)$.

38. (a) The function is defined for all values of x, so the domain is $(-\infty, \infty)$.
(b) The cube root attains all real values, so the range is $(-\infty, \infty)$.

39. (a) The function is defined for $-4 \leq x \leq 4$, so the domain is $[-4, 4]$.
(b) The function is equivalent to $y = \sqrt{|x|}$, $-4 \leq x \leq 4$, which attains values from 0 to 2 for x in the domain. The range is $[0, 2]$.

40. (a) The function is defined for $-2 \leq x \leq 2$, so the domain is $[-2, 2]$.
(b) The range is $[-1, 1]$.

41. First piece: Line through $(0, 1)$ and $(1, 0)$

$m = \frac{0-1}{1-0} = \frac{-1}{1} = -1$

$y = -x + 1$ or $1 - x$

Second piece: Line through $(1, 1)$ and $(2, 0)$

$m = \frac{0-1}{2-1} = \frac{-1}{1} = -1$

$y = -(x - 1) + 1$

$y = -x + 2$ or $2 - x$

$$f(x) = \begin{cases} 1 - x, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases}$$

42. First piece: Line through $(0, 0)$ and $(2, 5)$

$m = \frac{5-0}{2-0} = \frac{5}{2}$

$y = \frac{5}{2}x$

Second piece: Line through $(2, 5)$ and $(4, 0)$

$m = \frac{0-5}{4-2} = \frac{-5}{2} = -\frac{5}{2}$

$y = -\frac{5}{2}(x - 2) + 5$

$y = -\frac{5}{2}x + 10$ or $10 - \frac{5x}{2}$

$$f(x) = \begin{cases} \frac{5x}{2}, & 0 \leq x < 2 \\ 10 - \frac{5x}{2}, & 2 \leq x \leq 4 \end{cases}$$

(Note: x = 2 can be included on either piece.)

43. (a) $(f \circ g)(-1) = f(g(-1)) = f\left(\frac{1}{\sqrt{-1+2}}\right) = f(1) = \frac{1}{1} = 1$

(b) $(g \circ f)(2) = g(f(2)) = g\left(\frac{1}{2}\right) = \frac{1}{\sqrt{1/2+2}} = \frac{1}{\sqrt{2.5}}$ or $\sqrt{\frac{2}{5}}$

(c) $(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{1/x} = x,\ x \neq 0$

(d) $(g \circ g)(x) = g(g(x)) = g\left(\frac{1}{\sqrt{x+2}}\right) = \frac{1}{\sqrt{1/\sqrt{x+2}+2}}$

$$= \frac{\sqrt[4]{x+2}}{\sqrt{1+2\sqrt{x+2}}}$$

44. (a) $(f \circ g)(-1) = f(g(-1))$

$$= f\left(\sqrt[3]{-1+1}\right)$$

$$= f(0) = 2 - 0 = 2$$

(b) $(g \circ f)(2)g(f(2)) = g(2-2) = g(0) = \sqrt[3]{0+1} = 1$

(c) $(f \circ f)(x) = f(f(x)) = f(2-x) = 2-(2-x) = x$

(d) $(g \circ g)(x) = g(g(x)) = g\left(\sqrt[3]{x+1}\right) = \sqrt[3]{\sqrt[3]{x+1}+1}$

45. (a) $(f \circ g)(x) = f(g(x))$

$$= f\left(\sqrt{x+2}\right)$$

$$= 2 - \left(\sqrt{x+2}\right)^2$$

$$= -x,\ x \geq -2$$

$(g \circ f)(x) = g(f(x))$

$$= g\left(2-x^2\right)$$

$$= \sqrt{(2-x^2)+2} = \sqrt{4-x^2}$$

(b) Domain of $f \circ g$: $[-2, \infty)$

Domain of $g \circ f$: $[-2, 2]$

(c) Range of $f \circ g$: $(-\infty, 2]$

Range of $g \circ f$: $[0, 2]$

46. (a) $(f \circ g)(x) = f(g(x))$

$$= f\left(\sqrt{1-x}\right)$$

$$= \sqrt{\sqrt{1-x}}$$

$$= \sqrt[4]{1-x}$$

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{1-\sqrt{x}}$$

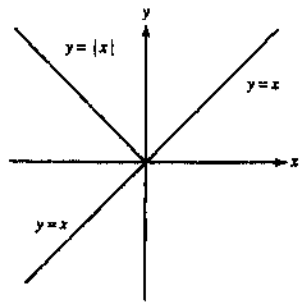
(b) Domain of $f \circ g$: $(-\infty, 1]$

Domain of $g \circ f$: $[0, 1]$

(c) Range of $f \circ g$: $[0, \infty)$

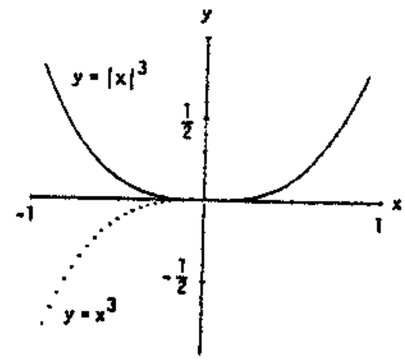
Range of $g \circ f$: $[0, 1]$

47.



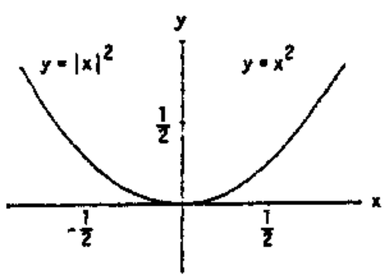
The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x)$, $x \geq 0$ across the y-axis.

48.



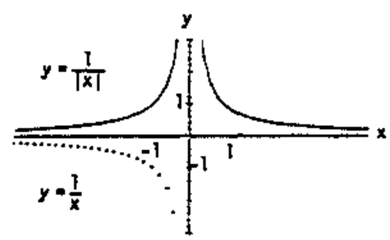
The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x)$, $x \geq 0$ across the y-axis.

49.



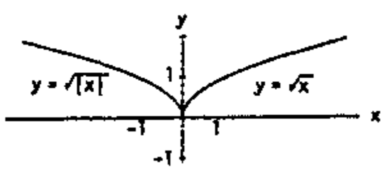
It does not change the graph.

50.



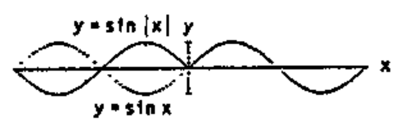
The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x)$, $x \geq 0$ across the y-axis.

51.



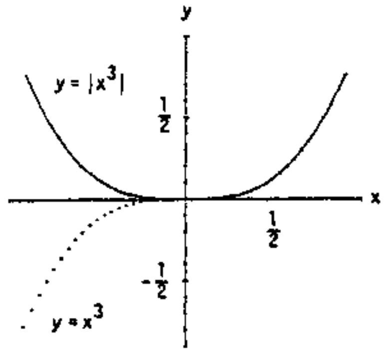
The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x)$, $x \geq 0$ across the y-axis.

52.



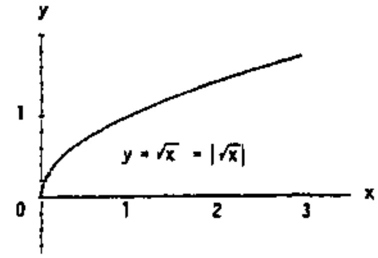
The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x)$, $x \geq 0$ across the y-axis.

53.



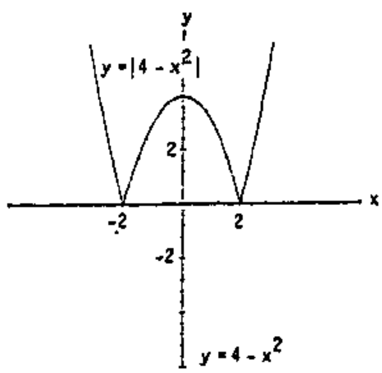
Whenever $g_1(x)$ is positive, the graph of $y = g_2(x) = |g_1(x)|$ is the same as the graph of $y = g_1(x)$. When $g_1(x)$ is negative, the graph of $y = g_2(x)$ is the reflection of the graph of $y = g_1(x)$ across the x-axis.

54.



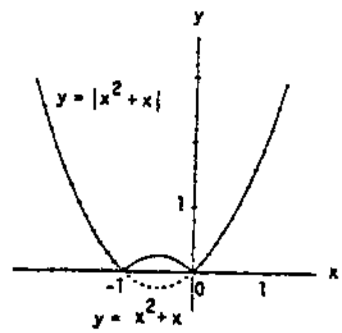
It does not change the graph.

55.



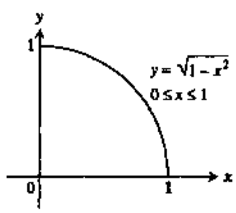
Whenever $g_1(x)$ is positive, the graph of $y = g_2(x) = |g_1(x)|$ is the same as the graph of $y = g_1(x)$. When $g_1(x)$ is negative, the graph of $y = g_2(x)$ is the reflection of the graph of $y = g_1(x)$ across the x-axis.

56.



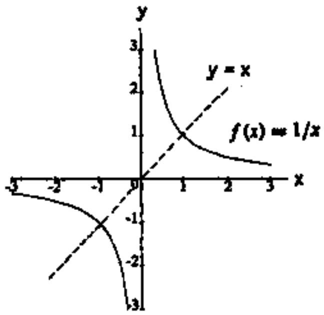
Whenever $g_1(x)$ is positive, the graph of $y = g_2(x) = |g_1(x)|$ is the same as the graph of $y = g_1(x)$. When $g_1(x)$ is negative, the graph of $y = g_2(x)$ is the reflection of the graph of $y = g_1(x)$ across the x-axis.

57. (a) The graph is symmetric about $y = x$.



(b) $y = \sqrt{1 - x^2} \Rightarrow y^2 = 1 - x^2 \Rightarrow x^2 = 1 - y^2 \Rightarrow x = \sqrt{1 - y^2} \Rightarrow y = \sqrt{1 - x^2} = f^{-1}(x)$

58. The graph is symmetric about $y = x$.



(b) $y = \frac{1}{x} \Rightarrow x = \frac{1}{y} \Rightarrow y = \frac{1}{x} = f^{-1}(x)$

59. (a) $y = 2 - 3x \to 3x = 2 - y \to x = \frac{2-y}{3}$.

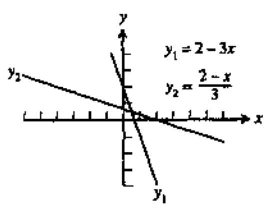
Interchange x and y: $y = \frac{2-x}{3} \to f^{-1}(x) = \frac{2-x}{3}$

Verify.

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{2-x}{3}\right) = 2 - 3\left(\frac{2-x}{3}\right) = 2 - (2-x) = x$

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(2-3x) = \frac{2-(2-3x)}{3} = \frac{3x}{3} = x$

(b)



60. (a) $y = (x+2)^2$, $x \ge -2 \to \sqrt{y} = x + 2 \to x = \sqrt{y} - 2$.

Interchange x and y: $y = \sqrt{x} - 2 \to f^{-1}(x) = \sqrt{x} - 2$

Verify.

For $x \ge 0$ (the domain of f^{-1})

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(\sqrt{x} - 2) = [(\sqrt{x} - 2) + 2]^2 = (\sqrt{x})^2 = x$

For $x \ge -2$ (the domain of f),

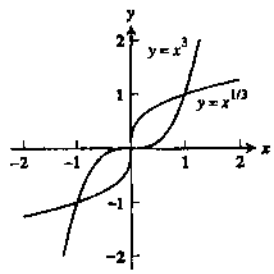
$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}((x+2)^2) = \sqrt{(x+2)^2} - 2 = |x+2| - 2 = (x+2) - 2 = x$

(b)

[−6, 12] by [−4, 8]

61. (a) $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x$, $g(f(x)) = \sqrt[3]{x^3} = x$

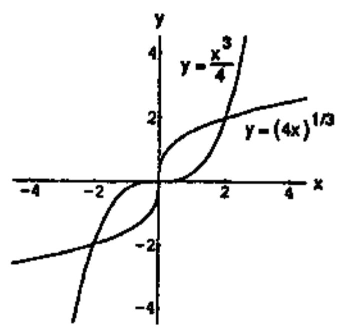
(b)



62. (a) $h(k(x)) = \frac{1}{4}\left((4x)^{1/3}\right)^3 = x$,

$k(h(x)) = \left(4 \cdot \frac{x^3}{4}\right)^{1/3} = x$

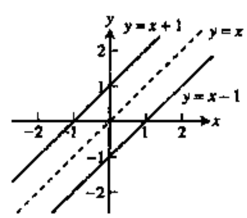
(b)



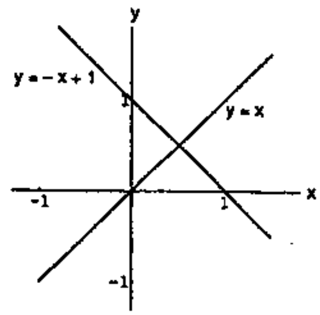
63. (a) $y = x + 1 \Rightarrow x = y - 1 \Rightarrow f^{-1}(x) = x - 1$

(b) $y = x + b \Rightarrow x = y - b \Rightarrow f^{-1}(x) = x - b$

(c) Their graphs will be parallel to one another and lie on opposite sides of the line $y = x$ equidistant from that line.



64. (a) $y = -x + 1 \Rightarrow x = -y + 1 \Rightarrow f^{-1}(x) = 1 - x$;
the lines intersect at a right angle
(b) $y = -x + b \Rightarrow x = -y + b \Rightarrow f^{-1}(x) = b - x$;
the lines intersect at a right angle
(c) f is its own inverse



65. $x = 2.71828182846$ (using a TI-92 Plus calculator).

66. $e^{\ln x} = x$ and $\ln(e^x) = x$ for all $x > 0$

67. (a) $e^{\ln 7.2} = 7.2$ (b) $e^{-\ln x^2} = \frac{1}{e^{\ln x^2}} = \frac{1}{x^2}$ (c) $e^{\ln x - \ln y} = e^{\ln(x/y)} = \frac{x}{y}$

68. (a) $e^{\ln(x^2+y^2)} = x^2 + y^2$ (b) $e^{-\ln 0.3} = \frac{1}{e^{\ln 0.3}} = \frac{1}{0.3} = \frac{10}{3}$ (c) $e^{\ln \pi x - \ln 2} = e^{\ln(\pi x/2)} = \frac{\pi x}{2}$

69. (a) $2 \ln \sqrt{e} = 2 \ln e^{1/2} = (2)\left(\frac{1}{2}\right) \ln e = 1$ (b) $\ln(\ln e^e) = \ln(e \ln e) = \ln e = 1$
(c) $\ln e^{(-x^2-y^2)} = (-x^2 - y^2) \ln e = -x^2 - y^2$

70. (a) $\ln(e^{\sec \theta}) = (\sec \theta)(\ln e) = \sec \theta$ (b) $\ln e^{(e^x)} = (e^x)(\ln e) = e^x$
(c) $\ln(e^{2 \ln x}) = \ln\left(e^{\ln x^2}\right) = \ln x^2 = 2 \ln x$

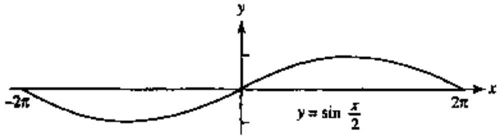
71. Using a calculator, $\sin^{-1}(0.6) \approx 0.6435$ radians or $36.8699°$.

72. Using a calculator, $\tan^{-1}(-2.3) \approx -1.1607$ radians or $-66.5014°$.

73. Since $\cos \theta = \frac{3}{7}$ and $0 \le \theta \le \pi$, $\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \left(\frac{3}{7}\right)^2} = \sqrt{\frac{40}{49}} = \frac{\sqrt{40}}{7}$. Therefore,
$\sin \theta = \frac{\sqrt{40}}{7}$, $\cos \theta = \frac{3}{7}$, $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{40}}{3}$, $\cot \theta = \frac{1}{\tan \theta}, = \frac{3}{\sqrt{40}}$, $\sec \theta = \frac{1}{\cos \theta} = \frac{7}{3}$, $\csc \theta = \frac{1}{\sin \theta} = \frac{7}{\sqrt{40}}$

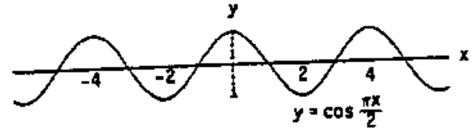
74. (a) Note that $\sin^{-1}(-0.2) \approx -0.2014$. In $[0, 2\pi)$, the solutions are $x = \pi - \sin^{-1}(-0.2) \approx 3.3430$ and $x = \sin^{-1}(-0.2) + 2\pi \approx 6.0818$.
(b) Since the period of sin x is 2π, the solutions are $x \approx 3.3430 + 2k\pi$ and $x \approx 6.0818 + 2k\pi$, k any integer.

75.



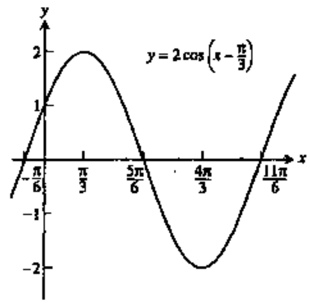
period $= 4\pi$

76.



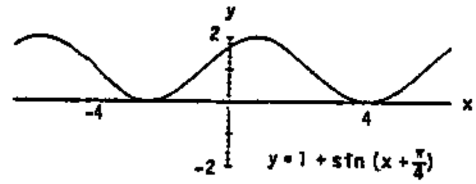
period $= 4$

77.



period $= 2\pi$

78.



period $= 2\pi$

79. (a) $\sin B = \sin \frac{\pi}{3} = \frac{b}{c} = \frac{b}{2} \Rightarrow b = 2 \sin \frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$. By the theorem of Pythagoras,
$a^2 + b^2 = c^2 \Rightarrow a = \sqrt{c^2 - b^2} = \sqrt{4 - 3} = 1.$

(b) $\sin B = \sin \frac{\pi}{3} = \frac{b}{c} = \frac{2}{c} \Rightarrow c = \frac{2}{\sin \frac{\pi}{3}} = \frac{2}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{4}{\sqrt{3}}$. Thus, $a = \sqrt{c^2 - b^2} = \sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - (2)^2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$.

80. (a) $\sin A = \frac{a}{c} \Rightarrow a = c \sin A$

(b) $\tan A = \frac{a}{b} \Rightarrow a = b \tan A$

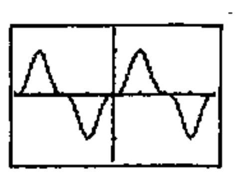
81. (a) $\tan B = \frac{b}{a} \Rightarrow a = \frac{b}{\tan B}$

(b) $\sin A = \frac{a}{c} \Rightarrow c = \frac{a}{\sin A}$

82. (a) $\sin A = \frac{a}{c}$

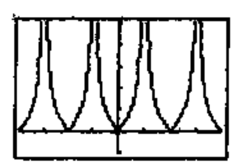
(c) $\sin A = \frac{a}{c} = \frac{\sqrt{c^2 - b^2}}{c}$

83. Since sin x has period 2π, $\sin^3(x + 2\pi) = \sin^3(x)$. This function has period 2π. A graph shows that no smaller number works for the period.



$[-2\pi, 2\pi]$ by $[-1.5, 1.5]$

84. Since tan x has period π, $|\tan(x + \pi)| = |\tan x|$. This function has period π. A graph shows that no smaller number works for the period.



$[-2\pi, 2\pi]$ by $[-1, 5]$

85. $\cos\left(x + \frac{\pi}{2}\right) = \cos x \cos\left(\frac{\pi}{2}\right) - \sin x \sin\left(\frac{\pi}{2}\right) = (\cos x)(0) - (\sin x)(1) = -\sin x$

86. $\sin\left(x - \frac{\pi}{2}\right) = \sin x \cos\left(-\frac{\pi}{2}\right) + \cos x \sin\left(-\frac{\pi}{2}\right) = (\sin x)(0) + (\cos x)(-1) = -\cos x$

87. $\sin \frac{7\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \sin \frac{\pi}{4} \cos \frac{\pi}{3} + \cos \frac{\pi}{4} \sin \frac{\pi}{3} = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}$

88. $\cos \frac{11\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) = \cos \frac{\pi}{4} \cos \frac{2\pi}{3} - \sin \frac{\pi}{4} \sin \frac{2\pi}{3} = \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{2} + \sqrt{6}}{4}$

89. (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$ (c) $\frac{\pi}{3}$

90. (a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{5\pi}{6}$

91. (a) $\frac{\pi}{4}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{3}$

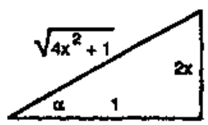
92. (a) $\frac{\pi}{4}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{3}$

93. $\sec\left(\cos^{-1}\frac{1}{2}\right)=\sec\left(\frac{\pi}{3}\right)=2$

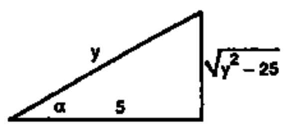
94. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)=\cot\left(-\frac{\pi}{3}\right)=-\frac{1}{\sqrt{3}}$

95. $\tan\left(\sec^{-1}1\right)+\sin\left(\csc^{-1}(-2)\right)=\tan\left(\cos^{-1}\frac{1}{1}\right)+\sin\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)=\tan(0)+\sin\left(-\frac{\pi}{6}\right)=0+\left(-\frac{1}{2}\right)=-\frac{1}{2}$

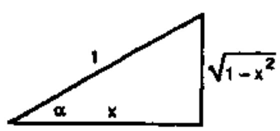
96. $\sec\left(\tan^{-1}1+\csc^{-1}1\right)=\sec\left(\frac{\pi}{4}+\sin^{-1}\frac{1}{1}\right)=\sec\left(\frac{\pi}{4}+\frac{\pi}{2}\right)=\sec\left(\frac{3\pi}{4}\right)=-\sqrt{2}$

97. $\alpha=\tan^{-1}2x$ indicates the diagram 

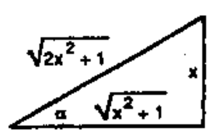
$\Rightarrow \sec\left(\tan^{-1}2x\right)=\sec\alpha=\sqrt{4x^2+1}$

98. $\alpha=\sec^{-1}\frac{y}{5}$ indicates the diagram 

$\Rightarrow \tan\left(\sec^{-1}\frac{y}{5}\right)=\tan\alpha=\frac{\sqrt{y^2-25}}{5}$

99. $\alpha=\cos^{-1}x$ indicates the diagram 

$\Rightarrow \tan\left(\cos^{-1}x\right)=\tan\alpha=\frac{\sqrt{1-x^2}}{x}$

100. $\alpha=\tan^{-1}\frac{x}{\sqrt{x^2+1}}$ indicates the diagram 

$\Rightarrow \sin\left(\tan^{-1}\frac{x}{\sqrt{x^2+1}}\right)=\sin\alpha=\frac{x}{\sqrt{2x^2+1}}$

101. (a) Defined; there is an angle whose tangent is 2.
(b) Not defined; there is no angle whose cosine is 2.

102. (a) Not defined; there is no angle whose cosecant is $\frac{1}{2}$.
(b) Defined; there is an angle whose cosecant is 2.

103. (a) Not defined; there is no angle whose secant is 0.
(b) Not defined; there is no angle whose sine is $\sqrt{2}$.

104. (a) Defined; there is an angle whose cotangent is $-\frac{1}{2}$.

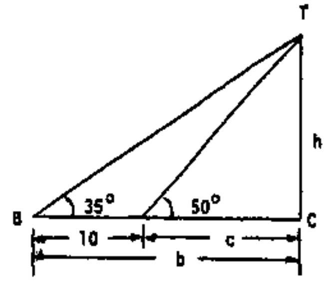
(b) Not defined; there is no angle whose cosine is -5.

105. Let h = height of vertical pole, and let b and c denote the distances of points B and C from the base of the pole, measured along the flat ground, respectively. Then, $\tan 50° = \frac{h}{c}$, $\tan 35° = \frac{h}{b}$, and $b - c = 10$. Thus, $h = c \tan 50°$ and $h = b \tan 35° = (c + 10) \tan 35°$

$\Rightarrow c \tan 50° = (c + 10) \tan 35° \Rightarrow c\,(\tan 50° - \tan 35°) = 10 \tan 35°$

$\Rightarrow c = \frac{10 \tan 35°}{\tan 50° - \tan 35°} \Rightarrow h = c \tan 50° = \frac{10 \tan 35° \tan 50°}{\tan 50° - \tan 35°}$

≈ 16.98 m.



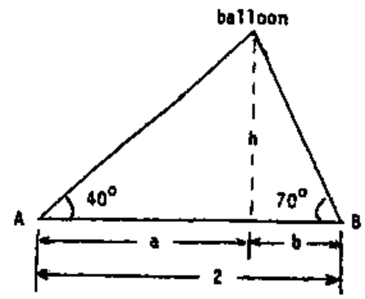
106. Let h = height of balloon above ground. From the figure at the right, $\tan 40° = \frac{h}{a}$, $\tan 70° = \frac{h}{b}$, and $a + b = 2$. Thus,

$h = b \tan 70° \Rightarrow h = (2 - a) \tan 70°$ and $h = a \tan 40°$

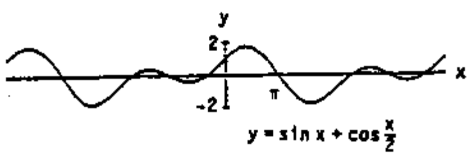
$\Rightarrow (2 - a) \tan 70° = a \tan 40° \Rightarrow a(\tan 40° + \tan 70°) = 2 \tan 70°$

$\Rightarrow a = \frac{2 \tan 70°}{\tan 40° + \tan 70°} \Rightarrow h = a \tan 40° = \frac{2 \tan 70° \tan 40°}{\tan 40° + \tan 70°}$

≈ 1.3 km.



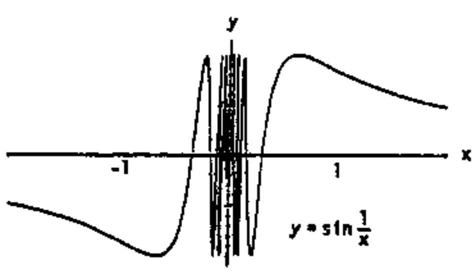
107. (a)



(b) The period appears to be 4π.

(c) $f(x + 4\pi) = \sin(x + 4\pi) + \cos\left(\frac{x + 4\pi}{2}\right) = \sin(x + 2\pi) + \cos\left(\frac{x}{2} + 2\pi\right) = \sin x + \cos\frac{x}{2}$ since the period of sine and cosine is 2π. Thus, f(x) has period 4π.

108. (a)



(b) $D = (-\infty, 0) \cup (0, \infty)$; $R = [-1, 1]$

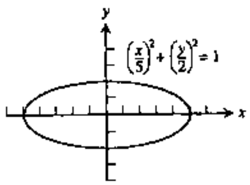
(c) f is not periodic. Suppose f has period p. Then $f\left(\frac{1}{2\pi} + kp\right) = f\left(\frac{1}{2\pi}\right) = \sin 2\pi = 0$ for all integers k.

Choose k so large that $\frac{1}{2\pi} + kp > \frac{1}{\pi} \Rightarrow 0 < \frac{1}{(1/2\pi) + kp} < \pi$. But then $f\left(\frac{1}{2\pi} + kp\right) = \sin\left(\frac{1}{(1/2\pi) + kp}\right) > 0$

which is a contradiction. Thus f has no period, as claimed.

109. (a) Substituting $\cos t = \frac{x}{5}$ and $\sin t = \frac{y}{2}$ in the identity $\cos^2 t + \sin^2 t = 1$ gives the Cartesian equation $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$. The entire ellipse is traced by the curve.

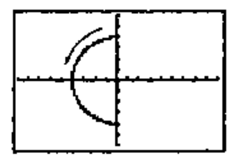
(b)



Initial point: $(5,0)$
Terminal point: $(5,0)$
The ellipse is traced exactly once in a counterclockwise direction starting and ending at the point $(5,0)$.

110. (a) Substituting $\cos t = \frac{x}{4}$ and $\sin t = \frac{y}{4}$ in the identity $\cos^2 t + \sin^2 t = 1$ gives the Cartesian equation $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$, or $x^2 + y^2 = 16$. The left half of the circle is traced by the parametrized curve.

(b)

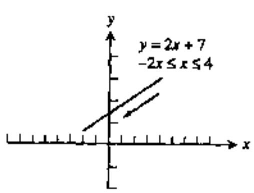


$[-9, 9]$ by $[-6, 6]$

Initial point: $(0,4)$
Terminal point: None (since the endpoint $\frac{3\pi}{2}$ is not included in the t-interval)
The semicircle is traced in a counterclockwise direction starting at $(0,4)$ and extending to, but not including, $(0,-4)$.

111. (a) Substituting $t = 2 - x$ into $y = 11 - 2t$ gives the Cartesian equation $y = 11 - 2(2 - x)$, or $y = 2x + 7$. The part of the line from $(4,15)$ to $(-2,3)$ is traced by the parametrized curve.

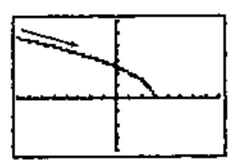
(b)



Initial point: $(4,15)$
Terminal point: $(-2,3)$
The line segment is traced from right to left starting at $(4,15)$ and ending at $(-2,3)$.

112. (a) Substituting $t = x - 1$ into $y = (t-1)^2$ gives the Cartesian equation $y = (x - 1 - 1)^2$, or $y = (x-2)^2$. The part of the parabola for $x \le 2$ is traced by the parametrized curve.

(b)



$[-8, 8]$ by $[-4, 6]$

Initial point: None
Terminal point: $(3,0)$
The curve is traced from left to right ending at the point $(3,0)$.

113. (a) For simplicity, we assume that x and y are linear functions of t, and that the point (x,y) starts at $(-2,5)$ for $t = 0$ and ends at $(4,3)$ for $t = 1$. Then $x = f(t)$, where $f(0) = -2$ and $f(1) = 4$. Since slope $= \frac{\Delta x}{\Delta t} = \frac{4-(-2)}{1-0} = 6$, $x = f(t) = 6t - 2 = -2 + 6t$. Also, $y = g(t)$, where $g(0) = 5$ and $g(1) = 3$.

Since slope $= \frac{\Delta y}{\Delta t} = \frac{3-5}{1-0} = -2$, $y = g(t) = -2t + 5 = 5 - 2t$. One possible parametrization is: $x = -2 + 6t$, $y = 5 - 2t$, $0 \le t \le 1$.

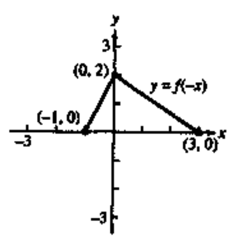
114. For simplicity, we assume that x and y are linear functions of t, and that the point (x,y) passes through $(-3,-2)$ for $t = 0$ and $(4,-1)$ for $t = 1$. Then $x = f(t)$, where $f(0) = -3$ and $f(1) = 4$. Since slope $= \frac{\Delta x}{\Delta t} = \frac{4-(-3)}{1-0} = 7$, $x = f(t) = 7t - 3 = -3 + 7t$. Also, $y = g(t)$, where $g(0) = -2$ and $g(1) = -1$. Since slope $= \frac{\Delta y}{\Delta t} = \frac{-1-(-2)}{1-0} = 1$, $y = g(t) = t - 2 = -2 + t$. One possible parametrization is: $x = -3 + 7t$, $y = -2 + t$, $-\infty < t < \infty$.

115. For simplicity, we assume that x and y are linear functions of t, and that the point (x,y) starts at (2,5) for $t = 0$ and passes through $(-1,0)$ for $t = 1$. Then $x = f(t)$, where $f(0) = 2$ and $f(1) = -1$. Since slope $= \frac{\Delta x}{\Delta t} = \frac{-1-2}{1-0} = -3$, $x = f(t) = -3t + 2 = 2 - 3t$. Also, $y = g(t)$, where $g(0) = 5$ and $g(1) = 0$. Since slope $= \frac{\Delta y}{\Delta t} = \frac{0-5}{1-0} = -5$, $y = g(t) = -5t + 5 = 5 - 5t$. One possible parametrization is: $x = 2 - 3t$, $y = 5 - 5t$, $t \ge 0$.

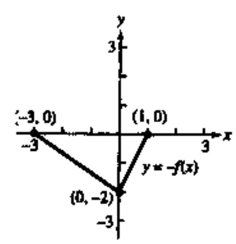
116. One possible parametrization is: $x = t$, $y = t(t-4)$, $t \le 2$.

PRELIMINARY CHAPTER ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

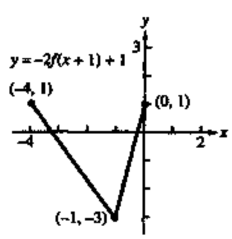
1. (a) The given graph is reflected about the y-axis.



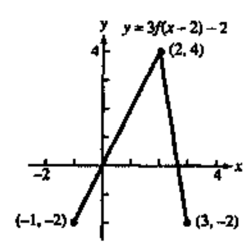
(b) The given graph is reflected about the x-axis.



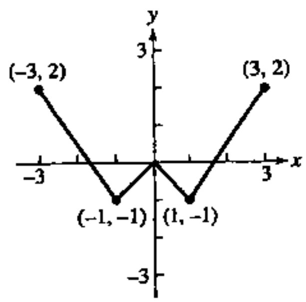
(c) The given graph is shifted left 1 unit, stretched vertically by a factor of 2, reflected about the x-axis, and then shifted upward 1 unit.



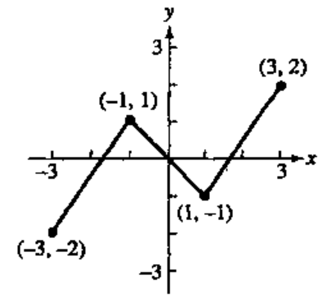
(d) The given graph is shifted right 2 units, stretched vertically by a factor of 3, and then shifted downward 2 units.



2. (a)

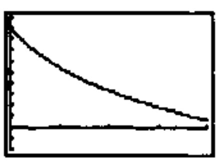


(b)



3. (a) $y = 100{,}000 - 10{,}000x,\ 0 \le x \le 10$
(b)
$$\begin{aligned} y &= 55{,}000 \\ 100{,}000 - 10{,}000x &= 55{,}000 \\ -10{,}000x &= -45{,}000 \\ x &= 4.5 \end{aligned}$$
The value is \$55,000 after 4.5 years.

4. (a) $f(0) = 90$ units
(b) $f(2) = 90 - 52 \ln 3 \approx 32.8722$ units
(c)

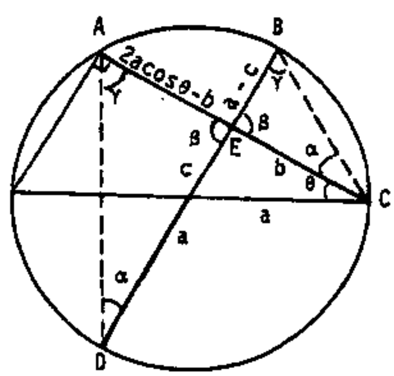


[0, 4] by [−20, 100]

5. $1500(1.08)^t = 5000 \to 1.08^t = \frac{5000}{1500} = \frac{10}{3} \to \ln(1.08)^t = \ln \frac{10}{3} \to t \ln 1.08 = \ln \frac{10}{3} \to t = \frac{\ln(10/3)}{\ln 1.08} \approx 15.6439$

It will take about 15.6439 years. (If the bank only pays interest at the end of the year, it will take 16 years.)

6. The angles labeled γ in the accompanying figure are equal since both angles subtend arc CD. Similarly, the two angles labeled α are equal since they both subtend arc AB. Thus, triangles AED and BEC are similar which implies $\frac{a-c}{b} = \frac{2a\cos\theta - b}{a+c} \Rightarrow (a-c)(a+c) = b(2a\cos\theta - b)$ $\Rightarrow a^2 - c^2 = 2ab\cos\theta - b^2 \Rightarrow c^2 = a^2 + b^2 - 2ab\cos\theta.$



7. As in the proof of the law of sines of Section P.5, Exercise 35, $ah = bc \sin A = ab \sin C = ac \sin B$ $\Rightarrow$ the area of ABC $= \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}ah = \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B.$

8. (a) The coordinates of P are $\left(\frac{a+0}{2}, \frac{b+0}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$. Thus the slope of OP $= \frac{\Delta y}{\Delta x} = \frac{b/2}{a/2} = \frac{b}{a}$.

(b) The slope of AB $= \frac{b-0}{0-a} = -\frac{b}{a}$. The line segments AB and OP are perpendicular when the product of their slopes is $-1 = \left(\frac{b}{a}\right)\left(-\frac{b}{a}\right) = -\frac{b^2}{a^2}$. Thus, $b^2 = a^2 \Rightarrow a = b$ (since both are positive). Therefore, AB is perpendicular to OP when $a = b$.

9. Triangle ABD is an isosceles right triangle with its right angle at B and an angle of measure $\frac{\pi}{4}$ at A. We therefore have $\frac{\pi}{4} = \angle DAB = \angle DAE + \angle CAB = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}$.

10. $\ln x^{(x^x)} = x^x \ln x$ and $\ln\left(x^x\right)^x = x \ln x^x = x^2 \ln x$; then, $x^x \ln x = x^2 \ln x \Rightarrow x^x = x^2 \Rightarrow x \ln x = 2 \ln x$ $\Rightarrow x = 2$. Therefore, $x^{(x^x)} = \left(x^x\right)^x$ when $x = 2$.

11. (a) If f is even, then $f(x) = f(-x)$ and $h(-x) = g(f(-x)) = g(f(x)) = h(x)$.
If f is odd, then $f(-x) = -f(x)$ and $h(-x) = g(f(-x)) = g(-f(x)) = g(f(x)) = h(x)$ because g is even.
If f is neither, then h may not be even. For example, if $f(x) = x^2 + x$ and $g(x) = x^2$, then $h(x) = x^4 + 2x^3 + x^2$ and $h(-x) = x^4 - 2x^3 + x^2 \neq h(x)$. Therefore, h need not be even.

(b) No, h is not always odd. Let $g(t) = t$ and $f(x) = x^2$. Then, $h(x) = g(f(x)) = f(x) = x^2$ is even although g is odd.
If f is odd, then $f(-x) = -f(x)$ and $h(-x) = g(f(-x)) = g(-f(x)) = -g(f(x)) = -h(x)$ because g is odd.
In this case, h is odd. However, if f is even, as in the above counterexample, we see that h need not be odd.

12. $A(t) = A_0e^{rt}$; $A(t) = 2A_0 \Rightarrow 2A_0 = A_0e^{rt} \Rightarrow e^{rt} = 2 \Rightarrow rt = \ln 2 \Rightarrow t = \frac{\ln 2}{r} \Rightarrow t \approx \frac{.7}{r} = \frac{70}{100r} = \frac{70}{(r\%)}$

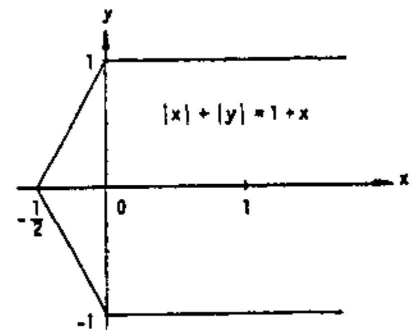
13. There are (infinitely) many such function pairs. For example, $f(x) = 3x$ and $g(x) = 4x$ satisfy $f(g(x)) = f(4x) = 3(4x) = 12x = 4(3x) = g(3x) = g(f(x))$.

14. Yes, there are many such function pairs. For example, if $g(x) = (2x+3)^3$ and $f(x) = x^{1/3}$, then $(f \circ g)(x) = f(g(x)) = f\left((2x+3)^3\right) = \left((2x+3)^3\right)^{1/3} = 2x+3$.

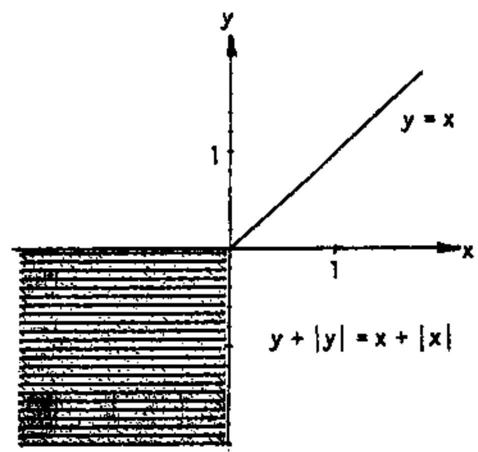
15. If f is odd and defined at x, then $f(-x) = -f(x)$. Thus $g(-x) = f(-x) - 2 = -f(x) - 2$ whereas $-g(x) = -(f(x) - 2) = -f(x) + 2$. Then g cannot be odd because $g(-x) = -g(x) \Rightarrow -f(x) - 2 = -f(x) + 2$ $\Rightarrow 4 = 0$, which is a contradiction. Also, g(x) is not even unless $f(x) = 0$ for all x. On the other hand, if f is even, then $g(x) = f(x) - 2$ is also even: $g(-x) = f(-x) - 2 = f(x) - 2 = g(x)$.

16. If g is odd and g(0) is defined, then $g(0) = g(-0) = -g(0)$. Therefore, $2g(0) = 0 \Rightarrow g(0) = 0$.

17. For (x,y) in the 1st quadrant, $|x|+|y| = 1+x$ $\Leftrightarrow x+y = 1+x \Leftrightarrow y = 1$. For (x,y) in the 2nd quadrant, $|x|+|y| = x+1 \Leftrightarrow -x+y = x+1$ $\Leftrightarrow y = 2x+1$. In the 3rd quadrant, $|x|+|y| = x+1$ $\Leftrightarrow -x-y = x+1 \Leftrightarrow y = -2x-1$. In the 4th quadrant, $|x|+|y| = x+1 \Leftrightarrow x+(-y) = x+1$ $\Leftrightarrow y = -1$. The graph is given at the right.



18. We use reasoning similar to Exercise 17.
    (1) 1st quadrant: $y+|y| = x+|x|$
    $\Leftrightarrow 2y = 2x \Leftrightarrow y = x$.
    (2) 2nd quadrant: $y+|y| = x+|x|$
    $\Leftrightarrow 2y = x+(-x) = 0 \Leftrightarrow y = 0$.
    (3) 3rd quadrant: $y+|y| = x+|x|$
    $\Leftrightarrow y+(-y) = x+(-x) \Leftrightarrow 0 = 0$
    $\Rightarrow$ all points in the 3rd quadrant satisfy the equation.
    (4) 4th quadrant: $y+|y| = x+|x|$
    $\Leftrightarrow y+(-y) = 2x \Leftrightarrow 0 = x$. Combining these results we have the graph given at the right:



19. If f is even and odd, then $f(-x) = -f(x)$ and $f(-x) = f(x) \Rightarrow f(x) = -f(x)$ for all x in the domain of f. Thus $2f(x) = 0 \Rightarrow f(x) = 0$.

20. (a) As suggested, let $E(x) = \frac{f(x)+f(-x)}{2} \Rightarrow E(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(x)+f(-x)}{2} = E(x) \Rightarrow E$ is an even function. Define $O(x) = f(x) - E(x) = f(x) - \frac{f(x)+f(-x)}{2} = \frac{f(x)-f(-x)}{2}$. Then $O(-x) = \frac{f(-x)-f(-(-x))}{2} = \frac{f(-x)-f(x)}{2} = -\left(\frac{f(x)-f(-x)}{2}\right) = -O(x) \Rightarrow O$ is an odd function $\Rightarrow f(x) = E(x) + O(x)$ is the sum of an even and an odd function.

    (b) Part (a) shows that $f(x) = E(x) + O(x)$ is the sum of an even and an odd function. If also $f(x) = E_1(x) + O_1(x)$, where E_1 is even and O_1 is odd, then $f(x) - f(x) = 0 = (E_1(x) + O_1(x)) - (E(x) + O(x))$. Thus, $E(x) - E_1(x) = O_1(x) - O(x)$ for all x in the domain of f (which is the same as the domain of $E - E_1$ and $O - O_1$). Now $(E - E_1)(-x) = E(-x) - E_1(-x) = E(x) - E_1(x)$ (since E and E_1 are even) $= (E - E_1)(x) \Rightarrow E - E_1$ is even. Likewise, $(O_1 - O)(-x) = O_1(-x) - O(-x) = -O_1(x) - (-O(x))$ (since O and O_1 are odd) $= -(O_1(x) - O(x)) = -(O_1 - O)(x) \Rightarrow O_1 - O$ is odd. Therefore, $E - E_1$ and

$O_1 - O$ are both even and odd so they must be zero at each x in the domain of f by Exercise 19. That is, $E_1 = E$ and $O_1 = O$, so the decomposition of f found in part (a) is unique.

21. If the graph of $f(x)$ passes the horizontal line test, so will the graph of $g(x) = -f(x)$ since it's the same graph reflected about the x-axis.
Alternate answer: If $g(x_1) = g(x_2)$ then $-f(x_1) = -f(x_2)$, $f(x_1) = f(x_2)$, and $x_1 = x_2$ since f is one-to-one.

22. Suppose that $g(x_1) = g(x_2)$. Then $\frac{1}{f(x_1)} = \frac{1}{f(x_2)}$, $f(x_1) = f(x_2)$, and $x_1 = x_2$ since f is one-to-one.

23. (a) The expression $a(b^{c-x}) + d$ is defined for all values of x, so the domain is $(-\infty, \infty)$. Since b^{c-x} attains all positive values, the range is (d, ∞) if $a > 0$ and the range is $(-\infty, d)$ if $a < 0$.
(b) The expression $a \log_b (x - c) + d$ is defined when $x - c > 0$, so the domain is (c, ∞).
Since $a \log_b (x - c) + d$ attains every real value for some value of x, the range is $(-\infty, \infty)$.

24. (a) Suppose $f(x_1) = f(x_2)$. Then:

$$\frac{ax_1 + b}{cx_1 + d} = \frac{ax_2 + b}{cx_2 + d}$$

$$(ax_1 + b)(cx_2 + d) = (ax_2 + b)(cx_1 + d)$$

$$acx_1x_2 + adx_1 + bcx_2 + bd = acx_1x_2 + adx_2 + bcx_1 + bd$$

$$adx_1 + bcx_2 = adx_2 + bcx_1$$

$$(ad - bc)x_1 = (ad - bc)x_2$$

Since $ad - bc \neq 0$, this means that $x_1 = x_2$.

(b) $y = \frac{ax + b}{cx + d}$

$$cxy + dy = ax + b$$

$$(cy - a)x = -dy + b$$

$$x = \frac{-dy + b}{cy - a}$$

Interchange x and y.

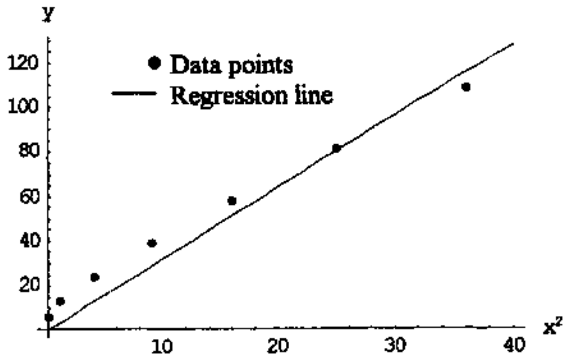
$$y = \frac{-dx + b}{cx - a}$$

$$f^{-1}(x) = \frac{-dx + b}{cx - a}$$

(c) As $x \to \pm\infty$, $f(x) = \frac{ax + b}{cx + d} \to \frac{a}{c}$, so the horizontal asymptote is $y = \frac{a}{c}$ $(c \neq 0)$. Since $f(x)$ is undefined at $x = -\frac{d}{c}$, the vertical asymptote is $x = -\frac{d}{c}$ provided $c \neq 0$.

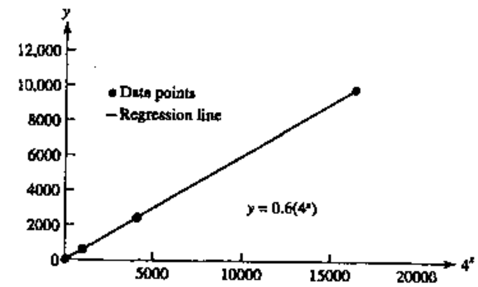
(d) As $x \to \pm\infty$, $f^{-1}(x) = \frac{-dx + b}{cx - a} \to -\frac{d}{c}$, so the horizontal asymptote is $y = -\frac{d}{c}$ $(c \neq 0)$. Since $f^{-1}(x)$ is undefined at $x = \frac{a}{c}$, the vertical asymptote is $x = \frac{a}{c}$. The horizontal asymptote of f becomes the vertical asymptote of f^{-1} and vice versa due to the reflection of the graph about the line $y = x$.

25. (a)



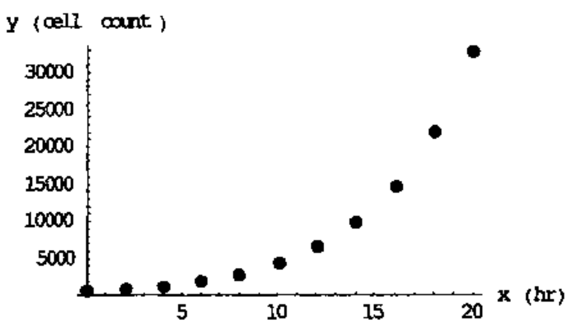
The graph does not support the assumption that $y \propto x^2$

(b)

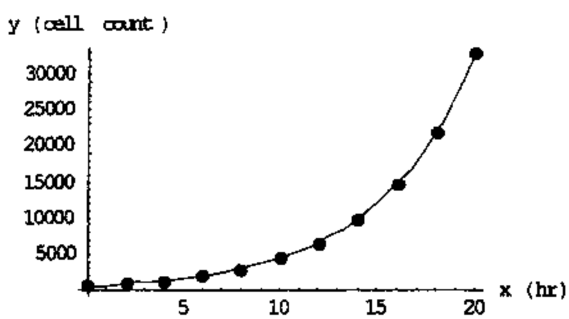


The graph supports the assumption that $y \propto 4^x$. The constant of proportionality is estimated from the slope of the regression line, which is 0.6, therefore, $y = 0.6(4^x)$.

26. Plot the data.



The graph suggests that an exponential relationship might be appropriate. The exponential regression function on the TI-92 Plus calculator gives $y = 599e^{0.2x}$ and the following graph shows the exponential regression curve superimposed on the graph of the data points.



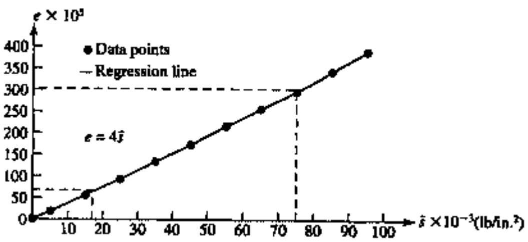
The curve appears to fit the data very well.

The cell count reaches 50,000 when $50{,}000 = 599e^{0.2x} \Rightarrow x = 5 \ln \frac{50{,}000}{599} \approx 22.123$ hours

$\approx$ 22 hours 7.4 minutes.

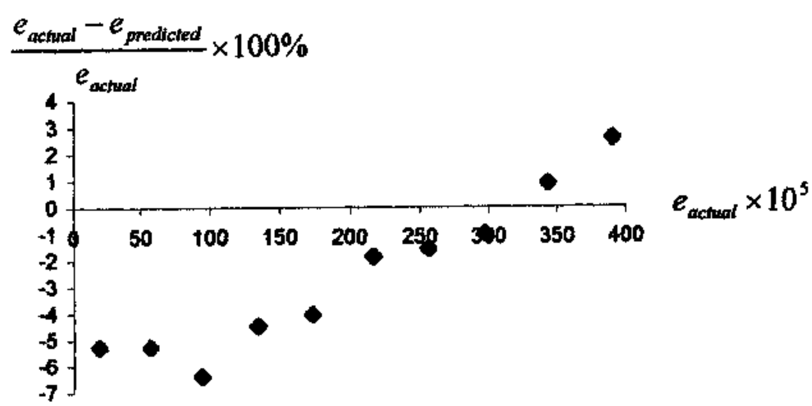
27. (a) Since the elongation of the spring is zero when the stress is $5(10^{-3})(\text{lb/in.}^2)$, the data should be adjusted by subtracting this amount from each of the stress data values. This gives the following table, where $\bar{s} = s - 5(10^{-3})$.

$\bar{s} \times 10^{-3}$	0	5	15	25	35	45	55	65	75	85	95
$e \times 10^5$	0	19	57	94	134	173	216	256	297	343	390



The slope of the graph is $\dfrac{(297-57)(10^5)}{(75-15)(10^{-3})} = 4.00(10^8)$ and the model is $e = 4(10^8)\bar{s}$ or $e = 4(10^8)(s - 5(10^{-3}))$.

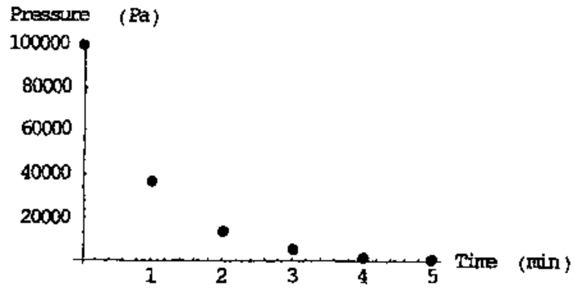
(b) As show in the following graph, the largest relative error is about 6.4%



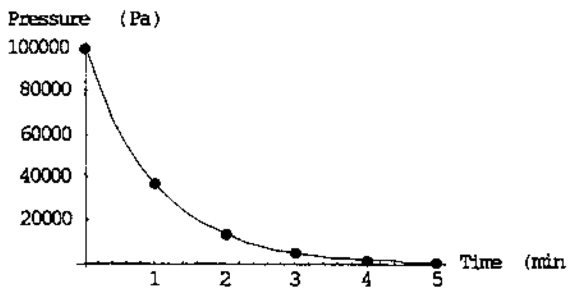
The model fits the data well. There does appear to be a pattern in the errors (i.e., they are not random) indicating that a refinement of the model is possible.

(c) $e = 4(10^8)(200 - 5)(10^{-3}) = 780(10^5)(\text{in./in.})$. Since $s = 200(10^{-3})(\text{lb/in.}^2)$ is well outside the range of the data used for the model, one should not feel comfortable with this prediction without further testing of the spring.

28. Plot the data.



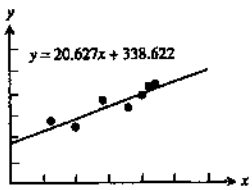
(a) The data suggests a decaying exponential relationship. The exponential regression function on the TI-92 Plus calculator gives $p = 100{,}085e^{-t}$ where p is the pressure in pascals and t is the elapsed time in minutes. The next graph superimposes the exponential regression curve on the data points.



(b) The graph shows that the exponential regression fits the data very well.

(c) The pressure reaches 200 Pa when $200 = 100{,}085e^{-t} \Rightarrow t = -\ln\left(\frac{200}{100{,}085}\right) \approx 6.22$ minutes ≈ 6 minutes 13 seconds.

29. (a) $y = 20.627x + 338.622$



(b) When $x = 30$, $y = 957.445$. According to the regression equation, about 957 degrees will be earned.

(c) The slope is 20.627. It represents the approximate annual increase in the number of doctorates earned by Hispanic Americans per year.

30. (a) $y = 14.60175 \cdot 1.00232^x$

(b) Solving $y = 25$ graphically, we obtain $x \approx 232$. According to the regression equation, the population will reach 25 million in the year 2132.

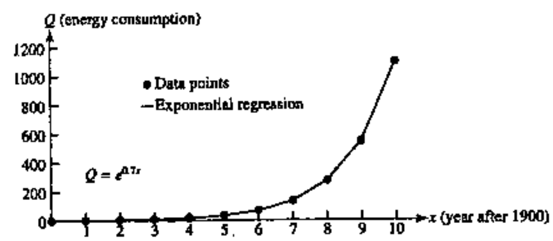
(c) 0.232%

31. (a) The TI-92 Plus calculator gives $f(x) = 2.000268 \sin(2.999187x - 1.000966) + 3.999881$.

(b) $f(x) = 2 \sin(3x - 1) + 4$

32. (a) $y = -590.969 + 152.817 \ln x$, where x is the number of years after 1960.
   (b) When $x = 85$, $y \approx 87.94$.
   About 87.94 million metric tons were produced.
   (c)
$$\begin{aligned} -590.969 + 152.817 \ln x &= 120 \\ 152.817 \ln x &= 710.969 \\ \ln x &= \frac{710.969}{152.817} \\ x &= e^{\frac{710.969}{152.817}} \approx 104.84 \end{aligned}$$
   According to the regression equation, oil production will reach 120 million metric tons when $x \approx 104.84$, in about 2005.

33. (a) The TI-93 Plus calculator gives $Q = 1.00(2.0138^x) = 1.00e^{0.7x}$



   (b) For 1996, $x = 9.6 \Rightarrow Q(9.6) = e^{0.7(9.6)} = 828.82$ units of energy consumed that year as estimated by the exponential regression. The exponential regression shows that energy consumption has doubled (i.e., increased by 100%) each decade during the 20th century. The annual rate of increase during this time is $e^{0.7(0.1)} - e^{0.7(0)} = 0.0725 = 7.25\%$.

NOTES:

CHAPTER 1 LIMITS AND CONTINUITY

1.1 RATE OF CHANGE AND LIMITS

1. (a) $\frac{\Delta f}{\Delta x} = \frac{f(3) - f(2)}{3 - 2} = \frac{28 - 9}{1} = 19$ (b) $\frac{\Delta f}{\Delta x} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1$

2. $\frac{\Delta R}{\Delta \theta} = \frac{R(2) - R(0)}{2 - 0} = \frac{\sqrt{8+1} - \sqrt{1}}{2} = \frac{3 - 1}{2} = 1$

3. (a) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{3\pi}{4}\right) - h\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-1 - 1}{\frac{\pi}{2}} = -\frac{4}{\pi}$ (b) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{\pi}{2}\right) - h\left(\frac{\pi}{6}\right)}{\frac{\pi}{2} - \frac{\pi}{6}} = \frac{0 - \sqrt{3}}{\frac{\pi}{3}} = \frac{-3\sqrt{3}}{\pi}$

4. (a) $\frac{\Delta g}{\Delta t} = \frac{g(\pi) - g(0)}{\pi - 0} = \frac{(2 - 1) - (2 + 1)}{\pi - 0} = -\frac{2}{\pi}$ (b) $\frac{\Delta g}{\Delta t} = \frac{g(\pi) - g(-\pi)}{\pi - (-\pi)} = \frac{(2 - 1) - (2 - 1)}{2\pi} = 0$

5. (a)

Q	Slope of PQ $= \frac{\Delta p}{\Delta t}$
$Q_1(10, 225)$	$\frac{650 - 225}{20 - 10} = 42.5$ m/sec
$Q_2(14, 375)$	$\frac{650 - 375}{20 - 14} = 45.83$ m/sec
$Q_3(16.5, 475)$	$\frac{650 - 475}{20 - 16.5} = 50.00$ m/sec
$Q_4(18, 550)$	$\frac{650 - 550}{20 - 18} = 50.00$ m/sec

(b) At $t = 20$, the Cobra was traveling approximately 50 m/sec or 180 km/h.

6. (a)

Q	Slope of PQ $= \frac{\Delta p}{\Delta t}$
$Q_1(5, 20)$	$\frac{80 - 20}{10 - 5} = 12$ m/sec
$Q_2(7, 39)$	$\frac{80 - 39}{10 - 7} = 13.7$ m/sec
$Q_3(8.5, 58)$	$\frac{80 - 58}{10 - 8.5} = 14.7$ m/sec
$Q_4(9.5, 72)$	$\frac{80 - 72}{10 - 9.5} = 16$ m/sec

(b) Approximately 16 m/sec

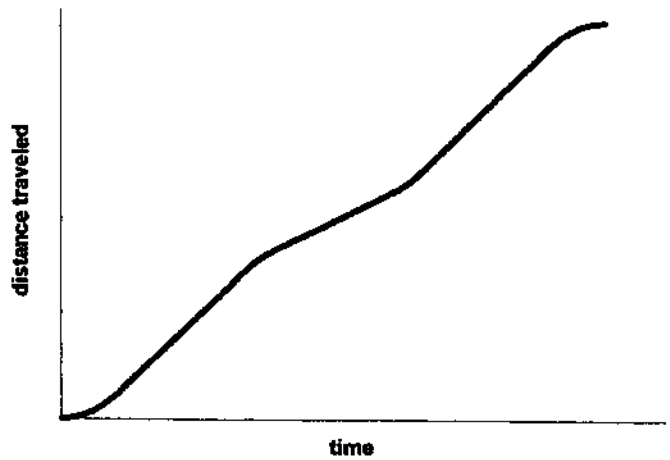
7. A plot of the data shows that the slope of the secant between $t = 0.8$ sec and $t = 1.0$ sec underestimates the instantaneous velocity (i.e., the slope of the tangent) at $t = 1.0$ sec, whereas the slope of the secant between $t = 1.0$ sec and $t = 1.2$ sec overestimates it.

Lower bound: $a = \frac{13.10 - 8.39}{1.0 - 0.8} = 23.55$ ft/sec

Upper bound: $b = \frac{18.87 - 13.10}{1.2 - 1.0} = 28.85$ ft/sec

$v(1) \approx \frac{a+b}{2} = \frac{23.55 + 28.85}{2} = 26.20$ ft/sec

8. There are many graphs that would be correct. One possible solution looks like this:



9. (a) Does not exist. As x approaches 1 from the right, g(x) approaches 0. As x approaches 1 from the left, g(x) approaches 1. There is no single number L that all the values g(x) get arbitrarily close to as $x \to 1$.
   (b) 1
   (c) 0

10. (a) 0
    (b) -1
    (c) Does not exist. As t approaches 0 from the left, f(t) approaches -1. As t approaches 0 from the right, f(t) approaches 1. There is no single number L that f(t) gets arbitrarily close to as $t \to 0$.

11. (a) True (b) True (c) False
    (d) False (e) False (f) True

12. (a) False (b) False (c) True
    (d) True (e) True

13. $\lim_{x \to 0} \frac{x}{|x|}$ does not exist because $\frac{x}{|x|} = \frac{x}{x} = 1$ if $x > 0$ and $\frac{x}{|x|} = \frac{x}{-x} = -1$ if $x < 0$. As x approaches 0 from the left, $\frac{x}{|x|}$ approaches -1. As x approaches 0 from the right, $\frac{x}{|x|}$ approaches 1. There is no single number L that all the function values get arbitrarily close to as $x \to 0$.

14. As x approaches 1 from the left, the values of $\frac{1}{x-1}$ become increasingly large and negative. As x approaches 1 from the right, the values become increasingly large and positive. There is no one number L that all the function values get arbitrarily close to as $x \to 1$, so $\lim_{x \to 1} \frac{1}{x-1}$ does not exist.

15. Nothing can be said about $\lim_{x \to x_0} f(x)$ because the existence of a limit as $x \to x_0$ does not depend on how the function is defined at x_0. In order for a limit to exist, f(x) must be arbitrarily close to a single real number L when x is close enough to x_0. That is, the existence of a limit depends on the values of f(x) for x <u>near</u> x_0, not on the definition of f(x) at x_0 itself.

16. Nothing can be said. In order for $\lim_{x \to 0} f(x)$ to exist, f(x) must close to a single value for x near 0 regardless of the value f(0) itself.

17. No, the definition does not require that f be defined at $x = 1$ in order for a limiting value to exist there. If f(1) is defined, it can be any real number, so we can conclude nothing about f(1) from $\lim_{x \to 1} f(x) = 5$.

18. No, because the existence of a limit depends on the values of f(x) when x is near 1, not on f(1) itself. If $\lim_{x \to 1} f(x)$ exists, its value may be some number other than $f(1) = 5$. We can conclude nothing about $\lim_{x \to 1} f(x)$, whether it exists or what its value is if it does exist, from knowing the value of f(1) alone.

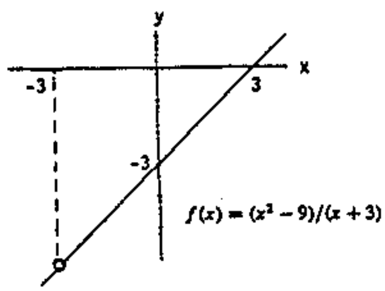
19. (a) $f(x) = (x^2 - 9)/(x + 3)$

x	−3.1	−3.01	−3.001	−3.0001	−3.00001	−3.000001
f(x)	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001

x	−2.9	−2.99	−2.999	−2.9999	−2.99999	−2.999999
f(x)	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999

The estimate is $\lim_{x \to -3} f(x) = -6$.

(b)



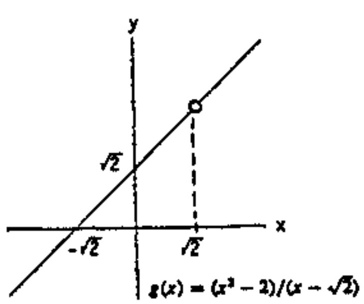
(c) $f(x) = \frac{x^2 - 9}{x + 3} = \frac{(x + 3)(x - 3)}{x + 3} = x - 3$ if $x \neq -3$, and $\lim_{x \to -3} (x - 3) = -3 - 3 = -6$.

20. (a) $g(x) = (x^2 - 2)/(x - \sqrt{2})$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
g(x)	2.81421	2.82421	2.82821	2.828413	2.828423	2.828426

The estimate is $\lim_{x \to \sqrt{2}} g(x) = 2\sqrt{2}$.

(b)



(c) $g(x) = \dfrac{x^2 - 2}{x - \sqrt{2}} = \dfrac{(x+\sqrt{2})(x-\sqrt{2})}{(x-\sqrt{2})} = x + \sqrt{2}$ if $x \neq \sqrt{2}$, and $\lim\limits_{x \to \sqrt{2}} (x + \sqrt{2}) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$.

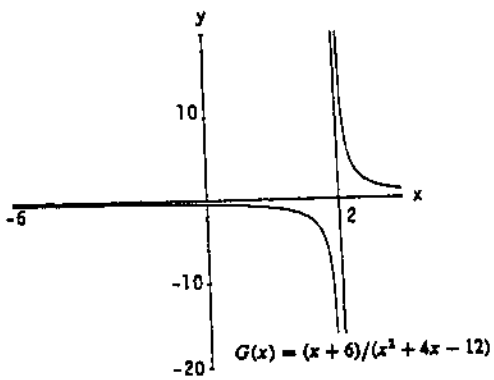
21. (a) $G(x) = (x+6)/(x^2 + 4x - 12)$

x	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999
G(x)	−0.126582	−0.1251564	−0.1250156	−0.1250016	−0.12500016	−0.12500002

x	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001
G(x)	−0.123457	−0.1248439	−0.1249844	−0.1249984	−0.12499984	−0.12499998

The estimate is $\lim\limits_{x \to -6} G(x) = -0.125$.

(b)



(c) $G(x) = \dfrac{x+6}{(x^2+4x-12)} = \dfrac{x+6}{(x+6)(x-2)} = \dfrac{1}{x-2}$ if $x \neq -6$, and $\lim\limits_{x \to -6} \dfrac{1}{x-2} = \dfrac{1}{-6-2} = -\frac{1}{8} = -0.125$.

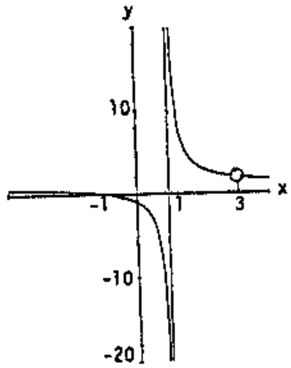
22. (a) $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$

x	2.9	2.99	2.999	2.9999	2.99999	2.999999
h(x)	2.052631	2.005025	2.000500	2.000050	2.000005	2.0000005

x	3.1	3.01	3.001	3.0001	3.00001	3.000001
h(x)	1.952380	1.995024	1.999500	1.999950	1.999995	1.999999

The estimate is $\lim\limits_{x \to 3} h(x) = 2$.

(b)



$h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$

(c) $h(x) = \frac{x^2 - 2x - 3}{x^2 - 4x + 3} = \frac{(x-3)(x+1)}{(x-3)(x-1)} = \frac{x+1}{x-1}$ if $x \neq 3$, and $\lim\limits_{x \to 3} \frac{x+1}{x-1} = \frac{3+1}{3-1} = \frac{4}{2} = 2.$

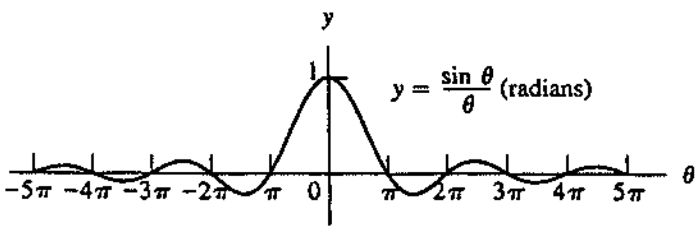
23. (a) $g(\theta) = (\sin \theta)/\theta$

θ	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

θ	−.1	−.01	−.001	−.0001	−.00001	−.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim\limits_{\theta \to 0} g(\theta) = 1$

(b)



NOT TO SCALE

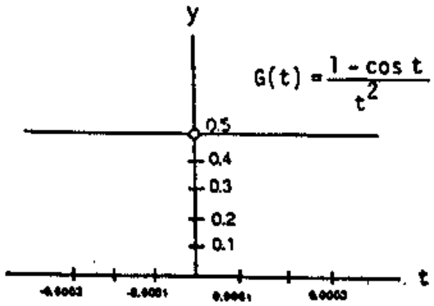
24. (a) $G(t) = (1 - \cos t)/t^2$

t	.1	.01	.001	.0001	.00001	.000001
G(t)	.499583	.499995	.499999	.5	.5	.5

t	−.1	−.01	−.001	−.0001	−.00001	−.000001
G(t)	.499583	.499995	.499999	.5	.5	.5

$\lim\limits_{t \to 0} G(t) = 0.5$

(b)



Graph is NOT TO SCALE

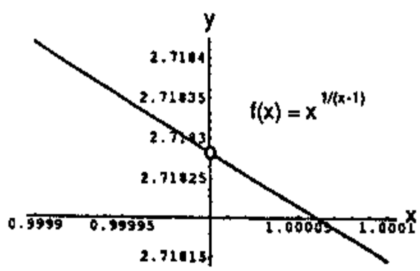
25. (a) $f(x) = x^{1/(1-x)}$

x	.9	.99	.999	.9999	.99999	.999999
f(x)	.348678	.366032	.367695	.367861	.367878	.367879

x	1.1	1.01	1.001	1.0001	1.00001	1.000001
f(x)	.385543	.369711	.368063	.367898	.367881	.367880

$\lim\limits_{x \to 1} f(x) \approx 0.36788$

(b)



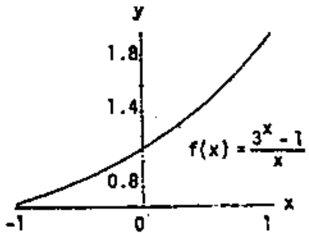
26. (a) $f(x) = (3^x - 1)/x$

x	.1	.01	.001	.0001	.00001	.000001
f(x)	1.161231	1.104669	1.099215	1.098672	1.098618	1.098612

x	−.1	−.01	−.001	−.0001	−.00001	−.000001
f(x)	1.040415	1.092599	1.098009	1.098551	1.098606	1.098611

$\lim\limits_{x \to 0} f(x) \approx 1.0986$

(b)



27. Step 1: $|x-5| < \delta \Rightarrow -\delta < x-5 < \delta \Rightarrow -\delta+5 < x < \delta+5$
Step 2: From the graph, $-\delta+5 = 4.9 \Rightarrow \delta = 0.1$, or $\delta+5 = 5.1 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$ in either case.

28. Step 1: $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3$
Step 2: From the graph, $-\delta-3 = -3.1 \Rightarrow \delta = 0.1$, or $\delta-3 = -2.9 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$.

29. Step 1: $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow -\delta+1 < x < \delta+1$
Step 2: From the graph, $-\delta+1 = \frac{9}{16} \Rightarrow \delta = \frac{7}{16}$, or $\delta+1 = \frac{25}{16} \Rightarrow \delta = \frac{9}{16}$; thus $\delta = \frac{7}{16}$.

30. Step 1: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$
Step 2: From the graph, $-\delta+2 = \sqrt{3} \Rightarrow \delta = 2-\sqrt{3} \approx 0.2679$, or $\delta+2 = \sqrt{5} \Rightarrow \delta = \sqrt{5}-2 \approx 0.2361$; thus $\delta = \sqrt{5}-2$.

31. Step 1: $|(x+1)-5| < 0.01 \Rightarrow |x-4| < 0.01 \Rightarrow -0.01 < x-4 < 0.01 \Rightarrow 3.99 < x < 4.01$
Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4 \Rightarrow \delta = 0.01$.

32. Step 1: $|(2x-2)-(-6)| < 0.02 \Rightarrow |2x+4| < 0.02 \Rightarrow -0.02 < 2x+4 < 0.02 \Rightarrow -4.02 < 2x < -3.98$
$\Rightarrow -2.01 < x < -1.99$
Step 2: $|x-(-2)| < \delta \Rightarrow -\delta < x+2 < \delta \Rightarrow -\delta-2 < x < \delta-2 \Rightarrow \delta = 0.01$.

33. Step 1: $\left|\sqrt{x+1}-1\right| < 0.1 \Rightarrow -0.1 < \sqrt{x+1}-1 < 0.1 \Rightarrow 0.9 < \sqrt{x+1} < 1.1 \Rightarrow 0.81 < x+1 < 1.21$
$\Rightarrow -0.19 < x < 0.21$
Step 2: $|x-0| < \delta \Rightarrow -\delta < x < \delta \Rightarrow \delta = 0.19$.

34. Step 1: $\left|\sqrt{19-x}-3\right| < 1 \Rightarrow -1 < \sqrt{19-x}-3 < 1 \Rightarrow 2 < \sqrt{19-x} < 4 \Rightarrow 4 < 19-x < 16$
$\Rightarrow -4 > x-19 > -16 \Rightarrow 15 > x > 3$ or $3 < x < 15$
Step 2: $|x-10| < \delta \Rightarrow -\delta < x-10 < \delta \Rightarrow -\delta+10 < x < \delta+10$.
Then $-\delta+10 = 3 \Rightarrow \delta = 7$, or $\delta+10 = 15 \Rightarrow \delta = 5$; thus $\delta = 5$.

35. Step 1: $\left|\frac{1}{x}-\frac{1}{4}\right| < 0.05 \Rightarrow -0.05 < \frac{1}{x}-\frac{1}{4} < 0.05 \Rightarrow 0.2 < \frac{1}{x} < 0.3 \Rightarrow \frac{10}{2} > x > \frac{10}{3}$ or $\frac{10}{3} < x < 5$.
Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4$.
Then $-\delta+4 = \frac{10}{3}$ or $\delta = \frac{2}{3}$, or $\delta+4 = 5$ or $\delta = 1$; thus $\delta = \frac{2}{3}$.

36. Step 1: $|x^2 - 3| < 0.1 \Rightarrow -0.1 < x^2 - 3 < 0.1 \Rightarrow 2.9 < x^2 < 3.1 \Rightarrow \sqrt{2.9} < x < \sqrt{3.1}$

Step 2: $|x - \sqrt{3}| < \delta \Rightarrow -\delta < x - \sqrt{3} < \delta \Rightarrow -\delta + \sqrt{3} < x < \delta + \sqrt{3}$.

Then $-\delta + \sqrt{3} = \sqrt{2.9} \Rightarrow \delta = \sqrt{3} - \sqrt{2.9} \approx 0.0291$, or $\delta + \sqrt{3} = \sqrt{3.1} \Rightarrow \delta = \sqrt{3.1} - \sqrt{3} \approx 0.0286$; thus $\delta = 0.0286$.

37. $|A - 9| \le 0.01 \Rightarrow -0.01 \le \pi\left(\frac{x}{2}\right)^2 - 9 \le 0.01 \Rightarrow 8.99 \le \frac{\pi x^2}{4} \le 9.01 \Rightarrow \frac{4}{\pi}(8.99) \le x^2 \le \frac{4}{\pi}(9.01)$

$\Rightarrow 2\sqrt{\frac{8.99}{\pi}} \le x \le 2\sqrt{\frac{9.01}{\pi}}$ or $3.384 \le x \le 3.387$. To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

38. $V = RI \Rightarrow \frac{V}{R} = I \Rightarrow \left|\frac{V}{R} - 5\right| \le 0.1 \Rightarrow -0.1 \le \frac{120}{R} - 5 \le 0.1 \Rightarrow 4.9 \le \frac{120}{R} \le 5.1 \Rightarrow \frac{10}{49} \ge \frac{R}{120} \ge \frac{10}{51} \Rightarrow$

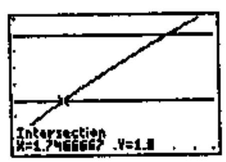
$\frac{(120)(10)}{51} \le R \le \frac{(120)(10)}{49} \Rightarrow 23.53 \le R \le 24.49$.

To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

39. (a) The limit can be found by substitution.

$\lim_{x \to 2} f(x) = f(2) = \sqrt{3(2) - 2} = \sqrt{4} = 2$

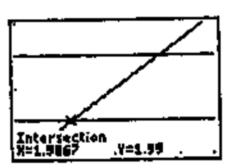
(b) The graphs of $y_1 = f(x)$, $y_2 = 1.8$, and $y_3 = 2.2$ are shown.



[1.5, 2.5] by [1.5, 2.3]

The intersections of y_1 with y_2 and y_3 are at $x \approx 1.7467$ and $x = 2.28$, respectively, so we may choose any value of a in $[1.7467, 2)$ (approximately) and any value of b in $[2, 2.28]$.
One possible answer: $a = 1.75$, $b = 2.28$.

(c) The graphs of $y_1 = f(x)$, $y_2 = 1.99$, and $y_3 = 2.01$ are shown.

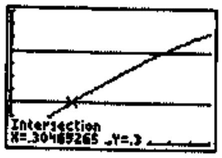


[1.97, 2.03] by [1.98, 2.02]

The intersections of y_1 with y_2 and y_3 are at $x = 1.9867$ and $x \approx 2.0134$, respectively, so we may choose any value of a in $[1.9867, 2)$ and any value of b in $[2, 2.0134]$ (approximately).
One possible answer: $a = 1.99$, $b = 2.01$.

40. (a) $f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

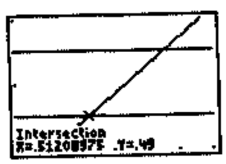
(b) The graphs of $y_1 = f(x)$, $y_2 = 0.3$, and $y_3 = 0.7$ are shown.



[0, 1] by [0, 1]

The intersections of y_1 with y_2 and y_3 are at $x \approx 0.3047$ and $x \approx 0.7754$, respectively, so we may choose any value of a in $\left[0.3047, \frac{\pi}{6}\right)$ and any value of b in $\left(\frac{\pi}{6}, 0.7754\right]$, where the interval endpoints are approximate. One possible answer: $a = 0.305$, $b = 0.775$.

(c) The graphs of $y_1 = f(x)$, $y_2 = 0.49$, and $y_3 = 0.51$ are shown.



[0.49, 0.55] by [0.48, 0.52]

The intersections of y_1 with y_2 and y_3 are at $x \approx 0.5121$ and $x \approx 0.5352$, respectively, so we may choose any value of a in $\left[0.5121, \frac{\pi}{6}\right)$ and any value of b in $\left(\frac{\pi}{6}, 0.5352\right]$, where the interval endpoints are approximate. One possible answer: $a = 0.513$, $b = 5.35$.

41. (a) In three seconds, the ball falls $4.9(3)^2 = 44.1$ m, so its average speed is $\frac{44.1}{3} = 14.7$ m/sec.

(b) The average speed over the interval from time $t = 3$ to time $3 + h$ is

$$\frac{\Delta y}{\Delta t} = \frac{4.9(3+h)^2 - 4.9(3)^2}{(3+h) - 3} = \frac{4.9(6h + h^2)}{h} = 29.4 + 4.9h$$

Since $\lim_{h \to 0}(29.4 + 4.9h) = 29.4$, the instantaneous speed is 29.4 m/sec.

42. (a) $y = gt^2 \to 20 = g(4^2) \to g = \frac{20}{16} = \frac{5}{4} = 1.25$

(b) Average speed $= \frac{20}{4} = 5$ m/sec.

(c) If the rock had not been stopped, its average speed over the interval from time $t = 4$ to time $t = 4 + h$ is

$$\frac{\Delta y}{\Delta t} = \frac{1.25(4+h)^2 - 1.25(4)^2}{(4+h) - 4} = \frac{1.25(8h + h^2)}{h} = 10 + 1.25h$$

Since $\lim_{h \to 0}(10 + 1.25h) = 10$, the instantaneous speed is 10 m/sec.

43. (a)

x	−0.1	−0.01	−0.001	−0.0001
f(x)	−0.054402	−0.005064	−0.000827	−0.000031

(b)

x	0.1	0.01	0.001	0.0001
f(x)	−0.054402	−0.005064	−0.000827	−0.000031

The limit appears to be 0.

44. (a)

x	−0.1	−0.01	−0.001	−0.0001
f(x)	0.5440	−0.5064	−0.8269	0.3056

(b)

x	0.1	0.01	0.001	0.0001
f(x)	−0.5440	−0.5064	0.8269	−0.3056

There is no clear indication of a limit.

45. (a)

x	−0.1	−0.01	−0.001	−0.0001
f(x)	2.0567	2.2763	2.2999	2.3023

(b)

x	0.1	0.01	0.001	0.0001
f(x)	2.5893	2.3293	2.3052	2.3029

The limit appears to be approximately 2.3.

46. (a)

x	−0.1	−0.01	−0.001	−0.0001
f(x)	0.074398	−0.009943	0.000585	0.000021

(b)

x	0.1	0.01	0.001	0.0001
f(x)	−0.074398	0.009943	−0.000585	−0.000021

The limit appears to be 0.

47-50. Example CAS commands:

<u>Maple</u>:

```
f:=x -> (x ∧ 4 − 81)/(x − 3);
plot (f(x), x=2.9..3.1);
limit (f(x), x=−1);
```

<u>Mathematica</u>:

```
x0=3; f=(x ∧ 4 − 81)/(x − 3)
Plot [f,{x,x0−0.1,x0 + 0.1}]
Limit [f,x - > x0]
```

51-54. (values of del may vary for a specified eps):

<u>Maple</u>:

```
f:=x -> (x ∧ 4 − 81)/(x − 3);
x0:='x0': eps :='eps':L:='L':del:='del':
```

```
y1:=x -> L - eps: y2:=x -> L + eps:
x0:=3: L=limit(f(x),x=x0);
eps:=0.1: del:= 0.16:
xmin:= x0 - 2*del: xmax :=x0 + 2*del:
ymin:=L - 2*eps: ymax:=L + 2*eps:
plot({f(x),y1(x),y2(x)}, x=x0-del..x0+del,view = [xmin..xmax,ymin..ymax]);
```

Mathematica:

```
Clear [f,x,L,eps,del]
y1 := L - eps; y2 := L + eps;
x0 = 3; f = (x ∧ 4 - 81)/(x-3)
Plot [f, {x,x0 - 0.2,x0 + 0.2}]
L = Limit[f, x -> x0]
eps = 0.1; del = 0.0015;
Plot [{f,y1,y2}, {x,x0 - del,x0 + del},
  PlotRange -> {{x0 - del,x0 + del}, {L - eps,L + eps)}}]
```

1.2 RULES FOR FINDING LIMITS

1. (a) $\lim_{x\to 3^-} f(x) = 3$

   (b) $\lim_{x\to 3^+} f(x) = -2$

   (c) $\lim_{x\to 3} f(x)$ does not exist, because the left- and right-hand limits are not equal.

   (d) $f(3) = 1$

2. (a) $\lim_{t\to -4^-} g(t) = 5$

   (b) $\lim_{t\to -4^+} g(t) = 2$

   (c) $\lim_{t\to -4} g(t)$ does not exist, because the left- and right-hand limits are not equal.

   (d) $g(-4) = 2$

3. (a) $\lim_{h\to 0^-} f(h) = -4$

   (b) $\lim_{h\to 0^+} f(h) = -4$

   (c) $\lim_{h\to 0} f(h) = -4$

   (d) $f(0) = -4$

4. (a) $\lim_{x\to -2^-} p(s) = 3$

   (b) $\lim_{x\to -2^+} p(s) = 3$

   (c) $\lim_{x\to -2} p(s) = 3$

   (d) $p(-2) = 3$

5. (a) $\lim_{x\to 0^-} F(x) = 4$

   (b) $\lim_{x\to 0^+} F(x) = -3$

   (c) $\lim_{x\to 0} F(x) =$ does not exist because the left- and right-hand limits are not equal.

   (d) $F(0) = 4$

6. (a) $\lim_{x\to 2^-} G(x) = 1$

   (b) $\lim_{x\to 2^+} G(x) = 1$

   (c) $\lim_{x\to 2} G(x) = 1$

   (d) $G(2) = 3$

7. (a) quotient rule
   (b) difference and power rules
   (c) sum and constant multiple rules

8. (a) quotient rule
   (b) power and product rules
   (c) difference and constant multiple rules

9. (a) $\lim\limits_{x\to c} f(x)\,g(x) = \left[\lim\limits_{x\to c} f(x)\right]\left[\lim\limits_{x\to c} g(x)\right] = (5)(-2) = -10$

(b) $\lim\limits_{x\to c} 2f(x)\,g(x) = 2\left[\lim\limits_{x\to c} f(x)\right]\left[\lim\limits_{x\to c} g(x)\right] = 2(5)(-2) = -20$

(c) $\lim\limits_{x\to c} [f(x) + 3g(x)] = \lim\limits_{x\to c} f(x) + 3\lim\limits_{x\to c} g(x) = 5 + 3(-2) = -1$

(d) $\lim\limits_{x\to c} \frac{f(x)}{f(x) - g(x)} = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} f(x) - \lim\limits_{x\to c} g(x)} = \frac{5}{5-(-2)} = \frac{5}{7}$

10. (a) $\lim\limits_{x\to 4} [g(x) + 3] = \lim\limits_{x\to 4} g(x) + \lim\limits_{x\to 4} 3 = -3 + 3 = 0$

(b) $\lim\limits_{x\to 4} xf(x) = \lim\limits_{x\to 4} x \cdot \lim\limits_{x\to 4} f(x) = (4)(0) = 0$

(c) $\lim\limits_{x\to 4} [g(x)]^2 = \left[\lim\limits_{x\to 4} g(x)\right]^2 = [-3]^2 = 9$

(d) $\lim\limits_{x\to 4} \frac{g(x)}{f(x) - 1} = \frac{\lim\limits_{x\to 4} g(x)}{\lim\limits_{x\to 4} f(x) - \lim\limits_{x\to 4} 1} = \frac{-3}{0-1} = 3$

11. (a) $\lim\limits_{x\to -7} (2x + 5) = 2(-7) + 5 = -14 + 5 = -9$

(b) $\lim\limits_{t\to 6} 8(t-5)(t-7) = 8(6-5)(6-7) = -8$

(c) $\lim\limits_{y\to 2} \frac{y+2}{y^2 + 5y + 6} = \frac{2+2}{(2)^2 + 5(2) + 6} = \frac{4}{4 + 10 + 6} = \frac{4}{20} = \frac{1}{5}$

(d) $\lim\limits_{h\to 0} \frac{3}{\sqrt{3h+1} + 1} = \frac{3}{\sqrt{3(0)+1} + 1} = \frac{3}{\sqrt{1} + 1} = \frac{3}{2}$

12. (a) $\lim\limits_{r\to -2} \left(r^3 - 2r^2 + 4r + 8\right) = (-2)^3 - 2(-2)^2 + 4(-2) + 8 = -8 - 8 - 8 + 8 = -16$

(b) $\lim\limits_{x\to 2} \frac{x+3}{x+6} = \frac{2+3}{2+6} = \frac{5}{8}$

(c) $\lim\limits_{y\to -3} (5-y)^{4/3} = [5-(-3)]^{4/3} = (8)^{4/3} = \left((8)^{1/3}\right)^4 = 2^4 = 16$

(d) $\lim\limits_{\theta\to 5} \frac{\theta - 5}{\theta^2 - 25} = \lim\limits_{\theta\to 5} \frac{\theta - 5}{(\theta+5)(\theta-5)} = \lim\limits_{\theta\to 5} \frac{1}{\theta + 5} = \frac{1}{5+5} = \frac{1}{10}$

13. (a) $\lim\limits_{t\to -5} \frac{t^2 + 3t - 10}{t+5} = \lim\limits_{t\to -5} \frac{(t+5)(t-2)}{t+5} = \lim\limits_{t\to -5} (t-2) = -5 - 2 = -7$

(b) $\lim\limits_{x\to -2} \frac{-2x - 4}{x^3 + 2x^2} = \lim\limits_{x\to -2} \frac{-2(x+2)}{x^2(x+2)} = \lim\limits_{x\to -2} \frac{-2}{x^2} = \frac{-2}{4} = -\frac{1}{2}$

(c) $\lim\limits_{y\to 1} \frac{y-1}{\sqrt{y+3} - 2} = \lim\limits_{y\to 1} \frac{(y-1)\left(\sqrt{y+3}+2\right)}{\left(\sqrt{y+3}-2\right)\left(\sqrt{y+3}+2\right)} = \lim\limits_{y\to 1} \frac{(y-1)\left(\sqrt{y+3}+2\right)}{(y+3)-4} = \lim\limits_{y\to 1} \left(\sqrt{y+3}+2\right)$

$= \sqrt{4} + 2 = 4$

14. (a) $\lim\limits_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim\limits_{x\to -1} \frac{\left(\sqrt{x^2+8}-3\right)\left(\sqrt{x^2+8}+3\right)}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim\limits_{x\to -1} \frac{(x^2+8)-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$

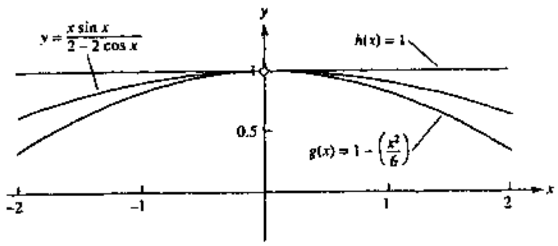
$= \lim\limits_{x\to -1} \frac{(x+1)(x-1)}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim\limits_{x\to -1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = -\frac{1}{3}$

(b) $\lim\limits_{\theta\to 1} \frac{\theta^4-1}{\theta^3-1} = \lim\limits_{\theta\to 1} \frac{(\theta^2+1)(\theta+1)(\theta-1)}{(\theta^2+\theta+1)(\theta-1)} = \lim\limits_{\theta\to 1} \frac{(\theta^2+1)(\theta+1)}{\theta^2+\theta+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}$

(c) $\lim\limits_{t\to 9} \frac{3-\sqrt{t}}{9-t} = \lim\limits_{t\to 9} \frac{\sqrt{t}-3}{\left(\sqrt{t}-3\right)\left(\sqrt{t}+3\right)} = \lim\limits_{t\to 9} \frac{1}{\sqrt{t}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

15. (a) $\lim\limits_{x\to 0}\left(1-\frac{x^2}{6}\right) = 1-\frac{0}{6} = 1$ and $\lim\limits_{x\to 0} 1 = 1$; by the sandwich theorem, $\lim\limits_{x\to 0} \frac{x \sin x}{2-2\cos x} = 1$

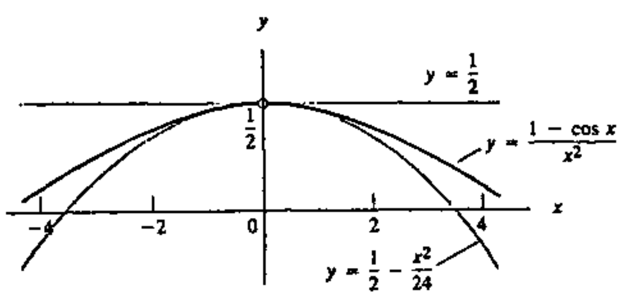
(b) For $x \neq 0$, $y = (x \sin x)/(2-2\cos x)$ lies between the other two graphs in the figure, and the graphs converge as $x \to 0$.



16. (a) $\lim\limits_{x\to 0}\left(\frac{1}{2}-\frac{x^2}{24}\right) = \lim\limits_{x\to 0} \frac{1}{2} - \lim\limits_{x\to 0} \frac{x^2}{24} = \frac{1}{2} - 0 = \frac{1}{2}$ and $\lim\limits_{x\to 0} \frac{1}{2} = \frac{1}{2}$; by the sandwich theorem,

$\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

(b) For all $x \neq 0$, the graph of $f(x) = (1-\cos x)/x^2$ lies between the line $y = \frac{1}{2}$ and the parabola $y = \frac{1}{2} - x^2/24$, and the graphs converge as $x \to 0$.



17. $\lim\limits_{h\to 0} \frac{(1+h)^2-1^2}{h} = \lim\limits_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim\limits_{h\to 0} \frac{h(2+h)}{h} = \lim\limits_{h\to 0} (2+h) = 2$

18. $\lim\limits_{h\to 0} \frac{[3(2+h)-4]-[3(2)-4]}{h} = \lim\limits_{h\to 0} \frac{3h}{h} = 3$

19. $\lim\limits_{h\to 0} \dfrac{\left(\frac{1}{-2+h}\right)-\left(\frac{1}{-2}\right)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{-2}{-2+h}-1}{-2h} = \lim\limits_{h\to 0} \dfrac{-2-(-2+h)}{-2h(-2+h)} = \lim\limits_{h\to 0} \dfrac{-h}{h(4-2h)} = -\frac{1}{4}$

20. $\lim\limits_{h\to 0} \dfrac{\sqrt{7+h}-\sqrt{7}}{h} = \lim\limits_{h\to 0} \dfrac{\left(\sqrt{7+h}-\sqrt{7}\right)\left(\sqrt{7+h}+\sqrt{7}\right)}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim\limits_{h\to 0} \dfrac{(7+h)-7}{h\left(\sqrt{7+h}+\sqrt{7}\right)}$

$= \lim\limits_{h\to 0} \dfrac{h}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim\limits_{h\to 0} \dfrac{1}{\sqrt{7+h}+\sqrt{7}} = \dfrac{1}{2\sqrt{7}}$

21. (a) True (b) True (c) False (d) True
(e) True (f) True (g) False (h) False
(i) False (j) False (k) True (l) False

22. (a) $\lim\limits_{x\to 2^+} f(x) = \frac{2}{2}+1 = 2$, $\lim\limits_{x\to 2^-} f(x) = 3-2 = 1$

(b) No, $\lim\limits_{x\to 2} f(x)$ does not exist because $\lim\limits_{x\to 2^+} f(x) \neq \lim\limits_{x\to 2^-} f(x)$

(c) $\lim\limits_{x\to 4^-} f(x) = \frac{4}{2}+1 = 3$, $\lim\limits_{x\to 4^+} f(x) = \frac{4}{2}+1 = 3$

(d) Yes, $\lim\limits_{x\to 4} f(x) = 3$ because $3 = \lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^+} f(x)$

23. (a) No, $\lim\limits_{x\to 0^+} f(x)$ does not exist since $\sin\left(\frac{1}{x}\right)$ does not approach any single value as x approaches 0

(b) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 0 = 0$

(c) $\lim\limits_{x\to 0} f(x)$ does not exist because $\lim\limits_{x\to 0^+} f(x)$ does not exist

24. (a) Yes, $\lim\limits_{x\to 0^+} g(x) = 0$ by the sandwich theorem since $-\sqrt{x} \le g(x) \le \sqrt{x}$ when $x > 0$

(b) No, $\lim\limits_{x\to 0^-} g(x)$ does not exist since $\sqrt{x}$ does not exist, and therefore the function is not defined, for $x < 0$

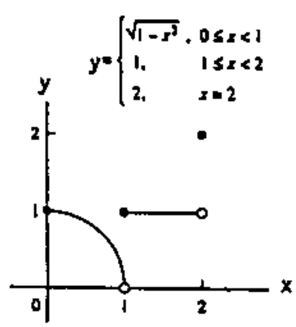
(c) No, $\lim\limits_{x\to 0} g(x)$ does not exist since $\lim\limits_{x\to 0^-} g(x)$ does not exist

25. (a) domain: $0 \le x \le 2$
range: $0 < y \le 1$ and $y = 2$

(b) $\lim\limits_{x\to c} f(x)$ exists for c belonging to $(0,1) \cup (1,2)$

(c) $x = 2$

(d) $x = 0$

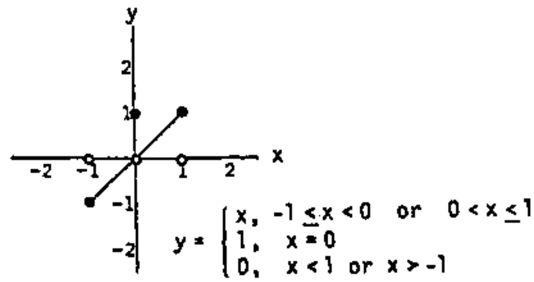


26. (a) domain: $-\infty < x < \infty$
range: $-1 \le y \le 1$

(b) $\lim\limits_{x \to c} f(x)$ exists for c belonging to
$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(c) none

(d) none



27. $\lim\limits_{x \to -0.5^-} \sqrt{\frac{x+2}{x+1}} = \sqrt{\frac{-0.5+2}{-0.5+1}} = \sqrt{\frac{3/2}{1/2}} = \sqrt{3}$

28. $\lim\limits_{x \to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right) = (2)\left(\frac{1}{2}\right) = 1$

29. $\lim\limits_{h \to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h} = \lim\limits_{h \to 0^+} \left(\frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}\right)\left(\frac{\sqrt{h^2+4h+5}+\sqrt{5}}{\sqrt{h^2+4h+5}+\sqrt{5}}\right)$

$= \lim\limits_{h \to 0^+} \frac{(h^2+4h+5)-5}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} = \lim\limits_{h \to 0^+} \frac{h(h+4)}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} = \frac{0+4}{\sqrt{5}+\sqrt{5}} = \frac{2}{\sqrt{5}}$

30. $\lim\limits_{h \to 0^-} \frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h} = \lim\limits_{h \to 0^-} \left(\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}\right)\left(\frac{\sqrt{6}+\sqrt{5h^2+11h+6}}{\sqrt{6}+\sqrt{5h^2+11h+6}}\right)$

$= \lim\limits_{h \to 0^-} \frac{6-(5h^2+11h+6)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \lim\limits_{h \to 0^-} \frac{-h(5h+11)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \frac{-(0+11)}{\sqrt{6}+\sqrt{6}} = -\frac{11}{2\sqrt{6}}$

31. (a) $\lim\limits_{x \to -2^+} (x+3)\frac{|x+2|}{x+2} = \lim\limits_{x \to -2^+} (x+3)\frac{(x+2)}{(x+2)}$ $\quad (|x+2| = x+2 \text{ for } x > -2)$

$= \lim\limits_{x \to -2^+} (x+3) = (-2)+3 = 1$

(b) $\lim\limits_{x \to -2^-} (x+3)\frac{|x+2|}{x+2} = \lim\limits_{x \to -2^-} (x+3)\left[\frac{-(x+2)}{(x+2)}\right]$ $\quad (|x+2| = -(x+2) \text{ for } x < -2)$

$= \lim\limits_{x \to -2^-} (x+3)(-1) = -(-2+3) = -1$

32. (a) $\lim\limits_{x \to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim\limits_{x \to 1^+} \frac{\sqrt{2x}(x-1)}{(x-1)}$ $\quad (|x-1| = x-1 \text{ for } x > 1)$

$= \lim\limits_{x \to 1^+} \sqrt{2x} = \sqrt{2}$

(b) $\lim\limits_{x \to 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim\limits_{x \to 1^-} \frac{\sqrt{2x}(x-1)}{-(x-1)}$ $\quad (|x-1| = -(x-1) \text{ for } x < 1)$

$= \lim\limits_{x \to 1^-} -\sqrt{2x} = -\sqrt{2}$

33. $\lim\limits_{x\to c} f(x)$ exists at those points c where $\lim\limits_{x\to c} x^4 = \lim\limits_{x\to c} x^2$. Thus, $c^4 = c^2 \Rightarrow c^2(1-c^2) = 0$ $\Rightarrow c = 0, 1$, or -1. Moreover, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x^2 = 0$ and $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to 1} f(x) = 1$.

34. Nothing can be concluded about the values of f, g, and h at $x = 2$. Yes, f(2) could be 0. Since the conditions of the sandwich theorem are satisfied, $\lim\limits_{x\to 2} f(x) = -5 \neq 0$.

35. (a) $1 = \lim\limits_{x\to -2} \frac{f(x)}{x^2} = \frac{\lim\limits_{x\to -2} f(x)}{\lim\limits_{x\to -2} x^2} = \frac{\lim\limits_{x\to -2} f(x)}{4} \Rightarrow \lim\limits_{x\to -2} f(x) = 4$.

(b) $1 = \lim\limits_{x\to -2} \frac{f(x)}{x^2} = \left[\lim\limits_{x\to -2} \frac{f(x)}{x}\right]\left[\lim\limits_{x\to -2} \frac{1}{x}\right] = \left[\lim\limits_{x\to -2} \frac{f(x)}{x}\right]\left(\frac{1}{-2}\right) \Rightarrow \lim\limits_{x\to -2} \frac{f(x)}{x} = -2$.

36. (a) $0 = 3\cdot 0 = \left[\lim\limits_{x\to 2} \frac{f(x)-5}{x-2}\right]\left[\lim\limits_{x\to 2} (x-2)\right] = \lim\limits_{x\to 2} \left[\left(\frac{f(x)-5}{x-2}\right)(x-2)\right] = \lim\limits_{x\to 2} [f(x)-5] = \lim\limits_{x\to 2} f(x) - 5$

$\Rightarrow \lim\limits_{x\to 2} f(x) = 5$.

(b) $0 = 4\cdot 0 = \left[\lim\limits_{x\to 2} \frac{f(x)-5}{x-2}\right]\left[\lim\limits_{x\to 2} (x-2)\right] \Rightarrow \lim\limits_{x\to 2} f(x) = 5$ as in part (a).

37. Yes. If $\lim\limits_{x\to a^+} f(x) = L = \lim\limits_{x\to a^-} f(x)$, then $\lim\limits_{x\to a} f(x) = L$. If $\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x)$, then $\lim\limits_{x\to a} f(x)$ does not exist.

38. Since $\lim\limits_{x\to c} f(x) = L$ if and only if $\lim\limits_{x\to c^+} f(x) = L$ and $\lim\limits_{x\to c^-} f(x) = L$, then $\lim\limits_{x\to c} f(x)$ can be found by calculating $\lim\limits_{x\to c^+} f(x)$.

39. $I = (5, 5+\delta) \Rightarrow 5 < x < 5+\delta$. Also, $\sqrt{x-5} < \epsilon \Rightarrow x - 5 < \epsilon^2 \Rightarrow x < 5 + \epsilon^2$. Choose $\delta = \epsilon^2$ $\Rightarrow \lim\limits_{x\to 5^+} \sqrt{x-5} = 0$.

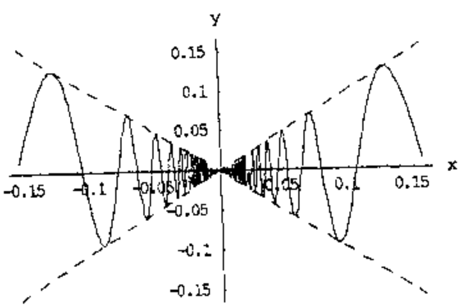
40. $I = (4-\delta, 4) \Rightarrow 4 - \delta < x < 4$. Also, $\sqrt{4-x} < \epsilon \Rightarrow 4 - x < \epsilon^2 \Rightarrow x > 4 - \epsilon^2$. Choose $\delta = \epsilon^2$ $\Rightarrow \lim\limits_{x\to 4^-} \sqrt{4-x} = 0$.

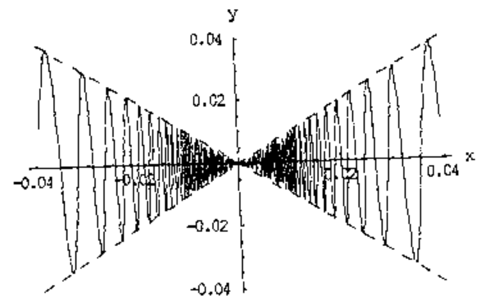
41. If f is an odd function of x, then $f(-x) = -f(x)$. Given $\lim\limits_{x\to 0^+} f(x) = 3$, then $\lim\limits_{x\to 0^-} f(x) = -3$.

42. If f is an even function of x, then $f(-x) = f(x)$. Given $\lim\limits_{x\to 2^-} f(x) = 7$ then $\lim\limits_{x\to -2^+} f(x) = 7$. However, nothing can be said about $\lim\limits_{x\to -2^-} f(x)$ because we don't know $\lim\limits_{x\to 2^+} f(x)$.

43. (a) $g(x) = x \sin\left(\frac{1}{x}\right)$

$-\frac{\pi}{20} \le x \le \frac{\pi}{20}$ $\qquad\qquad$ $-\frac{\pi}{80} \le x \le \frac{\pi}{180}$

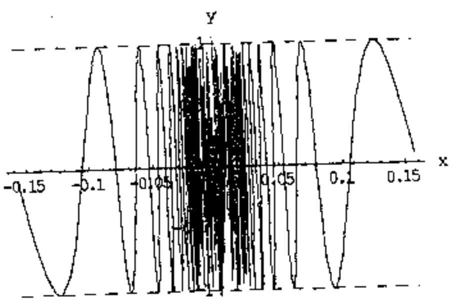


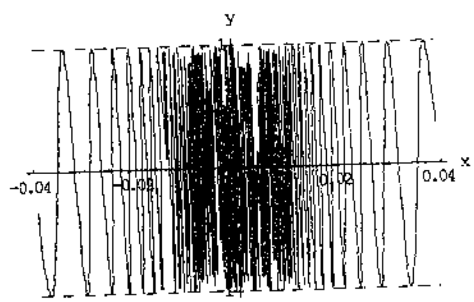


The graphs suggest that $\lim_{x\to 0} g(x) = 0$.

(b) $k(x) = \sin\left(\frac{1}{x}\right)$

$-\frac{\pi}{20} \le x \le \frac{\pi}{20}$ $\qquad$ $-\frac{\pi}{80} \le x \le \frac{\pi}{80}$

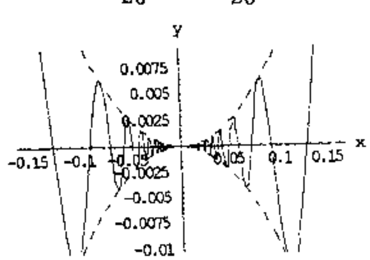


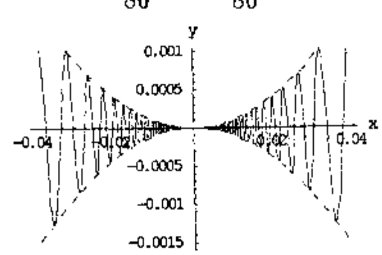


The graphs suggest that $\lim_{x\to 0} k(x)$ does not exist.
For both $g(x)$ and $k(x)$, the frequency of the oscillations increases without bound as $x \to 0$. For $g(x)$, the sandwich theorem can be applied. If $x > 0$, $-x \le x \sin\left(\frac{1}{x}\right) \le x \Rightarrow \lim_{x\to 0^+} g(x) = 0$ and if $x < 0$, $x \le x \sin\left(\frac{1}{x}\right) \le -x \Rightarrow \lim_{x\to 0^-} g(x) = 0$. Therefore, $\lim_{x\to 0} g(x) = 0$ since the left- and right-hand limits are both 0. For $k(x)$, the amplitude of the oscillations remains equal to one. Therefore, $k(x)$ cannot be kept arbitrarily close to any number by keeping x sufficiently close to 0.

44. (a) $h(x) = x^2 \cos\left(\frac{1}{x}\right)$

$-\frac{\pi}{20} \le x \le \frac{\pi}{20}$ $\qquad$ $-\frac{\pi}{80} \le x \le \frac{\pi}{80}$

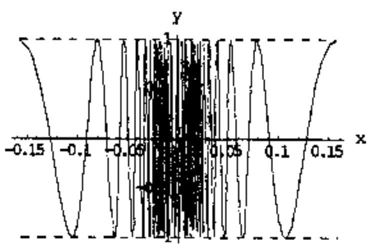




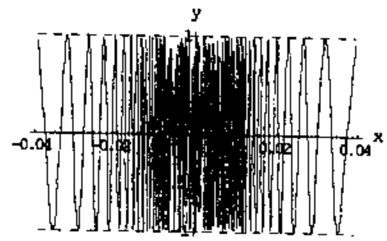
The graphs suggest that $\lim_{x\to 0} h(x) = 0$.

(b) $k(x) = \cos\left(\frac{1}{x}\right)$

$-\frac{\pi}{20} \le x \le \frac{\pi}{20}$



$-\frac{\pi}{80} \le x \le \frac{\pi}{80}$



The graphs suggest that $\lim_{x \to 0} k(x)$ does not exist.
For both h(x) and k(x), the frequency of the oscillations increases without bound as $x \to 0$. For h(x), the sandwich theorem can be applied: $-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2 \Rightarrow \lim_{x \to 0} g(x) = 0$. For k(x), the amplitude of the oscillations remains equal to one. Therefore, k(x) cannot be kept arbitrarily close to any number by keeping x sufficiently close to 0.

1.3 LIMITS INVOLVING INFINITY

Note: In these exercises we use the result $\lim_{x \to \pm\infty} \frac{1}{x^{m/n}} = 0$ whenever $\frac{m}{n} > 0$. This result follows immediately from Example 1 and the power rule in Theorem 7: $\lim_{x \to \pm\infty} \left(\frac{1}{x^{m/n}}\right) = \lim_{x \to \pm\infty} \left(\frac{1}{x}\right)^{m/n} = \left(\lim_{x \to \pm\infty} \frac{1}{x}\right)^{m/n} = 0^{m/n}$ $= 0$.

1. (a) π (b) π

2. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$

3. (a) $-\frac{5}{3}$ (b) $-\frac{5}{3}$

4. (a) $\frac{3}{4}$ (b) $\frac{3}{4}$

5. $-\frac{1}{x} \le \frac{\sin 2x}{x} \le \frac{1}{x} \Rightarrow \lim_{x \to \infty} \frac{\sin 2x}{x} = 0$ by the Sandwich Theorem

6. $\lim_{t \to -\infty} \frac{2 - t + \sin t}{t + \cos t} = \lim_{t \to -\infty} \frac{\frac{2}{t} - 1 + \left(\frac{\sin t}{t}\right)}{1 + \left(\frac{\cos t}{t}\right)} = \frac{0 - 1 + 0}{1 + 0} = -1$

7. (a) $\lim\limits_{x\to\infty} \frac{2x+3}{5x+7} = \lim\limits_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2}{5}$ (b) $\frac{2}{5}$ (same process as part (a))

8. (a) $\lim\limits_{x\to\infty} \frac{x+1}{x^2+3} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} = 0$ (b) 0 (same process as part (a))

9. (a) $\lim\limits_{x\to\infty} \frac{1-12x^3}{4x^2+12} = \lim\limits_{x\to\infty} \frac{\frac{1}{x^2}-12x}{4+\frac{12}{x^2}} = -\infty$ (b) $\lim\limits_{x\to-\infty} \frac{1-12x^3}{4x^2+12} = \lim\limits_{x\to-\infty} \frac{\frac{1}{x^2}-12x}{4+\frac{12}{x^2}} = \infty$

10. (a) $\lim\limits_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim\limits_{x\to\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} = 7$ (b) 7 (same process as part (a))

11. (a) $\lim\limits_{x\to\infty} \frac{3x^2-6x}{4x-8} = \lim\limits_{x\to\infty} \frac{3x-6}{4-\frac{8}{x}} = \infty$ (b) $\lim\limits_{x\to-\infty} \frac{3x^2-6x}{4x-8} = \lim\limits_{x\to-\infty} \frac{3x-6}{4-\frac{8}{x}} = -\infty$

12. (a) $\lim\limits_{x\to\infty} \frac{2x^5+3}{-x^2+x} = \lim\limits_{x\to\infty} \frac{2x^3+\frac{3}{x^2}}{-1+\frac{1}{x}} = -\infty$ (b) $\lim\limits_{x\to-\infty} \frac{2x^5+3}{-x^2+x} = \lim\limits_{x\to-\infty} \frac{2x^3+\frac{3}{x^2}}{-1+\frac{1}{x}} = \infty$

13. (a) $\lim\limits_{x\to\infty} \frac{-2x^3-2x+3}{3x^3+3x^2-5x} = \lim\limits_{x\to\infty} \frac{-2-\frac{2}{x^2}+\frac{3}{x^3}}{3+\frac{3}{x}-\frac{5}{x^2}} = -\frac{2}{3}$

(b) $-\frac{2}{3}$ (same process as part (a))

14. (a) $\lim\limits_{x\to\infty} \frac{-x^4}{x^4-7x^3+7x^2+9} = \lim\limits_{x\to\infty} \frac{-1}{1-\frac{7}{x}+\frac{7}{x^2}+\frac{9}{x^4}} = -1$

(b) -1 (same process as part (a))

15. $\lim\limits_{x\to\infty} \frac{2\sqrt{x}+x^{-1}}{3x-7} = \lim\limits_{x\to\infty} \frac{\left(\frac{2}{x^{1/2}}\right)+\left(\frac{1}{x^2}\right)}{3-\frac{7}{x}} = 0$ 16. $\lim\limits_{x\to\infty} \frac{2+\sqrt{x}}{2-\sqrt{x}} = \lim\limits_{x\to\infty} \frac{\left(\frac{2}{x^{1/2}}\right)+1}{\left(\frac{2}{x^{1/2}}\right)-1} = -1$

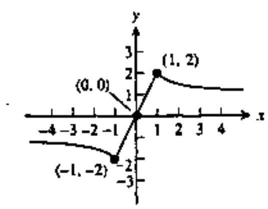
17. $\lim\limits_{x\to-\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}} = \lim\limits_{x\to-\infty} \frac{1-x^{(1/5)-(1/3)}}{1+x^{(1/5)-(1/3)}} = \lim\limits_{x\to-\infty} \frac{1-\left(\frac{1}{x^{2/15}}\right)}{1+\left(\frac{1}{x^{2/15}}\right)} = 1$

18. $\lim\limits_{x\to\infty} \frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}} = \lim\limits_{x\to\infty} \frac{x+\frac{1}{x^2}}{1-\frac{1}{x}} = \infty$

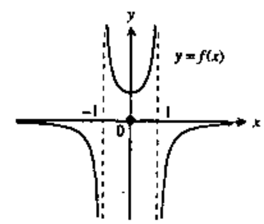
19. $\lim\limits_{x\to\infty} \frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}} = \lim\limits_{x\to\infty} \frac{2x^{1/15}-\frac{1}{x^{19/15}}+\frac{7}{x^{8/5}}}{1+\frac{3}{x^{3/5}}+\frac{1}{x^{11/10}}} = \infty$

20. $\lim\limits_{x\to-\infty} \frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4} = \lim\limits_{x\to-\infty} \frac{\frac{1}{x^{2/3}}-5+\frac{3}{x}}{2+\frac{1}{x^{1/3}}-\frac{4}{x}} = -\frac{5}{2}$

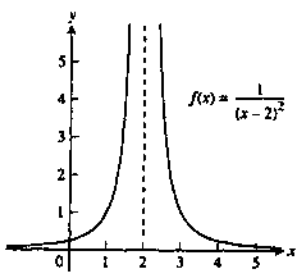
21. Here is one possibility.



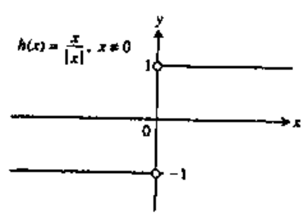
22. Here is one possibility.



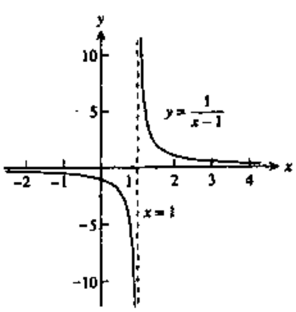
23. Here is one possibility.



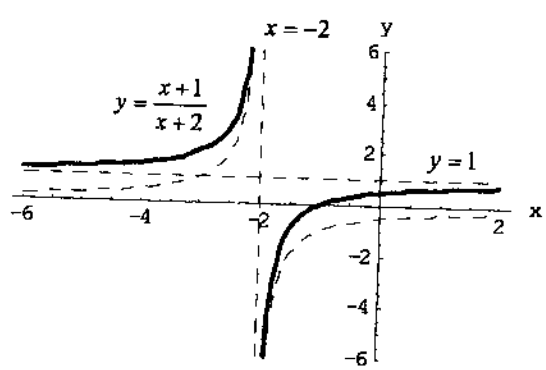
24. Here is one possibility.



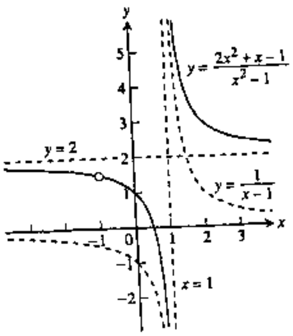
25. $y = \frac{1}{x-1}$



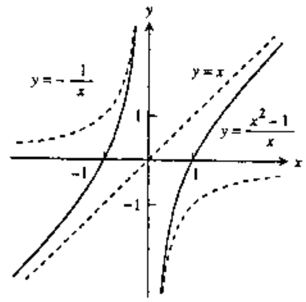
26. $y = \frac{x+1}{x+2} = 1 - \frac{1}{x+2}$



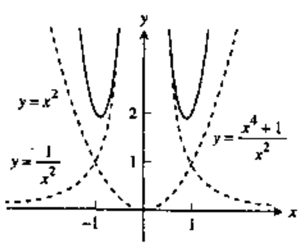
27. $y = \frac{2x^2 + x - 1}{x^2 - 1}$



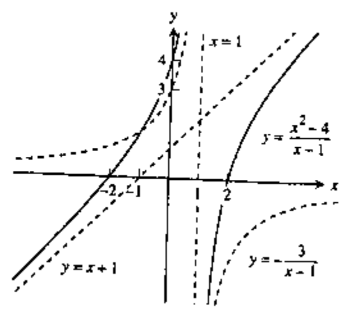
28. $y = \frac{x^2 - 1}{x} = x - \frac{1}{x}$



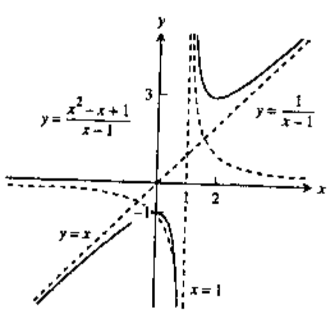
29. $y = \frac{x^4 + 1}{x^2} = x^2 + \frac{1}{x^2}$



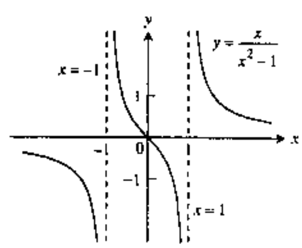
30. $y = \frac{x^2 - 4}{x - 1} = x + 1 - \frac{3}{x - 1}$



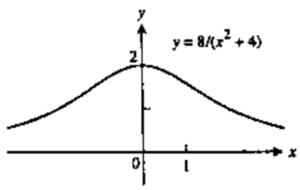
31. $y = \frac{x^2 - x + 1}{x - 1} = x + \frac{1}{x - 1}$



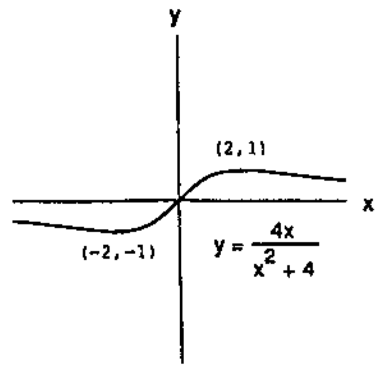
32. $y = \frac{x}{x^2 - 1}$



33. $y = \dfrac{8}{x^2+4}$



34. $y = \dfrac{4x}{x^2+4}$



35. An end behavior model is $\frac{2x^3}{x} = 2x^2$. (a)

36. An end behavior model is $\frac{x^5}{2x^2} = 0.5x^3$. (c)

37. An end behavior model is $\frac{2x^4}{-x} = -2x^3$. (d)

38. An end behavior model is $\frac{x^4}{-x^2} = -x^2$. (b)

39. (a) The function $y = e^x$ is a right end behavior model because $\lim\limits_{x\to\infty} \frac{e^x - 2x}{e^x} = \lim\limits_{x\to\infty}\left(1 - \frac{2x}{e^x}\right) = 1 - 0 = 1.$

(b) The function $y = -2x$ is a left end behavior model because $\lim\limits_{x\to-\infty} \frac{e^x - 2x}{-2x} = \lim\limits_{x\to-\infty}\left(-\frac{e^x}{2x} + 1\right) = 0 + 1 = 1.$

40. (a) The function $y = x^2$ is a right end behavior model because $\lim\limits_{x\to\infty} \dfrac{x^2 + e^{-x}}{x^2} = \lim\limits_{x\to\infty}\left(1 + \dfrac{e^{-x}}{x^2}\right) = 1 - 0 = 1.$

(b) The function $y = e^{-x}$ is a left end behavior model because $\lim\limits_{x\to-\infty} \dfrac{x^2 + e^{-x}}{e^{-x}} = \lim\limits_{x\to-\infty}\left(\dfrac{x^2}{e^{-x}} + 1\right)$

$= \lim\limits_{x\to-\infty}(x^2e^x + 1) = 0 + 1 = 1.$

41. (a, b) The function $y = x$ is both a right end behavior model and a left end behavior model because

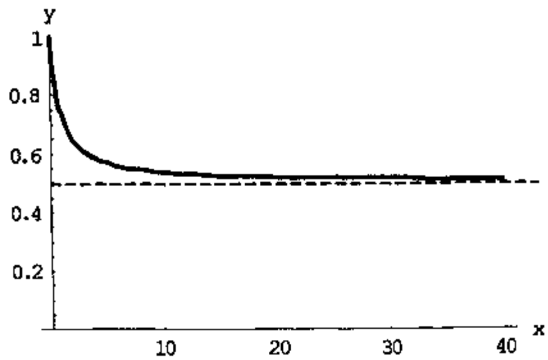
$$\lim_{x\to\pm\infty}\left(\frac{x + \ln|x|}{x}\right) = \lim_{x\to\pm\infty}\left(1 + \frac{\ln|x|}{x}\right) = 1 - 0 = 1.$$

42. (a, b) The function $y = x^2$ is both a right end behavior model and a left end behavior model because

$$\lim_{x\to\pm\infty}\left(\frac{x^2 + \sin x}{x^2}\right) = \lim_{x\to\pm\infty}\left(1 + \frac{\sin x}{x^2}\right) = 1.$$

43. $f(x) = \sqrt{x^2 + x + 1} - x$

(a)



The graph suggests that $\lim_{x\to\infty} f(x) = \frac{1}{2}$.

(b)

x	f(x) to 6 decimal places
0	1.000000
10	0.535654
100	0.503731
1000	0.500375
10000	0.500037
100000	0.500004
1000000	0.500000

The table of values also suggest that $\lim_{x\to\infty} f(x) = \frac{1}{2}$

Proof: $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right) = \lim_{x\to\infty}\left[\left(\sqrt{x^2+x+1}-x\right)\left(\frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x}\right)\right] = \lim_{x\to\infty}\left(\frac{1+x}{\sqrt{x^2+x+1}+x}\right)$

$= \lim_{x\to\infty}\left(\frac{1+1/x}{\sqrt{1+1/x+1/x^2}+1}\right) = \frac{1}{2}$

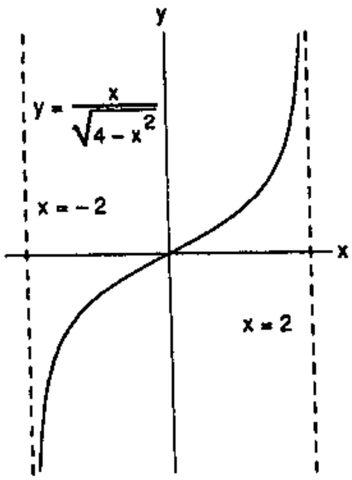
44. $\lim_{x\to\infty} \sqrt{x^2+x}-\sqrt{x^2-x} = \lim_{x\to\infty}\left[\sqrt{x^2+x}-\sqrt{x^2-x}\right]\cdot\left[\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}\right] = \lim_{x\to\infty} \frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}$

$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}} = \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}} = \frac{2}{1+1} = 1$

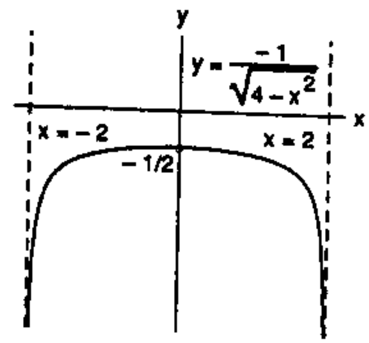
45. At most 2 horizontal asymptotes: one for $x \to \infty$ and possibly another for $x \to -\infty$.

46. At most the degree of the denominator, which is zero at a vertical asymptote. A polynomial of degree n has at most n real roots (or zeros).

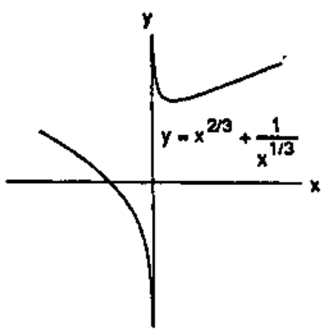
47. $y = \frac{x}{\sqrt{4-x^2}}$



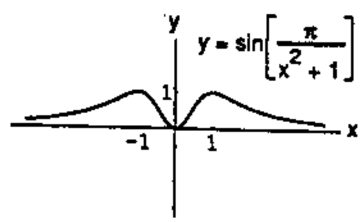
48. $y = \frac{-1}{\sqrt{4-x^2}}$



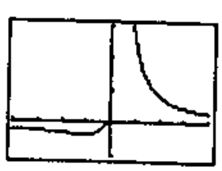
49. $y = x^{2/3} + \frac{1}{x^{1/3}}$



50. $y = \sin\left(\frac{\pi}{x^2+1}\right)$



51.



[−4, 4] by [−1, 3]

The graph of $y = f\left(\frac{1}{x}\right) = \frac{1}{x}e^{1/x}$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = \infty$

$\lim_{x\to -\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 0$

52.

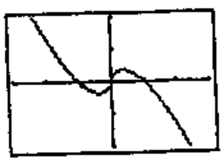
[−4, 4] by [−1, 3]

The graph of $y = f\left(\frac{1}{x}\right) = \frac{1}{x^2}e^{-1/x}$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 0$

$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = \infty$

53.



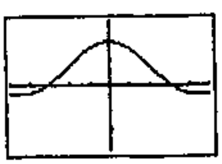
[−3, 3] by [−2, 2]

The graph of $y = f\left(\frac{1}{x}\right) = x \ln\left|\frac{1}{x}\right|$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 0$

$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 0$

54.



[−5, 5] by [−1.5, 1.5]

The graph of $y = f\left(\frac{1}{x}\right) = \frac{\sin x}{x}$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 1$

$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 1$

55. $\lim\limits_{x\to-\infty} \dfrac{\cos\frac{1}{x}}{1+\frac{1}{x}} = \lim\limits_{\theta\to0^-} \dfrac{\cos\theta}{1+\theta} = \frac{1}{1} = 1, \quad \left(\theta = \frac{1}{x}\right)$

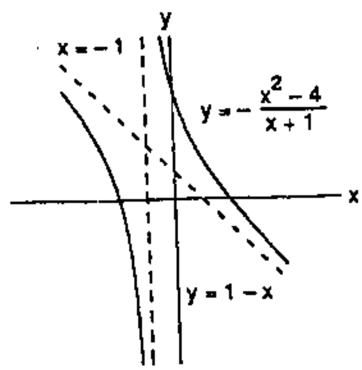
56. $\lim\limits_{x\to\infty} \left(\frac{1}{x}\right)^{1/x} = \lim\limits_{z\to0^+} z^z = 1, \quad \left(z = \frac{1}{x}\right)$

57. $\lim\limits_{x\to\pm\infty} \left(3+\frac{2}{x}\right)\left(\cos\frac{1}{x}\right) = \lim\limits_{\theta\to0} (3+2\theta)(\cos\theta) = (3)(1) = 3, \quad \left(\theta = \frac{1}{x}\right)$

58. $\lim\limits_{x\to\infty} \left(\frac{3}{x^2} - \cos\frac{1}{x}\right)\left(1+\sin\frac{1}{x}\right) = \lim\limits_{\theta\to0^+} \left(3\theta^2 - \cos\theta\right)(1+\sin\theta) = (0-1)(1+0) = -1, \quad \left(\theta = \frac{1}{x}\right)$

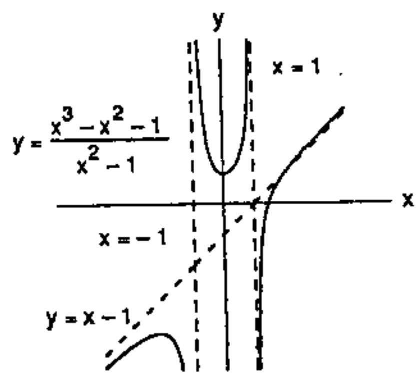
59. $y = -\dfrac{x^2-4}{x+1} = 1 - x + \dfrac{3}{x+1}$

The graph of the function mimics each term as it becomes dominant.

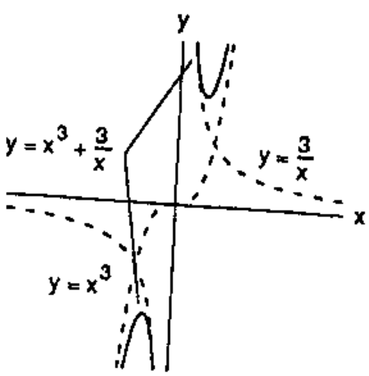


60. $y = \dfrac{x^3-x^2-1}{x^2-1} = x - 1 + \dfrac{x-2}{x^2-1}$

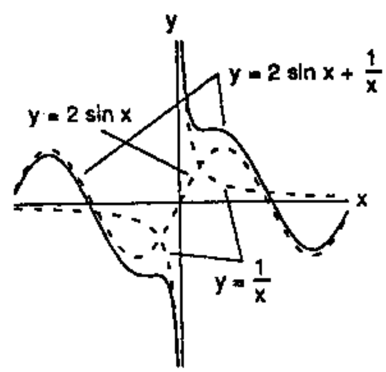
The graph of the function mimics each term as it becomes dominant.



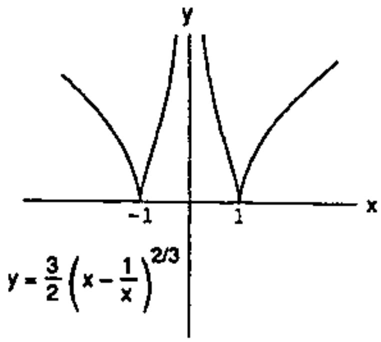
61. The graph of the function mimics each term as it becomes dominant.



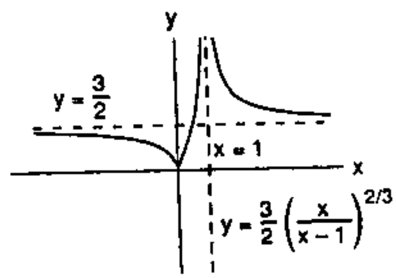
62. The graph of the function mimics each term as it becomes dominant.



63. (a) $y \to \infty$ (see the accompanying graph)
    (b) $y \to \infty$ (see the accompanying graph)
    (c) cusps at $x = \pm 1$ (see the accompanying graph)



64. (a) $y \to 0$ and a cusp at $x = 0$ (see the accompanying graph)
    (b) $y \to \frac{3}{2}$ (see the accompanying graph)
    (c) a vertical asymptote at $x = 1$ and contains the point $\left(-1, \frac{3}{2\sqrt[3]{4}}\right)$ (see the accompanying graph)



1.4 CONTINUITY

1. No, discontinuous at $x = 2$, not defined at $x = 2$

2. No, discontinuous at $x = 3$, $1 = \lim\limits_{x \to 3^-} g(x) \neq g(3) = 1.5$

3. Continuous on $[-1, 3]$

4. No, discontinuous at $x = 1$, $1.5 = \lim\limits_{x \to 1^-} k(x) \neq \lim\limits_{x \to 1^+} k(x) = 0$

5. (a) Yes (b) Yes, $\lim\limits_{x \to -1^+} f(x) = 0$
   (c) Yes (d) Yes

6. (a) Yes, $f(1) = 1$ (b) Yes, $\lim\limits_{x \to 1} f(x) = 2$
   (c) No (d) No

7. (a) No (b) No

8. $[-1,0) \cup (0,1) \cup (1,2) \cup (2,3)$

9. $f(2) = 0$, since $\lim_{x\to 2^-} f(x) = -2(2) + 4 = 0 = \lim_{x\to 2^+} f(x)$

10. $f(1)$ should be changed to $2 = \lim_{x\to 1} f(x)$

11. The function $f(x)$ is not continuous at $x = 0$ because $\lim_{x\to 0} f(x) = 0$, $f(0) = 1$ and, therefore, $\lim_{x\to 0} f(x) \neq f(0)$. The function $f(x)$ is not continuous at $x = 1$ because $\lim_{x\to 1} f(x)$ does not exist since $\lim_{x\to 1^-} f(x) = -1$ and $\lim_{x\to 1^+} f(x) = 0$. The discontinuity at $x = 0$ is removable because the function would be continuous there if the value of $f(0)$ were 0 instead of 1. The discontinuity at $x = 1$ is not removable because $\lim_{x\to 1} f(x)$ does not exist and the discontinuity cannot be removed by defining or redefining $f(1)$.

12. The function $f(x)$ is not continuous at $x = 1$ because $\lim_{x\to 1} f(x)$ does not exist since $\lim_{x\to 1^-} f(x) = -2$ and $\lim_{x\to 1^+} f(x) = 0$. The function $f(x)$ is not continuous at $x = 2$ because $\lim_{x\to 2} f(x) = 1$, $f(2) = 0$ and, therefore, $\lim_{x\to 2} f(x) \neq f(2)$. The discontinuity at $x = 1$ is not removable because $\lim_{x\to 1} f(x)$ does not exist and the discontinuity cannot be removed by defining or redefining $f(1)$. The discontinuity at $x = 2$ is removable because the function would be continuous there if the value of $f(2)$ were 1 instead of 0.

13. Discontinuous only when $x - 2 = 0 \Rightarrow x = 2$ $\Rightarrow$ continuous on $(-\infty, 2) \cup (2, \infty)$

14. Discontinuous only when $(x+2)^2 = 0 \Rightarrow x = -2$ $\Rightarrow$ continuous on $(-\infty, -2) \cup (-2, \infty)$

15. Discontinuous only when $t^2 - 4t + 3 = 0 \Rightarrow (t-3)(t-1) = 0 \Rightarrow t = 3$ or $t = 1 \Rightarrow$ continuous on $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$

16. Continuous everywhere. ($|t| + 1 \neq 0$ for all t; limits exist and are equal to function values.)

17. Discontinuous only at $\theta = 0 \Rightarrow$ continuous on $(-\infty, 0) \cup (0, \infty)$

18. Discontinuous when $\frac{\pi\theta}{2}$ is an odd integer multiple of $\frac{\pi}{2}$, i.e., $\frac{\pi\theta}{2} = (2n-1)\frac{\pi}{2}$, n an integer $\Rightarrow \theta = 2n - 1$, n an integer (i.e., θ is an odd integer). Continuous everywhere else $\Rightarrow$ continuous on $((2n-1)\pi/2, (2n+1)\pi/2)$ for n an integer.

19. Discontinuous when $2v + 3 < 0$ or $v < -\frac{3}{2} \Rightarrow$ continuous on the interval $\left[-\frac{3}{2}, \infty\right)$.

20. Discontinuous when $3x - 1 < 0$ or $x < \frac{1}{3} \Rightarrow$ continuous on the interval $\left[\frac{1}{3}, \infty\right)$.

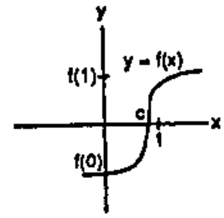
21. $\lim_{x\to\pi} \sin(x - \sin x) = \sin(\pi - \sin \pi) = \sin(\pi - 0) = \sin \pi = 0$; continuous at $x = \pi$

22. $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right) = \sin\left(\frac{\pi}{2}\cos(\tan(0))\right) = \sin\left(\frac{\pi}{2}\cos(0)\right) = \sin\left(\frac{\pi}{2}\right) = 1$; continuous at $t = 0$

23. $\lim_{y\to 1} \sec\left(y\sec^2 y - \tan^2 y - 1\right) = \lim_{y\to 1} \sec\left(y\sec^2 y - \sec^2 y\right) = \lim_{y\to 1} \sec\left((y-1)\sec^2 y\right) = \sec\left((1-1)\sec^2 1\right)$ $= \sec 0 = 1$; continuous at $y = 1$

24. $\lim_{\theta\to 0} \tan\left[\frac{\pi}{4}\cos\left(\sin \theta^{1/3}\right)\right] = \tan\left[\frac{\pi}{4}\cos(\sin(0))\right] = \tan\left(\frac{\pi}{4}\cos(0)\right) = \tan\left(\frac{\pi}{4}\right) = 1$; continuous at $\theta = 0$.

25. f(x) is continuous on $[0,1]$ and $f(0) < 0$, $f(1) > 0$ $\Rightarrow$ by the Intermediate Value Theorem f(x) takes on every value between f(0) and f(1) $\Rightarrow$ the equation $f(x) = 0$ has at least one solution between $x = 0$ and $x = 1$.



26. $\cos x = x \Rightarrow (\cos x) - x = 0$. If $x = -\frac{\pi}{2}$, $\cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0$. If $x = \frac{\pi}{2}$, $\cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} < 0$. Thus $\cos x - x = 0$ for some x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ according to the Intermediate Value Theorem.

27. All five statements ask for the same information because of the intermediate value property of continuous functions.

   (a) A root of $f(x) = x^3 - 3x - 1$ is a point c where $f(c) = 0$. The roots are approximately $x_1 = -1.53$, $x_2 = -0.347$, $x_3 = 1.88$, the points where f(x) changes sign.

   (b) The points where $y = x^3$ crosses $y = 3x + 1$ have the same y-coordinate, or $y = x^3 = 3x + 1 \Rightarrow y = f(x) = x^3 - 3x - 1 = 0$.

   (c) $x^3 - 3x = 1 \Rightarrow x^3 - 3x - 1 = 0$. The solutions to the equation are the roots of $f(x) = x^3 - 3x - 1$.

   (d) The points where $y = x^3 - 3x$ crosses $y = 1$ have common y-coordinates, or $y = x^3 - 3x = 1 \Rightarrow y = f(x) = x^3 - 3x - 1 = 0$.

   (e) The solutions of $x^3 - 3x - 1 = 0$ are those points where $f(x) = x^3 - 3x - 1$ has value 0.

28. Answers may vary. Note that f is continuous for every value of x.

   (a) $f(0) = 10$, $f(1) = 1^3 - 8(1) + 10 = 3$. Since $3 < \pi < 10$, by the Intermediate Value Theorem, there exists a c so that $0 < c < 1$ and $f(c) = \pi$.

   (b) $f(0) = 10$, $f(-4) = (-4)^3 - 8(-4) + 10 = -22$. Since $-22 < -\sqrt{3} < 10$, by the Intermediate Value Theorem, there exists a c so that $-4 < c < 0$ and $f(c) = -\sqrt{3}$.

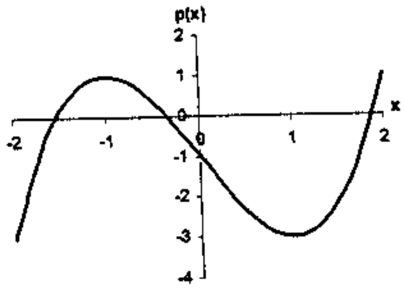
   (c) $f(0) = 10$, $f(1000) = (1000)^3 - 8(1000) + 10 = 999{,}992{,}010$. Since $10 < 5{,}000{,}000 < 999{,}992{,}010$, by the Intermediate Value Theorem, there exists a c so that $0 < c < 1000$ and $f(c) = 5{,}000{,}000$.

29. Answers may vary. For example, $f(x) = \frac{\sin(x-2)}{x-2}$ is discontinuous at $x = 2$ because it is not defined there. However, the discontinuity can be removed because f has a limit (namely 1) as $x \to 2$.

30. Answers may vary. For example, $g(x) = \frac{1}{x+1}$ has a discontinuity at $x = -1$ because $\lim\limits_{x \to -1} g(x)$ does not exist. ($\lim\limits_{x \uparrow -1} g(x) = -\infty$ and $\lim\limits_{x \downarrow -1} g(x) = +\infty$.)

31. Noting that $r = 0$ is triple zero, the polynomial can be rewritten as $x^3(x^2 - x - 5)$. Therefore, the roots of the quintic polynomial are $r_1 = \frac{1 - \sqrt{21}}{2} \approx -1.791$, $r_2 = r_3 = r_4 = 0$, and $r_5 = \frac{1 + \sqrt{21}}{2} \approx 2.791$.

32. The graph shows that the polynomial has three zeros between -2 and 2, any one a candidate for r. By zooming in, the choices for r are estimated at -1.532, -0.347, or 1.879.



33. (a) Suppose x_0 is rational $\Rightarrow f(x_0) = 1$. Choose $\epsilon = \frac{1}{2}$. For any $\delta > 0$ there is an irrational number x (actually infinitely many) in the interval $(x_0 - \delta, x_0 + \delta) \Rightarrow f(x) = 0$. Then $0 < |x - x_0| < \delta$ but $|f(x) - f(x_0)| = 1 > \frac{1}{2} = \epsilon$, so $\lim_{x \to x_0} f(x)$ fails to exist $\Rightarrow$ f is discontinuous at x_0 rational. On the other hand, x_0 irrational $\Rightarrow f(x_0) = 0$ and there is a rational number x in $(x_0 - \delta, x_0 + \delta) \Rightarrow f(x) = 1$. Again $\lim_{x \to x_0} f(x)$ fails to exist $\Rightarrow$ f is discontinuous at x_0 irrational. That is, f is discontinuous at every point.

(b) f is neither right-continuous nor left-continuous at any point x_0 because in every interval $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$ there exist both rational and irrational real numbers. Thus neither limits $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exist by the same arguments used in part (a).

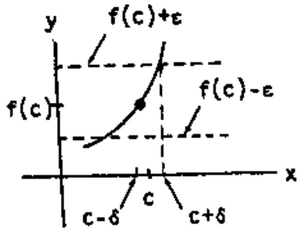
34. Yes. Both $f(x) = x$ and $g(x) = x - \frac{1}{2}$ are continuous on $[0, 1]$. However $\frac{f(x)}{g(x)}$ is undefined at $x = \frac{1}{2}$ since $g\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{f(x)}{g(x)}$ is discontinuous at $x = \frac{1}{2}$.

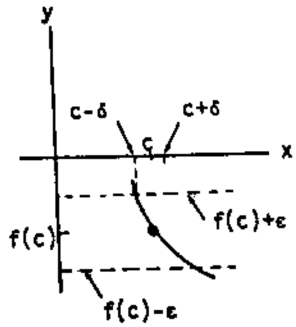
35. Yes, because of the Intermediate Value Theorem. If f(a) and f(b) did have different signs then f would have to equal zero at some point between a and b since f is continuous on [a, b].

36. Let f(x) be the new position of point x and let $d(x) = f(x) - x$. The displacement function d is negative if x is the left-hand point of the rubber band and positive if x is the right-hand point of the rubber band. By the Intermediate Value Theorem, $d(x) = 0$ for some point in between. That is, $f(x) = x$ for some point x, which is then in its original position.

37. If $f(0) = 0$ or $f(1) = 1$, we are done (i.e., $c = 0$ or $c = 1$ in those cases). Then let $f(0) = a > 0$ and $f(1) = b < 1$ because $0 \le f(x) \le 1$. Define $g(x) = f(x) - x \Rightarrow$ g is continuous on $[0, 1]$. Moreover, $g(0) = f(0) - 0 = a > 0$ and $g(1) = f(1) - 1 = b - 1 < 0 \Rightarrow$ by the Intermediate Value Theorem there is a number c in $(0, 1)$ such that $g(c) = 0 \Rightarrow f(c) - c = 0$ or $f(c) = c$.

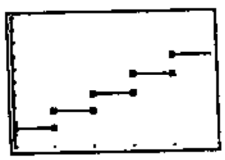
38. Let $\epsilon = \frac{|f(c)|}{2} > 0$. Since f is continuous at $x = c$ there is a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$ $\Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$.
If $f(c) > 0$, then $\epsilon = \frac{1}{2}f(c) \Rightarrow \frac{1}{2}f(c) < f(x) < \frac{3}{2}f(c) \Rightarrow f(x) > 0$ on the interval $(c - \delta, c + \delta)$.
If $f(c) < 0$, then $\epsilon = -\frac{1}{2}f(c) \Rightarrow \frac{3}{2}f(c) < f(x) < \frac{1}{2}f(c) \Rightarrow f(x) < 0$ on the interval $(c - \delta, c + \delta)$.





39. (a) Luisa's salary is $\$36{,}500 = \$36{,}500(1.035)^0$ for the first year ($0 \le t < 1$), $\$36{,}500(1.035)$ for the second year ($1 \le t < 2$), $\$36{,}500(1.035)^2$ for the third year ($2 \le t < 3$), and so on. This corresponds to $y = 36{,}500(1.035)^{\text{int } t}$

(b)



[0, 4.98] by [35,000, 45,000]

The function is continuous at all points in the domain $[0, 5)$ except at $t = 1, 2, 3, 4$.

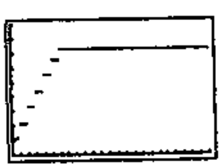
40. (a) We require:

$$f(x) = \begin{cases} 0, & x = 0 \\ 1.10, & 0 < x \le 1 \\ 2.20, & 1 < x \le 2 \\ 3.30, & 2 < x \le 3 \\ 4.40, & 3 < x \le 4 \\ 5.50, & 4 < x \le 5 \\ 6.60, & 5 < x \le 6 \\ 7.25, & 6 < x \le 24. \end{cases}$$

This may be written more compactly as

$$f(x) = \begin{cases} -1.10\text{int}(-x), & 0 \le x \le 6 \\ 7.25, & 6 < x \le 24 \end{cases}$$

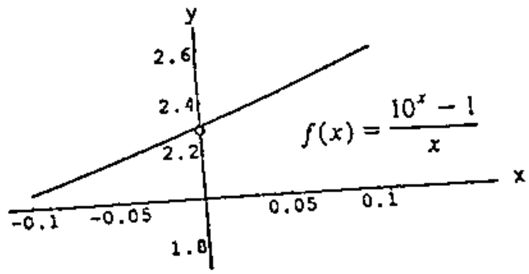
(b)



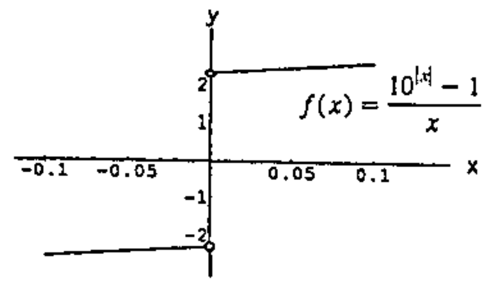
[0, 24] by [0, 9]

This is continuous for all values of x in the domain $[0, 24]$ except for $x = 0, 1, 2, 3, 4, 5, 6$.

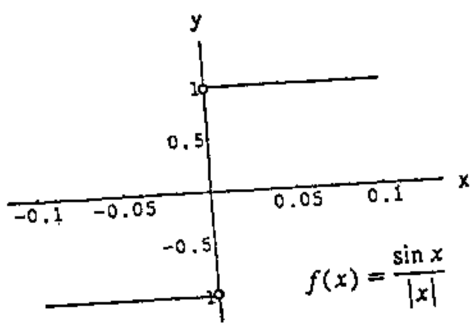
41. The function can be extended: $f(0) \approx 2.3$.



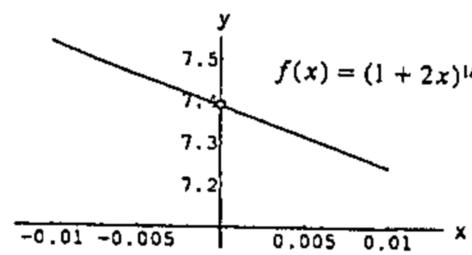
42. The function cannot be extended to be continuous at $x = 0$. If $f(0) \approx 2.3$, it will be continuous from the right. Or if $f(0) \approx -2.3$, it will be continuous from the left.



43. The function cannot be extended to be continuous at $x = 0$. If $f(0) = 1$, it will be continuous from the right. Or if $f(0) = -1$, it will be continuous from the left.



44. The function can be extended: $f(0) \approx 7.39$.



45. $x \approx 1.8794, -1.5321, -0.3473$

46. $x \approx 1.4516, -0.8546, 0.4030$

47. $x \approx 1.7549$

48. $x \approx 1.5596$

49. $x \approx 3.5156$

50. $x \approx -3.9059, 3.8392, 0.0667$

51. $x \approx 0.7391$

52. $x \approx -1.8955, 0, 1.8955$

1.5 TANGENT LINES

1. P_1: $m_1 = 1$, P_2: $m_2 = 5$

2. P_1: $m_1 = -2$, P_2: $m_2 = 0$

3. P_1: $m_1 = \frac{5}{2}$, P_2: $m_2 = -\frac{1}{2}$

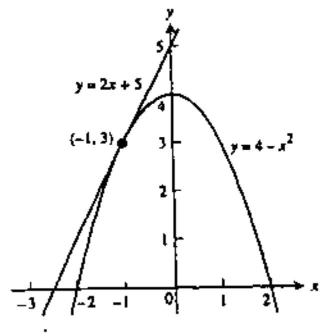
4. P_1: $m_1 = 3$, P_2: $m_2 = -3$

5. $m = \lim\limits_{h\to 0} \dfrac{[4-(-1+h)^2]-(4-(-1)^2)}{h}$

$= \lim\limits_{h\to 0} \dfrac{-(1-2h+h^2)+1}{h} = \lim\limits_{h\to 0} \dfrac{h(2-h)}{h} = 2;$

at $(-1,3)$: $y = 3+2(x-(-1)) \Rightarrow y = 2x+5$,

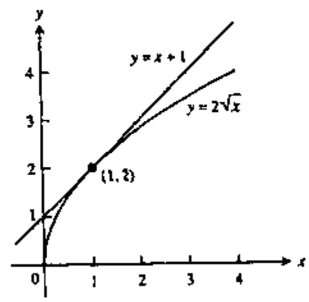
tangent line



6. $m = \lim\limits_{h\to 0} \dfrac{2\sqrt{1+h}-2\sqrt{1}}{h} = \lim\limits_{h\to 0} \dfrac{2\sqrt{1+h}-2}{h} \cdot \dfrac{2\sqrt{1+h}+2}{2\sqrt{1+h}+2}$

$= \lim\limits_{h\to 0} \dfrac{4(1+h)-4}{2h\left(\sqrt{1+h}+1\right)} = \lim\limits_{h\to 0} \dfrac{2}{\sqrt{1+h}+1} = 1;$

at $(1,2)$: $y = 2+1(x-1) \Rightarrow y = x+1$, tangent line

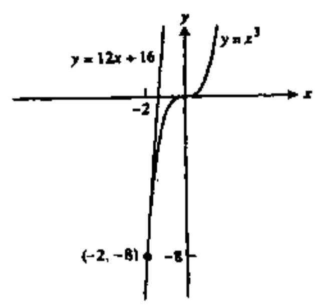


7. $m = \lim\limits_{h\to 0} \dfrac{(-2+h)^3-(-2)^3}{h} = \lim\limits_{h\to 0} \dfrac{-8+12h-6h^2+h^3+8}{h}$

$= \lim\limits_{h\to 0} \left(12-6h+h^2\right) = 12;$

at $(-2,-8)$: $y = -8+12(x-(-2)) \Rightarrow y = 12x+16$,

tangent line



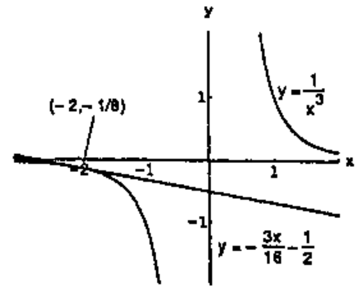
8. $m = \lim\limits_{h\to 0} \dfrac{\frac{1}{(-2+h)^3} - \frac{1}{(-2)^3}}{h} = \lim\limits_{h\to 0} \dfrac{-8-(-2+h)^3}{-8h(-2+h)^3}$

$= \lim\limits_{h\to 0} \dfrac{-(12h-6h^2+h^3)}{-8h(-2+h)^3} = \lim\limits_{h\to 0} \dfrac{12-6h+h^2}{8(-2+h)^3}$

$= \dfrac{12}{8(-8)} = -\dfrac{3}{16}$;

at $\left(-2,-\frac{1}{8}\right)$: $y = -\frac{1}{8} - \frac{3}{16}(x-(-2))$

$\Rightarrow y = -\frac{3}{16}x - \frac{1}{2}$, tangent line



9. $m = \lim\limits_{h\to 0} \dfrac{[(1+h)-2(1+h)^2]-(-1)}{h} = \lim\limits_{h\to 0} \dfrac{(1+h-2-4h-2h^2)+1}{h} = \lim\limits_{h\to 0} \dfrac{h(-3-2h)}{h} = -3$;

at $(1,-1)$: $y+1 = -3(x-1)$, tangent line

10. $m = \lim\limits_{h\to 0} \dfrac{[(1+h)^3+3(1+h)]-4}{h} = \lim\limits_{h\to 0} \dfrac{(1+3h+3h^2+h^3+3+3h)-4}{h} = \lim\limits_{h\to 0} \dfrac{h(6+3h+h^2)}{h} = 6$;

at $(1,4)$: $y-4 = 6(t-1)$, tangent line

11. $m = \lim\limits_{h\to 0} \dfrac{\frac{3+h}{(3+h)-2} - 3}{h} = \lim\limits_{h\to 0} \dfrac{(3+h)-3(h+1)}{h(h+1)} = \lim\limits_{h\to 0} \dfrac{-2h}{h(h+1)} = -2$;

at $(3,3)$: $y-3 = -2(u-3)$, tangent line

12. $m = \lim\limits_{h\to 0} \dfrac{\sqrt{(8+h)+1}-3}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{9+h}-3}{h} \cdot \dfrac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim\limits_{h\to 0} \dfrac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)} = \lim\limits_{h\to 0} \dfrac{h}{h\left(\sqrt{9+h}+3\right)}$

$= \dfrac{1}{\sqrt{9}+3} = \frac{1}{6}$; at $(8,3)$: $y-3 = \frac{1}{6}(x-8)$, tangent line

13. At $x=3$, $y=\frac{1}{2} \Rightarrow m = \lim\limits_{h\to 0} \dfrac{\frac{1}{(3+h)-1} - \frac{1}{2}}{h} = \lim\limits_{h\to 0} \dfrac{2-(2+h)}{2h(2+h)} = \lim\limits_{h\to 0} \dfrac{-h}{2h(2+h)} = -\frac{1}{4}$, slope

14. At $x=0$, $y=-1 \Rightarrow m = \lim\limits_{h\to 0} \dfrac{\frac{h-1}{h+1} - (-1)}{h} = \lim\limits_{h\to 0} \dfrac{(h-1)+(h+1)}{h(h+1)} = \lim\limits_{h\to 0} \dfrac{2h}{h(h+1)} = 2$, slope

15. At a horizontal tangent the slope $m = 0 \Rightarrow 0 = m = \lim\limits_{h\to 0} \dfrac{[(x+h)^2+4(x+h)-1]-(x^2+4x-1)}{h}$

$= \lim\limits_{h\to 0} \dfrac{(x^2+2xh+h^2+4x+4h-1)-(x^2+4x-1)}{h} = \lim\limits_{h\to 0} \dfrac{(2xh+h^2+4h)}{h} = \lim\limits_{h\to 0} (2x+h+4) = 2x+4$;

$2x+4=0 \Rightarrow x=-2$. Then $f(-2) = 4-8-1 = -5 \Rightarrow (-2,-5)$ is the point on the graph where there is a horizontal tangent.

16. $0 = m = \lim_{h\to 0} \frac{[(x+h)^3 - 3(x+h)] - (x^3 - 3x)}{h} = \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h) - (x^3 - 3x)}{h}$

$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} = \lim_{h\to 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3$; $3x^2 - 3 = 0 \Rightarrow x = -1$ or $x = 1$. Then $f(-1) = 2$ and $f(1) = -2 \Rightarrow (-1, 2)$ and $(1, -2)$ are the points on the graph where a horizontal tangent exists.

17. $-1 = m = \lim_{h\to 0} \frac{\frac{1}{(x+h)-1} - \frac{1}{x-1}}{h} = \lim_{h\to 0} \frac{(x-1)-(x+h-1)}{h(x-1)(x+h-1)} = \lim_{h\to 0} \frac{-h}{h(x-1)(x+h-1)} = -\frac{1}{(x-1)^2}$

$\Rightarrow (x-1)^2 = 1 \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$. If $x = 0$, then $y = -1$ and $m = -1$ $\Rightarrow y = -1 - (x - 0) = -(x+1)$. If $x = 2$, then $y = 1$ and $m = -1 \Rightarrow y = 1 - (x-2) = -(x-3)$.

18. $\frac{1}{4} = m = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$

$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$. Thus, $\frac{1}{4} = \frac{1}{2\sqrt{x}} \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4 \Rightarrow y = 2$. The tangent line is

$y = 2 + \frac{1}{4}(x - 4) = \frac{x}{4} + 1$.

19. $\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(100 - 4.9(2+h)^2) - (100 - 4.9(2)^2)}{h} = \lim_{h\to 0} \frac{-4.9(4 + 4h + h^2) + 4.9(4)}{h}$

$= \lim_{h\to 0} (-19.6 - 4.9h) = -19.6$. The minus sign indicates the object is falling downward at a speed of 19.6 m/sec.

20. $\lim_{h\to 0} \frac{f(10+h) - f(10)}{h} = \lim_{h\to 0} \frac{3(10+h)^2 - 3(10)^2}{h} = \lim_{h\to 0} \frac{3(20h + h^2)}{h} = 60$ ft/sec.

21. $\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\pi(3+h)^2 - \pi(3)^2}{h} = \lim_{h\to 0} \frac{\pi[9 + 6h + h^2 - 9]}{h} = \lim_{h\to 0} \pi(6+h) = 6\pi$

22. $\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{4\pi}{3}(2+h)^3 - \frac{4\pi}{3}(2)^3}{h} = \lim_{h\to 0} \frac{\frac{4\pi}{3}[12h + 6h^2 + h^3]}{h} = \lim_{h\to 0} \frac{4\pi}{3}[12 + 6h + h^2] = 16\pi$

23. $\lim_{h\to 0} \frac{s(1+h) - s(1)}{h} = \lim_{h\to 0} \frac{1.86(1+h)^2 - 1.86(1)^2}{h} = \lim_{h\to 0} \frac{1.86 + 3.72h + 1.86h^2 - 1.86}{h} = \lim_{h\to 0} (3.72 + 1.86h)$

$= 3.72$

24. $\lim_{h\to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h\to 0} \frac{11.44(2+h)^2 - 11.44(2)^2}{h} = \lim_{h\to 0} \frac{45.76 + 45.76h + 11.44h^2 - 45.76}{h}$

$= \lim_{h\to 0} (45.76 + 11.44h) = 45.76$

25. Slope at origin $= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0 \Rightarrow$ yes, $f(x)$ does have a tangent at the origin with slope 0.

26. $\lim\limits_{h \to 0} \frac{g(0+h)-g(0)}{h} = \lim\limits_{h \to 0} \frac{h \sin\left(\frac{1}{h}\right)}{h} = \lim\limits_{h \to 0} \sin \frac{1}{h}$. Since $\lim\limits_{h \to 0} \sin \frac{1}{h}$ does not exist, f(x) has no tangent at the origin.

27. $\lim\limits_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0^-} \frac{-1-0}{h} = \infty$, and $\lim\limits_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0^+} \frac{1-0}{h} = \infty$. Therefore, $\lim\limits_{h \to 0} \frac{f(0+h)-f(0)}{h} = \infty \Rightarrow$ yes, the graph of f has a vertical tangent at the origin.

28. $\lim\limits_{h \to 0^-} \frac{U(0+h)-U(0)}{h} = \lim\limits_{h \to 0^-} \frac{0-1}{h} = \infty$, and $\lim\limits_{h \to 0^+} \frac{U(0+h)-U(0)}{h} = \lim\limits_{h \to 0^+} \frac{1-1}{h} = 0 \Rightarrow$ no, the graph of f does not have a vertical tangent at (0, 1) because the limit does not exist.

29. (a) $\frac{\Delta f}{\Delta x} = \frac{f(0)-f(-2)}{0-(-2)} = \frac{1-e^{-2}}{2} \approx 0.432$ (b) $\frac{\Delta f}{\Delta x} = \frac{f(3)-f(1)}{3-1} = \frac{e^3-e}{2} \approx 8.684$

30. (a) $\frac{\Delta f}{\Delta x} = \frac{f(4)-f(1)}{4-1} = \frac{\ln 4 - 0}{3} = \frac{\ln 4}{3} \approx 0.462$

(b) $\frac{\Delta f}{\Delta x} = \frac{f(103)-f(100)}{103-100} = \frac{\ln 103 - \ln 100}{3} = \frac{1}{3} \ln \frac{103}{100} = \frac{1}{3} \ln 1.03 \approx 0.0099$

31. (a) $\frac{\Delta f}{\Delta t} = \frac{f(3\pi/4)-f(\pi/4)}{(3\pi/4)-(\pi/4)} = \frac{-1-1}{\pi/2} = -\frac{4}{\pi} \approx -1.273$

(b) $\frac{\Delta f}{\Delta t} = \frac{f(\pi/2)-f(\pi/6)}{(\pi/2)-(\pi/6)} = \frac{0-\sqrt{3}}{\pi/3} = -\frac{3\sqrt{3}}{\pi} \approx -1.654$

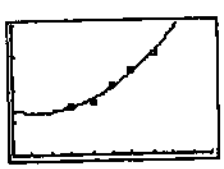
32. (a) $\frac{\Delta f}{\Delta t} = \frac{f(\pi)-f(0)}{\pi-0} = \frac{1-3}{\pi} = -\frac{2}{\pi} \approx -0.637$

(b) $\frac{\Delta f}{\Delta t} = \frac{f(\pi)-f(-\pi)}{\pi-(-\pi)} = \frac{1-1}{2\pi} = 0$

33. (a) $\frac{2.1-1.5}{1995-1993} = 0.3$
The rate of change was 0.3 billion dollars per year.

(b) $\frac{3.1-2.1}{1997-1995} = 0.5$
The rate of change was 0.5 billion dollars per year.

(c) $y = 0.0571x^2 - 0.1514x + 1.3943$



[0, 10] by [0, 4]

(d) $\frac{y(5)-y(3)}{5-3} \approx 0.31$

$\frac{y(7)-y(5)}{7-5} \approx 0.53$

According to the regression equation, the rates were 0.31 billion dollars per year and 0.53 billion dollars per year.

(e) $$\lim_{h\to 0} \frac{y(7+h)-y(7)}{h} = \lim_{h\to 0} \frac{[0.0571(7+h)^2 - 0.1514(7+h) + 1.3943] - [0.0571(7)^2 - 0.1514(7) + 1.3943]}{h}$$

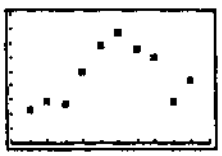
$$= \lim_{h\to 0} \frac{0.0571(14h+h^2) - 0.1514h}{h}$$

$$= \lim_{h\to 0} [0.0571(14) - 0.1514 + 0.0571h]$$

$$\approx 0.65$$

The funding was growing at a rate of about 0.65 billion dollars per year.

34. (a)



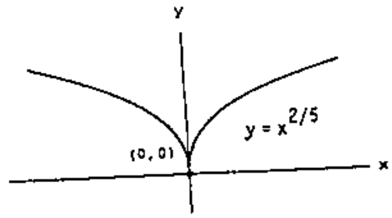
[7, 18] by [0, 900]

(b)

Q from year	Slope
1988	$\frac{440-225}{17-8} \approx 23.9$
1989	$\frac{440-289}{17-9} \approx 18.9$
1990	$\frac{440-270}{17-10} \approx 24.3$
1991	$\frac{440-493}{17-11} \approx -8.8$
1992	$\frac{440-684}{17-12} \approx -48.8$
1993	$\frac{440-763}{17-13} \approx -80.8$
1994	$\frac{440-651}{17-14} \approx -70.3$
1995	$\frac{440-600}{17-15} \approx -80.0$
1996	$\frac{440-296}{17-16} \approx 144.0$

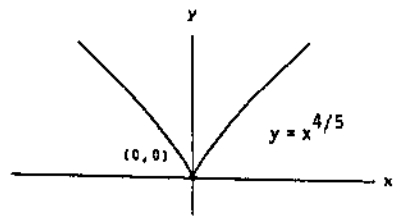
(c) As Q gets closer to 1997, the slopes do not seem to be approaching a limit value. The years 1995–97 seem to be very unusual and unpredictable.

35. (a) The graph appears to have a cusp at $x = 0$.



(b) $\lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^-} \frac{h^{2/5}-0}{h} = \lim\limits_{h\to 0^-} \frac{1}{h^{3/5}} = -\infty$ and $\lim\limits_{h\to 0^+} \frac{1}{h^{3/5}} = \infty \Rightarrow$ limit does not exist $\Rightarrow$ the graph of $y = x^{2/5}$ does not have a vertical tangent at $x = 0$.

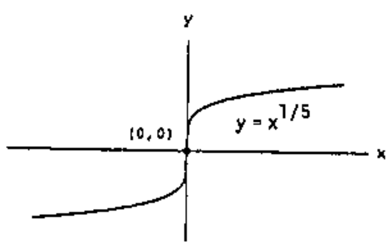
36. (a) The graph appears to have a cusp at $x = 0$.



(b) $\lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^-} \frac{h^{4/5}-0}{h} = \lim\limits_{h\to 0^-} \frac{1}{h^{1/5}} = -\infty$ and $\lim\limits_{h\to 0^+} \frac{1}{h^{1/5}} = \infty \Rightarrow$ limit does not exist

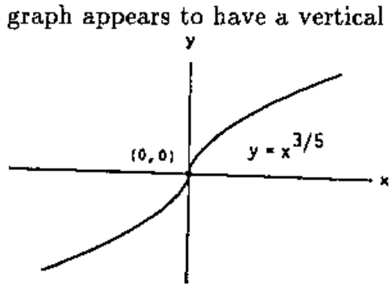
$\Rightarrow y = x^{4/5}$ does not have a vertical tangent at $x = 0$.

37. (a) The graph appears to have a vertical tangent at $x = 0$.



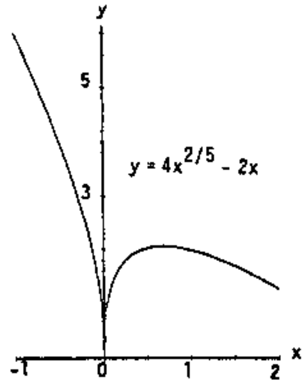
(b) $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{h^{1/5}-0}{h} = \lim\limits_{h\to 0} \frac{1}{h^{4/5}} = \infty \Rightarrow y = x^{1/5}$ has a vertical tangent at $x = 0$.

38. (a) The graph appears to have a vertical tangent at $x = 0$.



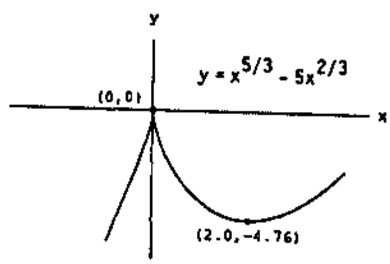
(b) $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{h^{3/5}-0}{h} = \lim\limits_{h\to 0} \frac{1}{h^{2/5}} = \infty \Rightarrow$ the graph of $y = x^{3/5}$ has a vertical tangent at $x = 0$.

39. (a) The graph appears to have a cusp at $x = 0$.



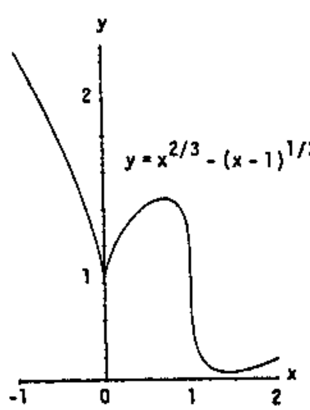
(b) $\lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^-} \frac{4h^{2/5}-2h}{h} = \lim\limits_{h\to 0^-} \frac{4h^{2/5}-2h}{h} = \lim\limits_{h\to 0^-} \frac{4}{h^{3/5}} - 2 = -\infty$ and $\lim\limits_{h\to 0^+} \frac{4}{h^{3/5}} - 2$ $= \infty \Rightarrow$ limit does not exist $\Rightarrow$ the graph of $y = 4x^{2/5} - 2x$ does not have a vertical tangent at $x = 0$.

40. (a) The graph appears to have a cusp at $x = 0$.



(b) $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{h^{5/3}-5h^{2/3}}{h} = \lim\limits_{h\to 0} h^{2/3} - \frac{5}{h^{1/3}} = 0 - \lim\limits_{h\to 0} \frac{5}{h^{1/3}}$ does not exist $\Rightarrow$ the graph of $y = x^{5/3} - 5x^{2/3}$ does not have a vertical tangent.

41. (a) The graph appears to have a vertical tangent at $x = 1$ and a cusp at $x = 0$.



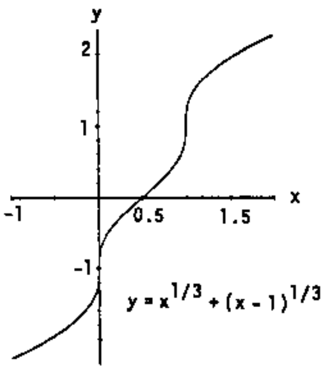
(b) $x = 1$: $\lim\limits_{h\to 0} \frac{(1+h)^{2/3} - (1+h-1)^{1/3} - 1}{h} = \lim\limits_{h\to 0} \frac{(1+h)^{2/3} - h^{1/3} - 1}{h} = -\infty$

$\Rightarrow y = x^{2/3} - (x-1)^{1/3}$ has a vertical tangent at $x = 1$;

$x = 0$: $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{h^{2/3}-(h-1)^{1/3}-(-1)^{1/3}}{h} = \lim\limits_{h\to 0}\left[\frac{1}{h^{1/3}} - \frac{(h-1)^{1/3}}{h} + \frac{1}{h}\right]$

does not exist $\Rightarrow y = x^{2/3} - (x-1)^{1/3}$ does not have a vertical tangent at $x = 0$.

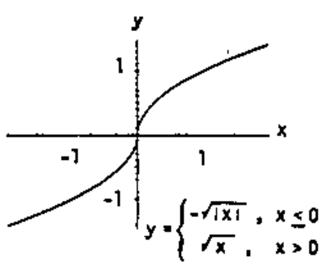
42. (a) The graph appears to have vertical tangents at $x = 0$ and $x = 1$.



(b) $x = 0$: $\lim\limits_{h\to 0} \frac{(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{h^{1/3}+(h-1)^{1/3}-(-1)^{1/3}}{h} = \infty \Rightarrow y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at $x = 0$;

$x = 1$: $\lim\limits_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0} \frac{(1+h)^{1/3}+(1+h-1)^{1/3}-1}{h} = \infty \Rightarrow y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at $x = 1$.

43. (a) The graph appears to have a vertical tangent at $x = 0$.

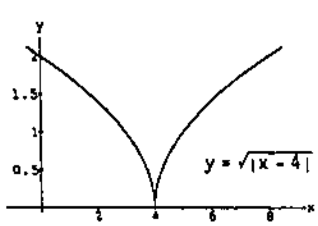


(b) $\lim\limits_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim\limits_{x\to 0^+} \frac{\sqrt{h}-0}{h} = \lim\limits_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty$;

$\lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^-} \frac{-\sqrt{|h|}-0}{h} = \lim\limits_{h\to 0^-} \frac{-\sqrt{|h|}}{-|h|} = \lim\limits_{h\to 0^-} \frac{1}{\sqrt{|h|}} = \infty$

$\Rightarrow$ y has a vertical tangent at $x = 0$.

44. (a) The graph appears to have a cusp at $x = 4$.



(b) $\lim\limits_{h\to0^+} \frac{f(4+h)-f(4)}{h} = \lim\limits_{h\to0^+} \frac{\sqrt{|4-(4+h)|}-0}{h} = \lim\limits_{h\to0^+} \frac{\sqrt{|h|}}{h} = \lim\limits_{h\to0^+} \frac{1}{\sqrt{h}} = \infty;$

$\lim\limits_{h\to0^-} \frac{f(4+h)-f(4)}{h} = \lim\limits_{h\to0^-} \frac{\sqrt{|4-(4+h)|}}{h} = \lim\limits_{h\to0^-} \frac{\sqrt{|h|}}{-\sqrt{h^2}} = -\infty$

$\Rightarrow y = \sqrt{4-x}$ does not have a vertical tangent at $x = 4$.

45-48. Example CAS commands:

Maple:

```
f:=x -> cos(x) + 4*sin(2*x);
x0:=Pi:
dq:=h -> (f(x0 + h) - f(x0))/h;
slope:=limit(dq(h),h=0);
L:=x -> f(x0) + slope*(x - x0);
y1:=f(x0) + dq(3)*(x - x0):
y2:=f(x0) + dq(2)*(x - x0):
y3:=f(x0) + dq(1)*(x - x0):
plot ({f(x),y1,y2,y3,L(x)},x = x0 - 1..x0 + 3);
```

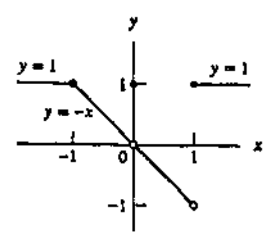
Mathematica:

```
Clear [f,m,x,y]
x0 = Pi; f[x_] := Cos[x] + 4 Sin[2x]
Plot[ f[x], {x,x0 - 1,x0 + 3) ]
dq[h_] := (f[x0 + h] - f[x0])/h
m = Limit[ dq[h], h -> 0 ]
y := f[x0] + m (x - x0)
y1 := f[x0] + dq[1] (x - x0)
y2 := f[x0] + dq[2] (x - x0)
y3 := f[x0] + dq[3] (x - x0)
Plot[ {f[x],y,y1,y2,y3}, {x,x0 - 1,x0 + 3} ]
```

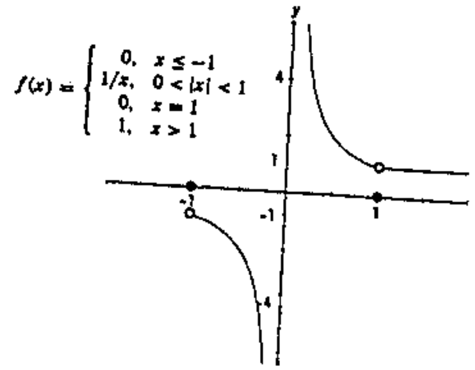
CHAPTER 1 PRACTICE EXERCISES

1. At $x = -1$: $\lim\limits_{x\to-1^-} f(x) = \lim\limits_{x\to-1^+} f(x) = 1 \Rightarrow \lim\limits_{x\to-1} f(x)$
$= 1 = f(-1) \Rightarrow$ f is continuous at $x = -1$.
At $x = 0$: $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^+} f(x) = 0 \Rightarrow \lim\limits_{x\to0} f(x) = 0$.
But $f(0) = 1 \neq \lim\limits_{x\to0} f(x) \Rightarrow$ f is discontinuous at $x = 0$.
At $x = 1$: $\lim\limits_{x\to1^-} f(x) = -1$ and $\lim\limits_{x\to1^+} f(x) = 1 \Rightarrow \lim\limits_{x\to1} f(x)$
does not exist $\Rightarrow$ f is discontinuous at $x = 1$.

$$f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1. \end{cases}$$



2. At $x = -1$: $\lim\limits_{x\to-1^-} f(x) = 0$ and $\lim\limits_{x\to-1^+} f(x) = -1 \Rightarrow \lim\limits_{x\to-1} f(x)$
does not exist $\Rightarrow$ f is discontinuous at $x = -1$.



At $x = 0$: $\lim\limits_{x\to 0^-} f(x) = -\infty$ and $\lim\limits_{x\to 0^+} f(x) = \infty \Rightarrow \lim\limits_{x\to 0} f(x)$ does not exist $\Rightarrow$ f is discontinuous at $x = 0$.

At $x = 1$: $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = 1 \Rightarrow \lim\limits_{x\to 1} f(x) = 1$. But $f(1) = 0 \neq \lim\limits_{x\to 1} f(x) \Rightarrow$ f is discontinuous at $x = 1$.

3. (a) $\lim\limits_{t\to t_0} (3f(t)) = 3\lim\limits_{t\to t_0} f(t) = 3(-7) = -21$

(b) $\lim\limits_{t\to t_0} (f(t))^2 = \left(\lim\limits_{t\to t_0} f(t)\right)^2 = (-7)^2 = 49$

(c) $\lim\limits_{t\to t_0} (f(t)\cdot g(t)) = \lim\limits_{t\to t_0} f(t)\cdot \lim\limits_{t\to t_0} g(t) = (-7)(0) = 0$

(d) $\lim\limits_{t\to t_0} \dfrac{f(t)}{g(t)-7} = \dfrac{\lim\limits_{t\to t_0} f(t)}{\lim\limits_{t\to t_0} (g(t)-7)} = \dfrac{-7}{\lim\limits_{t\to t_0} g(t) - \lim\limits_{t\to t_0} 7} = \dfrac{-7}{0-7} = 1$

(e) $\lim\limits_{t\to t_0} \cos(g(t)) = \cos\left(\lim\limits_{t\to t_0} g(t)\right) = \cos 0 = 1$

(f) $\lim\limits_{t\to t_0} |f(t)| = \left|\lim\limits_{t\to t_0} f(t)\right| = |-7| = 7$

(g) $\lim\limits_{t\to t_0} (f(t)+g(t)) = \lim\limits_{t\to t_0} f(t) + \lim\limits_{t\to t_0} g(t) = -7 + 0 = -7$

(h) $\lim\limits_{t\to t_0} (1/f(t)) = \dfrac{1}{\lim\limits_{t\to t_0} f(t)} = \dfrac{1}{-7} = -\frac{1}{7}$

4. (a) $\lim\limits_{x\to 0} -g(x) = -\lim\limits_{x\to 0} g(x) = -\sqrt{2}$

(b) $\lim\limits_{x\to 0} (g(x)\cdot f(x)) = \lim\limits_{x\to 0} g(x)\cdot \lim\limits_{x\to 0} f(x) = (\sqrt{2})\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{2}$

(c) $\lim\limits_{x\to 0} (f(x)+g(x)) = \lim\limits_{x\to 0} f(x) + \lim\limits_{x\to 0} g(x) = \frac{1}{2} + \sqrt{2}$

(d) $\lim\limits_{x\to 0} \dfrac{1}{f(x)} = \dfrac{1}{\lim\limits_{x\to 0} f(x)} = \dfrac{1}{\frac{1}{2}} = 2$

(e) $\lim\limits_{x\to 0} (x + f(x)) = \lim\limits_{x\to 0} x + \lim\limits_{x\to 0} f(x) = 0 + \frac{1}{2} = \frac{1}{2}$

(f) $\lim\limits_{x\to 0} \dfrac{f(x)\cdot\cos x}{x-1} = \dfrac{\lim\limits_{x\to 0} f(x)\cdot \lim\limits_{x\to 0}\cos x}{\lim\limits_{x\to 0} x - \lim\limits_{x\to 0} 1} = \dfrac{\left(\frac{1}{2}\right)(1)}{0-1} = -\frac{1}{2}$

5. Since $\lim\limits_{x\to 0} x = 0$ we must have that $\lim\limits_{x\to 0} (4-g(x)) = 0$. Otherwise, if $\lim\limits_{x\to 0} (4-g(x))$ is a finite positive number, we would have $\lim\limits_{x\to 0^-} \left[\frac{4-g(x)}{x}\right] = -\infty$ and $\lim\limits_{x\to 0^+} \left[\frac{4-g(x)}{x}\right] = \infty$ so the limit could not equal 1 as

$x \to 0$. Similar reasoning holds if $\lim\limits_{x\to 0} (4 - g(x))$ is a finite negative number. We conclude that $\lim\limits_{x\to 0} g(x) = 4$.

6. $2 = \lim\limits_{x\to -4} \left[x \lim\limits_{x\to 0} g(x)\right] = \lim\limits_{x\to -4} x \cdot \lim\limits_{x\to -4} \left[\lim\limits_{x\to 0} g(x)\right] = -4 \lim\limits_{x\to -4} \left[\lim\limits_{x\to 0} g(x)\right] = -4 \lim\limits_{x\to 0} g(x)$

(since $\lim\limits_{x\to 0} g(x)$ is a constant) $\Rightarrow \lim\limits_{x\to 0} g(x) = \frac{2}{-4} = -\frac{1}{2}$.

7. (a) $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} x^{1/3} = c^{1/3} = f(c)$ for every real number $c \Rightarrow f$ is continuous on $(-\infty, \infty)$

(b) $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} x^{3/4} = c^{3/4} = g(c)$ for every nonnegative real number $c \Rightarrow g$ is continuous on $[0, \infty)$

(c) $\lim\limits_{x\to c} h(x) = \lim\limits_{x\to c} x^{-2/3} = \frac{1}{c^{2/3}} = h(c)$ for every nonzero real number $c \Rightarrow h$ is continuous on $(-\infty, 0)$ and $(0, \infty)$

(d) $\lim\limits_{x\to c} k(x) = \lim\limits_{x\to c} x^{-1/6} = \frac{1}{c^{1/6}} = k(c)$ for every positive real number $c \Rightarrow h$ is continuous on $(0, \infty)$

8. (a) $\bigcup\limits_{n\in I} \left(\left(n - \frac{1}{2}\right)\pi, \left(n + \frac{1}{2}\right)\pi\right)$, where I = the set of all integers.

(b) $\bigcup\limits_{n\in I} (n\pi, (n+1)\pi)$, where I = the set of all integers.

(c) $(-\infty, \infty)$

(d) $(-\infty, 0) \cup (0, \infty)$

9. (a) $\lim\limits_{x\to 0} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x} = \lim\limits_{x\to 0} \frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim\limits_{x\to 0} \frac{x-2}{x(x+7)}$, $x \neq 2$; the limit does not exist because

$\lim\limits_{x\to 0^-} \frac{x-2}{x(x+7)} = \infty$ and $\lim\limits_{x\to 0^+} \frac{x-2}{x(x+7)} = -\infty$

(b) $\lim\limits_{x\to 2} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x} = \lim\limits_{x\to 2} \frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim\limits_{x\to 2} \frac{x-2}{x(x+7)}$, $x \neq 2 = \frac{0}{2(9)} = 0$

10. (a) $\lim\limits_{x\to 0} \frac{x^2 + x}{x^5 + 2x^4 + x^3} = \lim\limits_{x\to 0} \frac{x(x+1)}{x^3(x^2 + 2x + 1)} = \lim\limits_{x\to 0} \frac{x+1}{x^2(x+1)(x+1)} = \lim\limits_{x\to 0} \frac{1}{x^2(x+1)}$, $x \neq 0$ and $x \neq -1$.

Now $\lim\limits_{x\to 0^-} \frac{1}{x^2(x+1)} = \infty$ and $\lim\limits_{x\to 0^+} \frac{1}{x^2(x+1)} = \infty \Rightarrow \lim\limits_{x\to 0} \frac{x^2 + x}{x^5 + 2x^4 + x^3} = \infty$.

(b) $\lim\limits_{x\to -1} \frac{x^2 + x}{x^5 + 2x^4 + x^3} = \lim\limits_{x\to -1} \frac{x(x+1)}{x^3(x^2 + 2x + 1)} = \lim\limits_{x\to -1} \frac{1}{x^2(x+1)}$, $x \neq 0$ and $x \neq -1$. The limit does not exist because $\lim\limits_{x\to -1^-} \frac{1}{x^2(x+1)} = -\infty$ and $\lim\limits_{x\to -1^+} \frac{1}{x^2(x+1)} = \infty$.

11. $\lim\limits_{x\to 1} \frac{1 - \sqrt{x}}{1 - x} = \lim\limits_{x\to 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})} = \lim\limits_{x\to 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{2}$

12. $\lim\limits_{x\to a} \frac{x^2 - a^2}{x^4 - a^4} = \lim\limits_{x\to a} \frac{(x^2 - a^2)}{(x^2 + a^2)(x^2 - a^2)} = \lim\limits_{x\to a} \frac{1}{x^2 + a^2} = \frac{1}{2a^2}$

13. $\lim\limits_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim\limits_{h\to 0} \frac{(x^2+2hx+h^2)-x^2}{h} = \lim\limits_{h\to 0} (2x+h) = 2x$

14. $\lim\limits_{x\to 0} \frac{(x+h)^2 - x^2}{h} = \lim\limits_{x\to 0} \frac{(x^2+2hx+h^2)-x^2}{h} = \lim\limits_{x\to 0} (2x+h) = h$

15. $\lim\limits_{x\to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim\limits_{x\to 0} \frac{2-(2+x)}{2x(2+x)} = \lim\limits_{x\to 0} \frac{-1}{4+2x} = -\frac{1}{4}$

16. $\lim\limits_{x\to 0} \frac{(2+x)^3 - 8}{x} = \lim\limits_{x\to 0} \frac{(x^3+6x^2+12x+8)-8}{x} = \lim\limits_{x\to 0} (x^2+6x+12) = 12$

17. $\lim\limits_{x\to\infty} \frac{2x+3}{5x+7} = \lim\limits_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2+0}{5+0} = \frac{2}{5}$

18. $\lim\limits_{x\to-\infty} \frac{2x^2+3}{5x^2+7} = \lim\limits_{x\to-\infty} \frac{2+\frac{3}{x^2}}{5+\frac{7}{x^2}} = \frac{2+0}{5+0} = \frac{2}{5}$

19. $\lim\limits_{x\to-\infty} \frac{x^2-4x+8}{3x^3} = \lim\limits_{x\to-\infty} \left(\frac{1}{3x} - \frac{4}{3x^2} + \frac{8}{3x^3}\right) = 0 - 0 + 0 = 0$

20. $\lim\limits_{x\to\infty} \frac{1}{x^2-7x+1} = \lim\limits_{x\to\infty} \left(\frac{\frac{1}{x^2}}{1-\frac{7}{x}+\frac{1}{x^2}}\right) = \frac{0}{1-0+0} = 0$

21. $\lim\limits_{x\to-\infty} \frac{x^2-7x}{x+1} = \lim\limits_{x\to-\infty} \left(\frac{x-7}{1+\frac{1}{x}}\right) = -\infty$

22. $\lim\limits_{x\to\infty} \frac{x^4+x^3}{12x^3+128} = \lim\limits_{x\to\infty} \left(\frac{x+1}{12+\frac{128}{x^3}}\right) = \infty$

23. $\lim\limits_{x\to\infty} \frac{|\sin x|}{\text{int } x} \le \lim\limits_{x\to\infty} \frac{1}{\text{int } x} = 0$ since $\text{int } x \to \infty$ as $x \to \infty$

24. $\lim\limits_{\theta\to\infty} \frac{|\cos\theta - 1|}{\theta} \le \lim\limits_{\theta\to\infty} \frac{|-2|}{\theta} = 0$

25. $\lim\limits_{x\to\infty} \frac{x+\sin x+2\sqrt{x}}{x+\sin x} = \lim\limits_{x\to\infty} \left(\frac{1+\frac{\sin x}{x}+\frac{2}{\sqrt{x}}}{1+\frac{\sin x}{x}}\right) = \frac{1+0+0}{1+0} = 1$

26. $\lim\limits_{x\to\infty} \frac{x^{2/3}+x^{-1}}{x^{2/3}+\cos^2 x} = \lim\limits_{x\to\infty} \left(\frac{1+x^{-5/3}}{1+\frac{\cos^2 x}{x^{2/3}}}\right) = \frac{1+0}{1+0} = 1$

27. $\lim\limits_{x\to\infty} e^{-x^2} = \lim\limits_{x\to\infty} \frac{1}{e^{x^2}} = 0$

28. Letting $u = \frac{1}{x}$ gives $\lim\limits_{x\to-\infty} e^{1/x} = \lim\limits_{u\to 0^-} e^u = 1$.

29. (a) $f(-1) = -1$ and $f(2) = 5 \Rightarrow$ f has a root between -1 and 2 by the Intermediate Value Theorem.
(b), (c) root is 1.32471795724

30. (a) $f(-2) = -2$ and $f(0) = 2 \Rightarrow f$ has a root between -2 and 0 by the Intermediate Value Theorem.
(b), (c) root is -1.76929235424

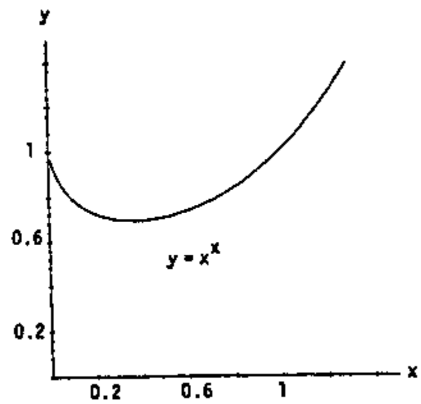
CHAPTER 1 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. (a)

x	0.1	0.01	0.001	0.0001	0.00001
x^x	0.7943	0.9550	0.9931	0.9991	0.9999

Apparently, $\lim\limits_{x\to 0^+} x^x = 1$

(b)

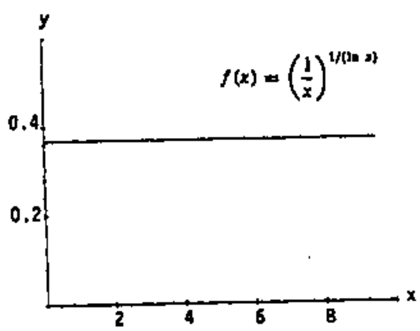


2. (a)

x	10	100	1000
$\left(\frac{1}{x}\right)^{1/(\ln x)}$	0.3678	0.3678	0.3678

Apparently, $\lim\limits_{x\to\infty} \left(\frac{1}{x}\right)^{1/(\ln x)} = 0.3678 = \frac{1}{e}$

(b)



3. $\lim\limits_{v\to c^-} L = \lim\limits_{v\to c^-} L_0\sqrt{1-\frac{v^2}{c^2}} = L_0\sqrt{1-\frac{\lim\limits_{v\to c^-} v^2}{c^2}} = L_0\sqrt{1-\frac{c^2}{c^2}} = 0$

The left-hand limit was needed because the function L is undefined if $v > c$ (the rocket cannot move faster than the speed of light).

4. $\left|\frac{\sqrt{x}}{2}-1\right|<0.2 \Rightarrow -0.2<\frac{\sqrt{x}}{2}-1<0.2 \Rightarrow 0.8<\frac{\sqrt{x}}{2}<1.2 \Rightarrow 1.6<\sqrt{x}<2.4 \Rightarrow 2.56<x<5.76.$

$\left|\frac{\sqrt{x}}{2}-1\right|<0.1 \Rightarrow -0.1<\frac{\sqrt{x}}{2}-1<0.1 \Rightarrow 0.9<\frac{\sqrt{x}}{2}<1.1 \Rightarrow 1.8<\sqrt{x}<2.2 \Rightarrow 3.24<x<4.84.$

5. $\left|10+(t-70)\times 10^{-4}-10\right|<0.0005 \Rightarrow \left|(t-70)\times 10^{-4}\right|<0.0005 \Rightarrow -0.0005<(t-70)\times 10^{-4}<0.0005$
$\Rightarrow -5<t-70<5 \Rightarrow 65^\circ<t<75^\circ \Rightarrow$ Within 5° F.

6. Yes. Let R be the radius of the equator (earth) and suppose at a fixed instant of time we label noon as the zero point, 0, on the equator $\Rightarrow 0+\pi R$ represents the midnight point (at the same exact time). Suppose x_1 is a point on the equator "just after" noon $\Rightarrow x_1+\pi R$ is simultaneously "just after" midnight. It seems reasonable that the temperature T at a point just after noon is hotter than it would be at the diametrically opposite point just after midnight: That is, $T(x_1)-T(x_1+\pi R)>0$. At exactly the same moment in time pick x_2 to be a point just before midnight $\Rightarrow x_2+\pi R$ is just before noon. Then $T(x_2)-T(x_2+\pi R)<0$. Assuming the temperature function T is continuous along the equator (which is reasonable), the Intermediate Value Theorem says there is a point c between 0 (noon) and πR (simultaneously midnight) such that $T(c)-T(c+\pi R)=0$; i.e., there is always a pair of antipodal points on the earth's equator where the temperatures are the same.

7. (a) At $x=0$: $\lim\limits_{a\to 0} r_+(a)=\lim\limits_{a\to 0}\frac{-1+\sqrt{1+a}}{a}=\lim\limits_{a\to 0}\left(\frac{-1+\sqrt{1+a}}{a}\right)\left(\frac{-1-\sqrt{1+a}}{-1-\sqrt{1+a}}\right)$

$=\lim\limits_{a\to 0}\frac{1-(1+a)}{a\left(-1-\sqrt{1+a}\right)}=\frac{-1}{-1-\sqrt{1+0}}=\frac{1}{2}$

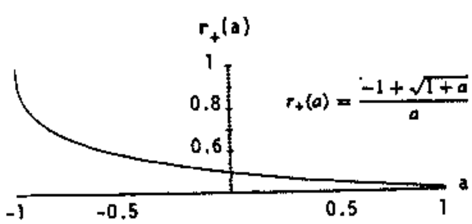
At $x=-1$: $\lim\limits_{a\to -1^+} r_+(a)=\lim\limits_{a\to -1^+}\frac{1-(1+a)}{a\left(-1-\sqrt{1+a}\right)}=\lim\limits_{a\to -1}\frac{-a}{a\left(-1-\sqrt{1+a}\right)}=\frac{-1}{-1-\sqrt{0}}=1$

(b) At $x=0$: $\lim\limits_{a\to 0^-} r_-(a)=\lim\limits_{a\to 0^-}\frac{-1-\sqrt{1+a}}{a}=\lim\limits_{a\to 0^-}\left(\frac{-1-\sqrt{1+a}}{a}\right)\left(\frac{-1+\sqrt{1+a}}{-1+\sqrt{1+a}}\right)$

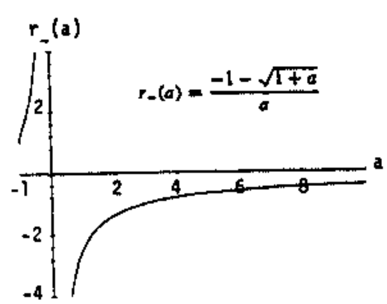
$=\lim\limits_{a\to 0^-}\frac{1-(1+a)}{a\left(-1+\sqrt{1+a}\right)}=\lim\limits_{a\to 0^-}\frac{a}{a\left(-1+\sqrt{1+a}\right)}=\lim\limits_{a\to 0^-}\frac{-1}{-1+\sqrt{1+a}}=\infty$ (because the denominator is always negative); $\lim\limits_{a\to 0^+} r_-(a)=\lim\limits_{a\to 0^+}\frac{-1}{-1+\sqrt{1+a}}=-\infty$ (because the denominator is always positive). Therefore, $\lim\limits_{a\to 0} r_-(a)$ does not exist.

At $x=-1$: $\lim\limits_{a\to -1^+} r_-(a)=\lim\limits_{a\to -1^+}\frac{-1-\sqrt{1+a}}{a}=\lim\limits_{a\to -1^+}\frac{-1}{-1+\sqrt{1+a}}=1$

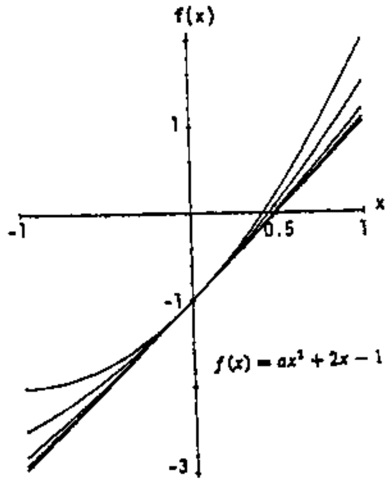
(c)

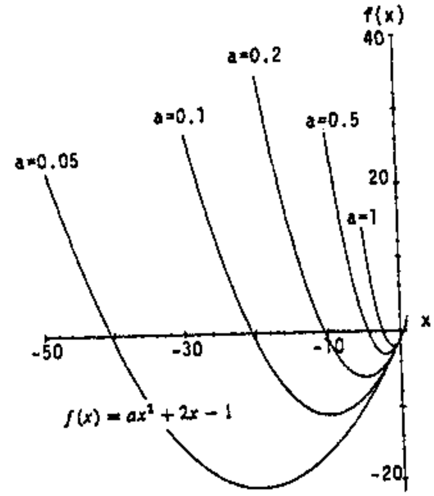


Graph not to scale



(d)





8. (a) Since $x \to 0^+$, $0 < x^3 < x < 1 \Rightarrow (x^3 - x) \to 0^- \Rightarrow \lim\limits_{x\to 0^+} f(x^3 - x) = \lim\limits_{y \to 0^-} f(y) = B$ where $y = x^3 - x$.

   (b) Since $x \to 0^-$, $-1 < x < x^3 < 0 \Rightarrow (x^3 - x) \to 0^+ \Rightarrow \lim\limits_{x\to 0^-} f(x^3 - x) = \lim\limits_{y \to 0^+} f(y) = A$ where $y = x^3 - x$.

   (c) Since $x \to 0^+$, $0 < x^4 < x^2 < 1 \Rightarrow (x^2 - x^4) \to 0^+ \Rightarrow \lim\limits_{x\to 0^+} f(x^2 - x^4) = \lim\limits_{y \to 0^+} f(y) = A$ where $y = x^2 - x^4$.

   (d) Since $x \to 0^-$, $-1 < x < 0 \Rightarrow 0 < x^4 < x^2 < 1 \Rightarrow (x^2 - x^4) \to 0^+ \Rightarrow \lim\limits_{x\to 0^+} f(x^2 - x^4) = A$ as in part (c).

9. (a) True, because if $\lim\limits_{x\to a} (f(x) + g(x))$ exists then $\lim\limits_{x\to a} (f(x) + g(x)) - \lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} [(f(x) + g(x)) - f(x)]$

   $= \lim\limits_{x\to a} g(x)$ exists, contrary to assumption.

   (b) False; for example take $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$. Then neither $\lim\limits_{x\to 0} f(x)$ nor $\lim\limits_{x\to 0} g(x)$ exists, but $\lim\limits_{x\to 0} (f(x) + g(x)) = \lim\limits_{x\to 0} \left(\frac{1}{x} - \frac{1}{x}\right) = \lim\limits_{x\to 0} 0 = 0$ exists.

   (c) True, because $g(x) = |x|$ is continuous $\Rightarrow g(f(x)) = |f(x)|$ is continuous (it is the composite of continuous functions).

   (d) False; for example let $f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases} \Rightarrow f(x)$ is discontinuous at $x = 0$. However $|f(x)| = 1$ is continuous at $x = 0$.

10. $f(x) = x + 2\cos x \Rightarrow f(0) = 0 + 2\cos 0 = 2 > 0$ and $f(-\pi) = -\pi + 2\cos(-\pi) = -\pi - 2 < 0$. Since $f(x)$ is continuous on $[-\pi, 0]$, by the Intermediate Value Theorem, $f(x)$ must take on every value between $[-\pi - 2, 2]$. Thus there is some number c in $[-\pi, 0]$ such that $f(c) = 0$; i.e., c is a solution to $x + 2\cos x = 0$.

11. Show $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} (x^2 - 7) = -6 = f(1)$.

    Step 1: $|(x^2 - 7) + 6| < \epsilon \Rightarrow -\epsilon < x^2 - 1 < \epsilon \Rightarrow 1 - \epsilon < x^2 < 1 + \epsilon \Rightarrow \sqrt{1-\epsilon} < x < \sqrt{1+\epsilon}$.

    Step 2: $|x - 1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow -\delta + 1 < x < \delta + 1$.

    Then $-\delta + 1 = \sqrt{1-\epsilon}$ or $\delta + 1 = \sqrt{1+\epsilon}$. Choose $\delta = \min\left\{1 - \sqrt{1-\epsilon}, \sqrt{1+\epsilon} - 1\right\}$, then

$0 < |x-1| < \delta \Rightarrow \left|(x^2-7)-6\right| < \epsilon$ and $\lim\limits_{x\to 1} f(x) = -6$. By the continuity test, f(x) is continuous at $x = 1$.

12. Show $\lim\limits_{x\to \frac{1}{4}} g(x) = \lim\limits_{x\to \frac{1}{4}} \frac{1}{2x} = 2 = g\left(\frac{1}{4}\right)$.

Step 1: $\left|\frac{1}{2x} - 2\right| < \epsilon \Rightarrow -\epsilon < \frac{1}{2x} - 2 < \epsilon \Rightarrow 2 - \epsilon < \frac{1}{2x} < 2 + \epsilon \Rightarrow \frac{1}{4-2\epsilon} > x > \frac{1}{4+2\epsilon}$.

Step 2: $\left|x - \frac{1}{4}\right| < \delta \Rightarrow -\delta < x - \frac{1}{4} < \delta \Rightarrow -\delta + \frac{1}{4} < x < \delta + \frac{1}{4}$.

Then $-\delta + \frac{1}{4} = \frac{1}{4+2\epsilon} \Rightarrow \delta = \frac{1}{4} - \frac{1}{4+2\epsilon} = \frac{\epsilon}{4(2+\epsilon)}$, or $\delta + \frac{1}{4} = \frac{1}{4-2\epsilon} \Rightarrow \delta = \frac{1}{4-2\epsilon} - \frac{1}{4} = \frac{\epsilon}{4(2-\epsilon)}$.

Choose $\delta = \frac{\epsilon}{4(2+\epsilon)}$, the smaller of the two values. Then $0 < \left|x - \frac{1}{4}\right| < \delta \Rightarrow \left|\frac{1}{2x} - 2\right| < \epsilon$ and $\lim\limits_{x\to \frac{1}{4}} \frac{1}{2x} = 2$.

By the continuity test, g(x) is continuous at $x = \frac{1}{4}$.

13. Show $\lim\limits_{x\to 2} h(x) = \lim\limits_{x\to 2} \sqrt{2x-3} = 1 = h(2)$.

Step 1: $\left|\sqrt{2x-3} - 1\right| < \epsilon \Rightarrow -\epsilon < \sqrt{2x-3} - 1 < \epsilon \Rightarrow 1 - \epsilon < \sqrt{2x-3} < 1 + \epsilon \Rightarrow \frac{(1-\epsilon)^2+3}{2} < x < \frac{(1+\epsilon)^2+3}{2}$.

Step 2: $|x-2| < \delta \Rightarrow -\delta < x - 2 < \delta$ or $-\delta + 2 < x < \delta + 2$.

Then $-\delta + 2 = \frac{(1-\epsilon)^2+3}{2} \Rightarrow \delta = 2 - \frac{(1-\epsilon)^2+3}{2} = \frac{1-(1-\epsilon)^2}{2} = \epsilon - \frac{\epsilon^2}{2}$, or $\delta + 2 = \frac{(1+\epsilon)^2+3}{2}$

$\Rightarrow \delta = \frac{(1+\epsilon)^2+3}{2} - 2 = \frac{(1+\epsilon)^2-1}{2} = \epsilon + \frac{\epsilon^2}{2}$. Choose $\delta = \epsilon - \frac{\epsilon^2}{2}$, the smaller of the two values. Then,

$0 < |x-2| < \delta \Rightarrow \left|\sqrt{2x-3} - 1\right| < \epsilon$, so $\lim\limits_{x\to 2} \sqrt{2x-3} = 1$. By the continuity test, h(x) is continuous at $x = 2$.

14. Show $\lim\limits_{x\to 5} F(x) = \lim\limits_{x\to 5} \sqrt{9-x} = 2 = F(5)$.

Step 1: $\left|\sqrt{9-x} - 2\right| < \epsilon \Rightarrow -\epsilon < \sqrt{9-x} - 2 < \epsilon \Rightarrow 9 - (2-\epsilon)^2 > x > 9 - (2+\epsilon)^2$.

Step 2: $0 < |x-5| < \delta \Rightarrow -\delta < x - 5 < \delta \Rightarrow -\delta + 5 < x < \delta + 5$.

Then $-\delta + 5 = 9 - (2+\epsilon)^2 \Rightarrow \delta = (2+\epsilon)^2 - 4 = \epsilon^2 + 2\epsilon$, or $\delta + 5 = 9 - (2-\epsilon)^2 \Rightarrow \delta = 4 - (2-\epsilon)^2 = \epsilon^2 - 2\epsilon$.

Choose $\delta = \epsilon^2 - 2\epsilon$, the smaller of the two values. Then, $0 < |x-5| < \delta \Rightarrow \left|\sqrt{9-x} - 2\right| < \epsilon$, so $\lim\limits_{x\to 5} \sqrt{9-x} = 2$. By the continuity test, F(x) is continuous at $x = 5$.

15. (a) Let $\epsilon > 0$ be given. If x is rational, then $f(x) = x \Rightarrow \left|f(x) - 0\right| = |x - 0| < \epsilon \Leftrightarrow |x - 0| < \epsilon$; i.e., choose $\delta = \epsilon$. Then $|x-0| < \delta \Rightarrow \left|f(x) - 0\right| < \epsilon$ for x rational. If x is irrational, then $f(x) = 0 \Rightarrow \left|f(x) - 0\right| < \epsilon \Leftrightarrow 0 < \epsilon$ which is true no matter how close irrational x is to 0, so again we can choose $\delta = \epsilon$. In either case, given $\epsilon > 0$ there is a $\delta = \epsilon > 0$ such that $0 < |x - 0| < \delta \Rightarrow \left|f(x) - 0\right| < \epsilon$. Therefore, f is continuous at $x = 0$.

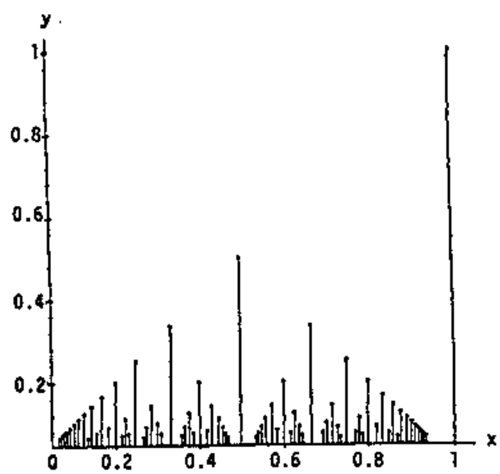
(b) Choose $x = c > 0$. Then within any interval $(c - \delta, c + \delta)$ there are both rational and irrational numbers. If c is rational, pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is an irrational number x in $(c - \delta, c + \delta) \Rightarrow \left|f(x) - f(c)\right| = |0 - c| = c > \frac{c}{2} = \epsilon$. That is, f is not continuous at any rational $c > 0$. On the other hand, suppose c is irrational $\Rightarrow f(c) = 0$. Again pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is a rational number x in $(c-\delta, c+\delta)$ with $|x-c| < \frac{c}{2} = \epsilon \Leftrightarrow \frac{c}{2} < x < \frac{3c}{2}$. Then $|f(x) - f(c)| = |x - 0|$ $= |x| > \frac{c}{2} = \epsilon \Rightarrow$ f is not continuous at any irrational $c > 0$.

If $x = c < 0$, repeat the argument picking $\epsilon = \frac{|c|}{2} = \frac{-c}{2}$. Therefore f fails to be continuous at any nonzero value $x = c$.

16. (a) Let $c = \frac{m}{n}$ be a rational number in $[0,1]$ reduced to lowest terms $\Rightarrow f(c) = \frac{1}{n}$. Pick $\epsilon = \frac{1}{2n}$. No matter how small $\delta > 0$ is taken, there is an irrational number x in the interval $(c-\delta, c+\delta) \Rightarrow |f(x) - f(c)| = \left|0 - \frac{1}{n}\right|$ $= \frac{1}{n} > \frac{1}{2n} = \epsilon$. Therefore f is discontinuous at $x = c$, a rational number.

(b) Now suppose c is an irrational number $\Rightarrow f(c) = 0$. Let $\epsilon > 0$ be given. Notice that $\frac{1}{2}$ is the only rational number reduced to lowest terms with denominator 2 and belonging to $[0,1]$; $\frac{1}{3}$ and $\frac{2}{3}$ the only rational with denominator 3 belonging to $[0,1]$; $\frac{1}{4}$ and $\frac{3}{4}$ with denominator 4 in $[0,1]$; $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ and $\frac{4}{5}$ with denominator 5 in $[0,1]$; etc. In general, choose N so that $\frac{1}{N} < \epsilon \Rightarrow$ there exist only finitely many rationals in $[0,1]$ having denominator $\le N$, say $r_1, r_2, \ldots, r_p$. Let $\delta = \min\{|c - r_i| : i = 1, \ldots, p\}$. Then the interval $(c-\delta, c+\delta)$ contains no rational numbers with denominator $\le N$. Thus, $0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| = |f(x) - 0|$ $= |f(x)| \le \frac{1}{N} < \epsilon \Rightarrow$ f is continuous at $x = c$ irrational.

(c) The graph looks like the markings on a typical ruler when the points $(x, f(x))$ on the graph of f(x) are connected to the x-axis with vertical lines.



$$f(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

NOTES:

CHAPTER 2 DERIVATIVES

2.1 THE DERIVATIVE AS A FUNCTION

1. Step 1: $f(x) = 4 - x^2$ and $f(x+h) = 4 - (x+h)^2$

   Step 2: $\frac{f(x+h)-f(x)}{h} = \frac{[4-(x+h)^2]-(4-x^2)}{h} = \frac{(4-x^2-2xh-h^2)-4+x^2}{h} = \frac{-2xh-h^2}{h} = \frac{h(-2x-h)}{h}$
   $= -2x - h$ if $h \neq 0$

   Step 3: $f'(x) = \lim\limits_{h\to 0} (-2x-h) = -2x$; $f'(-3) = 6$, $f'(0) = 0$

2. Step 1: $g(t) = \frac{1}{t^2}$ and $g(t+h) = \frac{1}{(t+h)^2}$

   Step 2: $\frac{g(t+h)-g(t)}{h} = \frac{\frac{1}{(t+h)^2}-\frac{1}{t^2}}{h} = \frac{\left(\frac{t^2-(t+h)^2}{(t+h)^2\cdot t^2}\right)}{h} = \frac{t^2-(t^2+2th+h^2)}{(t+h)^2\cdot t^2\cdot h} = \frac{-2th-h^2}{(t+h)^2\,t^2h}$
   $= \frac{h(-2t-h)}{(t+h)^2\,t^2h} = \frac{-2t-h}{(t+h)^2\,t^2}$ if $h \neq 0$

   Step 3: $g'(t) = \lim\limits_{h\to 0} \frac{-2t-h}{(t+h)^2\,t^2} = \frac{-2t}{t^2\cdot t^2} = \frac{-2}{t^3}$; $g'(-1) = 2$, $g'(2) = -\frac{1}{4}$

3. Step 1: $s(t) = t^3 - t^2$ and $s(t+h) = (t+h)^3 - (t+h)^3$

   Step 2: $\frac{s(t+h)-s(t)}{h} = \frac{[(t+h)^3-(t+h)^2]-(t^3-t^2)}{h}$
   $= \frac{(t^3+3t^2h+3th^2+h^3)+(t^2+2th+h^2)-(t^3-t^2)}{h}$
   $= \frac{h(3t^2+3th+h^2-2t-h)}{h} = 3t^2-2t+(3t-1)h+h^2$ if $h \neq 0$

   Step 3: $\frac{ds}{dt} = \lim\limits_{h\to 0} (3t^2-2t+(3t-1)h+h^2) = 3t^2-2t$; $\left.\frac{ds}{dt}\right|_{t=-1} = 5$

4. Step 1: $f(x) = x + \frac{9}{x}$ and $f(x+h) = (x+h) + \frac{9}{(x+h)}$

   Step 2: $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)+\frac{9}{(x+h)}-\left(x+\frac{9}{x}\right)}{h} = \frac{\frac{(x+h)^2+9}{(x+h)}-\left(\frac{x^2+9}{x}\right)}{h}$
   $= \frac{x(x^2+2xh+h^2+9)-(x+h)(x^2+9)}{xh(x+h)}$
   $= \frac{(x^3+2x^2h+xh^2+9x)-(x^3+x^2h+9x+9h)}{xh(x+h)} = \frac{h(x^2+xh-9)}{xh(x+h)}$

$$= \frac{x^2 + xh - 9}{x(x+h)} \text{ if } h \neq 0$$

Step 3: $f'(x) = \lim_{h \to 0} \frac{x^2 + xh - 9}{x(x+h)} = \frac{x^2 - 9}{x^2} = 1 - \frac{9}{x^2}$; $f'(-3) = 0$

5. Step 1: $p(\theta) = \sqrt{3\theta}$ and $p(\theta + h) = \sqrt{3(\theta + h)}$

Step 2: $\frac{p(\theta+h) - p(\theta)}{h} = \frac{\sqrt{3(\theta+h)} - \sqrt{3\theta}}{h} = \frac{\left(\sqrt{3\theta + 3h} - \sqrt{3\theta}\right)}{h} \cdot \frac{\left(\sqrt{3\theta+3h} + \sqrt{3\theta}\right)}{\left(\sqrt{3\theta+3h} + \sqrt{3\theta}\right)} = \frac{(3\theta + 3h) - 3\theta}{h\left(\sqrt{3\theta+3h} + \sqrt{3\theta}\right)}$

$$= \frac{3h}{h\left(\sqrt{3\theta+3h} + \sqrt{3\theta}\right)} = \frac{3}{\sqrt{3\theta+3h} + \sqrt{3\theta}}$$

Step 3: $p'(\theta) = \lim_{h \to 0} \frac{3}{\sqrt{3\theta + 3h} + \sqrt{3\theta}} = \frac{3}{\sqrt{3\theta} + \sqrt{3\theta}} = \frac{3}{2\sqrt{3\theta}}$; $p'(0.25) = \sqrt{3}$

6. $r = f(\theta) = \frac{2}{\sqrt{4-\theta}}$ and $f(\theta + h) = \frac{2}{\sqrt{4 - (\theta + h)}} \Rightarrow \frac{dr}{d\theta} = \lim_{h \to 0} \frac{f(\theta + h) - f(\theta)}{h} = \lim_{h \to 0} \frac{\frac{2}{\sqrt{4-\theta-h}} - \frac{2}{\sqrt{4-\theta}}}{h}$

$$= \lim_{h \to 0} \frac{2\sqrt{4-\theta} - 2\sqrt{4-\theta-h}}{h\sqrt{4-\theta}\sqrt{4-\theta-h}} = \lim_{h \to 0} \frac{2\sqrt{4-\theta} - 2\sqrt{4-\theta-h}}{h\sqrt{4-\theta}\sqrt{4-\theta-h}} \cdot \frac{\left(2\sqrt{4-\theta} + 2\sqrt{4-\theta-h}\right)}{\left(2\sqrt{4-\theta} + 2\sqrt{4-\theta-h}\right)}$$

$$= \lim_{h \to 0} \frac{4(4-\theta) - 4(4-\theta-h)}{2h\sqrt{4-\theta}\sqrt{4-\theta-h}\left(\sqrt{4-\theta} + \sqrt{4-\theta-h}\right)} = \lim_{h \to 0} \frac{2}{\sqrt{4-\theta}\sqrt{4-\theta-h}\left(\sqrt{4-\theta} + \sqrt{4-\theta-h}\right)}$$

$$= \frac{2}{(4-\theta)\left(2\sqrt{4-\theta}\right)} = \frac{1}{(4-\theta)\sqrt{4-\theta}} \Rightarrow \left.\frac{dr}{d\theta}\right|_{\theta=0} = \frac{1}{8}$$

7. $y = x^2 + x + 8 \Rightarrow \frac{dy}{dx} = 2x + 1 + 0 = 2x + 1 \Rightarrow \frac{d^2y}{dx^2} = 2$

8. $s = 5t^3 - 3t^5 \Rightarrow \frac{ds}{dt} = \frac{d}{dt}\left(5t^3\right) - \frac{d}{dt}\left(3t^5\right) = 15t^2 - 15t^4 \Rightarrow \frac{d^2s}{dt^2} = \frac{d}{dt}\left(15t^2\right) - \frac{d}{dt}\left(15t^4\right) = 30t - 60t^3$

9. $y = \frac{4x^3}{3} - 4 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{4}{3}x^3\right) - \frac{d}{dx}(4) = 4x^2 \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(4x^2\right) = 8x$

10. $y = \frac{x^3 + 7}{x} = x^2 + 7x^{-1} \Rightarrow \frac{dy}{dx} = 2x - 7x^{-2} \Rightarrow \frac{d^2y}{dx^2} = 2 + 14x^{-3}$

11. $y = \frac{1}{2}x^4 - \frac{3}{2}x^2 - x \Rightarrow y' = 2x^3 - 3x - 1 \Rightarrow y'' = 6x^2 - 3 \Rightarrow y''' = 12x \Rightarrow y^{(4)} = 12 \Rightarrow y^{(n)} = 0$ for all $n \geq 5$

12. $y = \frac{1}{120}x^5 \Rightarrow y' = \frac{1}{24}x^4 \Rightarrow y'' = \frac{1}{6}x^3 \Rightarrow y''' = \frac{1}{2}x^2 \Rightarrow y^{(4)} = x \Rightarrow y^{(5)} = 1 \Rightarrow y^{(n)} = 0$ for all $n \geq 6$

13. (a) $\frac{dy}{dx} = 3x^2 - 4 \Rightarrow m = \left.\frac{dy}{dx}\right|_{x=2} = 3(2)^2 - 4 = 8$

Therefore, the equation of the line tangent to the curve at the point $(2, 1)$ is $y - 1 = 8(x - 2)$ or $y = 8x - 15$.

(b) Since $x^2 \geq 0$ for all real values of x, it follows that $3x^2 \geq 0$ and $3x^2 - 4 \geq -4$. In addition, $3x^2 - 4 \to +\infty$ as $x \to \pm\infty$. Therefore, the range of values of the curve's slope is $[-4, \infty)$. The graph of the derivative is a parabola that opens upward and its vertex is at the point $(0, -4)$.

(c) The equation of one such tangent line is found in part (a) when $x = 2$. Also, $\frac{dy}{dx} = 8 \Rightarrow 3x^2 - 4 = 8$ $\Rightarrow x^2 = 4 \Rightarrow x = 2$ or $x = -2$. At $x = -2$, $y = (-2)^3 - 4(-2) + 1 = 1$. Therefore, the equation of the line tangent to the curve at the point $(-2, 1)$ is $y - 1 = 8(x - (-2))$ or $y = 8x + 17$.

14. (a) Set $\frac{dy}{dx} = 0$ and solve for x: $\frac{dy}{dx} = 1 - \frac{3}{2\sqrt{x}} = 0 \Rightarrow \sqrt{x} = \frac{3}{2} \Rightarrow x = \frac{9}{4}$. At $x = \frac{9}{4}$, the curve has value $y = \frac{9}{4} - 3\sqrt{\frac{9}{4}} = \frac{9}{4} - 3\left(\frac{3}{2}\right) = -\frac{9}{4}$. Therefore, an equation for the horizontal tangent to the curve at the point $\left(\frac{9}{4}, -\frac{9}{4}\right)$ is $y = -\frac{9}{4}$.

(b) The domain of the function $y = x - 3\sqrt{x}$ is $[0, \infty)$. The derivative, however, is undefined at $x = 0$. Therefore, to determine the range of values for the curve's slopes, consider $0 < x < \infty$. As $x \to \infty$, $\frac{dy}{dx} = 1 - \frac{3}{2\sqrt{x}} \to 1$ and, as $x \downarrow 0$, $\frac{dy}{dx} = 1 - \frac{3}{2\sqrt{x}} \to -\infty$. For all values of x between 0 and ∞, the function $\frac{dy}{dx} = 1 - \frac{3}{2\sqrt{x}}$ is increasing toward 1 as x increases. Therefore, the curve's slopes range from $-\infty$ near $x = 0$, to 1 but never reaching 1, as $x \to \infty$. That is, $-\infty < \frac{dy}{dx} < 1$ for $0 < x < \infty$.

15. Note that as x increases, the slope of the tangent line to the curve is first negative, then zero (when $x = 0$), then positive $\Rightarrow$ the slope is always increasing which matches (b).

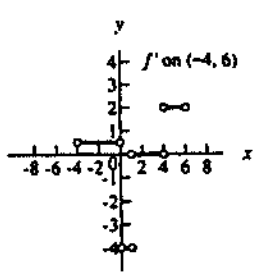
16. Note that the slope of the tangent line is never negative. For x negative, $f_2'(x)$ is positive but decreasing as x increases. When $x = 0$, the slope of the tangent line to x is 0. For $x > 0$, $f_2'(x)$ is positive and increasing. This graph matches (a).

17. $f_3(x)$ is an oscillating function like the cosine. Everywhere that the graph of f_3 has a horizontal tangent we expect f_3' to be zero, and (d) matches this condition.

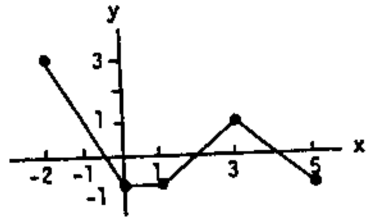
18. The graph matches with (c).

19. (a) f' is not defined at $x = 0, 1, 4$. At these points, the left-hand and right-hand derivatives do not agree. For example, $\lim\limits_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$ = slope of line joining $(-4, 0)$ and $(0, 2) = \frac{1}{2}$ but $\lim\limits_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$ = slope of line joining $(0, 2)$ and $(1, -2) = -4$. Since these values are not equal, $f'(0) = \lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist.

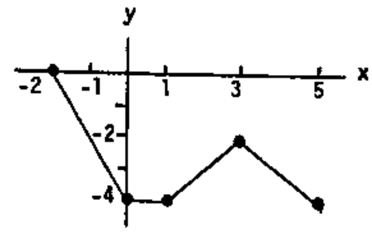
(b)



20. (a)



(b) Shift the graph in (a) down 3 units



21. Left-hand derivative: For $h < 0$, $f(0+h) = f(h) = h^2$ (using $y = x^2$ curve) $\Rightarrow \lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$

$= \lim\limits_{h\to 0^-} \frac{h^2-0}{h} = \lim\limits_{h\to 0^-} h = 0;$

Right-hand derivative: For $h > 0$, $f(0+h) = f(h) = h$ (using $y = x$ curve) $\Rightarrow \lim\limits_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$

$= \lim\limits_{h\to 0^+} \frac{h-0}{h} = \lim\limits_{h\to 0^+} 1 = 1;$

Then $\lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h} \neq \lim\limits_{h\to 0^+} \frac{f(0+h)-f(0)}{h} \Rightarrow$ the derivative $f'(0)$ does not exist.

22. Left-hand derivative: When $h < 0$, $1+h < 1 \Rightarrow f(1+h) = \sqrt{1+h} \Rightarrow \lim\limits_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$

$= \lim\limits_{h\to 0^-} \frac{\sqrt{1+h}-1}{h} = \lim\limits_{h\to 0^-} \frac{\left(\sqrt{1+h}-1\right)}{h} \cdot \frac{\left(\sqrt{1+h}+1\right)}{\left(\sqrt{1+h}+1\right)} = \lim\limits_{h\to 0^-} \frac{(1+h)-1}{h\left(\sqrt{1+h}+1\right)} = \lim\limits_{h\to 0^-} \frac{1}{\sqrt{1+h}+1} = \frac{1}{2};$

Right-hand derivative: When $h > 0$, $1+h > 1 \Rightarrow f(1+h) = 2(1+h)-1 = 2h+1 \Rightarrow \lim\limits_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$

$= \lim\limits_{h\to 0^+} \frac{(2h+1)-1}{h} = \lim\limits_{h\to 0^+} 2 = 2;$

Then $\lim\limits_{h\to 0^-} \frac{f(1+h)-f(1)}{h} \neq \lim\limits_{h\to 0^+} \frac{f(1+h)-f(1)}{h} \Rightarrow$ the derivative $f'(1)$ does not exist.

23. (a) The function is differentiable on its domain $-2 \le x \le 3$ (it is smooth)
(b) none
(c) none

24. (a) f is differentiable on $-2 \le x < -1$, $-1 < x < 0$, $0 < x < 2$, and $2 < x \le 3$
(b) f is continuous but not differentiable at $x = -1$: $\lim\limits_{x\to -1} f(x) = 0$ exists but there is a corner at $x = -1$ since

$\lim\limits_{h\to 0^-} \frac{f(-1+h)-f(-1)}{h} = -3$ and $\lim\limits_{h\to 0^+} \frac{f(-1+h)-f(-1)}{h} = 3 \Rightarrow f'(-1)$ does not exist

(c) f is neither continuous nor differentiable at $x = 0$ and $x = 2$:

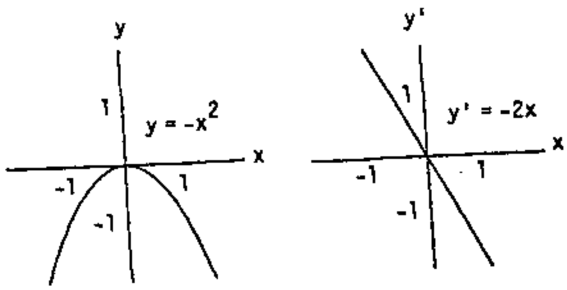
at $x = 0$, $\lim\limits_{x\to 0^-} f(x) = 3$ but $\lim\limits_{x\to 0^+} f(x) = 0 \Rightarrow \lim\limits_{x\to 0} f(x)$ does not exist;

at $x = 2$, $\lim\limits_{x\to 2} f(x)$ exists but $\lim\limits_{x\to 2} f(x) \neq f(2)$

25. (a) f is differentiable on $-1 \le x < 0$ and $0 < x \le 2$
(b) f is continuous but not differentiable at $x = 0$: $\lim\limits_{x \to 0} f(x) = 0$ exists but there is a cusp at $x = 0$, so $f'(0) = \lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist
(c) none

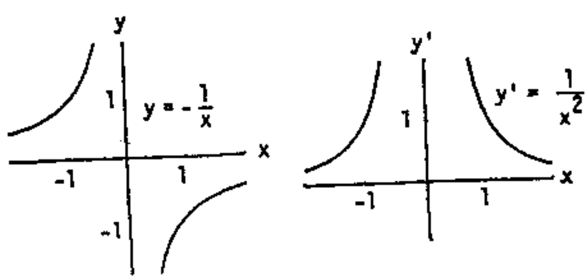
26. (a) f is differentiable on $-3 \le x < -2$, $-2 < x < 2$, and $2 < x \le 3$
(b) f is continuous but not differentiable at $x = -2$ and $x = 2$: there are corners at those points
(c) none

27. (a) $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{-(x+h)^2 - (-x^2)}{h} = \lim\limits_{h \to 0} \frac{-x^2 - 2xh - h^2 + x^2}{h} = \lim\limits_{h \to 0} (-2x - h) = -2x$
(b)



(c) $y' = -2x$ is positive for $x < 0$, y' is zero when $x = 0$, y' is negative when $x > 0$
(d) $y = -x^2$ is increasing for $-\infty < x < 0$ and decreasing for $0 < x < \infty$; the function is increasing on intervals where $y' > 0$ and decreasing on intervals where $y' < 0$

28. (a) $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{\left(\frac{-1}{x+h} - \frac{-1}{x}\right)}{h} = \lim\limits_{h \to 0} \frac{-x + (x+h)}{x(x+h)h} = \lim\limits_{h \to 0} \frac{1}{x(x+h)} = \frac{1}{x^2}$
(b)

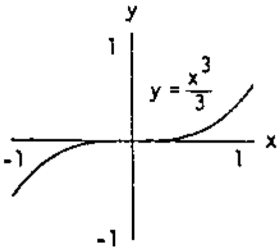


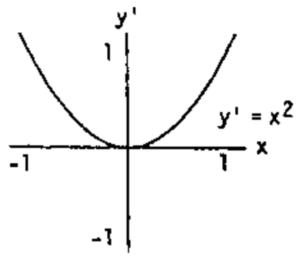
(c) y' is positive for all $x \neq 0$, y' is never 0, y' is never negative
(d) $y = -\frac{1}{x}$ is increasing for $-\infty < x < 0$ and $0 < x < \infty$

29. (a) Using the alternate formula for calculating derivatives: $f'(c) = \lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim\limits_{x \to c} \frac{\left(\frac{x^3}{3} - \frac{c^3}{3}\right)}{x - c}$

$$= \lim_{x \to c} \frac{x^3 - c^3}{3(x-c)} = \lim_{x \to c} \frac{(x-c)(x^2 + xc + c^2)}{3(x-c)} = \lim_{x \to c} \frac{x^2 + xc + c^2}{3} = c^2 \Rightarrow f'(x) = x^2$$

(b)





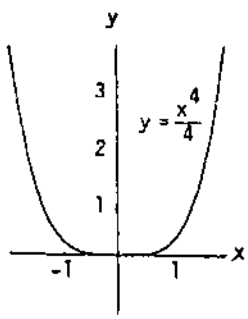
(c) y' is positive for all $x \neq 0$, and $y' = 0$ when $x = 0$; y' is never negative

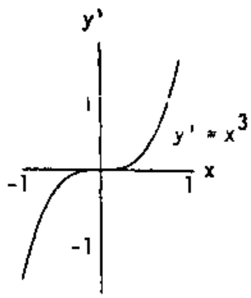
(d) $y = \frac{x^3}{3}$ is increasing for all x (the graph is horizontal at $x = 0$) because y is increasing where $y' > 0$; y is never decreasing

30. (a) Using the alternate form for calculating derivatives: $f'(c) = \lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim\limits_{x \to c} \frac{\left(\frac{x^4}{4} - \frac{c^4}{4}\right)}{x - c}$

$$= \lim_{x \to c} \frac{x^4 - c^4}{4(x - c)} = \lim_{x \to c} \frac{(x - c)\left(x^3 + cx^2 + c^2x + c^3\right)}{4(x - c)} = \lim_{x \to c} \frac{x^3 + cx^2 + c^2x + c^3}{4} = c^3 \Rightarrow f'(x) = x^3$$

(b)





(c) y' is positive for $x > 0$, y' is zero for $x = 0$, y' is negative for $x < 0$

(d) $y = \frac{x^4}{4}$ is increasing on $0 < x < \infty$ and decreasing on $-\infty < x < 0$

31. $y' = \lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim\limits_{x \to c} \frac{x^3 - c^3}{x - c} = \lim\limits_{x \to c} \frac{(x - c)\left(x^2 + xc + c^2\right)}{x - c} = \lim\limits_{x \to c} \left(x^2 + xc + c^2\right) = 3c^2.$

The slope of the curve $y = x^3$ at $x = c$ is $y' = 3c^2$. Notice that $3c^2 \geq 0$ for all $c \Rightarrow y = x^3$ never has a negative slope.

32. Horizontal tangents occur where $y' = 0$. Thus, $y' = \lim\limits_{h \to 0} \frac{2\sqrt{x + h} - 2\sqrt{x}}{h}$

$$= \lim_{h \to 0} \frac{2\left(\sqrt{x + h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x + h} + \sqrt{x}\right)}{\left(\sqrt{x + h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{2((x + h) - x))}{h\left(\sqrt{x + h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{2}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{\sqrt{x}}.$$

Then $y' = 0$ when $\frac{1}{\sqrt{x}} = 0$ which is never true $\Rightarrow$ the curve has no horizontal tangents.

33. $y' = \lim\limits_{h\to 0} \frac{\left(2(x+h)^2 - 13(x+h) + 5\right) - \left(2x^2 - 13x + 5\right)}{h} = \lim\limits_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 13x - 13h + 5 - 2x^2 + 13x - 5}{h}$

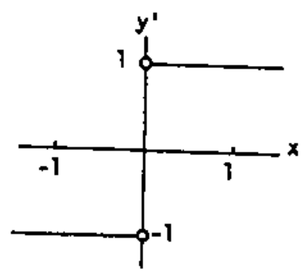
$= \lim\limits_{h\to 0} \frac{4xh + 2h^2 - 13h}{h} = \lim\limits_{h\to 0} (4x + 2h - 13) = 4x - 13$, slope at x. The slope is -1 when $4x - 13 = -1$ $\Rightarrow 4x = 12 \Rightarrow x = 3 \Rightarrow y = 2\cdot 3^2 - 13\cdot 3 + 5 = -16$. Thus the tangent line is $y + 16 = (-1)(x-3)$ and the point of tangency is $(3, -16)$.

34. For the curve $y = \sqrt{x}$, we have $y' = \lim\limits_{h\to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim\limits_{h\to 0} \frac{(x+h) - x}{\left(\sqrt{x+h} + \sqrt{x}\right)h}$

$= \lim\limits_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$. Suppose $\left(a, \sqrt{a}\right)$ is the point of tangency of such a line and $(-1, 0)$ is the point on the line where it crosses the x-axis. Then the slope of the line is $\frac{\sqrt{a} - 0}{a - (-1)} = \frac{\sqrt{a}}{a+1}$ which must also equal $\frac{1}{2\sqrt{a}}$; using the derivative formula at $x = a \Rightarrow \frac{\sqrt{a}}{a+1} = \frac{1}{2\sqrt{a}} \Rightarrow 2a = a + 1 \Rightarrow a = 1$. Thus such a line does exist: its point of tangency is $(1, 1)$, its slope is $\frac{1}{2\sqrt{a}} = \frac{1}{2}$; and an equation of the line is $y - 1 = \frac{1}{2}(x - 1)$.

35. No. Derivatives of functions have the intermediate value property. The function $f(x) = \text{int } x$ satisfies $f(0) = 0$ and $f(1) = 1$ but does not take on the value $\frac{1}{2}$ anywhere in $[0, 1] \Rightarrow$ f does not have the intermediate value property. Thus f cannot be the derivative of any function on $[0, 1] \Rightarrow$ f cannot be the derivative of any function on $(-\infty, \infty)$.

36. The graphs are the same. So we know that for $f(x) = |x|$, we have $f'(x) = \frac{|x|}{x}$.



37. Yes; the derivative of $-f$ is $-f'$ so that $f'(x_0)$ exists $\Rightarrow -f'(x_0)$ exists as well.

38. Yes; the derivative of $3g$ is $3g'$ so that $g'(7)$ exists $\Rightarrow 3g'(7)$ exists as well.

39. Yes, $\lim\limits_{t\to 0} \frac{g(t)}{h(t)}$ can exist but it need not equal zero. For example, let $g(t) = mt$ and $h(t) = t$. Then $g(0) = h(0) = 0$, but $\lim\limits_{t\to 0} \frac{g(t)}{h(t)} = \lim\limits_{t\to 0} \frac{mt}{t} = \lim\limits_{t\to 0} m = m$, which need not be zero.

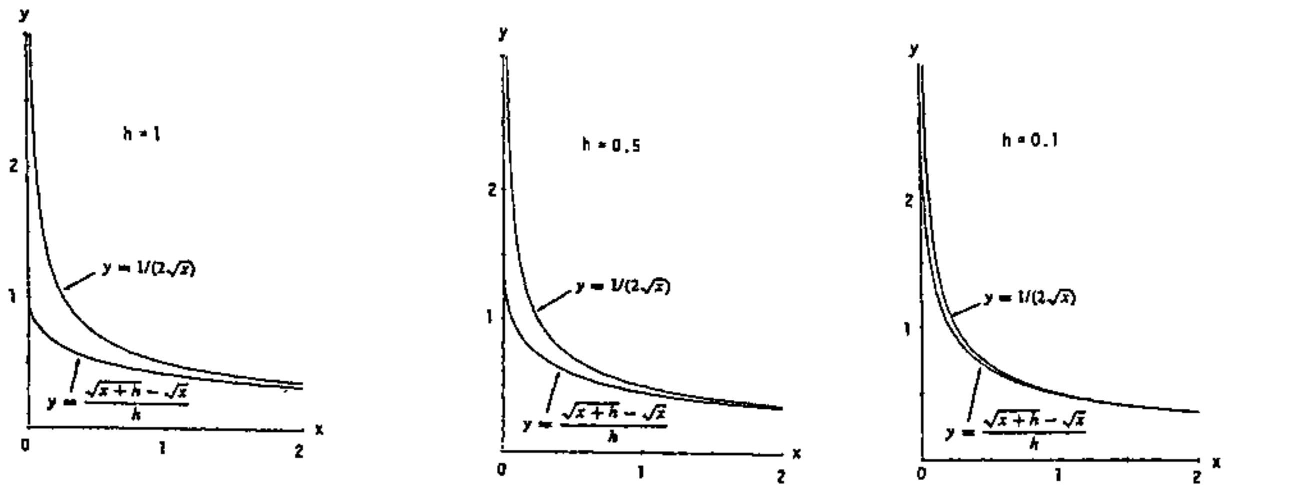
40. (a) Suppose $|f(x)| \le x^2$ for $-1 \le x \le 1$. Then $|f(0)| \le 0^2 \Rightarrow f(0) = 0$. Then $f'(0) = \lim\limits_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim\limits_{h\to 0} \frac{f(h) - 0}{h} = \lim\limits_{h\to 0} \frac{f(h)}{h}$. For $|h| \le 1$, $-h^2 \le f(h) \le h^2 \Rightarrow -h \le \frac{f(h)}{h} \le h \Rightarrow f'(0) = \lim\limits_{h\to 0} \frac{f(h)}{h} = 0$

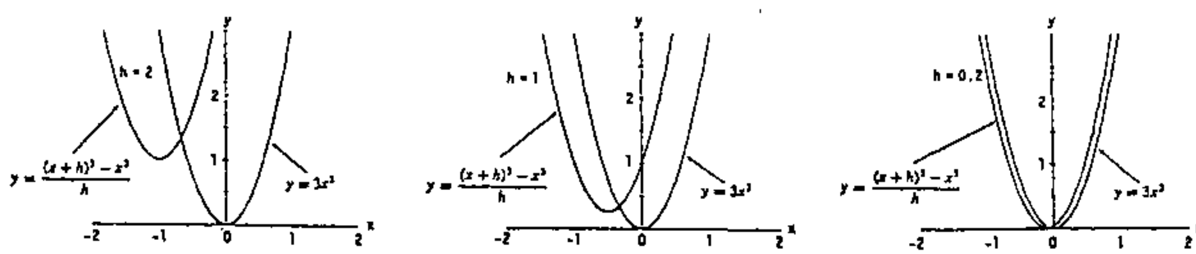
by the Sandwich Theorem for limits.

(b) Note that for $x \neq 0$, $|f(x)| = \left|x^2 \sin \frac{1}{x}\right| = |x^2||\sin x| \le |x^2| \cdot 1 = x^2$ (since $-1 \le \sin x \le 1$). By part (a), f is differentiable at $x = 0$ and $f'(0) = 0$.

41. The graphs are shown below for $h = 1, 0.5, 0.1$. The function $y = \frac{1}{2\sqrt{x}}$ is the derivative of the function $y = \sqrt{x}$ so that $\frac{1}{2\sqrt{x}} = \lim\limits_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$. The graphs reveal that $y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$ gets closer to $y = \frac{1}{2\sqrt{x}}$ as h gets smaller and smaller.

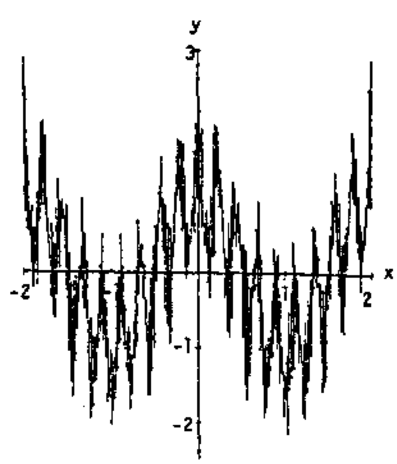


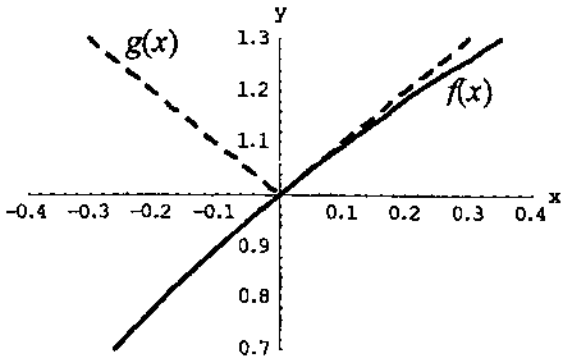
42. The graphs are shown below for $h = 2, 1, 0.2$. The function $y = 3x^2$ is the derivative of the function $y = x^3$ so that $3x^2 = \lim\limits_{h \to 0} \frac{(x+h)^3 - x^3}{h}$. The graphs reveal that $y = \frac{(x+h)^3 - x^3}{h}$ gets closer to $y = 3x^2$ as h gets smaller and smaller.



43. Weierstrass's nowhere differentiable continuous function.

$$g(x) = \cos(\pi x) + \left(\frac{2}{3}\right)^1 \cos(9\pi x) + \left(\frac{2}{3}\right)^2 \cos(9^2\pi x) + \left(\frac{2}{3}\right)^3 \cos(9^3\pi x) + \cdots + \left(\frac{2}{3}\right)^7 \cos(9^7\pi x)$$



44. 

The function f(x) is differentiable at (0, 1) because the graph of f(x) is smooth at the point (0, 1). Tracing along the graph of f(x), from left to right, the value of the function continually increases through the point (0, 1) with no sudden change in the rate of increase. The function g(x) is not differentiable at (0, 1) because the graph of g(x) has a sharp corner there. Tracing along the graph of g(x), from left to right, there is an abrupt change at the point (0, 1). To the left of the point the values of g(x) decrease at a constant rate and to the right the values increase at a constant rate. There is no derivative at x = 0 because

$\lim\limits_{h\to 0^-} \frac{(|0+h|+1)-(|0|+1)}{h} = \lim\limits_{h\to 0^-} \frac{-h}{h} = -1$ and $\lim\limits_{h\to 0^+} \frac{(|0+h|+1)-(|0|+1)}{h} = \lim\limits_{h\to 0^+} \frac{h}{h} = 1$. Consequently, $g'(0) = \lim\limits_{h\to 0} \frac{g(0+h)-g(0)}{h}$ does not exist since the right- and left-hand limits are not equal.

45-50. Example CAS commands:

Maple:

```
f:=x -> x∧2*cos(x);
q:=h -> (f(x + h) − f(x))/h;
slope:=limit(q(h),h=0);
fp:=unapply(%,x);
x0:=Pi/4;
L:=x -> f(x0) + fp(x0)*(x − x0);
plot({f(x),L(x)},x=x0 − 2..x0 + 1);
```

Mathematica:

```
Clear [f,m,x,y]
x0 = Pi/4; f[x_] = x∧2 Cos[x]
Plot[ f[x], {x,x0 − 3,x0 + 3} ]
q[x_,h_] = (f[x + h] − f[x])/h
m[x_] = Limit[ q[x,h], h -> 0 ]
y = f[x0] + m[x0] (x − x0)
Plot[ {f[x],y}, {x,x0 − 3,x0 + 3} ]
m[x0 − 1]//N
m[x0 + 1]//N
Plot[ {f[x],m[x]}, {x,x0 − 3,x0 + 3} ]
```

In Exercise 63, you could define

x0 = 1; f[x_] = x ∧ (1/3) + x ∧ (2/3)

However, Mathematica 4.0 uses a complex branch for odd roots of negative numbers (as does Maple 6), so the above will only work for positive x. To get the real roots for all x, you could force it as below, but this form is not good for taking derivatives:

x0 = 1; f[x_] = Sign[x] Abs[x] ∧ (1/3) + Abs[x] ∧ (2/3)

2.2 THE DERIVATIVE AS A RATE OF CHANGE

1. $s = t^2 - 3t + 2,\ 0 \le t \le 2$
   (a) displacement $= \Delta s = s(2) - s(0) = -2\text{m},\ v_{av} = \frac{\Delta s}{\Delta t} = \frac{-2\text{ m}}{2\text{ sec}} = -1$ m/sec
   (b) $v = \frac{ds}{dt} = 2t - 3,\ |v(0)| = |-3| = 3$ m/sec, $|v(2)| = |1| = 1$ m/sec; $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 2$,
   $a(0) = a(2) = 2\text{ m/sec}^2$
   (c) $v = 0 \Rightarrow 2t - 3 = 0 \Rightarrow t = \frac{3}{2}$ sec. For $0 \le t < \frac{3}{2}$, v is negative and s is decreasing, whereas for $\frac{3}{2} < t \le 2$, v is positive and s is increasing. Therefore, the body changes direction at $t = \frac{3}{2}$.

2. $s = 6t - t^2,\ 0 \le t \le 6$
   (a) displacement $= \Delta s = s(6) - s(0) = 0 - 0 = 0,\ v_{av} = \frac{\Delta s}{\Delta t} = \frac{0\text{ m}}{6\text{ sec}} = 0$ m/sec
   (b) $v = \frac{ds}{dt} = 6 - 2t,\ |v(0)| = |6| = 6$ m/sec, $|v(6)| = |-6| = 6$ m/sec; $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -2$,
   $a(0) = a(6) = -2\text{ m/sec}^2$
   (c) $v = 0 \Rightarrow 6 - 2t = 0 \Rightarrow t = 3$ sec. For $0 \le t < 3$, v is positive and s is increasing, whereas for $3 < t \le 6$, v is negative and s is decreasing. Therefore, the body changes direction at $t = 3$.

3. $s = -t^3 + 3t^2 - 3t,\ 0 \le t \le 3$
   (a) displacement $= \Delta s = s(3) - s(0) = -9$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{-9}{3} = -3$ m/sec
   (b) $v = \frac{ds}{dt} = -3t^2 + 6t - 3 \Rightarrow |v(0)| = |-3| = 3$ m/sec and $|v(3)| = |-12| = 12$ m/sec; $a = \frac{d^2s}{dt^2} = -6t + 6$
   $\Rightarrow a(0) = 6\text{ m/sec}^2$ and $a(3) = -12\text{ m/sec}^2$
   (c) $v = 0 \Rightarrow -3t^2 + 6t - 3 = 0 \Rightarrow t^2 - 2t + 1 = 0 \Rightarrow (t-1)^2 = 0 \Rightarrow t = 1$. For all other values of t in the interval the velocity v is negative (the graph of $v = -3t^2 + 6t - 3$ is a parabola with vertex at $t = 1$ which opens downward $\Rightarrow$ the body never changes direction.

4. $s = \frac{t^4}{4} - t^3 + t^2,\ 0 \le t \le 2$
   (a) $\Delta s = s(2) - s(0) = 0$ m, $v_{av} = \frac{\Delta s}{\Delta t} = 0$ m/sec
   (b) $v = t^3 - 3t^2 + 2t \Rightarrow |v(0)| = 0$ m/sec and $|v(2)| = 0$ m/sec; $a = 3t^2 - 6t + 2 \Rightarrow a(0) = 2\text{ m/sec}^2$ and $a(2) = 2\text{ m/sec}^2$

(c) $v = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-2)(t-1) = 0 \Rightarrow t = 0, 1, 2 \Rightarrow v = t(t-2)(t-1)$ is positive in the interval for $0 < t < 1$ and v is negative for $1 < t < 2 \Rightarrow$ the body changes direction at $t = 1$.

5. $s = t^3 - 6t^2 + 9t$ and let the positive direction be to the right on the s-axis.
   (a) $v = 3t^2 - 12t + 9$ so that $v = 0 \Rightarrow t^2 - 4t + 3 = (t-3)(t-1) = 0 \Rightarrow t = 1$ or 3; $a = 6t - 12 \Rightarrow a(1) = -6\ \text{m/sec}^2$ and $a(3) = 6\ \text{m/sec}^2$. Thus the body is motionless but being accelerated left when $t = 1$, and motionless but being accelerated right when $t = 3$.
   (b) $a = 0 \Rightarrow 6t - 12 = 0 \Rightarrow t = 2$ with speed $|v(2)| = |12 - 24 + 9| = 3$ m/sec
   (c) The body moves to the right or forward on $0 \le t < 1$, and to the left or backward on $1 < t < 2$. The positions are $s(0) = 0$, $s(1) = 4$ and $s(2) = 2 \Rightarrow$ total distance $= |s(1) - s(0)| + |s(2) - s(1)| = |4| + |-2| = 6$ m.

6. $v = t^2 - 4t + 3 \Rightarrow a = 2t - 4$
   (a) $v = 0 \Rightarrow t^2 - 4t + 3 = 0 \Rightarrow t = 1$ or $3 \Rightarrow a(1) = -2\ \text{m/sec}^2$ and $a(3) = 2\ \text{m/sec}^2$
   (b) $v > 0 \Rightarrow (t-3)(t-1) > 0 \Rightarrow 0 < t < 1$ or $t > 3$ and the body is moving forward; $v < 0 \Rightarrow (t-3)(t-1) < 0 \Rightarrow 1 < t < 3$ and the body is moving backward
   (c) velocity increasing $\Rightarrow a > 0 \Rightarrow 2t - 4 > 0 \Rightarrow t > 2$; velocity decreasing $\Rightarrow a < 0 \Rightarrow 2t - 4 < 0 \Rightarrow t < 2$

7. $s_m = 1.86t^2 \Rightarrow v_m = 3.72t$ and solving $3.72t = 27.8 \Rightarrow t \approx 7.5$ sec on Mars; $s_j = 11.44t^2 \Rightarrow v_j = 22.88t$ and solving $22.88t = 27.8 \Rightarrow t \approx 1.2$ sec on Jupiter.

8. (a) $v(t) = s'(t) = 24 - 1.6t$ m/sec, and $a(t) = v'(t) = s''(t) = -1.6\ \text{m/sec}^2$
   (b) Solve $v(t) = 0 \Rightarrow 24 - 1.6t = 0 \Rightarrow t = 15$ sec
   (c) $s(15) = 24(15) - .8(15)^2 = 180$ m
   (d) Solve $s(t) = 90 \Rightarrow 24t - .8t^2 = 90 \Rightarrow t = \frac{30 \pm 15\sqrt{2}}{2} \approx 4.39$ sec going up and 25.6 sec going down
   (e) Twice the time it took to reach its highest point or 30 sec

9. $s = 15t - \frac{1}{2}g_s t^2 \Rightarrow v = 15 - g_s t$ so that $v = 0 \Rightarrow 15 - g_s t = 0 \Rightarrow t = \frac{15}{g_s}$. Therefore $\frac{15}{g_s} = 20 \Rightarrow g_s = \frac{3}{4} = 0.75\ \text{m/sec}^2$

10. Solving $s_m = 832t - 2.6t^2 = 0 \Rightarrow t(832 - 2.6t) = 0 \Rightarrow t = 0$ or $320 \Rightarrow 320$ sec on the moon; solving $s_e = 832t - 16t^2 = 0 \Rightarrow t(832 - 16t) = 0 \Rightarrow t = 0$ or $52 \Rightarrow 52$ sec on the earth. Also, $v_m = 832 - 5.2t = 0 \Rightarrow t = 160$ and $s_m(160) \approx 66{,}560$ ft, the height it reaches above the moon's surface; $v_e = 832 - 32t = 0 \Rightarrow t = 26$ and $s_e(26) \approx 10{,}816$ ft, the height it reaches above the earth's surface.

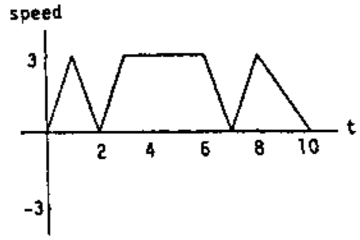
11. (a) $s = 179 - 16t^2 \Rightarrow v = -32t \Rightarrow$ speed $= |v| = 32t$ ft/sec and $a = -32\ \text{ft/sec}^2$
    (b) $s = 0 \Rightarrow 179 - 16t^2 = 0 \Rightarrow t = \sqrt{\frac{179}{16}} \approx 3.3$ sec
    (c) When $t = \sqrt{\frac{179}{16}}$, $v = -32\sqrt{\frac{179}{16}} = -8\sqrt{179} \approx -107.0$ ft/sec

12. (a) $\lim_{\theta \to \frac{\pi}{2}} v = \lim_{\theta \to \frac{\pi}{2}} 9.8(\sin\theta)t = 9.8t$ so we expect $v = 9.8t$ m/sec in free fall
    (b) $a = \frac{dv}{dt} = 9.8\ \text{m/sec}^2$

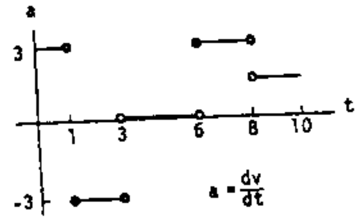
13. (a) at 2 and 7 seconds

(b) between 3 and 6 seconds: $3 \le t \le 6$

(c)

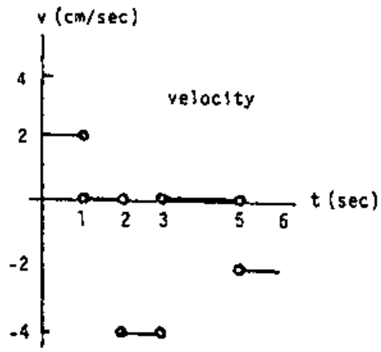


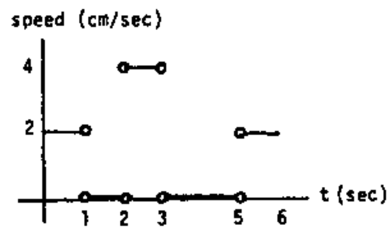
(d)



14. (a) P is moving to the left when $2 < t < 3$ or $5 < t < 6$; P is moving to the right when $0 < t < 1$; P is standing still when $1 < t < 2$ or $3 < t < 5$

(b)





15. (a) 190 ft/sec

(b) 2 sec

(c) at 8 sec, 0 ft/sec

(d) 10.8 sec, 90 ft/sec

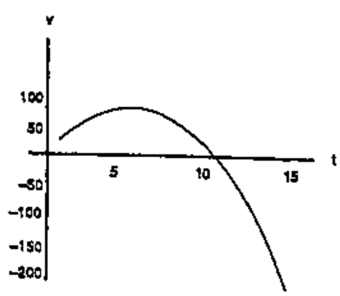
(e) From $t = 8$ until $t = 10.8$ sec, a total of 2.8 sec

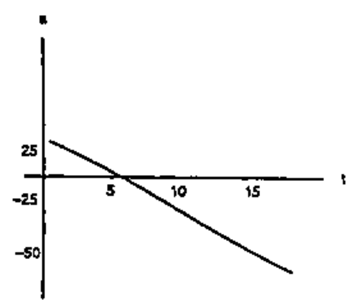
(f) Greatest acceleration happens 2 sec after launch

(g) From $t = 2$ to $t = 10.8$ sec; during this period, $a = \dfrac{v(10.8) - v(2)}{10.8 - 2} \approx -32\ \text{ft/sec}^2$

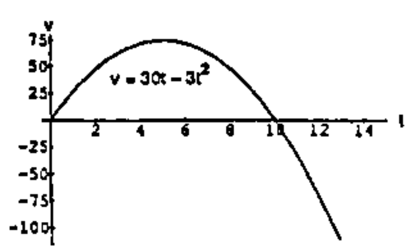
16. Answers will vary.

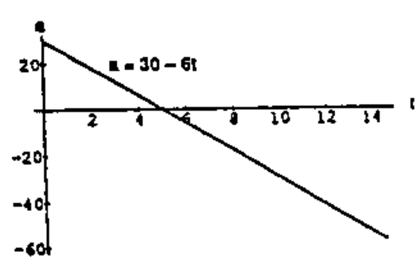
(a)





(b)





17. $s = 490t^2 \Rightarrow v = 980t \Rightarrow a = 980$

(a) Solving $160 = 490t^2 \Rightarrow t = \frac{4}{7}$ sec. The average velocity was $\frac{s(4/7) - s(0)}{4/7} = 280$ cm/sec.

(b) At the 160 cm mark the balls are falling at $v(4/7) = 560$ cm/sec. The acceleration at the 160 cm mark was 980 cm/sec^2.

(c) The light was flashing at a rate of $\frac{17}{4/7} = 29.75$ flashes per second.

18. C = position, A = velocity, and B = acceleration. Neither A nor C can be the derivative of B because B's derivative is constant. Graph C cannot be the derivative of A either, because A has some negative slopes while C has only positive values. So, C, being the derivative of neither A nor B must be the graph of position. Curve C has both positive and negative slopes, so its derivative, the velocity, must be A and not B. That leaves B for acceleration.

19. C = position, B = velocity, and A = acceleration. Curve C cannot be the derivative of either A or B because C has only negative values while both A and B have some positive slopes. So, C represents position. Curve C has no positive slopes, so its derivative, the velocity, must be B. That leaves A for acceleration. Indeed, A is negative where B has negative slopes and positive where B has positive slopes.

20. (a) $c(100) = 11{,}000 \Rightarrow c_{av} = \frac{11{,}000}{100} = \$110$; $c(x) = 2000 + 100x - .1x^2 \Rightarrow c'(x) = 100 - .2x$

(b) Marginal cost $= c'(x) \Rightarrow$ the marginal cost of producing 100 machines is $c'(100) = \$80$

(c) The cost of producing the 101st machine is $c(101) - c(100) = 100 - \frac{201}{10} = \$79.90$

21. (a) $r(x) = 20{,}000\left(1 - \frac{1}{x}\right) \Rightarrow r'(x) = \frac{20{,}000}{x^2} \Rightarrow r'(100) = \$2$/machine

(b) $\Delta r \approx r'(100) = \$2$

(c) $\lim\limits_{x\to\infty} r'(x) = \lim\limits_{x\to\infty} \frac{20{,}000}{x^2} = 0$. The increase in revenue as the number of items increases without bound will approach zero.

22. $b(t) = 10^6 + 10^4 t - 10^3 t^2 \Rightarrow b'(t) = 10^4 - (2)\left(10^3 t\right) = 10^3(10 - 2t)$

(a) $b'(0) = 10^4$ bacteria/hr

(b) $b'(5) = 0$ bacteria/hr

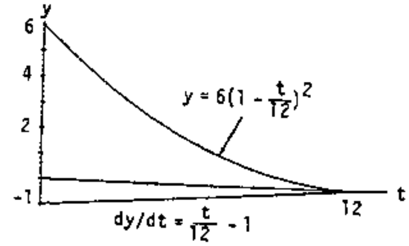
(c) $b'(10) = -10^4$ bacteria/hr

23. $Q(t) = 200(30 - t)^2 = 200\left(900 - 60t + t^2\right) \Rightarrow Q'(t) = 200(-60 + 2t) \Rightarrow Q'(10) = -8{,}000$ gallons/min is the rate the water is running at the end of 10 min. Then $\frac{Q(10) - Q(0)}{10} = -10{,}000$ gallons/min is the average rate the water flows during the first 10 min. The negative signs indicate water is <u>leaving</u> the tank.

24. (a) $y = 6\left(1 - \frac{t}{12}\right)^2 = 6\left(1 - \frac{t}{6} + \frac{t^2}{144}\right) \Rightarrow \frac{dy}{dt} = \frac{t}{12} - 1$

(b) The largest value of $\frac{dy}{dt}$ is 0 m/h when $t = 12$ and the fluid level is falling the slowest at that time. The smallest value of $\frac{dy}{dt}$ is -1 m/h, when $t = 0$, and the fluid level is falling the fastest at that time.

(c) In this situation, $\frac{dy}{dt} \le 0 \Rightarrow$ the graph of y is always decreasing. As $\frac{dy}{dt}$ increases in value, the slope of the graph of y increases from -1 to 0 over the interval $0 \le t \le 12$.



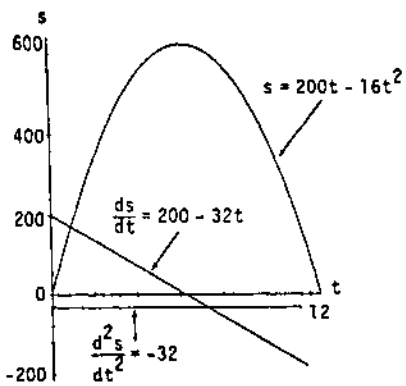
25. (a) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2 \Rightarrow \left.\frac{dV}{dr}\right|_{r=2} = 4\pi(2)^2 = 16\pi \text{ ft}^3/\text{ft}$

(b) When $r = 2$, $\frac{dV}{dr} = 16\pi$ so that when r changes by 1 unit, we expect V to change by approximately 16π. Therefore when r changes by 0.2 units V changes by approximately $(16\pi)(0.2) = 3.2\pi \approx 10.05 \text{ ft}^3$. Note that $V(2.2) - V(2) \approx 11.09 \text{ ft}^3$.

26. $200 \text{ km/hr} = 55\frac{5}{9} = \frac{500}{9}$ m/sec, and $D = \frac{10}{9}t^2 \Rightarrow V = \frac{20}{9}t$. Thus $V = \frac{500}{9} \Rightarrow \frac{20}{9}t = \frac{500}{9} \Rightarrow t = 25$ sec. When $t = 25$, $D = \frac{10}{9}(25)^2 = \frac{6250}{9}$ m

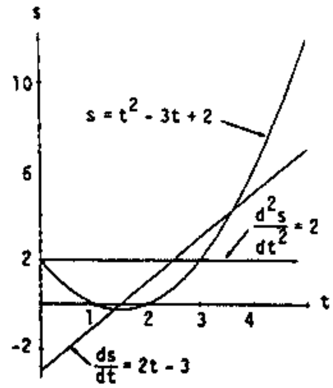
27. $s = v_0 t - 16t^2 \Rightarrow v = v_0 - 32t$; $v = 0 \Rightarrow t = \frac{v_0}{32}$; $1900 = v_0 t - 16t^2$ so that $t = \frac{v_0}{32} \Rightarrow 1900 = \frac{v_0^2}{32} - \frac{v_0^2}{64}$ $\Rightarrow v_0 = \sqrt{(64)(1900)} = 80\sqrt{19}$ ft/sec and, finally, $\frac{80\sqrt{19} \text{ ft}}{\text{sec}} \cdot \frac{60 \text{ sec}}{1 \text{ min}} \cdot \frac{60 \text{ min}}{1 \text{ hr}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \approx 238$ mph.

28.



(a) $v = 0$ when $t = 6.25$ sec
(b) $v > 0$ when $0 \le t < 6.25 \Rightarrow$ body moves up; $v < 0$ when $6.25 < t \le 12.5 \Rightarrow$ body moves down
(c) body changes direction at $t = 6.25$ sec
(d) body speeds up on $(6.25, 12.5]$ and slows down on $[0, 6.25)$
(e) The body is moving fastest at the endpoints $t = 0$ and $t = 12.5$ when it is traveling 200 ft/sec. It's moving slowest at $t = 6.25$ when the speed is 0.
(f) When $t = 6.25$ the body is $s = 625$ m from the origin and farthest away.

29.



(a) $v = 0$ when $t = \frac{3}{2}$ sec

(b) $v < 0$ when $0 \le t < 1.5 \Rightarrow$ body moves left; $v > 0$ when $1.5 < t \le 5 \Rightarrow$ body moves right

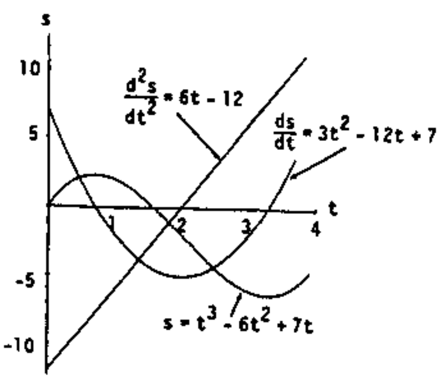
(c) body changes direction at $t = \frac{3}{2}$ sec

(d) body speeds up on $\left(\frac{3}{2}, 5\right]$ and slows down on $\left[0, \frac{3}{2}\right)$

(e) body is moving fastest at $t = 5$ when the speed $= |v(5)| = 7$ units/sec; it is moving slowest at $t = \frac{3}{2}$ when the speed is 0

(f) When $t = 5$ the body is $s = 10$ units from the origin and farthest away.

30.



(a) $v = 0$ when $t = \frac{6 \pm \sqrt{15}}{3}$ sec

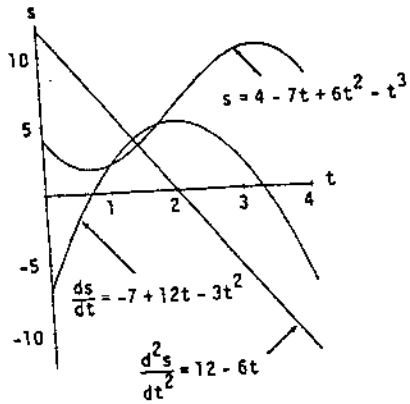
(b) $v < 0$ when $\frac{6 - \sqrt{15}}{3} < t < \frac{6 + \sqrt{15}}{3} \Rightarrow$ body moves left; $v > 0$ when $0 \le t < \frac{6 - \sqrt{15}}{3}$ or $\frac{6 + \sqrt{15}}{3} < t \le 4$ $\Rightarrow$ body moves right

(c) body changes direction at $t = \frac{6 \pm \sqrt{15}}{3}$ sec

(d) body speeds up on $\left(\frac{6 - \sqrt{15}}{3}, 2\right) \cup \left(\frac{6 + \sqrt{15}}{3}, 4\right]$ and slows down on $\left[0, \frac{6 - \sqrt{15}}{3}\right) \cup \left(2, \frac{6 + \sqrt{15}}{3}\right)$.

(e) The body is moving fastest at $t = 0$ and $t = 4$ when it is moving 7 units/sec and slowest at $t = \frac{6 \pm \sqrt{15}}{3}$ sec

(f) When $t = \frac{6 + \sqrt{15}}{3}$ the body is at position $s \approx -6.303$ units and farthest from the origin.

31. 

(a) $v = 0$ when $t = \frac{6 \pm \sqrt{15}}{3}$

(b) $v < 0$ when $0 \le t < \frac{6-\sqrt{15}}{3}$ or $\frac{6+\sqrt{15}}{3} < t \le 4 \Rightarrow$ body is moving left; $v > 0$ when $\frac{6-\sqrt{15}}{3} < t < \frac{6+\sqrt{15}}{3} \Rightarrow$ body is moving right

(c) body changes direction at $t = \frac{6 \pm \sqrt{15}}{3}$ sec

(d) body speeds up on $\left(\frac{6-\sqrt{15}}{3}, 2\right) \cup \left(\frac{6+\sqrt{15}}{3}, 4\right]$ and slows down on $\left[0, \frac{6-\sqrt{15}}{3}\right) \cup \left(2, \frac{6+\sqrt{15}}{3}\right)$

(e) The body is moving fastest at 7 units/sec when $t = 0$ and $t = 4$; it is moving slowest and stationary at $t = \frac{6 \pm \sqrt{15}}{3}$

(f) When $t = \frac{6+\sqrt{15}}{3}$ the position is $s \approx 10.303$ units and the body is farthest from the origin.

32. (a) It takes 135 seconds.

(b) Average speed $= \frac{\Delta F}{\Delta t} = \frac{5-0}{73-0} = \frac{5}{73} \approx 0.068$ furlongs/sec.

(c) Using a symmetric difference quotient, the horse's speed is approximately $\frac{\Delta F}{\Delta t} = \frac{4-2}{59-33} = \frac{2}{26} = \frac{1}{13} \approx 0.077$ furlongs/sec.

(d) The horse is running the fastest during the last furlong (between 9th and 10th furlong markers). This furlong takes only 11 seconds to run, which is the least amount of time for a furlong.

(e) The horse accelerates the fastest during the first furlong (between markers 0 and 1).

2.3 DERIVATIVES OF PRODUCTS, QUOTIENTS, AND NEGATIVE POWERS

1. $y = 6x^2 - 10x - 5x^{-2} \Rightarrow \frac{dy}{dx} = 12x - 10 + 10x^{-3} \Rightarrow \frac{d^2y}{dx^2} = 12 - 0 - 30x^{-4} = 12 - 30x^{-4}$

2. $w = 3z^{-3} - z^{-1} \Rightarrow \frac{dw}{dz} = -9z^{-4} + z^{-2} = -9z^{-4} + \frac{1}{z^2} \Rightarrow \frac{d^2w}{dz^2} = 36z^{-5} - 2z^{-3} = 36z^{-5} - \frac{2}{z^3}$

3. $r = \frac{1}{3}s^{-2} - \frac{5}{2}s^{-1} \Rightarrow \frac{dr}{ds} = -\frac{2}{3}s^{-3} + \frac{5}{2}s^{-2} = \frac{-2}{3s^3} + \frac{5}{2s^2} \Rightarrow \frac{d^2r}{ds^2} = 2s^{-4} - 5s^{-3} = \frac{2}{s^4} - \frac{5}{s^3}$

4. $r = 12\theta^{-1} - 4\theta^{-3} + \theta^{-4} \Rightarrow \frac{dr}{d\theta} = -12\theta^{-2} + 12\theta^{-4} - 4\theta^{-5} = \frac{-12}{\theta^2} + \frac{12}{\theta^4} - \frac{4}{\theta^5} \Rightarrow \frac{d^2r}{d\theta^2} = 24\theta^{-3} - 48\theta^{-5} + 20\theta^{-6}$
$= \frac{24}{\theta^3} - \frac{48}{\theta^5} + \frac{20}{\theta^6}$

5. (a) $y = (3 - x^2)(x^3 - x + 1) \Rightarrow y' = (3 - x^2)\cdot\frac{d}{dx}(x^3 - x + 1) + (x^3 - x + 1)\cdot\frac{d}{dx}(3 - x^2)$
$= (3 - x^2)(3x^2 - 1) + (x^3 - x + 1)(-2x) = -5x^4 + 12x^2 - 2x - 3$
(b) $y = -x^5 + 4x^3 - x^2 - 3x + 3 \Rightarrow y' = -5x^4 + 12x^2 - 2x - 3$

6. $y = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x} + 1\right)$

(a) $y' = (x + x^{-1})\cdot(1 + x^{-2}) + (x - x^{-1} + 1)(1 - x^{-2}) = 2x + 1 - \frac{1}{x^2} + \frac{2}{x^3}$

(b) $y = x^2 + x + \frac{1}{x} - \frac{1}{x^2} \Rightarrow y' = 2x + 1 - \frac{1}{x^2} + \frac{2}{x^3}$

7. $y = \frac{2x+5}{3x-2}$; use the quotient rule: $u = 2x + 5$ and $v = 3x - 2 \Rightarrow u' = 2$ and $v' = 3 \Rightarrow y' = \frac{vu' - uv'}{v^2}$

$= \frac{(3x-2)(2) - (2x+5)(3)}{(3x-2)^2} = \frac{6x - 4 - 6x - 15}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$

8. $g(x) = \frac{x^2 - 4}{x + 0.5}$; use the quotient rule: $u = x^2 - 4$ and $v = x + 0.5 \Rightarrow u' = 2x$ and $v' = 1 \Rightarrow g'(x) = \frac{vu' - uv'}{v^2}$
$= \frac{(x+0.5)(2x) - (x^2 - 4)(1)}{(x+0.5)^2} = \frac{2x^2 + x - x^2 + 4}{(x+0.5)^2} = \frac{x^2 + x + 4}{(x+0.5)^2}$

9. $f(t) = \frac{t^2 - 1}{t^2 + t - 2} \Rightarrow f'(t) = \frac{(t^2 + t - 2)(2t) - (t^2 - 1)(2t + 1)}{(t^2 + t - 2)^2} = \frac{(t-1)(t+2)(2t) - (t-1)(t+1)(2t+1)}{(t-1)^2(t+2)^2}$
$= \frac{(t+2)(2t) - (t+1)(2t+1)}{(t-1)(t+2)^2} = \frac{2t^2 + 4t - 2t^2 - 3t - 1}{(t-1)(t+2)^2} = \frac{t-1}{(t-1)(t+2)^2} = \frac{1}{(t+2)^2}$

10. $v = (1 - t)(1 + t^2)^{-1} = \frac{1 - t}{1 + t^2} \Rightarrow \frac{dv}{dt} = \frac{(1 + t^2)(-1) - (1 - t)(2t)}{(1 + t^2)^2} = \frac{-1 - t^2 - 2t + 2t^2}{(1 + t^2)^2} = \frac{t^2 - 2t - 1}{(1 + t^2)^2}$

11. $f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1} \Rightarrow f'(s) = \frac{(\sqrt{s} + 1)\left(\frac{1}{2\sqrt{s}}\right) - (\sqrt{s} - 1)\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s} + 1)^2} = \frac{(\sqrt{s} + 1) - (\sqrt{s} - 1)}{2\sqrt{s}(\sqrt{s} + 1)^2} = \frac{1}{\sqrt{s}(\sqrt{s} + 1)^2}$
NOTE: $\frac{d}{ds}(\sqrt{s}) = \frac{1}{2\sqrt{s}}$ from Example 1 in Section 2.1

12. $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right) \Rightarrow r' = 2\left(\frac{\sqrt{\theta}(0) - 1\left(\frac{1}{2\sqrt{\theta}}\right)}{\theta} + \frac{1}{2\sqrt{\theta}}\right) = -\frac{1}{\theta^{3/2}} + \frac{1}{\theta^{1/2}}$

13. $y = \frac{1}{(x^2-1)(x^2+x+1)}$; use the quotient rule: $u = 1$ and $v = (x^2-1)(x^2+x+1) \Rightarrow u' = 0$ and

$v' = (x^2-1)(2x+1) + (x^2+x+1)(2x) = 2x^3 + x^2 - 2x - 1 + 2x^3 + 2x^2 + 2x = 4x^3 + 3x^2 - 1$

$\Rightarrow \frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{0 - 1(4x^3 + 3x^2 - 1)}{(x^2-1)^2(x^2+x+1)^2} = \frac{-4x^3 - 3x^2 + 1}{(x^2-1)^2(x^2+x+1)^2}$

14. $y = \frac{(x+1)(x+2)}{(x-1)(x-2)} = \frac{x^2+3x+2}{x^2-3x+2} \Rightarrow y' = \frac{(x^2-3x+2)(2x+3) - (x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2} = \frac{-6x^2+12}{(x-1)^2(x-2)^2}$

$= \frac{-6(x^2-2)}{(x-1)^2(x-2)^2}$

15. $s = \frac{t^2+5t-1}{t^2} = 1 + \frac{5}{t} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2} \Rightarrow \frac{ds}{dt} = 0 - 5t^{-2} + 2t^{-3} = -5t^{-2} + 2t^{-3} \Rightarrow \frac{d^2s}{dt^2} = 10t^{-3} - 6t^{-4}$

16. $r = \frac{(\theta-1)(\theta^2+\theta+1)}{\theta^3} = \frac{\theta^3-1}{\theta^3} = 1 - \frac{1}{\theta^3} = 1 - \theta^{-3} \Rightarrow \frac{dr}{d\theta} = 0 + 3\theta^{-4} = 3\theta^{-4} \Rightarrow \frac{d^2r}{d\theta^2} = -12\theta^{-5}$

17. $w = \left(\frac{1+3z}{3z}\right)(3-z) = \left(\frac{1}{3}z^{-1} + 1\right)(3-z) = z^{-1} - \frac{1}{3} + 3 - z = z^{-1} + \frac{8}{3} - z \Rightarrow \frac{dw}{dz} = -z^{-2} + 0 - 1 = -z^{-2} - 1$

$\Rightarrow \frac{d^2w}{dz^2} = 2z^{-3} - 0 = 2z^{-3}$

18. $p = \left(\frac{q^2+3}{12q}\right)\left(\frac{q^4-1}{q^3}\right) = \frac{q^6 - q^2 + 3q^4 - 3}{12q^4} = \frac{1}{12}q^2 - \frac{1}{12}q^{-2} + \frac{1}{4} - \frac{1}{4}q^{-4} \Rightarrow \frac{dp}{dq} = \frac{1}{6}q + \frac{1}{6}q^{-3} + q^{-5}$

$\Rightarrow \frac{d^2p}{dq^2} = \frac{1}{6} - \frac{1}{2}q^{-4} - 5q^{-6}$

19. $u(0) = 5,\ u'(0) = 3,\ v(0) = -1,\ v'(0) = 2$

(a) $\frac{d}{dx}(uv) = uv' + vu' \Rightarrow \left.\frac{d}{dx}(uv)\right|_{x=0} = u(0)v'(0) + v(0)u'(0) = 5 \cdot 2 + (-1)(3) = 7$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \Rightarrow \left.\frac{d}{dx}\left(\frac{u}{v}\right)\right|_{x=0} = \frac{v(0)u'(0) - u(0)v'(0)}{(v(0))^2} = \frac{(-1)(3) - (5)(2)}{(-1)^2} = -13$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{uv' - vu'}{u^2} \Rightarrow \left.\frac{d}{dx}\left(\frac{v}{u}\right)\right|_{x=0} = \frac{u(0)v'(0) - v(0)u'(0)}{(u(0))^2} = \frac{(5)(2) - (-1)(3)}{(5)^2} = \frac{13}{25}$

(d) $\frac{d}{dx}(7v - 2u) = 7v' - 2u' \Rightarrow \left.\frac{d}{dx}(7v - 2u)\right|_{x=0} = 7v'(0) - 2u'(0) = 7 \cdot 2 - 2(3) = 8$

20. $u(1) = 2,\ u'(1) = 0,\ v(1) = 5,\ v'(1) = -1$

(a) $\frac{d}{dx}(uv)\Big|_{x=1} = u(1)v'(1) + v(1)u'(1) = 2\cdot(-1) + 5\cdot 0 = -2$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=1} = \frac{v(1)u'(1) - u(1)v'(1)}{(v(1))^2} = \frac{5\cdot 0 - 2\cdot(-1)}{(5)^2} = \frac{2}{25}$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=1} = \frac{u(1)v'(1) - v(1)u'(1)}{(u(1))^2} = \frac{2\cdot(-1) - 5\cdot 0}{(2)^2} = -\frac{1}{2}$

(d) $\frac{d}{dx}(7v - 2u)\Big|_{x=1} = 7v'(1) - 2u'(1) = 7\cdot(-1) - 2\cdot 0 = -7$

21. $y = \frac{4x}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} = \frac{4x^2 + 4 - 8x^2}{(x^2+1)^2} = \frac{4(-x^2+1)}{(x^2+1)^2}$. When $x = 0$, $y = 0$ and $y' = \frac{4(0+1)}{1}$ $= 4$, so the tangent to the curve at $(0,0)$ is the line $y = 4x$. When $x = 1$, $y = 2 \Rightarrow y' = 0$, so the tangent to the curve at $(1,2)$ is the line $y = 2$.

22. $y = \frac{8}{x^2+4} \Rightarrow y' = \frac{(x^2+4)(0) - 8(2x)}{(x^2+4)^2} = \frac{-16x}{(x^2+4)^2}$. When $x = 2$, $y = 1$ and $y' = \frac{-16(2)}{(2^2+4)^2} = -\frac{1}{2}$, so the tangent line to the curve at $(2,1)$ has the equation $y - 1 = -\frac{1}{2}(x-2)$, or $y = -\frac{x}{2} + 2$.

23. $y = ax^2 + bx + c$ passes through $(0,0) \Rightarrow 0 = a(0) + b(0) + c \Rightarrow c = 0$; $y = ax^2 + bx$ passes through $(1,2)$ $\Rightarrow 2 = a + b$; $y' = 2ax + b$ and since the curve is tangent to $y = x$ at the origin, its slope is 1 at $x = 0$ $\Rightarrow y' = 1$ when $x = 0 \Rightarrow 1 = 2a(0) + b \Rightarrow b = 1$. Then $a + b = 2 \Rightarrow a = 1$. In summary $a = b = 1$ and $c = 0$ so the curve is $y = x^2 + x$.

24. $y = cx - x^2$ passes through $(1,0) \Rightarrow 0 = c(1) - 1 \Rightarrow c = 1 \Rightarrow$ the curve is $y = x - x^2$. For this curve, $y' = 1 - 2x$ and $x = 1 \Rightarrow y' = -1$. Since $y = x - x^2$ and $y = x^2 + ax + b$ have common tangents at $x = 0$, $y = x^2 + ax + b$ must also have slope -1 at $x = 1$. Thus $y' = 2x + a \Rightarrow -1 = 2\cdot 1 + a \Rightarrow a = -3$ $\Rightarrow y = x^2 - 3x + b$. Since this last curve passes through $(1,0)$, we have $0 = 1 - 3 + b \Rightarrow b = 2$. In summary, $a = -3$, $b = 2$ and $c = 1$ so the curves are $y = x^2 - 3x + 2$ and $y = x - x^2$.

25. Let c be a constant $\Rightarrow \frac{dc}{dx} = 0 \Rightarrow \frac{d}{dx}(u\cdot c) = u\cdot\frac{dc}{dx} + c\cdot\frac{du}{dx} = u\cdot 0 + c\,\frac{du}{dx} = c\,\frac{du}{dx}$. Thus when one of the functions is a constant, the Product Rule is just the Constant Multiple Rule $\Rightarrow$ the Constant Multiple Rule is a special case of the Product Rule.

26. (a) We use the Quotient rule to derive the Reciprocal Rule (with $u = 1$): $\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{v\cdot 0 - 1\cdot\frac{dv}{dx}}{v^2} = \frac{-1\cdot\frac{dv}{dx}}{v^2}$ $= -\frac{1}{v^2}\cdot\frac{dv}{dx}$.

(b) Now, using the Reciprocal Rule and the Product Rule, we'll derive the Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx}\left(u \cdot \frac{1}{v}\right)$

$= u \cdot \frac{d}{dx}\left(\frac{1}{v}\right) + \frac{1}{v} \cdot \frac{du}{dx}$ (Product Rule) $= u \cdot \left(\frac{-1}{v^2}\right)\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx}$ (Reciprocal Rule) $\Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{-u\frac{dv}{dx} + v\frac{du}{dx}}{v^2}$

$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, the Quotient Rule.

27. (a) $\frac{d}{dx}(uvw) = \frac{d}{dx}((uv)\cdot w) = (uv)\frac{dw}{dx} + w \cdot \frac{d}{dx}(uv) = uv\frac{dw}{dx} + w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) = uv\frac{dw}{dx} + wu\frac{dv}{dx} + wv\frac{du}{dx}$

$= uvw' + uv'w + u'vw$

(b) $\frac{d}{dx}(u_1u_2u_3u_4) = \frac{d}{dx}((u_1u_2u_3)u_4) = (u_1u_2u_3)\frac{du_4}{dx} + u_4\frac{d}{dx}(u_1u_2u_3) \Rightarrow \frac{d}{dx}(u_1u_2u_3u_4)$

$= u_1u_2u_3\frac{du_4}{dx} + u_4\left(u_1u_2\frac{du_3}{dx} + u_3u_1\frac{du_2}{dx} + u_3u_2\frac{du_1}{dx}\right)$ (using (a) above)

$\Rightarrow \frac{d}{dx}(u_1u_2u_3u_4) = u_1u_2u_3\frac{du_4}{dx} + u_1u_2u_4\frac{du_3}{dx} + u_1u_3u_4\frac{du_2}{dx} + u_2u_3u_4\frac{du_1}{dx}$

$= u_1u_2u_3u_4' + u_1u_2u_3'u_4 + u_1u_2'u_3u_4 + u_1'u_2u_3u_4$

(c) Generalizing (a) and (b) above, $\frac{d}{dx}(u_1\cdots u_n) = u_1u_2\cdots u_{n-1}u_n' + u_1u_2\cdots u_{n-2}u_{n-1}'u_n + \ldots + u_1'u_2\cdots u_n$

28. In this problem we don't know the Power Rule works with fractional powers so we can't use it. Remember $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ (from Example 1 in Section 2.1)

(a) $\frac{d}{dx}(x^{3/2}) = \frac{d}{dx}(x \cdot x^{1/2}) = x \cdot \frac{d}{dx}(\sqrt{x}) + \sqrt{x}\frac{d}{dx}(x) = x \cdot \frac{1}{2\sqrt{x}} + \sqrt{x} \cdot 1 = \frac{\sqrt{x}}{2} + \sqrt{x} = \frac{3\sqrt{x}}{2} = \frac{3}{2}x^{1/2}$

(b) $\frac{d}{dx}(x^{5/2}) = \frac{d}{dx}(x^2 \cdot x^{1/2}) = x^2\frac{d}{dx}(\sqrt{x}) + \sqrt{x}\frac{d}{dx}(x^2) = x^2 \cdot \left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x} \cdot 2x = \frac{1}{2}x^{3/2} + 2x^{3/2} = \frac{5}{2}x^{3/2}$

(c) $\frac{d}{dx}(x^{7/2}) = \frac{d}{dx}(x^3 \cdot x^{1/2}) = x^3\frac{d}{dx}(\sqrt{x}) + \sqrt{x}\frac{d}{dx}(x^3) = x^3 \cdot \left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x} \cdot 3x^2 = \frac{1}{2}x^{5/2} + 3x^{5/2} = \frac{7}{2}x^{5/2}$

(d) We have $\frac{d}{dx}(x^{3/2}) = \frac{3}{2}x^{1/2}$, $\frac{d}{dx}(x^{5/2}) = \frac{5}{2}x^{3/2}$, $\frac{d}{dx}(x^{7/2}) = \frac{7}{2}x^{5/2}$ so it appears that $\frac{d}{dx}(x^{n/2}) = \frac{n}{2}x^{(n/2)-1}$ whenever n is an odd positive integer ≥ 3.

29. $P = \frac{nRT}{V - nb} - \frac{an^2}{V^2}$. We are holding T constant, and a, b, n, R are also constant so their derivatives are zero

$\Rightarrow \frac{dP}{dV} = \frac{(V - nb)\cdot 0 - (nRT)(1)}{(V - nb)^2} - \frac{V^2(0) - (an^2)(2V)}{(V^2)^2} = \frac{-nRT}{(V - nb)^2} + \frac{2an^2}{V^3}$

30. $A(q) = \frac{km}{q} + cm + \frac{hq}{2} = (km)q^{-1} + cm + \left(\frac{h}{2}\right)q$

$\frac{dA}{dq} = -(km)q^{-2} + \frac{h}{2} = -\frac{km}{q^2} + \frac{h}{2}$

$\frac{d^2A}{dq^2} = 2(km)q^{-3} = \frac{2km}{q^3}$

2.4 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1. $y = -10x + 3\cos x \Rightarrow \frac{dy}{dx} = -10 + 3\frac{d}{dx}(\cos x) = -10 - 3\sin x$

2. $y = \frac{3}{x} + 5\sin x \Rightarrow \frac{dy}{dx} = \frac{-3}{x^2} + 5\frac{d}{dx}(\sin x) = \frac{-3}{x^2} + 5\cos x$

3. $y = \csc x - 4\sqrt{x} + 7 \Rightarrow \frac{dy}{dx} = -\csc x \cot x - \frac{4}{2\sqrt{x}} + 0 = -\csc x \cot x - \frac{2}{\sqrt{x}}$

4. $y = x^2 \cot x - \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = x^2 \frac{d}{dx}(\cot x) + \cot x \cdot \frac{d}{dx}(x^2) + \frac{2}{x^3} = -x^2\csc^2 x + (\cot x)(2x) + \frac{2}{x^3}$
$= -x^2\csc^2 x + 2x\cot x + \frac{2}{x^3}$

5. $y = (\sec x + \tan x)(\sec x - \tan x) \Rightarrow \frac{dy}{dx} = (\sec x + \tan x)\frac{d}{dx}(\sec x - \tan x) + (\sec x - \tan x)\frac{d}{dx}(\sec x + \tan x)$
$= (\sec x + \tan x)(\sec x \tan x - \sec^2 x) + (\sec x - \tan x)(\sec x \tan x + \sec^2 x)$
$= (\sec^2 x \tan x + \sec x \tan^2 x - \sec^3 x - \sec^2 x \tan x) + (\sec^2 x \tan x - \sec x \tan^2 x + \sec^3 x - \tan x \sec^2 x) = 0.$
$\left(\text{Note also that } y = \sec^2 x - \tan^2 x = (\tan^2 x + 1) - \tan^2 x = 1 \Rightarrow \frac{dy}{dx} = 0.\right)$

6. $y = (\sin x + \cos x)\sec x \Rightarrow \frac{dy}{dx} = (\sin x + \cos x)\frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}(\sin x + \cos x)$
$= (\sin x + \cos x)(\sec x \tan x) + (\sec x)(\cos x - \sin x) = \frac{(\sin x + \cos x)\sin x}{\cos^2 x} + \frac{\cos x - \sin x}{\cos x}$
$= \frac{\sin^2 x + \cos x \sin x + \cos^2 x - \cos x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
$\left(\text{Note also that } y = \sin x \sec x + \cos x \sec x = \tan x + 1 \Rightarrow \frac{dy}{dx} = \sec^2 x.\right)$

7. $y = \frac{\cot x}{1 + \cot x} \Rightarrow \frac{dy}{dx} = \frac{(1 + \cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1 + \cot x)}{(1 + \cot x)^2} = \frac{(1 + \cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1 + \cot x)^2}$
$= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1 + \cot x)^2} = \frac{-\csc^2 x}{(1 + \cot x)^2}$

8. $y = \frac{\cos x}{1 + \sin x} \Rightarrow \frac{dy}{dx} = \frac{(1 + \sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2} = \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2}$
$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} = \frac{-\sin x - 1}{(1 + \sin x)^2} = \frac{-(1 + \sin x)}{(1 + \sin x)^2} = \frac{-1}{1 + \sin x}$

9. $y = \frac{4}{\cos x} + \frac{1}{\tan x} = 4\sec x + \cot x \Rightarrow \frac{dy}{dx} = 4\sec x \tan x - \csc^2 x$

10. $y = \frac{\cos x}{x} + \frac{x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{x(-\sin x) - (\cos x)(1)}{x^2} + \frac{(\cos x)(1) - x(-\sin x)}{\cos^2 x} = \frac{-x \sin x - \cos x}{x^2} + \frac{\cos x + x \sin x}{\cos^2 x}$

11. $y = x^2 \sin x + 2x \cos x - 2 \sin x \Rightarrow \frac{dy}{dx} = \left(x^2 \cos x + (\sin x)(2x)\right) + \left((2x)(-\sin x) + (\cos x)(2)\right) - 2 \cos x$
$= x^2 \cos x + 2x \sin x - 2x \sin x + 2 \cos x - 2 \cos x = x^2 \cos x$

12. $y = x^2 \cos x - 2x \sin x - 2 \cos x \Rightarrow \frac{dy}{dx} = \left(x^2(-\sin x) + (\cos x)(2x)\right) - \left(2x \cos x + (\sin x)(2)\right) - 2(-\sin x)$
$= -x^2 \sin x + 2x \cos x - 2x \cos x - 2 \sin x + 2 \sin x = -x^2 \sin x$

13. $s = \tan t - t \Rightarrow \frac{ds}{dt} = \frac{d}{dt}(\tan t) - 1 = \sec^2 t - 1$

14. $s = t^2 - \sec t + 1 \Rightarrow \frac{ds}{dt} = 2t - \frac{d}{dt}(\sec t) = 2t - \sec t \tan t$

15. $s = \frac{1 + \csc t}{1 - \csc t} \Rightarrow \frac{ds}{dt} = \frac{(1 - \csc t)(-\csc t \cot t) - (1 + \csc t)(\csc t \cot t)}{(1 - \csc t)^2}$
$= \frac{-\csc t \cot t + \csc^2 t \cot t - \csc t \cot t - \csc^2 t \cot t}{(1 - \csc t)^2} = \frac{-2 \csc t \cot t}{(1 - \csc t)^2}$

16. $s = \frac{\sin t}{1 - \cos t} \Rightarrow \frac{ds}{dt} = \frac{(1 - \cos t)(\cos t) - (\sin t)(\sin t)}{(1 - \cos t)^2} = \frac{\cos t - \cos^2 t - \sin^2 t}{(1 - \cos t)^2} = \frac{\cos t - 1}{(1 - \cos t)^2} = -\frac{1}{1 - \cos t}$
$= \frac{1}{\cos t - 1}$

17. $r = 4 - \theta^2 \sin \theta \Rightarrow \frac{dr}{d\theta} = -\left(\theta^2 \frac{d}{d\theta}(\sin \theta) + (\sin \theta)(2\theta)\right) = -\left(\theta^2 \cos \theta + 2\theta \sin \theta\right) = -\theta(\theta \cos \theta + 2 \sin \theta)$

18. $r = \theta \sin \theta + \cos \theta \Rightarrow \frac{dr}{d\theta} = (\theta \cos \theta + (\sin \theta)(1)) - \sin \theta = \theta \cos \theta$

19. $r = \sec \theta \csc \theta \Rightarrow \frac{dr}{d\theta} = (\sec \theta)(-\csc \theta \cot \theta) + (\csc \theta)(\sec \theta \tan \theta)$
$= \left(\frac{-1}{\cos \theta}\right)\left(\frac{1}{\sin \theta}\right)\left(\frac{\cos \theta}{\sin \theta}\right) + \left(\frac{1}{\sin \theta}\right)\left(\frac{1}{\cos \theta}\right)\left(\frac{\sin \theta}{\cos \theta}\right) = \frac{-1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} = \sec^2 \theta - \csc^2 \theta$

20. $r = (1 + \sec \theta) \sin \theta \Rightarrow \frac{dr}{d\theta} = (1 + \sec \theta) \cos \theta + (\sin \theta)(\sec \theta \tan \theta) = (\cos \theta + 1) + \tan^2 \theta = \cos \theta + \sec^2 \theta$

21. $p = 5 + \frac{1}{\cot q} = 5 + \tan q \Rightarrow \frac{dp}{dq} = \sec^2 q$

22. $p = (1 + \csc q) \cos q \Rightarrow \frac{dp}{dq} = (1 + \csc q)(-\sin q) + (\cos q)(-\csc q \cot q) = (-\sin q - 1) - \cot^2 q = -\sin q - \csc^2 q$

23. $p = \frac{\sin q + \cos q}{\cos q} \Rightarrow \frac{dp}{dq} = \frac{(\cos q)(\cos q - \sin q) - (\sin q + \cos q)(-\sin q)}{\cos^2 q}$
$= \frac{\cos^2 q - \cos q \sin q + \sin^2 q + \cos q \sin q}{\cos^2 q} = \frac{1}{\cos^2 q} = \sec^2 q$

24. $p = \frac{\tan q}{1 + \tan q} \Rightarrow \frac{dp}{dq} = \frac{(1 + \tan q)(\sec^2 q) - (\tan q)(\sec^2 q)}{(1 + \tan q)^2} = \frac{\sec^2 q + \tan q \sec^2 q - \tan q \sec^2 q}{(1 + \tan q)^2} = \frac{\sec^2 q}{(1 + \tan q)^2}$

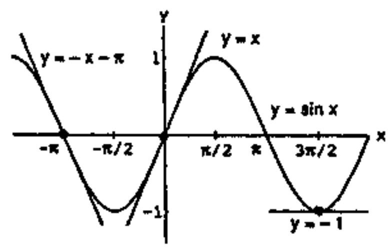
25. (a) $y = \csc x \Rightarrow y' = -\csc x \cot x \Rightarrow y'' = -\left((\csc x)\left(-\csc^2 x\right) + (\cot x)(-\csc x \cot x)\right) = \csc^3 x + \csc x \cot^2 x$
$= (\csc x)\left(\csc^2 x + \cot^2 x\right) = (\csc x)\left(\csc^2 x + \csc^2 x - 1\right) = 2 \csc^3 x - \csc x$

(b) $y = \sec x \Rightarrow y' = \sec x \tan x \Rightarrow y'' = (\sec x)\left(\sec^2 x\right) + (\tan x)(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$
$= (\sec x)\left(\sec^2 x + \tan^2 x\right) = (\sec x)\left(\sec^2 x + \sec^2 x - 1\right) = 2 \sec^3 x - \sec x$

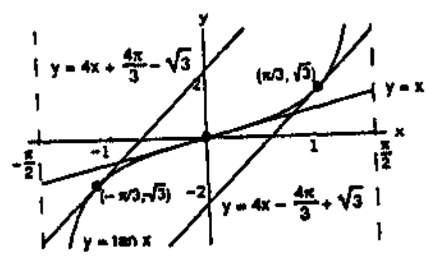
26. (a) $y = -2 \sin x \Rightarrow y' = -2 \cos x \Rightarrow y'' = -2(-\sin x) = 2 \sin x \Rightarrow y''' = 2 \cos x \Rightarrow y^{(4)} = -2 \sin x$

(b) $y = 9 \cos x \Rightarrow y' = -9 \sin x \Rightarrow y'' = -9 \cos x \Rightarrow y''' = -9(-\sin x) = 9 \sin x \Rightarrow y^{(4)} = 9 \cos x$

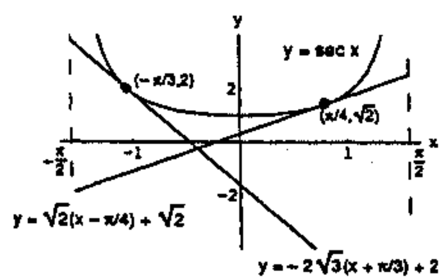
27. $y = \sin x \Rightarrow y' = \cos x \Rightarrow$ slope of tangent at $x = -\pi$ is $y'(-\pi) = \cos(-\pi) = -1$; slope of tangent at $x = 0$ is $y'(0) = \cos(0) = 1$; and slope of tangent at $x = \frac{3\pi}{2}$ is $y'\left(\frac{3\pi}{2}\right) = \cos \frac{3\pi}{2} = 0$. The tangent at $(-\pi, 0)$ is $y - 0 = -1(x + \pi)$, or $y = -x - \pi$; the tangent at $(0,0)$ is $y - 0 = 1(x - 0)$, or $y = x$; and the tangent at $\left(\frac{3\pi}{2}, -1\right)$ is $y = -1$.



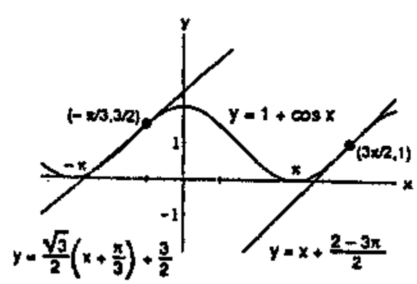
28. $y = \tan x \Rightarrow y' = \sec^2 x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $\sec^2\left(-\frac{\pi}{3}\right) = 4$; slope of tangent at $x = 0$ is $\sec^2(0) = 1$; and slope of tangent at $x = \frac{\pi}{3}$ is $\sec^2\left(\frac{\pi}{3}\right) = 4$. The tangent at $\left(-\frac{\pi}{3}, \tan\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, -\sqrt{3}\right)$ is $y + \sqrt{3} = 4\left(x + \frac{\pi}{3}\right)$; the tangent at $(0,0)$ is $y = x$; and the tangent at $\left(\frac{\pi}{3}, \tan\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \sqrt{3}\right)$ is $y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right)$.



29. $y = \sec x \Rightarrow y' = \sec x \tan x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $\sec\left(-\frac{\pi}{3}\right)\tan\left(-\frac{\pi}{3}\right) = -2\sqrt{3}$; slope of tangent at $x = \frac{\pi}{4}$ is $\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \sqrt{2}$. The tangent at the point $\left(-\frac{\pi}{3}, \sec\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, 2\right)$ is $y - 2 = -2\sqrt{3}\left(x + \frac{\pi}{3}\right)$; the tangent at the point $\left(\frac{\pi}{4}, \sec\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, \sqrt{2}\right)$ is $y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$.



30. $y = 1 + \cos x \Rightarrow y' = -\sin x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $-\sin\left(-\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$; slope of tangent at $x = \frac{3\pi}{2}$ is $-\sin\left(\frac{3\pi}{2}\right) = 1$. The tangent at the point $\left(-\frac{\pi}{3}, 1 + \cos\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, \frac{3}{2}\right)$ is $y - \frac{3}{2} = \frac{\sqrt{3}}{2}\left(x + \frac{\pi}{3}\right)$; the tangent at the point



$\left(\frac{3\pi}{2}, 1+\cos\left(\frac{3\pi}{2}\right)\right)=\left(\frac{3\pi}{2}, 1\right)$ is $y-1=x-\frac{3\pi}{2}$

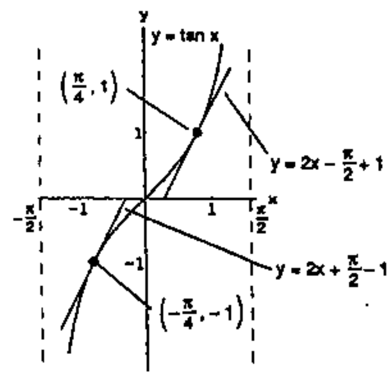
31. Yes, $y=x+\sin x \Rightarrow y'=1+\cos x$; horizontal tangent occurs where $1+\cos x=0 \Rightarrow \cos x=-1$ $\Rightarrow x=\pi$

32. No, $y=2x+\sin x \Rightarrow y'=2+\cos x$; horizontal tangent occurs where $2+\cos x=0 \Rightarrow \cos x=-2$. But there are no x-values for which $\cos x=-2$.

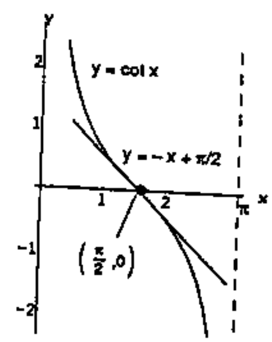
33. No, $y=x-\cot x \Rightarrow y'=1+\csc^2 x$; horizontal tangent occurs where $1+\csc^2 x=0 \Rightarrow \csc^2 x=-1$. But there are no x-values for which $\csc^2 x=-1$.

34. Yes, $y=x+2\cos x \Rightarrow y'=1-2\sin x$; horizontal tangent occurs where $1-2\sin x=0 \Rightarrow 1=2\sin x$ $\Rightarrow \frac{1}{2}=\sin x \Rightarrow x=\frac{\pi}{6}$ or $x=\frac{5\pi}{6}$

35. We want all points on the curve where the tangent line has slope 2. Thus, $y=\tan x \Rightarrow y'=\sec^2 x$ so that $y'=2 \Rightarrow \sec^2 x=2 \Rightarrow \sec x=\pm\sqrt{2}$ $\Rightarrow x=\pm\frac{\pi}{4}$. Then the tangent line at $\left(\frac{\pi}{4}, 1\right)$ has equation $y-1=2\left(x-\frac{\pi}{4}\right)$; the tangent line at $\left(-\frac{\pi}{4}, -1\right)$ has equation $y+1=2\left(x+\frac{\pi}{4}\right)$.



36. We want all points on the curve $y=\cot x$ where the tangent line has slope -1. Thus $y=\cot x$ $\Rightarrow y'=-\csc^2 x$ so that $y'=-1 \Rightarrow -\csc^2 x=-1$ $\Rightarrow \csc^2 x=1 \Rightarrow \csc x=\pm 1 \Rightarrow x=\frac{\pi}{2}$. The tangent line at $\left(\frac{\pi}{2}, 0\right)$ is $y=-x+\frac{\pi}{2}$.



37. $y=4+\cot x-2\csc x \Rightarrow y'=-\csc^2 x+2\csc x\cot x=-\left(\frac{1}{\sin x}\right)\left(\frac{1-2\cos x}{\sin x}\right)$

(a) When $x=\frac{\pi}{2}$, then $y'=-1$; the tangent line is $y=-x+\frac{\pi}{2}+2$.

(b) To find the location of the horizontal tangent set $y'=0 \Rightarrow 1-2\cos x=0 \Rightarrow x=\frac{\pi}{3}$ radians. When $x=\frac{\pi}{3}$, then $y=4-\sqrt{3}$ is the horizontal tangent.

38. $y=1+\sqrt{2}\csc x+\cot x \Rightarrow y'=-\sqrt{2}\csc x\cot x-\csc^2 x=-\left(\frac{1}{\sin x}\right)\left(\frac{\sqrt{2}\cos x+1}{\sin x}\right)$

(a) If $x=\frac{\pi}{4}$, then $y'=-4$; the tangent line is $y=-4x+\pi+4$.

(b) To find the location of the horizontal tangent set $y' = 0 \Rightarrow \sqrt{2}\cos x + 1 = 0 \Rightarrow x = \frac{3\pi}{4}$ radians. When $x = \frac{3\pi}{4}$, then $y = 2$ is the horizontal tangent.

39. $s = 2 - 2\sin t \Rightarrow v = \frac{ds}{dt} = -2\cos t \Rightarrow a = \frac{dv}{dt} = 2\sin t \Rightarrow j = \frac{da}{dt} = 2\cos t$. Therefore, velocity $= v\left(\frac{\pi}{4}\right)$ $= -\sqrt{2}$ m/sec; speed $= \left|v\left(\frac{\pi}{4}\right)\right| = \sqrt{2}$ m/sec; acceleration $= a\left(\frac{\pi}{4}\right) = \sqrt{2}$ m/sec^2; jerk $= j\left(\frac{\pi}{4}\right) = \sqrt{2}$ m/sec^3.

40. $s = \sin t + \cos t \Rightarrow v = \frac{ds}{dt} = \cos t - \sin t \Rightarrow a = \frac{dv}{dt} = -\sin t - \cos t \Rightarrow j = \frac{da}{dt} = -\cos t + \sin t$. Therefore velocity $= v\left(\frac{\pi}{4}\right) = 0$ m/sec; speed $= \left|v\left(\frac{\pi}{4}\right)\right| = 0$ m/sec; acceleration $= a\left(\frac{\pi}{4}\right) = -\sqrt{2}$ m/sec^2; jerk $= j\left(\frac{\pi}{4}\right) = 0$ m/sec^3.

41. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{\sin^2 3x}{x^2} = \lim\limits_{x\to 0} 9\left(\frac{\sin 3x}{3x}\right)\left(\frac{\sin 3x}{3x}\right) = 9$ so that f is continuous at $x = 0 \Rightarrow \lim\limits_{x\to 0} f(x) = f(0)$ $\Rightarrow c = 9$.

42. $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-} (x + b) = b$ and $\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} \cos x = 1$ so that g is continuous at $x = 0 \Rightarrow \lim\limits_{x\to 0^-} g(x)$ $= \lim\limits_{x\to 0^+} g(x) \Rightarrow b = 1$. Now g is not differentiable at $x = 0$: At $x = 0$, the left-hand derivative is $\frac{d}{dx}(x+b)\Big|_{x=0} = 1$, but the right-hand derivative is $\frac{d}{dx}(\cos x)\Big|_{x=0} = -\sin 0 = 0$. The left- and right-hand derivatives can never agree at $x = 0$, so g is not differentiable at $x = 0$ for any value of b (including $b = 1$).

43. $\frac{d^{999}}{dx^{999}}(\cos x) = \sin x$ because $\frac{d^4}{dx^4}(\cos x) = \cos x \Rightarrow$ the derivative of cos x any number of times that is a multiple of 4 is cos x. Thus, dividing 999 by 4 gives $999 = 249 \cdot 4 + 3 \Rightarrow \frac{d^{999}}{dx^{999}}(\cos x)$ $= \frac{d^3}{dx^3}\left[\frac{d^{249\cdot 4}}{dx^{249\cdot 4}}(\cos x)\right] = \frac{d^3}{dx^3}(\cos x) = \sin x$.

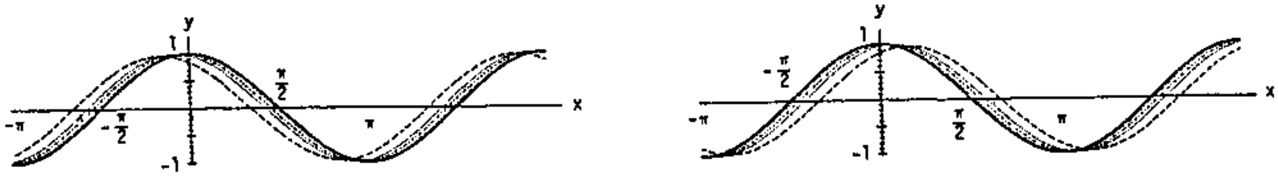
44. (a) $y = \sec x = \frac{1}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x$ $\Rightarrow \frac{d}{dx}(\sec x) = \sec x \tan x$

(b) $y = \csc x = \frac{1}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = \frac{-\cos x}{\sin^2 x} = \left(\frac{-1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = -\csc x \cot x$ $\Rightarrow \frac{d}{dx}(\csc x) = -\csc x \cot x$

(c) $y = \cot x = \frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$

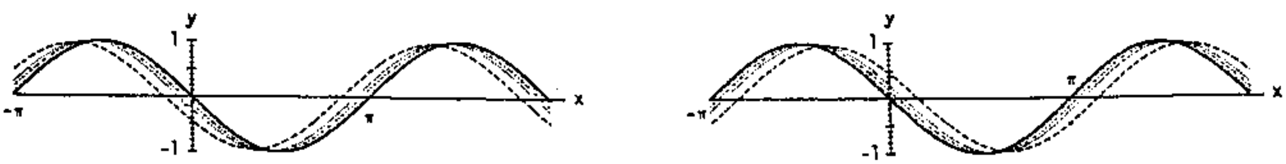
$\Rightarrow \frac{d}{dx}(\cot x) = -\csc^2 x$

45.



As h takes on the values of 1, 0.5, 0.3 and 0.1 the corresponding dashed curves of $y = \frac{\sin(x+h) - \sin x}{h}$ get closer and closer to the black curve $y = \cos x$ because $\frac{d}{dx}(\sin x) = \lim\limits_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x$. The same is true as h takes on the values of -1, -0.5, -0.3 and -0.1.

46.

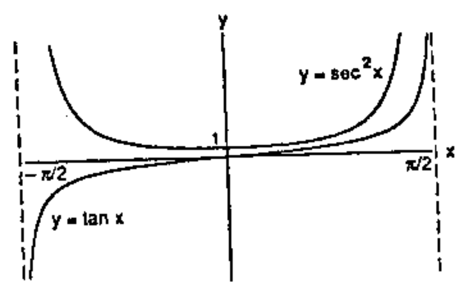


As h takes on the values of 1, 0.5, 0.3, and 0.1 the corresponding dashed curves of $y = \frac{\cos(x+h) - \cos x}{h}$ get closer and closer to the black curve $y = -\sin x$ because $\frac{d}{dx}(\cos x) = \lim\limits_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = -\sin x$. The same is true as h takes on the values of -1, -0.5, -0.3, and -0.1.

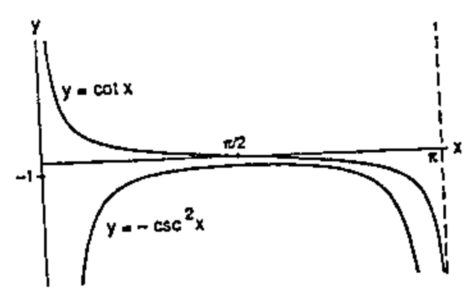
47. This is a grapher exercise. Compare your graphs with Exercises 45 and 46.

48. $\lim\limits_{h \to 0} \frac{|0+h| - |0-h|}{2h} = \lim\limits_{x \to 0} \frac{|h| - |h|}{2h} = \lim\limits_{h \to 0} 0 = 0 \Rightarrow$ the limits of the centered difference quotient exists even though the derivative of $f(x) = |x|$ does not exist at $x = 0$.

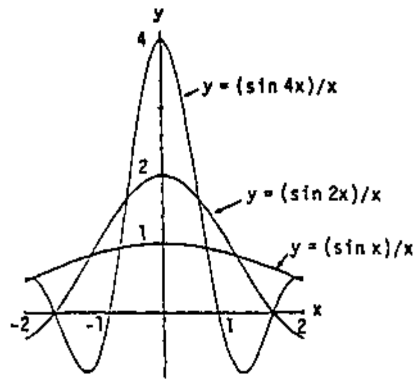
49. $y = \tan x \Rightarrow y' = \sec^2 x$, so the smallest value $y' = \sec^2 x$ takes on is $y' = 1$ when $x = 0$; y' has no maximum value since $\sec^2 x$ has no largest value on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; y' is never negative since $\sec^2 x \geq 1$.



50. $y = \cot x \Rightarrow y' = -\csc^2 x$ so y' has no smallest value since $-\csc^2 x$ has no minimum value on $(0, \pi)$; the largest value of y' is -1, when $x = \frac{\pi}{2}$; the slope is never positive since the largest value $y' = -\csc x$ takes on is the negative value -1.



51. $y = \frac{\sin x}{x}$ appears to cross the y-axis at $y = 1$, since $\lim\limits_{x\to0} \frac{\sin x}{x} = 1$; $y = \frac{\sin 2x}{x}$ appears to cross the y-axis at $y = 2$, since $\lim\limits_{x\to0} \frac{\sin 2x}{x} = 2$; $y = \frac{\sin 4x}{x}$ appears to cross the y-axis at $y = 4$, since $\lim\limits_{x\to0} \frac{\sin 4x}{x} = 4$. However, none of these graphs actually cross the y-axis since $x = 0$ is not in the domain of the functions. Also, $\lim\limits_{x\to0} \frac{\sin 5x}{x} = 5$, $\lim\limits_{x\to0} \frac{\sin(-3x)}{x} = -3$, and $\lim\limits_{x\to0} \frac{\sin kx}{x} = k \Rightarrow$ the graphs of $y = \frac{\sin 5x}{x}$, $y = \frac{\sin(-3x)}{x}$, and $y = \frac{\sin kx}{x}$ approach 5, -3, and k, respectively, as $x \to 0$. However, the graphs do not actually cross the y-axis.



52. (a)

h	$\frac{\sin h}{h}$	$\left(\frac{\sin h}{h}\right)\left(\frac{180}{\pi}\right)$
1	.017453283	.999999492
0.01	.017453292	1
0.001	.017453292	1
0.0001	.017453292	1

$$\lim_{h\to0} \frac{\sin h^\circ}{h} = \lim_{x\to0} \frac{\sin\left(h\cdot\frac{\pi}{180}\right)}{h} = \lim_{h\to0} \frac{\frac{\pi}{180}\sin\left(h\cdot\frac{\pi}{180}\right)}{\frac{\pi}{180}\cdot h} = \lim_{\theta\to0} \frac{\frac{\pi}{180}\sin\theta}{\theta} = \frac{\pi}{180} \qquad \left(\theta = h\cdot\frac{\pi}{180}\right)$$

(converting to radians)

(b)

h	$\frac{\cos h - 1}{h}$
1	-0.0001523
0.01	-0.0000015
0.001	-0.0000001
0.0001	0

$\lim\limits_{h\to0} \frac{\cos h - 1}{h} = 0$, whether h is measured in degrees or radians.

(c) In degrees, $\frac{d}{dx}(\sin x) = \lim\limits_{h\to0} \frac{\sin(x+h) - \sin x}{h} = \lim\limits_{h\to0} \frac{(\sin x\cos h + \cos x\sin h) - \sin x}{h}$

$$= \lim_{h\to0} \left(\sin x\cdot\frac{\cos h - 1}{h}\right) + \lim_{h\to0} \left(\cos x\cdot\frac{\sin h}{h}\right) = (\sin x)\cdot\lim_{h\to0} \left(\frac{\cos h - 1}{h}\right) + (\cos x)\cdot\lim_{h\to0} \left(\frac{\sin h}{h}\right)$$

$$= (\sin x)(0) + (\cos x)\left(\frac{\pi}{180}\right) = \frac{\pi}{180}\cos x$$

(d) In degrees, $\frac{d}{dx}(\cos x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$

$= \lim_{h\to 0} \frac{(\cos x)(\cos h - 1) - \sin x \sin h}{h} = \lim_{h\to 0}\left(\cos x \cdot \frac{\cos h - 1}{h}\right) - \lim_{h\to 0}\left(\sin x \cdot \frac{\sin h}{h}\right)$

$= (\cos x) \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) - (\sin x) \lim_{h\to 0}\left(\frac{\sin h}{h}\right) = (\cos x)(0) - (\sin x)\left(\frac{\pi}{180}\right) = -\frac{\pi}{180}\sin x$

(e) $\frac{d^2}{dx^2}(\sin x) = \frac{d}{dx}\left(\frac{\pi}{180}\cos x\right) = -\left(\frac{\pi}{180}\right)^2 \sin x$; $\frac{d^3}{dx^3}(\sin x) = \frac{d}{dx}\left(-\left(\frac{\pi}{180}\right)^2 \sin x\right) = -\left(\frac{\pi}{180}\right)^3 \cos x$;

$\frac{d^2}{dx^2}(\cos x) = \frac{d}{dx}\left(-\frac{\pi}{180}\sin x\right) = -\left(\frac{\pi}{180}\right)^2 \cos x$; $\frac{d^3}{dx^3}(\cos x) = \frac{d}{dx}\left(-\left(\frac{\pi}{180}\right)^2 \cos x\right) = \left(\frac{\pi}{180}\right)^3 \sin x$

2.5 THE CHAIN RULE

1. $f(u) = 6u - 9 \Rightarrow f'(u) = 6 \Rightarrow f'(g(x)) = 6$; $g(x) = \frac{1}{2}x^4 \Rightarrow g'(x) = 2x^3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$ $= 6 \cdot 2x^3 = 12x^3$

2. $f(u) = 2u^3 \Rightarrow f'(u) = 6u^2 \Rightarrow f'(g(x)) = 6(8x-1)^2$; $g(x) = 8x - 1 \Rightarrow g'(x) = 8$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$ $= 6(8x-1)^2 \cdot 8 = 48(8x-1)^2$

3. $f(u) = \sin u \Rightarrow f'(u) = \cos u \Rightarrow f'(g(x)) = \cos(3x+1)$; $g(x) = 3x + 1 \Rightarrow g'(x) = 3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$ $= (\cos(3x+1))(3) = 3\cos(3x+1)$

4. $f(u) = \cos u \Rightarrow f'(u) = -\sin u \Rightarrow f'(g(x)) = -\sin(\sin x)$; $g(x) = \sin x \Rightarrow g'(x) = \cos x$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = -(\sin(\sin x))\cos x$

5. $f(u) = \tan u \Rightarrow f'(u) = \sec^2 u \Rightarrow f'(g(x)) = \sec^2(10x - 5)$; $g(x) = 10x - 5 \Rightarrow g'(x) = 10$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = \left(\sec^2(10x-5)\right)(10) = 10\sec^2(10x-5)$

6. $f(u) = -\sec u \Rightarrow f'(u) = -\sec u \tan u \Rightarrow f'(g(x)) = -\sec(x^2+7x)\tan(x^2+7x)$; $g(x) = x^2 + 7x$ $\Rightarrow g'(x) = 2x + 7$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = -(2x+7)\sec(x^2+7x)\tan(x^2+7x)$

7. With $u = (4 - 3x)$, $y = u^9$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 9u^8 \cdot (-3) = -27(4-3x)^8$

8. With $u = \left(1 - \frac{x}{7}\right)$, $y = u^{-7}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -7u^{-8} \cdot \left(-\frac{1}{7}\right) = \left(1 - \frac{x}{7}\right)^{-8}$

9. With $u = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)$, $y = u^4$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4u^3 \cdot \left(\frac{x}{4} + 1 + \frac{1}{x^2}\right) = 4\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3 \left(\frac{x}{4} + 1 + \frac{1}{x^2}\right)$

10. With $u = \tan x$, $y = \sec u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec u \tan u)(\sec^2 x) = (\sec(\tan x)\tan(\tan x))\sec^2 x$

11. With $u = \pi - \frac{1}{x}$, $y = \cot u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-\csc^2 u)\left(\frac{1}{x^2}\right) = -\frac{1}{x^2}\csc^2\left(\pi - \frac{1}{x}\right)$

12. With $u = \sin x$, $y = u^3$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2 \cos x = 3(\sin^2 x)(\cos x)$

13. $q = \sqrt{2r - r^2} = (2r - r^2)^{1/2} \Rightarrow \frac{dq}{dr} = \frac{1}{2}(2r - r^2)^{-1/2} \cdot \frac{d}{dr}(2r - r^2) = \frac{1}{2}(2r - r^2)^{-1/2}(2 - 2r) = \frac{1 - r}{\sqrt{2r - r^2}}$

14. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right) \Rightarrow \frac{ds}{dt} = \cos\left(\frac{3\pi t}{2}\right) \cdot \frac{d}{dt}\left(\frac{3\pi t}{2}\right) - \sin\left(\frac{3\pi t}{2}\right) \cdot \frac{d}{dt}\left(\frac{3\pi t}{2}\right) = \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right) - \frac{3\pi}{2}\sin\left(\frac{3\pi t}{2}\right)$

$= \frac{3\pi}{2}\left(\cos\frac{3\pi t}{2} - \sin\frac{3\pi t}{2}\right)$

15. $r = (\csc\theta + \cot\theta)^{-1} \Rightarrow \frac{dr}{d\theta} = -(\csc\theta + \cot\theta)^{-2}\frac{d}{d\theta}(\csc\theta + \cot\theta) = \frac{\csc\theta\cot\theta + \csc^2\theta}{(\csc\theta + \cot\theta)^2} = \frac{\csc\theta(\cot\theta + \csc\theta)}{(\csc\theta + \cot\theta)^2}$

$= \frac{\csc\theta}{\csc\theta + \cot\theta}$

16. $r = -(\sec\theta + \tan\theta)^{-1} \Rightarrow \frac{dr}{d\theta} = (\sec\theta + \tan\theta)^{-2}\frac{d}{d\theta}(\sec\theta + \tan\theta) = \frac{\sec\theta\tan\theta + \sec^2\theta}{(\sec\theta + \tan\theta)^2} = \frac{\sec\theta(\tan\theta + \sec\theta)}{(\sec\theta + \tan\theta)^2}$

$= \frac{\sec\theta}{\sec\theta + \tan\theta}$

17. $y = x^2\sin^4 x + x\cos^{-2} x \Rightarrow \frac{dy}{dx} = x^2\frac{d}{dx}(\sin^4 x) + \sin^4 x \cdot \frac{d}{dx}(x^2) + x\frac{d}{dx}(\cos^{-2} x) + \cos^{-2} x \cdot \frac{d}{dx}(x)$

$= x^2\left(4\sin^3 x\frac{d}{dx}(\sin x)\right) + 2x\sin^4 x + x\left(-2\cos^{-3} x \cdot \frac{d}{dx}(\cos x)\right) + \cos^{-2} x$

$= x^2(4\sin^3 x\cos x) + 2x\sin^4 x + x\left((-2\cos^{-3} x)(-\sin x)\right) + \cos^{-2} x$

$= 4x^2\sin^3 x\cos x + 2x\sin^4 x + 2x\sin x\cos^{-3} x + \cos^{-2} x$

18. $y = \frac{1}{x}\sin^{-5} x - \frac{x}{3}\cos^3 x \Rightarrow y' = \frac{1}{x}\frac{d}{dx}(\sin^{-5} x) + \sin^{-5} x \cdot \frac{d}{dx}\left(\frac{1}{x}\right) - \frac{x}{3}\frac{d}{dx}(\cos^3 x) - \cos^3 x \cdot \frac{d}{dx}\left(\frac{x}{3}\right)$

$= \frac{1}{x}(-5\sin^{-6} x\cos x) + (\sin^{-5} x)\left(-\frac{1}{x^2}\right) - \frac{x}{3}\left((3\cos^2 x)(-\sin x)\right) - (\cos^3 x)\left(\frac{1}{3}\right)$

$= -\frac{5}{x}\sin^{-6} x\cos x - \frac{1}{x^2}\sin^{-5} x + x\cos^2 x\sin x - \frac{1}{3}\cos^3 x$

19. $y = \frac{1}{21}(3x - 2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1} \Rightarrow \frac{dy}{dx} = \frac{7}{21}(3x - 2)^6 \cdot \frac{d}{dx}(3x - 2) + (-1)\left(4 - \frac{1}{2x^2}\right)^{-2} \cdot \frac{d}{dx}\left(4 - \frac{1}{2x^2}\right)$

$= \frac{7}{21}(3x - 2)^6 \cdot 3 + (-1)\left(4 - \frac{1}{2x^2}\right)^{-2}\left(\frac{1}{x^3}\right) = (3x - 2)^6 - \frac{1}{x^3\left(4 - \frac{1}{2x^2}\right)^2}$

20. $y = (4x + 3)^4(x + 1)^{-3} \Rightarrow \frac{dy}{dx} = (4x + 3)^4(-3)(x + 1)^{-4} \cdot \frac{d}{dx}(x + 1) + (x + 1)^{-3}(4)(4x + 3)^3 \cdot \frac{d}{dx}(4x + 3)$

$= (4x + 3)^4(-3)(x + 1)^{-4}(1) + (x + 1)^{-3}(4)(4x + 3)^3(4) = -3(4x + 3)^4(x + 1)^{-4} + 16(4x + 3)^3(x + 1)^{-3}$

$= \frac{(4x + 3)^3}{(x + 1)^4}\left[-3(4x + 3) + 16(x + 1)\right] = \frac{(4x + 3)^3(4x + 7)}{(x + 1)^4}$

21. $h(x) = x \tan\left(2\sqrt{x}\right) + 7 \Rightarrow h'(x) = x \frac{d}{dx}\left(\tan\left(2x^{1/2}\right)\right) + \tan\left(2x^{1/2}\right) \cdot \frac{d}{dx}(x) + 0$
$= x \sec^2\left(2x^{1/2}\right) \cdot \frac{d}{dx}\left(2x^{1/2}\right) + \tan\left(2x^{1/2}\right) = x \sec^2\left(2\sqrt{x}\right) \cdot \frac{1}{\sqrt{x}} + \tan\left(2\sqrt{x}\right) = \sqrt{x} \sec^2\left(2\sqrt{x}\right) + \tan\left(2\sqrt{x}\right)$

22. $k(x) = x^2 \sec\left(\frac{1}{x}\right) \Rightarrow k'(x) = x^2 \frac{d}{dx}\left(\sec \frac{1}{x}\right) + \sec\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left(x^2\right) = x^2 \sec\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left(\frac{1}{x}\right) + 2x \sec\left(\frac{1}{x}\right)$
$= x^2 \sec\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) + 2x \sec\left(\frac{1}{x}\right) = 2x \sec\left(\frac{1}{x}\right) - \sec\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right)$

23. $f(\theta) = \left(\frac{\sin \theta}{1 + \cos \theta}\right)^2 \Rightarrow f'(\theta) = 2\left(\frac{\sin \theta}{1 + \cos \theta}\right) \cdot \frac{d}{d\theta}\left(\frac{\sin \theta}{1 + \cos \theta}\right) = \frac{2 \sin \theta}{1 + \cos \theta} \cdot \frac{(1 + \cos \theta)(\cos \theta) - (\sin \theta)(-\sin \theta)}{(1 + \cos \theta)^2}$
$= \frac{(2 \sin \theta)\left(\cos \theta + \cos^2 \theta + \sin^2 \theta\right)}{(1 + \cos \theta)^3} = \frac{(2 \sin \theta)(\cos \theta + 1)}{(1 + \cos \theta)^3} = \frac{2 \sin \theta}{(1 + \cos \theta)^2}$

24. $r = \sin\left(\theta^2\right) \cos(2\theta) \Rightarrow \frac{dr}{d\theta} = \sin\left(\theta^2\right)(-\sin 2\theta) \frac{d}{d\theta}(2\theta) + \cos(2\theta)\left(\cos\left(\theta^2\right)\right) \cdot \frac{d}{d\theta}\left(\theta^2\right)$
$= \sin\left(\theta^2\right)(-\sin 2\theta)(2) + (\cos 2\theta)\left(\cos\left(\theta^2\right)\right)(2\theta) = -2 \sin\left(\theta^2\right) \sin(2\theta) + 2\theta \cos(2\theta) \cos\left(\theta^2\right)$

25. $r = \left(\sec \sqrt{\theta}\right) \tan\left(\frac{1}{\theta}\right) \Rightarrow \frac{dr}{d\theta} = \left(\sec \sqrt{\theta}\right)\left(\sec^2 \frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right) + \tan\left(\frac{1}{\theta}\right)\left(\sec \sqrt{\theta} \tan \sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)$
$= -\frac{1}{\theta^2} \sec \sqrt{\theta} \sec^2\left(\frac{1}{\theta}\right) + \frac{1}{2\sqrt{\theta}} \tan\left(\frac{1}{\theta}\right) \sec \sqrt{\theta} \tan \sqrt{\theta} = \left(\sec \sqrt{\theta}\right)\left[\frac{\tan \sqrt{\theta} \tan\left(\frac{1}{\theta}\right)}{2\sqrt{\theta}} - \frac{\sec^2\left(\frac{1}{\theta}\right)}{\theta^2}\right]$

26. $q = \sin\left(\frac{t}{\sqrt{t+1}}\right) \Rightarrow \frac{dq}{dt} = \cos\left(\frac{t}{\sqrt{t+1}}\right) \cdot \frac{d}{dt}\left(\frac{t}{\sqrt{t+1}}\right) = \cos\left(\frac{t}{\sqrt{t+1}}\right) \cdot \frac{\sqrt{t+1}(1) - t \cdot \frac{d}{dt}\left(\sqrt{t+1}\right)}{\left(\sqrt{t+1}\right)^2}$
$= \cos\left(\frac{t}{\sqrt{t+1}}\right) \cdot \frac{\sqrt{t+1} - \frac{t}{2\sqrt{t+1}}}{t+1} = \cos\left(\frac{t}{\sqrt{t+1}}\right)\left(\frac{2(t+1) - t}{2(t+1)^{3/2}}\right) = \left(\frac{t+2}{2(t+1)^{3/2}}\right) \cos\left(\frac{t}{\sqrt{t+1}}\right)$

27. $y = \sin^2(\pi t - 2) \Rightarrow \frac{dy}{dt} = 2 \sin(\pi t - 2) \cdot \frac{d}{dt} \sin(\pi t - 2) = 2 \sin(\pi t - 2) \cdot \cos(\pi t - 2) \cdot \frac{d}{dt}(\pi t - 2)$
$= 2\pi \sin(\pi t - 2) \cos(\pi t - 2)$

28. $y = (1 + \cos 2t)^{-4} \Rightarrow \frac{dy}{dt} = -4(1 + \cos 2t)^{-5} \cdot \frac{d}{dt}(1 + \cos 2t) = -4(1 + \cos 2t)^{-5}(-\sin 2t) \cdot \frac{d}{dt}(2t) = \frac{8 \sin 2t}{(1 + \cos 2t)^5}$

29. $y = \left(1 + \cot\left(\frac{t}{2}\right)\right)^{-2} \Rightarrow \frac{dy}{dt} = -2\left(1 + \cot\left(\frac{t}{2}\right)\right)^{-3} \cdot \frac{d}{dt}\left(1 + \cot\left(\frac{t}{2}\right)\right) = -2\left(1 + \cot\left(\frac{t}{2}\right)\right)^{-3} \cdot \left(-\csc^2\left(\frac{t}{2}\right)\right) \cdot \frac{d}{dt}\left(\frac{t}{2}\right)$
$= \frac{\csc^2\left(\frac{t}{2}\right)}{\left(1 + \cot\left(\frac{t}{2}\right)\right)^3}$

30. $y = \sin(\cos(2t-5)) \Rightarrow \frac{dy}{dt} = \cos(\cos(2t-5)) \cdot \frac{d}{dt}\cos(2t-5) = \cos(\cos(2t-5)) \cdot (-\sin(2t-5)) \cdot \frac{d}{dt}(2t-5)$
$= -2\cos(\cos(2t-5))(\sin(2t-5))$

31. $y = \left[1+\tan^4\left(\frac{t}{12}\right)\right]^3 \Rightarrow \frac{dy}{dt} = 3\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2 \cdot \frac{d}{dt}\left[1+\tan^4\left(\frac{t}{12}\right)\right] = 3\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[4\tan^3\left(\frac{t}{12}\right) \cdot \frac{d}{dt}\tan\left(\frac{t}{12}\right)\right]$
$= 12\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[\tan^3\left(\frac{t}{12}\right)\sec^2\left(\frac{t}{12}\right) \cdot \frac{1}{12}\right] = \left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[\tan^3\left(\frac{t}{12}\right)\sec^2\left(\frac{t}{12}\right)\right]$

32. $y = \left(1+\cos\left(t^2\right)\right)^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2}\left(1+\cos\left(t^2\right)\right)^{-1/2} \cdot \frac{d}{dt}\left(1+\cos\left(t^2\right)\right) = \frac{1}{2}\left(1+\cos\left(t^2\right)\right)^{-1/2}\left(-\sin\left(t^2\right) \cdot \frac{d}{dt}\left(t^2\right)\right)$
$= -\frac{1}{2}\left(1+\cos\left(t^2\right)\right)^{-1/2}\left(\sin\left(t^2\right)\right) \cdot 2t = -\frac{t\sin\left(t^2\right)}{\sqrt{1+\cos\left(t^2\right)}}$

33. $t = \frac{\pi}{4} \Rightarrow x = 2\cos\frac{\pi}{4} = \sqrt{2}$, $y = 2\sin\frac{\pi}{4} = \sqrt{2}$; $\frac{dx}{dt} = -2\sin t$, $\frac{dy}{dt} = 2\cos t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-2\sin t} = -\cot t$
$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -\cot\frac{\pi}{4} = -1$; tangent line is $y - \sqrt{2} = -1\left(x-\sqrt{2}\right)$ or $y = -x + 2\sqrt{2}$; $\frac{dy'}{dt} = \csc^2 t$
$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\csc^2 t}{-2\sin t} = -\frac{1}{2\sin^3 t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = -\sqrt{2}$

34. $t = \frac{2\pi}{3} \Rightarrow x = \cos\frac{2\pi}{3} = -\frac{1}{2}$, $y = \sqrt{3}\cos\frac{2\pi}{3} = -\frac{\sqrt{3}}{2}$; $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = -\sqrt{3}\sin t \Rightarrow \frac{dy}{dx} = \frac{-\sqrt{3}\sin t}{-\sin t} = \sqrt{3}$
$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}} = \sqrt{3}$; tangent line is $y - \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}\left[x - \left(-\frac{1}{2}\right)\right]$ or $y = \sqrt{3}\,x$; $\frac{dy'}{dt} = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{0}{-\sin t} = 0$
$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{2\pi}{3}} = 0$

35. $t = \frac{1}{4} \Rightarrow x = \frac{1}{4}$, $y = \frac{1}{2}$; $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = \frac{1}{2\sqrt{t}} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2\sqrt{t}} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{1}{4}} = \frac{1}{2\sqrt{\frac{1}{4}}} = 1$; tangent line is
$y - \frac{1}{2} = 1 \cdot \left(x - \frac{1}{4}\right)$ or $y = x + \frac{1}{4}$; $\frac{dy'}{dt} = -\frac{1}{4}t^{-3/2} \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = -\frac{1}{4}t^{-3/2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{4}} = -2$

36. $t = 3 \Rightarrow x = -\sqrt{3+1} = -2$, $y = \sqrt{3(3)} = 3$; $\frac{dx}{dt} = -\frac{1}{2}(t+1)^{-1/2}$, $\frac{dy}{dt} = \frac{3}{2}(3t)^{-1/2} \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{3}{2}\right)(3t)^{-1/2}}{\left(-\frac{1}{2}\right)(t+1)^{-1/2}}$
$= -\frac{3\sqrt{t+1}}{\sqrt{3t}} = \left.\frac{dy}{dx}\right|_{t=3} = \frac{-3\sqrt{3+1}}{\sqrt{3(3)}} = -2$; tangent line is $y - 3 = -2[x - (-2)]$ or $y = -2x - 1$;
$\frac{dy'}{dt} = \frac{\sqrt{3t}\left[-\frac{3}{2}(t+1)^{-1/2}\right] + 3\sqrt{t+1}\left[\frac{3}{2}(3t)^{-1/2}\right]}{3t} = \frac{3}{2t\sqrt{3t}\,\sqrt{t+1}} \Rightarrow \frac{d^2y}{dx^2} = \frac{\left(\frac{3}{2t\sqrt{3t}\,\sqrt{t+1}}\right)}{\left(\frac{-1}{2\sqrt{t+1}}\right)} = -\frac{3}{t\sqrt{3t}}$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=3} = -\frac{1}{3}$

37. $t=-1 \Rightarrow x=5,\ y=1;\ \frac{dx}{dt}=4t,\ \frac{dy}{dt}=4t^3 \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{4t^3}{4t}=t^2 \Rightarrow \left.\frac{dy}{dx}\right|_{t=-1}=(-1)^2=1$; tangent line is

$y-1=1\cdot(x-5)$ or $y=x-4;\ \frac{dy'}{dt}=2t \Rightarrow \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=\frac{2t}{4t}=\frac{1}{2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-1}=\frac{1}{2}$

38. $t=\frac{\pi}{3} \Rightarrow x=\frac{\pi}{3}-\sin\frac{\pi}{3}=\frac{\pi}{3}-\frac{\sqrt{3}}{2},\ y=1-\cos\frac{\pi}{3}=1-\frac{1}{2}=\frac{1}{2};\ \frac{dx}{dt}=1-\cos t,\ \frac{dy}{dt}=\sin t \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}$

$=\frac{\sin t}{1-\cos t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}}=\frac{\sin\left(\frac{\pi}{3}\right)}{1-\cos\left(\frac{\pi}{3}\right)}=\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}=\sqrt{3}$; tangent line is $y-\frac{1}{2}=\sqrt{3}\left(x-\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$

$\Rightarrow y=\sqrt{3}x-\frac{\pi\sqrt{3}}{3}+2;\ \frac{dy'}{dt}=\frac{(1-\cos t)(\cos t)-(\sin t)(\sin t)}{(1-\cos t)^2}=\frac{-1}{1-\cos t} \Rightarrow \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=\frac{\left(\frac{-1}{1-\cos t}\right)}{1-\cos t}$

$=\frac{-1}{(1-\cos t)^2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}}=-4$

39. $t=\frac{\pi}{2} \Rightarrow x=\cos\frac{\pi}{2}=0,\ y=1+\sin\frac{\pi}{2}=2;\ \frac{dx}{dt}=-\sin t,\ \frac{dy}{dt}=\cos t \Rightarrow \frac{dy}{dx}=\frac{\cos t}{-\sin t}=-\cot t$

$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}}=-\cot\frac{\pi}{2}=0$; tangent line is $y=2;\ \frac{dy'}{dt}=\csc^2 t \Rightarrow \frac{d^2y}{dx^2}=\frac{\csc^2 t}{-\sin t}=-\csc^3 t \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}}=-1$

40. $t=-\frac{\pi}{4} \Rightarrow x=\sec^2\left(-\frac{\pi}{4}\right)-1=1,\ y=\tan\left(-\frac{\pi}{4}\right)=-1;\ \frac{dx}{dt}=2\sec^2 t\tan t,\ \frac{dy}{dt}=\sec^2 t$

$\Rightarrow \frac{dy}{dx}=\frac{\sec^2 t}{2\sec^2 t\tan t}=\frac{1}{2\tan t}=\frac{1}{2}\cot t \Rightarrow \left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{4}}=\frac{1}{2}\cot\left(-\frac{\pi}{4}\right)=-\frac{1}{2}$; tangent line is

$y-(-1)=-\frac{1}{2}(x-1)$ or $y=-\frac{1}{2}x-\frac{1}{2};\ \frac{dy'}{dt}=-\frac{1}{2}\csc^2 t \Rightarrow \frac{d^2y}{dx^2}=\frac{-\frac{1}{2}\csc^2 t}{2\sec^2 t\tan t}=-\frac{1}{4}\cot^3 t$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-\frac{\pi}{4}}=\frac{1}{4}$

41. $y=\left(1+\frac{1}{x}\right)^3 \Rightarrow y'=3\left(1+\frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right)=-\frac{3}{x^2}\left(1+\frac{1}{x}\right)^2 \Rightarrow y''=\left(-\frac{3}{x^2}\right)\cdot\frac{d}{dx}\left(1+\frac{1}{x}\right)^2-\left(1+\frac{1}{x}\right)^2\cdot\frac{d}{dx}\left(\frac{3}{x^2}\right)$

$=\left(-\frac{3}{x^2}\right)\left(2\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right)+\left(\frac{6}{x^3}\right)\left(1+\frac{1}{x}\right)^2=\frac{6}{x^4}\left(1+\frac{1}{x}\right)+\frac{6}{x^3}\left(1+\frac{1}{x}\right)^2=\frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(\frac{1}{x}+1+\frac{1}{x}\right)$

$=\frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)$

42. $y=\left(1-\sqrt{x}\right)^{-1} \Rightarrow y'=-\left(1-\sqrt{x}\right)^{-2}\left(-\frac{1}{2}x^{-1/2}\right)=\frac{1}{2}\left(1-\sqrt{x}\right)^{-2}x^{-1/2}$

$\Rightarrow y''=\frac{1}{2}\left[\left(1-\sqrt{x}\right)^{-2}\left(-\frac{1}{2}x^{-3/2}\right)+x^{-1/2}(-2)\left(1-\sqrt{x}\right)^{-3}\left(-\frac{1}{2}x^{-1/2}\right)\right]$

$= \frac{1}{2}\left[\frac{-1}{2}x^{-3/2}\left(1-\sqrt{x}\right)^{-2} + x^{-1}\left(1-\sqrt{x}\right)^{-3}\right] = \frac{1}{2}x^{-1}\left(1-\sqrt{x}\right)^{-3}\left[-\frac{1}{2}x^{-1/2}\left(1-\sqrt{x}\right)+1\right]$

$= \frac{1}{2x}\left(1-\sqrt{x}\right)^{-3}\left(-\frac{1}{2\sqrt{x}}+\frac{1}{2}+1\right) = \frac{1}{2x}\left(1-\sqrt{x}\right)^{-3}\left(\frac{3}{2}-\frac{1}{2\sqrt{x}}\right)$

43. $y = \frac{1}{9}\cot(3x-1) \Rightarrow y' = -\frac{1}{9}\csc^2(3x-1)(3) = -\frac{1}{3}\csc^2(3x-1) \Rightarrow y'' = \left(-\frac{2}{3}\right)\left(\csc(3x-1)\cdot\frac{d}{dx}\csc(3x-1)\right)$
$= -\frac{2}{3}\csc(3x-1)\left(-\csc(3x-1)\cot(3x-1)\cdot\frac{d}{dx}(3x-1)\right) = 2\csc^2(3x-1)\cot(3x-1)$

44. $y = 9\tan\left(\frac{x}{3}\right) \Rightarrow y' = 9\left(\sec^2\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right) = 3\sec^2\left(\frac{x}{3}\right) \Rightarrow y'' = 3\cdot 2\sec\left(\frac{x}{3}\right)\left(\sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right) = 2\sec^2\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)$

45. $g(x) = \sqrt{x} \Rightarrow g'(x) = \frac{1}{2\sqrt{x}} \Rightarrow g(1) = 1$ and $g'(1) = \frac{1}{2}$; $f(u) = u^5 + 1 \Rightarrow f'(u) = 5u^4 \Rightarrow f'(g(1)) = f'(1) = 5$; therefore, $(f\circ g)'(1) = f'(g(1))\cdot g'(1) = 5\cdot\frac{1}{2} = \frac{5}{2}$

46. $g(x) = (1-x)^{-1} \Rightarrow g'(x) = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2} \Rightarrow g(-1) = \frac{1}{2}$ and $g'(-1) = \frac{1}{4}$; $f(u) = 1 - \frac{1}{u}$
$\Rightarrow f'(u) = \frac{1}{u^2} \Rightarrow f'(g(-1)) = f'\left(\frac{1}{2}\right) = 4$; therefore, $(f\circ g)'(-1) = f'(g(-1))g'(-1) = 4\cdot\frac{1}{4} = 1$

47. $g(x) = 5\sqrt{x} \Rightarrow g'(x) = \frac{5}{2\sqrt{x}} \Rightarrow g(1) = 5$ and $g'(1) = \frac{5}{2}$; $f(u) = \cot\left(\frac{\pi u}{10}\right) \Rightarrow f'(u) = -\csc^2\left(\frac{\pi u}{10}\right)\left(\frac{\pi}{10}\right)$
$= \frac{-\pi}{10}\csc^2\left(\frac{\pi u}{10}\right) \Rightarrow f'(g(1)) = f'(5) = -\frac{\pi}{10}\csc^2\left(\frac{\pi}{2}\right) = -\frac{\pi}{10}$; therefore, $(f\circ g)'(1) = f'(g(1))g'(1) = -\frac{\pi}{10}\cdot\frac{5}{2}$
$= -\frac{\pi}{4}$

48. $g(x) = \pi x \Rightarrow g'(x) = \pi \Rightarrow g\left(\frac{1}{4}\right) = \frac{\pi}{4}$ and $g'\left(\frac{1}{4}\right) = \pi$; $f(u) = u + \sec^2 u \Rightarrow f'(u) = 1 + 2\sec u\cdot\sec u\tan u$
$= 1 + 2\sec^2 u\tan u \Rightarrow f'\left(g\left(\frac{1}{4}\right)\right) = f'\left(\frac{\pi}{4}\right) = 1 + 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 5$; therefore, $(f\circ g)'\left(\frac{1}{4}\right) = f'\left(g\left(\frac{1}{4}\right)\right)g'\left(\frac{1}{4}\right) = 5\pi$

49. $g(x) = 10x^2 + x + 1 \Rightarrow g'(x) = 20x + 1 \Rightarrow g(0) = 1$ and $g'(0) = 1$; $f(u) = \frac{2u}{u^2+1} \Rightarrow f'(u) = \frac{\left(u^2+1\right)(2) - (2u)(2u)}{\left(u^2+1\right)^2}$
$= \frac{-2u^2+2}{\left(u^2+1\right)^2} \Rightarrow f'(g(0)) = f'(1) = 0$; therefore, $(f\circ g)'(0) = f'(g(0))g'(0) = 0\cdot 1 = 0$

50. $g(x) = \frac{1}{x^2} - 1 \Rightarrow g'(x) = -\frac{2}{x^3} \Rightarrow g(-1) = 0$ and $g'(-1) = 2$; $f(u) = \left(\frac{u-1}{u+1}\right)^2 \Rightarrow f'(u) = 2\left(\frac{u-1}{u+1}\right)\frac{d}{du}\left(\frac{u-1}{u+1}\right)$
$= 2\left(\frac{u-1}{u+1}\right)\cdot\frac{(u+1)(1)-(u-1)(1)}{(u+1)^2} = \frac{2(u-1)(2)}{(u+1)^3} = \frac{4(u-1)}{(u+1)^3} \Rightarrow f'(g(-1)) = f'(0) = -4$; therefore,
$(f\circ g)'(-1) = f'(g(-1))g'(-1) = (-4)(2) = -8$

51. (a) $y = 2f(x) \Rightarrow \frac{dy}{dx} = 2f'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = 2f'(2) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$

(b) $y = f(x) + g(x) \Rightarrow \frac{dy}{dx} = f'(x) + g'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=3} = f'(3) + g'(3) = 2\pi + 5$

(c) $y = f(x) \cdot g(x) \Rightarrow \frac{dy}{dx} = f(x)g'(x) + g(x)f'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=3} = f(3)g'(3) + g(3)f'(3) = 3 \cdot 5 + (-4)(2\pi) = 15 - 8\pi$

(d) $y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(2)\left(\frac{1}{3}\right) - (8)(-3)}{2^2} = \frac{37}{6}$

(e) $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x))g'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = f'(g(2))g'(2) = f'(2)(-3) = \frac{1}{3}(-3) = -1$

(f) $y = (f(x))^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(f(x))^{-1/2} \cdot f'(x) = \frac{f'(x)}{2\sqrt{f(x)}} \Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = \frac{f'(2)}{2\sqrt{f(2)}} = \frac{\left(\frac{1}{3}\right)}{2\sqrt{8}} = \frac{1}{6\sqrt{8}} = \frac{1}{12\sqrt{2}} = \frac{\sqrt{2}}{24}$

(g) $y = (g(x))^{-2} \Rightarrow \frac{dy}{dx} = -2(g(x))^{-3} \cdot g'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=3} = -2(g(3))^{-3}g'(3) = -2(-4)^{-3} \cdot 5 = \frac{5}{32}$

(h) $y = \left((f(x))^2 + (g(x))^2\right)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left((f(x))^2 + (g(x))^2\right)^{-1/2}\left(2f(x) \cdot f'(x) + 2g(x) \cdot g'(x)\right)$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{2}\left((f(2))^2 + (g(2))^2\right)^{-1/2}\left(2f(2)f'(2) + 2g(2)g'(2)\right) = \frac{1}{2}\left(8^2 + 2^2\right)^{-1/2}\left(2 \cdot 8 \cdot \frac{1}{3} + 2 \cdot 2 \cdot (-3)\right)$

$= -\frac{5}{3\sqrt{17}}$

52. (a) $y = 5f(x) - g(x) \Rightarrow \frac{dy}{dx} = 5f'(x) - g'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = 5f'(1) - g'(1) = 5\left(-\frac{1}{3}\right) - \left(\frac{-8}{3}\right) = 1$

(b) $y = f(x)(g(x))^3 \Rightarrow \frac{dy}{dx} = f(x)\left(3(g(x))^2g'(x)\right) + (g(x))^3f'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 3f(0)(g(0))^2g'(0) + (g(0))^3f'(0)$

$= 3(1)(1)^2\left(\frac{1}{3}\right) + (1)^3(5) = 6$

(c) $y = \frac{f(x)}{g(x)+1} \Rightarrow \frac{dy}{dx} = \frac{(g(x)+1)f'(x) - f(x)\,g'(x)}{(g(x)+1)^2} \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = \frac{(g(1)+1)f'(1) - f(1)g'(1)}{(g(1)+1)^2}$

$= \frac{(-4+1)\left(-\frac{1}{3}\right) - (3)\left(-\frac{8}{3}\right)}{(-4+1)^2} = 1$

(d) $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x))g'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = f'(g(0))g'(0) = f'(1)\left(\frac{1}{3}\right) = \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{1}{9}$

(e) $y = g(f(x)) \Rightarrow \frac{dy}{dx} = g'(f(x))f'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = g'(f(0))f'(0) = g'(1)(5) = \left(-\frac{8}{3}\right)(5) = -\frac{40}{3}$

(f) $y = \left(x^{11} + f(x)\right)^{-2} \Rightarrow \frac{dy}{dx} = -2\left(x^{11} + f(x)\right)^{-3}\left(11x^{10} + f'(x)\right) \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = -2(1 + f(1))^{-3}\left(11 + f'(1)\right)$

$= -2(1+3)^{-3}\left(11 - \frac{1}{3}\right) = \left(-\frac{2}{4^3}\right)\left(\frac{32}{3}\right) = -\frac{1}{3}$

(g) $y = f(x + g(x)) \Rightarrow \frac{dy}{dx} = f'(x + g(x))\left(1 + g'(x)\right) \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = f'(0 + g(0))\left(1 + g'(0)\right) = f'(1)\left(1 + \frac{1}{3}\right)$
$= \left(-\frac{1}{3}\right)\left(\frac{4}{3}\right) = -\frac{4}{9}$

53. $\frac{ds}{dt} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dt}$: $s = \cos\theta \Rightarrow \frac{ds}{d\theta} = -\sin\theta \Rightarrow \left.\frac{ds}{d\theta}\right|_{\theta=\frac{3\pi}{2}} = -\sin\left(\frac{3\pi}{2}\right) = 1$ so that $\frac{ds}{dt} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dt} = 1 \cdot 5 = 5$

54. $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$: $y = x^2 + 7x - 5 \Rightarrow \frac{dy}{dx} = 2x + 7 \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = 9$ so that $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 9 \cdot \frac{1}{3} = 3$

55. With $y = x$, we should get $\frac{dy}{dx} = 1$ for both (a) and (b):

(a) $y = \frac{u}{5} + 7 \Rightarrow \frac{dy}{du} = \frac{1}{5}$; $u = 5x - 35 \Rightarrow \frac{du}{dx} = 5$; therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{5} \cdot 5 = 1$, as expected

(b) $y = 1 + \frac{1}{u} \Rightarrow \frac{dy}{du} = -\frac{1}{u^2}$; $u = (x-1)^{-1} \Rightarrow \frac{du}{dx} = -(x-1)^{-2}(1) = \frac{-1}{(x-1)^2}$; therefore $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$= \frac{-1}{u^2} \cdot \frac{-1}{(x-1)^2} = \frac{-1}{\left((x-1)^{-1}\right)^2} \cdot \frac{-1}{(x-1)^2} = (x-1)^2 \cdot \frac{1}{(x-1)^2} = 1$, again as expected

56. With $y = x^{3/2}$, we should get $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$ for both (a) and (b):

(a) $y = u^3 \Rightarrow \frac{dy}{du} = 3u^2$; $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}}$; therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot \frac{1}{2\sqrt{x}} = 3\left(\sqrt{x}\right)^2 \cdot \frac{1}{2\sqrt{x}} = \frac{3}{2}\sqrt{x}$, as expected.

(b) $y = \sqrt{u} \Rightarrow \frac{dy}{du} = \frac{1}{2\sqrt{u}}$; $u = x^3 \Rightarrow \frac{du}{dx} = 3x^2$; therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 3x^2 = \frac{1}{2\sqrt{x^3}} \cdot 3x^2 = \frac{3}{2}x^{1/2}$, again as expected.

57. $y = 2\tan\left(\frac{\pi x}{4}\right) \Rightarrow \frac{dy}{dx} = \left(2\sec^2\frac{\pi x}{4}\right)\left(\frac{\pi}{4}\right) = \frac{\pi}{2}\sec^2\frac{\pi x}{4}$

(a) $\left.\frac{dy}{dx}\right|_{x=1} = \frac{\pi}{2}\sec^2\left(\frac{\pi}{4}\right) = \pi \Rightarrow$ slope of tangent is 2; thus, $y(1) = 2\tan\left(\frac{\pi}{4}\right) = 2$ and $y'(1) = \pi \Rightarrow$ tangent line is given by $y - 2 = \pi(x-1) \Rightarrow y = \pi x + 2 - \pi$

(b) $y' = \frac{\pi}{2}\sec^2\left(\frac{\pi x}{4}\right)$ and the smallest value the secant function can have in $-2 < x < 2$ is $1 \Rightarrow$ the minimum value of y' is $\frac{\pi}{2}$ and that occurs when $\frac{\pi}{2} = \frac{\pi}{2}\sec^2\left(\frac{\pi x}{4}\right) \Rightarrow 1 = \sec^2\left(\frac{\pi x}{4}\right) \Rightarrow \pm 1 = \sec\left(\frac{\pi x}{4}\right) \Rightarrow x = 0$.

58. (a) $y = \sin 2x \Rightarrow y' = 2\cos 2x \Rightarrow y'(0) = 2\cos(0) = 2 \Rightarrow$ tangent to $y = \sin 2x$ at the origin is $y = 2x$; $y = -\sin\left(\frac{x}{2}\right) \Rightarrow y' = -\frac{1}{2}\cos\left(\frac{x}{2}\right) \Rightarrow y'(0) = -\frac{1}{2}\cos 0 = -\frac{1}{2} \Rightarrow$ tangent to $y = -\sin\left(\frac{x}{2}\right)$ at the origin is $y = -\frac{1}{2}x$. The tangents are perpendicular to each other at the origin since the product of their slopes is -1.

(b) $y = \sin(mx) \Rightarrow y' = m\cos(mx) \Rightarrow y'(0) = m\cos 0 = m$; $y = -\sin\left(\frac{x}{m}\right) \Rightarrow y' = -\frac{1}{m}\cos\left(\frac{x}{m}\right)$
$\Rightarrow y'(0) = -\frac{1}{m}\cos(0) = -\frac{1}{m}$. Since $m \cdot \left(-\frac{1}{m}\right) = -1$, the tangent lines are perpendicular at the origin.

(c) $y = \sin(mx) \Rightarrow y' = m\cos(mx)$. The largest value $\cos(mx)$ can attain is 1 at $x = 0 \Rightarrow$ the largest value y' can attain is $|m|$ because $|y'| = |m\cos(mx)| = |m||\cos mx| \le |m| \cdot 1 = |m|$. Also, $y = -\sin\left(\frac{x}{m}\right)$ $\Rightarrow y' = -\frac{1}{m}\cos\left(\frac{x}{m}\right) \Rightarrow |y'| = \left|\frac{-1}{m}\cos\left(\frac{x}{m}\right)\right| \le \left|\frac{1}{m}\right|\left|\cos\left(\frac{x}{m}\right)\right| \le \frac{1}{|m|} \Rightarrow$ the largest value y' can attain is $\left|\frac{1}{m}\right|$.

(d) $y = \sin(mx) \Rightarrow y' = m\cos(mx) \Rightarrow y'(0) = m \Rightarrow$ slope of curve at the origin is m. Also, $\sin(mx)$ completes m periods on $[0, 2\pi]$. Therefore the slope of the curve $y = \sin(mx)$ at the origin is the same as the number of periods it completes on $[0, 2\pi]$. In particular, for large m, we can think of "compressing" the graph of $y = \sin x$ horizontally which gives more periods completed on $[0, 2\pi]$, but also increases the slope of the graph at the origin.

59. $s = A\cos(2\pi bt) \Rightarrow v = \frac{ds}{dt} = -A\sin(2\pi bt)(2\pi b) = -2\pi bA\sin(2\pi bt)$. If we replace b with 2b to double the frequency, the velocity formula gives $v = -4\pi bA\sin(4\pi bt) \Rightarrow$ doubling the frequency causes the velocity to double. Also $v = -2\pi bA\sin(2\pi bt) \Rightarrow a = \frac{dv}{dt} = -4\pi^2b^2A\cos(2\pi bt)$. If we replace b with 2b in the acceleration formula, we get $a = -16\pi^2b^2A\cos(4\pi bt) \Rightarrow$ doubling the frequency causes the acceleration to quadruple. Finally, $a = -4\pi^2b^2A\cos(2\pi bt) \Rightarrow j = \frac{da}{dt} = 8\pi^3b^3A\sin(2\pi bt)$. If we replace b with 2b in the jerk formula, we get $j = 64\pi^3b^3A\sin(2\pi bt) \Rightarrow$ doubling the frequency multiplies the jerk by a factor of 8.

60. (a) $y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25 \Rightarrow y' = 37\cos\left[\frac{2\pi}{365}(x - 101)\right]\left(\frac{2\pi}{365}\right) = \frac{74\pi}{365}\cos\left[\frac{2\pi}{365}(x - 101)\right]$.

The temperature is increasing the fastest when y' is as large as possible. The largest value of $\cos\left[\frac{2\pi}{365}(x - 101)\right]$ is 1 and occurs when $\frac{2\pi}{365}(x - 101) = 0 \Rightarrow x = 101 \Rightarrow$ on day 101 of the year (~ April 11), the temperature is increasing the fastest.

(b) $y'(101) = \frac{74\pi}{365}\cos\left[\frac{2\pi}{365}(101 - 101)\right] = \frac{74\pi}{365}\cos(0) = \frac{74\pi}{365} \approx 0.64$ °F/day

61. $s = (1 + 4t)^{1/2} \Rightarrow v = \frac{ds}{dt} = \frac{1}{2}(1 + 4t)^{-1/2}(4) = 2(1 + 4t)^{-1/2} \Rightarrow v(6) = 2(1 + 4 \cdot 6)^{-1/2} = \frac{2}{5}$ m/sec;
$v = 2(1 + 4t)^{-1/2} \Rightarrow a = \frac{dv}{dt} = -\frac{1}{2} \cdot 2(1 + 4t)^{-3/2}(4) = -4(1 + 4t)^{-3/2} \Rightarrow a(6) = -4(1 + 4 \cdot 6)^{-3/2} = -\frac{4}{125}$ m/sec^2

62. We need to show $a = \frac{dv}{dt}$ is constant: $a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt}$ and $\frac{dv}{ds} = \frac{d}{ds}(k\sqrt{s}) = \frac{k}{2\sqrt{s}} \Rightarrow a = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v$ $= \frac{k}{2\sqrt{s}} \cdot k\sqrt{s} = \frac{k^2}{2}$ which is a constant.

63. v proportional to $\frac{1}{\sqrt{s}} \Rightarrow v = \frac{k}{\sqrt{s}}$ for some constant $k \Rightarrow \frac{dv}{ds} = -\frac{k}{2s^{3/2}}$. Thus, $a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v$ $= -\frac{k}{2s^{3/2}} \cdot \frac{k}{\sqrt{s}} = -\frac{k^2}{2}\left(\frac{1}{s^2}\right) \Rightarrow$ acceleration is a constant times $\frac{1}{s^2}$ so a is proportional to $\frac{1}{s^2}$.

64. Let $\frac{dx}{dt} = f(x)$. Then, $a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot f(x) = \frac{d}{dx}\left(\frac{dx}{dt}\right) \cdot f(x) = \frac{d}{dx}(f(x)) \cdot f(x) = f'(x)f(x)$, as required.

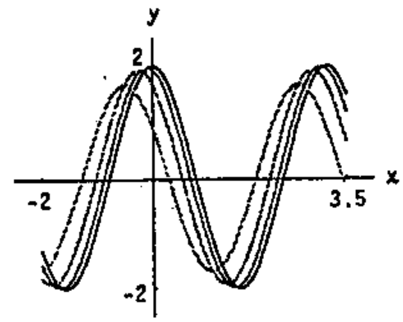
65. $T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow \frac{dT}{dL} = 2\pi \cdot \frac{1}{2\sqrt{\frac{L}{g}}} \cdot \frac{1}{g} = \frac{\pi}{g\sqrt{\frac{L}{g}}} = \frac{\pi}{\sqrt{gL}}$. Therefore, $\frac{dT}{du} = \frac{dT}{dL} \cdot \frac{dL}{du} = \frac{\pi}{\sqrt{gL}} \cdot kL = \frac{\pi k\sqrt{L}}{\sqrt{g}} = \frac{1}{2} \cdot 2\pi k\sqrt{\frac{L}{g}}$ $= \frac{kT}{2}$, as required.

66. No. The chain rule says that when g is differentiable at 0 and f is differentiable at g(0), then $f \circ g$ is differentiable at 0. But the chain rule says nothing about what happens when g is not differentiable at 0 so there is no contradiction.

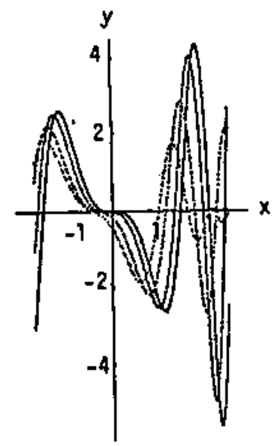
67. The graph of $y = (f \circ g)(x)$ has a horizontal tangent at $x = 1$ provided that $(f \circ g)'(1) = 0 \Rightarrow f'(g(1))g'(1) = 0$ $\Rightarrow$ either $f'(g(1)) = 0$ or $g'(1) = 0$ (or both) $\Rightarrow$ either the graph of f has a horizontal tangent at $u = g(1)$, or the graph of g has a horizontal tangent at $x = 1$ (or both).

68. $(f \circ g)'(-5) < 0 \Rightarrow f'(g(-5)) \cdot g'(-5) < 0 \Rightarrow f'(g(-5))$ and $g'(-5)$ are both nonzero and have opposite signs. That is, either $[f'(g(-5)) > 0$ and $g'(-5) < 0]$ or $[f'(g(-5)) < 0$ and $g'(-5) > 0]$.

69. As $h \to 0$, the graph of $y = \frac{\sin 2(x+h) - \sin 2x}{h}$ approaches the graph of $y = 2\cos 2x$ because $\lim_{h \to 0} \frac{\sin 2(x+h) - \sin 2x}{h} = \frac{d}{dx}(\sin 2x) = 2\cos 2x.$



70. As $h \to 0$, the graph of $y = \frac{\cos[(x+h)^2] - \cos(x^2)}{h}$ approaches the graph of $y = -2x\sin(x^2)$ because $\lim_{h \to 0} \frac{\cos[(x+h)^2] - \cos(x^2)}{h} = \frac{d}{dx}[\cos(x^2)] = -2x\sin(x^2).$



71. $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = 2\cos 2t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos 2t}{\cos t} = \frac{2(2\cos^2 t - 1)}{\cos t}$; then $\frac{dy}{dx} = 0 \Rightarrow \frac{2(2\cos^2 t - 1)}{\cos t} = 0$ $\Rightarrow 2\cos^2 t - 1 = 0 \Rightarrow \cos t = \pm\frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. In the 1st quadrant: $t = \frac{\pi}{4} \Rightarrow x = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $y = \sin 2\left(\frac{\pi}{4}\right) = 1 \Rightarrow \left(\frac{\sqrt{2}}{2}, 1\right)$ is the point where the tangent line is horizontal. At the origin: $x = 0$ and $y = 0$ $\Rightarrow \sin t = 0 \Rightarrow t = 0$ or $t = \pi$ and $\sin 2t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$; thus $t = 0$ and $t = \pi$ give the tangent lines at the origin. Tangents at origin: $\left.\frac{dy}{dx}\right|_{t=0} = 2 \Rightarrow y = 2x$ and $\left.\frac{dy}{dx}\right|_{t=\pi} = -2 \Rightarrow y = -2x$

72. $\frac{dx}{dt} = 2\cos 2t$ and $\frac{dy}{dt} = 3\cos 3t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos 3t}{2\cos 2t} = \frac{3(\cos 2t\cos t - \sin 2t \sin t)}{2(2\cos^2 t - 1)}$

$= \frac{3\left[(2\cos^2 t - 1)(\cos t) - 2\sin t\cos t \sin t\right]}{2(2\cos^2 t - 1)} = \frac{(3\cos t)(2\cos^2 t - 1 - 2\sin^2 t)}{2(2\cos^2 t - 1)} = \frac{(3\cos t)(4\cos^2 t - 3)}{2(2\cos^2 t - 1)}$; then

$\frac{dy}{dx} = 0 \Rightarrow \frac{(3\cos t)(4\cos^2 t - 3)}{2(2\cos^2 t - 1)} = 0 \Rightarrow 3\cos t = 0$ or $4\cos^2 t - 3 = 0$: $3\cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$ and

$4\cos^2 t - 3 = 0 \Rightarrow \cos t = \pm\frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. In the 1st quadrant: $t = \frac{\pi}{6} \Rightarrow x = \sin 2\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

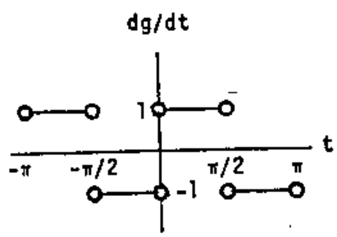
and $y = \sin 3\left(\frac{\pi}{6}\right) = 1 \Rightarrow \left(\frac{\sqrt{3}}{2}, 1\right)$ is the point where the graph has a horizontal tangent. At the origin: $x = 0$

and $y = 0 \Rightarrow \sin 2t = 0$ and $\sin 3t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3} \Rightarrow t = 0$ and $t = \pi$ give

the tangent lines at the origin. Tangents at the origin: $\left.\frac{dy}{dx}\right|_{t=0} = \frac{3\cos 0}{2\cos 0} = \frac{3}{2} \Rightarrow y = \frac{3}{2}x$, and $\left.\frac{dy}{dx}\right|_{t=\pi}$

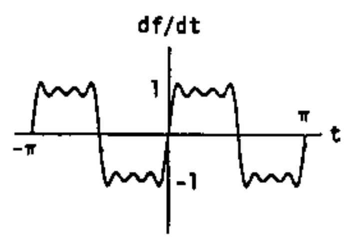
$= \frac{3\cos(3\pi)}{2\cos(2\pi)} = -\frac{3}{2} \Rightarrow y = -\frac{3}{2}x$

73. (a)

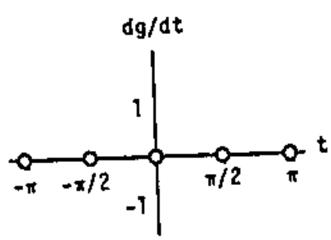


(b) $\frac{df}{dt} = 1.27324\sin 2t + 0.42444\sin 6t + 0.2546\sin 10t + 0.18186\sin 14t$

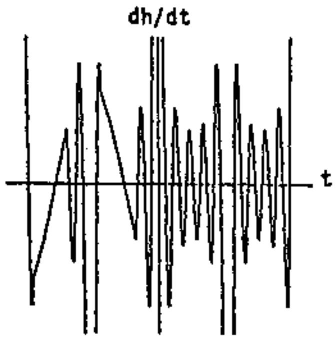
(c) The curve of $y = \frac{df}{dt}$ approximates $y = \frac{dg}{dt}$ the best when t is not $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}$, nor π.



74. (a)



(b) $\frac{dh}{dt} = 2.5464\cos(2t) + 2.5464\cos(6t) + 2.5465\cos(10t) + 2.54604\cos(14t) + 2.54646\cos(18t)$

(c) 

75-80. Example CAS commands:

Maple:

```
x:= t -> exp(t) - t^2;
y:= t -> t + exp(-t);
plot([x(t), y(t), t = -1..2]);
diff(x(t),t);
dx:= unapply(%,t);
diff(y(t),t);
dy:= unapply(%,t);
dy(t)/dx(t);
dydx:= unapply(%,t);
diff(dydx(t),t);
simplify(%): dy1:= unapply(%,t);
dy1(t)/dx(t);
d2ydx2:= unapply(%,t);
t0:=1: evalf(d2ydx2(t0));
tanline:= t -> y(t0) + (dy(t0)/dx(t0))*(t - x(t(0));
plot({[x(t), y(t), t = -1..2], [t, tanline(t), t=t0-1..t0+2]});
```

Mathematica:

```
Clear[x,y,t]
{a,b} = {-Pi,Pi}; t0 = Pi/4;
x[t_] = t - Cos[t]
y[t_] = 1 + Sin[t]
p1 = ParametricPlot[ {x[t],y[t]}, {t,a,b} ]
yp[t_] = y'[t]/x'[t]
ypp[t_] = yp'[t]/x'[t]
yp[t0] // N
ypp[t0] // N
tanline[x_] = y[t0] + yp[t0]*(x-x[t0])
p2 = Plot[ tanline[x], {x,0,0.2} ]
Show[ {p1,p2} ]
```

2.6 IMPLICIT DIFFERENTIATION

1. $y = x^{9/4} \Rightarrow \frac{dy}{dx} = \frac{9}{4}x^{5/4}$

2. $y = \sqrt[3]{2x} = (2x)^{1/3} \Rightarrow \frac{dy}{dx} = \frac{1}{3}(2x)^{-2/3} \cdot 2 = \frac{2^{1/3}}{3x^{2/3}}$

3. $y = 7\sqrt{x+6} = 7(x+6)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{7}{2}(x+6)^{-1/2} = \frac{7}{2\sqrt{x+6}}$

4. $y = (1-6x)^{2/3} \Rightarrow \frac{dy}{dx} = \frac{2}{3}(1-6x)^{-1/3}(-6) = -4(1-6x)^{-1/3}$

5. $y = x\left(x^2+1\right)^{1/2} \Rightarrow y' = (1)\left(x^2+1\right)^{1/2} + \left(\frac{x}{2}\right)\left(x^2+1\right)^{-1/2}(2x) = \frac{2x^2+1}{\sqrt{x^2+1}}$

6. $y = x\left(x^2+1\right)^{-1/2} \Rightarrow y' = (1)\left(x^2+1\right)^{-1/2} + (x)\left(-\frac{1}{2}\right)\left(x^2+1\right)^{-3/2}(2x) = \left(x^2+1\right)^{-3/2}\left[\left(x^2+1\right) - x^2\right]$

$= \frac{1}{\left(x^2+1\right)^{3/2}}$

7. $s = \sqrt[7]{t^2} = t^{2/7} \Rightarrow \frac{ds}{dt} = \frac{2}{7}t^{-5/7}$

8. $r = \sqrt[4]{\theta^{-3}} = \theta^{-3/4} \Rightarrow \frac{dr}{d\theta} = -\frac{3}{4}\theta^{-7/4}$

9. $y = \sin\left((2t+5)^{-2/3}\right) \Rightarrow \frac{dy}{dt} = \cos\left((2t+5)^{-2/3}\right)\cdot\left(-\frac{2}{3}\right)(2t+5)^{-5/3}\cdot 2 = -\frac{4}{3}(2t+5)^{-5/3}\cos\left((2t+5)^{-2/3}\right)$

10. $f(x) = \sqrt{1-\sqrt{x}} = \left(1-x^{1/2}\right)^{1/2} \Rightarrow f'(x) = \frac{1}{2}\left(1-x^{1/2}\right)^{-1/2}\left(-\frac{1}{2}x^{-1/2}\right) = \frac{-1}{4\left(\sqrt{1-\sqrt{x}}\right)\sqrt{x}} = \frac{-1}{4\sqrt{x\left(1-\sqrt{x}\right)}}$

11. $g(x) = 2\left(2x^{-1/2}+1\right)^{-1/3} \Rightarrow g'(x) = -\frac{2}{3}\left(2x^{-1/2}+1\right)^{-4/3}\cdot(-1)x^{-3/2} = \frac{2}{3}\left(2x^{-1/2}+1\right)^{-4/3}x^{-3/2}$

12. $h(\theta) = \sqrt[3]{1+\cos(2\theta)} = (1+\cos 2\theta)^{1/3} \Rightarrow h'(\theta) = \frac{1}{3}(1+\cos 2\theta)^{-2/3}\cdot(-\sin 2\theta)\cdot 2 = -\frac{2}{3}(\sin 2\theta)(1+\cos 2\theta)^{-2/3}$

13. $x^2y + xy^2 = 6$:

Step 1: $\left(x^2\frac{dy}{dx} + y\cdot 2x\right) + \left(x\cdot 2y\frac{dy}{dx} + y^2\cdot 1\right) = 0$

Step 2: $x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$

Step 3: $\frac{dy}{dx}\left(x^2+2xy\right) = -2xy - y^2$

Step 4: $\frac{dy}{dx} = \frac{-2xy-y^2}{x^2+2xy}$

14. $2xy + y^2 = x + y$:

Step 1: $\left(2x\frac{dy}{dx} + 2y\right) + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$

Step 2: $2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y$

Step 3: $\frac{dy}{dx}(2x+2y-1) = 1 - 2y$

Step 4: $\frac{dy}{dx} = \frac{1-2y}{2x+2y-1}$

15. $x^3 - xy + y^3 = 1 \Rightarrow 3x^2 - y - x\,\frac{dy}{dx} + 3y^2\,\frac{dy}{dx} = 0 \Rightarrow \left(3y^2 - x\right)\frac{dy}{dx} = y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}$

16. $x^2(x-y)^2 = x^2 - y^2$:

Step 1: $x^2\left[2(x-y)\left(1 - \frac{dy}{dx}\right)\right] + (x-y)^2(2x) = 2x - 2y\,\frac{dy}{dx}$

Step 2: $-2x^2(x-y)\,\frac{dy}{dx} + 2y\,\frac{dy}{dx} = 2x - 2x^2(x-y) - 2x(x-y)^2$

Step 3: $\frac{dy}{dx}\left[-2x^2(x-y) + 2y\right] = 2x\left[1 - x(x-y) - (x-y)^2\right]$

Step 4: $\frac{dy}{dx} = \frac{2x\left[1 - x(x-y) - (x-y)^2\right]}{-2x^2(x-y) + 2y} = \frac{x\left[1 - x(x-y) - (x-y)^2\right]}{y - x^2(x-y)} = \frac{x\left(1 - x^2 + xy - x^2 + 2xy - y^2\right)}{x^2y - x^3 + y}$

$= \frac{x - 2x^3 + 3x^2y - xy^2}{x^2y - x^3 + y}$

17. $y^2 = \frac{x-1}{x+1} \Rightarrow 2y\,\frac{dy}{dx} = \frac{(x+1) - (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{y(x+1)^2}$

18. $x^2 = \frac{x-y}{x+y} \Rightarrow x^3 + x^2y = x - y \Rightarrow 3x^2 + 2xy + x^2y' = 1 - y' \Rightarrow \left(x^2+1\right)y' = 1 - 3x^2 - 2xy \Rightarrow y' = \frac{1 - 3x^2 - 2xy}{x^2 + 1}$

19. $x = \tan y \Rightarrow 1 = \left(\sec^2 y\right)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$

20. $x + \sin y = xy \Rightarrow 1 + (\cos y)\,\frac{dy}{dx} = y + x\,\frac{dy}{dx} \Rightarrow (\cos y - x)\,\frac{dy}{dx} = y - 1 \Rightarrow \frac{dy}{dx} = \frac{y-1}{\cos y - x}$

21. $y\,\sin\left(\frac{1}{y}\right) = 1 - xy \Rightarrow y\left[\cos\left(\frac{1}{y}\right)\cdot(-1)\,\frac{1}{y^2}\cdot\frac{dy}{dx}\right] + \sin\left(\frac{1}{y}\right)\cdot\frac{dy}{dx} = -x\,\frac{dy}{dx} - y \Rightarrow \frac{dy}{dx}\left[-\frac{1}{y}\cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) + x\right] = -y$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{-\frac{1}{y}\cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) + x} = \frac{-y^2}{y\,\sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right) + xy}$

22. $y^2\cos\left(\frac{1}{y}\right) = 2x + 2y \Rightarrow y^2\left[-\sin\left(\frac{1}{y}\right)\cdot(-1)\,\frac{1}{y^2}\cdot\frac{dy}{dx}\right] + \cos\left(\frac{1}{y}\right)\cdot 2y\,\frac{dy}{dx} = 2 + 2\,\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left[\sin\left(\frac{1}{y}\right) + 2y\,\cos\left(\frac{1}{y}\right) - 2\right] = 2$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sin\left(\frac{1}{y}\right) + 2y\,\cos\left(\frac{1}{y}\right) - 2}$

23. $\theta^{1/2} + r^{1/2} = 1 \Rightarrow \frac{1}{2}\theta^{-1/2} + \frac{1}{2}r^{-1/2}\cdot\frac{dr}{d\theta} = 0 \Rightarrow \frac{dr}{d\theta}\left[\frac{1}{2\sqrt{r}}\right] = \frac{-1}{2\sqrt{\theta}} \Rightarrow \frac{dr}{d\theta} = -\frac{2\sqrt{r}}{2\sqrt{\theta}} = -\frac{\sqrt{r}}{\sqrt{\theta}}$

24. $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4} \Rightarrow \frac{dr}{d\theta} - \theta^{-1/2} = \theta^{-1/3} + \theta^{-1/4} \Rightarrow \frac{dr}{d\theta} = \theta^{-1/2} + \theta^{-1/3} + \theta^{-1/4}$

25. $\sin(r\theta) = \frac{1}{2} \Rightarrow [\cos(r\theta)]\left(r + \theta\,\frac{dr}{d\theta}\right) = 0 \Rightarrow \frac{dr}{d\theta}[\theta\cos(r\theta)] = -r\cos(r\theta) \Rightarrow \frac{dr}{d\theta} = \frac{-r\cos(r\theta)}{\theta\cos(r\theta)} = -\frac{r}{\theta}$, $\cos(r\theta) \neq 0$

26. $\cos r + \cos\theta = r\theta \Rightarrow (-\sin r)\,\frac{dr}{d\theta} - \sin\theta = r + \theta\,\frac{dr}{d\theta} \Rightarrow \frac{dr}{d\theta}[-\theta - \sin r] = r + \sin\theta \Rightarrow \frac{dr}{d\theta} = \frac{-(r+\sin\theta)}{\theta + \sin r}$

27. $x^{2/3} + y^{2/3} = 1 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\,\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}\left[\frac{2}{3}y^{-1/3}\right] = -\frac{2}{3}x^{-1/3} \Rightarrow y' = \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$;

Differentiating again, $y'' = \frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)y' + y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}} = \frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)\left(-\frac{y^{1/3}}{x^{1/3}}\right) + y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{3}x^{-2/3}y^{-1/3} + \frac{1}{3}y^{1/3}x^{-4/3} = \frac{y^{1/3}}{3x^{4/3}} + \frac{1}{3y^{1/3}x^{2/3}}$

28. $y^2 = x^2 + 2x \Rightarrow 2yy' = 2x + 2 \Rightarrow y' = \frac{2x+2}{2y} = \frac{x+1}{y}$; then $y'' = \frac{y - (x+1)y'}{y^2} = \frac{y - (x+1)\left(\frac{x+1}{y}\right)}{y^2}$

$\Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{y^2 - (x+1)^2}{y^3}$

29. $2\sqrt{y} = x - y \Rightarrow y^{-1/2}y' = 1 - y' \Rightarrow y'\left(y^{-1/2} + 1\right) = 1 \Rightarrow \frac{dy}{dx} = y' = \frac{1}{y^{-1/2}+1} = \frac{\sqrt{y}}{\sqrt{y}+1}$; we can

differentiate the equation $y'\left(y^{-1/2}+1\right) = 1$ again to find y'': $y'\left(-\frac{1}{2}y^{-3/2}y'\right) + \left(y^{-1/2}+1\right)y'' = 0$

$\Rightarrow \left(y^{-1/2}+1\right)y'' = \frac{1}{2}[y']^2 y^{-3/2} \Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{\frac{1}{2}\left(\frac{1}{y^{-1/2}+1}\right)^2 y^{-3/2}}{\left(y^{-1/2}+1\right)} = \frac{1}{2y^{3/2}\left(y^{-1/2}+1\right)^3} = \frac{1}{2\left(1+\sqrt{y}\right)^3}$

30. $xy + y^2 = 1 \Rightarrow xy' + y + 2yy' = 0 \Rightarrow xy' + 2yy' = -y \Rightarrow y'(x+2y) = -y \Rightarrow y' = \frac{-y}{(x+2y)}$; $\frac{d^2y}{dx^2} = y''$

$= \frac{-(x+2y)y' + y(1+2y')}{(x+2y)^2} = \frac{-(x+2y)\left[\frac{-y}{(x+2y)}\right] + y\left[1 + 2\left(\frac{-y}{(x+2y)}\right)\right]}{(x+2y)^2} = \frac{\frac{1}{(x+2y)}\left[y(x+2y) + y(x+2y) - 2y^2\right]}{(x+2y)^2}$

$= \frac{2y(x+2y) - 2y^2}{(x+2y)^3} = \frac{2y^2 + 2xy}{(x+2y)^3} = \frac{2y(x+y)}{(x+2y)^3}$

31. $x^3 + y^3 = 16 \Rightarrow 3x^2 + 3y^2y' = 0 \Rightarrow 3y^2y' = -3x^2 \Rightarrow y' = -\frac{x^2}{y^2}$; we differentiate $y^2y' = -x^2$ to find y'':

$y^2y'' + y'[2y\cdot y'] = -2x \Rightarrow y^2y'' = -2x - 2y[y']^2 \Rightarrow y'' = \frac{-2x - 2y\left(-\frac{x^2}{y^2}\right)^2}{y^2} = \frac{-2x - \frac{2x^4}{y^3}}{y^2}$

$= \frac{-2xy^3 - 2x^4}{y^5} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(2,2)} = \frac{-32-32}{32} = -2$

32. $xy + y^2 = 1 \Rightarrow xy' + y + 2yy' = 0 \Rightarrow y'(x+2y) = -y \Rightarrow y' = \frac{-y}{(x+2y)} \Rightarrow y'' = \frac{(x+2y)(-y') - (-y)(1+2y')}{(x+2y)^2}$;

since $y'\big|_{(0,-1)} = -\frac{1}{2}$ we obtain $y''\big|_{(0,-1)} = \frac{(-2)\left(\frac{1}{2}\right) - (1)(0)}{4} = -\frac{1}{4}$

33. $x^2 - 2tx + 2t^2 = 4 \Rightarrow 2x\frac{dx}{dt} - 2x - 2t\frac{dx}{dt} + 4t = 0 \Rightarrow (2x-2t)\frac{dx}{dt} = 2x - 4t \Rightarrow \frac{dx}{dt} = \frac{2x-4t}{2x-2t} = \frac{x-2t}{x-t}$;

$2y^3 - 3t^2 = 4 \Rightarrow 6y^2\frac{dy}{dt} - 6t = 0 \Rightarrow \frac{dy}{dt} = \frac{6t}{6y^2} = \frac{t}{y^2}$; thus $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{t}{y^2}\right)}{\left(\frac{x-2t}{x-t}\right)} = \frac{t(x-t)}{y^2(x-2t)}$; $t = 2$

$\Rightarrow x^2 - 2(2)x + 2(2)^2 = 4 \Rightarrow x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0 \Rightarrow x = 2$; $t = 2 \Rightarrow 2y^3 - 3(2)^2 = 4$

$\Rightarrow 2y^3 = 16 \Rightarrow y^3 = 8 \Rightarrow y = 2$; therefore $\frac{dy}{dx}\Big|_{t=2} = \frac{2(2-2)}{(2)^2(2-2(2))} = 0$

34. $x = \sqrt{5-\sqrt{t}} \Rightarrow \frac{dx}{dt} = \frac{1}{2}\left(5-\sqrt{t}\right)^{-1/2}\left(-\frac{1}{2}t^{-1/2}\right) = \frac{-1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}$; $y(t-1) = \ln y \Rightarrow \frac{dy}{dt}(t-1) + y = \left(\frac{1}{y}\right)\frac{dy}{dt}$

$\Rightarrow \left(t - 1 - \frac{1}{y}\right)\frac{dy}{dt} = -y \Rightarrow \frac{dy}{dt} = \frac{-y}{\left(t-1-\frac{1}{y}\right)} = \frac{-y^2}{ty-y-1}$; thus $\frac{dy}{dx} = \frac{\left(\frac{-y^2}{ty-y-1}\right)}{\left(\frac{-1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}\right)} = \frac{4y^2\sqrt{t}\sqrt{5-\sqrt{t}}}{ty-y-1}$;

$t = 1 \Rightarrow y(1-1) = \ln y \Rightarrow 0 = \ln y \Rightarrow y = 1$; therefore $\frac{dy}{dx}\Big|_{t=1} = \frac{4(1)^2\sqrt{1}\sqrt{5-\sqrt{1}}}{(1)(1)-1-1} = -8$

35. $x + 2x^{3/2} = t^2 + t \Rightarrow \frac{dx}{dt} + 3x^{1/2}\frac{dx}{dt} = 2t + 1 \Rightarrow \left(1 + 3x^{1/2}\right)\frac{dx}{dt} = 2t + 1 \Rightarrow \frac{dx}{dt} = \frac{2t+1}{1+3x^{1/2}}$; $y\sqrt{t+1} + 2t\sqrt{y} = 4$

$\Rightarrow \frac{dy}{dt}\sqrt{t+1} + y\left(\frac{1}{2}\right)(t+1)^{-1/2} + 2\sqrt{y} + 2t\left(\frac{1}{2}y^{-1/2}\right)\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt}\sqrt{t+1} + \frac{y}{2\sqrt{t+1}} + 2\sqrt{y} + \left(\frac{t}{\sqrt{y}}\right)\frac{dy}{dt} = 0$

$\Rightarrow \left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right)\frac{dy}{dt} = \frac{-y}{2\sqrt{t+1}} - 2\sqrt{y} \Rightarrow \frac{dy}{dt} = \frac{\left(\frac{-y}{2\sqrt{t+1}} - 2\sqrt{y}\right)}{\left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right)} = \frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2\sqrt{y}(t+1) + 2t\sqrt{t+1}}$; thus

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2\sqrt{y}(t+1) + 2t\sqrt{t+1}}\right)}{\left(\frac{2t+1}{1+3x^{1/2}}\right)}$; $t = 0 \Rightarrow x + 2x^{3/2} = 0 \Rightarrow x\left(1 + 2x^{1/2}\right) = 0 \Rightarrow x = 0$; $t = 0$

$\Rightarrow y\sqrt{0+1} + 2(0)\sqrt{y} = 4 \Rightarrow y = 4$; therefore $\frac{dy}{dx}\Big|_{t=0} = \frac{\left(\frac{-4\sqrt{4} - 4(4)\sqrt{0+1}}{2\sqrt{4}(0+1) + 2(0)\sqrt{0+1}}\right)}{\left(\frac{2(0)+1}{1+3(0)^{1/2}}\right)} = -6$

36. $x \sin t + 2x = t \Rightarrow \frac{dx}{dt} \sin t + x \cos t + 2\frac{dx}{dt} = 1 \Rightarrow (\sin t + 2)\frac{dx}{dt} = 1 - x \cos t \Rightarrow \frac{dx}{dt} = \frac{1 - x \cos t}{\sin t + 2}$;
$t \sin t - 2t = y \Rightarrow \sin t + t \cos t - 2 = \frac{dy}{dt}$; thus $\frac{dy}{dx} = \frac{\sin t + t \cos t - 2}{\left(\frac{1 - x \cos t}{\sin t + 2}\right)}$; $t = \pi \Rightarrow x \sin \pi + 2x = \pi$
$\Rightarrow x = \frac{\pi}{2}$; therefore $\left.\frac{dy}{dx}\right|_{t=\pi} = \frac{\sin \pi + \pi \cos \pi - 2}{\left[\frac{1 - \left(\frac{\pi}{2}\right)\cos \pi}{\sin \pi + 2}\right]} = \frac{-4\pi - 8}{2 + \pi} = -4$

37. $y^2 + x^2 = y^4 - 2x$ at $(-2,1)$ and $(-2,-1) \Rightarrow 2y\frac{dy}{dx} + 2x = 4y^3\frac{dy}{dx} - 2 \Rightarrow 2y\frac{dy}{dx} - 4y^3\frac{dy}{dx} = -2 - 2x$
$\Rightarrow \frac{dy}{dx}(2y - 4y^3) = -2 - 2x \Rightarrow \frac{dy}{dx} = \frac{x+1}{2y^3 - y} \Rightarrow \left.\frac{dy}{dx}\right|_{(-2,1)} = -1$ and $\left.\frac{dy}{dx}\right|_{(-2,-1)} = 1$

38. $(x^2 + y^2)^2 = (x - y)^2$ at$(1,0)$ and $(1,-1) \Rightarrow 2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 2(x - y)\left(1 - \frac{dy}{dx}\right)$
$\Rightarrow \frac{dy}{dx}\left[2y(x^2 + y^2) + (x - y)\right] = -2x(x^2 + y^2) + (x - y) \Rightarrow \frac{dy}{dx} = \frac{-2x(x^2 + y^2) + (x - y)}{2y(x^2 + y^2) + (x - y)} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,0)} = -1$
and $\left.\frac{dy}{dx}\right|_{(1,-1)} = 1$

39. $x^2 + xy - y^2 = 1 \Rightarrow 2x + y + xy' - 2yy' = 0 \Rightarrow (x - 2y)y' = -2x - y \Rightarrow y' = \frac{2x + y}{2y - x}$;
(a) the slope of the tangent line $m = y'|_{(2,3)} = \frac{7}{4} \Rightarrow$ the tangent line is $y - 3 = \frac{7}{4}(x - 2) \Rightarrow y = \frac{7}{4}x - \frac{1}{2}$
(b) the normal line is $y - 3 = -\frac{4}{7}(x - 2) \Rightarrow y = -\frac{4}{7}x + \frac{29}{7}$

40. $x^2y^2 = 9 \Rightarrow 2xy^2 + 2x^2yy' = 0 \Rightarrow x^2yy' = -xy^2 \Rightarrow y' = -\frac{y}{x}$;
(a) the slope of the tangent line $m = y'|_{(-1,3)} = -\left.\frac{y}{x}\right|_{(-1,3)} = 3 \Rightarrow$ the tangent line is $y - 3 = 3(x + 1)$
$\Rightarrow y = 3x + 6$
(b) the normal line is $y - 3 = -\frac{1}{3}(x + 1) \Rightarrow y = -\frac{1}{3}x + \frac{8}{3}$

41. $y^2 - 2x - 4y - 1 = 0 \Rightarrow 2yy' - 2 - 4y' = 0 \Rightarrow 2(y - 2)y' = 2 \Rightarrow y' = \frac{1}{y - 2}$;
(a) the slope of the tangent line $m = y'|_{(-2,1)} = -1 \Rightarrow$ the tangent line is $y - 1 = -1(x + 2) \Rightarrow y = -x - 1$
(b) the normal line is $y - 1 = 1(x + 2) \Rightarrow y = x + 3$

42. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0 \Rightarrow 12x + 3y + 3xy' + 4yy' + 17y' = 0 \Rightarrow y'(3x + 4y + 17) = -12x - 3y$
$\Rightarrow y' = \frac{-12x - 3y}{3x + 4y + 17}$;
(a) the slope of the tangent line $m = y'|_{(-1,0)} = \left.\frac{-12x - 3y}{3x + 4y + 17}\right|_{(-1,0)} = \frac{6}{7} \Rightarrow$ the tangent line is $y - 0 = \frac{6}{7}(x + 1)$
$\Rightarrow y = \frac{6}{7}x + \frac{6}{7}$
(b) the normal line is $y - 0 = -\frac{7}{6}(x + 1) \Rightarrow y = -\frac{7}{6}x - \frac{7}{6}$

43. $2xy + \pi \sin y = 2\pi \Rightarrow 2xy' + 2y + \pi(\cos y)y' = 0 \Rightarrow y'(2x + \pi \cos y) = -2y \Rightarrow y' = \frac{-2y}{2x + \pi \cos y}$;

(a) the slope of the tangent line $m = y'\big|_{\left(1, \frac{\pi}{2}\right)} = \frac{-2y}{2x + \pi \cos y}\Big|_{\left(1, \frac{\pi}{2}\right)} = -\frac{\pi}{2} \Rightarrow$ the tangent line is

$y - \frac{\pi}{2} = -\frac{\pi}{2}(x - 1) \Rightarrow y = -\frac{\pi}{2}x + \pi$

(b) the normal line is $y - \frac{\pi}{2} = \frac{2}{\pi}(x - 1) \Rightarrow y = \frac{2}{\pi}x - \frac{2}{\pi} + \frac{\pi}{2}$

44. $x \sin 2y = y \cos 2x \Rightarrow x(\cos 2y)2y' + \sin 2y = -2y \sin 2x + y' \cos 2x \Rightarrow y'(2x \cos 2y - \cos 2x)$

$= -\sin 2y - 2y \sin 2x \Rightarrow y' = \frac{\sin 2y + 2y \sin 2x}{\cos 2x - 2x \cos 2y}$;

(a) the slope of the tangent line $m = y'\big|_{\left(\frac{\pi}{4}, \frac{\pi}{2}\right)} = \frac{\sin 2y + 2y \sin 2x}{\cos 2x - 2x \cos 2y}\Big|_{\left(\frac{\pi}{4}, \frac{\pi}{2}\right)} = \frac{\pi}{\frac{\pi}{2}} = 2 \Rightarrow$ the tangent line is

$y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{4}\right) \Rightarrow y = 2x$

(b) the normal line is $y - \frac{\pi}{2} = -\frac{1}{2}\left(x - \frac{\pi}{4}\right) \Rightarrow y = -\frac{1}{2}x + \frac{5\pi}{8}$

45. $y = 2 \sin(\pi x - y) \Rightarrow y' = 2[\cos(\pi x - y)] \cdot (\pi - y') \Rightarrow y'[1 + 2 \cos(\pi x - y)] = 2\pi \cos(\pi x - y)$

$\Rightarrow y' = \frac{2\pi \cos(\pi x - y)}{1 + 2 \cos(\pi x - y)}$;

(a) the slope of the tangent line $m = y'\big|_{(1,0)} = \frac{2\pi \cos(\pi x - y)}{1 + 2 \cos(\pi x - y)}\Big|_{(1,0)} = 2\pi \Rightarrow$ the tangent line is

$y - 0 = 2\pi(x - 1) \Rightarrow y = 2\pi x - 2\pi$

(b) the normal line is $y - 0 = -\frac{1}{2\pi}(x - 1) \Rightarrow y = -\frac{x}{2\pi} + \frac{1}{2\pi}$

46. $x^2 \cos^2 y - \sin y = 0 \Rightarrow x^2(2 \cos y)(-\sin y)y' + 2x \cos^2 y - y' \cos y = 0 \Rightarrow y'\left[-2x^2 \cos y \sin y - \cos y\right]$

$= -2x \cos^2 y \Rightarrow y' = \frac{2x \cos^2 y}{2x^2 \cos y \sin y + \cos y}$;

(a) the slope of the tangent line $m = y'\big|_{(0,\pi)} = \frac{2x \cos^2 y}{2x^2 \cos y \sin y + \cos y}\Big|_{(0,\pi)} = 0 \Rightarrow$ the tangent line is $y = \pi$

(b) the normal line is $x = 0$

47. Solving $x^2 + xy + y^2 = 7$ and $y = 0 \Rightarrow x^2 = 7 \Rightarrow x = \pm\sqrt{7} \Rightarrow \left(-\sqrt{7}, 0\right)$ and $\left(\sqrt{7}, 0\right)$ are the points where the curve crosses the x-axis. Now $x^2 + xy + y^2 = 7 \Rightarrow 2x + y + xy' + 2yy' = 0 \Rightarrow (x + 2y)y' = -2x - y$ $\Rightarrow y' = -\frac{2x + y}{x + 2y} \Rightarrow m = -\frac{2x + y}{x + 2y} \Rightarrow$ the slope at $\left(-\sqrt{7}, 0\right)$ is $m = -\frac{-2\sqrt{7}}{-\sqrt{7}} = -2$ and the slope at $\left(\sqrt{7}, 0\right)$ is $m = -\frac{2\sqrt{7}}{\sqrt{7}} = -2$. Since the slope is -2 in each case, the corresponding tangents must be parallel.

48. $x^2 + xy + y^2 = 7 \Rightarrow 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \Rightarrow (x + 2y)\frac{dy}{dx} = -2x - y \Rightarrow \frac{dy}{dx} = \frac{-2x - y}{x + 2y}$ and $\frac{dx}{dy} = \frac{x + 2y}{-2x - y}$;

(a) Solving $\frac{dy}{dx} = 0 \Rightarrow -2x - y = 0 \Rightarrow y = -2x$ and substitution into the original equation gives

$x^2 + x(-2x) + (-2x)^2 = 7 \Rightarrow 3x^2 = 7 \Rightarrow x = \pm\sqrt{\frac{7}{3}}$ and $y = \mp 2\sqrt{\frac{7}{3}}$ when the tangents are parallel to the x-axis.

(b) Solving $\frac{dx}{dy} = 0 \Rightarrow x + 2y = 0 \Rightarrow y = -\frac{x}{2}$ and substitution gives $x^2 + x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 = 7 \Rightarrow \frac{3x^2}{4} = 7$

$\Rightarrow x = \pm 2\sqrt{\frac{7}{3}}$ and $y = \mp\sqrt{\frac{7}{3}}$ when the tangents are parallel to the y-axis.

49. $y^4 = y^2 - x^2 \Rightarrow 4y^3y' = 2yy' - 2x \Rightarrow 2\left(2y^3 - y\right)y' = -2x \Rightarrow y' = \frac{x}{y - 2y^3}$; the slope of the tangent line at

$\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$ is $\frac{x}{y-2y^3}\Big|_{\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)} = \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} - \frac{6\sqrt{3}}{8}} = \frac{\frac{1}{4}}{\frac{1}{2} - \frac{3}{4}} = \frac{1}{2-3} = -1$; the slope of the tangent line at $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)$

is $\frac{x}{y-2y^3}\Big|_{\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)} = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - \frac{2}{8}} = \frac{2\sqrt{3}}{4-2} = \sqrt{3}$

50. $y^2(2-x) = x^3 \Rightarrow 2yy'(2-x) + y^2(-1) = 3x^2 \Rightarrow y' = \frac{y^2 + 3x^2}{2y(2-x)}$; the slope of the tangent line is

$m = \frac{y^2 + 3x^2}{2y(2-x)}\Big|_{(1,1)} = \frac{4}{2} = 2 \Rightarrow$ the tangent line is $y - 1 = 2(x-1) \Rightarrow y = 2x - 1$; the normal line is

$y - 1 = -\frac{1}{2}(x-1) \Rightarrow y = -\frac{1}{2}x + \frac{3}{2}$

51. $y^4 - 4y^2 = x^4 - 9x^2 \Rightarrow 4y^3y' - 8yy' = 4x^3 - 18x \Rightarrow y'\left(4y^3 - 8y\right) = 4x^3 - 18x \Rightarrow y' = \frac{4x^3 - 18x}{4y^3 - 8y} = \frac{2x^3 - 9x}{2y^3 - 4y}$

$= \frac{x\left(2x^2 - 9\right)}{y\left(2y^2 - 4\right)} = m$; $(-3, 2)$: $m = \frac{(-3)(18-9)}{2(8-4)} = -\frac{27}{8}$; $(-3, -2)$: $m = \frac{27}{8}$; $(3, 2)$: $m = \frac{27}{8}$; $(3, -2)$: $m = -\frac{27}{8}$

52. $x^3 + y^3 - 9xy = 0 \Rightarrow 3x^2 + 3y^2y' - 9xy' - 9y = 0 \Rightarrow y'\left(3y^2 - 9x\right) = 9y - 3x^2 \Rightarrow y' = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$

(a) $y'\big|_{(4,2)} = \frac{5}{4}$ and $y'\big|_{(2,4)} = \frac{4}{5}$;

(b) $y' = 0 \Rightarrow \frac{3y - x^2}{y^2 - 3x} = 0 \Rightarrow 3y - x^2 = 0 \Rightarrow y = \frac{x^2}{3} \Rightarrow x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0 \Rightarrow x^6 - 54x^3 = 0$

$\Rightarrow x^3\left(x^3 - 54\right) = 0 \Rightarrow x = 0$ or $x = \sqrt[3]{54} = 3\sqrt[3]{2} \Rightarrow$ there is a horizontal tangent at $x = 3\sqrt[3]{2}$. To find the corresponding y-value, we will use part (c).

(c) $\frac{dx}{dy} = 0 \Rightarrow \frac{y^2 - 3x}{3y - x^2} = 0 \Rightarrow y^2 - 3x = 0 \Rightarrow y = \pm\sqrt{3x}$; $y = \sqrt{3x} \Rightarrow x^3 + \left(\sqrt{3x}\right)^3 - 9x\sqrt{3x} = 0$

$\Rightarrow x^3 - 6\sqrt{3}\,x^{3/2} = 0 \Rightarrow x^{3/2}\left(x^{3/2} - 6\sqrt{3}\right) = 0 \Rightarrow x^{3/2} = 0$ or $x^{3/2} = 6\sqrt{3} \Rightarrow x = 0$ or $x = \sqrt[3]{108} = 3\sqrt[3]{4}$.

Since the equation $x^3 + y^3 - 9xy = 0$ is symmetric in x and y, the graph is symmetric about the line $y = x$. That is, if (a, b) is a point on the folium, then so is (b, a). Moreover, if $y'\big|_{(a,b)} = m$, then $y'\big|_{(b,a)} = \frac{1}{m}$. Thus, if the folium has a horizontal tangent at (a, b), it has a vertical tangent at (b, a) so one might expect that with a horizontal tangent at $x = \sqrt[3]{54}$ and a vertical tangent at $x = 3\sqrt[3]{4}$, the points of tangency are $\left(\sqrt[3]{54}, 3\sqrt[3]{4}\right)$ and $\left(3\sqrt[3]{4}, \sqrt[3]{54}\right)$, respectively. One can check that these points do satisfy the equation $x^3 + y^3 - 9xy = 0$.

53. (a) if $f(x) = \frac{3}{2}x^{2/3} - 3$, then $f'(x) = x^{-1/3}$ and $f''(x) = -\frac{1}{3}x^{-4/3}$ so the claim $f''(x) = x^{-1/3}$ is false

(b) if $f(x) = \frac{9}{10}x^{5/3} - 7$, then $f'(x) = \frac{3}{2}x^{2/3}$ and $f''(x) = x^{-1/3}$ is true

(c) $f''(x) = x^{-1/3} \Rightarrow f'''(x) = -\frac{1}{3}x^{-4/3}$ is true

(d) if $f'(x) = \frac{3}{2}x^{2/3} + 6$, then $f''(x) = x^{-1/3}$ is true

54. $2x^2 + 3y^2 = 5 \Rightarrow 4x + 6yy' = 0 \Rightarrow y' = -\frac{2x}{3y} \Rightarrow y'\big|_{(1,1)} = -\frac{2x}{3y}\Big|_{(1,1)} = -\frac{2}{3}$ and $y'\big|_{(1,-1)} = -\frac{2x}{3y}\Big|_{(1,-1)} = \frac{2}{3}$;

also, $y^2 = x^3 \Rightarrow 2yy' = 3x^2 \Rightarrow y' = \frac{3x^2}{2y} \Rightarrow y'\big|_{(1,1)} = \frac{3x^2}{2y}\Big|_{(1,1)} = \frac{3}{2}$ and $y'\big|_{(1,-1)} = \frac{3x^2}{2y}\Big|_{(1,-1)} = -\frac{3}{2}$. Therefore

the tangents to the curves are perpendicular at $(1,1)$ and $(1,-1)$ (i.e., the curves are orthogonal at these two points of intersection).

55. $x^2 + 2xy - 3y^2 = 0 \Rightarrow 2x + 2xy' + 2y - 6yy' = 0 \Rightarrow y'(2x - 6y) = -2x - 2y \Rightarrow y' = \frac{x+y}{3y-x} \Rightarrow$ the slope of the

tangent line $m = y'\big|_{(1,1)} = \frac{x+y}{3y-x}\Big|_{(1,1)} = 1 \Rightarrow$ the equation of the normal line at $(1,1)$ is $y - 1 = -1(x-1)$ $\Rightarrow y = -x + 2$. To find where the normal line intersects the curve we substitute into its equation: $x^2 + 2x(2-x) - 3(2-x)^2 = 0 \Rightarrow x^2 + 4x - 2x^2 - 3(4 - 4x + x^2) = 0 \Rightarrow -4x^2 + 16x - 12 = 0 \Rightarrow x^2 - 4x + 3 = 0$ $\Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 3$ and $y = -x + 2 = -1$. Therefore, the normal to the curve at $(1,1)$ intersects the curve at the point $(3,-1)$. Note that it also intersects the curve at $(1,1)$.

56. $xy + 2x - y = 0 \Rightarrow x\frac{dy}{dx} + y + 2 - \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{1-x}$; the slope of the line $2x + y = 0$ is -2. In order to be parallel, the normal lines must also have slope of -2. Since a normal is perpendicular to a tangent, the slope of the tangent is $\frac{1}{2}$. Therefore, $\frac{y+2}{1-x} = \frac{1}{2} \Rightarrow 2y + 4 = 1 - x \Rightarrow x = -3 - 2y$. Substituting in the original equation, $y(-3-2y) + 2(-3-2y) - y = 0 \Rightarrow y^2 + 4y + 3 = 0 \Rightarrow y = -3$ or $y = -1$. If $y = -3$, then $x = 3$ and $y + 3 = -2(x-3) \Rightarrow y = -2x + 3$. If $y = -1$, then $x = -1$ and $y + 1 = -2(x+1) \Rightarrow y = -2x - 3$.

57. $y^2 = x \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$. If a normal is drawn from $(a,0)$ to (x_1,y_1) on the curve its slope satisfies $\frac{y_1 - 0}{x_1 - a} = -2y_1$ $\Rightarrow y_1 = -2y_1(x_1 - a)$ or $a = x_1 + \frac{1}{2}$. Since $x_1 \ge 0$ on the curve, we must have that $a \ge \frac{1}{2}$. By symmetry, the two points on the parabola are $(x_1, \sqrt{x_1})$ and $(x_1, -\sqrt{x_1})$. For the normal to be perpendicular,

$\left(\frac{\sqrt{x_1}}{x_1 - a}\right)\left(\frac{\sqrt{x_1}}{a - x_1}\right) = -1 \Rightarrow \frac{x_1}{(a-x_1)^2} = 1 \Rightarrow x_1 = (a - x_1)^2 \Rightarrow x_1 = \left(x_1 + \frac{1}{2} - x_1\right)^2 \Rightarrow x_1 = \frac{1}{4}$ and $y_1 = \pm\frac{1}{2}$.

Therefore, $\left(\frac{1}{4}, \pm\frac{1}{2}\right)$ and $a = \frac{3}{4}$.

58. Ex. 5a.) $y = x^{1/2}$ has no derivative at $x = 0$ because the slope of the graph becomes vertical at $x = 0$.

Ex. 5b.) $y = x^{2/3}$ has no derivative at $x = 0$ because the slope of the graph becomes vertical at $x = 0$.

Ex. 6a.) $y = (1 - x^2)^{1/4}$ has a derivative only on $(-1, 1)$ because the function is defined only on $[-1, 1]$ and the slope of the tangent becomes vertical at both $x = -1$ and $x = 1$.

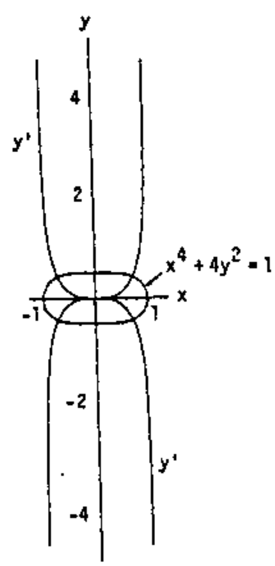
59. $xy^3 + x^2y = 6 \Rightarrow x\left(3y^2 \frac{dy}{dx}\right) + y^3 + x^2 \frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx}(3xy^2 + x^2) = -y^3 - 2xy \Rightarrow \frac{dy}{dx} = \frac{-y^3 - 2xy}{3xy^2 + x^2}$
$= -\frac{y^3 + 2xy}{3xy^2 + x^2}$; also, $xy^3 + x^2y = 6 \Rightarrow x(3y^2) + y^3 \frac{dx}{dy} + x^2 + y\left(2x \frac{dx}{dy}\right) = 0 \Rightarrow \frac{dx}{dy}(y^3 + 2xy) = -3xy^2 - x^2$
$\Rightarrow \frac{dx}{dy} = -\frac{3xy^2 + x^2}{y^3 + 2xy}$; thus $\frac{dx}{dy}$ appears to equal $\frac{1}{\frac{dy}{dx}}$. The two different treatments view the graphs as functions symmetric across the line $y = x$, so their slopes are reciprocals of one another at the corresponding points (a, b) and (b, a).

60. $x^3 + y^2 = \sin^2 y \Rightarrow 3x^2 + 2y \frac{dy}{dx} = (2 \sin y)(\cos y) \frac{dy}{dx} \Rightarrow \frac{dy}{dx}(2y - 2 \sin y \cos y) = -3x^2 \Rightarrow \frac{dy}{dx} = \frac{-3x^2}{2y - 2 \sin y \cos y}$
$= \frac{3x^2}{2 \sin y \cos y - 2y}$; also, $x^3 + y^2 = \sin^2 y \Rightarrow 3x^2 \frac{dx}{dy} + 2y = 2 \sin y \cos y \Rightarrow \frac{dx}{dy} = \frac{2 \sin y \cos y - 2y}{3x^2}$; thus $\frac{dx}{dy}$ appears to equal $\frac{1}{\frac{dy}{dx}}$. The two different treatments view the graphs as functions symmetric across the line $y = x$ so their slopes are reciprocals of one another at the corresponding points (a, b) and (b, a).

61. $x^4 + 4y^2 = 1$:

(a) $y^2 = \frac{1 - x^4}{4} \Rightarrow y = \pm\frac{1}{2}\sqrt{1 - x^4} \Rightarrow \frac{dy}{dx} = \pm\frac{1}{4}(1 - x^4)^{-1/2}(-4x^3) = \frac{\pm x^3}{(1 - x^4)^{1/2}}$; differentiating implicitly, we find, $4x^3 + 8y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-4x^3}{8y} = \frac{-4x^3}{8\left(\pm\frac{1}{2}\sqrt{1 - x^4}\right)} = \frac{\pm x^3}{(1 - x^4)^{1/2}}$.

(b)

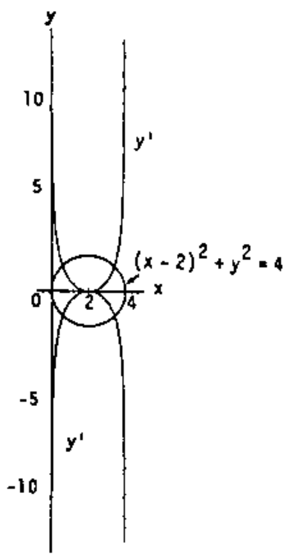


62. $(x-2)^2+y^2=4$:

(a) $y = \pm\sqrt{4-(x-2)^2} \Rightarrow \frac{dy}{dx} = \pm\frac{1}{2}\left(4-(x-2)^2\right)^{-1/2}(-2(x-2)) = \frac{\pm(x-2)}{\left[4-(x-2)^2\right]^{1/2}}$; differentiating implicitly,

$$2(x-2)+2y\,\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-2(x-2)}{2y}=\frac{-(x-2)}{y}=\frac{-(x-2)}{\pm\left[4-(x-2)^2\right]^{1/2}}=\frac{\pm(x-2)}{\left[4-(x-2)^2\right]^{1/2}}.$$

(b)



63-70. Example CAS commands:

Maple:

```
with(plots):
eq1 := x + tan(y/x) = 2;
x0 := 1: y0 := Pi/4:
subs({x=x0, y=y0}, eq1);
implicitplot(eq1, x=x0 - 3..x0 + 3, y=y0 - 3..y0 + 3);
subs(y=y(x),eq1):
diff(%,x);
solve(%,diff(y(x),x));
m:=subs({x=x0,y(x)=y0},%);
tanline := y = y0 + m*(x-x0);
implicitplot({eq1,tanline}, x=x0 - 2..x0 + 2,y=y0 - 3..y0 + 2);
```

Mathematica:

```
Graphics 'ImplicitPlot'
Clear[x,y]
{x0,y0} = {1,Pi/4}; eqn = x + Tan[y/x] == 2
ImplicitPlot[ eqn, {x,x0 - 3,x0 + 3}, {y,y0 - 3,y0 + 3} ]
eqn /. {x -> x0, y -> y0}
eqn /. {y -> y[x]}
D[%,x]
Solve[%, y'[x] ]
slope = y'[x] /. First[%]
m = slope /. {x -> x0, y[x] -> y0}
tanline = y == y0 + m (x - x0)
ImplicitPlot[{eqn, tanline}, {x,x0 - 3,x0 + 3}, {y,y0 - 3,y0 + 3} ]
```

2.7 RELATED RATES

1. $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

2. $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

3. (a) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$ (b) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = 2\pi rh \frac{dr}{dt}$

   (c) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi rh \frac{dr}{dt}$

4. (a) $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt}$ (b) $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{2}{3}\pi rh \frac{dr}{dt}$

   (c) $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi rh \frac{dr}{dt}$

5. (a) $\frac{dV}{dt} = 1$ volt/sec (b) $\frac{dI}{dt} = -\frac{1}{3}$ amp/sec

   (c) $\frac{dV}{dt} = R\left(\frac{dI}{dt}\right) + I\left(\frac{dR}{dt}\right) \Rightarrow \frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - R\frac{dI}{dt}\right) \Rightarrow \frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - \frac{V}{I}\frac{dI}{dt}\right)$

   (d) $\frac{dR}{dt} = \frac{1}{2}\left[1 - \frac{12}{2}\left(-\frac{1}{3}\right)\right] = \left(\frac{1}{2}\right)(3) = \frac{3}{2}$ ohms/sec, R is increasing

6. (a) $P = Ri^2 \Rightarrow \frac{dP}{dt} = i^2 \frac{dR}{dt} + 2Ri \frac{di}{dt}$

   (b) $P = Ri^2 \Rightarrow 0 = \frac{dP}{dt} = i^2 \frac{dR}{dt} + 2Ri \frac{di}{dt} \Rightarrow \frac{dR}{dt} = -\frac{2Ri}{i^2}\frac{di}{dt} = -\frac{2\left(\frac{P}{i}\right)}{i^2}\frac{di}{dt} = -\frac{2P}{i^3}\frac{di}{dt}$

7. (a) $s = \sqrt{x^2+y^2} = \left(x^2+y^2\right)^{1/2} \Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt}$

   (b) $s = \sqrt{x^2+y^2} = \left(x^2+y^2\right)^{1/2} \Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dt}$

   (c) $s = \sqrt{x^2+y^2} \Rightarrow s^2 = x^2+y^2 \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow 2s \cdot 0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$

8. (a) $s = \sqrt{x^2+y^2+z^2} \Rightarrow s^2 = x^2+y^2+z^2 \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}$

   $\Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{dz}{dt}$

   (b) From part (a) with $\frac{dx}{dt} = 0 \Rightarrow \frac{ds}{dt} = \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{dz}{dt}$

   (c) From part (a) with $\frac{ds}{dt} = 0 \Rightarrow 0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} \Rightarrow \frac{dx}{dt} + \frac{y}{x}\frac{dy}{dt} + \frac{z}{x}\frac{dz}{dt} = 0$

9. (a) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}ab\cos\theta\frac{d\theta}{dt}$ (b) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}ab\cos\theta\frac{d\theta}{dt} + \frac{1}{2}b\sin\theta\frac{da}{dt}$

(c) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}ab\cos\theta\,\frac{d\theta}{dt} + \frac{1}{2}b\sin\theta\,\frac{da}{dt} + \frac{1}{2}a\sin\theta\,\frac{db}{dt}$

10. Given $A = \pi r^2$ $\frac{dr}{dt} = 0.01$ cm/sec, and $r = 50$ cm. Since $\frac{dA}{dt} = 2\pi r\,\frac{dr}{dt}$, then $\left.\frac{dA}{dt}\right|_{r=50} = 2\pi(50)\left(\frac{1}{100}\right)$ $= \pi$ cm^2/min.

11. Given $\frac{d\ell}{dt} = -2$ cm/sec, $\frac{dw}{dt} = 2$ cm/sec, $\ell = 12$ cm and $w = 5$ cm.

(a) $A = \ell w \Rightarrow \frac{dA}{dt} = \ell\,\frac{dw}{dt} + w\,\frac{d\ell}{dt} \Rightarrow \frac{dA}{dt} = 12(2) + 5(-2) = 14$ cm^2/sec, increasing

(b) $P = 2\ell + 2w \Rightarrow \frac{dP}{dt} = 2\,\frac{d\ell}{dt} + 2\,\frac{dw}{dt} = 2(-2) + 2(2) = 0$ cm/sec, constant

(c) $D = \sqrt{w^2+\ell^2} = \left(w^2+\ell^2\right)^{1/2} \Rightarrow \frac{dD}{dt} = \frac{1}{2}\left(w^2+\ell^2\right)^{-1/2}\left(2w\,\frac{dw}{dt} + 2\ell\,\frac{d\ell}{dt}\right) \Rightarrow \frac{dD}{dt} = \frac{w\,\frac{dw}{dt} + \ell\,\frac{d\ell}{dt}}{\sqrt{w^2+\ell^2}}$

$= \frac{(5)(2) + (12)(-2)}{\sqrt{25+144}} = -\frac{14}{13}$ cm/sec, decreasing

12. (a) $V = xyz \Rightarrow \frac{dV}{dt} = yz\,\frac{dx}{dt} + xz\,\frac{dy}{dt} + xy\,\frac{dz}{dt} \Rightarrow \left.\frac{dV}{dt}\right|_{(4,3,2)} = (3)(2)(1) + (4)(2)(-2) + (4)(3)(1) = 2$ m^3/sec

(b) $S = 2xy + 2xz + 2yz \Rightarrow \frac{dS}{dt} = (2y+2z)\,\frac{dx}{dt} + (2x+2z)\,\frac{dy}{dt} + (2x+2y)\,\frac{dz}{dt}$

$\Rightarrow \left.\frac{dS}{dt}\right|_{(4,3,2)} = (10)(1) + (12)(-2) + (14)(1) = 0$ m^2/sec

(c) $\ell = \sqrt{x^2+y^2+z^2} = \left(x^2+y^2+z^2\right)^{1/2} \Rightarrow \frac{d\ell}{dt} = \frac{x}{\sqrt{x^2+y^2+z^2}}\,\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\frac{dz}{dt}$

$\Rightarrow \left.\frac{d\ell}{dt}\right|_{(4,3,2)} = \left(\frac{4}{\sqrt{29}}\right)(1) + \left(\frac{3}{\sqrt{29}}\right)(-2) + \left(\frac{2}{\sqrt{29}}\right)(1) = 0$ m/sec

13. Given: $\frac{dx}{dt} = 5$ ft/sec, the ladder is 13 ft long, and $x = 12$, $y = 5$ at the instant of time

(a) Since $x^2 + y^2 = 169 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\,\frac{dx}{dt} = -\left(\frac{12}{5}\right)(5) = -12$ ft/sec, the ladder is sliding down the wall

(b) The area of the triangle formed by the ladder and walls is $A = \frac{1}{2}xy \Rightarrow \frac{dA}{dt} = \left(\frac{1}{2}\right)\left(x\,\frac{dy}{dt} + y\,\frac{dx}{dt}\right)$. The area is changing at $\frac{1}{2}[12(-12) + 5(5)] = -\frac{119}{2} = -59.5$ ft^2/sec.

(c) $\cos\theta = \frac{x}{13} \Rightarrow -\sin\theta\,\frac{d\theta}{dt} = \frac{1}{13}\cdot\frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{1}{13\sin\theta}\cdot\frac{dx}{dt} = -\left(\frac{1}{5}\right)(5) = -1$ rad/sec

14. $s^2 = y^2 + x^2 \Rightarrow 2s\,\frac{ds}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{s}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) \Rightarrow \frac{ds}{dt} = \frac{1}{\sqrt{169}}[5(-442) + 12(-481)]$ $= -614$ knots

15. Let s represent the distance between the girl and the kite and x represents the horizontal distance between the girl and kite $\Rightarrow s^2 = (300)^2 + x^2 \Rightarrow \frac{ds}{dt} = \frac{x}{s}\,\frac{dx}{dt} = \frac{400(25)}{500} = 20$ ft/sec.

16. When the diameter is 3.8 in., the radius is 1.9 in. and $\frac{dr}{dt} = \frac{1}{3000}$ in/min. Also $V = 6\pi r^2 \Rightarrow \frac{dV}{dt} = 12\pi r \frac{dr}{dt}$ $\Rightarrow \frac{dV}{dt} = 12\pi(1.9)\left(\frac{1}{3000}\right) = 0.0076\pi$. The volume is changing at about 0.0239 in^3/min.

17. $V = \frac{1}{3}\pi r^2 h$, $h = \frac{3}{8}(2r) = \frac{3r}{4} \Rightarrow r = \frac{4h}{3} \Rightarrow V = \frac{1}{3}\pi\left(\frac{4h}{3}\right)^2 h = \frac{16\pi h^3}{27} \Rightarrow \frac{dV}{dt} = \frac{16\pi h^2}{9}\frac{dh}{dt}$

(a) $\left.\frac{dh}{dt}\right|_{h=4} = \left(\frac{9}{16\pi 4^2}\right)(10) = \frac{90}{256\pi} \approx 0.1119$ m/sec $= 11.19$ cm/sec

(b) $r = \frac{4h}{3} \Rightarrow \frac{dr}{dt} = \frac{4}{3}\frac{dh}{dt} = \frac{4}{3}\left(\frac{90}{256\pi}\right) = \frac{15}{32\pi} \approx 0.1492$ m/sec $= 14.92$ cm/sec

18. (a) $V = \frac{1}{3}\pi r^2 h$ and $r = \frac{15h}{2} \Rightarrow V = \frac{1}{3}\pi\left(\frac{15h}{2}\right)^2 h = \frac{75\pi h^3}{4} \Rightarrow \frac{dV}{dt} = \frac{225\pi h^2}{4}\frac{dh}{dt} \Rightarrow \left.\frac{dh}{dt}\right|_{h=5} = \frac{4(-50)}{225\pi(5)^2} = \frac{-8}{225\pi}$ ≈ -0.0113 m/min $= -1.13$ cm/min

(b) $r = \frac{15h}{2} \Rightarrow \frac{dr}{dt} = \frac{15}{2}\frac{dh}{dt} \Rightarrow \left.\frac{dr}{dt}\right|_{h=5} = \left(\frac{15}{2}\right)\left(\frac{-8}{225\pi}\right) = \frac{-4}{15\pi} \approx -0.0849$ m/sec $= -8.49$ cm/sec

19. (a) $V = \frac{\pi}{3}y^2(3R - y) \Rightarrow \frac{dV}{dt} = \frac{\pi}{3}\left[2y(3R - y) + y^2(-1)\right]\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \left[\frac{\pi}{3}\left(6Ry - 3y^2\right)\right]^{-1}\frac{dV}{dt} \Rightarrow$ at $R = 13$ and $y = 8$ we have $\frac{dy}{dt} = \frac{1}{144\pi}(-6) = \frac{-1}{24\pi}$ m/min

(b) The hemisphere is on the circle $r^2 + (13 - y)^2 = 169 \Rightarrow r = \sqrt{26y - y^2}$ m

(c) $r = \left(26y - y^2\right)^{1/2} \Rightarrow \frac{dr}{dt} = \frac{1}{2}\left(26y - y^2\right)^{-1/2}(26 - 2y)\frac{dy}{dt} \Rightarrow \frac{dr}{dt} = \frac{13 - y}{\sqrt{26y - y^2}}\frac{dy}{dt} \Rightarrow \left.\frac{dr}{dt}\right|_{y=8} = \frac{13 - 8}{\sqrt{26 \cdot 8 - 64}}\left(\frac{-1}{24\pi}\right)$ $= \frac{-5}{288\pi}$ m/min

20. If $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$, and $\frac{dV}{dt} = kS = 4k\pi r^2$, then $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Rightarrow 4k\pi r^2 = 4\pi r^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = k$, a constant. Therefore, the radius is increasing at a constant rate.

21. If $V = \frac{4}{3}\pi r^3$, $r = 5$, and $\frac{dV}{dt} = 100\pi$ ft^3/min, then $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = 1$ ft/min. Then $S = 4\pi r^2 \Rightarrow \frac{dS}{dt}$ $= 8\pi r\frac{dr}{dt} = 8\pi(5)(1) = 40\pi$ ft^2/min, the rate at which the area is increasing.

22. Let s represent the length of the rope and x the horizontal distance of the boat from the dock.

(a) We have $s^2 = x^2 + 36 \Rightarrow \frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt} = \frac{s}{\sqrt{s^2 - 36}}\frac{ds}{dt}$. Therefore, the boat is approaching the dock at $\left.\frac{dx}{dt}\right|_{s=10} = \frac{10}{\sqrt{10^2 - 36}}(2) = 2.5$ ft/sec.

(b) $\cos\theta = \frac{6}{r} \Rightarrow -\sin\theta\frac{d\theta}{dt} = -\frac{6}{r^2}\frac{dr}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{6}{r^2\sin\theta}\frac{dr}{dt}$. Thus, $r = 10$, $x = 8$, and $\sin\theta = \frac{8}{10}$ $\Rightarrow \frac{d\theta}{dt} = \frac{6}{10^2\left(\frac{8}{10}\right)} \cdot (-2) = -\frac{3}{20}$ rad/sec

23. Let s represent the distance between the bicycle and balloon, h the height of the balloon and x the horizontal distance between the balloon and the bicycle. The relationship between the variables is $s^2 = h^2 + x^2$ $\Rightarrow \frac{ds}{dt} = \frac{1}{s}\left(h\,\frac{dh}{dt} + x\,\frac{dx}{dt}\right) \Rightarrow \frac{ds}{dt} = \frac{1}{85}[68(1) + 51(17)] = 11$ ft/sec.

24. (a) Let h be the height of the coffee in the pot. Since the radius of the pot is 3, the volume of the coffee is $V = 9\pi h \Rightarrow \frac{dV}{dt} = 9\pi\,\frac{dh}{dt} \Rightarrow$ the rate the coffee is rising is $\frac{dh}{dt} = \frac{1}{9\pi}\,\frac{dV}{dt} = \frac{10}{9\pi}$ in/min.

(b) Let h be the height of the coffee in the pot. From the figure, the radius of the filter $r = \frac{h}{2} \Rightarrow V = \frac{1}{3}\pi r^2 h$ $= \frac{\pi h^3}{12}$, the volume of the filter. The rate the coffee is falling is $\frac{dh}{dt} = \frac{4}{\pi h^2}\,\frac{dV}{dt} = \frac{4}{25\pi}(-10) = -\frac{8}{5\pi}$ in/min.

25. $y = QD^{-1} \Rightarrow \frac{dy}{dt} = D^{-1}\,\frac{dQ}{dt} - QD^{-2}\,\frac{dD}{dt} = \frac{1}{41}(0) - \frac{233}{(41)^2}(-2) = \frac{466}{1681}$ L/min $\Rightarrow$ increasing about 0.2772 L/min

26. (a) $\frac{dc}{dt} = \left(3x^2 - 12x + 15\right)\frac{dx}{dt} = \left(3(2)^2 - 12(2) + 15\right)(0.1) = 0.3$, $\frac{dr}{dt} = 9\,\frac{dx}{dt} = 9(0.1) = 0.9$, $\frac{dp}{dt} = 0.9 - 0.3 = 0.6$

(b) $\frac{dc}{dt} = \left(3x^2 - 12x - 45x^{-2}\right)\frac{dx}{dt} = \left(3(1.5)^2 - 12(1.5) - 45(1.5)^{-2}\right)(0.05) = -1.5625$, $\frac{dr}{dt} = 70\,\frac{dx}{dt} = 70(0.05) = 3.5$, $\frac{dp}{dt} = 3.5 - (-1.5625) = 5.0625$

27. Let P(x, y) represent a point on the curve $y = x^2$ and θ the angle of inclination of a line containing P and the origin. Consequently, $\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{x^2}{x} = x \Rightarrow \sec^2\theta\,\frac{d\theta}{dt} = \frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = \cos^2\theta\,\frac{dx}{dt}$. Since $\frac{dx}{dt} = 10$ m/sec and $\cos^2\theta\big|_{x=3} = \frac{x^2}{y^2 + x^2} = \frac{3^2}{9^2 + 3^2} = \frac{1}{10}$, we have $\left.\frac{d\theta}{dt}\right|_{x=3} = 1$ rad/sec.

28. $y = (-x)^{1/2}$ and $\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{(-x)^{1/2}}{x} \Rightarrow \sec^2\theta\,\frac{d\theta}{dt} = \frac{\left(\frac{1}{2}\right)(-x)^{-1/2}(-1)x - (-x)^{1/2}(1)}{x^2}\,\frac{dx}{dt}$

$\Rightarrow \frac{d\theta}{dt} = \left(\frac{\frac{-x}{2\sqrt{-x}} - \sqrt{-x}}{x^2}\right)\left(\cos^2\theta\right)\left(\frac{dx}{dt}\right)$. Now, $\tan\theta = \frac{2}{-4} = -\frac{1}{2} \Rightarrow \cos\theta = -\frac{2}{\sqrt{5}} \Rightarrow \cos^2\theta = \frac{4}{5}$. Then

$\frac{d\theta}{dt} = \left(\frac{\frac{4}{4} - 2}{16}\right)\left(\frac{4}{5}\right)(-8) = \frac{2}{5}$ rad/sec.

29. The distance from the origin is $s = \sqrt{x^2 + y^2}$ and we wish to find $\left.\frac{ds}{dt}\right|_{(5,12)}$

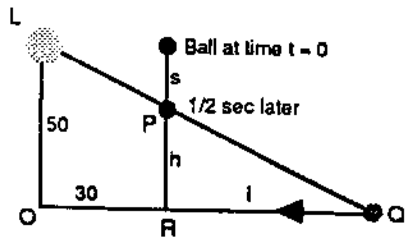
$= \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}\left.\left(2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt}\right)\right|_{(5,12)} = \frac{(5)(-1) + (12)(-5)}{\sqrt{25 + 144}} = -5$ m/sec

30. When s represents the length of the shadow and x the distance of the man from the streetlight, then $s = \frac{3}{5}x$.

(a) If I represents the distance of the tip of the shadow from the streetlight, then $I = s + x \Rightarrow \frac{dI}{dt} = \frac{ds}{dt} + \frac{dx}{dt}$ (which is velocity not speed) $\Rightarrow \left|\frac{dI}{dt}\right| = \left|\frac{3}{5}\,\frac{dx}{dt} + \frac{dx}{dt}\right| = \left|\frac{8}{5}\right|\left|\frac{dx}{dt}\right| = \frac{8}{5}|-5| = 8$ ft/sec, the speed the tip of the shadow is moving along the ground.

(b) $\frac{ds}{dt} = \frac{3}{5}\,\frac{dx}{dt} = \frac{3}{5}(-5) = -3$ ft/sec, so the length of the shadow is <u>decreasing</u> at a rate of 3 ft/sec.

31. Let $s = 16t^2$ represent the distance the ball has fallen, h the distance between the ball and the ground, and I the distance between the shadow and the point directly beneath the ball. Accordingly, $s + h = 50$ and since the triangle LOQ and triangle PRQ are similar we have



$I = \frac{30h}{50 - h} \Rightarrow h = 50 - 16t^2$ and $I = \frac{30(50 - 16t^2)}{50 - (50 - 16t^2)}$

$= \frac{1500}{16t^2} - 30 \Rightarrow \frac{dI}{dt} = -\frac{1500}{8t^3} \Rightarrow \left.\frac{dI}{dt}\right|_{t=\frac{1}{2}} = -1500$ ft/sec.

32. Let s = distance of car from foot of perpendicular in the textbook diagram $\Rightarrow \tan\theta = \frac{s}{132} \Rightarrow \sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{132}\frac{ds}{dt}$ $\Rightarrow \frac{d\theta}{dt} = \frac{\cos^2\theta}{132}\frac{ds}{dt}$; $\frac{ds}{dt} = -264$ and $\theta = 0 \Rightarrow \frac{d\theta}{dt} = -2$ rad/sec. A half second later the car has traveled 132 ft right of the perpendicular $\Rightarrow |\theta| = \frac{\pi}{4}$, $\cos^2\theta = \frac{1}{2}$, and $\frac{ds}{dt} = 264$ (since s increases) $\Rightarrow \frac{d\theta}{dt} = \frac{\left(\frac{1}{2}\right)}{132}(264) = 1$ rad/sec.

33. The volume of the ice is $V = \frac{4}{3}\pi r^3 - \frac{4}{3}\pi 4^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Rightarrow \left.\frac{dr}{dt}\right|_{r=6} = -\frac{5}{72\pi}$ in/min when $\frac{dV}{dt} = -10\ \text{in}^3/\text{min}$.

The surface area is $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r\frac{dr}{dt} \Rightarrow \left.\frac{dS}{dt}\right|_{r=6} = 48\pi\left(-\frac{5}{72\pi}\right) = -\frac{10}{3}\ \text{in}^2/\text{min}$.

34. Let s represent the horizontal distance between the car and plane while r is the line-of-sight distance between the car and plane $\Rightarrow 9 + s^2 = r^2 \Rightarrow \frac{ds}{dt} = \frac{r}{\sqrt{r^2 - 9}}\frac{dr}{dt} \Rightarrow \left.\frac{ds}{dt}\right|_{r=5} = \frac{5}{\sqrt{16}}(-160) = -200$ mph

$\Rightarrow$ speed of plane + speed of car = 200 mph $\Rightarrow$ the speed of the car is 80 mph.

35. When x represents the length of the shadow, then $\tan\theta = \frac{80}{x} \Rightarrow \sec^2\theta\,\frac{d\theta}{dt} = -\frac{80}{x^2}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{-x^2\sec^2\theta}{80}\frac{d\theta}{dt}$. We are given that $\frac{d\theta}{dt} = 0.27^\circ = \frac{3\pi}{2000}$ rad/min. At $x = 60$, $\cos\theta = \frac{3}{5} \Rightarrow$

$$\left|\frac{dx}{dt}\right| = \left|\frac{-x^2\sec^2\theta}{80}\frac{d\theta}{dt}\right|\Bigg|_{\left(\frac{d\theta}{dt} = \frac{3\pi}{2000}\text{ and }\sec\theta = \frac{5}{3}\right)} = \frac{3\pi}{16}\text{ ft/min} \approx 0.589\text{ ft/min} \approx 7.1\text{ in/min.}$$

36. $\tan\theta = \frac{A}{B} \Rightarrow \sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{B}\frac{dA}{dt} - \frac{A}{B^2}\frac{dB}{dt} \Rightarrow$ at $A = 10$ m and $B = 20$ m we have $\cos\theta = \frac{20}{10\sqrt{5}} = \frac{2}{\sqrt{5}}$ and

$$\frac{d\theta}{dt} = \left[\left(\frac{1}{20}\right)(-2) - \left(\frac{10}{400}(1)\right)\right]\left(\frac{4}{5}\right) = \left(\frac{-1}{10} - \frac{1}{40}\right)\left(\frac{4}{5}\right) = -\frac{1}{10}\text{ rad/sec} = -\frac{18^\circ}{\pi}/\text{sec} \approx -6^\circ/\text{sec}$$

37. Let x represent distance of the player from second base and s the distance to third base. Then $\frac{dx}{dt} = -16$ ft/sec

(a) $s^2 = x^2 + 8100 \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$. When the player is 30 ft from first base, $x = 60$

$\Rightarrow s = 30\sqrt{13}$ and $\frac{ds}{dt} = \frac{60}{30\sqrt{13}}(-16) = \frac{-32}{\sqrt{13}} \approx -8.875$ ft/sec

(b) $\cos\theta_1 = \frac{90}{s} \Rightarrow -\sin\theta_1\frac{d\theta_1}{dt} = -\frac{90}{s^2}\cdot\frac{ds}{dt} \Rightarrow \frac{d\theta_1}{dt} = \frac{90}{s^2\sin\theta_1}\cdot\frac{ds}{dt} = \frac{90}{sx}\cdot\frac{ds}{dt}$. Therefore, $x = 60$ and $s = 30\sqrt{13}$

$\Rightarrow \frac{d\theta_1}{dt} = \frac{90}{(30\sqrt{13})(60)}\cdot\left(\frac{-32}{\sqrt{13}}\right) = \frac{-8}{65}$ rad/sec; $\sin\theta_2 = \frac{90}{s} \Rightarrow \cos\theta_2\frac{d\theta_2}{dt} = -\frac{90}{s^2}\cdot\frac{ds}{dt} \Rightarrow \frac{d\theta_2}{dt} = \frac{-90}{s^2\cos\theta_2}\cdot\frac{ds}{dt}$

$= \frac{-90}{sx}\cdot\frac{ds}{dt}$. Therefore, $x = 60$ and $s = 30\sqrt{13} \Rightarrow \frac{d\theta_2}{dt} = \frac{8}{65}$ rad/sec.

(c) $\frac{d\theta_1}{dt} = \frac{90}{s^2 \sin\theta_1}\cdot\frac{ds}{dt} = \frac{90}{\left(s^2\cdot\frac{x}{s}\right)}\cdot\left(\frac{x}{s}\right)\cdot\left(\frac{dx}{dt}\right) = \left(\frac{90}{s^2}\right)\left(\frac{dx}{dt}\right) = \left(\frac{90}{x^2+8100}\right)\frac{dx}{dt} \Rightarrow \lim\limits_{x\to 0}\frac{d\theta_1}{dt}$

$= \lim\limits_{x\to 0}\left(\frac{90}{x^2+8100}\right)(-15) = -\frac{1}{6}$ rad/sec; $\frac{d\theta_2}{dt} = \frac{-90}{s^2\cos\theta_2}\cdot\frac{ds}{dt} = \left(\frac{-90}{s^2\cdot\frac{x}{s}}\right)\left(\frac{x}{s}\right)\left(\frac{dx}{dt}\right) = \left(\frac{-90}{s^2}\right)\left(\frac{dx}{dt}\right)$

$= \left(\frac{-90}{x^2+8100}\right)\frac{dx}{dt} \Rightarrow \lim\limits_{x\to 0}\frac{d\theta_2}{dt} = \frac{1}{6}$ rad/sec

38. Let a represent the distance between point O and ship A, b the distance between point O and ship B, and D the distance between the ships. By the Law of Cosines, $D^2 = a^2 + b^2 - 2ab\cos 120°$
$\Rightarrow \frac{dD}{dt} = \frac{1}{2D}\left[2a\frac{da}{dt} + 2b\frac{db}{dt} + a\frac{db}{dt} + b\frac{da}{dt}\right]$. When $a = 5$, $\frac{da}{dt} = 14$, $b = 3$, and $\frac{db}{dt} = 21$, then $\frac{dD}{dt} = \frac{413}{2D}$ where $D = 7$. The ships are moving $\frac{dD}{dt} = 29.5$ knots apart.

CHAPTER 2 PRACTICE EXERCISES

1. $y = x^5 - 0.125x^2 + 0.25x \Rightarrow \frac{dy}{dx} = 5x^4 - 0.25x + 0.25$

2. $y = 3 - 0.7x^3 + 0.3x^7 \Rightarrow \frac{dy}{dx} = -2.1x^2 + 2.1x^6$

3. $y = x^3 - 3(x^2 + \pi^2) \Rightarrow \frac{dy}{dx} = 3x^2 - 3(2x+0) = 3x^2 - 6x = 3x(x-2)$

4. $y = x^7 + \sqrt{7}x - \frac{1}{\pi+1} \Rightarrow \frac{dy}{dx} = 7x^6 + \sqrt{7}$

5. $y = (x+1)^2(x^2+2x) \Rightarrow \frac{dy}{dx} = (x+1)^2(2x+2) + (x^2+2x)(2(x+1)) = 2(x+1)\left[(x+1)^2 + x(x+2)\right]$
$= 2(x+1)(2x^2+4x+1)$

6. $y = (2x-5)(4-x)^{-1} \Rightarrow \frac{dy}{dx} = (2x-5)(-1)(4-x)^{-2}(-1) + (4-x)^{-1}(2) = (4-x)^{-2}\left[(2x-5) + 2(4-x)\right]$
$= 3(4-x)^{-2}$

7. $y = (\theta^2 + \sec\theta + 1)^3 \Rightarrow \frac{dy}{d\theta} = 3(\theta^2 + \sec\theta + 1)^2(2\theta + \sec\theta\tan\theta)$

8. $y = \left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)^2 \Rightarrow \frac{dy}{d\theta} = 2\left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)\left(\frac{\csc\theta\cot\theta}{2} - \frac{\theta}{2}\right) = \left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)(\csc\theta\cot\theta - \theta)$

9. $s = \frac{\sqrt{t}}{1+\sqrt{t}} \Rightarrow \frac{ds}{dt} = \frac{(1+\sqrt{t})\cdot\frac{1}{2\sqrt{t}} - \sqrt{t}\left(\frac{1}{2\sqrt{t}}\right)}{(1+\sqrt{t})^2} = \frac{(1+\sqrt{t}) - \sqrt{t}}{2\sqrt{t}(1+\sqrt{t})^2} = \frac{1}{2\sqrt{t}(1+\sqrt{t})^2}$

10. $s = \frac{1}{\sqrt{t}-1} \Rightarrow \frac{ds}{dt} = \frac{(\sqrt{t}-1)(0) - 1\left(\frac{1}{2\sqrt{t}}\right)}{(\sqrt{t}-1)^2} = \frac{-1}{2\sqrt{t}(\sqrt{t}-1)^2}$

11. $y = 2\tan^2 x - \sec^2 x \Rightarrow \frac{dy}{dx} = (4\tan x)(\sec^2 x) - (2\sec x)(\sec x \tan x) = 2\sec^2 x \tan x$

12. $y = \frac{1}{\sin^2 x} - \frac{2}{\sin x} = \csc^2 x - 2\csc x \Rightarrow \frac{dy}{dx} = (2\csc x)(-\csc x \cot x) - 2(-\csc x \cot x) = (2\csc x \cot x)(1 - \csc x)$

13. $s = \cos^4(1-2t) \Rightarrow \frac{ds}{dt} = 4\cos^3(1-2t)(-\sin(1-2t))(-2) = 8\cos^3(1-2t)\sin(1-2t)$

14. $s = \cot^3\left(\frac{2}{t}\right) \Rightarrow \frac{ds}{dt} = 3\cot^2\left(\frac{2}{t}\right)\left(-\csc^2\left(\frac{2}{t}\right)\right)\left(\frac{-2}{t^2}\right) = \frac{6}{t^2}\cot^2\left(\frac{2}{t}\right)\csc^2\left(\frac{2}{t}\right)$

15. $s = (\sec t + \tan t)^5 \Rightarrow \frac{ds}{dt} = 5(\sec t + \tan t)^4(\sec t \tan t + \sec^2 t) = 5(\sec t)(\sec t + \tan t)^5$

16. $s = \csc^5(1 - t + 3t^2) \Rightarrow \frac{ds}{dt} = 5\csc^4(1 - t + 3t^2)(-\csc(1 - t + 3t^2)\cot(1 - t + 3t^2))(-1 + 6t)$
$= -5(6t-1)\csc^5(1 - t + 3t^2)\cot(1 - t + 3t^2)$

17. $r = \sqrt{2\theta \sin\theta} = (2\theta\sin\theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(2\theta\sin\theta)^{-1/2}(2\theta\cos\theta + 2\sin\theta) = \frac{\theta\cos\theta + \sin\theta}{\sqrt{2\theta\sin\theta}}$

18. $r = 2\theta\sqrt{\cos\theta} = 2\theta(\cos\theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = 2\theta\left(\frac{1}{2}\right)(\cos\theta)^{-1/2}(-\sin\theta) + 2(\cos\theta)^{1/2} = \frac{-\theta\sin\theta}{\sqrt{\cos\theta}} + 2\sqrt{\cos\theta}$
$= \frac{2\cos\theta - \theta\sin\theta}{\sqrt{\cos\theta}}$

19. $r = \sin\sqrt{2\theta} = \sin(2\theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = \cos(2\theta)^{1/2}\left(\frac{1}{2}(2\theta)^{-1/2}(2)\right) = \frac{\cos\sqrt{2\theta}}{\sqrt{2\theta}}$

20. $r = \sin\left(\theta + \sqrt{\theta+1}\right) \Rightarrow \frac{dr}{d\theta} = \cos\left(\theta + \sqrt{\theta+1}\right)\left(1 + \frac{1}{2\sqrt{\theta+1}}\right) = \frac{2\sqrt{\theta+1}+1}{2\sqrt{\theta+1}}\cos\left(\theta + \sqrt{\theta+1}\right)$

21. $y = \frac{1}{2}x^2\csc\frac{2}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^2\left(-\csc\frac{2}{x}\cot\frac{2}{x}\right)\left(\frac{-2}{x^2}\right) + \left(\csc\frac{2}{x}\right)\left(\frac{1}{2}\cdot 2x\right) = \csc\frac{2}{x}\cot\frac{2}{x} + x\csc\frac{2}{x}$

22. $y = 2\sqrt{x}\sin\sqrt{x} \Rightarrow \frac{dy}{dx} = 2\sqrt{x}(\cos\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right) + (\sin\sqrt{x})\left(\frac{2}{2\sqrt{x}}\right) = \cos\sqrt{x} + \frac{\sin\sqrt{x}}{\sqrt{x}}$

23. $y = x^{-1/2}\sec(2x)^2 \Rightarrow \frac{dy}{dx} = x^{-1/2}\sec(2x)^2\tan(2x)^2(2(2x)\cdot 2) + \sec(2x)^2\left(-\frac{1}{2}x^{-3/2}\right)$
$= 8x^{1/2}\sec(2x)^2\tan(2x)^2 - \frac{1}{2}x^{-3/2}\sec(2x)^2 = \frac{1}{2}x^{1/2}\sec(2x)^2\left[16\tan(2x)^2 - x^{-2}\right]$

24. $y = \sqrt{x}\csc(x+1)^3 = x^{1/2}\csc(x+1)^3$
$\Rightarrow \frac{dy}{dx} = x^{1/2}\left(-\csc(x+1)^3\cot(x+1)^3\right)\left(3(x+1)^2\right) + \csc(x+1)^3\left(\frac{1}{2}x^{-1/2}\right)$

$$= -3\sqrt{x}\,(x+1)^2 \csc(x+1)^3 \cot(x+1)^3 + \frac{\csc(x+1)^3}{2\sqrt{x}} = \frac{1}{2}\sqrt{x}\,\csc(x+1)^3\left[\frac{1}{x} - 6(x+1)^2 \cot(x+1)^3\right]$$

25. $y = 5\cot x^2 \Rightarrow \frac{dy}{dx} = 5\left(-\csc^2 x^2\right)(2x) = -10x\csc^2\left(x^2\right)$

26. $y = x^2 \cot 5x \Rightarrow \frac{dy}{dx} = x^2\left(-\csc^2 5x\right)(5) + (\cot 5x)(2x) = -5x^2\csc^2 5x + 2x\cot 5x$

27. $y = x^2\sin^2\left(2x^2\right) \Rightarrow \frac{dy}{dx} = x^2\left(2\sin\left(2x^2\right)\right)\left(\cos\left(2x^2\right)\right)(4x) + \sin^2\left(2x^2\right)(2x) = 8x^3\sin\left(2x^2\right)\cos\left(2x^2\right) + 2x\sin^2\left(2x^2\right)$

28. $y = x^{-2}\sin^2\left(x^3\right) \Rightarrow \frac{dy}{dx} = x^{-2}\left(2\sin\left(x^3\right)\right)\left(\cos\left(x^3\right)\right)\left(3x^2\right) + \sin^2\left(x^3\right)\left(-2x^{-3}\right) = 6\sin\left(x^3\right)\cos\left(x^3\right) - 2x^{-3}\sin^2\left(x^3\right)$

29. $s = \left(\frac{4t}{t+1}\right)^{-2} \Rightarrow \frac{ds}{dt} = -2\left(\frac{4t}{t+1}\right)^{-3}\left(\frac{(t+1)(4)-(4t)(1)}{(t+1)^2}\right) = -2\left(\frac{4t}{t+1}\right)^{-3}\frac{4}{(t+1)^2} = -\frac{(t+1)}{8t^3}$

30. $s = \frac{-1}{15(15t-1)^3} = -\frac{1}{15}(15t-1)^{-3} \Rightarrow \frac{ds}{dt} = -\frac{1}{15}(-3)(15t-1)^{-4}(15) = \frac{3}{(15t-1)^4}$

31. $y = \left(\frac{\sqrt{x}}{x+1}\right)^2 \Rightarrow \frac{dy}{dx} = 2\left(\frac{\sqrt{x}}{x+1}\right)\cdot\frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right)-\left(\sqrt{x}\right)(1)}{(x+1)^2} = \frac{(x+1)-2x}{(x+1)^3} = \frac{1-x}{(x+1)^3}$

32. $y = \left(\frac{2\sqrt{x}}{2\sqrt{x}+1}\right)^2 \Rightarrow \frac{dy}{dx} = 2\left(\frac{2\sqrt{x}}{2\sqrt{x}+1}\right)\left(\frac{\left(2\sqrt{x}+1\right)\left(\frac{1}{\sqrt{x}}\right)-\left(2\sqrt{x}\right)\left(\frac{1}{\sqrt{x}}\right)}{\left(2\sqrt{x}+1\right)^2}\right) = \frac{4\sqrt{x}\left(\frac{1}{\sqrt{x}}\right)}{\left(2\sqrt{x}+1\right)^3} = \frac{4}{\left(2\sqrt{x}+1\right)^3}$

33. $y = \sqrt{\frac{x^2+x}{x^2}} = \left(1+\frac{1}{x}\right)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(1+\frac{1}{x}\right)^{-1/2}\left(-\frac{1}{x^2}\right) = -\frac{1}{2x^2\sqrt{1+\frac{1}{x}}}$

34. $y = 4x\sqrt{x+\sqrt{x}} = 4x\left(x+x^{1/2}\right)^{1/2} \Rightarrow \frac{dy}{dx} = 4x\left(\frac{1}{2}\right)\left(x+x^{1/2}\right)^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right) + \left(x+x^{1/2}\right)^{1/2}(4)$

$$= \left(x+\sqrt{x}\right)^{-1/2}\left[2x\left(1+\frac{1}{2\sqrt{x}}\right) + 4\left(x+\sqrt{x}\right)\right] = \left(x+\sqrt{x}\right)^{-1/2}\left(2x+\sqrt{x}+4x+4\sqrt{x}\right) = \frac{6x+5\sqrt{x}}{\sqrt{x+\sqrt{x}}}$$

35. $r = \left(\frac{\sin\theta}{\cos\theta-1}\right)^2 \Rightarrow \frac{dr}{d\theta} = 2\left(\frac{\sin\theta}{\cos\theta-1}\right)\left[\frac{(\cos\theta-1)(\cos\theta)-(\sin\theta)(-\sin\theta)}{(\cos\theta-1)^2}\right]$

$$= 2\left(\frac{\sin\theta}{\cos\theta-1}\right)\left(\frac{\cos^2\theta-\cos\theta+\sin^2\theta}{(\cos\theta-1)^2}\right) = \frac{(2\sin\theta)(1-\cos\theta)}{(\cos\theta-1)^3} = \frac{-2\sin\theta}{(\cos\theta-1)^2}$$

36. $r = \left(\frac{\sin\theta+1}{1-\cos\theta}\right)^2 \Rightarrow \frac{dr}{d\theta} = 2\left(\frac{\sin\theta+1}{1-\cos\theta}\right)\left[\frac{(1-\cos\theta)(\cos\theta)-(\sin\theta+1)(\sin\theta)}{(1-\cos\theta)^2}\right]$

$$= \frac{2(\sin\theta+1)}{(1-\cos\theta)^3}\left(\cos\theta-\cos^2\theta-\sin^2\theta-\sin\theta\right) = \frac{2(\sin\theta+1)(\cos\theta-\sin\theta-1)}{(1-\cos\theta)^3}$$

37. $y = (2x+1)\sqrt{2x+1} = (2x+1)^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}(2x+1)^{1/2}(2) = 3\sqrt{2x+1}$

38. $y = 20(3x-4)^{1/4}(3x-4)^{-1/5} = 20(3x-4)^{1/20} \Rightarrow \frac{dy}{dx} = 20\left(\frac{1}{20}\right)(3x-4)^{-19/20}(3) = \frac{3}{(3x-4)^{19/20}}$

39. $y = 3\left(5x^2+\sin 2x\right)^{-3/2} \Rightarrow \frac{dy}{dx} = 3\left(-\frac{3}{2}\right)\left(5x^2+\sin 2x\right)^{-5/2}[10x+(\cos 2x)(2)] = \frac{-9(5x+\cos 2x)}{\left(5x^2+\sin 2x\right)^{5/2}}$

40. $y = \left(3+\cos^3 3x\right)^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}\left(3+\cos^3 3x\right)^{-4/3}\left(3\cos^2 3x\right)(-\sin 3x)(3) = \frac{3\cos^2 3x\sin 3x}{\left(3+\cos^3 3x\right)^{4/3}}$

41. $xy+2x+3y=1 \Rightarrow \left(x\frac{dy}{dx}+y\right)+2+3\frac{dy}{dx}=0 \Rightarrow x\frac{dy}{dx}+3\frac{dy}{dx}=-2-y \Rightarrow \frac{dy}{dx}(x+3)=-2-y \Rightarrow \frac{dy}{dx}=-\frac{y+2}{x+3}$

42. $x^2+xy+y^2-5x=2 \Rightarrow 2x+\left(x\frac{dy}{dx}+y\right)+2y\frac{dy}{dx}-5=0 \Rightarrow x\frac{dy}{dx}+2y\frac{dy}{dx}=5-2x-y \Rightarrow \frac{dy}{dx}(x+2y)$
$= 5-2x-y \Rightarrow \frac{dy}{dx}=\frac{5-2x-y}{x+2y}$

43. $x^3+4xy-3y^{4/3}=2x \Rightarrow 3x^2+\left(4x\frac{dy}{dx}+4y\right)-4y^{1/3}\frac{dy}{dx}=2 \Rightarrow 4x\frac{dy}{dx}-4y^{1/3}\frac{dy}{dx}=2-3x^2-4y$
$\Rightarrow \frac{dy}{dx}\left(4x-4y^{1/3}\right)=2-3x^2-4y \Rightarrow \frac{dy}{dx}=\frac{2-3x^2-4y}{4x-4y^{1/3}}$

44. $5x^{4/5}+10y^{6/5}=15 \Rightarrow 4x^{-1/5}+12y^{1/5}\frac{dy}{dx}=0 \Rightarrow 12y^{1/5}\frac{dy}{dx}=-4x^{-1/5} \Rightarrow \frac{dy}{dx}=-\frac{1}{3}x^{-1/5}y^{-1/5}=-\frac{1}{3(xy)^{1/5}}$

45. $(xy)^{1/2}=1 \Rightarrow \frac{1}{2}(xy)^{-1/2}\left(x\frac{dy}{dx}+y\right)=0 \Rightarrow x^{1/2}y^{-1/2}\frac{dy}{dx}=-x^{-1/2}y^{1/2} \Rightarrow \frac{dy}{dx}=-x^{-1}y \Rightarrow \frac{dy}{dx}=-\frac{y}{x}$

46. $x^2y^2=1 \Rightarrow x^2\left(2y\frac{dy}{dx}\right)+y^2(2x)=0 \Rightarrow 2x^2y\frac{dy}{dx}=-2xy^2 \Rightarrow \frac{dy}{dx}=-\frac{y}{x}$

47. $y^2=\frac{x}{x+1} \Rightarrow 2y\frac{dy}{dx}=\frac{(x+1)(1)-(x)(1)}{(x+1)^2} \Rightarrow \frac{dy}{dx}=\frac{1}{2y(x+1)^2}$

48. $y^2=\left(\frac{1+x}{1-x}\right)^{1/2} \Rightarrow y^4=\frac{1+x}{1-x} \Rightarrow 4y^3\frac{dy}{dx}=\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2} \Rightarrow \frac{dy}{dx}=\frac{1}{2y^3(1-x)^2}$

49. $p^3+4pq-3q^2=2 \Rightarrow 3p^2\frac{dp}{dq}+4\left(p+q\frac{dp}{dq}\right)-6q=0 \Rightarrow 3p^2\frac{dp}{dq}+4q\frac{dp}{dq}=6q-4p \Rightarrow \frac{dp}{dq}\left(3p^2+4q\right)=6q-4p$
$\Rightarrow \frac{dp}{dq}=\frac{6q-4p}{3p^2+4q}$

50. $q = (5p^2+2p)^{-3/2} \Rightarrow 1 = -\frac{3}{2}(5p^2+2p)^{-5/2}\left(10p\,\frac{dp}{dq} + 2\,\frac{dp}{dq}\right) \Rightarrow -\frac{2}{3}(5p^2+2p)^{5/2} = \frac{dp}{dq}(10p+2)$

$\Rightarrow \frac{dp}{dq} = -\frac{(5p^2+2p)^{5/2}}{3(5p+1)}$

51. $r\cos 2s + \sin^2 s = \pi \Rightarrow r(-\sin 2s)(2) + (\cos 2s)\left(\frac{dr}{ds}\right) + 2\sin s\cos s = 0 \Rightarrow \frac{dr}{ds}(\cos 2s) = 2r\sin 2s - 2\sin s\cos s$

$\Rightarrow \frac{dr}{ds} = \frac{2r\sin 2s - \sin 2s}{\cos 2s} = \frac{(2r-1)(\sin 2s)}{\cos 2s} = (2r-1)(\tan 2s)$

52. $2rs - r - s + s^2 = -3 \Rightarrow 2\left(r + s\,\frac{dr}{ds}\right) - \frac{dr}{ds} - 1 + 2s = 0 \Rightarrow \frac{dr}{ds}(2s-1) = 1 - 2s - 2r \Rightarrow \frac{dr}{ds} = \frac{1-2s-2r}{2s-1}$

53. (a) $x^3 + y^3 = 1 \Rightarrow 3x^2 + 3y^2\,\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{y^2(-2x) - (-x^2)\left(2y\,\frac{dy}{dx}\right)}{y^4}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2xy^2 + (2yx^2)\left(-\frac{x^2}{y^2}\right)}{y^4} = \frac{-2xy^2 - \frac{2x^4}{y}}{y^4} = \frac{-2xy^3 - 2x^4}{y^5}$

(b) $y^2 = 1 - \frac{2}{x} \Rightarrow 2y\,\frac{dy}{dx} = \frac{2}{x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{yx^2} \Rightarrow \frac{dy}{dx} = (yx^2)^{-1} \Rightarrow \frac{d^2y}{dx^2} = -(yx^2)^{-2}\left[y(2x) + x^2\,\frac{dy}{dx}\right]$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2xy - x^2\left(\frac{1}{yx^2}\right)}{y^2x^4} = \frac{-2xy^2 - 1}{y^3x^4}$

54. (a) $x^2 - y^2 = 1 \Rightarrow 2x - 2y\,\frac{dy}{dx} = 0 \Rightarrow -2y\,\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

(b) $\frac{dy}{dx} = \frac{x}{y} \Rightarrow \frac{d^2y}{dx^2} = \frac{y(1) - x\,\frac{dy}{dx}}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2} = \frac{y^2 - x^2}{y^3} = \frac{-1}{y^3}$ (since $y^2 - x^2 = -1$)

55. (a) Let $h(x) = 6f(x) - g(x) \Rightarrow h'(x) = 6f'(x) - g'(x) \Rightarrow h'(1) = 6f'(1) - g'(1) = 6\left(\frac{1}{2}\right) - (-4) = 7$

(b) Let $h(x) = f(x)g^2(x) \Rightarrow h'(x) = f(x)\big(2g(x)\big)g'(x) + g^2(x)f'(x) \Rightarrow h'(0) = 2f(0)g(0)g'(0) + g^2(0)f'(0)$

$= 2(1)(1)\left(\frac{1}{2}\right) + (1)^2(-3) = -2$

(c) Let $h(x) = \frac{f(x)}{g(x)+1} \Rightarrow h'(x) = \frac{(g(x)+1)f'(x) - f(x)g'(x)}{(g(x)+1)^2} \Rightarrow h'(1) = \frac{(g(1)+1)f'(1) - f(1)g'(1)}{(g(1)+1)^2}$

$= \frac{(5+1)\left(\frac{1}{2}\right) - 3(-4)}{(5+1)^2} = \frac{5}{12}$

(d) Let $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x) \Rightarrow h'(0) = f'(g(0))g'(0) = f'(1)\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

(e) Let $h(x) = g(f(x)) \Rightarrow h'(x) = g'(f(x))f'(x) \Rightarrow h'(0) = g'(f(0))f'(0) = g'(1)f'(0) = (-4)(-3) = 12$

(f) Let $h(x) = (x + f(x))^{3/2} \Rightarrow h'(x) = \frac{3}{2}(x + f(x))^{1/2}\big(1 + f'(x)\big) \Rightarrow h'(1) = \frac{3}{2}(1 + f(1))^{1/2}\big(1 + f'(1)\big)$

$= \frac{3}{2}(1+3)^{1/2}\left(1 + \frac{1}{2}\right) = \frac{9}{2}$

(g) Let $h(x) = f(x + g(x)) \Rightarrow h'(x) = f'(x + g(x))(1 + g'(x)) \Rightarrow h'(0) = f'(g(0))(1 + g'(0))$
$= f'(1)\left(1 + \frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = \frac{3}{4}$

56. (a) Let $h(x) = \sqrt{x}\, f(x) \Rightarrow h'(x) = \sqrt{x}\, f'(x) + f(x) \cdot \frac{1}{2\sqrt{x}} \Rightarrow h'(1) = \sqrt{1}\, f'(1) + f(1) \cdot \frac{1}{2\sqrt{1}} = \frac{1}{5} + (-3)\left(\frac{1}{2}\right) = -\frac{13}{10}$

(b) Let $h(x) = (f(x))^{1/2} \Rightarrow h'(x) = \frac{1}{2}(f(x))^{-1/2}(f'(x)) \Rightarrow h'(0) = \frac{1}{2}(f(0))^{-1/2} f'(0) = \frac{1}{2}(9)^{-1/2}(-2) = -\frac{1}{3}$

(c) Let $h(x) = f(\sqrt{x}) \Rightarrow h'(x) = f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \Rightarrow h'(1) = f'(\sqrt{1}) \cdot \frac{1}{2\sqrt{1}} = \frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$

(d) Let $h(x) = f(1 - 5\tan x) \Rightarrow h'(x) = f'(1 - 5\tan x)(-5\sec^2 x) \Rightarrow h'(0) = f'(1 - 5\tan 0)(-5\sec^2 0)$
$= f'(1)(-5) = \frac{1}{5}(-5) = -1$

(e) Let $h(x) = \frac{f(x)}{2 + \cos x} \Rightarrow h'(x) = \frac{(2 + \cos x)f'(x) - f(x)(-\sin x)}{(2 + \cos x)^2} \Rightarrow h'(0) = \frac{(2+1)f'(0) - f(0)(0)}{(2+1)^2} = \frac{3(-2)}{9} = -\frac{2}{3}$

(f) Let $h(x) = 10\sin\left(\frac{\pi x}{2}\right)f^2(x) \Rightarrow h'(x) = 10\sin\left(\frac{\pi x}{2}\right)\left(2f(x)f'(x)\right) + f^2(x)\left(10\cos\left(\frac{\pi x}{2}\right)\right)\left(\frac{\pi}{2}\right)$
$\Rightarrow h'(1) = 10\sin\left(\frac{\pi}{2}\right)\left(2f(1)f'(1)\right) + f^2(1)\left(10\cos\left(\frac{\pi}{2}\right)\right)\left(\frac{\pi}{2}\right) = 20(-3)\left(\frac{1}{5}\right) + 0 = -12$

57. $x = t^2 + \pi \Rightarrow \frac{dx}{dt} = 2t$; $y = 3\sin 2x \Rightarrow \frac{dy}{dx} = 3(\cos 2x)(2) = 6\cos 2x = 6\cos(2t^2 + 2\pi) = 6\cos(2t^2)$; thus,
$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 6\cos(2t^2) \cdot 2t \Rightarrow \left.\frac{dy}{dt}\right|_{t=0} = 6\cos(0) \cdot 0 = 0$

58. $t = (u^2 + 2u)^{1/3} \Rightarrow \frac{dt}{du} = \frac{1}{3}(u^2 + 2u)^{-2/3}(2u + 2) = \frac{2}{3}(u^2 + 2u)^{-2/3}(u + 1)$; $s = t^2 + 5t \Rightarrow \frac{ds}{dt} = 2t + 5$
$= 2(u^2 + 2u)^{1/3} + 5$; thus $\frac{ds}{du} = \frac{ds}{dt} \cdot \frac{dt}{du} = \left[2(u^2 + 2u)^{1/3} + 5\right]\left(\frac{2}{3}\right)(u^2 + 2u)^{-2/3}(u + 1)$
$\Rightarrow \left.\frac{ds}{du}\right|_{u=2} = \left[2(2^2 + 2(2))^{1/3} + 5\right]\left(\frac{2}{3}\right)(2^2 + 2(2))^{-2/3}(2 + 1) = 2(2 \cdot 8^{1/3} + 5)(8^{-2/3}) = 2(2 \cdot 2 + 5)\left(\frac{1}{4}\right) = \frac{9}{2}$

59. $r = 8\sin\left(s + \frac{\pi}{6}\right) \Rightarrow \frac{dr}{ds} = 8\cos\left(s + \frac{\pi}{6}\right)$; $w = \sin(\sqrt{r} - 2) \Rightarrow \frac{dw}{dr} = \cos(\sqrt{r} - 2)\left(\frac{1}{2\sqrt{r}}\right)$
$= \frac{\cos\sqrt{8\sin\left(s + \frac{\pi}{6}\right)} - 2}{2\sqrt{8\sin\left(s + \frac{\pi}{6}\right)}}$; thus, $\frac{dw}{ds} = \frac{dw}{dr} \cdot \frac{dr}{ds} = \frac{\cos\left(\sqrt{8\sin\left(s + \frac{\pi}{6}\right)} - 2\right)}{2\sqrt{8\sin\left(s + \frac{\pi}{6}\right)}} \cdot \left[8\cos\left(s + \frac{\pi}{6}\right)\right]$
$\Rightarrow \left.\frac{dw}{ds}\right|_{s=0} = \frac{\cos\left(\sqrt{8\sin\left(\frac{\pi}{6}\right)} - 2\right) \cdot 8\cos\left(\frac{\pi}{6}\right)}{2\sqrt{8\sin\left(\frac{\pi}{6}\right)}} = \frac{(\cos 0)(8)\left(\frac{\sqrt{3}}{2}\right)}{2\sqrt{4}} = \sqrt{3}$

60. $\theta^2 t + \theta = 1 \Rightarrow \left(\theta^2 + t\left(2\theta\,\frac{d\theta}{dt}\right)\right) + \frac{d\theta}{dt} = 0 \Rightarrow \frac{d\theta}{dt}(2\theta t + 1) = -\theta^2 \Rightarrow \frac{d\theta}{dt} = \frac{-\theta^2}{2\theta t + 1}$; $r = (\theta^2 + 7)^{1/3}$
$\Rightarrow \frac{dr}{d\theta} = \frac{1}{3}(\theta^2 + 7)^{-2/3}(2\theta) = \frac{2}{3}\theta(\theta^2 + 7)^{-2/3}$; now $t = 0$ and $\theta^2 t + \theta = 1 \Rightarrow \theta = 1$ so that $\left.\frac{d\theta}{dt}\right|_{t=0,\ \theta=1} = \frac{-1}{1} = -1$

and $\left.\frac{dr}{d\theta}\right|_{\theta=1} = \frac{2}{3}(1+7)^{-2/3} = \frac{1}{6} \Rightarrow \left.\frac{dr}{dt}\right|_{t=0} = \left.\frac{dr}{d\theta}\right|_{t=0} \cdot \left.\frac{d\theta}{dt}\right|_{t=0} = \left(\frac{1}{6}\right)(-1) = -\frac{1}{6}$

61. $y^3 + y = 2\cos x \Rightarrow 3y^2\frac{dy}{dx} + \frac{dy}{dx} = -2\sin x \Rightarrow \frac{dy}{dx}(3y^2+1) = -2\sin x \Rightarrow \frac{dy}{dx} = \frac{-2\sin x}{3y^2+1} \Rightarrow \left.\frac{dy}{dx}\right|_{(0,1)}$

$= \frac{-2\sin(0)}{3+1} = 0$; $\frac{d^2y}{dx^2} = \frac{(3y^2+1)(-2\cos x) - (-2\sin x)\left(6y\frac{dy}{dx}\right)}{(3y^2+1)^2}$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(0,1)} = \frac{(3+1)(-2\cos 0) - (-2\sin 0)(6\cdot 0)}{(3+1)^2} = -\frac{1}{2}$

62. $x^{1/3} + y^{1/3} = 4 \Rightarrow \frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^{2/3}}{x^{2/3}} \Rightarrow \left.\frac{dy}{dx}\right|_{(8,8)} = -1$; $\frac{dy}{dx} = \frac{-y^{2/3}}{x^{2/3}}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{(x^{2/3})\left(-\frac{2}{3}y^{-1/3}\frac{dy}{dx}\right) - \left(-y^{2/3}\right)\left(\frac{2}{3}x^{-1/3}\right)}{(x^{2/3})^2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(8,8)} = \frac{(8^{2/3})\left[-\frac{2}{3}\cdot 8^{-1/3}\cdot(-1)\right] + (8^{2/3})\left(\frac{2}{3}\cdot 8^{-1/3}\right)}{8^{4/3}}$

$= \frac{\frac{1}{3}+\frac{1}{3}}{8^{2/3}} = \frac{\frac{2}{3}}{4} = \frac{1}{6}$

63. $f(t) = \frac{1}{2t+1}$ and $f(t+h) = \frac{1}{2(t+h)+1} \Rightarrow \frac{f(t+h)-f(t)}{h} = \frac{\frac{1}{2(t+h)+1} - \frac{1}{2t+1}}{h} = \frac{2t+1-(2t+2h+1)}{(2t+2h+1)(2t+1)h}$

$= \frac{-2h}{(2t+2h+1)(2t+1)h} = \frac{-2}{(2t+2h+1)(2t+1)} \Rightarrow f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \frac{-2}{(2t+2h+1)(2t+1)}$

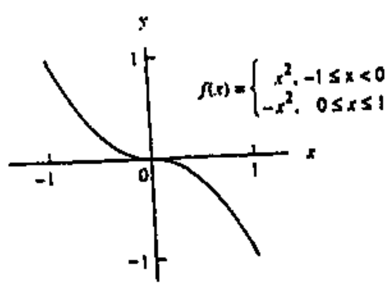
$= \frac{-2}{(2t+1)^2}$

64. $g(x) = 2x^2 + 1$ and $g(x+h) = 2(x+h)^2 + 1 = 2x^2 + 4xh + 2h^2 + 1 \Rightarrow \frac{g(x+h)-g(x)}{h}$

$= \frac{(2x^2+4xh+2h^2+1) - (2x^2+1)}{h} = \frac{4xh+2h^2}{h} = 4x + 2h \Rightarrow g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}(4x+2h)$

$= 4x$

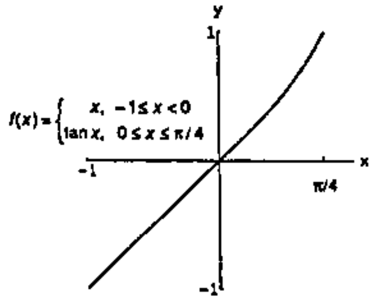
65. (a)



(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} -x^2 = 0 \Rightarrow \lim_{x\to 0} f(x) = 0$. Since $\lim_{x\to 0} f(x) = 0 = f(0)$ it follows that f is continuous at $x = 0$.

(c) $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} (2x) = 0$ and $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (-2x) = 0 \Rightarrow \lim_{x\to 0} f'(x) = 0$. Since this limit exists, it follows that f is differentiable at $x = 0$.

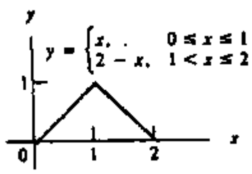
66. (a)



(b) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} x = 0$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \tan x = 0 \Rightarrow \lim\limits_{x\to 0} f(x) = 0$. Since $\lim\limits_{x\to 0} f(x) = 0 = f(0)$, it follows that f is continuous at $x = 0$.

(c) $\lim\limits_{x\to 0^-} f'(x) = \lim\limits_{x\to 0^-} 1 = 1$ and $\lim\limits_{x\to 0^+} f'(x) = \lim\limits_{x\to 0^+} \sec^2 x = 1 \Rightarrow \lim\limits_{x\to 0} f'(x) = 1$. Since this limit exists it follows that f is differentiable at $x = 0$.

67. (a)



(b) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x = 1$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (2 - x) = 1 \Rightarrow \lim\limits_{x\to 1} f(x) = 1$. Since $\lim\limits_{x\to 1} f(x) = 1 = f(1)$, it follows that f is continuous at $x = 1$.

(c) $\lim\limits_{x\to 1^-} f'(x) = \lim\limits_{x\to 1^-} 1 = 1$ and $\lim\limits_{x\to 1^+} f'(x) = \lim\limits_{x\to 1^+} -1 = -1 \Rightarrow \lim\limits_{x\to 1^-} f'(x) \neq \lim\limits_{x\to 1^+} f'(x)$, so $\lim\limits_{x\to 1} f'(x)$ does not exist $\Rightarrow$ f is not differentiable at $x = 1$.

68. (a) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \sin 2x = 0$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} mx = 0 \Rightarrow \lim\limits_{x\to 0} f(x) = 0$, independent of m; since $f(0) = 0 = \lim\limits_{x\to 0} f(x)$ it follows that f is continuous at $x = 0$ for all values of m.

(b) $\lim\limits_{x\to 0^-} f'(x) = \lim\limits_{x\to 0^-} (\sin 2x)' = \lim\limits_{x\to 0^-} 2\cos 2x = 2$ and $\lim\limits_{x\to 0^+} f'(x) = \lim\limits_{x\to 0^+} (mx)' = \lim\limits_{x\to 0^+} m = m \Rightarrow$ f is differentiable at $x = 0$ provided that $\lim\limits_{x\to 0^-} f'(x) = \lim\limits_{x\to 0^+} f'(x) \Rightarrow m = 2$.

69. $y = \frac{x}{2} + \frac{1}{2x-4} = \frac{1}{2}x + (2x-4)^{-1} \Rightarrow \frac{dy}{dx} = \frac{1}{2} - 2(2x-4)^{-2}$; the slope of the tangent is $-\frac{3}{2} \Rightarrow -\frac{3}{2} = \frac{1}{2} - 2(2x-4)^{-2} \Rightarrow -2 = -2(2x-4)^{-2} \Rightarrow 1 = \frac{1}{(2x-4)^2} \Rightarrow (2x-4)^2 = 1 \Rightarrow 4x^2 - 16x + 16 = 1$ $\Rightarrow 4x^2 - 16x + 15 = 0 \Rightarrow (2x-5)(2x-3) = 0 \Rightarrow x = \frac{5}{2}$ or $x = \frac{3}{2} \Rightarrow \left(\frac{5}{2}, \frac{9}{4}\right)$ and $\left(\frac{3}{2}, -\frac{1}{4}\right)$ are points on the curve where the slope is $-\frac{3}{2}$.

70. $y = x - \frac{1}{2x} \Rightarrow \frac{dy}{dx} = 1 + \frac{2}{(2x)^2} = 1 + \frac{1}{2x^2}$; the slope of the tangent is 3 $\Rightarrow 3 = 1 + \frac{1}{2x^2} \Rightarrow 2 = \frac{1}{2x^2} \Rightarrow x^2 = \frac{1}{4}$
$\Rightarrow x = \pm\frac{1}{2} \Rightarrow \left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$ are points on the curve where the slope is 3.

71. $y = 2x^3 - 3x^2 - 12x + 20 \Rightarrow \frac{dy}{dx} = 6x^2 - 6x - 12$; the tangent is parallel to the x-axis when $\frac{dy}{dx} = 0$
$\Rightarrow 6x^2 - 6x - 12 = 0 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2$ or $x = -1 \Rightarrow (2,0)$ and $(-1,27)$ are points on the curve where the tangent is parallel to the x-axis.

72. $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \left.\frac{dy}{dx}\right|_{(-2,-8)} = 12$; an equation of the tangent line at $(-2,-8)$ is $y + 8 = 12(x+2)$
$\Rightarrow y = 12x + 16$; x-intercept: $0 = 12x + 16 \Rightarrow x = -\frac{4}{3} \Rightarrow \left(-\frac{4}{3}, 0\right)$; y-intercept: $y = 12(0) + 16 = 16 \Rightarrow (0,16)$

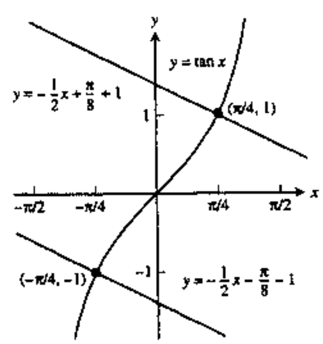
73. $y = 2x^3 - 3x^2 - 12x + 20 \Rightarrow \frac{dy}{dx} = 6x^2 - 6x - 12$

(a) The tangent is perpendicular to the line $y = 1 - \frac{x}{24}$ when $\frac{dy}{dx} = -\left(\frac{1}{-\left(\frac{1}{24}\right)}\right) = 24$; $6x^2 - 6x - 12 = 24$
$\Rightarrow x^2 - x - 2 = 4 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = -2$ or $x = 3 \Rightarrow (-2,16)$ and $(3,11)$ are points where the tangent is perpendicular to $y = 1 - \frac{x}{24}$.

(b) The tangent is parallel to the line $y = \sqrt{2} - 12x$ when $\frac{dy}{dx} = -12 \Rightarrow 6x^2 - 6x - 12 = -12 \Rightarrow x^2 - x = 0$
$\Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or $x = 1 \Rightarrow (0,20)$ and $(1,7)$ are points where the tangent is parallel to $y = \sqrt{2} - 12x$.

74. $y = \frac{\pi \sin x}{x} \Rightarrow \frac{dy}{dx} = \frac{x(\pi \cos x) - (\pi \sin x)(1)}{x^2} \Rightarrow m_1 = \left.\frac{dy}{dx}\right|_{x=\pi} = \frac{-\pi^2}{\pi^2} = -1$ and $m_2 = \left.\frac{dy}{dx}\right|_{x=-\pi} \frac{\pi^2}{\pi^2} = 1$.
Since $m_1 = -\frac{1}{m_2}$ the tangents intersect at right angles.

75. $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \sec^2 x$; now the slope of $y = -\frac{x}{2}$ is $-\frac{1}{2} \Rightarrow$ the normal line is parallel to $y = -\frac{x}{2}$ when $\frac{dy}{dx} = 2$. Thus, $\sec^2 x = 2 \Rightarrow \frac{1}{\cos^2 x} = 2$
$\Rightarrow \cos^2 x = \frac{1}{2} \Rightarrow \cos x = \frac{\pm 1}{\sqrt{2}} \Rightarrow x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$
for $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow \left(-\frac{\pi}{4}, -1\right)$ and $\left(\frac{\pi}{4}, 1\right)$ are points where the normal is parallel to $y = -\frac{x}{2}$.

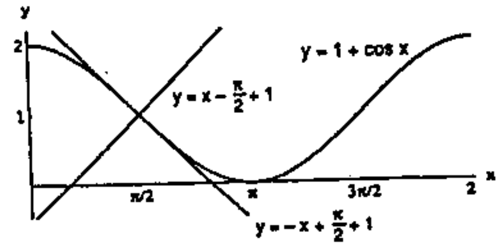


76. $y = 1 + \cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow \left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{2}, 1\right)} = -1$

$\Rightarrow$ the tangent at $\left(\frac{\pi}{2}, 1\right)$ is the line $y - 1 = -\left(x - \frac{\pi}{2}\right)$

$\Rightarrow y = -x + \frac{\pi}{2} + 1$; the normal at $\left(\frac{\pi}{2}, 1\right)$ is

$y - 1 = (1)\left(x - \frac{\pi}{2}\right) \Rightarrow y = x - \frac{\pi}{2} + 1$



77. $y = x^2 + C \Rightarrow \frac{dy}{dx} = 2x$ and $y = x \Rightarrow \frac{dy}{dx} = 1$; the parabola is tangent to $y = x$ when $2x = 1 \Rightarrow x = \frac{1}{2} \Rightarrow y = \frac{1}{2}$; thus, $\frac{1}{2} = \left(\frac{1}{2}\right)^2 + C \Rightarrow C = \frac{1}{4}$

78. $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \left.\frac{dy}{dx}\right|_{x=a} = 3a^2 \Rightarrow$ the tangent line at (a, a^3) is $y - a^3 = 3a^2(x - a)$. The tangent line intersects $y = x^3$ when $x^3 - a^3 = 3a^2(x - a) \Rightarrow (x - a)(x^2 + xa + a^2) = 3a^2(x - a) \Rightarrow (x - a)(x^2 + xa - 2a^2) = 0$ $\Rightarrow (x - a)^2(x + 2a) = 0 \Rightarrow x = a$ or $x = -2a$. Now $\left.\frac{dy}{dx}\right|_{x=-2a} = 3(-2a)^2 = 12a^2 = 4(3a^2)$, so the slope at $x = -2a$ is 4 times as large as the slope at (a, a^3) where $x = a$.

79. The line through $(0, 3)$ and $(5, -2)$ has slope $m = \frac{3 - (-2)}{0 - 5} = -1 \Rightarrow$ the line through $(0, 3)$ and $(5, -2)$ is $y = -x + 3$; $y = \frac{c}{x+1} \Rightarrow \frac{dy}{dx} = \frac{-c}{(x+1)^2}$, so the curve is tangent to $y = -x + 3 \Rightarrow \frac{dy}{dx} = -1 = \frac{-c}{(x+1)^2}$ $\Rightarrow (x + 1)^2 = c$, $x \neq -1$. Moreover, $y = \frac{c}{x+1}$ intersects $y = -x + 3 \Rightarrow \frac{c}{x+1} = -x + 3$, $x \neq -1$ $\Rightarrow c = (x + 1)(-x + 3)$, $x \neq -1$. Thus $c = c \Rightarrow (x + 1)^2 = (x + 1)(-x + 3) \Rightarrow (x + 1)[x + 1 - (-x + 3)]$ $= 0$, $x \neq -1 \Rightarrow (x + 1)(2x - 2) = 0 \Rightarrow x = 1$ (since $x \neq -1$) $\Rightarrow c = 4$.

80. Let $\left(b, \pm\sqrt{a^2 - b^2}\right)$ be a point on the circle $x^2 + y^2 = a^2$. Then $x^2 + y^2 = a^2 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$ $\Rightarrow \left.\frac{dy}{dx}\right|_{x=b} = \frac{-b}{\pm\sqrt{a^2 - b^2}} \Rightarrow$ normal line through $\left(b, \pm\sqrt{a^2 - b^2}\right)$ has slope $\frac{\mp\sqrt{a^2 - b^2}}{b} \Rightarrow$ normal line is $y - \left(\mp\sqrt{a^2 - b^2}\right) = \frac{\mp\sqrt{a^2 - b^2}}{b}(x - b) \Rightarrow y \pm \sqrt{a^2 - b^2} = \frac{\mp\sqrt{a^2 - b^2}}{b}x \pm \sqrt{a^2 - b^2} \Rightarrow y = \mp\frac{\sqrt{a^2 - b^2}}{b}x$ which passes through the origin.

81. $x^2 + 2y^2 = 9 \Rightarrow 2x + 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{2y} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,2)} = -\frac{1}{4} \Rightarrow$ the tangent line is $y = 2 - \frac{1}{4}(x - 1)$ $= -\frac{1}{4}x + \frac{9}{4}$ and the normal line is $y = 2 + 4(x - 1) = 4x - 2$.

82. $x^3 + y^2 = 2 \Rightarrow 3x^2 + 2y\,\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-3x^2}{2y} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{3}{2} \Rightarrow$ the tangent line is $y = 1 + \frac{-3}{2}(x-1)$ $= -\frac{3}{2}x + \frac{5}{2}$ and the normal line is $y = 1 + \frac{2}{3}(x-1) = \frac{2}{3}x + \frac{1}{3}$.

83. $xy + 2x - 5y = 2 \Rightarrow \left(x\,\frac{dy}{dx} + y\right) + 2 - 5\,\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}(x-5) = -y - 2 \Rightarrow \frac{dy}{dx} = \frac{-y-2}{x-5} \Rightarrow \left.\frac{dy}{dx}\right|_{(3,2)} = 2$ $\Rightarrow$ the tangent line is $y = 2 + 2(x-3) = 2x - 4$ and the normal line is $y = 2 + \frac{-1}{2}(x-3) = -\frac{1}{2}x + \frac{7}{2}$.

84. $(y-x)^2 = 2x + 4 \Rightarrow 2(y-x)\left(\frac{dy}{dx} - 1\right) = 2 \Rightarrow (y-x)\,\frac{dy}{dx} = 1 + (y-x) \Rightarrow \frac{dy}{dx} = \frac{1+y-x}{y-x} \Rightarrow \left.\frac{dy}{dx}\right|_{(6,2)} = \frac{3}{4}$ $\Rightarrow$ the tangent line is $y = 2 + \frac{3}{4}(x-6) = \frac{3}{4}x - \frac{5}{2}$ and the normal line is $y = 2 - \frac{4}{3}(x-6) = -\frac{4}{3}x + 10$.

85. $x + \sqrt{xy} = 6 \Rightarrow 1 + \frac{1}{2\sqrt{xy}}\left(x\,\frac{dy}{dx} + y\right) = 0 \Rightarrow x\,\frac{dy}{dx} + y = -2\sqrt{xy} \Rightarrow \frac{dy}{dx} = \frac{-2\sqrt{xy} - y}{x} \Rightarrow \left.\frac{dy}{dx}\right|_{(4,1)} = \frac{-5}{4}$ $\Rightarrow$ the tangent line is $y = 1 - \frac{5}{4}(x-4) = -\frac{5}{4}x + 6$ and the normal line is $y = 1 + \frac{4}{5}(x-4) = \frac{4}{5}x - \frac{11}{5}$.

86. $x^{3/2} + 2y^{3/2} = 17 \Rightarrow \frac{3}{2}x^{1/2} + 3y^{1/2}\,\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x^{1/2}}{2y^{1/2}} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,4)} = -\frac{1}{4} \Rightarrow$ the tangent line is $y = 4 - \frac{1}{4}(x-1) = -\frac{1}{4}x + \frac{17}{4}$ and the normal line is $y = 4 + 4(x-1) = 4x$.

87. $x^3y^3 + y^2 = x + y \Rightarrow \left[x^3\left(3y^2\,\frac{dy}{dx}\right) + y^3(3x^2)\right] + 2y\,\frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow 3x^3y^2\,\frac{dy}{dx} + 2y\,\frac{dy}{dx} - \frac{dy}{dx} = 1 - 3x^2y^3$ $\Rightarrow \frac{dy}{dx}\left(3x^3y^2 + 2y - 1\right) = 1 - 3x^2y^3 \Rightarrow \frac{dy}{dx} = \frac{1 - 3x^2y^3}{3x^3y^2 + 2y - 1} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{2}{4}$, but $\left.\frac{dy}{dx}\right|_{(1,-1)}$ is undefined. Therefore, the curve has slope $-\frac{1}{2}$ at $(1,1)$ but the slope is undefined at $(1,-1)$.

88. $y = \sin(x - \sin x) \Rightarrow \frac{dy}{dx} = [\cos(x - \sin x)](1 - \cos x)$; $y = 0 \Rightarrow \sin(x - \sin x) = 0 \Rightarrow x - \sin x = k\pi$, $k = -2, -1, 0, 1, 2$ (for our interval) $\Rightarrow \cos(x - \sin x) = \cos(k\pi) = \pm 1$. Therefore, $\frac{dy}{dx} = 0$ and $y = 0$ when $1 - \cos x = 0$ and $x = k\pi$. For $-2\pi \le x \le 2\pi$, these equations hold when $k = -2$, 0, and 2 (since $\cos(-\pi) = \cos\pi = -1$). Thus the curve has horizontal tangents at the x-axis for the x-values -2π, 0, and 2π (which are even integer multiples of π) $\Rightarrow$ the curve has an infinite number of horizontal tangents.

89. $x = \frac{1}{2}\tan t,\ y = \frac{1}{2}\sec t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2}\sec t\tan t}{\frac{1}{2}\sec^2 t} = \frac{\tan t}{\sec t} = \sin t \Rightarrow \left.\frac{dy}{dx}\right|_{t=\pi/3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$; $t = \frac{\pi}{3}$ $\Rightarrow x = \frac{1}{2}\tan\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $y = \frac{1}{2}\sec\frac{\pi}{3} = 1 \Rightarrow y = \frac{\sqrt{3}}{2}x + \frac{1}{4}$; $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\cos t}{\frac{1}{2}\sec^2 t} = 2\cos^3 t \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\pi/3}$ $= 2\cos^3\left(\frac{\pi}{3}\right) = \frac{1}{4}$

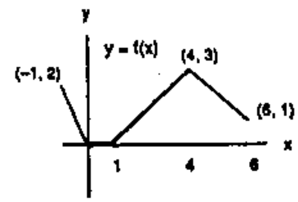
90. $x = 1 + \frac{1}{t^2}$, $y = 1 - \frac{3}{t} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{3}{t^2}\right)}{\left(-\frac{2}{t^3}\right)} = -\frac{3}{2}t \Rightarrow \left.\frac{dy}{dx}\right|_{t=2} = -\frac{3}{2}(2) = -3$; $t = 2 \Rightarrow x = 1 + \frac{1}{2^2} = \frac{5}{4}$ and

$y = 1 - \frac{3}{2} = -\frac{1}{2} \Rightarrow y = -3x + \frac{13}{4}$; $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left(-\frac{3}{2}\right)}{\left(-\frac{2}{t^3}\right)} = \frac{3}{4}t^3 \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{3}{4}(2)^3 = 6$

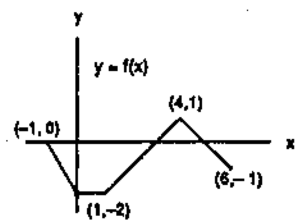
91. B = graph of f, A = graph of f′. Curve B cannot be the derivative of A because A has only negative slopes while some of B's values are positive.

92. A = graph of f, B = graph of f′. Curve A cannot be the derivative of B because B has only negative slopes while A has positive values for $x > 0$.

93.



94.



95. (a) 0, 0

(b) largest 1700, smallest about 1400

96. rabbits/day and foxes/day

97. (a) $S = 2\pi r^2 + 2\pi rh$ and h constant $\Rightarrow \frac{dS}{dt} = 4\pi r \frac{dr}{dt} + 2\pi h \frac{dr}{dt} = (4\pi r + 2\pi h)\frac{dr}{dt}$

(b) $S = 2\pi r^2 + 2\pi rh$ and r constant $\Rightarrow \frac{dS}{dt} = 2\pi r \frac{dh}{dt}$

(c) $S = 2\pi r^2 + 2\pi rh \Rightarrow \frac{dS}{dt} = 4\pi r \frac{dr}{dt} + 2\pi\left(r\frac{dh}{dt} + h\frac{dr}{dt}\right) = (4\pi r + 2\pi h)\frac{dr}{dt} + 2\pi r\frac{dh}{dt}$

(d S constant $\Rightarrow \frac{dS}{dt} = 0 \Rightarrow 0 = (4\pi r + 2\pi h)\frac{dr}{dt} + 2\pi r\frac{dh}{dt} \Rightarrow (2r + h)\frac{dr}{dt} = -r\frac{dh}{dt} \Rightarrow \frac{dr}{dt} = \frac{-r}{2r + h}\frac{dh}{dt}$

98. $S = \pi r\sqrt{r^2 + h^2} \Rightarrow \frac{dS}{dt} = \pi r \cdot \frac{\left(r\frac{dr}{dt} + h\frac{dh}{dt}\right)}{\sqrt{r^2 + h^2}} + \pi\sqrt{r^2 + h^2}\frac{dr}{dt}$;

(a) h constant $\Rightarrow \frac{dh}{dt} = 0 \Rightarrow \frac{dS}{dt} = \frac{\pi r^2 \frac{dr}{dt}}{\sqrt{r^2 + h^2}} + \pi\sqrt{r^2 + h^2}\frac{dr}{dt} = \left[\pi\sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}}\right]\frac{dr}{dt}$

(b) r constant $\Rightarrow \frac{dr}{dt} = 0 \Rightarrow \frac{dS}{dt} = \frac{\pi rh}{\sqrt{r^2 + h^2}}\frac{dh}{dt}$

(c) In general, $\frac{dS}{dt} = \left[\pi\sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}}\right]\frac{dr}{dt} + \frac{\pi rh}{\sqrt{r^2 + h^2}}\frac{dh}{dt}$

99. $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$; so $r = 10$ and $\frac{dr}{dt} = -\frac{2}{\pi}$ m/sec $\Rightarrow \frac{dA}{dt} = (2\pi)(10)\left(-\frac{2}{\pi}\right) = -40$ m^2/sec

100. $V = s^3 \Rightarrow \frac{dV}{dt} = 3s^2 \cdot \frac{ds}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{3s^2}\frac{dV}{dt}$; so $s = 20$ and $\frac{dV}{dt} = 1200$ cm^3/min $\Rightarrow \frac{ds}{dt} = \frac{1}{3(20)^2}(1200) = 1$ cm/min

101. $\frac{dR_1}{dt} = -1$ ohm/sec, $\frac{dR_2}{dt} = 0.5$ ohm/sec; and $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow \frac{-1}{R^2}\frac{dR}{dt} = \frac{-1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt}$. Also, $R_1 = 75$ ohms and $R_2 = 50$ ohms $\Rightarrow \frac{1}{R} = \frac{1}{75} + \frac{1}{50} \Rightarrow R = 30$ ohms. Therefore, from the derivative equation, $\frac{-1}{(30)^2}\frac{dR}{dt} = \frac{-1}{(75)^2}(-1) - \frac{1}{(50)^2}(0.5) = \left(\frac{1}{5625} - \frac{1}{5000}\right) \Rightarrow \frac{dR}{dt} = (-900)\left(\frac{5000-5625}{5625 \cdot 5000}\right) = \frac{9(625)}{50(5625)} = \frac{1}{50}$ $= 0.02$ ohm/sec.

102. $\frac{dR}{dt} = 3$ ohms/sec and $\frac{dX}{dt} = -2$ ohms/sec; $Z = \sqrt{R^2+X^2} \Rightarrow \frac{dZ}{dt} = \frac{R\frac{dR}{dt} + X\frac{dX}{dt}}{\sqrt{R^2+X^2}}$ so that $R = 10$ ohms and $X = 20$ ohms $\Rightarrow \frac{dZ}{dt} = \frac{(10)(3)+(20)(-2)}{\sqrt{10^2+20^2}} = \frac{-1}{\sqrt{5}} \approx -0.45$ ohm/sec.

103. Given $\frac{dx}{dt} = 10$ m/sec and $\frac{dy}{dt} = 5$ m/sec, let D be the distance from the origin $\Rightarrow D^2 = x^2 + y^2 \Rightarrow 2D\frac{dD}{dt}$ $= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow D\frac{dD}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$. When $(x,y) = (3,-4)$, $D = \sqrt{3^2+(-4)^2} = 5$ and $5\frac{dD}{dt} = (5)(10) + (12)(5) \Rightarrow \frac{dD}{dt} = \frac{110}{5} = 22$. Therefore, the particle is moving <u>away</u> <u>from</u> the origin at 22 m/sec (because the distance D is increasing).

104. Let D be the distance from the origin. We are given that $\frac{dD}{dt} = 11$ units/sec. Then $D^2 = x^2 + y^2$ $= x^2 + \left(x^{3/2}\right)^2 = x^2 + x^3 \Rightarrow 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 3x^2\frac{dx}{dt} = x(2+3x)\frac{dx}{dt}$; $x = 3 \Rightarrow D = \sqrt{3^2+3^3} = 6$ and substitution in the derivative equation gives $(2)(6)(11) = (3)(2+9)\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 4$ units/sec.

105. (a) From the diagram we have $\frac{10}{h} = \frac{4}{r} \Rightarrow r = \frac{2}{5}h$.

(b) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4\pi h^3}{75} \Rightarrow \frac{dV}{dt} = \frac{4\pi h^2}{25}\frac{dh}{dt}$, so $\frac{dV}{dt} = -5$ and $h = 6 \Rightarrow \frac{dh}{dt} = -\frac{125}{144\pi}$ ft/min.

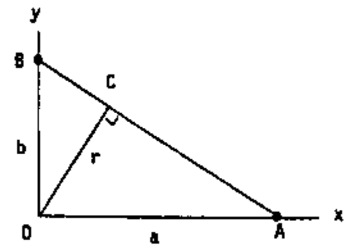
106. From the sketch in the text, $s = r\theta \Rightarrow \frac{ds}{dt} = r\frac{d\theta}{dt} + \theta\frac{dr}{dt}$. Also $r = 1.2$ is constant $\Rightarrow \frac{dr}{dt} = 0$ $\Rightarrow \frac{ds}{dt} = r\frac{d\theta}{dt} = (1.2)\frac{d\theta}{dt}$. Therefore, $\frac{ds}{dt} = 6$ ft/sec and $r = 1.2$ ft $\Rightarrow \frac{d\theta}{dt} = 5$ rad/sec

107. (a) From the sketch in the text, $\frac{d\theta}{dt} = -0.6$ rad/sec and $x = \tan\theta$. Also $x = \tan\theta \Rightarrow \frac{dx}{dt} = \sec^2\theta\frac{d\theta}{dt}$; at point A, $x = 0 \Rightarrow \theta = 0 \Rightarrow \frac{dx}{dt} = \left(\sec^2 0\right)(-0.6) = -0.6$. Therefore the speed of the light is $0.6 = \frac{3}{5}$ km/sec when it reaches point A.

(b) $\frac{(3/5)\text{ rad}}{\text{sec}} \cdot \frac{1\text{ rev}}{2\pi\text{ rad}} \cdot \frac{60\text{ sec}}{\text{min}} = \frac{18}{\pi}$ revs/min

108. From the figure, $\frac{a}{r} = \frac{b}{BC} \Rightarrow \frac{a}{r} = \frac{b}{\sqrt{b^2-r^2}}$. We are given that r is constant. Differentiation gives,

$\frac{1}{r}\cdot\frac{da}{dt}=\frac{\left(\sqrt{b^2-r^2}\right)\left(\frac{db}{dt}\right)-(b)\left(\frac{b}{\sqrt{b^2-r^2}}\right)\left(\frac{db}{dt}\right)}{b^2-r^2}$. Then,



$b=2r$ and $\frac{db}{dt}=-0.3r$

$$\Rightarrow \frac{da}{dt}=r\left[\frac{\sqrt{(2r)^2-r^2}\,(-0.3r)-(2r)\left(\frac{2r(-0.3r)}{\sqrt{(2r)^2-r^2}}\right)}{(2r)^2-r^2}\right]$$

$=\frac{\sqrt{3r^2}\,(-0.3r)+\frac{4r^2(0.3r)}{\sqrt{3r^2}}}{3r}=\frac{\left(3r^2\right)(-0.3r)+\left(4r^2\right)(0.3r)}{3\sqrt{3}\,r^2}=\frac{0.3r}{3\sqrt{3}}=\frac{r}{10\sqrt{3}}$ m/sec. Since $\frac{da}{dt}$ is positive, the distance OA is increasing when OB = 2r, and B is moving toward 0 at the rate of 0.3r m/sec.

CHAPTER 2 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. (a) $\sin 2\theta=2\sin\theta\cos\theta \Rightarrow \frac{d}{d\theta}(\sin 2\theta)=\frac{d}{d\theta}(2\sin\theta\cos\theta) \Rightarrow 2\cos 2\theta=2[(\sin\theta)(-\sin\theta)+(\cos\theta)(\cos\theta)]$
$\Rightarrow \cos 2\theta=\cos^2\theta-\sin^2\theta$
   (b) $\cos 2\theta=\cos^2\theta-\sin^2\theta \Rightarrow \frac{d}{d\theta}(\cos 2\theta)=\frac{d}{d\theta}\left(\cos^2\theta-\sin^2\theta\right) \Rightarrow -2\sin 2\theta=(2\cos\theta)(-\sin\theta)-(2\sin\theta)(\cos\theta)$
$\Rightarrow \sin 2\theta=\cos\theta\sin\theta+\sin\theta\cos\theta \Rightarrow \sin 2\theta=2\sin\theta\cos\theta$

2. The derivative of $\sin(x+a)=\sin x\cos a+\cos x\sin a$ with respect to x is $\cos(x+a)=\cos x\cos a-\sin x\sin a$, which is also an identity. This principle does not apply to the equation $x^2-2x-8=0$, since $x^2-2x-8=0$ is not an identity: it holds for 2 values of x (−2 and 4), but not for all x.

3. (a) $f(x)=\cos x \Rightarrow f'(x)=-\sin x \Rightarrow f''(x)=-\cos x$, and $g(x)=a+bx+cx^2 \Rightarrow g'(x)=b+2cx \Rightarrow g''(x)=2c$; also, $f(0)=g(0) \Rightarrow \cos(0)=a \Rightarrow a=1$; $f'(0)=g'(0) \Rightarrow -\sin(0)=b \Rightarrow b=0$; $f''(0)=g''(0)$ $\Rightarrow -\cos(0)=2c \Rightarrow c=-\frac{1}{2}$. Therefore, $g(x)=1-\frac{1}{2}x^2$.
   (b) $f(x)=\sin(x+a) \Rightarrow f'(x)=\cos(x+a)$, and $g(x)=b\sin x+c\cos x \Rightarrow g'(x)=b\cos x-c\sin x$; also, $f(0)=g(0) \Rightarrow \sin(a)=b\sin(0)+c\cos(0) \Rightarrow c=\sin a$; $f'(0)=g'(0) \Rightarrow \cos(a)=b\cos(0)-c\sin(0)$ $\Rightarrow b=\cos a$. Therefore, $g(x)=\sin x\cos a+\cos x\sin a$.
   (c) When $f(x)=\cos x$, $f'''(x)=\sin x$ and $f^{(4)}(x)=\cos x$; when $g(x)=1-\frac{1}{2}x^2$, $g'''(x)=0$ and $g^{(4)}(x)=0$. Thus $f'''(0)=0=g'''(0)$ so the third derivatives agree at $x=0$. However, the fourth derivatives do not agree since $f^{(4)}(0)=1$ but $g^{(4)}(0)=0$. In case (b), when $f(x)=\sin(x+a)$ and $g(x)$ $=\sin x\cos a+\cos x\sin a$, notice that $f(x)=g(x)$ for all x, not just $x=0$. Since this is an identity, we have $f^{(n)}(x)=g^{(n)}(x)$ for any x and any positive integer n.

4. (a) $y=\sin x \Rightarrow y'=\cos x \Rightarrow y''=-\sin x \Rightarrow y''+y=-\sin x+\sin x=0$; $y=\cos x \Rightarrow y'=-\sin x$ $\Rightarrow y''=-\cos x \Rightarrow y''+y=-\cos x+\cos x=0$; $y=a\cos x+b\sin x \Rightarrow y'=-a\sin x+b\cos x$ $\Rightarrow y''=-a\cos x-b\sin x \Rightarrow y''+y=(-a\cos x-b\sin x)+(a\cos x+b\sin x)=0$

(b) $y = \sin(2x) \Rightarrow y' = 2\cos(2x) \Rightarrow y'' = -4\sin(2x) \Rightarrow y'' + 4y = -4\sin(2x) + 4\sin(2x) = 0$. Similarly, $y = \cos(2x)$ and $y = a\cos(2x) + b\sin(2x)$ satisfy the differential equation $y'' + 4y = 0$. In general, $y = \cos(mx)$, $y = \sin(mx)$ and $y = a\cos(mx) + b\sin(mx)$ satisfy the differential equation $y'' + m^2y = 0$.

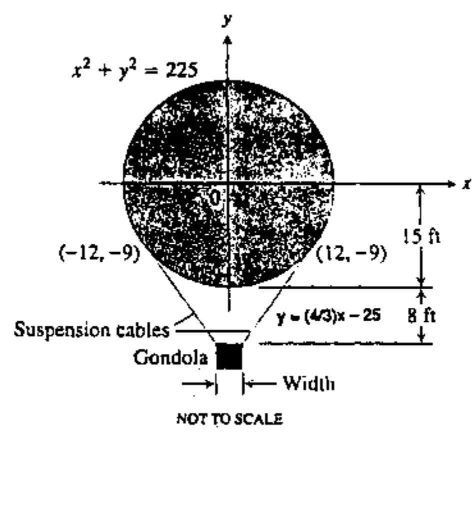
5. If the circle $(x-h)^2 + (y-k)^2 = a^2$ and $y = x^2 + 1$ are tangent at $(1,2)$, then the slope of this tangent is $m = 2x|_{(1,2)} = 2$ and the tangent line is $y = 2x$. The line containing (h,k) and $(1,2)$ is perpendicular to $y = 2x \Rightarrow \frac{k-2}{h-1} = -\frac{1}{2} \Rightarrow h = 5 - 2k \Rightarrow$ the location of the center is $(5-2k, k)$. Also, $(x-h)^2 + (y-k)^2 = a^2$ $\Rightarrow x - h + (y-k)y' = 0 \Rightarrow 1 + (y')^2 + (y-k)y'' = 0 \Rightarrow y'' = \frac{1+(y')^2}{k-y}$. At the point $(1,2)$ we know $y' = 2$ from the tangent line and that $y'' = 2$ from the parabola. Since the second derivatives are equal at $(1,2)$ we obtain $2 = \frac{1+(2)^2}{k-2} \Rightarrow k = \frac{9}{2}$. Then $h = 5 - 2k = -4 \Rightarrow$ the circle is $(x+4)^2 + \left(y - \frac{9}{2}\right)^2 = a^2$. Since $(1,2)$ lies on the circle we have that $a = \frac{5\sqrt{5}}{2}$.

6. The total revenue is the number of people times the price of the fare: $r(x) = xp = x\left(3 - \frac{x}{40}\right)^2$, where $0 \le x \le 60$. The marginal revenue is $\frac{dr}{dx} = \left(3 - \frac{x}{40}\right)^2 + 2x\left(3 - \frac{x}{40}\right)\left(-\frac{1}{40}\right) \Rightarrow \frac{dr}{dx} = \left(3 - \frac{x}{40}\right)\left[\left(3 - \frac{x}{40}\right) - \frac{2x}{40}\right]$ $= 3\left(3 - \frac{x}{40}\right)\left(1 - \frac{x}{40}\right)$. Then $\frac{dr}{dx} = 0 \Rightarrow x = 40$ (since $x = 120$ does not belong to the domain). When 40 people are on the bus the marginal revenue is zero and the fare is $p(40) = \left(3 - \frac{x}{40}\right)^2\Big|_{x=40} = \$4.00$.

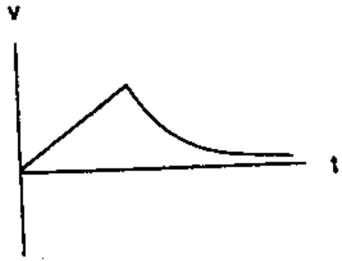
7. (a) $y = uv \Rightarrow \frac{dy}{dt} = \frac{du}{dt}v + u\frac{dv}{dt} = (0.04u)v + u(0.05v) = 0.09uv = 0.09y$

(b) If $\frac{du}{dt} = -0.02u$ and $\frac{dv}{dt} = 0.03v$, then $\frac{dy}{dt} = (-0.02u)v + (0.03v)u = 0.01uv = 0.01y$, increasing at 1% per year.

8. When $x^2 + y^2 = 225$, then $y' = -\frac{x}{y}$. The tangent line to the balloon at $(12,-9)$ is $y + 9 = \frac{4}{3}(x - 12)$ $\Rightarrow y = \frac{4}{3}x - 25$. The top of the gondola is $15 + 8$ $= 23$ ft below the center of the balloon. The intersection of $y = -23$ and $y = \frac{4}{3}x - 25$ is at the far right edge of the gondola $\Rightarrow -23 = \frac{4}{3}x - 25$ $\Rightarrow x = \frac{3}{2}$. Thus the gondola is $2x = 3$ ft wide.



9. Answers will vary. Here is one possibility



10. $s(t) = 10\cos\left(t+\frac{\pi}{4}\right) \Rightarrow v(t) = \frac{ds}{dt} = -10\sin\left(t+\frac{\pi}{4}\right) \Rightarrow a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -10\cos\left(t+\frac{\pi}{4}\right)$

(a) $s(0) = 10\cos\left(\frac{\pi}{4}\right) = \frac{10}{\sqrt{2}}$

(b) Left: -10, Right: 10

(c) Solving $10\cos\left(t+\frac{\pi}{4}\right) = -10 \Rightarrow \cos\left(t+\frac{\pi}{4}\right) = -1 \Rightarrow t = \frac{3\pi}{4}$ when the particle is farthest to the left. Solving $10\cos\left(t+\frac{\pi}{4}\right) = 10 \Rightarrow \cos\left(t+\frac{\pi}{4}\right) = 1 \Rightarrow t = -\frac{\pi}{4}$, but $t \ge 0 \Rightarrow t = 2\pi + \frac{-\pi}{4} = \frac{7\pi}{4}$ when the particle is farthest to the right. Thus, $v\left(\frac{3\pi}{4}\right) = 0$, $v\left(\frac{7\pi}{4}\right) = 0$, $a\left(\frac{3\pi}{4}\right) = 10$, and $a\left(\frac{7\pi}{4}\right) = -10$.

(d) Solving $10\cos\left(t+\frac{\pi}{4}\right) = 0 \Rightarrow t = \frac{\pi}{4} \Rightarrow v\left(\frac{\pi}{4}\right) = -10$, $\left|v\left(\frac{\pi}{4}\right)\right| = 10$ and $a\left(\frac{\pi}{4}\right) = 0$.

11. (a) $s(t) = 64t - 16t^2 \Rightarrow v(t) = \frac{ds}{dt} = 64 - 32t = 32(2-t)$. The maximum height is reached when $v(t) = 0$ $\Rightarrow t = 2$ sec. The velocity when it leaves the hand is $v(0) = 64$ ft/sec.

(b) $s(t) = 64t - 2.6t^2 \Rightarrow v(t) = \frac{ds}{dt} = 64 - 5.2t$. The maximum height is reached when $v(t) = 0 \Rightarrow t \approx 12.31$ sec. The maximum height is about $s(12.31) = 393.85$ ft.

12. $s_1 = 3t^3 - 12t^2 + 18t + 5$ and $s_2 = -t^3 + 9t^2 - 12t \Rightarrow v_1 = 9t^2 - 24t + 18$ and $v_2 = -3t^2 + 18t - 12$; $v_1 = v_2$ $\Rightarrow 9t^2 - 24t + 18 = -3t^2 + 18t - 12 \Rightarrow 2t^2 - 7t + 5 = 0 \Rightarrow (t-1)(2t-5) = 0 \Rightarrow t = 1$ sec and $t = 2.5$ sec.

13. $m\left(v^2 - v_0^2\right) = k\left(x_0^2 - x^2\right) \Rightarrow m\left(2v\,\frac{dv}{dt}\right) = k\left(-2x\,\frac{dx}{dt}\right) \Rightarrow m\,\frac{dv}{dt} = k\left(-\frac{2x}{2v}\right)\frac{dx}{dt} \Rightarrow m\,\frac{dv}{dt} = -kx\left(\frac{1}{v}\right)\frac{dx}{dt}$. Then substituting $\frac{dx}{dt} = v \Rightarrow m\,\frac{dv}{dt} = -kx$, as claimed.

14. (a) $x = At^2 + Bt + C$ on $[t_1, t_2] \Rightarrow v = \frac{dx}{dt} = 2At + B \Rightarrow v\left(\frac{t_1+t_2}{2}\right) = 2A\left(\frac{t_1+t_2}{2}\right) + B = A\left(t_1+t_2\right) + B$ is the instantaneous velocity at the midpoint. The average velocity over the time interval is $v_{av} = \frac{\Delta x}{\Delta t}$
$$= \frac{\left(At_2^2 + Bt_2 + C\right) - \left(At_1^2 + Bt_1 + C\right)}{t_2 - t_1} = \frac{(t_2 - t_1)\left[A\left(t_2 + t_1\right) + B\right]}{t_2 - t_1} = A\left(t_2 + t_1\right) + B.$$

(b) On the graph of the parabola $x = At^2 + Bt + C$, the slope of the curve at the midpoint of the interval $[t_1, t_2]$ is the same as the average slope of the curve over the interval.

15. (a) To be continuous at $x = \pi$ requires that $\lim\limits_{x\to\pi^-} \sin x = \lim\limits_{x\to\pi^+} (mx + b) \Rightarrow 0 = m\pi + b \Rightarrow m = -\frac{b}{\pi}$;

(b) If $y' = \begin{cases} \cos x, & x < \pi \\ m, & x \geq \pi \end{cases}$ is differentiable at $x = \pi$, then $\lim\limits_{x\to\pi^-} \cos x = m \Rightarrow m = -1$ and $b = \pi$.

16. $f'(0) = \lim\limits_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \frac{\frac{1-\cos x}{x} - 0}{x} = \lim\limits_{x\to 0} \left(\frac{1-\cos x}{x^2}\right)\left(\frac{1+\cos x}{1+\cos x}\right) = \lim\limits_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \left(\frac{1}{1+\cos x}\right) = \frac{1}{2}$.

Therefore $f'(0)$ exists with value $\frac{1}{2}$.

17. (a) For all a, b and for all $x \neq 2$, f is differentiable at x. Next, f differentiable at $x = 2 \Rightarrow$ f continuous at $x = 2 \Rightarrow \lim\limits_{x\to 2^-} f(x) = f(2) \Rightarrow 2a = 4a - 2b + 3 \Rightarrow 2a - 2b + 3 = 0$. Also, f differentiable at $x \neq 2$ $\Rightarrow f'(x) = \begin{cases} a, & x < 2 \\ 2ax - b, & x > 2 \end{cases}$. In order that $f'(2)$ exist we must have $a = 2a(2) - b \Rightarrow a = 4a - b \Rightarrow 3a = b$.

Then $2a - 2b + 3 = 0$ and $3a = b \Rightarrow a = \frac{3}{4}$ and $b = \frac{9}{4}$.

(b) For $x < 2$, the graph of f is a straight line having a slope of $\frac{3}{4}$ and passing through the origin for $x \geq 2$, the graph of f is a parabola. At $x = 2$, the value of the y-coordinate on the parabola is $\frac{3}{2}$ which matches the y-coordinate of the point on the straight line at $x = 2$. In addition, the slope of the parabola at the match up point is $\frac{3}{4}$ which is equal to the slope of the straight line. Therefore, since the graph is differentiable at the match up point, the graph is smooth there.

18. (a) For any a, b and for any $x \neq -1$, g is differentiable at x. Next, g differentiable at $x = -1 \Rightarrow$ g continuous at $x = -1 \Rightarrow \lim\limits_{x\to -1^+} g(x) = g(-1) \Rightarrow -a - 1 + 2b = -a + b \Rightarrow b = 1$. Also, g differentiable at $x \neq -1$ $\Rightarrow g'(x) = \begin{cases} a, & x < -1 \\ 3ax^2 + 1, & x > -1 \end{cases}$. In order that $g'(-1)$ exist we must have $a = 3a(-1)^2 + 1 \Rightarrow a = 3a + 1$ $\Rightarrow a = -\frac{1}{2}$.

(b) For $x \leq -1$, the graph of f is a straight line having a slope of $-\frac{1}{2}$ and a y-intercept of 1. For $x > -1$, the graph of f is a parabola. At $x = -1$, the value of the y-coordinate on the parabola is $\frac{3}{2}$ which matches the y-coordinate of the point on the straight line at $x = -1$. In addition, the slope of the parabola at the up point is $-\frac{1}{2}$ which is equal to the slope of the straight line. Therefore, since the graph is differentiable at the match up point, the graph is smooth there.

19. f odd $\Rightarrow f(-x) = -f(x) \Rightarrow \frac{d}{dx}(f(-x)) = \frac{d}{dx}(-f(x)) \Rightarrow f'(-x)(-1) = -f'(x) \Rightarrow f'(-x) = f'(x) \Rightarrow f'$ is even.

20. f even $\Rightarrow f(-x) = f(x) \Rightarrow \frac{d}{dx}(f(-x)) = \frac{d}{dx}(f(x)) \Rightarrow f'(-x)(-1) = f'(x) \Rightarrow f'(-x) = -f'(x) \Rightarrow f'$ is odd.

21. Let $h(x) = (fg)(x) = f(x)\,g(x) \Rightarrow h'(x) = \lim\limits_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim\limits_{x \to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x_0)}{x - x_0}$

$= \lim\limits_{x \to x_0} \frac{f(x)\,g(x) - f(x)\,g(x_0) + f(x)\,g(x_0) - f(x_0)\,g(x_0)}{x - x_0} = \lim\limits_{x \to x_0} \left[f(x)\left[\frac{g(x) - g(x_0)}{x - x_0}\right]\right] + \lim\limits_{x \to x_0} \left[g(x_0)\left[\frac{f(x) - f(x_0)}{x - x_0}\right]\right]$

$= f(x_0) \lim\limits_{x \to x_0} \left[\frac{g(x) - g(x_0)}{x - x_0}\right] + g(x_0)\,f'(x_0) = 0 \cdot \lim\limits_{x \to x_0} \left[\frac{g(x) - g(x_0)}{x - x_0}\right] + g(x_0)\,f'(x_0) = g(x_0)\,f'(x_0)$, if g is continuous at x_0. Therefore $(fg)(x)$ is differentiable at x_0 if $f(x_0) = 0$, and $(fg)'(x_0) = g(x_0)\,f'(x_0)$.

22. From Exercise 21 we have that fg is differentiable at 0 if f is differentiable at 0, $f(0) = 0$ and g is continuous at 0.

(a) If $f(x) = \sin x$ and $g(x) = |x|$, then $|x| \sin x$ is differentiable because $f'(0) = \cos(0) = 1$, $f(0) = \sin(0) = 0$ and $g(x) = |x|$ is continuous at $x = 0$.

(b) If $f(x) = \sin x$ and $g(x) = x^{2/3}$, then $x^{2/3} \sin x$ is differentiable because $f'(0) = \cos(0) = 1$, $f(0) = \sin(0) = 0$ and $g(x) = x^{2/3}$ is continuous at $x = 0$.

(c) If $f(x) = 1 - \cos x$ and $g(x) = \sqrt[3]{x}$, then $\sqrt[3]{x}\,(1 - \cos x)$ is differentiable because $f'(0) = \sin(0) = 0$, $f(0) = 1 - \cos(0) = 0$ and $g(x) = x^{1/3}$ is continuous at $x = 0$.

(d) If $f(x) = x$ and $g(x) = x \sin\left(\frac{1}{x}\right)$, then $x^2 \sin\left(\frac{1}{x}\right)$ is differentiable because $f'(0) = 1$, $f(0) = 0$ and

$\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = \lim\limits_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim\limits_{t \to \infty} \frac{\sin t}{t} = 0$ (so g is continuous at $x = 0$).

23. If $f(x) = x$ and $g(x) = x \sin\left(\frac{1}{x}\right)$, then $x^2 \sin\left(\frac{1}{x}\right)$ is differentiable at $x = 0$ because $f'(0) = 1$, $f(0) = 0$ and

$\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = \lim\limits_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim\limits_{t \to \infty} \frac{\sin t}{t} = 0$ (so g is continuous at $x = 0$). In fact, from Exercise 21, $h'(0) = g(0)\,f'(0) = 0$. However, for $x \neq 0$, $h'(x) = \left[x^2 \cos\left(\frac{1}{x}\right)\right]\left(-\frac{1}{x^2}\right) + 2x \sin\left(\frac{1}{x}\right)$. But $\lim\limits_{x \to 0} h'(x) = \lim\limits_{x \to 0} \left[-\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)\right]$ does not exist because $\cos\left(\frac{1}{x}\right)$ has no limit as $x \to 0$. Therefore, the derivative is not continuous at $x = 0$ because it has no limit there.

24. From the given conditions we have $f(x + h) = f(x)\,f(h)$, $f(h) - 1 = hg(h)$ and $\lim\limits_{h \to 0} g(h) = 1$. Therefore,

$f'(x) = \lim\limits_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{f(x)\,f(h) - f(x)}{h} = \lim\limits_{h \to 0} f(x)\left[\frac{f(h) - 1}{h}\right] = f(x)\left[\lim\limits_{h \to 0} g(h)\right] = f(x) \cdot 1 = f(x)$

$\Rightarrow f'(x) = f(x)$ exists.

25. Step 1: The formula holds for $n = 2$ (a single product) since $y = u_1 u_2 \Rightarrow \frac{dy}{dx} = \frac{du_1}{dx}\,u_2 + u_1\,\frac{du_2}{dx}$.

Step 2: Assume the formula holds for $n = k$:

$$y = u_1 u_2 \cdots u_k \Rightarrow \frac{dy}{dx} = \frac{du_1}{dx}\,u_2 u_3 \cdots u_k + u_1\,\frac{du_2}{dx}\,u_3 \cdots u_k + \ldots + u_1 u_2 \cdots u_{k-1}\,\frac{du_k}{dx}.$$

If $y = u_1 u_2 \cdots u_k u_{k+1} = (u_1 u_2 \cdots u_k) u_{k+1}$, then $\frac{dy}{dx} = \frac{d(u_1 u_2 \cdots u_k)}{dx} u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}$

$= \left(\frac{du_1}{dx} u_2 u_3 \cdots u_k + u_1 \frac{du_2}{dx} u_3 \cdots u_k + \cdots + u_1 u_2 \cdots u_{k-1} \frac{du_k}{dx} \right) u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}$

$= \frac{du_1}{dx} u_2 u_3 \cdots u_{k+1} + u_1 \frac{du_2}{dx} u_3 \cdots u_{k+1} + \cdots + u_1 u_2 \cdots u_{k-1} \frac{du_k}{dx} u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}.$

Thus the original formula holds for $n = (k+1)$ whenever it holds for $n = k$.

NOTES:

CHAPTER 3 APPLICATIONS OF DERIVATIVES

3.1 EXTREME VALUES OF FUNCTIONS

1. An absolute minimum at $x = c_2$, an absolute maximum at $x = b$. Theorem 1 guarantees the existence of such extreme values because h is continuous on $[a, b]$.

2. An absolute minimum at $x = b$, an absolute maximum at $x = c$. Theorem 1 guarantees the existence of such extreme values because f is continuous on $[a, b]$.

3. No absolute minimum. An absolute maximum at $x = c$. Since the function's domain is an open interval, the function does not satisfy the hypotheses of Theorem 1 and need not have absolute extreme values.

4. No absolute extrema. The function is neither continuous nor defined on a closed interval, so it need not fulfill the conclusions of Theorem 1.

5. An absolute minimum at $x = a$ and an absolute maximum at $x = c$. Note that $y = g(x)$ is not continuous but still has extrema. When the hypothesis of Theorem 1 is satisfied then extrema are guaranteed, but when the hypothesis is not satisfied, absolute extrema may or may not occur.

6. Absolute minimum at $x = c$ and an absolute maximum at $x = a$. Note that $y = g(x)$ is not continuous but still has absolute extrema. When the hypothesis of Theorem 1 is satisfied then extrema are guaranteed, but when the hypothesis is not satisfied, absolute extrema may or may not occur.

7. Local minimum at $(-1, 0)$, local maximum at $(1, 0)$

8. Minima at $(-2, 0)$ and $(2, 0)$, maximum at $(0, 2)$

9. Maximum at $(0, 5)$. Note that there is no minimum since the endpoint $(2, 0)$ is excluded from the graph.

10. Local maximum at $(-3, 0)$, local minimum at $(2, 0)$, maximum at $(1, 2)$, minimum at $(0, -1)$

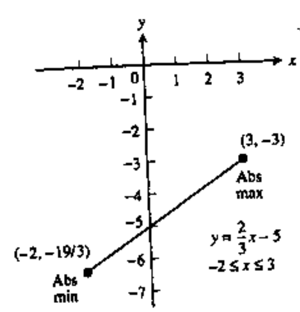
11. Graph (c), since this is the only graph that has positive slope at c.

12. Graph (b), since this is the only graph that represents a differentiable function at a and b and has negative slope at c.

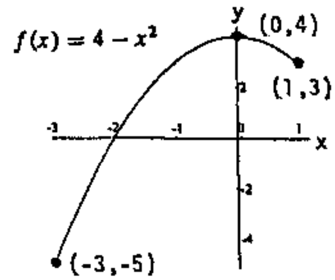
13. Graph (d), since this is the only graph representing a function that is differentiable at b but not at a.

14. Graph (a), since this is the only graph that represents a function that is not differentiable at a or b.

15. $f(x) = \frac{2}{3}x - 5 \Rightarrow f'(x) = \frac{2}{3} \Rightarrow$ no critical points;
$f(-2) = -\frac{19}{3}$, $f(3) = -3 \Rightarrow$ the absolute maximum is -3 at $x = 3$ and the absolute minimum is $-\frac{19}{3}$ at $x = -2$



16. $f(x) = 4 - x^2 \Rightarrow f'(x) = -2x \Rightarrow$ a critical point at $x = 0$; $f(-3) = -5$, $f(0) = 4$, $f(1) = 3 \Rightarrow$ the absolute maximum is 4 at $x = 0$ and the absolute minimum is -5 at $x = -3$



17. The first derivative of $f'(x) = \cos\left(x + \frac{\pi}{4}\right)$, has zeros at $x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$.

Critical point values: $x = \frac{\pi}{4}$ $\quad f(x) = 1$

$x = \frac{5\pi}{4}$ $\quad f(x) = -1$

Endpoint values: $x = 0$ $\quad f(x) = \frac{1}{\sqrt{2}}$

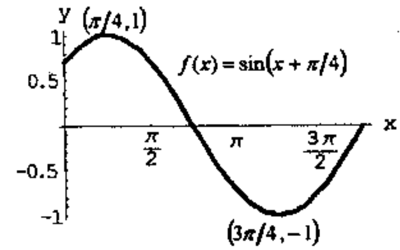
$x = \frac{7\pi}{4}$ $\quad f(x) = 0$

Maximum value is 1 at $x = \frac{\pi}{4}$;

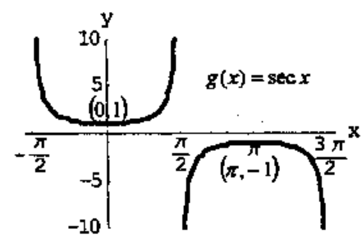
minimum value is -1 at $x = \frac{5\pi}{4}$;

local minimum at $\left(0, \frac{1}{\sqrt{2}}\right)$;

local maximum at $\left(\frac{7\pi}{4}, 0\right)$



Graph for Exercise 17



Graph for Exercise 18

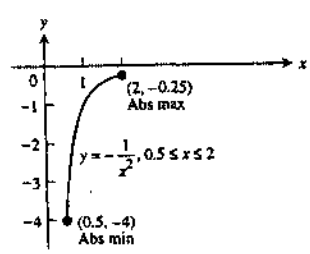
18. The first derivative $g'(x) = \sec x \tan x$ has zeros at $x = 0$ and $x = \pi$ and is undefined at $x = \frac{\pi}{2}$. Since $g(x) = \sec x$ is also undefined at $x = \frac{\pi}{2}$, the critical points occur only at $x = 0$ and $x = \pi$.

Critical point values: $x = 0$ $\quad g(x) = 1$

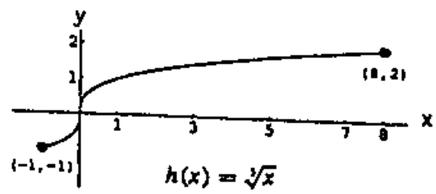
$x = \pi$ $\quad g(x) = -1$

Since the range of $g(x)$ is $(-\infty, -1] \cup [1, \infty)$, these values must be a local minimum and local maximum, respectively. Local minimum at $(0, 1)$; local maximum at $(\pi, -1)$. There are no absolute extrema on the interval $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

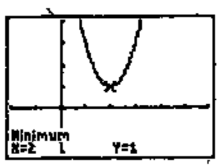
19. $F(x) = -\frac{1}{x^2} = -x^{-2} \Rightarrow F'(x) = 2x^{-3} = \frac{2}{x^3}$, however $x = 0$ is not a critical point since 0 is not in the domain; $F(0.5) = -4$, $F(2) = -0.25 \Rightarrow$ the absolute maximum is -0.25 at $x = 2$ and the absolute minimum is -4 at $x = 0.5$



20. $h(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow h'(x) = \frac{1}{3}x^{-2/3} \Rightarrow$ a critical point at $x = 0$; $h(-1) = -1$, $h(0) = 0$, $h(8) = 2 \Rightarrow$ the absolute maximum is 2 at $x = 8$ and the absolute minimum is -1 at $x = -1$



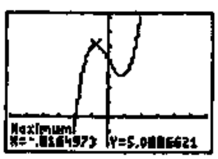
21.



[−2, 6] by [−2, 4]

Minimum value is 1 at $x = 2$.

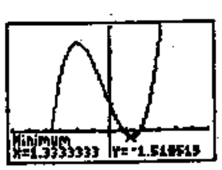
22.



[−6, 6] by [−2, 7]

To find the exact values, note that $y' = 3x^2 - 2$, which is zero when $x = \pm\sqrt{\frac{2}{3}}$. Local maximum at $\left(-\sqrt{\frac{2}{3}}, 4 + \frac{4+\sqrt{6}}{9}\right) \approx (-0.816, 5.089)$; local minimum at $\left(\sqrt{\frac{2}{3}}, 4 - \frac{4\sqrt{6}}{9}\right) \approx (0.816, 2.911)$

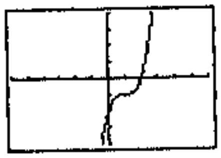
23.



[−6, 6] by [−5, 20]

To find the exact values, note that $y' = 3x^2 + 2x - 8 = (3x - 4)(x + 2)$, which is zero when $x = -2$ or $x = \frac{4}{3}$. Local maximum at $(-2, 17)$; local minimum at $\left(\frac{4}{3}, -\frac{41}{27}\right)$

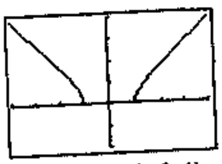
24.



[−6, 6] by [−4, 4]

Note that $y' = 3x^2 - 6x + 3 = 3(x-1)^2$, which is zero at $x = 1$. The graph shows that the function assumes lower values to the left and higher values to the right of this point, so the function has no local or global extreme values.

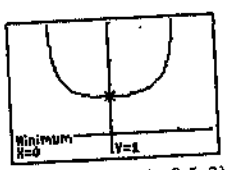
25.



[−4, 4] by [−2, 4]

Minimum value is 0 at $x = -1$ and at $x = 1$.

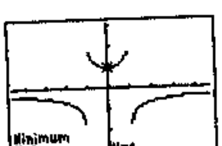
26.



[−1.5, 1.5] by [−0.5, 3]

The minimum value is 1 at $x = 0$.

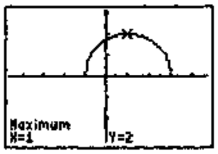
27.



[−4.7, 4.7] by [−3.1, 3.1]

The actual graph of the function has asymptotes at $x = \pm 1$, so there are no extrema near these values. (This is an example of *grapher failure*.) There is a local minimum at $(0, 1)$.

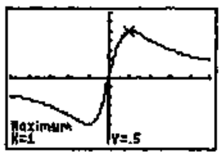
28.



[−4.7, 4.7] by [−3.1, 3.1]

Maximum value is 2 at $x = 1$;
minimum value is 0 at $x = -1$ and at $x = 3$.

29.

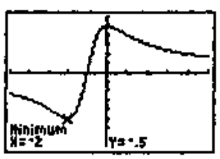


[−5, 5] by [−0.7, 0.7]

Maximum value is $\frac{1}{2}$ at $x = 1$;

minimum value is $-\frac{1}{2}$ at $x = -1$.

30.

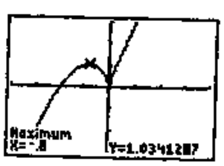


[−5, 5] by [−0.8, 0.6]

Maximum value is $\frac{1}{2}$ at $x = 0$;

minimum value is $-\frac{1}{2}$ at $x = -2$.

31.

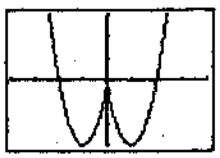


[−4, 4] by [−3, 3]

$$y' = x^{2/3}(1) + \frac{2}{3}x^{-1/3}(x+2) = \frac{5x+4}{3\sqrt[3]{x}}$$

crit. pt.	derivative	extremum	value
$x = -\frac{4}{5}$	0	local max	$\frac{12}{25}10^{1/3} = 1.034$
$x = 0$	undefined	local min	0

32.

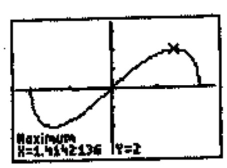


[−4, 4] by [−3, 3]

$$y' = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}(x^2-4) = \frac{8x^2-8}{3\sqrt[3]{x}}$$

crit. pt.	derivative	extremum	value
$x = -1$	0	minimum	-3
$x = 0$	undefined	local max	0
$x = 1$	0	minimum	-3

33.

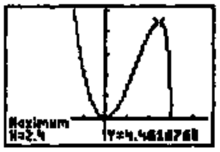


[−2.35, 2.35] by [−3.5, 3.5]

$$y' = x \cdot \frac{1}{2\sqrt{4-x^2}}(-2x) + (1)\sqrt{4-x^2} = \frac{-x^2 + (4-x^2)}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}}$$

crit. pt.	derivative	extremum	value
$x = -2$	undefined	local max	0
$x = -\sqrt{2}$	0	minimum	-2
$x = \sqrt{2}$	0	maximum	2
$x = 2$	undefined	local min	0

34.

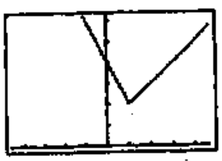


[−4.7, 4.7] by [−1, 5]

$$y' = x^2 \cdot \frac{1}{2\sqrt{3-x}}(-1) + 2x\sqrt{3-x} = \frac{-x^2 + (4x)(3-x)}{2\sqrt{3-x}} = \frac{-5x^2 + 12x}{2\sqrt{3-x}}$$

crit. pt.	derivative	extremum	value
$x = 0$	0	minimum	0
$x = \frac{12}{5}$	0	local max	$\frac{144}{125}15^{1/2} \approx 4.462$
$x = 3$	undefined	minimum	0

35.



[−4.7, 4.7] by [0, 6.2]

$$y' = \begin{cases} -2, & x < 1 \\ 1, & x > 1 \end{cases}$$

crit. pt.	derivative	extremum	value
$x = 1$	undefined	minimum	2

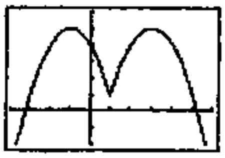
36.

[−4, 4] by [−1, 6]

$$y' = \begin{cases} -1, & x < 0 \\ 2 - 2x, & x > 0 \end{cases}$$

crit. pt.	derivative	extremum	value
$x = 0$	undefined	local min	3
$x = 1$	0	local max	4

37.

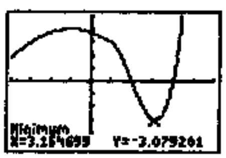


[−4, 6] by [−2, 6]

$$y' = \begin{cases} -2x - 2, & x < 1 \\ -2x + 6, & x > 1 \end{cases}$$

crit. pt.	derivative	extremum	value
$x = -1$	0	maximum	5
$x = 1$	undefined	local min	1
$x = 3$	0	maximum	5

38.



[−4, 6] by [−5, 5]

We begin by determining whether $f'(x)$ is defined at $x = 1$, where

$$f(x) = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$$

Left-hand derivative:

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} &= \lim_{h\to 0^-} \frac{-\frac{1}{4}(1+h)^2 - \frac{1}{2}(1+h) + \frac{15}{4} - 3}{h} \\ &= \lim_{h\to 0^-} \frac{-h^2 - 4h}{4h} \\ &= \lim_{h\to 0^-} \frac{1}{4}(-h - 4) \\ &= -1 \end{aligned}$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{(1+h)^3-6(1+h)^2+8(1+h)-3}{h}$$
$$=\lim_{h\to 0^+}\frac{h^3-3h^2-h}{h}$$
$$=\lim_{h\to 0^+}(h^2-3h-1)$$
$$=-1$$

Thus $f'(x)=\begin{cases}-\frac{1}{2}x-\frac{1}{2}, & x\le 1\\ 3x^2-12x+8, & x>1\end{cases}$

Note that $-\frac{1}{2}x-\frac{1}{2}=0$ when $x=-1$, and $3x^2-12x+8=0$ when $x=\dfrac{12\pm\sqrt{12^2-4(3)(8)}}{2(3)}$ $=\dfrac{12\pm\sqrt{48}}{6}=2\pm\dfrac{2\sqrt{3}}{3}$. But $2-\dfrac{2\sqrt{3}}{3}\approx 0.845<1$, so the only critical points occur at $x=-1$ and $x=2+\dfrac{2\sqrt{3}}{3}\approx 3.155$.

crit. pt.	derivative	extremum	value
$x=-1$	0	local max	4
$x\approx 3.155$	0	local max	≈ -3.079

39. (a) No, since $f'(x)=\frac{2}{3}(x-2)^{-1/3}$, which is undefined at $x=2$.

(b) The derivative is defined and nonzero for all $x\ne 2$. Also, $f(2)=0$ and $f(x)>0$ for all $x\ne 2$.

(c) No, f(x) need not have a global maximum because its domain is all real numbers. Any restriction of f to a closed interval of the form [a, b] would have both a maximum value and a minimum value on the interval.

(d) The answers are the same as (a) and (b) with 2 replaced by a.

40. Note that $f(x)=\begin{cases}-x^3+9x, & x\le -3 \text{ or } 0\le x<3\\ x^3-9x, & -3<x<0 \text{ or } x\ge 3\end{cases}$

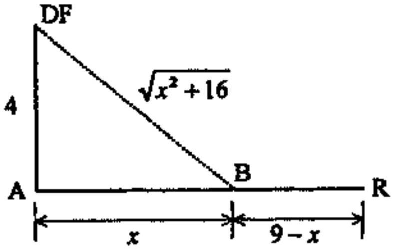
Therefore, $f'(x)=\begin{cases}-3x^2+9, & x<-3 \text{ or } 0<x<3\\ 3x^2-9, & -3<x<0 \text{ or } x>3\end{cases}$

(a) No, since the left- and right-hand derivatives at $x=0$ are -9 and 9, respectively.

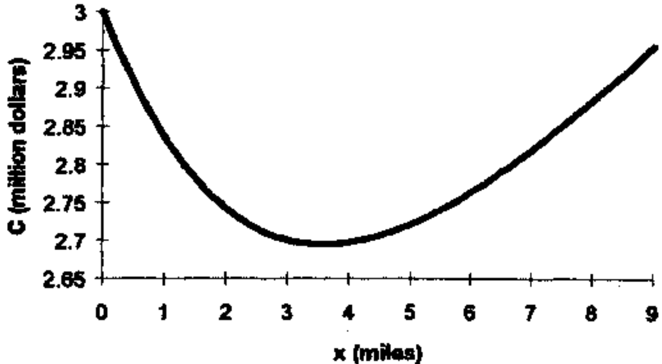
(b) No, since the left- and right-hand derivatives at $x=3$ are -18 and 18, respectively.

(c) No, since the left- and right-hand derivatives at $x=-3$ are -18 and 18, respectively.

(d) The critical points occur when $f'(x)=0$ (at $x=\pm\sqrt{3}$) and when $f'(x)$ is undefined (at $x=0$ and $x=\pm 3$). The minimum value is 0 at $x=-3$, at $x=0$, and at $x=3$; local maxima occur at $(-\sqrt{3},6\sqrt{3})$ and $(\sqrt{3},6\sqrt{3})$.

41. 

(a) The construction cost is $C(x) = 0.3\sqrt{16+x^2} + 0.2(9-x)$ million dollars, where $0 \le x \le 9$ miles. The following is a graph of $C(x)$.

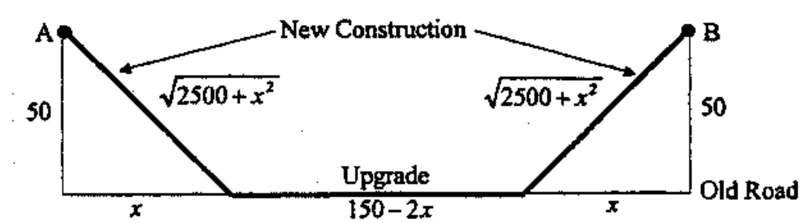


Solving $C'(x) = \frac{0.3x}{\sqrt{16+x^2}} - 0.2 = 0$ gives $x = \pm\frac{8\sqrt{5}}{5} \approx \pm 3.58$ miles, but only $x = 3.58$ miles is a critical point in the specified domain. Evaluating the costs at the critical and endpoints gives $C(0) = \$3$ million, $C\left(8\sqrt{5}/5\right) \approx \$2.694$ million, and $C(9) \approx \$2.955$ million. Therefore, to minimize the cost of construction, the pipeline should be placed from the docking facility to point B, 3.58 miles along the shore from point A, and then along the shore from B to the refinery.

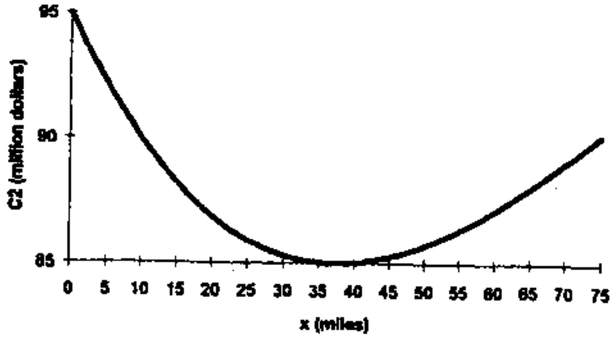
(b) If the per mile cost of underwater construction is p, then $C(x) = p\sqrt{16+x^2} + 0.2(9-x)$ and $C'(x) = \frac{px}{\sqrt{16+x^2}} - 0.2 = 0$ gives $x_c = \frac{0.8}{\sqrt{p^2 - 0.04}}$, which minimizes the construction cost provided $x_c \le 9$. The value of p that gives $x_c = 9$ miles is 0.218864. Consequently, if the underwater construction costs \$218,864 per mile or less, then running the pipeline along a straight line directly from the docking facility to the refinery will minimize the cost of construction.

In theory, p would have to be infinite to justify running the pipe directly from the docking facility to point A (i.e., for x_c to be zero). For all values of $p > 0.218864$ there is always an $x_c \in (0, 9)$ that will give a minimum value for C. This is proved by looking at $C''(x_c) = \frac{16p}{\left(16 + x_c^2\right)^{3/2}}$ which is always positive for $p > 0$.

42. There are two options to consider. The first is to build a new road straight from village A to village B. The second is to build a new highway segment from village A to the Old Road, reconstruct a segment of Old Road, and build a new highway segment from Old Road to village B, as shown in the figure. The cost of the first option is $C_1 = 0.5(150) = \$75$ million.

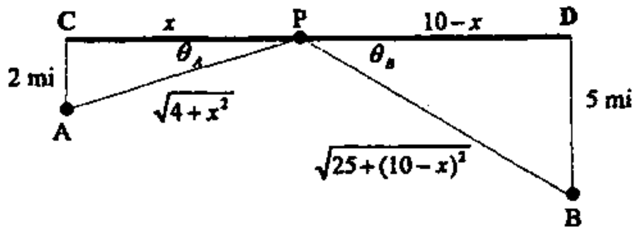


The construction cost for the second option is $C_2(x) = 0.5\left(2\sqrt{2500+x^2}\right) + 0.3(150-2x)$ million dollars for $0 \le x \le 75$ miles. The following is a graph of $C_2(x)$.

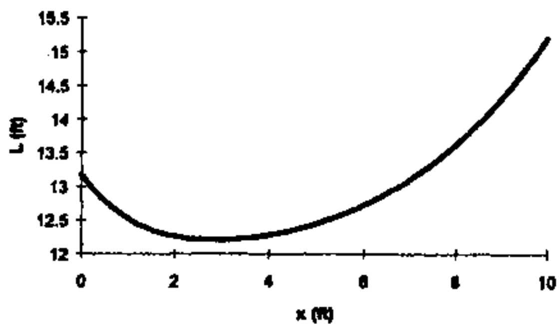


Solving $C_2'(x) = \frac{x}{\sqrt{2500+x^2}} - 0.6 = 0$ gives $x = \pm 37.5$ miles, but only $x = 37.5$ miles is in the specified domain. In summary, $C_1 = \$75$ million, $C_2(0) = \$95$ million, $C_2(37.5) = \$85$ million, and $C_2(75) = \$90.139$ million. Consequently, a new road straight from village A to village B is the least expensive option.

43.



The length of pipeline is $L(x) = \sqrt{4+x^2} + \sqrt{25+(10-x)^2}$ for $0 \le x \le 10$. The following is a graph of $L(x)$.



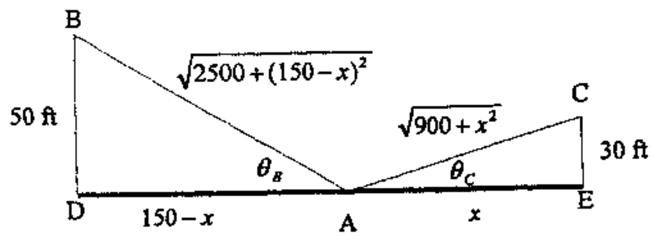
Setting the derivative of $L(x)$ equal to zero gives $L'(x) = \frac{x}{\sqrt{4+x^2}} - \frac{(10-x)}{\sqrt{25+(10-x)^2}} = 0$. Note that

$\frac{x}{\sqrt{4+x^2}} = \cos\theta_A$ and $\frac{10-x}{\sqrt{25-(10-x)^2}} = \cos\theta_B$, therefore, $L'(x) = 0$ when $\cos\theta_A = \cos\theta_B$, or

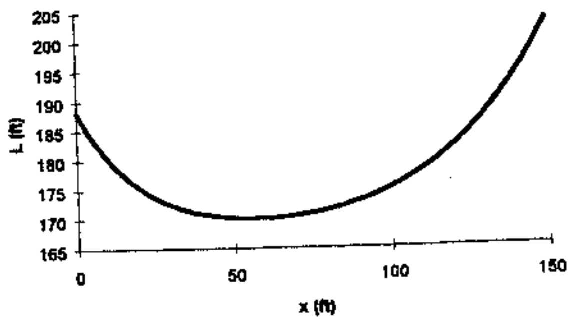
$\theta_A = \theta_B$ and ΔACP is similar to ΔBDP. Use simple proportions to determine x as follows:

$\frac{x}{2} = \frac{10-x}{5} \Rightarrow x = \frac{20}{7} \approx 2.857$ miles along the coast from town A to town B.

If the two towns were on opposite sides of the river, the obvious solution would be to place the pump station on a straight line (the shortest distance) between the two towns, again forcing $\theta_A = \theta_B$. The shortest length of pipe is the same regardless of whether the towns are on the same or opposite sides of the river.

44. 

(a) The length of guy wire is $L(x) = \sqrt{900 + x^2} + \sqrt{2500 + (150 - x)^2}$ for $0 \le x \le 150$. The following is a graph of $L(x)$:



Setting $L'(x)$ equal to zero gives $L'(x) = \dfrac{x}{\sqrt{900 + x^2}} - \dfrac{(150 - x)}{\sqrt{2500 + (150 - x)^2}} = 0$. Note that

$\dfrac{x}{\sqrt{900 + x^2}} = \cos\theta_C$ and $\dfrac{150 - x}{\sqrt{2500 - (150 - x)^2}} = \cos\theta_B$. Therefore, $L'(x) = 0$ when $\cos\theta_C = \cos\theta_B$,

or $\theta_C = \theta_B$ and ΔACE is similar to ΔABD. Use simple proportions to determine x: $\frac{x}{30} = \frac{150 - x}{50}$

$\Rightarrow x = \frac{225}{4} = 56.25$ feet.

(b) If the heights of the towers are h_B and h_C, and the horizontal distance between them is s, then

$L(x) = \sqrt{h_C^2 + x^2} + \sqrt{h_B^2 + (s - x)^2}$ and $L'(x) = \dfrac{x}{\sqrt{h_C^2 + x^2}} - \dfrac{(s - x)}{\sqrt{h_B^2 + (s - x)^2}}$ However, $\dfrac{x}{\sqrt{h_C^2 + x^2}} = \cos\theta_C$

and $\dfrac{(s - x)}{\sqrt{h_B^2 + (s - x)^2}} = \cos\theta_B$. Therefore, $L'(x) = 0$ when $\cos\theta_C = \cos\theta_B$, or $\theta_C = \theta_B$, and ΔACE is similar

to ΔABD. Simple proportions can again be used to determine the optimum x: $\frac{x}{h_C} = \frac{s - x}{h_B}$

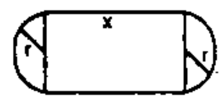
$\Rightarrow x = \left(\dfrac{h_C}{h_B + h_C}\right)s.$

45. (a) $V(x) = 160x - 52x^2 + 4x^3$
$V'(x) = 160 - 104x + 12x^2 = 4(x - 2)(3x - 20)$
The only critical point in the interval $(0, 5)$ is at $x = 2$. The maximum value of $V(x)$ is 144 at $x = 2$.
(b) The largest possible volume of the box is 144 cubic units, and it occurs when $x = 2$.

46. (a) $P'(x) = 2 - 200x^{-2}$
The only critical point in the interval $(0, \infty)$ is at $x = 10$. The minimum value of $P(x)$ is 40 at $x = 10$.
(b) The smallest possible perimeter of the rectangle is 40 units and it occurs at $x = 10$, which makes the rectangle a 10 by 10 square.

47. Let x represent the length of the base and $\sqrt{25-x^2}$ the height of the triangle. The area of the triangle is represented by $A(x) = \frac{x}{2}\sqrt{25-x^2}$ where $0 \le x \le 5$. Consequently, solving $A'(x) = 0 \Rightarrow \frac{25-2x^2}{2\sqrt{25-x^2}} = 0$ $\Rightarrow x = \frac{5}{\sqrt{2}}$. Since $A(0) = A(5) = 0$, $A(x)$ is maximized at $x = \frac{5}{\sqrt{2}}$. The largest possible area is $A\left(\frac{5}{\sqrt{2}}\right) = \frac{25}{4}\ \text{cm}^2$.

48. (a) From the diagram the perimeter $P = 2x + 2\pi r = 400$ $\Rightarrow x = 200 - \pi r$. We wish to maximize the area $A = 2rx$ $\Rightarrow A(r) = 400r - 2\pi r^2$



(b) $A'(r) = 400 - 4\pi r$ and $A''(r) = -4\pi$. The critical point is $r = \frac{100}{\pi}$ and $A''\left(\frac{100}{\pi}\right) = -4\pi < 0$. There is a maximum at $r = \frac{100}{\pi}$. The values $x = 100$ m and $r = \frac{100}{\pi} \approx 31.83$ m maximize the area of the rectangle.

49. $s = -\frac{1}{2}gt^2 + v_0t + s_0 \Rightarrow \frac{ds}{dt} = -gt + v_0 = 0 \Rightarrow t = \frac{v_0}{g}$. Then $s\left(\frac{v_0}{g}\right) = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\left(\frac{v_0}{g}\right) + s_0$ $= \frac{v_0^2}{2g} + s_0$ is the maximum height since $\frac{d^2s}{dt^2} = -g < 0$.

50. $\frac{di}{dt} = -2\sin t + 2\cos t$, solving $\frac{di}{dt} = 0 \Rightarrow \tan t = 1 \Rightarrow t = \frac{\pi}{4} + n\pi$ where n is a nonnegative integer (in this Exercise t is never negative) $\Rightarrow$ the peak current is $2\sqrt{2}$ amps

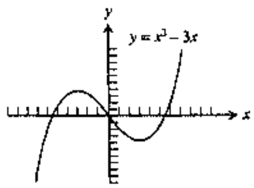
51. Yes, since $f(x) = |x| = \sqrt{x^2} = \left(x^2\right)^{1/2} \Rightarrow f'(x) = \frac{1}{2}\left(x^2\right)^{-1/2}(2x) = \frac{x}{\left(x^2\right)^{1/2}} = \frac{x}{|x|}$ is not defined at $x = 0$. Thus it is not required that f' be zero at a local extreme point since f' may be undefined there.

52. If $f(c)$ is a local maximum value of f, then $f(x) \le f(c)$ for all x in some open interval (a, b) containing c. Since f is even, $f(-x) = f(x) \le f(c) = f(-c)$ for all $-x$ in the open interval $(-b, -a)$ containing $-c$. That is, f assumes a local maximum at the point $-c$. This is also clear from the graph of f because the graph of an even function is symmetric about the y-axis.

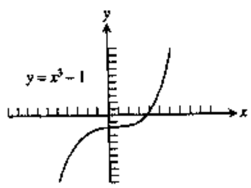
53. If $g(c)$ is a local minimum value of g, then $g(x) \ge g(c)$ for all x in some open interval (a, b) containing c. Since g is odd, $g(-x) = -g(x) \le -g(c) = g(-c)$ for all $-x$ in the open interval $(-b, -a)$ containing $-c$. That is, g assumes a local maximum at the point $-c$. This is also clear from the graph of g because the graph of an odd function is symmetric about the origin.

54. If there are no boundary points or critical points the function will have no extreme values in its domain. Such functions do indeed exist, for example $f(x) = x$ for $-\infty < x < \infty$. (Any other linear function $f(x) = mx + b$ with $m \ne 0$ will do as well.)

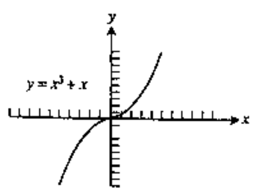
55. (a) $f'(x) = 3ax^2 + 2bx + c$ is a quadratic, so it can have 0, 1, or 2 zeros, which would be the critical points of f. Examples:



The function $f(x) = x^3 - 3x$ has two critical points at $x = -1$ and $x = 1$.



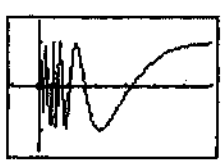
The function $f(x) = x^3 - 1$ has one critical point at $x = 0$.



The function $f(x) = x^3 + x$ has no critical points.

(b) The function can have either two local extreme values or no extreme values. (If there is only one critical point, the cubic function has no extreme values.)

56. (a)



$[-0.1, 0.6]$ by $[-1.5, 1.5]$

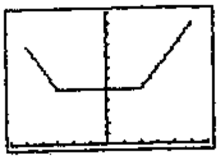
$f(0) = 0$ is not a local extreme value because in any open interval containing $x = 0$, there are infinitely many points where $f(x) = 1$ and where $f(x) = -1$.

(b) One possible answer, on the interval $[0, 1]$:

$$f(x) = \begin{cases} (1-x)\cos\frac{1}{1-x}, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

This function has no local extreme value at $x = 1$. Note that it is continuous on $[0, 1]$.

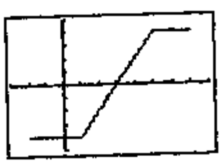
57.



$[-6, 6]$ by $[0, 12]$

Maximum value is 11 at $x = 5$;
minimum value is 5 on the interval $[-3, 2]$;
local maximum at $(-5, 9)$

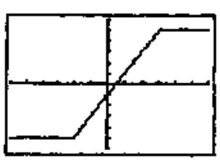
58.



$[-3, 8]$ by $[-5, 5]$

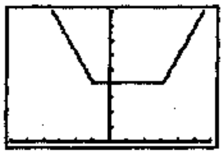
Maximum value is 4 on the interval $[5, 7]$;
minimum value is -4 on the interval $[-2, 1]$.

59.



$[-6, 6]$ by $[-6, 6]$

Maximum value is 5 on the interval $[3, \infty)$;
minimum value is -5 on the interval $(-\infty, -2]$.

60. 

[−6, 6] by [0, 9]

Minimum value is 4 on the interval $[-1, 3]$

61-70. Example CAS commands:

Maple:

```
f:=x -> 2 + 2*x - 3*(x ^ 2) ^ (1/3);
plot(f(x), x=-1..10/3);
fp:=diff(f(x),x);
solve(fp=0,x);
simplify(fp);
den:=denom(%);
solve(denom(fp)=0,x);
evalf([f(-1),f(0),f(1),f(10/3)]);
```

Mathematica:

```
Note: Here, use (x ^ 2) ^ (1/3) instead of x ^ (2/3), to avoid complex roots for negative x
a = -1; b = 10/3; f[x_] = 2 + 2 x - 3 (x ^ 2) ^ (1/3)
f'[x]
Plot[ {f[x], f'[x]}, {x,a,b} ]
NSolve[f'[x]==0]
Note: include critical point x=0
{f[a], f[0], f[x] /. %, f[b]} // N
```

3.2 THE MEAN VALUE THEOREM AND DIFFERENTIAL EQUATIONS

1. Does not; $f(x)$ is not differentiable at $x = 0$ in $(-1, 8)$.

2. Does; $g(x)$ is continuous for every point of $[0, 1]$ and differentiable for every point in $(0, 1)$.

3. Does; $s(t)$ is continuous for every point of $[0, 1]$ and differentiable for every point in $(0, 1)$.

4. Does not; $f(\theta)$ is not continuous at $\theta = 0$ because $\lim\limits_{\theta \to 0} f(\theta) = 1 \neq 0 = f(0)$.

5. Since $f(x)$ is not continuous on $0 \le x \le 1$, Rolle's Theorem does not apply because $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x \to 1^-} x = 1$ $\neq 0 = f(1)$ and $f(x)$ is not continuous at $x = 1$.

6. Since $f(x)$ must be continuous at $x = 0$ and $x = 1$ we have $\lim\limits_{x\to 0^+} f(x) = a = f(0) \Rightarrow a = 3$ and $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) \Rightarrow -1 + 3 + a = m + b \Rightarrow 5 = m + b$. Since $f(x)$ must also be differentiable at $x = 1$ we have $\lim\limits_{x\to 1^-} f'(x) = \lim\limits_{x\to 1^+} f'(x) \Rightarrow -2x + 3\big|_{x=1} = m\Big|_{x=1} \Rightarrow 1 = m$. Therefore, $a = 3$, $m = 1$ and $b = 4$.

7. By Corollary 1, $f'(x) = 0$ for all $x \Rightarrow f(x) = C$, where C is a constant. Since $f(-1) = 3$ we have $C = 3$ $\Rightarrow f(x) = 3$ for all x.

8. $g(t) = 2t + 5 \Rightarrow g'(t) = 2 = f'(t)$ for all t. By Corollary 2, $f(t) = g(t) + C$ for some constant C. Then $f(0) = g(0) + C \Rightarrow 5 = 5 + C \Rightarrow C = 0 \Rightarrow f(t) = g(t) = 2t + 5$ for all t.

9. (a) $y = \frac{x^2}{2} + C$ (b) $y = \frac{x^3}{3} + C$ (c) $y = \frac{x^4}{4} + C$

10. (a) $y = x^2 + C$ (b) $y = x^2 - x + C$ (c) $y = x^3 + x^2 - x + C$

11. (a) $r' = -\theta^{-2} \Rightarrow r = \frac{1}{\theta} + C$ (b) $r = \theta + \frac{1}{\theta} + C$ (c) $r = 5\theta - \frac{1}{\theta} + C$

12. (a) $y' = \frac{1}{2}t^{-1/2} \Rightarrow y = t^{1/2} + C \Rightarrow y = \sqrt{t} + C$ (b) $y = 2\sqrt{t} + C$
(c) $y = 2t^2 - 2\sqrt{t} + C$

13. $f(x) = x^2 - x + C$; $0 = f(0) = 0^2 - 0 + C \Rightarrow C = 0 \Rightarrow f(x) = x^2 - x$

14. $g(x) = -\frac{1}{x} + x^2 + C$; $1 = g(-1) = -\frac{1}{-1} + (-1)^2 + C \Rightarrow C = -1 \Rightarrow g(x) = -\frac{1}{x} + x^2 - 1$

15. $r(\theta) = 8\theta + \cot\theta + C$; $0 = r\left(\frac{\pi}{4}\right) = 8\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right) + C \Rightarrow 0 = 2\pi + 1 + C \Rightarrow C = -2\pi - 1$
$\Rightarrow r(\theta) = 8\theta + \cot\theta - 2\pi - 1$

16. $r(t) = \sec t - t + C$; $0 = r(0) = \sec(0) - 0 + C \Rightarrow C = -1 \Rightarrow r(t) = \sec t - t - 1$

17. $v = \frac{ds}{dt} = 9.8t + 5 \Rightarrow s = 4.9t^2 + 5t + C$; at $s = 10$ and $t = 0$ we have $C = 10 \Rightarrow s = 4.9t^2 + 5t + 10$

18. $v = \frac{ds}{dt} = 32t - 2 \Rightarrow s = 16t^2 - 2t + C$; at $s = 4$ and $t = \frac{1}{2}$ we have $C = 1 \Rightarrow s = 16t^2 - 2t + 1$

19. $v = \frac{ds}{dt} = \sin(\pi t) \Rightarrow s = -\frac{1}{\pi}\cos(\pi t) + C$; at $s = 0$ and $t = 0$ we have $C = \frac{1}{\pi} \Rightarrow s = \frac{1 - \cos(\pi t)}{\pi}$

20. $v = \frac{ds}{dt} = \frac{2}{\pi}\cos\left(\frac{2t}{\pi}\right) \Rightarrow s = \sin\left(\frac{2t}{\pi}\right) + C$; at $s = 1$ and $t = \pi^2$ we have $C = 1 \Rightarrow s = \sin\left(\frac{2t}{\pi}\right) + 1$

21. $a = 32 \Rightarrow v = 32t + C_1$; at $v = 20$ and $t = 0$ we have $C_1 = 20 \Rightarrow v = 32t + 20 \Rightarrow s = 16t^2 + 20t + C_2$; at $s = 5$ and $t = 0$ we have $C_2 = 5 \Rightarrow s = 16t^2 + 20t + 5$

22. $a = 9.8 \Rightarrow v = 9.8t + C_1$; at $v = -3$ and $t = 0$ we have $C_1 = -3 \Rightarrow v = 9.8t - 3 \Rightarrow s = 4.9t^2 - 3t + C_2$; at $s = 0$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = 4.9t^2 - 3t$

23. $a = -4\sin(2t) \Rightarrow v = 2\cos(2t) + C_1$; at $v = 2$ and $t = 0$ we have $C_1 = 0 \Rightarrow v = 2\cos(2t)$
$\Rightarrow s = \sin(2t) + C_2$; at $s = -3$ and $t = 0$ we have $C_2 = -3 \Rightarrow s = \sin(2t) - 3$

24. $a = \frac{9}{\pi^2}\cos\left(\frac{3t}{\pi}\right) \Rightarrow v = \frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right) + C_1$; at $v = 0$ and $t = 0$ we have $C_1 = 0 \Rightarrow v = \frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right)$
$\Rightarrow s = -\cos\left(\frac{3t}{\pi}\right) + C_2$; at $s = -1$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = -\cos\left(\frac{3t}{\pi}\right)$

25. $a(t) = v'(t) = 1.6 \Rightarrow v(t) = 1.6t + C$; at $(0,0)$ we have $C = 0 \Rightarrow v(t) = 1.6t$. When $t = 30$, then $v(30) = 48$ m/sec.

26. $a(t) = v'(t) = 20 \Rightarrow v(t) = 20t + C$; at $(0,0)$ we have $C = 0 \Rightarrow v(t) = 20t$. When $t = 60$, then $v(60) = 20(60) = 1200$ m/sec.

27. $a(t) = v'(t) = 9.8 \Rightarrow v(t) = 9.8t + C_1$; at $(0,0)$ we have $C_1 = 0 \Rightarrow s'(t) = v(t) = 9.8t \Rightarrow s(t) = 4.9t^2 + C_2$; at $(0,0)$ we have $C_2 = 0 \Rightarrow s(t) = 4.9t^2$. Then $s(t) = 10 \Rightarrow t^2 = \frac{10}{4.9} \Rightarrow t = \sqrt{\frac{10}{4.9}}$, and $v\left(\sqrt{\frac{10}{4.9}}\right) = 9.8\sqrt{\frac{10}{4.9}}$ $= \frac{2(4.9)\sqrt{10}}{\sqrt{4.9}} = (2)\sqrt{4.9}\sqrt{10} = 14$ m/sec.

28. $a(t) = v'(t) = -3.72 \Rightarrow v(t) = -3.72t + C_1$; at $(0,93)$ we have $C_1 = 93 \Rightarrow s'(t) = v(t) = -3.72t + 93$ $\Rightarrow s(t) = -1.86t^2 + 93t + C_2$; at $(0,0)$ we have $C_2 = 0 \Rightarrow s(t) = -1.86t^2 + 93t$. Then $v(t) = 0 \Rightarrow t = \frac{93}{3.72} = 25$ so the maximum height of the rock is $s(25) = 1162.5$ m.

29. (a) $\frac{d^2s}{dt^2} = 15t^{1/2} - 3t^{-1/2} \Rightarrow \frac{ds}{dt} = 10t^{3/2} - 6t^{1/2} + C$; $\frac{ds}{dt}(1) = 4 \Rightarrow 4 = 10(1)^{3/2} - 6(1)^{1/2} + C$ $\Rightarrow C = 0 \Rightarrow v = 10t^{3/2} - 6t^{1/2}$

(b) $v = \frac{ds}{dt} = 10t^{3/2} - 6t^{1/2} \Rightarrow s = 4t^{5/2} - 4t^{3/2} + C$; $s(1) = 0 \Rightarrow 0 = 4(1)^{5/2} - 4(1)^{3/2} + C$ $\Rightarrow C = 0 \Rightarrow s = 4t^{5/2} - 4t^{3/2}$

30. (a) $\frac{ds}{dt} = 9.8t - 3 \Rightarrow s = 4.9t^2 - 3t + C$; i) at $s = 5$ and $t = 0$ we have $C = 5 \Rightarrow s = 4.9t^2 - 3t + 5$; displacement $= s(3) - s(1) = [(4.9)(9) - 9 + 5] - (4.9 - 3 + 5) = 33.2$ units; ii) at $s = -2$ and $t = 0$ we have $C = -2 \Rightarrow s = 4.9t^2 - 3t - 2$; displacement $= s(3) - s(1) = ((4.9)(9) - 9 - 2) - (4.9 - 3 - 2) = 33.2$ units; iii) at $s = s_0$ and $t = 0$ we have $C = s_0 \Rightarrow s = 4.9t^2 - 3t + s_0$; displacement $= s(3) - s(1)$ $= ((4.9)(9) - 9 + s_0) - (4.9 - 3 + s_0) = 33.2$ units

(b) True. Given an antiderivative $f(t)$ of the velocity function, we know that the body's position function is $s = f(t) + C$ for some constant C. Therefore, the displacement from $t = a$ to $t = b$ is $(f(b) + C) - (f(a) + C)$ $= f(b) - f(a)$. Thus we can find the displacement from any antiderivative f as the numerical difference $f(b) - f(a)$ without knowing the exact values of C and s.

31. If $T(t)$ is the temperature of the thermometer at time t, then $T(0) = -19°$ C and $T(14) = 100°$ C. From the Mean Value Theorem there exists a $0 < t_0 < 14$ such that $\frac{T(14) - T(0)}{14 - 0} = 8.5°$ C/sec $= T'(t_0)$, the rate at which the temperature was changing at $t = t_0$ as measured by the rising mercury on the thermometer.

32. Because the trucker's average speed was 79.5 mph, and by the Mean Value Theorem, the trucker must have been going that speed at least once during the trip.

33. Because its average speed was approximately 7.667 knots, and by the Mean Value Theorem, it must have been going that speed at least once during the trip.

34. The runner's average speed for the marathon was approximately 11.909 mph. Therefore, by the Mean Value Theorem, the runner must have been going that speed at least once during the marathon. Since the initial speed and final speed are both 0 mph and the runner's speed is continuous, by the Intermediate Value Theorem, the runner's speed must have been 11 mph at least twice.

35. The conclusion of the Mean Value Theorem yields $\frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{c^2} \Rightarrow c^2\left(\frac{a-b}{ab}\right) = a-b \Rightarrow c = \sqrt{ab}$.

36. The conclusion of the Mean Value Theorem yields $\frac{b^2-a^2}{b-a} = 2c \Rightarrow c = \frac{a+b}{2}$.

37. $f'(x) = [\cos x \sin(x+2) + \sin x \cos(x+2)] - 2\sin(x+1)\cos(x+1) = \sin(x+x+2) - \sin 2(x+1)$

$= \sin(2x+2) - \sin(2x+2) = 0$. Therefore, the function has the constant value $f(0) = -\sin^2 1 \approx -0.7081$ which explains why the graph is a horizontal line.

38. Example CAS commands:

Maple:
```
(x + 2)*(x + 1)*x*(x - 1)*(x - 2);
expand(%);
f:=unapply(%,x);
plot({f(x),diff(f(x),x)},x=-2..2);
```

Mathematica:
```
(x + 2) (x + 1)x(x - 1) (x - 2)
Expand[%]
f[x_] = %
Plot[ {f[x],f'[x]}, {x,-2,2} ]
```

39. $f(x)$ must be zero at least once between a and b by the Intermediate Value Theorem. Now suppose that $f(x)$ is zero twice between a and b. Then by the Mean Value Theorem, $f'(x)$ would have to be zero at least once between the two zeros of $f(x)$, but this can't be true since we are given that $f'(x) \neq 0$ on this interval. Therefore, $f(x)$ is zero once and only once between a and b.

40. Consider the function $k(x) = f(x) - g(x)$. $k(x)$ is continuous and differentiable on $[a,b]$, and since $k(a) = f(a) - g(a) = 0$ and $k(b) = f(b) - g(b) = 0$, by the Mean Value Theorem, there must be a point c in (a,b) where $k'(c) = 0$. But since $k'(c) = f'(c) - g'(c)$, this means that $f'(c) = g'(c)$, and c is a point where the graphs of f and g have parallel or identical tangent lines.

41. Yes. By Corollary 2 we have $f(x) = g(x) + C$ since $f'(x) = g'(x)$. If the graphs start at the same point $x = a$, then $f(a) = g(a) \Rightarrow C = 0 \Rightarrow f(x) = g(x)$.

42. Let $f(x) = \sin x$ for $a \le x \le b$. From the Mean Value Theorem there exists a c between a and b such that $\frac{\sin b - \sin a}{b-a} = \cos c \Rightarrow -1 \le \frac{\sin b - \sin a}{b-a} \le 1 \Rightarrow \left|\frac{\sin b - \sin a}{b-a}\right| \le 1 \Rightarrow |\sin b - \sin a| \le |b-a|$.

43. By the Mean Value Theorem, $\frac{f(b)-f(a)}{b-a} = f'(c)$ for some point c between a and b. Since $b - a > 0$ and $f(b) < f(a)$, we have $f(b) - f(a) < 0 \Rightarrow f'(c) < 0$.

44. The condition is that f' should be continuous over $[a,b]$. The Mean Value Theorem then guarantees the existence of a point c in (a,b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$. If f' is continuous, then it has a minimum and

maximum value on $[a,b]$, and $\min f' \le f'(c) \le \max f'$, as required.

45. $f'(x) = \left(1 + x^4 \cos x\right)^{-1} \Rightarrow f''(x) = -\left(1 + x^4 \cos x\right)^{-2}\left(4x^3 \cos x - x^4 \sin x\right)$
$= -x^3\left(1 + x^4 \cos x\right)^{-2}(4 \cos x - x \sin x) < 0$ for $0 \le x \le 0.1 \Rightarrow f'(x)$ is decreasing when $0 \le x \le 0.1$
$\Rightarrow \min f' \approx 0.9999$ and $\max f' = 1$. Now we have $0.9999 \le \frac{f(0.1) - 1}{0.1} \le 1 \Rightarrow 0.09999 \le f(0.1) - 1 \le 0.1$
$\Rightarrow 1.09999 \le f(0.1) \le 1.1$.

46. $f'(x) = \left(1 - x^4\right)^{-1} \Rightarrow f''(x) = -\left(1 - x^4\right)^{-2}\left(-4x^3\right) = \frac{4x^3}{\left(1 - x^4\right)^2} > 0$ for $0 < x \le 0.1 \Rightarrow f'(x)$ is increasing when
$0 \le x \le 0.1 \Rightarrow \min f' = 1$ and $\max f' = 1.0001$. Now we have $1 \le \frac{f(0.1) - 2}{0.1} \le 1.0001$
$\Rightarrow 0.1 \le f(0.1) - 2 \le 0.10001 \Rightarrow 2.1 \le f(0.1) \le 2.10001$.

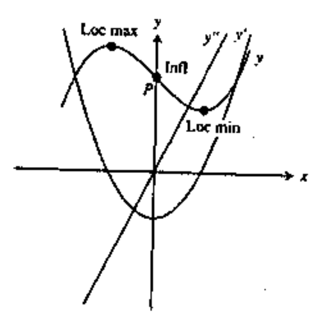
47-50. Example CAS commands

Maple:
```
with(plots): with(DEtools):
a:=0;b:=1;
eq:= D(y) (x)=x*sqrt(1-x);
sol:= dsolve({eq},y(x));
tograph:={seq(subs(_C1=i,sol),i={-2,-1,-,1,2})};
plot1:= implicitplot(tograph,x=a..b,y=-6..6):
display({plot1});
partsol:=dsolve({eq,y(1/2)=1},y(x));
implicitplot(partsol,x=a..b,y=-6..6,scaling=CONSTRAINED);
```

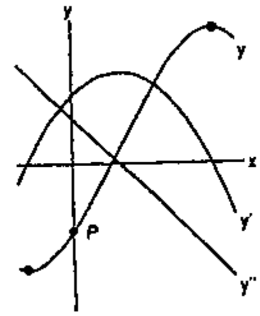
Mathematica:
```
a=0;b=1;
eq=D[y[x],x] = x*Sqrt[1-x]
sol=Flatten[DSolve[eq,y[x],x]]
cvals={-2,-1,1,2};
tograph=Table[y[x] /. (sol /. C[1] → cvals[[i]]), {i,1,4}]
Plot[Evaluate[tograph],{x,a,b}];
partsol = DSolve[{eq,y[1/2]=1},y[x],x]//Flatten
Plot[y[x] /. partsol,{x,a,b}]
```

3.3 THE SHAPE OF A GRAPH

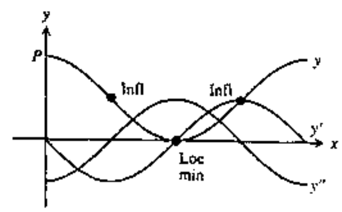
1. The graph of $y = f''(x) \Rightarrow$ the graph of $y = f(x)$ is concave up on $(0, \infty)$, concave down on $(-\infty, 0) \Rightarrow$ a point of inflection at $x = 0$; the graph of $y = f'(x) \Rightarrow y' = +++ \mid --- \mid +++ \Rightarrow$ the graph $y = f(x)$ has both a local maximum and a local minimum



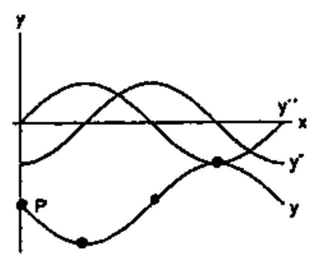
2. The graph of $y = f''(x) \Rightarrow y'' = +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has a point of inflection, the graph of $y = f'(x)$ $\Rightarrow y' = --- \mid +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has both a local maximum and a local minimum



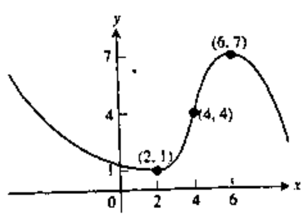
3. The graph of $y = f''(x) \Rightarrow y'' = --- \mid +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has two points of inflection, the graph of $y = f'(x) \Rightarrow y' = --- \mid +++ \Rightarrow$ the graph of $y = f(x)$ has a local minimum



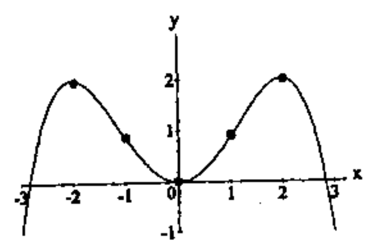
4. The graph of $y = f''(x) \Rightarrow y'' = +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has a point of inflection; the graph of $y = f'(x)$ $\Rightarrow y' = --- \mid +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has both a local maximum and a local minimum



5.



6.



7. (a) Zero: $x = \pm 1$;
   positive: $(-\infty, -1)$ and $(1, \infty)$;
   negative: $(-1, 1)$
   (b) Zero: $x = 0$;
   positive: $(0, \infty)$;
   negative: $(-\infty, 0)$

8. (a) Zero: $x \approx 0, \ \pm 1.25$;
   positive: $(-1.25, 0)$ and $(1.25, \infty)$;
   negative: $(-\infty, -1.25)$ and $(0, 1.25)$
   (b) Zero: $x \approx \pm 0.7$;
   positive: $(-\infty, -0.7)$ and $(0.7, \infty)$;
   negative: $(-0.7, 0.7)$

9. (a) $(-\infty, -2]$ and $[0, 2]$
   (b) $[-2, 0]$ and $[2, \infty)$
   (c) Local maxima: $x = -2$ and $x = 2$;
   local minimum: $x = 0$

10. (a) $[-2, 2]$
   (b) $(-\infty, -2]$ and $[2, \infty)$
   (c) Local maximum: $x = 2$;
   local minimum: $x = -2$

11. (a) $[0,1]$, $[3,4]$, and $[5.5,6]$
    (b) $[1,3]$ and $[4,5.5]$
    (c) Local maxima: $x = 1$, $x = 4$ (if f is continuous at $x = 4$), and $x = 6$;
    local minima: $x = 0$, $x = 3$, and $x = 5.5$

12. If f is continuous on the interval $[0,3]$;
    (a) $[0,3]$
    (b) Nowhere
    (c) Local maximum: $x = 3$;
    local minimum: $x = 0$

13. (a) $f'(x) = (x-1)(x+2) \Rightarrow$ critical points at -2 and 1
    (b) $f' = +++\underset{-2}{|}---\underset{1}{|}+++ \Rightarrow$ increasing on $(-\infty,-2]$ and $[1,\infty)$, decreasing on $[-2,1]$
    (c) Local maximum at $x = -2$ and a local minimum at $x = 1$

14. (a) $f'(x) = (x-1)^2(x+2) \Rightarrow$ critical points at -2 and 1
    (b) $f' = ---\underset{-2}{|}+++\underset{1}{|}+++ \Rightarrow$ increasing on $[-2,1]$ and $[1,\infty)$, decreasing on $(-\infty,-2]$
    (c) No local maximum and a local minimum at $x = -2$

15. (a) $f'(x) = (x-1)(x+2)(x-3) \Rightarrow$ critical points at -2, 1 and 3
    (b) $f' = ---\underset{-2}{|}+++\underset{1}{|}---\underset{3}{|}+++ \Rightarrow$ increasing on $[-2,1]$ and $[3,\infty)$, decreasing on $(-\infty,-2]$ and $[1,3]$
    (c) Local maximum at $x = 1$, local minima at $x = -2$ and $x = 3$

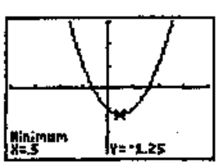
16. (a) $f'(x) = x^{-1/3}(x+2) \Rightarrow$ critical points at -2 and 0
    (b) $f' = +++\underset{-2}{|}---\underset{0}{)(}+++ \Rightarrow$ increasing on $(-\infty,-2]$ and $[0,\infty)$, decreasing on $[-2,0]$
    (c) Local maximum at $x = -2$, local minimum at $x = 0$

17. $y' = 2x - 1$

Intervals	$x < \frac{1}{2}$	$x > \frac{1}{2}$
Sign of y'	$-$	$+$
Behavior of y	Decreasing	Increasing

$y'' = 2$ (always positive: concave up)

Graphical support:



$[-4, 4]$ by $[-3, 3]$

(a) $\left[\frac{1}{2}, \infty\right)$

(b) $\left(-\infty, \frac{1}{2}\right]$

(c) $(-\infty, \infty)$

(d) Nowhere

(e) Local (and absolute) minimum at $\left(\frac{1}{2}, -\frac{5}{4}\right)$

(f) None

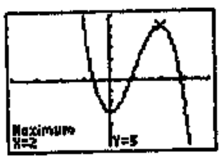
18. $y' = -6x^2 + 12x = -6x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y'	$-$	$+$	$-$
Behavior of y	Decreasing	Increasing	Decreasing

$y'' = -12x + 12 = -12(x-1)$

Intervals	$x < 1$	$x > 1$
Sign of y''	$+$	$-$
Behavior of y	Concave up	Concave down

Graphical support:



[−4, 4] by [−6, 6]

(a) $[0, 2]$

(b) $(-\infty, 0]$ and $[2, \infty)$

(c) $(-\infty, 1)$

(d) $(1, \infty)$

(e) Local maximum: $(2, 5)$;
local minimum: $(0, -3)$

(f) At $(1, 1)$

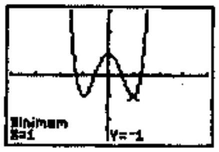
19. $y' = 8x^3 - 8x = 8x(x-1)(x+1)$

Intervals	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	$-$	$+$	$-$	$+$
Behavior of y	Decreasing	Increasing	Decreasing	Increasing

$y'' = 24x^2 - 8 = 8(\sqrt{3}x - 1)(\sqrt{3}x + 1)$

Intervals	$x < -\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}} < x$
Sign of y''	$+$	$-$	$+$
Behavior of y	Concave up	Concave down	Concave up

Graphical support:



$[-4, 4]$ by $[-3, 3]$

(a) $[-1,0]$ and $[1,\infty)$

(b) $(-\infty,1]$ and $[0,1]$

(c) $\left(-\infty,-\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}},\infty\right)$

(d) $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

(e) Local maximum: $(0,1)$;
local (and absolute) minima: $(-1,-1)$ and $(1,-1)$

(f) $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{9}\right)$

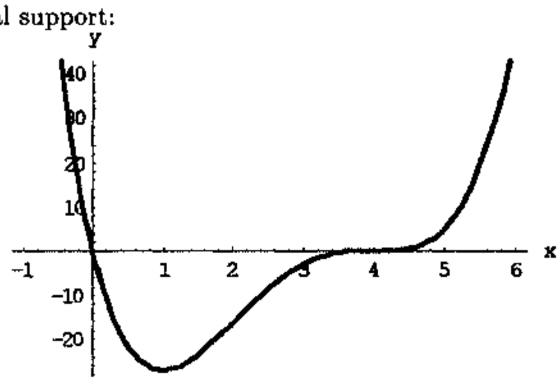
20. $y' = 4x^3 - 36x^2 + 96x - 64 = 4(x-1)(x-4)^2$

Intervals	$x < 1$	$1 < x < 4$	$x > 4$
Sign of y'	$-$	$+$	$+$
Behavior of y	Decreasing	Increasing	Increasing

$y'' = 12x^2 - 72x + 96 = 12(x-2)(x-4)$

Intervals	$x < 2$	$2 < x < 4$	$x > 4$
Sign of y''	$+$	$-$	$+$
Behavior of y	Concave up	Concave down	Concave up

Graphical support:



(a) $(1,\infty)$

(b) $(-\infty,1)$

(c) $(-\infty,2)$ and $(4,\infty)$

(d) $(2,4)$

(e) Local (and absolute) minimum at $(1,-27)$

(f) Inflection points at $(2,-16)$ and $(4,0)$

21. $y' = x\dfrac{1}{2\sqrt{8-x^2}}(-2x) + \left(\sqrt{8-x^2}\right)(1) = \dfrac{8-2x^2}{\sqrt{8-x^2}}$

Intervals	$-\sqrt{8} < x < -2$	$-2 < x < 2$	$2 < x < \sqrt{8}$
Sign of y'	$-$	$+$	$-$
Behavior of y	Decreasing	Increasing	Decreasing

$$y'' = \frac{\left(\sqrt{8-x^2}\right)(-4x) - (8-2x^2)\dfrac{1}{2\sqrt{8-x^2}}(-2x)}{\left(\sqrt{8-x^2}\right)^2} = \frac{2x^3-24x}{(8-x^2)^{3/2}} = \frac{2x(x^2-12)}{(8-x^2)^{3/2}}$$

Intervals	$-\sqrt{8} < x < 0$	$0 < x < \sqrt{8}$
Sign of y'	$+$	$-$
Behavior of y	Concave up	Concave down

Graphical support:

$[-3.02, 3.02]$ by $[-6.5, 6.5]$

(a) $[-2, 2]$

(b) $\left[-\sqrt{8}, -2\right]$ and $\left[2, \sqrt{8}\right]$

(c) $\left(-\sqrt{8}, 0\right)$

(d) $\left(0, \sqrt{8}\right)$

(e) Local maxima: $\left(-\sqrt{8}, 0\right)$ and $(2, 4)$;
local minima: $(-2, -4)$ and $\left(\sqrt{8}, 0\right)$
Note that the local extrema at $x = \pm 2$ are also absolute extrema

(f) $(0, 0)$

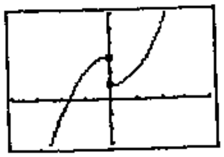
22. $y' = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases}$

Intervals	$x < 0$	$x > 0$
Sign of y'	$+$	$+$
Behavior of y	Increasing	Increasing

$y'' = \begin{cases} -2, & x < 0 \\ 2, & x > 0 \end{cases}$

Intervals	$x < 0$	$x > 0$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

Graphical support:



$[-4, 4]$ by $[-3, 6]$

(a) $(-\infty, \infty)$

(b) None

(c) $(0, \infty)$

(d) $(-\infty, 0)$

(e) Local minimum: $(0, 1)$

(f) Note that $(0, 1)$ is not an inflection point because the graph has no tangent line at this point. There are no inflection points.

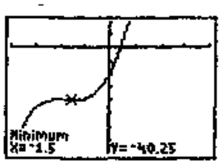
23. $y' = 12x^2 + 42x + 36 = 6(x+2)(2x+3)$

Intervals	$x < -2$	$-2 < x < -\frac{3}{2}$	$-\frac{3}{2} < x$
Sign of y'	$+$	$-$	$+$
Behavior of y	Increasing	Decreasing	Increasing

$y'' = 24x + 42 = 6(4x + 7)$

Intervals	$x < -\frac{7}{4}$	$-\frac{7}{4} < x$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

Graphical support:



$[-4, 4]$ by $[-80, 20]$

(a) $(-\infty, -2]$ and $\left[-\frac{3}{2}, \infty\right)$

(b) $\left[-2, -\frac{3}{2}\right]$

(c) $\left(-\frac{7}{4}, \infty\right)$

(d) $\left(-\infty, -\frac{7}{4}\right)$

(e) Local maximum: $(-2, -40)$; local minimum: $\left(-\frac{3}{2}, -\frac{161}{4}\right)$

(f) $\left(-\frac{7}{4}, -\frac{321}{8}\right)$

24. $y' = -4x^3 + 12x^2 - 4$

Using grapher techniques, the zeros of y' are $x \approx -0.53$, $x \approx 0.65$, and $x \approx 2.88$.

Intervals	$x < -0.53$	$-0.53 < x < 0.65$	$0.65 < x < 2.88$	$2.88 < x$
Sign of y'	+	−	+	−
Behavior of y	Increasing	Decreasing	Increasing	Decreasing

$y'' = -12x^2 + 24x = -12x(x - 2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y''	−	+	−
Behavior of y	Concave down	Concave up	Concave down

Graphical support:

[−2, 4] by [−20, 20]

(a) $(-\infty, -0.53]$ and $[0.65, 2.88]$

(b) $[-0.53, 0.65]$ and $[2.88, \infty)$

(c) $(0, 2)$

(d) $(-\infty, 0)$ and $(2, \infty)$

(e) Local maxima: $(-0.53, 2.45)$ and $(2.88, 16.23)$; local minimum: $(0.65, -0.68)$
Note that the local maximum at $x \approx 2.88$ is also an absolute maximum.

(f) $(0, 1)$ and $(2, 9)$

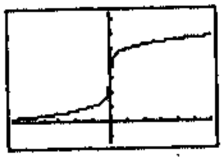
25. $y' = \frac{2}{5}x^{-4/5}$

Intervals	$x < 0$	$0 < x$
Sign of y'	+	+
Behavior of y	Increasing	Increasing

$y'' = -\frac{8}{25}x^{-9/5}$

Intervals	$x < 0$	$0 < x$
Sign of y''	+	−
Behavior of y	Concave up	Concave down

Graphical support:



[−6, 6] by [−1.5, 7.5]

(a) $(-\infty,\infty)$
(b) None
(c) $(-\infty,0)$
(d) $(0,\infty)$
(e) None
(f) $(0,3)$

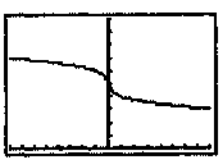
26. $y' = -\frac{1}{3}x^{-2/3}$

Intervals	$x < 0$	$0 < x$
Sign of y'	−	−
Behavior of y	Decreasing	Decreasing

$y'' = \frac{2}{9}x^{-5/3}$

Intervals	$x < 0$	$0 < x$
Sign of y''	−	+
Behavior of y	Concave down	Concave up

Graphical support:



[−8, 8] by [0, 10]

(a) None
(b) $(-\infty,\infty)$
(c) $(0,\infty)$
(d) $(-\infty,0)$
(e) None
(f) $(0,5)$

27. $y = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$

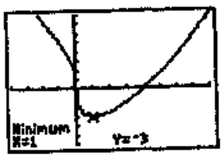
$y' = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4x-4}{3x^{2/3}}$

Intervals	$x<0$	$0<x<1$	$1<x$
Sign of y'	$-$	$-$	$+$
Behavior of y	Decreasing	Decreasing	Increasing

$y''=\frac{4}{9}x^{-2/3}+\frac{8}{9}x^{-5/3}=\frac{4x+8}{9x^{5/3}}$

Intervals	$x<-2$	$-2<x<0$	$0<x$
Sign of y''	$+$	$-$	$+$
Behavior of y	Concave up	Concave down	Concave up

Graphical support:



$[-4, 8]$ by $[-6, 8]$

(a) $[1,\infty)$

(b) $(-\infty,1]$

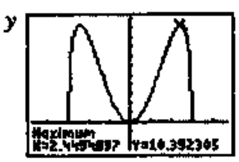
(c) $(-\infty,-2)$ and $(0,\infty)$

(d) $(-2,0)$

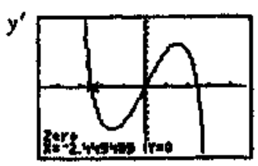
(e) Local minimum: $(1,-3)$

(f) $\left(-2,6\sqrt[3]{2}\right)\approx(-2,7.56)$ and $(0,0)$

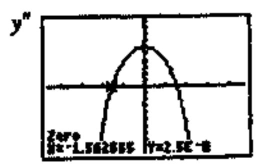
28. This problem can be solved using either graphical or analytic methods. The following is a graphical solution.



$[-4.7, 4.7]$ by $[-3, 11]$



$[-4.7, 4.7]$ by $[-10, 10]$



$[-4.7, 4.7]$ by $[-10, 10]$

An analytic solution follows.

$$y'=x^2\frac{1}{2\sqrt{9-x^2}}(-2x)+\sqrt{9-x^2}\,(2x)=\frac{-3x^3+18x}{\sqrt{9-x^2}}=\frac{-3x(x^2-6)}{\sqrt{9-x^2}}$$

Intervals	$-3<x<-\sqrt{6}$	$-\sqrt{6}<x<0$	$0<x<\sqrt{6}$	$\sqrt{6}<x<3$
Sign of y'	$+$	$-$	$+$	$-$
Behavior of y	Increasing	Decreasing	Increasing	Decreasing

$$y''=\frac{\left(\sqrt{9-x^2}\right)\left(-9x^2+18\right)-\left(-3x^3+18x\right)\frac{1}{2\sqrt{9-x^2}}(-2x)}{\left(\sqrt{9-x^2}\right)^2}=\frac{(9-x^2)(-9x^2+18)+(-3x^3+18x)(x)}{(9-x^2)^{3/2}}$$

$$=\frac{6x^4-81x^2+162}{(9-x^2)^{3/2}}$$

Find the zeros of y'':

$$\frac{3(2x^4 - 27x^2 + 54)}{(9 - x^2)^{3/2}} = 0$$

$$2x^4 - 27x^2 + 54 = 0$$

$$x^2 = \frac{27 \pm \sqrt{27^2 - 4(2)(54)}}{2(2)} = \frac{27 \pm 3\sqrt{33}}{4}$$

$$x = \pm\sqrt{\frac{27 - 3\sqrt{33}}{4}} \approx \pm 1.56$$

Note that we do not use $x = \pm\sqrt{\frac{27 + 3\sqrt{33}}{4}} \approx \pm 3.33$, because these values are outside of the domain.

Intervals	$-3 < x < -1.56$	$-1.56 < x < 1.56$	$1.56 < x < 3$
Sign of y''	$-$	$+$	$-$
Behavior of y	Concave down	Concave up	Concave down

(a) $[-3, -\sqrt{6}]$ and $[0, \sqrt{6}]$ or, $\approx [-3, -2.45]$ and $[0, 2.45]$

(b) $[-\sqrt{6}, 0]$ and $[\sqrt{6}, 3]$ or, $\approx [-2.45, 0]$ and $[2.45, 3]$

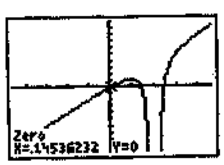
(c) Approximately $(-1.56, 1.56)$

(d) Approximately $(-3, -1.56)$ and $(1.56, 3)$

(e) Local maxima: $(\pm\sqrt{6}, 6\sqrt{3}) \approx (\pm 2.45, 10.39)$;
local minima: $(0, 0)$ and $(\pm 3, 0)$

(f) $\approx (\pm 1.56, 6.25)$

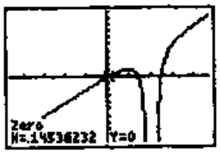
29. We use a combination of analytic and grapher techniques to solve this problem. Depending on the viewing window chosen, graphs obtained using the nderiv function on a TI-92 calculator may exhibit strange behavior near $x = 2$ because, for example, $\text{nderiv}(y,x) \mid x = 2 \approx 1{,}000{,}000$ while y' is actually undefined at $x = 2$. The graph of $y = \frac{x^3 - 2x^2 + x - 1}{x - 2}$ is shown below:



$[-4.7, 4.7]$ by $[-5, 15]$

$$y' = \frac{(x-2)(3x^2 - 4x + 1) - (x^3 - 2x^2 + x - 1)(1)}{(x-2)^2} = \frac{2x^3 - 8x^2 + 8x - 1}{(x-2)^2}$$

The graph of y' is shown below:



$[-4.7, 4.7]$ by $[-10, 10]$

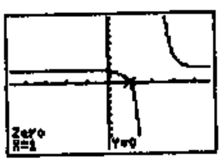
The zeros of y' are $x \approx 0.15$, $x \approx 1.40$, and $x \approx 2.45$.

Intervals	$x < 0.15$	$0.15 < x < 1.40$	$1.40 < x < 2$	$2 < x < 2.45$	$2.45 < x$
Sign of y'	$-$	$+$	$-$	$-$	$+$
Behavior of y	Decreasing	Increasing	Decreasing	Decreasing	Increasing

$$y'' = \frac{(x-2)^2(6x^2-16x+8)-(2x^3-8x^2+8x-1)(x-2)}{(x-2)^4}$$

$$= \frac{(x-2)(6x^2-16x+8)-2(2x^3-8x^2-1)}{(x-2)^3} = \frac{2x^3-12x^2+24x-14}{(x-2)^3} = \frac{2(x-1)(x^2-5x+7)}{(x-2)^3}$$

The graph of y'' is shown below.



$[-4.7, 4.7]$ by $[-10, 10]$

Note that the discriminant of x^2-5x+7 is $(-5)^2-4(1)(7) = -3$, so the only solution of $y'' = 0$ is $x = 1$.

Intervals	$x < 1$	$1 < x < 2$	$2 < x$
Sign of y''	$+$	$-$	$+$
Behavior of y	Concave up	Concave down	Concave up

(a) Approximately $[0.15, 1.40]$ and $[2.45, \infty)$

(b) Approximately $(-\infty, 0.15]$, $[1.40, 2)$, and $(2, 2.45]$

(c) $(-\infty, 1)$ and $(2, \infty)$

(d) $(1, 2)$

(e) Local maximum: $\approx (1.40, 1.29)$; local minima: $\approx (0.15, 0.48)$ and $(2.45, 9.22)$

(f) $(1, 1)$

30. $y = x^{3/4}(5-x) = 5x^{3/4} - x^{7/4}$

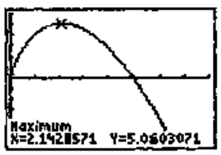
$$y' = \frac{15}{4}x^{-1/4} - \frac{7}{4}x^{3/4} = \frac{15-7x}{4x^{1/4}}$$

Intervals	$0 < x < \frac{15}{7}$	$\frac{15}{7} < x$
Sign of y'	$+$	$-$
Behavior of y	Increasing	Decreasing

$$y'' = -\frac{15}{16}x^{-5/4} - \frac{21}{16}x^{-1/4} = \frac{-3(7x+5)}{16x^{5/4}}$$

Since $y'' < 0$ for all $x > 0$, the graph of y is concave down for $x > 0$.

Graphical support:



[0, 8] by [−6, 6]

(a) $\left[0, \frac{15}{7}\right)$

(b) $\left[\frac{15}{7}, \infty\right)$

(c) None

(d) $(0, \infty)$

(e) Local (and absolute) maximum:
$\left(\frac{15}{7}, \left(\frac{15}{7}\right)^{3/4} \cdot \frac{20}{7}\right) \approx \left(\frac{15}{7}, 5.06\right)$;
local minimum: $(0, 0)$

(f) None

31. $y = x^{1/4}(x+3) = x^{5/4} + 3x^{1/4}$

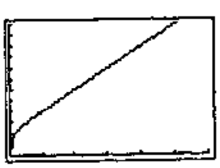
$$y' = \frac{5}{4}x^{1/4} + \frac{3}{4}x^{-3/4} = \frac{5x+3}{4x^{3/4}}$$

Since $y' > 0$ for all $x > 0$, y is always increasing on its domain $x \geq 0$.

$$y'' = \frac{5}{16}x^{-3/4} - \frac{9}{16}x^{-7/4} = \frac{5x-9}{16x^{7/4}}$$

Intervals	$0 < x < \frac{9}{5}$	$\frac{9}{5} < x$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

Graphical support:


[0, 6] by [0, 12]

(a) $[0, \infty)$

(b) None

(c) $\left(\frac{9}{5}, \infty\right)$

(d) $\left(0, \frac{9}{5}\right)$

(e) Local (and absolute) minimum: $(0, 0)$

(f) $\left(\frac{9}{5}, \frac{24}{5} \cdot \sqrt[4]{\frac{9}{5}}\right) \approx (1.8, 5.56)$

32. $y' = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{-x^2+1}{(x^2+1)^2}$

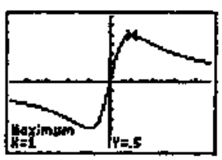
Intervals	$x < -1$	$-1 < x < 1$	$1 < x$
Sign of y'	$-$	$+$	$-$
Behavior of y	Decreasing	Increasing	Decreasing

$$y'' = \frac{(x^2+1)^2(-2x) - (-x^2+1)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{(x^2+1)(-2x) - 4x(-x^2+1)}{(x^3+1)^3}$$

$$= \frac{2x^3 - 6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

Intervals	$x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$\sqrt{3} < x$
Sign of y''	$-$	$+$	$-$	$+$
Behavior of y	Concave down	Concave up	Concave down	Concave up

Graphical support:



[−4.7, 4.7] by [−0.7, 0.7]

(a) $[-1, 1]$

(b) $(-\infty, -1]$ and $[1, \infty)$

(c) $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$

(d) $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$

(e) Local maximum: $\left(1, \frac{1}{2}\right)$;

local minimum: $\left(-1, -\frac{1}{2}\right)$

(f) $(0, 0)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$, and $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$

33. $y' = (x-1)^2(x-2)$

Intervals	$x < 1$	$1 < x < 2$	$2 < x$
Sign of y'	$-$	$-$	$+$
Behavior of y	Decreasing	Decreasing	Increasing

$$y'' = (x-1)^2(1) + (x-2)(2)(x-1) = (x-1)[(x-1) + 2(x-2)] = (x-1)(3x-5)$$

Intervals	$x < 1$	$1 < x < \frac{5}{3}$	$\frac{5}{3} < x$
Sign of y''	+	−	+
Behavior of y	Concave up	Concave down	Concave up

(a) There are no local maxima.

(b) There is a local (and absolute) minimum at $x = 2$.

(c) There are points of inflection at $x = 1$ and at $x = \frac{5}{3}$.

34. $y' = (x-1)^2(x-2)(x-4)$

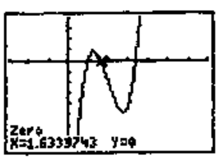
Intervals	$x < 1$	$1 < x < 2$	$2 < x < 4$	$4 < x$
Sign of y'	+	+	−	+
Behavior of y	Increasing	Increasing	Decreasing	Increasing

$$y'' = \frac{d}{dx}[(x-1)^2(x^2-6x+8)] = (x-1)^2(2x-6) + (x^2-6x+8)(2)(x-1)$$
$$= (x-1)[(x-1)(2x-6) + 2(x^2-6x+8)] = (x-1)(4x^2-20x+22)$$
$$= 2(x-1)(2x^2-10x+11)$$

Note that the zeros of y'' are $x = 1$ and

$$x = \frac{10 \pm \sqrt{10^2 - 4(2)(11)}}{4} = \frac{10 \pm \sqrt{12}}{4} = \frac{5 \pm \sqrt{3}}{2} \approx 1.63 \text{ or } 3.37.$$

The zeros of y'' can also be found graphically, as shown.



[−3, 7] by [−8, 4]

Intervals	$x < 1$	$1 < x < 1.63$	$1.63 < x < 3.37$	$3.37 < x$
Sign of y''	−	+	−	+
Behavior of y	Concave down	Concave up	Concave down	Concave up

(a) Local maximum at $x = 2$

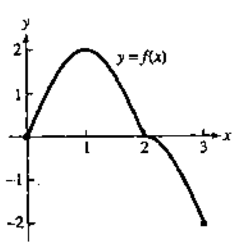
(b) Local minimum at $x = 4$

(c) Points of inflection at $x = 1$, at $x \approx 1.63$, and at $x \approx 3.37$.

35. (a) Absolute maximum at $(1, 2)$; absolute minimum at $(3, -2)$

(b) None

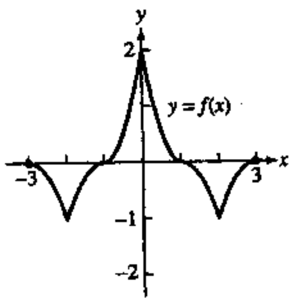
(c) One possible answer



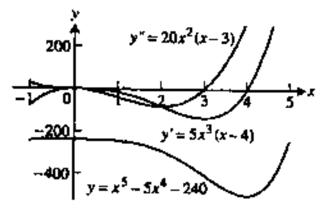
36. (a) Absolute maximum at $(0, 2)$; absolute minimum at $(2, -1)$ and $(-2, -1)$

(b) At $(1, 0)$ and $(-1, 0)$

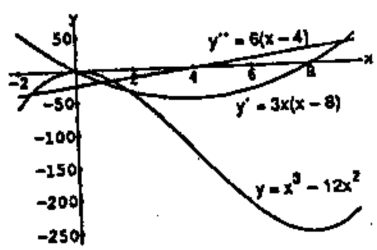
(c) One possible answer:



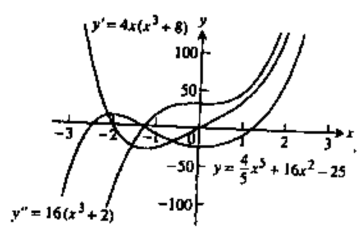
37. If $y = x^5 - 5x^4 - 240$, then $y' = 5x^3(x - 4)$ and $y'' = 20x^2(x - 3)$. The zeros of y' are extrema of y. The right-hand zero of y'' is a point of inflection for y. Inflection at $x = 3$, local maximum at $x = 0$, local minimum at $x = 4$.



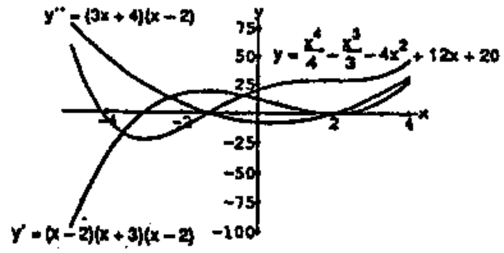
38. If $y = x^3 - 12x^2$, then $y' = 3x(x - 8)$ and $y'' = 6(x - 4)$. The zeros of y' and y'' are extrema and points of inflection, respectively.



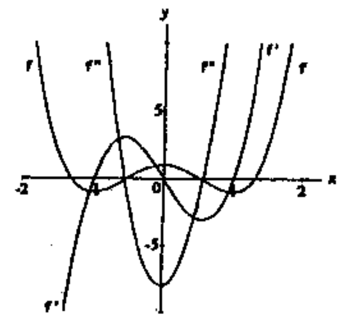
39. If $y = \frac{4}{5}x^5 + 16x^2 - 25$, then $y' = 4x\left(x^3 + 8\right)$ and $y'' = 16\left(x^3 + 2\right)$. The zeros of y' and y'' are extrema and points of inflection, respectively.



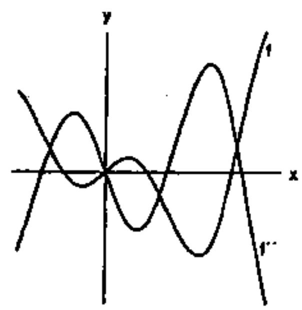
40. If $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$, then $y' = (x-2)^2(x+3)$ and $y'' = (3x+4)(x-2)$. The zeros of y' and y'' are extrema and points of inflection, respectively.



41. The graph of f falls where $f' < 0$, rises where $f' > 0$, and has horizontal tangents where $f' = 0$. It has local minima at points where f' changes from negative to positive and local maxima where f' changes from positive to negative. The graph of f is concave down where $f'' < 0$ and concave up where $f'' > 0$. It has points of inflection at values of x where f'' changes sign and a tangent line exists.



42. The graph f is concave down where $f'' < 0$, and concave up where $f'' > 0$. It has an inflection point each time f'' changes sign, provided a tangent line exists there.



43. (a) $v(t) = s'(t) = 2t - 4$ (b) $a(t) = v'(t) = 2$
(c) It begins at position 3 moving in a negative direction. It moves to position -1 when $t = 2$, and then changes direction, moving in a positive direction thereafter.

44. (a) $v(t) = s'(t) = -2 - 2t$ (b) $a(t) = v'(t) = -2$
(c) It begins at position 6 moving in the negative direction thereafter.

45. (a) $v(t) = s'(t) = 3t^2 - 3$ (b) $a(t) = v'(t) = 6t$
(c) It begins at position 3 moving in a negative direction. It moves to position 1 when $t = 1$, and then changes direction, moving in a positive direction thereafter.

46. (a) $v(t) = s'(t) = 6t - 6t^2$ (b) $a(t) = v'(t) = 6 - 12t$
(c) It begins at position 0. It starts moving in the positive direction until it reaches position 1 when $t = 1$, and then it changes direction. It moves in the negative direction thereafter.

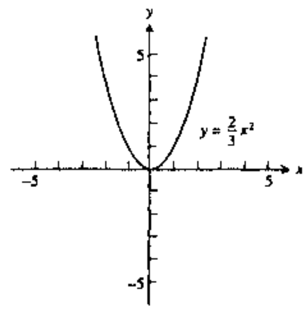
47. (a) The velocity is zero when the tangent line is horizontal, at approximately $t = 2.2$, $t = 6$, and $t = 9.8$.
(b) The acceleration is zero at the inflection points, approximately $t = 4$, $t = 8$, and $t = 12$.

48. (a) The velocity is zero when the tangent line is horizontal, at approximately $t = -0.2$, $t = 4$, and $t = 12$.
(b) The acceleration is zero at the inflection points, approximately $t = 1.5$, $t = 5.2$, $t = 8$, $t = 11$, and $t = 13$.

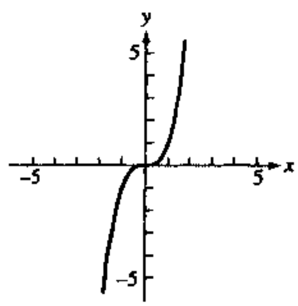
49. No. f must have a horizontal tangent at that point, but f could be increasing (or decreasing), and there would be no local extremum. For example, if $f(x) = x^3$, $f'(0) = 0$ but there is no local extremum at $x = 0$.

50. No. $f''(x)$ could still be positive (or negative) on both sides of $x = c$, in which case the concavity of the function would not change at $x = c$. For example, if $f(x) = x^4$, then $f''(0) = 0$, but f has no inflection point at $x = 0$.

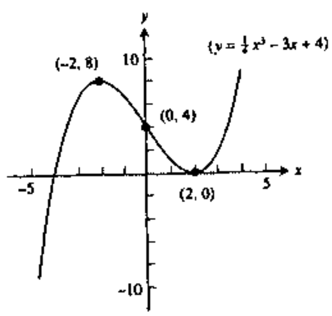
51. One possible answer:



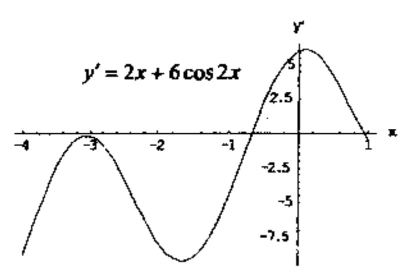
52. One possible answer:



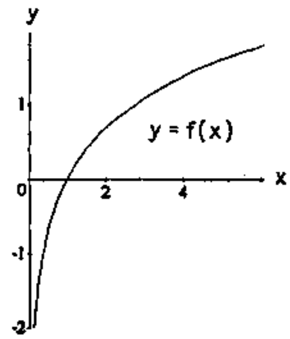
53. One possible answer:



54. No: $y = x^2 + 3\sin(2x) \Rightarrow y' = 2x + 6\cos(2x)$. The graph of y' does not touch the x-axis near $x = -3$ indicating that there is no horizontal tangent near $x = -3$.



55. The graph must be concave down for $x > 0$ because $f''(x) = -\frac{1}{x^2} < 0$.



56. The second derivative, being continuous and never zero, cannot change sign. Therefore the graph will always be concave up or concave down so it will have no inflection points.

57. A quadratic curve never has an inflection point. If $y = ax^2 + bx + c$ where $a \neq 0$, then $y' = 2ax + b$ and $y'' = 2a$. Since 2a is a constant, it is not possible for y'' to change signs.

58. A cubic curve always has exactly one inflection point. If $y = ax^3 + bx^2 + cx + d$ where $a \neq 0$, then $y' = 3ax^2 + 2bx + c$ and $y'' = 6ax + 2b$. Since $\frac{-b}{3a}$ is a solution of $y'' = 0$, we have that y'' changes its sign at $x = -\frac{b}{3a}$ and y' exists everywhere (so there is a tangent at $x = -\frac{b}{3a}$). Thus the curve has an inflection point at $x = -\frac{b}{3a}$. There are no other inflection points because y'' changes sign only at this zero.

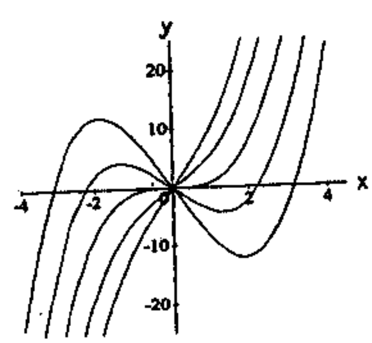
59. With $f(-2) = 11 > 0$ and $f(-1) = -1 < 0$ we conclude from the Intermediate Value Theorem that $f(x) = x^4 + 3x + 1$ has at least one zero between -2 and -1. Then $-2 < x < -1 \Rightarrow -8 < x^3 < -1$ $\Rightarrow -32 < 4x^3 < -4 \Rightarrow -29 < 4x^3 + 3 < -1 \Rightarrow f'(x) < 0$ for $-2 < x < -1 \Rightarrow f(x)$ is decreasing on $[-2, -1]$ $\Rightarrow f(x) = 0$ has exactly one solution in the interval $(-2, -1)$.

60. $g(t) = \sqrt{t} + \sqrt{t+1} - 4 \Rightarrow g'(t) = \frac{1}{2\sqrt{t}} + \frac{1}{2\sqrt{t+1}} > 0 \Rightarrow g(t)$ is increasing for t in $(0, \infty)$; $g(3) = \sqrt{3} - 2 < 0$ and $g(15) = \sqrt{15} > 0 \Rightarrow g(t)$ has exactly one zero in $(0, \infty)$.

61. $r(\theta) = \theta + \sin^2\left(\frac{\theta}{3}\right) - 8 \Rightarrow r'(\theta) = 1 + \frac{2}{3}\sin\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right) = 1 + \frac{1}{3}\sin\left(\frac{2\theta}{3}\right) > 0$ on $(-\infty, \infty) \Rightarrow r(\theta)$ is increasing on $(-\infty, \infty)$; $r(0) = -8$ and $r(8) = \sin^2\left(\frac{8}{3}\right) > 0 \Rightarrow r(\theta)$ has exactly one zero in $(-\infty, \infty)$.

62. $r(\theta) = \tan\theta - \cot\theta - \theta \Rightarrow r'(\theta) = \sec^2\theta + \csc^2\theta - 1 = \sec^2\theta + \cot^2\theta > 0$ on $\left(0, \frac{\pi}{2}\right) \Rightarrow r(\theta)$ is increasing on $\left(0, \frac{\pi}{2}\right)$; $r\left(\frac{\pi}{4}\right) = -\frac{\pi}{4} < 0$ and $r(1.57) \approx 1254.2 \Rightarrow r(\theta)$ has exactly one zero in $\left(0, \frac{\pi}{2}\right)$.

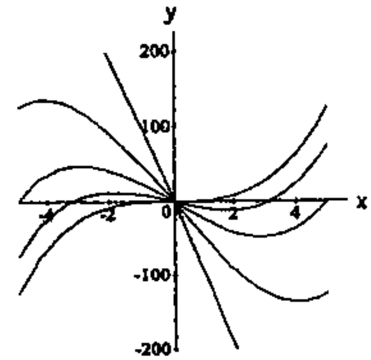
63 (a) It appears to control the number and magnitude of the local extrema. If $k < 0$, there is a local maximum to the left of the origin and a local minimum to the right. The larger the magnitude of k ($k < 0$), the greater the magnitude of the extrema. If $k > 0$, the graph has only positive slopes and lies entirely in the first and third



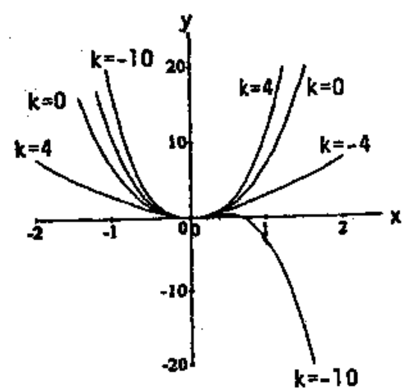
quadrants with no local extrema. The graph becomes increasingly steep and straight as $k \to \infty$.

(b) $f'(x) = 3x^2 + k \Rightarrow$ the discriminant $0^2 - 4(3)(k) = -12k$ is positive for $k < 0$, zero for $k = 0$, and negative for $k > 0$; f' has two zeros $x = \pm\sqrt{-\frac{k}{3}}$ when $k < 0$, one zero $x = 0$ when $k = 0$ and no real zeros when $k > 0$; the sign of k controls the number of local extrema.

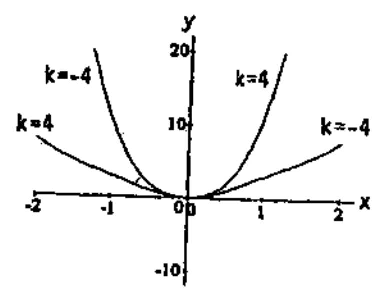
(c) As $k \to \infty$, $f'(x) \to \infty$ and the graph becomes increasingly steep and straight. As $k \to -\infty$, the crest of the graph (local maximum) in the second quadrant becomes increasingly high and the trough (local minimum) in the fourth quadrant becomes increasingly deep.



64. (a) It appears to control the concavity and the number of local extrema.



(b) $f(x) = x^4 + kx^3 + 6x^2 \Rightarrow f'(x) = 4x^3 + 3kx^2 + 12x$ $\Rightarrow f''(x) = 12x^2 + 6kx + 12 \Rightarrow$ the discriminant is $36k^2 - 4(12)(12) = 36(k+4)(k-4)$, so the sign line of the discriminant is $+++\underset{-4}{|}---\underset{4}{|}+++ \Rightarrow$ the discriminant is positive when $|k| > 4$, zero when $k = \pm 4$, and negative when $|k| < 4$; $f''(x) = 0$ has two zeros when $|k| > 4$, one zero when $k = \pm 4$, and no real zeros for $|k| < 4$; the value of k controls the number of possible points of inflection.



65. (a) $f'(x) = \dfrac{(1 + ae^{-bx})(0) - (c)(-abe^{-bx})}{(1 + ae^{-bx})^2} = \dfrac{abce^{-bx}}{(1 + ae^{-bx})^2} = \dfrac{abce^{bx}}{(e^{bx} + a)^2}$

so the sign of $f'(x)$ is the same as the sign of abc.

(b) $f''(x) = \dfrac{(e^{bx}+a)^2(ab^2ce^{bx}) - (abce^{bx})2(e^{bx}+a)(be^{bx})}{(e^{bx}+a)^4} = \dfrac{(e^{bx}+a)(ab^2ce^{bx}) - (abce^{bx})(2be^{bx})}{(e^{bx}+a)^3 2}$

$= -\dfrac{ab^2ce^{bx}(e^{bx}-a)}{(e^{bx}+a)^3}$

Since $a > 0$, this changes sign when $x = \frac{\ln a}{b}$ due to the $e^{bx} - a$ factor in the numerator, and f(x) has a point of inflection at that location.

66. (a) $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$f''(x) = 12ax^2 + 6bx + 2c$

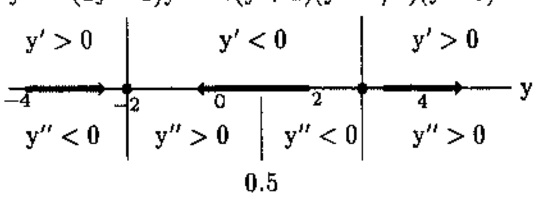
Since $f''(x)$ is quadratic, it must have 0, 1, or 2 zeros. If $f''(x)$ has 0 or 1 zeros, it will not change sign and the concavity of f(x) will not change, so there is no point of inflection. If $f''(x)$ has 2 zeros, it will change sign twice, and f(x) will have 2 points of inflection.

(b) If f has no points of inflection, then $f''(x)$ has 0 or 1 zeros, so the discriminant of $f''(x)$ is ≤ 0. This gives $(6b)^2 - 4(12a)(2c) \leq 0$, or $3b^2 \leq 8ac$.

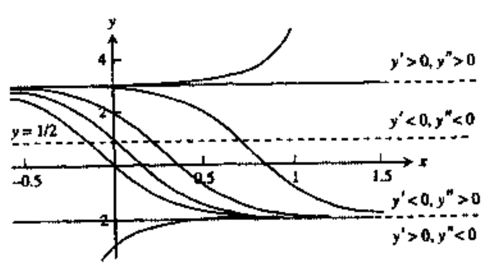
If f has 2 points of inflection, then $f''(x)$ has 2 zeros and the inequality is reversed, so $3b^2 > 8ac$. In summary, f has 2 points of inflection if and only if $3b^2 > 8ac$.

3.4 GRAPHICAL SOLUTIONS TO DIFFERENTIAL EQUATIONS

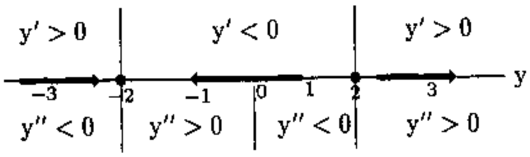
1. $y' = (y+2)(y-3)$
   (a) $y = -2$ is a stable equilibrium value and $y = 3$ is an unstable equilibrium.
   (b) $y'' = (2y-1)y' = 2(y+2)(y-1/2)(y-3)$



   (c)



2. $y' = (y+2)(y-2)$
   (a) $y = -2$ is a stable equilibrium value and $y = 2$ is an unstable equilibrium value.
   (b) $y'' = 2yy' = 2(y+2)y(y-2)$



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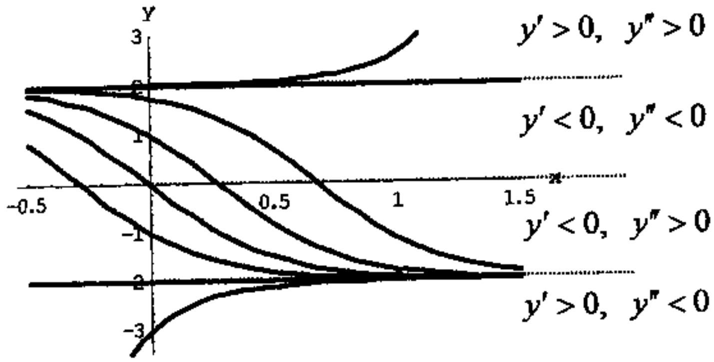
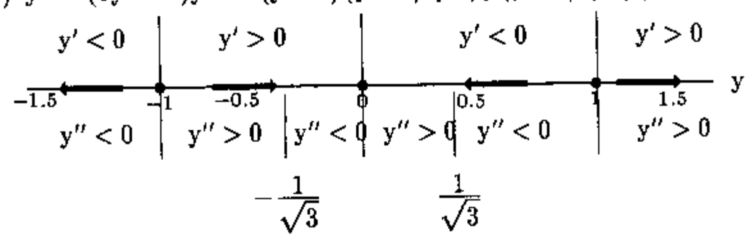


3. $y' = y^3 - y = (y+1)y(y-1)$
   (a) $y = -1$ and $y = 1$ are unstable equilibria and $y = 0$ is a stable equilibrium.
   (b) $y'' = (3y^2 - 1)y' = 3(y+1)\left(y + 1/\sqrt{3}\right)y\left(y - 1/\sqrt{3}\right)(y-1)$



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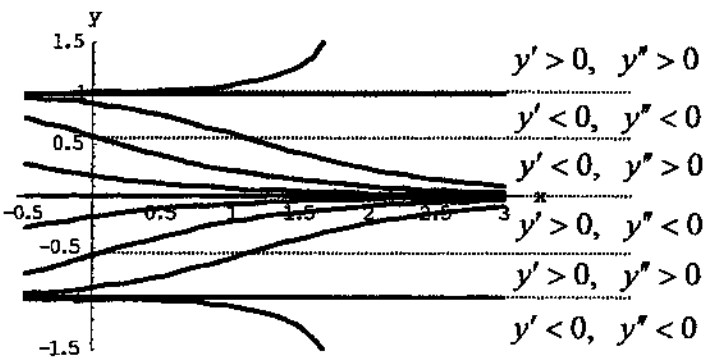
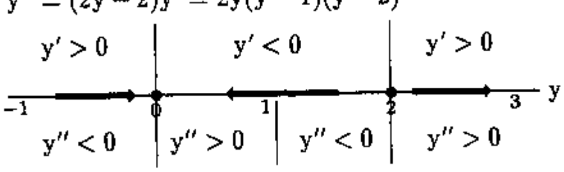


4. $y' = y(y-2)$
   (a) $y = 0$ is a stable equilibrium and $y = 2$ is an unstable equilibrium.
   (b) $y'' = (2y-2)y' = 2y(y-1)(y-2)$



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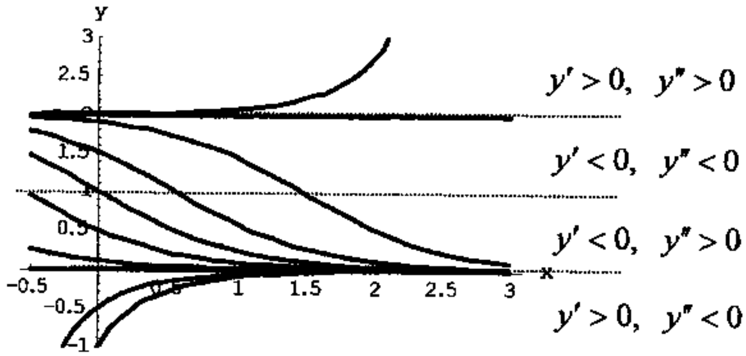
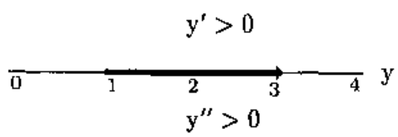


5. $y' = \sqrt{y}$, $y > 0$

(a) There are no equilibrium values.

(b) $y'' = \frac{1}{2\sqrt{y}} y' = \frac{1}{2\sqrt{y}} \cdot \sqrt{y} = \frac{1}{2}$



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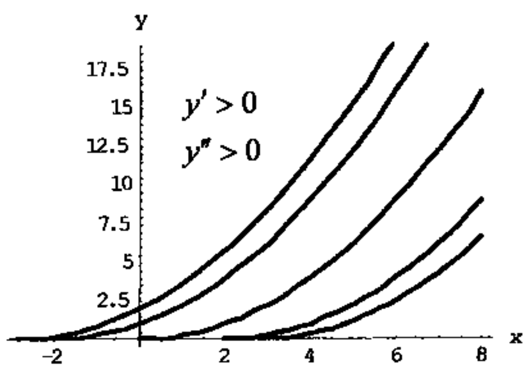
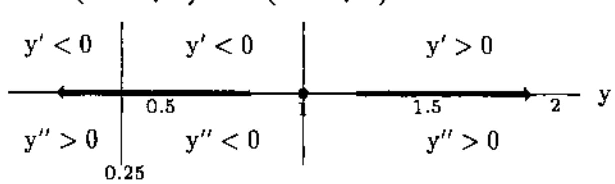


6. $y' = y - \sqrt{y}$, $y > 0$

(a) $y = 1$ is an unstable equilibrium value

(b) $y'' = \left(1 - \frac{1}{2\sqrt{y}}\right) y' = \left(1 - \frac{1}{2\sqrt{y}}\right)(y - \sqrt{y}) = \left(\sqrt{y} - \frac{1}{2}\right)(\sqrt{y} - 1)$



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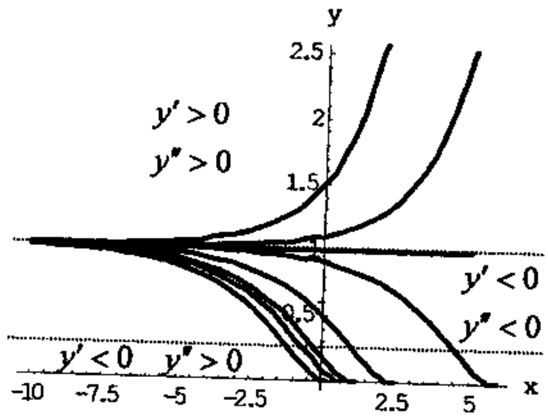
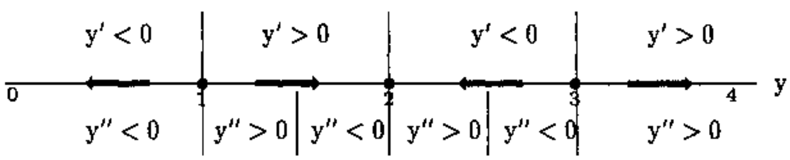


7. $y' = (y-1)(y-2)(y-3)$

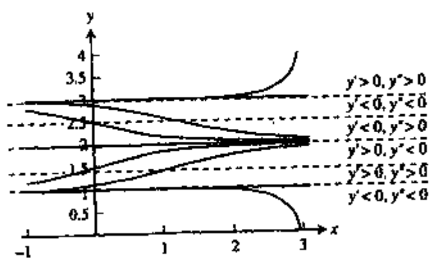
   (a) $y = 1$ and $y = 3$ are unstable equilibria and $y = 2$ is a stable equilibrium.

   (b) $y'' = (3y^2 - 12y + 11)(y-1)(y-2)(y-3) = 3(y-1)\left(y - \frac{6-\sqrt{3}}{3}\right)(y-2)\left(y - \frac{6+\sqrt{3}}{3}\right)(y-3)$



$$\frac{6-\sqrt{3}}{3} \approx 1.42 \qquad \frac{6+\sqrt{3}}{3} \approx 2.58$$

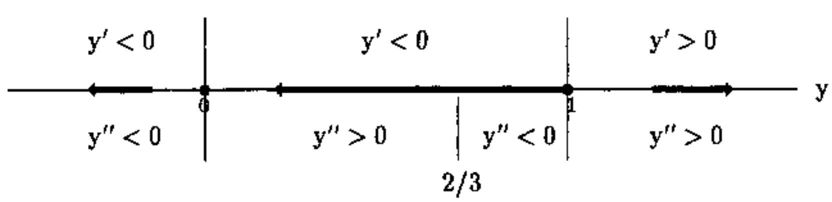
   (c)



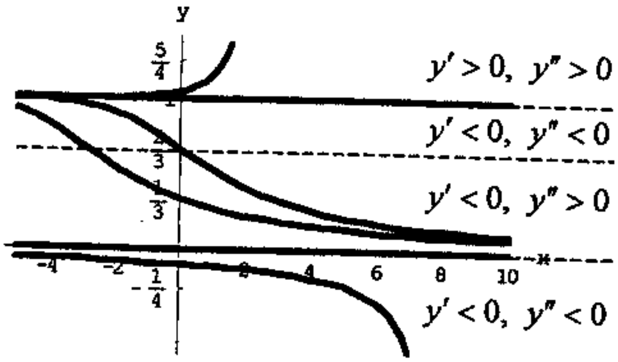
8. $y' = y^3 - y^2 = y^2(y-1)$

   (a) $y = 0$ and $y = 1$ are unstable equilibria.

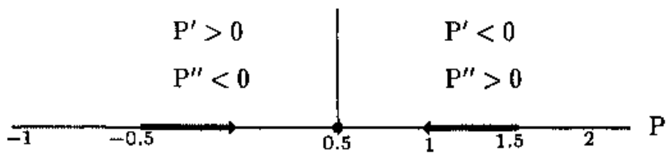
   (b) $y'' = (3y^2 - 2y)(y^3 - y^2) = y^3(3y-2)(y-1)$

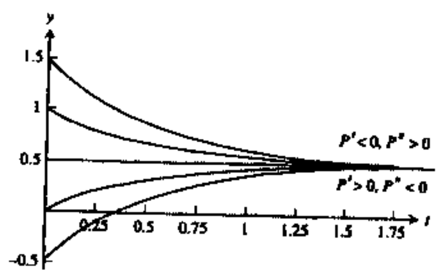


(c)



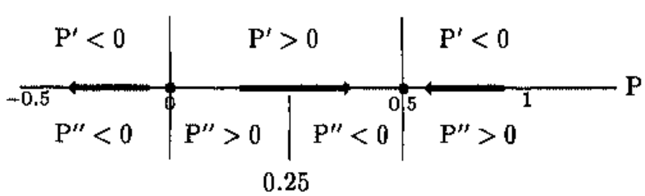
9. $\frac{dP}{dt} = 1 - 2P$ has a stable equilibrium at $P = \frac{1}{2}$. $\frac{d^2P}{dt^2} = -2\,\frac{dP}{dt} = -2(1 - 2P)$

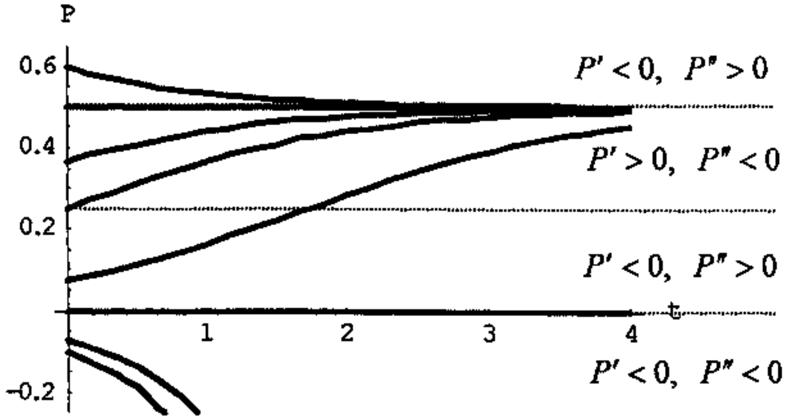




10. $\frac{dP}{dt} = P(1 - 2P)$ has an unstable equilibrium at $P = 0$ and a stable equilibrium at $P = \frac{1}{2}$.

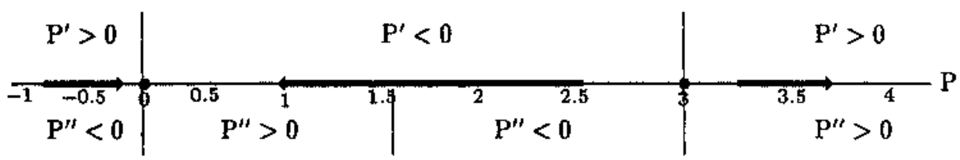
$\frac{d^2P}{dt^2} = (1 - 4P)\,\frac{dP}{dt} = P(1 - 4P)(1 - 2P)$

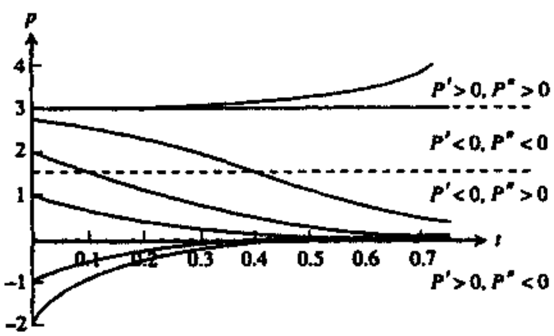




11. $\frac{dP}{dt} = 2P(P-3)$ has a stable equilibrium at $P = 0$ and an unstable equilibrium at $P = 3$.

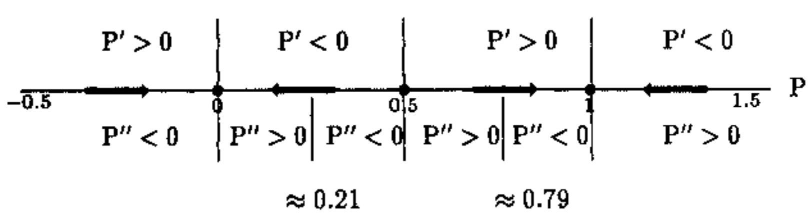
$\frac{d^2P}{dt^2} = 2(2P-3)\,\frac{dP}{dt} = 4P(2P-3)(P-3)$

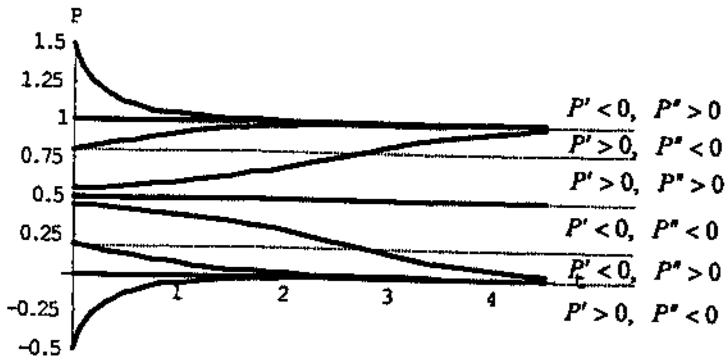




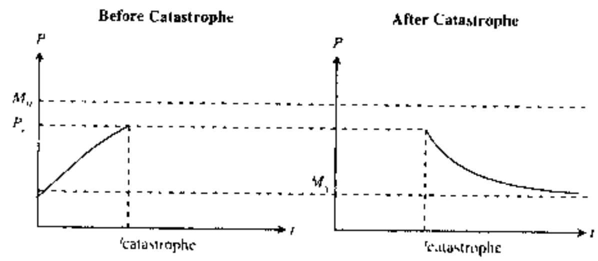
12. $\frac{dP}{dt} = 3P(1-P)\left(P-\frac{1}{2}\right)$ has stable equilibria at $P = 0$ and $P = 1$ and an unstable equilibrium at $P = \frac{1}{2}$.

$\frac{d^2P}{dt^2} = -\frac{3}{2}(6P^2-6P+1)\,\frac{dP}{dt} = \frac{3}{2}P\left(P-\frac{3-\sqrt{3}}{6}\right)\left(P-\frac{1}{2}\right)\left(P-\frac{3+\sqrt{3}}{6}\right)(P-1)$



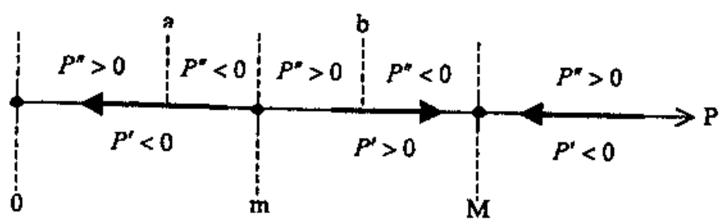


13.



Before the catastrophe, the population exhibits logistic growth and $P(t) \to M_0$, the stable equilibrium. After the catastrophe, the population declines logistically and $P(t) \to M_1$, the new stable equilibrium.

14. $\frac{dP}{dt} = rP(M - P)(P - m),\ \ r,\ M,\ m > 0$



The model has 3 equilibrium points. The rest points $P = 0$, $P = M$ are asymptotically stable while $P = m$ is unstable. For initial populations greater than m, the model predicts P approaches M for large t. For initial populations less than m, the model predicts extinction. Points of inflection occur at $P = a$ and $P = b$ where $a = \frac{1}{3}\left[M + m - \sqrt{M^2 - mM + m^2}\right]$ and $b = \frac{1}{3}\left[M + m + \sqrt{M^2 - mM + m^2}\right]$.

(a) The model is reasonable in the sense that if $P < m$, then $P \to 0$ as $t \to \infty$; if $m < P < M$, then $P \to M$ as $t \to \infty$; if $P > M$, then $P \to M$ as $t \to \infty$.

(b) It is different if the population falls below m, for then $P \to 0$ as $t \to \infty$ (extinction). It is probably a more realistic model for that reason because we know some populations have become extinct after the population level became too low.

(c) For $P > M$ we see that $\frac{dP}{dt} = rP(M-P)(P-m)$ is negative. Thus the curve is everywhere decreasing. Moreover, $P \equiv M$ is a solution to the differential equation. Since the equation satisfies the existence and uniqueness conditions, solution trajectories cannot cross. Thus, $P \to M$ as $t \to \infty$.

(d) See the initial discussion above.

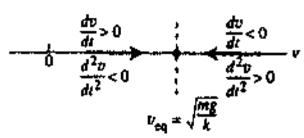
(e) See the initial discussion above.

15. $\frac{dv}{dt} = g - \frac{k}{m}v^2$, g, k, $m > 0$ and $v(t) \geq 0$

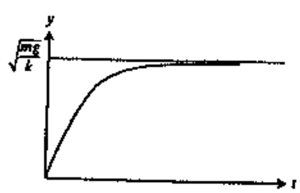
Equilibrium: $\frac{dv}{dt} = g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{mg}{k}}$

Concavity: $\frac{d^2v}{dt^2} = -2\left(\frac{k}{m}v\right)\frac{dv}{dt} = -2\left(\frac{k}{m}v\right)\left(g - \frac{k}{m}v^2\right)$

(a)



(b)



(c) $v_{terminal} = \sqrt{\frac{160}{0.005}} = 178.9\ \frac{ft}{s} = 122$ mph

16. $F = F_p - F_r$

$ma = mg - k\sqrt{v}$

$\frac{dv}{dt} = g - \frac{k}{m}\sqrt{v}$, $v(0) = v_0$

Thus, $\frac{dv}{dt} = 0$ implies $v = \left(\frac{mg}{k}\right)^2$, the terminal velocity. If $v_0 < \left(\frac{mg}{k}\right)^2$, the object will fall faster and faster, approaching the terminal velocity; if $v_0 > \left(\frac{mg}{k}\right)^2$, the object will slow down to the terminal velocity.

17. $F = F_p - F_r$

$ma = 50 - 5|v|$

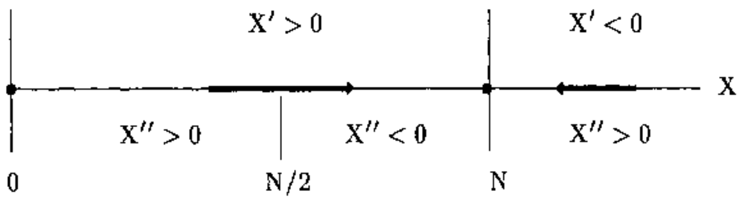
$\frac{dv}{dt} = \frac{1}{m}(50 - 5|v|)$

The maximum velocity occurs when $\frac{dv}{dt} = 0$ or $v = 10$ ft/sec.

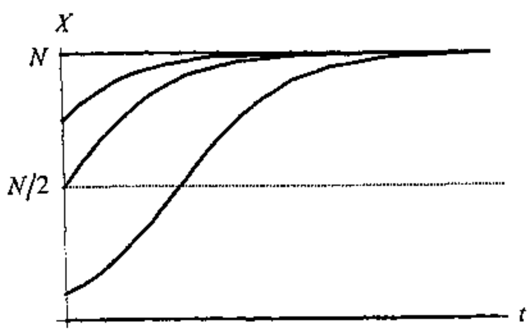
18. (a) The model seems reasonable because the rate of spread of a piece of information, an innovation, or a cultural fad is proportional to the product of the number of individuals who have it (X) and those who do not (N − X). When X is small, there are only a few individuals to spread the item so the rate of spread is slow. On the other hand, when (N − X) is small the rate of spread will be slow because there are only a few individuals who can receive it during an interval of time. The rate of spread will be fastest when both X and (N − X) are large because then there are a lot of individuals to spread the item and a lot of individuals to receive it.

(b) There is a stable equilibrium at $X = N$ and an unstable equilibrium at $X = 0$.

$\frac{d^2X}{dt^2} = k\frac{dX}{dt}(N - X) - kX\frac{dX}{dt} = k^2X(N - X)(N - 2X) \Rightarrow$ inflection points at $X = 0$, $X = \frac{N}{2}$, and $X = N$.



(c)



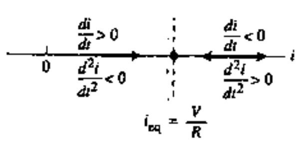
(d) The spread rate is most rapid when $X = \frac{N}{2}$. Eventually all of the people will receive the item.

19. $L\frac{di}{dt} + Ri = V \Rightarrow \frac{di}{dt} = \frac{V}{L} - \frac{R}{L}i = \frac{R}{L}\left(\frac{V}{R} - i\right)$, $V, L, R > 0$

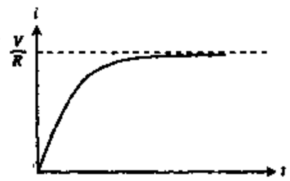
Equilibrium: $\frac{di}{dt} = \frac{R}{L}\left(\frac{V}{R} - i\right) = 0 \Rightarrow i = \frac{V}{R}$,

Concavity: $\frac{d^2i}{dt^2} = -\left(\frac{R}{L}\right)\frac{di}{dt} = -\left(\frac{R}{L}\right)^2\left(\frac{V}{R} - i\right)$

Phase Line:

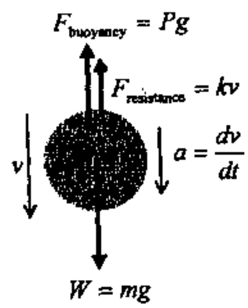


If the switch is closed at $t = 0$, then $i(0) = 0$, and the graph of the solution looks like this:



As $t \to \infty$, $i(t) \to i_{\text{steady state}} = \frac{V}{R}$. (In the steady state condition, the self-inductance acts like a simple wire connector and, as a result, the current through the resistor can be calculated using the familiar version of Ohm's Law.)

20. (a) Free body diagram of the pearl:

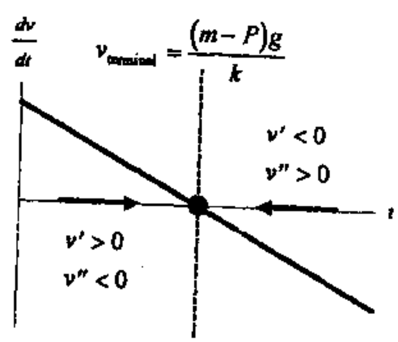


(b) Use Newton's Second Law, summing forces in the direction of the acceleration:

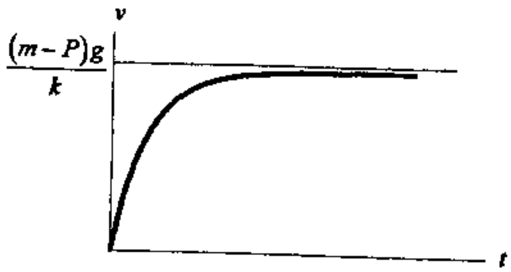
$$mg - Pg - kv = ma \Rightarrow \frac{dv}{dt} = \left(\frac{m - P}{m}\right)g - \frac{k}{m}v.$$

(c) Equilibrium: $\frac{dv}{dt} = \frac{k}{m}\left(\frac{(m-P)g}{k} - v\right) = 0 \Rightarrow v_{\text{terminal}} = \frac{(m-P)g}{k}$

Concavity: $\frac{d^2v}{dt^2} = -\left(\frac{k}{m}\right)\frac{dv}{dt} = -\left(\frac{k}{m}\right)^2\left(\frac{(m-P)g}{k} - v\right)$



(d)



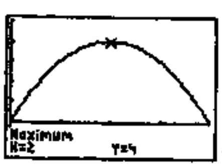
(e) The terminal velocity of the pearl is $\frac{(m-P)g}{k}$.

3.5 MODELING AND OPTIMIZATION

1. Let ℓ and w represent the length and width of the rectangle, respectively. With an area of 16 in.^{2}, we have that $(\ell)(w) = 16 \Rightarrow w = 16\ell^{-1} \Rightarrow$ the perimeter is $P = 2\ell + 2w = 2\ell + 32\ell^{-1}$ and $P'(\ell) = 2 - \frac{32}{\ell^2} = \frac{2(\ell^2 - 16)}{\ell^2}$. Solving $P'(\ell) = 0 \Rightarrow \frac{2(\ell+4)(\ell-4)}{\ell^2} = 0 \Rightarrow \ell = -4, 4$. Since $\ell > 0$ for the length of a rectangle, ℓ must be 4 and $w = 4 \Rightarrow$ the perimeter is 16 in., a minimum since $P''(\ell) = \frac{64}{\ell^3} > 0$.

2. Let x represent the length of the rectangle in meters $(0 < x < 4)$. Then the width is $4 - x$ and the area is $A(x) = x(4 - x) = 4x - x^2$. Since $A'(x) = 4 - 2x$, the critical point occurs at $x = 2$. Since $A'(x) > 0$ for $0 < x < 2$ and $A'(x) < 0$ for $2 < x < 4$, this critical point corresponds to the maximum area. The rectangle with the largest area measures 2 m by $4 - 2 = 2$m, so it is a square.

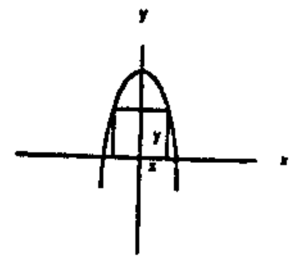
   Graphical support:



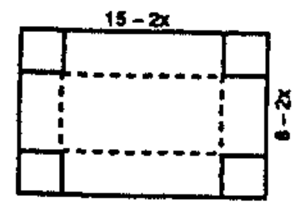
[0, 4] by [−1.5, 5]

3. (a) The line containing point P also contains the points $(0,1)$ and $(1,0) \Rightarrow$ the line containing P is $y = 1 - x$ $\Rightarrow$ a general point on that line is $(x, 1 - x)$.
(b) The area $A(x) = 2x(1 - x)$, where $0 \le x \le 1$.
(c) When $A(x) = 2x - 2x^2$, then $A'(x) = 0 \Rightarrow 2 - 4x = 0 \Rightarrow x = \frac{1}{2}$. Since $A(0) = 0$ and $A(1) = 0$, we conclude that $A\left(\frac{1}{2}\right) = \frac{1}{2}$ sq units is the largest area. The dimensions are 1 unit by $\frac{1}{2}$ unit.

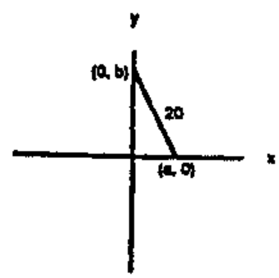
4. The area of the rectangle is $A = 2xy = 2x(12 - x^2)$, where $0 \le x \le \sqrt{12}$. Solving $A'(x) = 0 \Rightarrow 24 - 6x^2 = 0 \Rightarrow x = -2$ or 2. Now -2 is not in the domain, and since $A(0) = 0$ and $A(\sqrt{12}) = 0$, we conclude that $A(2) = 32$ square units is the maximum area. The dimensions are 4 units by 8 units.



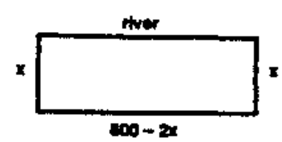
5. The volume of the box is $V(x) = x(15 - 2x)(8 - 2x)$ $= 120x - 46x^2 + 4x^3$, where $0 \le x \le 4$. Solving $V'(x) = 0$ $\Rightarrow 120 - 92x + 12x^2 = 4(6 - x)(5 - 3x) = 0 \Rightarrow x = \frac{5}{3}$ or 6, but 6 is not in the domain. Since $V(0) = V(4) = 0$, $V\left(\frac{5}{3}\right) = \frac{2450}{27} \approx 91$ square units must be the maximum volume of the box with dimensions $\frac{14}{3} \times \frac{35}{3} \times \frac{5}{3}$ inches.



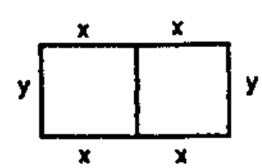
6. The area of the triangle is $A = \frac{1}{2}ba = \frac{b}{2}\sqrt{400 - b^2}$, where $0 \le b \le 20$. Then $\frac{dA}{db} = \frac{1}{2}\sqrt{400 - b^2} - \frac{b^2}{2\sqrt{400 - b^2}} = \frac{200 - b^2}{\sqrt{400 - b^2}}$ $= 0 \Rightarrow$ the interior critical point is $b = 10\sqrt{2}$. When $b = 0$ or 20, the area is zero $\Rightarrow A(10\sqrt{2})$ is the maximum area. When $a^2 + b^2 = 400$ and $b = 10\sqrt{2}$, the value of a is also $10\sqrt{2} \Rightarrow$ the maximum area occurs when $a = b$.



7. The area is $A(x) = x(800 - 2x)$, where $0 \le x \le 400$. Solving $A'(x) = 800 - 4x = 0 \Rightarrow x = 200$. With $A(0) = A(400) = 0$, the maximum area is $A(200) = 80{,}000\ m^2$. The dimensions are 200 m by 400 m.



8. The area is $2xy = 216 \Rightarrow y = \frac{108}{x}$. The perimeter is $P = 4x + 3y$ $= 4x + 324x^{-1}$, where $0 < x$; $\frac{dP}{dx} = 4 - \frac{324}{x^2} = 0 \Rightarrow x^2 - 81 = 0$ $\Rightarrow$ the critical points are 0 and ± 9, but 0 and -9 are not in the domain. Then $P''(9) > 0 \Rightarrow$ at $x = 9$ there is a minimum $\Rightarrow$ the dimensions of the outer rectangle are 18 m by 12 m $\Rightarrow$ 72 meters of fence will be needed.



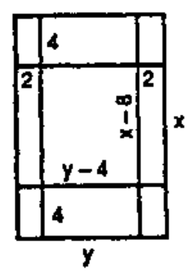
9. (a) We minimize the weight $= tS$ where S is the surface area, and t is the thickness of the steel walls of the tank. The surface area is $S = x^2 + 4xy$ where x is the length of a side of the square base of the tank, and y is its depth. The volume of the tank must be 500 $ft^3 \Rightarrow y = \frac{500}{x^2}$. Therefore, the weight of the tank is $w(x) = t\left(x^2 + \frac{2000}{x}\right)$. Treating the thickness as a constant gives $w'(x) = t\left(2x - \frac{2000}{x^2}\right)$ for $x > 0$. The critical value is at $x = 10$. Since $w''(10) = t\left(2 + \frac{4000}{10^3}\right) > 0$, there is a minimum at $x = 10$. Therefore, the optimum dimensions of the tank are 10 ft on the base edges and 5 ft deep.

   (b) Minimizing the surface area of the tank minimizes its weight for a given wall thickness. The thickness of the steel walls would likely be determined by other considerations such as structural requirements.

10. (a) With the volume of the tank being 1125 ft^3, we have that $yx^2 = 1125 \Rightarrow y = \frac{1125}{x^2}$. The cost of building the tank is $c(x) = 5x^2 + 30x\left(\frac{1125}{x^2}\right)$, where $0 < x$. Then $c'(x) = 10x - \frac{33{,}750}{x^2} = 0 \Rightarrow$ the critical points are 0 and 15, but 0 is not in the domain. Thus $c''(15) > 0 \Rightarrow$ at $x = 15$ we have a minimum. The values of $x = 15$ ft and $y = 5$ ft will minimize the cost.

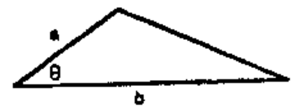
   (b) The cost function, $c = 5(x^2 + 4xy) + 10xy$, can be separated into two items: (1) the cost of materials and labor to fabricate the tank, and (2) the cost for the excavation. Since the area of the sides and bottom of the tank is $(x^2 + 4xy)$, it can be deduced that the unit cost to fabricate the tank is $\$5/ft^2$. Normally, excavation costs are per unit volume of excavated material. Consequently, the total excavation cost can be taken as $10xy = \left(\frac{10}{x}\right)(x^2y)$. This suggests that the unit cost of excavation is $\frac{\$10/ft^2}{x}$ where x is the length of a side of the square base of the tank in feet. For the least expensive tank, the unit cost for the excavation is $\frac{\$10/ft^2}{15\ ft} = \frac{\$0.67}{ft^3} = \frac{\$18}{yd^3}$. The total cost of the least expensive tank is \$3375, which is the sum of \$2625 for fabrication and \$750 for the excavation.

11. The area of the printing is $(y-4)(x-8) = 50$. Consequently, $y = \left(\frac{50}{x-8}\right) + 4$. The area of the paper is $A(x) = x\left(\frac{50}{x-8} + 4\right)$, where $8 < x$. Then $A'(x) = \left(\frac{50}{x-8} + 4\right) - x\left(\frac{50}{(x-8)^2}\right)$ $= \frac{4(x-8)^2 - 400}{(x-8)^2} = 0 \Rightarrow$ the critical points are -2 and 18, but -2 is not in the domain. Thus $A''(18) > 0 \Rightarrow$ at $x = 18$ we have a minimum. Therefore the dimensions 18 by 9 inches minimize the amount of paper.



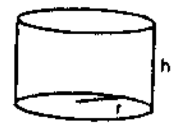
12. The volume of the cone is $V = \frac{1}{3}\pi r^2 h$, where $r = x = \sqrt{9 - y^2}$ and $h = y + 3$ (from the figure in the text). Thus, $V(y) = \frac{\pi}{3}(9 - y^2)(y + 3) = \frac{\pi}{3}(27 + 9y - 3y^2 - y^3) \Rightarrow V'(y) = \frac{\pi}{3}(9 - 6y - 3y^2) = \pi(1 - y)(3 + y)$. The critical points are -3 and 1, but -3 is not in the domain. Thus $V''(1) = \frac{\pi}{3}(-6 - 6(1)) < 0 \Rightarrow$ at $y = 1$ we have a maximum volume of $V(1) = \frac{\pi}{3}(8)(4) = \frac{32\pi}{3}$ cubic units.

13. The area of the triangle is $A(\theta) = \frac{ab \sin \theta}{2}$, where $0 < \theta < \pi$.



Solving $A'(\theta) = 0 \Rightarrow \frac{ab \cos \theta}{2} = 0 \Rightarrow \theta = \frac{\pi}{2}$. Since $A''(\theta)$ $= -\frac{ab \sin \theta}{2} \Rightarrow A''\left(\frac{\pi}{2}\right) < 0$, there is a maximum at $\theta = \frac{\pi}{2}$.

14. A volume $V = \pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$. The amount of material is the surface area given by the sides and bottom of the can $\Rightarrow S = 2\pi rh + \pi r^2 = \frac{2000}{r} + \pi r^2$, $0 < r$. Then $\frac{dS}{dr} = -\frac{2000}{r^2} + 2\pi r$

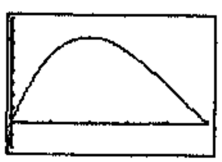


$= 0 \Rightarrow \frac{\pi r^3 - 1000}{r^2} = 0$. The critical points are 0 and $\frac{10}{\sqrt[3]{\pi}}$, but 0 is not in the domain. Since $\frac{d^2S}{dr^2} = \frac{4000}{r^3} + 2\pi$ > 0, we have a minimum surface area when $r = \frac{10}{\sqrt[3]{\pi}}$ cm and $h = \frac{1000}{\pi r^2} = \frac{10}{\sqrt[3]{\pi}}$ cm.

15. With a volume of 1000 cm and $V = \pi r^2 h$, then $h = \frac{1000}{\pi r^2}$. The amount of aluminum used per can is $A = 8r^2 + 2\pi rh = 8r^2 + \frac{2000}{r}$. Then $A'(r) = 16r - \frac{2000}{r^2} = 0 \Rightarrow \frac{8r^3 - 1000}{r^2} = 0 \Rightarrow$ the critical points are 0 and 5, but $r = 0$ results in no can. Since $A''(r) = 16 + \frac{4000}{r^3} > 0$ we have a minimum at $r = 5 \Rightarrow h = \frac{40}{\pi}$ and $h:r = 8:\pi$.

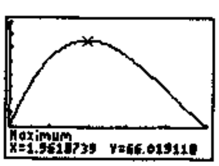
16. (a) The base measures $10 - 2x$ in. by $\frac{15 - 2x}{2}$ in., so the volume formula is
$V(x) = \frac{x(10 - 2x)(15 - 2x)}{2} = 2x^3 - 25x^2 + 75x$.

(b) We require $x > 0$, $2x < 10$, and $2x < 15$. Combining these requirements, the domain is the interval $(0, 5)$.



[0, 5] by [−20, 80]

(c)



[0, 5] by [−20, 80]

The maximum volume is approximately 66.02 in.^{3} when $x \approx 1.96$ in.

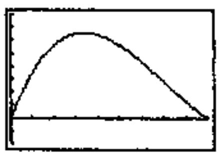
(d) $V'(x) = 6x^2 - 50x + 75$

The critical point occurs when $V'(x) = 0$, at $x = \frac{50 \pm \sqrt{(-50)^2 - 4(6)(75)}}{2(6)} = \frac{50 \pm \sqrt{700}}{12}$

$= \frac{25 \pm 5\sqrt{7}}{6}$, that is, $x \approx 1.96$ or $x \approx 6.37$. We discard the larger value because it is not in the domain. Since $V''(x) = 12x - 50$, which is negative when $x \approx 1.96$, the critical point corresponds to the maximum volume. The maximum volume occurs when $x = \frac{25 - 5\sqrt{7}}{6} \approx 1.96$, which confirms the result in (c).

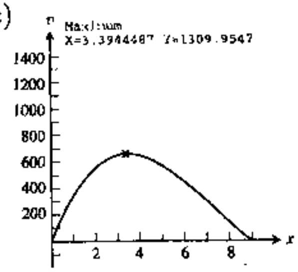
17. (a) The "sides" of the suitcase will measure $24 - 2x$ in. by $18 - 2x$ in. and will be $2x$ in. apart, so the volume formula is $V(x) = 2x(24 - 2x)(18 - 2x) = 8x^3 - 168x^2 + 864x$.

(b) We require $x > 0$, $2x < 18$, and $2x < 24$. Combining these requirements, the domain is the interval $(0, 9)$.



[0, 9] by [−400, 1600]

(c)



This maximum volume is approximately $1309.95\ in^3$ when $x \approx 3.39$ in.

(d) $V'(x) = 24x^2 - 336x + 864 = 24(x^2 - 14x + 36)$
The critical point is at

$$x = \frac{14 \pm \sqrt{(-14)^2 - 4(1)(36)}}{2(1)} = \frac{14 \pm \sqrt{52}}{2} = 7 \pm \sqrt{13},$$

that is, $x \approx 3.39$ or $x \approx 10.61$. We discard the larger value because it is not in the domain. Since $V''(x) = 24(2x - 14)$, which is negative when $x \approx 3.39$, the critical point corresponds to the maximum volume. The maximum value occurs at $x = 7 - \sqrt{13} \approx 3.39$, which confirms the results in (c).

(e) $8x^3 - 168x^2 + 864x = 1120$
$8(x^3 - 21x^2 + 108x - 140) = 0$
$8(x - 2)(x - 5)(x - 14) = 0$
Since 14 is not in the domain, the possible values of x are $x = 2$ in. or $x = 5$ in.

(f) The dimensions of the resulting box are $2x$ in., $(24 - 2x)$ in., and $(18 - 2x)$ in. Each of these measurements must be positive, so that gives the domain of $(0, 9)$.

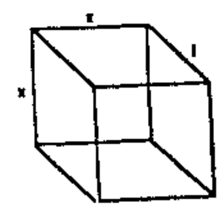
18. If the upper right vertex of the rectangle is located at $(x, 4\cos 0.5x)$ for $0 < x < \pi$, then the rectangle has width $2x$ and height $4\cos 0.5x$, so the area is $A(x) = 8x\cos 0.5x$. Then $A'(x) = 8x(-0.5\sin 0.5x) + 8(\cos 0.5x)(1)$ $= -4x\sin 0.5x + 8\cos 0.5x$. Solving $A'(x) = 0$ graphically for $0 < x < \pi$, we find that $x \approx 1.72$. Evaluating $2x$ and $4\cos 0.5x$ for $x \approx 1.72$, the dimensions of the rectangle are approximately 3.44 (width) by 2.61 (height), and the maximum area is approximately 8.98.

19. Let the radius of the cylinder be r cm, $0 < r < 10$. Then the height is $2\sqrt{100 - r^2}$ and the volume is $V(r) = 2\pi r^2\sqrt{100 - r^2}$ cm^3. Then

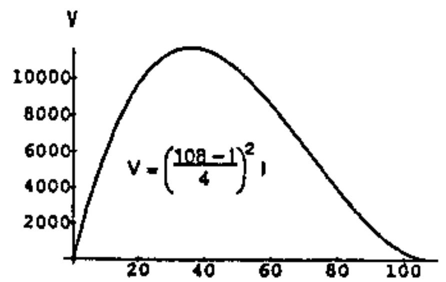
$$V'(r) = 2\pi r^2\left(\frac{1}{2\sqrt{100 - r^2}}\right)(-2r) + \left(2\pi\sqrt{100 - r^2}\right)(2r) = \frac{-2\pi r^3 + 4\pi r(100 - r^2)}{\sqrt{100 - r^2}} = \frac{2\pi r(200 - 3r^2)}{\sqrt{100 - r^2}}$$

The critical point for $0 < r < 10$ occurs at $r = \sqrt{\frac{200}{3}} = 10\sqrt{\frac{2}{3}}$. Since $V'(r) > 0$ for $0 < r < 10\sqrt{\frac{2}{3}}$ and $V'(r) < 0$ for $10\sqrt{\frac{2}{3}} < r < 10$, the critical point corresponds to the maximum volume. The dimensions are $r = 10\sqrt{\frac{2}{3}} \approx 8.16$ cm and $h = \frac{20}{\sqrt{3}} \approx 11.55$ cm, and the volume is $\frac{4000\pi}{3\sqrt{3}} \approx 2418.40$ cm^3.

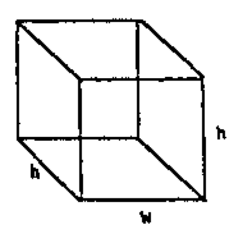
20. (a) From the diagram we have $4x + \ell = 108$ and $V = x^2\ell$. The volume of the box is $V(x) = x^2(108 - 4x)$, where $0 \le x < 27$. Then $V'(x) = 216x - 12x^2 = 12x(18 - x) = 0 \Rightarrow$ the critical points are 0 and 18, but $x = 0$ results in no box. Since $V''(x) = 216 - 24x < 0$ at $x = 18$ we have a maximum. The dimensions of the box are $18 \times 18 \times 36$ in.



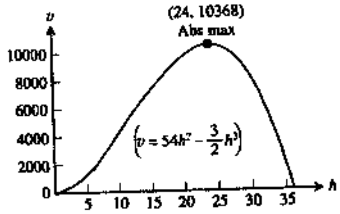
(b) In terms of length, $V(\ell) = x^2\ell = \left(\frac{108 - \ell}{4}\right)^2 \ell$. The graph indicates that the maximum volume occurs near $\ell = 36$, which is consistent with the result of part (a).



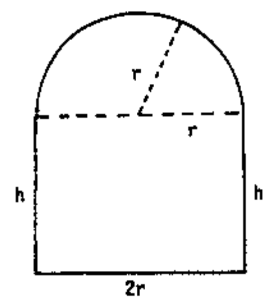
21. (a) From the diagram we have $3h + 2w = 108$ and $V = h^2w$ $\Rightarrow V(h) = h^2\left(54 - \frac{3}{2}h\right) = 54h^2 - \frac{3}{2}h^3$. Then $V'(h) = 108h - \frac{9}{2}h^2$ $= \frac{9}{2}h(24 - h) = 0 \Rightarrow h = 0$ or $h = 24$, but $h = 0$ results in no box. Since $V''(h) = 108 - 9h < 0$ at $h = 24$, we have a maximum volume at $h = 24$ inches and $w = 54 - \frac{3}{2}h = 18$ inches.



(b)



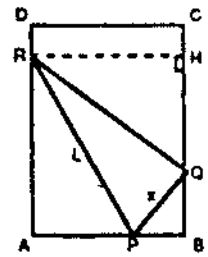
22. From the diagram the perimeter is $P = 2r + 2h + \pi r$, where r is the radius of the semicircle and h is the height of the rectangle. The amount of light transmitted is $A = 2rh + \frac{1}{4}\pi r^2 = r(P - 2r - \pi r) + \frac{1}{4}\pi r^2 = rP - 2r^2 - \frac{3}{4}\pi r^2$. Then $\frac{dA}{dr} = P - 4r - \frac{3}{2}\pi r = 0 \Rightarrow r = \frac{2P}{8 + 3\pi} \Rightarrow$ $2h = P - \frac{4P}{8 + 3\pi} - \frac{2\pi P}{8 + 3\pi} = \frac{(4 + \pi)P}{8 + 3\pi}$. Therefore, $\frac{2r}{h} = \frac{8}{4 + \pi}$ gives the proportions that admit the most light since $\frac{d^2A}{dr^2} = -4 - \frac{3}{2}\pi < 0$.



23. The fixed volume is $V = \pi r^2 h + \frac{2}{3}\pi r^3 \Rightarrow h = \frac{V}{\pi r^2} - \frac{2r}{3}$, where h is the height of the cylinder and r is the radius of the hemisphere. To minimize the cost we must minimize surface area of the cylinder added to twice the surface area of the hemisphere. Thus, we minimize $C = 2\pi rh + 4\pi r^2 = 2\pi r\left(\frac{V}{\pi r^2} - \frac{2r}{3}\right) + 4\pi r^2 = \frac{2V}{r} + \frac{8}{3}\pi r^2$. Then $\frac{dC}{dr} = -\frac{2V}{r^2} + \frac{16}{3}\pi r = 0 \Rightarrow V = \frac{8}{3}\pi r^3 \Rightarrow r = \left(\frac{3V}{8\pi}\right)^{1/3}$. From the volume equation, $h = \frac{V}{\pi r^2} - \frac{2r}{3}$ $= \frac{4V^{1/3}}{\pi^{1/3} \cdot 3^{2/3}} - \frac{2 \cdot 3^{1/3} \cdot V^{1/3}}{3 \cdot 2 \cdot \pi^{1/3}} = \frac{3^{1/3} \cdot 2 \cdot 4 \cdot V^{1/3} - 2 \cdot 3^{1/3} \cdot V^{1/3}}{3 \cdot 2 \cdot \pi^{1/3}} = \left(\frac{3V}{\pi}\right)^{1/3}$. Since $\frac{d^2C}{dr^2} = \frac{4V}{r^3} + \frac{16}{3}\pi > 0$, these dimensions do minimize the cost.

24. The volume of the trough is maximized when the area of the cross section is maximized. From the diagram the area of the cross section is $A(\theta) = \cos\theta + \sin\theta\cos\theta$, $0 < \theta < \frac{\pi}{2}$. Then $A'(\theta) = -\sin\theta + \cos^2\theta - \sin^2\theta$ $= -\left(2\sin^2\theta + \sin\theta - 1\right) = -(2\sin\theta - 1)(\sin\theta + 1)$ so $A'(\theta) = 0 \Rightarrow \sin\theta = \frac{1}{2}$ or $\sin\theta = -1 \Rightarrow \theta = \frac{\pi}{6}$ because $\sin\theta \neq -1$ when $0 < \theta < \frac{\pi}{2}$. Also, $A'(\theta) > 0$ for $0 < \theta < \frac{\pi}{6}$ and $A'(\theta) < 0$ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$. Therefore, at $\theta = \frac{\pi}{6}$ there is a maximum.

25. (a) From the diagram we have: $\overline{AP} = x$, $\overline{RA} = \sqrt{L^2 - x^2}$,



$\overline{PB} = 8.5 - x$, $\overline{CH} = \overline{DR} = 11 - \overline{RA} = 11 - \sqrt{L^2 - x^2}$,

$\overline{QB} = \sqrt{x^2 - (8.5 - x)^2}$, $\overline{HQ} = 11 - \overline{CH} - \overline{QB}$

$= 11 - \left[11 - \sqrt{L^2 - x^2} + \sqrt{x^2 - (8.5 - x)^2}\right]$

$= \sqrt{L^2 - x^2} - \sqrt{x^2 - (8.5 - x)^2}$,

$\overline{RQ}^2 = \overline{RH}^2 + \overline{HQ}^2 = (8.5)^2 + \left(\sqrt{L^2 - x^2} - \sqrt{x^2 - (8.5 - x)^2}\right)^2$. It follows that

$\overline{RP}^2 = \overline{PQ}^2 + \overline{RQ}^2 \Rightarrow L^2 = x^2 + \left(\sqrt{L^2 - x^2} - \sqrt{x^2 - (x - 8.5)^2}\right)^2 + (8.5)^2$

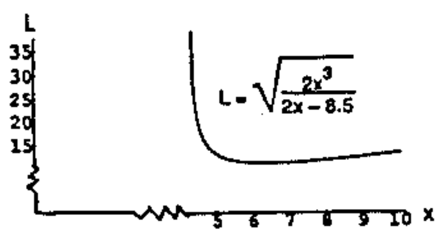
$\Rightarrow L^2 = x^2 + L^2 - x^2 - 2\sqrt{L^2 - x^2}\,\sqrt{17x - (8.5)^2} + 17x - (8.5)^2 + (8.5)^2$

$\Rightarrow 17^2x^2 = 4\left(L^2 - x^2\right)\left(17x - (8.5)^2\right) \Rightarrow L^2 = x^2 + \dfrac{17^2x^2}{4\left[17x - (8.5)^2\right]} = \dfrac{17x^3}{17x - (8.5)^2}$

$= \dfrac{17x^3}{17x - \left(\frac{17}{2}\right)^2} = \dfrac{4x^3}{4x - 17} = \dfrac{2x^3}{(2x - 8.5)}$.

(b) If $f(x) = \dfrac{4x^3}{4x - 17}$ is minimized, then L^2 is minimized. Now $f'(x) = \dfrac{4x^2(8x - 51)}{(4x - 17)^2} \Rightarrow f'(x) < 0$ when $x < \frac{51}{8}$ and $f'(x) > 0$ when $x > \frac{51}{8}$. Thus L^2 is minimized when $x = \frac{51}{8}$.

(c) When $x = \frac{51}{8}$, then $L \approx 11.0$ in.



26. (a) From the figure in the text we have $P = 2x + 2y \Rightarrow y = \frac{P}{2} - x$. If $P = 36$, then $y = 18 - x$. When the cylinder is formed, $x = 2\pi r \Rightarrow r = \frac{x}{2\pi}$ and $h = y \Rightarrow h = 18 - x$. The volume of the cylinder is $V = \pi r^2 h$ $\Rightarrow V(x) = \dfrac{18x^2 - x^3}{4\pi}$. Solving $V'(x) = \dfrac{3x(12 - x)}{4\pi} = 0 \Rightarrow x = 0$ or 12; but when $x = 0$, there is no cylinder. Then $V''(x) = \frac{3}{\pi}\left(3 - \frac{x}{2}\right) \Rightarrow V''(12) < 0 \Rightarrow$ there is a maximum at $x = 12$. The values of $x = 12$ cm and $y = 6$ cm give the largest volume.

(b) In this case $V(x) = \pi x^2(18 - x)$. Solving $V'(x) = 3\pi x(12 - x) = 0 \Rightarrow x = 0$ or 12; but $x = 0$ would result in no cylinder. Then $V''(x) = 6\pi(6 - x) \Rightarrow V''(12) < 0 \Rightarrow$ there is a maximum at $x = 12$. The values of $x = 12$ cm and $y = 6$ cm give the largest volume.

27. Note that $h^2 + r^2 = 3$ and so $r = \sqrt{3 - h^2}$. Then the volume is given by $V = \frac{\pi}{3}r^2h = \frac{\pi}{3}(3 - h^2)h = \pi h - \frac{\pi}{3}h^3$ for $0 < h < \sqrt{3}$, and so $\frac{dV}{dh} = \pi - \pi h^2 = \pi(1 - h^2)$. The critical point (for $h > 0$) occurs at $h = 1$. Since $\frac{dV}{dh} > 0$ for

$0 < h < 1$ and $\frac{dV}{dh} < 0$ for $1 < h < \sqrt{3}$, the critical point corresponds to the maximum volume. The cone of greatest volume has radius $\sqrt{2}$ m, height 1 m, and volume $\frac{2\pi}{3}$ m^3.

28. (a) $f(x) = x^2 + \frac{a}{x} \Rightarrow f'(x) = x^{-2}(2x^3 - a)$, so that $f'(x) = 0$ when $x = 2$ implies $a = 16$

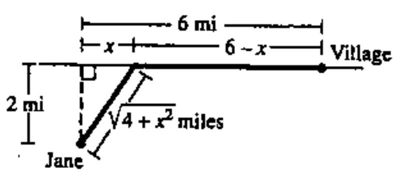
(b) $f(x) = x^2 + \frac{a}{x} \Rightarrow f''(x) = 2x^{-3}(x^3 + a)$, so that $f''(x) = 0$ when $x = 1$ implies $a = -1$

29. If $f(x) = x^2 + \frac{a}{x}$, then $f'(x) = 2x - ax^{-2}$ and $f''(x) = 2 + 2ax^{-3}$. The critical points are 0 and $\sqrt[3]{\frac{a}{2}}$, but $x \neq 0$. Now $f''\left(\sqrt[3]{\frac{a}{2}}\right) = 6 > 0 \Rightarrow$ at $x = \sqrt[3]{\frac{a}{2}}$ there is a local minimum. However, no local maximum exists for any a.

30. If $f(x) = x^3 + ax^2 + bx$, then $f'(x) = 3x^2 + 2ax + b$ and $f''(x) = 6x + 2a$.
(a) A local maximum at $x = -1$ and local minimum at $x = 3 \Rightarrow f'(-1) = 0$ and $f'(3) = 0 \Rightarrow 3 - 2a + b = 0$ and $27 + 6a + b = 0 \Rightarrow a = -3$ and $b = -9$.
(b) A local minimum at $x = 4$ and a point of inflection at $x = 1 \Rightarrow f'(4) = 0$ and $f''(1) = 0 \Rightarrow 48 + 8a + b = 0$ and $6 + 2a = 0 \Rightarrow a = -3$ and $b = -24$.

31. (a) $s(t) = -16t^2 + 96t + 112$
$v(t) = s'(t) = -32t + 96$
At $t = 0$, the velocity is $v(0) = 96$ ft/sec.
(b) The maximum height occurs when $v(t) = 0$, when $t = 3$. The maximum height is $s(3) = 256$ ft and it occurs at $t = 3$ sec.
(c) Note that $s(t) = -16t^2 + 96t + 112 = -16(t+1)(t-7)$, so $s = 0$ at $t = -1$ or $t = 7$. Choosing the positive value of t, the velocity when $s = 0$ is $v(7) = -128$ ft/sec.

32.



Let x be the distance from the point on the shoreline nearest Jane's boat to the point where she lands her boat. Then she needs to row $\sqrt{4 + x^2}$ mi at 2 mph and walk $6 - x$ mi at 5 mph. The total amount of time to reach the village is

$f(x) = \frac{\sqrt{4+x^2}}{2} + \frac{6-x}{5}$ hours $(0 \le x \le 6)$. Then $f'(x) = \frac{1}{2}\frac{1}{2\sqrt{4+x^2}}(2x) - \frac{1}{5} = \frac{x}{2\sqrt{4+x^2}} - \frac{1}{5}$. Solving $f'(x) = 0$, we have:

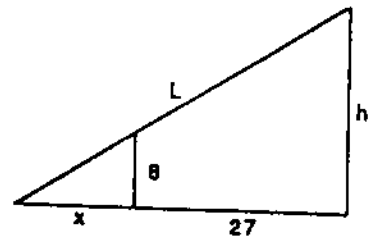
$$\frac{x}{2\sqrt{4+x^2}} = \frac{1}{5}$$
$$5x = 2\sqrt{4+x^2}$$
$$25x^2 = 4(4+x^2)$$
$$21x^2 = 16$$

$$x = \pm\frac{4}{\sqrt{21}}$$

We discard the negative value of x because it is not in the domain. Checking the endpoints and critical point, we have $f(0) = 2.2$, $f\left(\frac{4}{\sqrt{21}}\right) \approx 2.12$, and $f(6) \approx 3.16$. Jane should land her boat $\frac{4}{\sqrt{21}} \approx 0.87$ miles down the shoreline from the point nearest her boat.

33. $\frac{8}{x} = \frac{h}{x+27} \Rightarrow h = 8 + \frac{216}{x}$ and $L(x) = \sqrt{h^2 + (x+27)^2}$

$= \sqrt{\left(8 + \frac{216}{x}\right)^2 + (x+27)^2}$ when $x \geq 0$. Note that $L(x)$ is minimized when $f(x) = \left(8 + \frac{216}{x}\right)^2 + (x+27)^2$ is minimized.

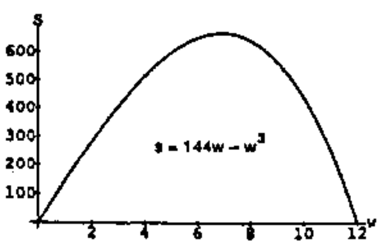


If $f'(x) = 0$, then $2\left(8 + \frac{216}{x}\right)\left(-\frac{216}{x^2}\right) + 2(x+27) = 0$

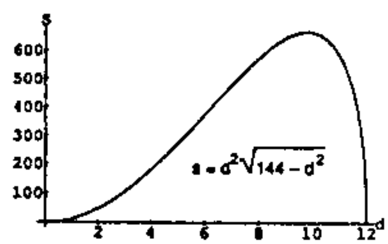
$\Rightarrow (x+27)\left(1 - \frac{1728}{x^3}\right) = 0 \Rightarrow x = -27$ (not acceptable since distance is never negative) or $x = 12$. Then $L(12) = \sqrt{2197} \approx 46.87$ ft.

34. (a) From the diagram we have $d^2 = 144 - w^2$. The strength of the beam is $S = kwd^2 = kw(144 - w^2)$ $\Rightarrow S = 144kw - kw^3 \Rightarrow S'(w) = 144k - 3kw^2 = 3k(48 - w^2)$ so $S'(w) = 0 \Rightarrow w = \pm 4\sqrt{3}$; $S''(4\sqrt{3}) < 0$ and $-4\sqrt{3}$ is not acceptable. Therefore $S(4\sqrt{3})$ is the maximum strength. The dimensions of the strongest beam are $4\sqrt{3}$ by $4\sqrt{6}$ inches.

(b)



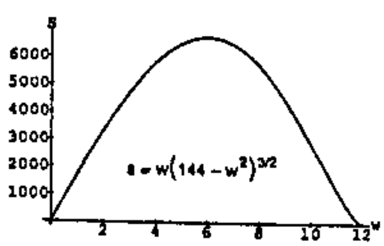
(c)



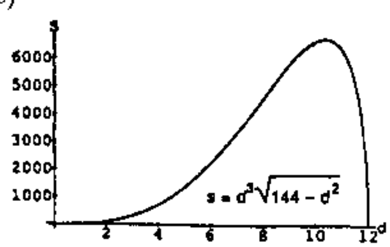
Both graphs indicate the same maximum value and are consistent with each other. Changing k does not change the dimensions that give the strongest beam (i.e., does not change the values of w and d that produce the strongest beam).

35. (a) From the situation we have $w^2 = 144 - d^2$. The stiffness of the beam is $S = kwd^3 = kd^3(144 - d^2)^{1/2}$, where $0 \leq d \leq 12$. Also, $S'(d) = \frac{4kd^2(108 - d^2)}{\sqrt{144 - d^2}} \Rightarrow$ critical points at 0, 12, and $6\sqrt{3}$. Both $d = 0$ and $d = 12$ cause $S = 0$. The maximum occurs at $d = 6\sqrt{3}$. The dimensions are 6 by $6\sqrt{3}$ inches.

(b)



(c)



Both graphs indicate the same maximum value and are consistent with each other. The changing of k has no effect.

36. (a) $s_1 = s_2 \Rightarrow \sin t = \sin\left(t + \frac{\pi}{3}\right) \Rightarrow \sin t = \sin t \cos \frac{\pi}{3} + \sin \frac{\pi}{3} \cos t \Rightarrow \sin t = \frac{1}{2} \sin t + \frac{\sqrt{3}}{2} \cos t \Rightarrow \tan t = \sqrt{3}$ $\Rightarrow t = \frac{\pi}{3}$ or $\frac{4\pi}{3}$

(b) The distance between the particles is $s(t) = |s_1 - s_2| = \left|\sin t - \sin\left(t + \frac{\pi}{3}\right)\right| = \frac{1}{2}\left|\sin t - \sqrt{3} \cos t\right|$

$\Rightarrow s'(t) = \frac{(\sin t - \sqrt{3} \cos t)(\cos t + \sqrt{3} \sin t)}{2\left|\sin t - \sqrt{3} \cos t\right|}$ since $\frac{d}{dx}|x| = \frac{x}{|x|} \Rightarrow$ critical times and endpoints are $0, \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, 2\pi$; then $s(0) = \frac{\sqrt{3}}{2}$, $s\left(\frac{\pi}{3}\right) = 0$, $s\left(\frac{5\pi}{6}\right) = 1$, $s\left(\frac{4\pi}{3}\right) = 0$, $s\left(\frac{11\pi}{6}\right) = 1$, $s(2\pi) = \frac{\sqrt{3}}{2} \Rightarrow$ the greatest distance between the particles is 1

(c) Since $s'(t) = \frac{(\sin t - \sqrt{3} \cos t)(\cos t + \sqrt{3} \sin t)}{2\left|\sin t - \sqrt{3} \cos t\right|}$ we can conclude that at $t = \frac{\pi}{3}$ and $\frac{4\pi}{3}$, $s'(t)$ has cusps and the distance between the particles is changing the fastest near these points

37. (a) $s = 10 \cos(\pi t) \Rightarrow v = -10\pi \sin(\pi t) \Rightarrow$ speed $= |10\pi \sin(\pi t)| = 10\pi|\sin(\pi t)| \Rightarrow$ the maximum speed is $10\pi \approx 31.42$ cm/sec since the maximum value of $|\sin(\pi t)|$ is 1; the cart is moving the fastest at $t = 0.5$ sec, 1.5 sec, 2.5 sec and 3.5 sec when $|\sin(\pi t)|$ is 1. At these times the distance is $s = 10 \cos\left(\frac{\pi}{2}\right) = 0$ cm and $a = -10\pi^2 \cos(\pi t) \Rightarrow |a| = 10\pi^2 |\cos(\pi t)| \Rightarrow |a| = 0$ cm/sec^2

(b) $|a| = 10\pi^2 |\cos(\pi t)|$ is greatest at $t = 0.0$ sec, 1.0 sec, 2.0 sec, 3.0 sec and 4.0 sec, and at these times the magnitude of the cart's position is $|s| = 10$ cm from the rest position and the speed is 0 cm/sec.

38. (a) $2 \sin t = \sin 2t \Rightarrow 2 \sin t - 2 \sin t \cos t = 0 \Rightarrow (2 \sin t)(1 - \cos t) = 0 \Rightarrow t = k\pi$ where k is a positive integer

(b) The vertical distance between the masses is $s(t) = |s_1 - s_2| = \left((s_1 - s_2)^2\right)^{1/2} = \left((\sin 2t - 2 \sin t)^2\right)^{1/2}$

$\Rightarrow s'(t) = \left(\frac{1}{2}\right)\left((\sin 2t - 2 \sin t)^2\right)^{-1/2}(2)(\sin 2t - 2 \sin t)(2 \cos 2t - 2 \cos t)$

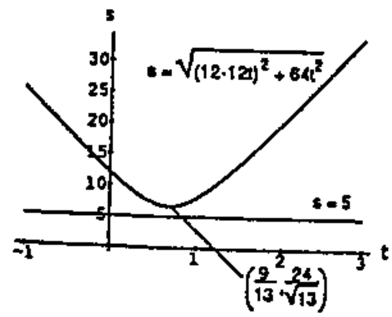
$= \frac{2(\cos 2t - \cos t)(\sin 2t - 2 \sin t)}{|\sin 2t - 2 \sin t|} = \frac{4(2 \cos t + 1)(\cos t - 1)(\sin t)(\cos t - 1)}{|\sin 2t - 2 \sin t|} \Rightarrow$ critical times at $0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$; then $s(0) = 0$, $s\left(\frac{2\pi}{3}\right) = \left|\sin\left(\frac{4\pi}{3}\right) - 2 \sin\left(\frac{2\pi}{3}\right)\right| = \frac{3\sqrt{3}}{2}$, $s(\pi) = 0$, $s\left(\frac{4\pi}{3}\right)$ $= \left|\sin\left(\frac{8\pi}{3}\right) - 2 \sin\left(\frac{4\pi}{3}\right)\right| = \frac{3\sqrt{3}}{2}$, $s(2\pi) = 0 \Rightarrow$ the greatest distance is $\frac{3\sqrt{3}}{2}$ at $t = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$

39. (a) $s = \sqrt{(12 - 12t)^2 + (8t)^2} = \left((12 - 12t)^2 + 64t^2\right)^{1/2}$

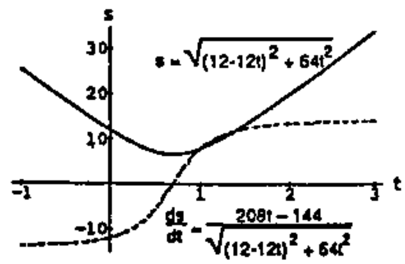
(b) $\frac{ds}{dt} = \frac{1}{2}\left((12 - 12t)^2 + 64t^2\right)^{-1/2}[2(12 - 12t)(-12) + 128t] = \frac{208t - 144}{\sqrt{(12 - 12t)^2 + 64t^2}}$

$\Rightarrow \left.\frac{ds}{dt}\right|_{t=0} = -12$ knots and $\left.\frac{ds}{dt}\right|_{t=1} = 8$ knots

(c) The graph indicates that the ships did not see each other because $s(t) > 5$ for all values of t.



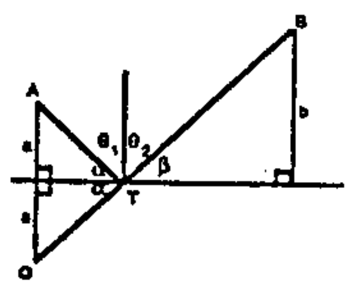
(d) The graph supports the conclusions in parts (b) and (c).



(e) $\lim\limits_{t\to\infty} \frac{ds}{dt} = \sqrt{\lim\limits_{t\to\infty} \frac{(208t-144)^2}{144(1-t)^2+64t^2}} = \sqrt{\lim\limits_{t\to\infty} \frac{\left(208-\frac{144}{t}\right)^2}{144\left(\frac{1}{t}-1\right)^2+64}} = \sqrt{\frac{208^2}{144+64}} = \sqrt{208} = 4\sqrt{13}$

which equals the square root of the sums of the squares of the individual speeds.

40. The distance $\overline{OT} + \overline{TB}$ is minimized when $\overline{OB}$ is a straight line. Hence $\angle\alpha = \angle\beta \Rightarrow \theta_1 = \theta_2$.



41. If $v = kax - kx^2$, then $v' = ka - 2kx$ and $v'' = -2k$, so $v' = 0 \Rightarrow x = \frac{a}{2}$. At $x = \frac{a}{2}$ there is a maximum since $v''\left(\frac{a}{2}\right) = -2k < 0$. The maximum value of v is $\frac{ka^2}{4}$.

42. (a) According to the graph, $y'(0) = 0$.

(b) According to the graph, $y'(-L) = 0$.

(c) $y(0) = 0$, so $d = 0$.

Now $y'(x) = 3ax^2 + 2bx + c$, so $y'(0) = 0$ implies that $c = 0$. Therefore, $y(x) = ax^3 + bx^2$ and $y'(x) = 3ax^2 + 2bx$. Then $y(-L) = -aL^3 + bL^2 = H$ and $y'(-L) = 3aL^2 - 2bL = 0$, so we have two linear equations in the two unknowns a and b. The second equation gives $b = \frac{3aL}{2}$. Substituting into the first equation, we have $-aL^3 + \frac{3aL^3}{2} = H$, or $\frac{aL^3}{2} = H$, so $a = 2\,\frac{H}{L^3}$. Therefore, $b = 3\,\frac{H}{L^2}$ and the equation for y is $y(x) = 2\,\frac{H}{L^3}x^3 + 3\,\frac{H}{L^2}x^2$, or $y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right]$.

43. The profit is $p = nx - nc = n(x-c) = \left[a(x-c)^{-1} + b(100-x)\right](x-c) = a + b(100-x)(x-c)$ $= a + (bc + 100b)x - 100bc - bx^2$. Then $p'(x) = bc + 100b - 2bx$ and $p''(x) = -2b$. Solving $p'(x) = 0 \Rightarrow$ $x = \frac{c}{2} + 50$. At $x = \frac{c}{2} + 50$ there is a maximum profit since $p''(x) = -2b < 0$ for all x.

44. Let x represent the number of people over 50. The profit is $p(x) = (50+x)(200-2x) - 32(50+x) - 6000$ $= -2x^2 + 68x + 2400$. Then $p'(x) = -4x + 68$ and $p'' = -4$. Solving $p'(x) = 0 \Rightarrow x = 17$. At $x = 17$ there is a maximum since $p''(17) < 0$. It would take 67 people to maximize the profit.

45. (a) $A(q) = kmq^{-1} + cm + \frac{h}{2}q$, where $q > 0 \Rightarrow A'(q) = -kmq^{-2} + \frac{h}{2} = \frac{hq^2 - 2km}{2q^2}$ and $A''(q) = 2kmq^{-3}$. The critical points are $-\sqrt{\frac{2km}{h}}$, 0, and $\sqrt{\frac{2km}{h}}$, but only $\sqrt{\frac{2km}{h}}$ is in the domain. Then $A''\left(\sqrt{\frac{2km}{h}}\right) > 0 \Rightarrow$ at $q = \sqrt{\frac{2km}{h}}$ there is a minimum average weekly cost.

(b) $A(q) = \frac{(k+bq)m}{q} + cm + \frac{h}{2}q = kmq^{-1} + bm + cm + \frac{h}{2}q$, where $q > 0 \Rightarrow A'(q) = 0$ at $q = \sqrt{\frac{2km}{h}}$ as in (a). Also $A''(q) = 2kmq^{-3} > 0$ so the most economical quantity to order is still $q = \sqrt{\frac{2km}{h}}$ which minimizes the average weekly cost.

46. We start with $c(x)$ = the cost of producing x items, $x > 0$, and $\frac{c(x)}{x}$ = the average cost of producing x items, assumed to be differentiable. If the average cost can be minimized, it will be at a production level at which $\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0 \Rightarrow \frac{xc'(x) - c(x)}{x^2} = 0$ (by the quotient rule) $\Rightarrow xc'(x) - c(x) = 0$ (multiply both sides by x^2) $\Rightarrow c'(x) = \frac{c(x)}{x}$ where $c'(x)$ is the marginal cost. This concludes the proof. (Note: The theorem does not assure a production level that will give a minimum average cost, but rather, it indicates where to look to see if there is one. Find the production levels where the average cost equals the marginal cost, then check to see if any of them give a minimum.)

47. The profit $p(x) = r(x) - c(x) = 6x - \left(x^3 - 6x^2 + 15x\right) = -x^3 + 6x^2 - 9x$, where $x \geq 0$. Then $p'(x) = -3x^2 + 12x - 9 = -3(x-3)(x-1)$ and $p''(x) = -6x + 12$. The critical points are 1 and 3. Thus $p''(1) = 6 > 0 \Rightarrow$ at $x = 1$ there is a local minimum, and $p''(3) = -6 < 0 \Rightarrow$ at $x = 3$ there is a local maximum. But $p(3) = 0 \Rightarrow$ the best you can do is break even.

48. The average cost of producing x items is is $\overline{c}(x) = \frac{c(x)}{x} = x^2 - 20x + 20{,}000 \Rightarrow \overline{c}'(x) = 2x - 20 = 0 \Rightarrow x = 10$, the only critical value. The average cost $\overline{c}(10) = \$19{,}900$ per item is a minimum average cost because $\overline{c}''(10) = 2 > 0$.

49. (a) The artisan should order px units of material in order to have enough until the next delivery.

(b) The average number of units in storage until the next delivery is $\frac{px}{2}$ and so the cost of storing them is $s\left(\frac{px}{2}\right)$ per day, and the total cost for x days is $\left(\frac{px}{2}\right)sx$. When added to the delivery cost, the total cost for delivery and storage for each cycle is: cost per cycle $= d + \frac{px}{2}sx$.

(c) The average cost per day for storage and delivery of materials is:

average cost per day $= \frac{\left(d + \frac{ps}{2}x^2\right)}{x} = \frac{d}{x} + \frac{ps}{2}x$. To minimize the average cost per day, set the derivative equal to zero. $\frac{d}{dx}\left(d(x)^{-1} + \frac{ps}{2}x\right) = -d(x)^{-2} + \frac{ps}{2} = 0 \Rightarrow x = \pm\sqrt{\frac{2d}{ps}}$. Only the positive root makes sense in this context so that $x^* = \sqrt{\frac{2d}{ps}}$. To verify that x^* gives a minimum, check the second derivative

$\left[\frac{d}{dx}\left(-dx^{-2} + \frac{ps}{2}\right)\right]\Bigg|_{\sqrt{\frac{2d}{ps}}} = \frac{2d}{x^3}\Bigg|_{\sqrt{\frac{2d}{ps}}} = \frac{2d}{\left(\sqrt{\frac{2d}{ps}}\right)^3} > 0 \Rightarrow$ a minimum. The amount to deliver is $px^* = \sqrt{\frac{2pd}{s}}$.

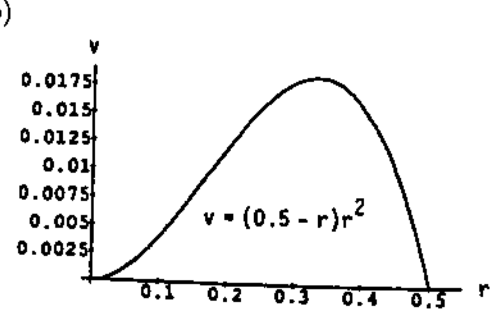
(d) The line and hyperbola intersect when $\frac{d}{x} = \frac{ps}{2}x$. Solving for x gives $x_{\text{intersection}} = \pm\sqrt{\frac{2d}{ps}}$. For $x > 0$, $x_{\text{intersection}} = \sqrt{\frac{2d}{ps}} = x^*$. From this result, the average cost per day is minimized when the average daily cost of delivery is equal to the average daily cost of storage.

50. Average Cost: $\frac{c(x)}{x} = \frac{2000}{x} + 96 + 4x^{1/2} \Rightarrow \frac{d}{dx}\left(\frac{c(x)}{x}\right) = -\frac{2000}{x^2} + 2x^{-1/2} = 0 \Rightarrow x = 100$. Check for a minimum: $\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right)\Big|_{x=100} = \frac{4000}{100^3} - 100^{-3/2} = 0.003 > 0 \Rightarrow$ a minimum at $x = 100$. At a production level of 100,000 units, the average cost will be minimized at \$156 per unit.

51. We have $\frac{dR}{dM} = CM - M^2$. Solving $\frac{d^2R}{dM^2} = C - 2M = 0 \Rightarrow M = \frac{C}{2}$. Also, $\frac{d^3R}{dM^3} = -2 < 0 \Rightarrow$ at $M = \frac{C}{2}$ there is a maximum.

52. (a) If $v = cr_0r^2 - cr^3$, then $v' = 2cr_0r - 3cr^2 = cr(2r_0 - 3r)$ and $v'' = 2cr_0 - 6cr = 2c(r_0 - 3r)$. The solution of $v' = 0$ is $r = 0$ or $\frac{2r_0}{3}$, but 0 is not in the domain. Also, $v' > 0$ for $r < \frac{2r_0}{3}$ and $v' < 0$ for $r > \frac{2r_0}{3} \Rightarrow$ at $r = \frac{2r_0}{3}$ there is a maximum.

(b)



53. If $x > 0$, then $(x-1)^2 \geq 0 \Rightarrow x^2 + 1 \geq 2x \Rightarrow \frac{x^2+1}{x} \geq 2$. In particular if a, b, c and d are positive integers, then $\left(\frac{a^2+1}{a}\right)\left(\frac{b^2+1}{b}\right)\left(\frac{c^2+1}{c}\right)\left(\frac{d^2+1}{d}\right) \geq 16$.

54. (a) $f(x) = \frac{x}{\sqrt{a^2+x^2}} \Rightarrow f'(x) = \frac{\left(a^2+x^2\right)^{1/2} - x^2\left(a^2+x^2\right)^{-1/2}}{\left(a^2+x^2\right)} = \frac{a^2+x^2-x^2}{\left(a^2+x^2\right)^{3/2}} = \frac{a^2}{\left(a^2+x^2\right)^{3/2}} > 0$

$\Rightarrow f(x)$ is an increasing function of x

(b) $g(x) = \frac{d-x}{\sqrt{b^2+(d-x)^2}} \Rightarrow g'(x) = \frac{-\left(b^2+(d-x)^2\right)^{1/2} + (d-x)^2\left(b^2+(d-x)^2\right)^{-1/2}}{b^2+(d-x)^2}$

$= \frac{-\left(b^2+(d-x)^2\right) + (d-x)^2}{\left(b^2+(d-x)^2\right)^{3/2}} = \frac{-b^2}{\left(b^2+(d-x)^2\right)^{3/2}} < 0 \Rightarrow g(x)$ is a decreasing function of x

(c) Since $c_1, c_2 > 0$, the derivative $\frac{dt}{dx}$ is an increasing function of x (from part (a)) minus a decreasing function of x (from part (b)): $\frac{dx}{dt} = \frac{1}{c_1}f(x) - \frac{1}{c_2}g(x) \Rightarrow \frac{d^2x}{dt^2} = \frac{1}{c_1}f'(x) - \frac{1}{c_2}g'(x) > 0$ since $f'(x) > 0$ and $g'(x) < 0 \Rightarrow \frac{dx}{dt}$ is an increasing function of x.

55. At $x = c$, the tangents to the curves are parallel. Justification: The vertical distance between the curves is $D(x) = f(x) - g(x)$, so $D'(x) = f'(x) - g'(x)$. The maximum value of D will occur at a point c where $D' = 0$. At such a point, $f'(c) - g'(c) = 0$, or $f'(c) = g'(c)$.

56. (a) $f(x) = 3 + 4\cos x + \cos 2x$ is a periodic function with period 2π

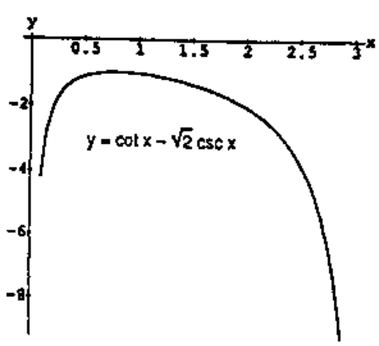
(b) No, $f(x) = 3 + 4\cos x + \cos 2x = 3 + 4\cos x + \left(2\cos^2 x - 1\right) = 2\left(1 + 2\cos x + \cos^2 x\right) = 2(1+\cos x)^2 \geq 0$

$\Rightarrow f(x)$ is never negative

57. (a) If $y = \cot x - \sqrt{2}\csc x$ where $0 < x < \pi$, then $y' = (\csc x)\left(\sqrt{2}\cot x - \csc x\right)$. Solving $y' = 0$

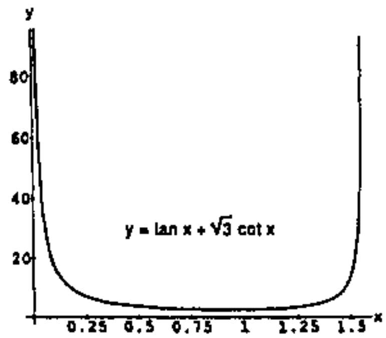
$\Rightarrow \cos x = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{\pi}{4}$. For $0 < x < \frac{\pi}{4}$ we have $y' > 0$, and $y' < 0$ when $\frac{\pi}{4} < x < \pi$. Therefore, at $x = \frac{\pi}{4}$ there is a maximum value of $y = -1$.

(b)



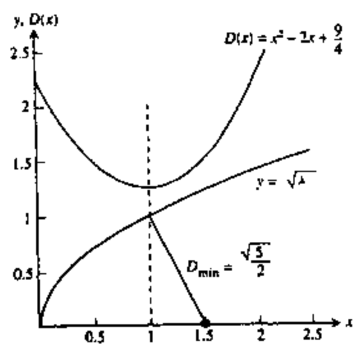
58. (a) If $y = \tan x + 3 \cot x$ where $0 < x < \frac{\pi}{2}$, then $y' = \sec^2 x - 3 \csc^2 x$. Solving $y' = 0 \Rightarrow \tan x = \pm\sqrt{3}$ $\Rightarrow x = \pm\frac{\pi}{3}$, but $-\frac{\pi}{3}$ is not in the domain. Also, $y'' = 2 \sec^2 x \tan x + 6 \csc^2 x \cot x > 0$ for all $0 < x < \frac{\pi}{2}$. Therefore at $x = \frac{\pi}{3}$ there is a minimum value of $y = 2\sqrt{3}$.

(b)



59. (a) The square of the distance is $D(x) = \left(x - \frac{3}{2}\right)^2 + \left(\sqrt{x} + 0\right)^2 = x^2 - 2x + \frac{9}{4}$, so $D'(x) = 2x - 2$ and the critical point occurs at $x = 1$. Since $D'(x) < 0$ for $x < 1$ and $D'(x) > 0$ for $x > 1$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(1)} = \frac{\sqrt{5}}{2}$.

(b)



The minimum distance is from the point $(3/2, 0)$ to the point $(1, 1)$ on the graph of $y = \sqrt{x}$, and this occurs at the value $x = 1$ where $D(x)$, the distance squared, has its minimum value.

60. (a) Calculus method:

The square of the distance from the point $(1, \sqrt{3})$ to $\left(x, \sqrt{16 - x^2}\right)$ is given by

$D(x) = (x - 1)^2 + \left(\sqrt{16 - x^2} - \sqrt{3}\right)^2 = x^2 - 2x + 1 + 16 - x^2 - 2\sqrt{48 - 3x^2} + 3 = -2x + 20 - 2\sqrt{48 - 3x^2}$.

Then $D'(x) = -2 - \frac{2}{2\sqrt{48 - 3x^2}}(-6x) = -2 + \frac{6x}{\sqrt{48 - 3x^2}}$. Solving $D'(x) = 0$ we have:

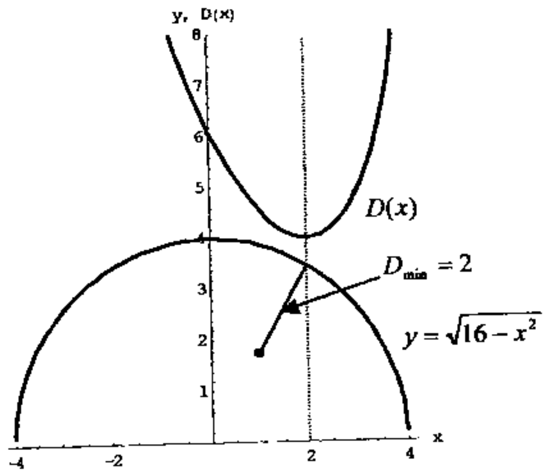
$$
\begin{aligned}
6x &= 2\sqrt{48 - 3x^2} \\
36x^2 &= 4(48 - 3x^2) \\
9x^2 &= 48 - 3x^2 \\
12x^2 &= 48 \\
x &= \pm 2
\end{aligned}
$$

We discard $x = -2$ as an extraneous solution, leaving $x = 2$. Since $D'(x) < 0$ for $-4 < x < 2$ and $D'(x) > 0$ for $2 < x < 4$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(2)} = 2$.

Geometry method:

The semicircle is centered at the origin and has radius 4. The distance from the origin to $(1, \sqrt{3})$ is $\sqrt{1^2 + (\sqrt{3})^2} = 2$. The shortest distance from the point to the semicircle is the distance along the radius containing the point $(1, \sqrt{3})$. That distance is $4 - 2 = 2$.

(b)



The minimum distance is from the point $(1, \sqrt{3})$ to the point $(2, 2\sqrt{2})$ on the graph of $y = \sqrt{16 - x^2}$, and this occurs at the value $x = 2$ where $D(x)$, the distance squared, has its minimum value.

61. (a) The base radius of the cone is $r = \frac{2\pi a - x}{2\pi}$ and so the height is $h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \left(\frac{2\pi a - x}{2\pi}\right)^2}$.

Therefore, $V(x) = \frac{\pi}{3} r^2 h = \frac{\pi}{3}\left(\frac{2\pi a - x}{2\pi}\right)^2 \sqrt{a^2 - \left(\frac{2\pi a - x}{2\pi}\right)^2}$.

(b) To simplify the calculations, we shall consider the volume as a function of r:

volume $= f(r) = \frac{\pi}{3} r^2 \sqrt{a^2 - r^2}$, where $0 < r < a$.

$$f'(r) = \frac{\pi}{3} \frac{d}{dr} \sqrt{r^2(a^2 - r^2)}$$

$$= \frac{\pi}{3}\left[r^2 \cdot \frac{1}{2\sqrt{a^2 - r^2}} \cdot (-2r) + \left(\sqrt{a^2 - r^2}\right)(2r)\right]$$

$$= \frac{\pi}{3}\left[\frac{-r^3 + 2r\left(a^2 - r^2\right)}{\sqrt{a^2 - r^2}}\right]$$

$$= \frac{\pi}{3}\left[\frac{2a^2r - 3r^3}{\sqrt{a^2 - r^2}}\right]$$

$$= \frac{\pi r(2a^2 - 3r^2)}{3\sqrt{a^2 - r^2}}$$

The critical point occurs when $r^2 = \frac{2a^2}{3}$, which gives $r = a\sqrt{\frac{2}{3}} = \frac{a\sqrt{6}}{3}$. Then

$h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \frac{2a^2}{3}} = \sqrt{\frac{a^2}{3}} = \frac{a\sqrt{3}}{3}$. Using $r = \frac{a\sqrt{6}}{3}$ and $h = \frac{a\sqrt{3}}{3}$, we may now find the values of r and h for the given values of a.

When $a = 4$: $r = \frac{4\sqrt{6}}{3}$, $h = \frac{4\sqrt{3}}{3}$;

When $a = 5$: $r = \frac{5\sqrt{6}}{3}$, $h = \frac{5\sqrt{3}}{3}$;

When $a = 6$: $r = 2\sqrt{6}$, $h = 2\sqrt{3}$;

When $a = 8$: $r = \frac{8\sqrt{6}}{3}$, $h = \frac{8\sqrt{3}}{3}$

(c) Since $r = \frac{a\sqrt{6}}{3}$ and $h = \frac{a\sqrt{3}}{3}$, the relationship is $\frac{r}{h} = \sqrt{2}$.

62. (a) Let x_0 represent the fixed value of x at point P, so that P has coordinates (x_0, a), and let $m = f'(x_0)$ be the slope of line RT. Then the equation of line RT is $y = m(x - x_0) + a$. The y-intercept of this line is $m(0 - x_0) + a = a - mx_0$, and the x-intercept is the solution of $m(x - x_0) + a = 0$, or $x = \frac{mx_0 - a}{m}$. Let O designate the origin. Then

(Area of triangle RST)

$= 2$(Area of triangle ORT)

$= 2 \cdot \frac{1}{2}$(x-intercept of line RT)(y-intercept of line RT)

$$= 2 \cdot \frac{1}{2}\left(\frac{mx_0 - a}{m}\right)(a - mx_0)$$

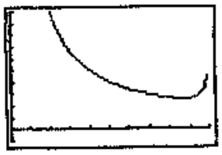
$$= -m\left(\frac{mx_0 - a}{m}\right)\left(\frac{mx_0 - a}{m}\right)$$

$$= -m\left(\frac{mx_0 - a}{m}\right)^2$$

$$= -m\left(x_0 - \frac{a}{m}\right)^2$$

Substituting x for x_0, $f'(x)$ for m, and $f(x)$ for a, we have $A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2$.

(b) The domain is the open interval $(0, 10)$. To graph, let $y_1 = f(x) = 5 + 5\sqrt{1 - \frac{x^2}{100}}$, $y_2 = f'(x) = \text{NDER}(y_1)$, and $y_3 = A(x) = -y_2\left(x - \frac{y_1}{y_2}\right)^2$. The graph of the area function $y_3 = A(x)$ is shown below.



[0, 10] by [−100, 1000]

The vertical asymptotes at $x = 0$ and $x = 10$ correspond to horizontal or vertical tangent lines, which do not form triangles.

(c) Using our expression for the y-intercept of the tangent line, the height of the triangle is

$$\begin{aligned} a - mx &= f(x) - f'(x) \cdot x \\ &= 5 + \tfrac{1}{2}\sqrt{100 - x^2} - \frac{-x}{2\sqrt{100 - x^2}}x \\ &= 5 + \tfrac{1}{2}\sqrt{100 - x^2} + \frac{x^2}{2\sqrt{100 - x^2}} \end{aligned}$$

We may use graphing methods or the analytic method in part (d) to find that the minimum value of $A(x)$ occurs at $x \approx 8.66$. Substituting this value into the expression above, the height of the triangle is 15. This is 3 times the y-coordinate of the center of the ellipse.

(d) Part (a) remains unchanged. The domain is $(0, C)$. To graph, note that

$f(x) = B + B\sqrt{1 - \frac{x^2}{C^2}} = B + \frac{B}{C}\sqrt{C^2 - x^2}$ and $f'(x) = \frac{B}{C}\frac{1}{2\sqrt{C^2 - x^2}}(-2x) = \frac{-Bx}{C\sqrt{C^2 - x^2}}$. Therefore we have

$$\begin{aligned} A(x) &= -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2 = \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{B + \frac{B}{C}\sqrt{C^2 - x^2}}{\frac{-Bx}{C\sqrt{C^2 - x^2}}}\right]^2 \\ &= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{\left(BC + B\sqrt{C^2 - x^2}\right)\left(\sqrt{C^2 - x^2}\right)}{-Bx}\right]^2 \\ &= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + \left(BC + B\sqrt{C^2 - x^2}\right)\left(\sqrt{C^2 - x^2}\right)\right]^2 \\ &= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + BC\sqrt{C^2 - x^2} + B\left(C^2 - x^2\right)\right]^2 \\ &= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[BC\left(C + \sqrt{C^2 - x^2}\right)\right]^2 \\ &= \frac{BC\left(C + \sqrt{C^2 - x^2}\right)^2}{x\sqrt{C^2 - x^2}} \end{aligned}$$

$$A'(x) = BC \cdot \frac{\left(x\sqrt{C^2-x^2}\right)(2)\left(C+\sqrt{C^2-x^2}\right)\left(\frac{-x}{\sqrt{C^2-x^2}}\right) - \left(C+\sqrt{C^2-x^2}\right)^2\left(x\frac{-x}{\sqrt{C^2-x^2}} + \sqrt{C^2-x^2}(1)\right)}{x^2\left(C^2-x^2\right)}$$

$$= \frac{BC\left(C+\sqrt{C^2-x^2}\right)}{x^2\left(C^2-x^2\right)}\left[-2x^2 - \left(C+\sqrt{C^2-x^2}\right)\left(\frac{-x^2}{\sqrt{C^2-x^2}} + \sqrt{C^2-x^2}\right)\right]$$

$$= \frac{BC\left(C+\sqrt{C^2-x^2}\right)}{x^2\left(C^2-x^2\right)}\left[-2x^2 + \frac{Cx^2}{\sqrt{C^2-x^2}} - C\sqrt{C^2-x^2} + x^2 - \left(C^2-x^2\right)\right]$$

$$= \frac{BC\left(C+\sqrt{C^2-x^2}\right)}{x^2\left(C^2-x^2\right)}\left(\frac{Cx^2}{\sqrt{C^2-x^2}} - C\sqrt{C^2-x^2} - C^2\right)$$

$$= \frac{BC\left(C+\sqrt{C^2-x^2}\right)}{x^2\left(C^2-x^2\right)^{3/2}}\left[Cx^2 - C\left(C^2-x^2\right) - C^2\sqrt{C^2-x^2}\right]$$

$$= \frac{BC^2\left(C+\sqrt{C^2-x^2}\right)}{x^2\left(C^2-x^2\right)^{3/2}}\left(2x^2 - C^2 - C\sqrt{C^2-x^2}\right)$$

To find the critical points for $0 < x < C$, we solve:

$$\begin{aligned} 2x^2 - C^2 &= C\sqrt{C^2-x^2} \\ 4x^4 - 4C^2x^2 + C^4 &= C^4 - C^2x^2 \\ 4x^4 - 3C^2x^2 &= 0 \\ x^2(4x^2 - 3C^2) &= 0 \end{aligned}$$

The minimum value of $A(x)$ for $0 < x < C$ occurs at the critical point $x = \frac{C\sqrt{3}}{2}$, or $x^2 = \frac{3C^2}{4}$. The corresponding triangle height is

$$\begin{aligned} a - mx &= f(x) - f'(x) \cdot x \\ &= B + \frac{B}{C}\sqrt{C^2-x^2} + \frac{Bx^2}{C\sqrt{C^2-x^2}} \\ &= B + \frac{B}{C}\sqrt{C^2 - \frac{3C^2}{4}} + \frac{B\left(\frac{3C^2}{4}\right)}{C\sqrt{C^2 - \frac{3C^2}{4}}} \\ &= B + \frac{B}{C}\left(\frac{C}{2}\right) + \frac{\frac{3BC^2}{4}}{\frac{C^2}{2}} \\ &= B + \frac{B}{2} + \frac{3B}{2} \\ &= 3B \end{aligned}$$

This shows that the triangle has minimum area when its height is 3B.

3.6 LINEARIZATION AND DIFFERENTIALS

1. $f(x) = x^3 - 2x + 3 \Rightarrow f'(x) = 3x^2 - 2 \Rightarrow L(x) = f'(2)(x-2) + f(2) = 10(x-2) + 7 \Rightarrow L(x) = 10x - 13$ at $x = 2$

2. $f(x) = \sqrt{x^2+9} = (x^2+9)^{1/2} \Rightarrow f'(x) = \left(\frac{1}{2}\right)(x^2+9)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+9}} \Rightarrow L(x) = f'(-4)(x+4) + f(-4)$
$= -\frac{4}{5}(x+4) + 5 \Rightarrow L(x) = -\frac{4}{5}x + \frac{9}{5}$ at $x = -4$

3. $f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - x^{-2} \Rightarrow L(x) = f(1) + f'(1)(x-1) = 2 + 0(x-1) = 2$

4. $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3x^{2/3}} \Rightarrow L(x) = f'(-8)(x-(-8)) + f(-8) = \frac{1}{12}(x+8) - 2 \Rightarrow L(x) = \frac{x}{12} - \frac{4}{3}$

5. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow L(x) = f(\pi) + f'(\pi)(x-\pi) = 0 + 1(x-\pi) = x - \pi$

6. (a) $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow L(x) = f'(0)(x-0) + f(0) = x \Rightarrow L(x) = x$
(b) $f(x) = \cos x \Rightarrow f'(x) = -\sin x \Rightarrow L(x) = f'(0)(x-0) + f(0) = 1 \Rightarrow L(x) = 1$
(c) $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow L(x) = f'(0)(x-0) + f(0) = x \Rightarrow L(x) = x$

7. $f'(x) = k(1+x)^{k-1}$
We have $f(0) = 1$ and $f'(0) = k$. $L(x) = f(0) + f'(0)(x-0) = 1 + k(x-0) = 1 + kx$

8. (a) $f(x) = (1-x)^6 = [1+(-x)]^6 \approx 1 + 6(-x) = 1 - 6x$
(b) $f(x) = \frac{2}{1-x} = 2[1+(-x)]^{-1} \approx 2[1+(-1)(-x)] = 2 + 2x$
(c) $f(x) = (1+x)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)x = 1 - \frac{x}{2}$
(d) $f(x) = \sqrt{2+x^2} = \sqrt{2}\left(1+\frac{x^2}{2}\right)^{1/2} \approx \sqrt{2}\left(1+\frac{1}{2}\frac{x^2}{2}\right) = \sqrt{2}\left(1+\frac{x^2}{4}\right)$
(e) $f(x) = (4+3x)^{1/3} = 4^{1/3}\left(1+\frac{3x}{4}\right)^{1/3} \approx 4^{1/3}\left(1+\frac{1}{3}\frac{3x}{4}\right) = 4^{1/3}\left(1+\frac{x}{4}\right)$
(f) $f(x) = \left(1-\frac{1}{2+x}\right)^{2/3} = \left[1+\left(-\frac{1}{2+x}\right)\right]^{2/3} \approx 1 + \frac{2}{3}\left(-\frac{1}{2+x}\right) = 1 - \frac{2}{6+3x}$

9. Center $= -1$
$f'(x) = 4x + 4$
We have $f(-1) = -5$ and $f'(-1) = 0$
$L(x) = f(-1) + f'(-1)(x-(-1)) = -5 + 0(x+1) = -5$

10. Center $= 8$
$f'(x) = \frac{1}{3}x^{-2/3}$
We have $f(8) = 2$ and $f'(8) = \frac{1}{12}$.
$L(x) = f(8) + f'(8)(x-8) = 2 + \frac{1}{12}(x-8) = \frac{x}{12} + \frac{4}{3}$

11. Center = 1

$f'(x) = \frac{(x+1)(1) - (x)(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$

We have $f(1) = \frac{1}{2}$ and $f'(1) = \frac{1}{4}$

$L(x) = f(1) + f'(1)(x-1) = \frac{1}{2} + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{1}{4}$

Alternate solution:

Using center $= \frac{3}{2}$, we have $f\left(\frac{3}{2}\right) = \frac{3}{5}$ and $f'\left(\frac{3}{2}\right) = \frac{4}{25}$.

$L(x) = f\left(\frac{3}{2}\right) + f'\left(\frac{3}{2}\right)\left(x - \frac{3}{2}\right) = \frac{3}{5} + \frac{4}{25}\left(x - \frac{3}{2}\right) = \frac{4}{25}x + \frac{9}{25}$

12. Center $= \frac{\pi}{2}$

$f'(x) = -\sin x$

We have $f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = -1$.

$L(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) = 0 - 1\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}$

13. (a) $(1.0002)^{50} = (1 + 0.0002)^{50} \approx 1 + 50(0.0002) = 1 + .01 = 1.01$

(b) $\sqrt[3]{1.009} = (1 + 0.009)^{1/3} \approx 1 + \left(\frac{1}{3}\right)(0.009) = 1 + 0.003 = 1.003$

14. $f(x) = \sqrt{x+1} + \sin x = (x+1)^{1/2} + \sin x \Rightarrow f'(x) = \left(\frac{1}{2}\right)(x+1)^{-1/2} + \cos x \Rightarrow L_f(x) = f'(0)(x-0) + f(0)$ $= \frac{3}{2}(x-0) + 1 \Rightarrow L_f(x) = \frac{3}{2}x + 1$, the linearization of f(x); $g(x) = \sqrt{x+1} = (x+1)^{1/2} \Rightarrow g'(x)$ $= \left(\frac{1}{2}\right)(x+1)^{-1/2} \Rightarrow L_g(x) = g'(0)(x-0) + g(0) = \frac{1}{2}(x-0) + 1 \Rightarrow L_g(x) = \frac{1}{2}x + 1$, the linearization of g(x); $h(x) = \sin x \Rightarrow h'(x) = \cos x \Rightarrow L_h(x) = h'(0)(x-0) + h(0) = (1)(x-0) + 0 \Rightarrow L_h(x) = x$, the linearization of h(x). $L_f(x) = L_g(x) + L_h(x)$ implies that the linearization of a sum is equal to the sum of the linearizations.

15. $y = x^3 - 3\sqrt{x} = x^3 - 3x^{1/2} \Rightarrow dy = \left(3x^2 - \frac{3}{2}x^{-1/2}\right)dx \Rightarrow dy = \left(3x^2 - \frac{3}{2\sqrt{x}}\right)dx$

16. $y = x\sqrt{1-x^2} = x\left(1-x^2\right)^{1/2} \Rightarrow dy = \left[(1)\left(1-x^2\right)^{1/2} + (x)\left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)\right]dx$

$= \left(1-x^2\right)^{-1/2}\left[\left(1-x^2\right) - x^2\right]dx = \frac{\left(1-2x^2\right)}{\sqrt{1-x^2}}\,dx$

17. $y = \frac{2x}{1+x^2} \Rightarrow dy = \left(\frac{(2)\left(1+x^2\right) - (2x)(2x)}{\left(1+x^2\right)^2}\right)dx = \frac{2-2x^2}{\left(1+x^2\right)^2}\,dx$

18. $y = \frac{2\sqrt{x}}{3(1+\sqrt{x})} = \frac{2x^{1/2}}{3(1+x^{1/2})} \Rightarrow dy = \left(\frac{x^{-1/2}(3(1+x^{1/2})) - 2x^{1/2}\left(\frac{3}{2}x^{-1/2}\right)}{9(1+x^{1/2})^2}\right) dx = \frac{3x^{-1/2}+3-3}{9(1+x^{1/2})^2}\, dx$

$\Rightarrow dy = \frac{1}{3\sqrt{x}(1+\sqrt{x})^2}\, dx$

19. $2y^{3/2} + xy - x = 0 \Rightarrow 3y^{1/2}\, dy + y\, dx + x\, dy - dx = 0 \Rightarrow \left(3y^{1/2} + x\right) dy = (1-y)\, dx \Rightarrow dy = \frac{1-y}{3\sqrt{y}+x}\, dx$

20. $xy^2 - 4x^{3/2} - y = 0 \Rightarrow y^2\, dx + 2xy\, dy - 6x^{1/2}\, dx - dy = 0 \Rightarrow (2xy-1)\, dy = \left(6x^{1/2} - y^2\right) dx$

$\Rightarrow dy = \frac{6\sqrt{x} - y^2}{2xy-1}\, dx$

21. $y = \sin(5\sqrt{x}) = \sin(5x^{1/2}) \Rightarrow dy = (\cos(5x^{1/2}))\left(\frac{5}{2}x^{-1/2}\right) dx \Rightarrow dy = \frac{5\cos(5\sqrt{x})}{2\sqrt{x}}\, dx$

22. $y = \cos(x^2) \Rightarrow dy = [-\sin(x^2)](2x)\, dx = -2x\sin(x^2)\, dx$

23. $y = 4\tan\left(\frac{x^3}{3}\right) \Rightarrow dy = 4\left(\sec^2\left(\frac{x^3}{3}\right)\right)(x^2)\, dx \Rightarrow dy = 4x^2\sec^2\left(\frac{x^3}{3}\right) dx$

24. $y = \sec(x^2-1) \Rightarrow dy = [\sec(x^2-1)\tan(x^2-1)](2x)\, dx = 2x[\sec(x^2-1)\tan(x^2-1)]\, dx$

25. (a) $\Delta f = f(0.1) - f(0) = 0.21 - 0 = 0.21$
(b) Since $f'(x) = 2x+2$, $f'(0) = 2$.
Therefore, $df = 2\, dx = 2(0.1) = 0.2$.
(c) $|\Delta f - df| = |0.21 - 0.2| = 0.01$

26. (a) $\Delta f = f(1.1) - f(1) = 0.231 - 0 = 0.231$
(b) Since $f'(x) = 3x^2 - 1$, $f'(1) = 2$.
Therefore, $df = 2\, dx = 2(0.1) = 0.2$.
(c) $|\Delta f - df| = |0.231 - 0.2| = 0.031$

27. (a) $\Delta f = f(0.55) - f(0.5) = \frac{20}{11} - 2 = -\frac{2}{11}$
(b) Since $f'(x) = -x^{-2}$, $f'(0.5) = -4$.
Therefore, $df = -4\, dx = -4(0.05) = -0.2 = -\frac{1}{5}$
(c) $|\Delta f - df| = \left|-\frac{2}{11} + \frac{1}{5}\right| = \frac{1}{55}$

28. (a) $\Delta f = f(1.01) - f(1) = 1.04060401 - 1 = 0.04060401$
(b) Since $f'(x) = 4x^3$, $f'(1) = 4$.
Therefore, $df = 4\, dx = 4(0.01) = 0.04$.
(c) $|\Delta f - df| = |0.04060401 - 0.04| = 0.00060401$

29. Note that $\frac{dV}{dr} = 4\pi r^2$, so that $dV = 4\pi r^2\, dr$. When r changes from a to $a + dr$ the change in volume is approximately $4\pi a^2\, dr$.

30. Note that $\frac{dS}{dr} = 8\pi r$, so $dS = 8\pi r\ dr$. When r changes from a to $a + dr$, the change in surface area is approximately $8\pi a\ dr$.

31. Note that $\frac{dV}{dx} = 3x^2$, so $dV = 3x^2\ dx$. When x changes from a to $a + dx$, the change in volume is approximately $3a^2\ dx$.

32. Note that $\frac{dS}{dx} = 12x$, so $dS = 12x\ dx$. When x changes from a to $a + dx$, the change in surface area is approximately $12a\ dx$.

33. Given $r = 2$ m, $dr = .02$ m
(a) $A = \pi r^2 \Rightarrow dA = 2\pi r\ dr = 2\pi(2)(.02) = .08\pi\ m^2$
(b) $\left(\frac{.08\pi}{4\pi}\right)(100\%) = 2\%$

34. $C = 2\pi r$ and $dC = 2$ in. $\Rightarrow dC = 2\pi\ dr \Rightarrow dr = \frac{1}{\pi} \Rightarrow$ the diameter grew about $\frac{2}{\pi}$ in.; $A = \pi r^2 \Rightarrow dA = 2\pi r\ dr$ $= 2\pi(5)\left(\frac{1}{\pi}\right) = 10$ in.^{2}

35. The volume of a cylinder is $V = \pi r^2 h$. When h is held fixed, we have $\frac{dV}{dr} = 2\pi rh$, and so $dV = 2\pi rh\ dr$. For $h = 30$ in., $r = 6$ in., and $dr = 0.5$ in., the thickness of the shell is approximately $dV = 2\pi rh\ dr = 2\pi(6)(30)(0.5)$ $= 180\pi \approx 565.5$ in.^{3}

36. Let $\theta =$ angle of elevation and $h =$ height of building. Then $h = 30\tan\theta$, so $dh = 30\sec^2\theta\ d\theta$. We want $|dh| < 0.04h$, which gives:

$$\left|30\sec^2\theta\ d\theta\right| < 0.04\left|30\tan\theta\right|$$

$$\frac{1}{\cos^2\theta}\left|d\theta\right| < \frac{0.04\sin\theta}{\cos\theta} \Rightarrow \left|d\theta\right| < 0.04\sin\theta\cos\theta$$

$\left|d\theta\right| < 0.04\sin\frac{5\pi}{12}\cos\frac{5\pi}{12} = 0.01$ radian. The angle should be measured with an error of less than 0.01 radian (or approximately 0.57 degrees), which is a percentage error of approximately 0.76%

37. $V = \pi h^3 \Rightarrow dV = 3\pi h^2\ dh$; recall that $\Delta V \approx dV$. Then $|\Delta V| \le (1\%)(V) = \frac{(1)(\pi h^3)}{100} \Rightarrow |dV| \le \frac{(1)(\pi h^3)}{100}$ $\Rightarrow |3\pi h^2\ dh| \le \frac{(1)(\pi h^3)}{100} \Rightarrow |dh| \le \frac{1}{300}h = \left(\frac{1}{3}\%\right)h$. Therefore the greatest tolerated error in the measurement of h is $\frac{1}{3}\%$.

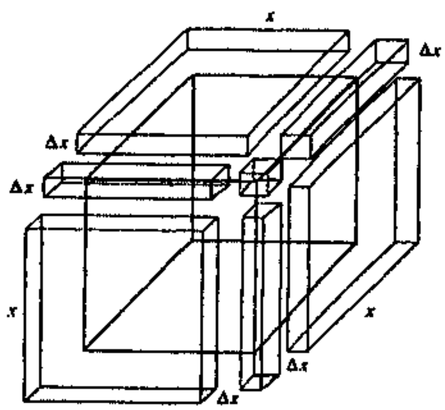
38. (a) Let D_i represent the inside diameter. Then $V = \pi r^2 h = \pi\left(\frac{D_i}{2}\right)^2 h = \frac{\pi D_i^2 h}{4}$ and $h = 10 \Rightarrow V = \frac{5\pi D_i^2}{2} \Rightarrow$ $dV = 5\pi D_i\ dD_i$. Recall that $\Delta V \approx dV$. We want $|\Delta V| \le (1\%)(V) \Rightarrow |dV| \le \left(\frac{1}{100}\right)\left(\frac{5\pi D_i^2}{2}\right) = \frac{\pi D_i^2}{40}$

$\Rightarrow |5\pi D_i\ dD_i| \le \frac{\pi D_i^2}{40} \Rightarrow |dD_i| \le \frac{D_i}{200} = \left(\frac{1}{2}\%\right) D_i \Rightarrow$ the measurement must have an error less than $\frac{1}{2}\%$.

(b) Let D_e represent the exterior diameter. Then $S = 2\pi rh = \frac{2\pi D_e h}{2} = \pi D_e h$, when $h = 10 \Rightarrow S = 10\pi D_e$ $\Rightarrow dS = 10\pi\ dD_e$. Recall that $\Delta S \approx dS$. We want $|\Delta S| \le (5\%)(S) \Rightarrow |dS| \le \left(\frac{5}{100}\right)(10\pi D_e) \Rightarrow |10\pi\ dD_e| \le \frac{\pi D_e}{2} \Rightarrow |dD_e| \le \frac{D_e}{20} = (5\%)\, D_e \Rightarrow$ the measurement must have an error less than 5%.

39. $V = \pi r^2 h$, h is constant $\Rightarrow dV = 2\pi rh\ dr$; recall that $\Delta V \approx dV$. We want $|\Delta V| \le \frac{1}{1000} V \Rightarrow |dV| \le \frac{\pi r^2 h}{1000}$ $\Rightarrow |2\pi rh\ dr| \le \frac{\pi r^2 h}{1000} \Rightarrow |dr| \le \frac{r}{2000} = (0.05\%)r \Rightarrow$ a 0.05% variation in the radius can be tolerated.

40. $\text{Volume} = (x + \Delta x)^3 = x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$



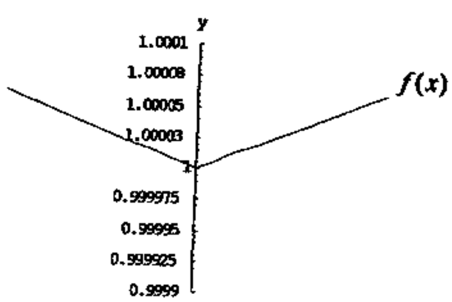
41. $W = a + \frac{b}{g} = a + bg^{-1} \Rightarrow dW = -bg^{-2}\ dg = -\frac{b\ dg}{g^2} \Rightarrow \frac{dW_{moon}}{dW_{earth}} = \frac{\left(-\frac{b\ dg}{(5.2)^2}\right)}{\left(-\frac{b\ dg}{(32)^2}\right)} = \left(\frac{32}{5.2}\right)^2 = 37.87$, so a change of gravity on the moon has about 38 times the effect that a change of the same magnitude has on Earth.

42. (a) $T = 2\pi\left(\frac{L}{g}\right)^{1/2} \Rightarrow dT = 2\pi\sqrt{L}\left(-\frac{1}{2}g^{-3/2}\right) dg = -\pi\sqrt{L}\, g^{-3/2}\ dg$

(b) If g increases, then $dg > 0 \Rightarrow dT < 0$. The period T decreases and the clock ticks more frequently. Both the pendulum speed and clock speed increase.

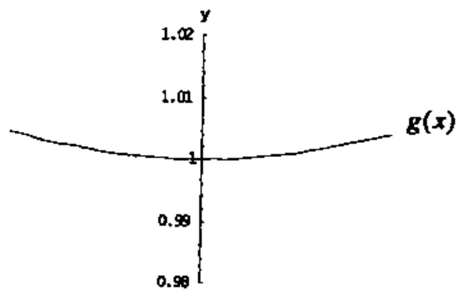
(c) $0.001 = -\pi\sqrt{100}\left(980^{-3/2}\right) dg \Rightarrow dg \approx -0.977$ cm/sec^2 $\Rightarrow$ the new $g \approx 979$ cm/sec^2

43. (a) Window: $-0.00006 \le x \le 0.00006$, $0.9999 \le y \le 1.0001$



After zooming in seven times, starting with the window $-1 \le x \le 1$ and $0 \le y \le 2$ on a TI-92 Plus calculator, the graph of f(x) shows no signs of straightening out.

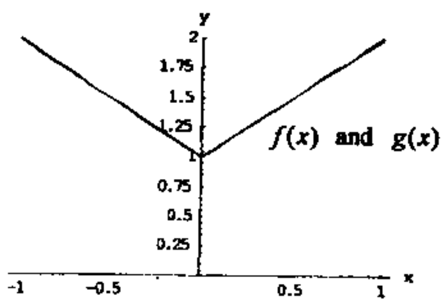
(b) Window: $-0.01 \le x \le 0.01$, $0.98 \le y \le 1.02$



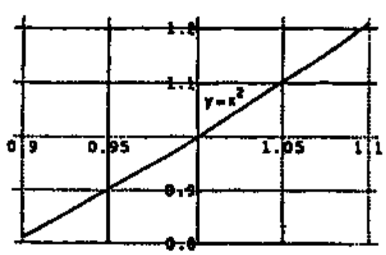
After zooming in only twice, starting with the window $-1 \le x \le 1$ and $0 \le y \le 2$ on a TI-92 Plus calculator, the graph of g(x) already appears to be smoothing toward a horizontal straight line.

(c) After seven zooms, starting with the window $-1 \le x \le 1$ and $0 \le y \le 2$ on a TI-92 Plus calculator, the graph of g(x) looks exactly like a horizontal straight line.

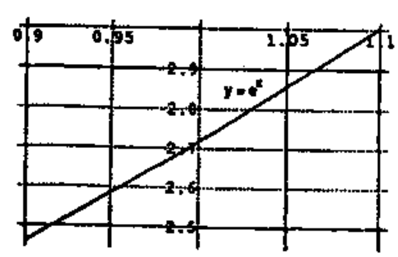
(d) Window: $-1 \le x \le 1$, $0 \le y \le 2$



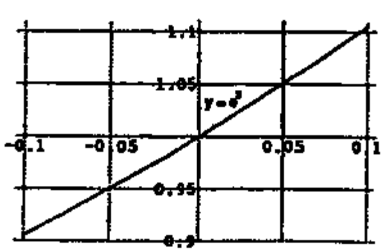
44. (a) $y = x^2$ at $x = 1$



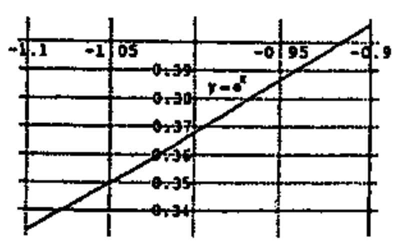
(b) $y = e^x$ at $x = 1$



(b) $y = e^x$ at $x = 0$



(b) $y = e^x$ at $x = -1$



45. $E(x) = f(x) - g(x) \Rightarrow E(x) = f(x) - m(x-a) - c$. Then $E(a) = 0 \Rightarrow f(a) - m(a-a) - c = 0 \Rightarrow c = f(a)$. Next we calculate m: $\lim_{x\to a} \frac{E(x)}{x-a} = 0 \Rightarrow \lim_{x\to a} \frac{f(x) - m(x-a) - c}{x-a} = 0 \Rightarrow \lim_{x\to a} \left[\frac{f(x)-f(a)}{x-a} - m\right] = 0$ (since $c = f(a)$) $\Rightarrow f'(a) - m = 0 \Rightarrow m = f'(a)$. Therefore, $g(x) = m(x-a) + c = f'(a)(x-a) + f(a)$ is the linear approximation, as claimed.

46. (a) i. $Q(a) = f(a)$ implies that $b_0 = f(a)$.

ii. Since $Q'(x) = b_1 + 2b_2(x-a)$, $Q'(a) = f'(a)$ implies that $b_1 = f'(a)$.

iii. Since $Q''(x) = 2b_2$, $Q''(a) = f''(a)$ implies that $b_2 = \frac{f''(a)}{2}$.

In summary, $b_0 = f(a)$, $b_1 = f'(a)$, and $b_2 = \frac{f''(a)}{2}$.

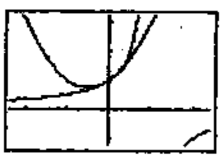
(b) $f(x) = (1-x)^{-1}$

$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$

$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$

Since $f(0) = 1$, $f'(0) = 1$, and $f''(0) = 2$, the coefficients are $b_0 = 1$, $b_1 = 1$, $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 + x + x^2$.

(c)



[−2.35, 2.35] by [−1.25, 5.25]

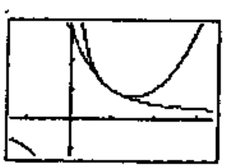
As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(d) $g(x) = x^{-1}$

$g'(x) = -x^{-2}$

$g''(x) = 2x^{-3}$

Since $g(1) = 1$, $g'(1) = -1$, and $g''(1) = 2$, the coefficients are $b_0 = 1$, $b_1 = -1$, and $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 - (x-1) + (x-1)^2$.



[−1.35, 3.35] by [−1.25, 3.25]

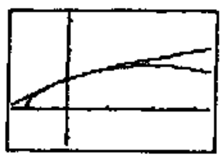
As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(e) $h(x) = (1+x)^{1/2}$

$h'(x) = \frac{1}{2}(1+x)^{-1/2}$

$h''(x) = -\frac{1}{4}(1+x)^{-3/2}$

Since $h(0) = 1$, $h'(0) = \frac{1}{2}$, and $h''(0) = -\frac{1}{4}$, the coefficients are $b_0 = 1$, $b_1 = \frac{1}{2}$, and $b_2 = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$. The quadratic approximation is $Q(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$.



$[-1.35, 3.35]$ by $[-1.25, 3.25]$

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(f) The linearization of any differentiable function $u(x)$ at $x = a$ is $L(x) = u(a) + u'(a)(x-a) = b_0 + b_1(x-a)$, where b_0 and b_1 are the coefficients of the constant and linear terms of the quadratic approximation. Thus, the linearization for $f(x)$ at $x = 0$ is $1 + x$; the linearization for $g(x)$ at $x = 1$ is $1 - (x-1)$ or $2 - x$; and the linearization for $h(x)$ at $x = 0$ is $1 + \frac{x}{2}$.

47. $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x}}{1+\frac{x}{2}} = \dfrac{\sqrt{1+0}}{1+\frac{0}{2}} = 1$

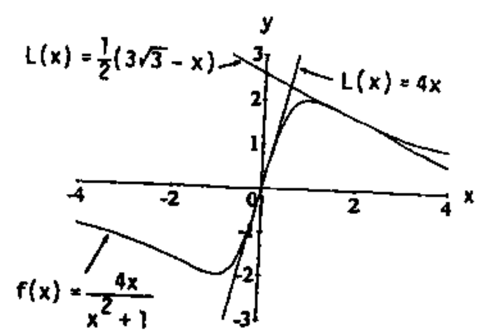
48. If f has a horizontal tangent at $x = a$, then $f'(a) = 0$ and the linearization of f at $x = a$ is $L(x) = f(a) + f'(a)(x-a) = f(a) + 0 \cdot (x-a) = f(a)$. The linearization is a constant.

49. $f(x) = \dfrac{4x}{x^2+1} \Rightarrow f'(x) = \dfrac{4(1-x^2)}{(x^2+1)^2}$;

At $x = 0$: $L(x) = f'(0)(x-0) + f(0) = 4x$;

At $x = \sqrt{3}$: $L(x) = f'(\sqrt{3})(x-\sqrt{3}) + f(\sqrt{3})$

$= \left(-\frac{1}{2}\right)(x-\sqrt{3}) + \sqrt{3} \Rightarrow L(x) = \frac{1}{2}(3\sqrt{3} - x)$



50. (a) $\sqrt{1+x} \approx 1 + \frac{x}{2}$ gives the following: $\sqrt{1+1} \approx 1 + \frac{1}{2} \Rightarrow \sqrt{\sqrt{1+1}} \approx \sqrt{1+\frac{1}{2}}$

$\approx 1 + \frac{1}{4} \Rightarrow \sqrt{\sqrt{\sqrt{1+1}}} \approx \sqrt{1+\frac{1}{4}} \approx 1 + \frac{1}{8}$, and so forth. That is, $\underbrace{\sqrt{\cdots\sqrt{\sqrt{1+1}}}}_{\text{n square roots}} \approx 1 + \frac{1}{2^n} \to 1$ as $n \to \infty$.

For successive tenth roots we obtain the approximation $1+\frac{1}{10n} \to 1$ as $n \to \infty$.

(b) Yes, you can use any positive number in place of 2. Repeating the argument in (a) gives

$\underbrace{\sqrt{\cdots\sqrt{\sqrt{1+x}}}}_{\text{n square roots}} \approx 1+\frac{x}{2n} \to 1+0$ as $n \to \infty$ provided that the number $1+x$ is positive.

51-54. Example CAS commands:

Maple:

```
with(plots):
a:= 1: f:=x -> x ∧ 3 + x ∧ 2 − 2*x;
plot(f(x), x=−1..2);
diff(f(x),x);
fp := unapply (%,x);
L:=x -> f(a) + fp(a)*(x − a);
plot({f(x), L(x)}, x=−1..2);
err:=x -> abs(f(x) − L(x));
plot(err(x), x=−1..2, title = `absolute error function`);
err(−1);
```

Mathematica:

```
Clear[x]
{x1,x2} = {−1,2}; a = 1; f[x_] = x ∧ 3 + x ∧ 2 − 2x
Plot[ f[x], {x,x1,x2} ]
L[x_] = f[a] + f'[a] (x − a)
Plot[ {f[x], L[x]}, {x,x1,x2} ]
err[x_] := Abs[f[x] − L[x]]
Plot[ err[x], {x,x1,x2} ]
err[x1] // N

eps = 0.5; del = 0.3;
Plot[ {err[x],eps} {x,a−del,a+del} ]

eps = 0.1; del = 0.15;
Plot[ {err[x],eps}, {x,a−del,a+del} ]

eps = 0.01; del = 0.05;
Plot[ {err[x],eps}, {x,a−del,a+del} ]
```

3.7 NEWTON'S METHOD

1. $y = x^2+x-1 \Rightarrow y' = 2x+1 \Rightarrow x_{n+1} = x_n - \frac{x_n^2+x_n-1}{2x_n+1}$; $x_0 = 1 \Rightarrow x_1 = 1 - \frac{1+1-1}{2+1} = \frac{2}{3}$

$\Rightarrow x_2 = \frac{2}{3} - \frac{\frac{4}{9}+\frac{2}{3}-1}{\frac{4}{3}+1} \Rightarrow x_2 = \frac{2}{3} - \frac{4+6-9}{12+9} = \frac{2}{3} - \frac{1}{21} = \frac{13}{21} \approx .61905$; $x_0 = -1 \Rightarrow x_1 = 1 - \frac{1-1-1}{-2+1} = -2$

$\Rightarrow x_2 = -2 - \frac{4-2-1}{-4+1} = -\frac{5}{3} \approx -1.66667$

2. $y = x^3 + 3x + 1 \Rightarrow y' = 3x^2 + 3 \Rightarrow x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}$; $x_0 = 0 \Rightarrow x_1 = 0 - \frac{1}{3} = -\frac{1}{3}$

$\Rightarrow x_2 = -\frac{1}{3} - \frac{-\frac{1}{27} - 1 + 1}{\frac{1}{3} + 3} = -\frac{1}{3} + \frac{1}{90} = -\frac{29}{90} \approx -0.32222$

3. $y = x^4 + x - 3 \Rightarrow y' = 4x^3 + 1 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 + x_n - 3}{4x_n^3 + 1}$; $x_0 = 1 \Rightarrow x_1 = 1 - \frac{1 + 1 - 3}{4 + 1} = \frac{6}{5}$

$\Rightarrow x_2 = \frac{6}{5} - \frac{\frac{1296}{625} + \frac{6}{5} - 3}{\frac{864}{125} + 1} = \frac{6}{5} - \frac{1296 + 750 - 1875}{4320 + 625} = \frac{6}{5} - \frac{171}{4945} = \frac{5763}{4945} \approx 1.16542$; $x_0 = -1 \Rightarrow x_1 = -1 - \frac{1 - 1 - 3}{-4 + 1}$

$= -2 \Rightarrow x_2 = -2 - \frac{16 - 2 - 3}{-32 + 1} = -2 + \frac{11}{31} = -\frac{51}{31} \approx -1.64516$

4. $y = 2x - x^2 + 1 \Rightarrow y' = 2 - 2x \Rightarrow x_{n+1} = x_n - \frac{2x_n - x_n^2 + 1}{2 - 2x_n}$; $x_0 = 0 \Rightarrow x_1 = 0 - \frac{0 - 0 + 1}{2 - 0} = -\frac{1}{2}$

$\Rightarrow x_2 = -\frac{1}{2} - \frac{-1 - \frac{1}{4} + 1}{2 + 1} = -\frac{1}{2} + \frac{1}{12} = -\frac{5}{12} \approx -.41667$; $x_0 = 2 \Rightarrow x_1 = 2 - \frac{4 - 4 + 1}{2 - 4} = \frac{5}{2} \Rightarrow x_2 = \frac{5}{2} - \frac{5 - \frac{25}{4} + 1}{2 - 5}$

$= \frac{5}{2} - \frac{20 - 25 + 4}{-12} = \frac{5}{2} - \frac{1}{12} = \frac{29}{12} \approx 2.41667$

5. $y = x^4 - 2 \Rightarrow y' = 4x^3 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 - 2}{4x_n^3}$; $x_0 = 1 \Rightarrow x_1 = 1 - \frac{1 - 2}{4} = \frac{5}{4} \Rightarrow x_2 = \frac{5}{4} - \frac{\frac{625}{256} - 2}{\frac{125}{16}} = \frac{5}{4} - \frac{625 - 512}{2000}$

$= \frac{5}{4} - \frac{113}{2000} = \frac{2500 - 113}{2000} = \frac{2387}{2000} \approx 1.1935$

6. From Exercise 5, $x_{n+1} = x_n - \frac{x_n^4 - 2}{4x_n^3}$; $x_0 = -1 \Rightarrow x_1 = -1 - \frac{1 - 2}{-4} = -1 - \frac{1}{4} = -\frac{5}{4} \Rightarrow x_2 = -\frac{5}{4} - \frac{\frac{625}{256} - 2}{-\frac{125}{16}}$

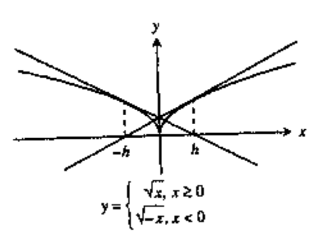
$= -\frac{5}{4} - \frac{625 - 512}{-2000} = -\frac{5}{4} + \frac{113}{2000} \approx -1.1935$

7. $f(x_0) = 0$ and $f'(x_0) \neq 0 \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ gives $x_1 = x_0 \Rightarrow x_2 = x_0 \Rightarrow x_n = x_0$ for all $n \geq 0$. That is, all of the approximations in Newton's method will be the root of $f(x) = 0$ as well as x_0.

8. It does matter. If you start too far away from $x = \frac{\pi}{2}$, the calculated values may approach some other root. Starting with $x_0 = -0.5$, for instance, leads to $x = -\frac{\pi}{2}$ as the root, not $x = \frac{\pi}{2}$.

9. If $x_0 = h > 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = h - \frac{f(h)}{f'(h)}$

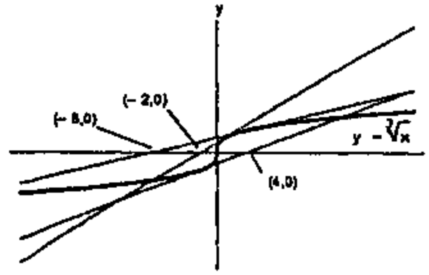
$= h - \frac{\sqrt{h}}{\left(\frac{1}{2\sqrt{h}}\right)} = h - \left(\sqrt{h}\right)\left(2\sqrt{h}\right) = -h$;



if $x_0 = -h < 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -h - \frac{f(-h)}{f'(-h)}$

$$= -h - \frac{\sqrt{h}}{\left(\frac{-1}{2\sqrt{h}}\right)} = -h + \left(\sqrt{h}\right)\left(2\sqrt{h}\right) = h.$$

10. $f(x) = x^{1/3} \Rightarrow f'(x) = \left(\frac{1}{3}\right)x^{-2/3} \Rightarrow x_{n+1} = x_n - \frac{x_n^{1/3}}{\left(\frac{1}{3}\right)x_n^{-2/3}}$
$= -2x_n$; $x_0 = 1 \Rightarrow x_1 = -2$, $x_2 = 4$, $x_3 = -8$, and $x_4 = 16$ and so forth. Since $|x_n| = 2x_{n-1}$ we may conclude that $n \to \infty \Rightarrow |x_n| \to \infty$.



11. The points of intersection of $y = x^3$ and $y = 3x + 1$, or of $y = x^3 - 3x$ and $y = 1$, have the same x-values as the roots of $f(x) = x^3 - 3x - 1$ or the solutions of $g'(x) = 0$.

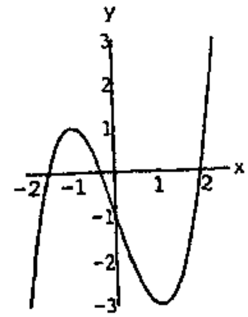
12. $f(x) = x - 1 - 0.5 \sin x \Rightarrow f'(x) = 1 - 0.5 \cos x \Rightarrow x_{n+1} = x_n - \frac{x_n - 1 - 0.5 \sin x_n}{1 - 0.5 \cos x_n}$; if $x_0 = 1.5$, then $x_1 = 1.49870$

13. The following commands are for the TI-92 Plus calculator. (Be sure your calculator is in approximate mode.) Go to the home screen and type the following:
(a) Define f(x) = x^3+3*x+1 (enter)
f(x) STO> y0 (enter)
nDeriv(f(x),x) STO> yp (enter)
(b) −0.3 STO> x (enter)
(c) x − y0 ÷ yp STO> x (enter)(enter)(enter)
After executing the last command two times the value, $x = -0.322185$, does not change in the sixth decimal place thereafter.
(d) Now try x0 = 0 by typing the following commands:
0 STO> x (enter)
x − y0 ÷ yp STO> x (enter)(enter)(enter)(enter)
After executing the last command three times the value, $x = -0.322185$, does not change in the sixth decimal place thereafter.
(e) Try $f(x) = \sin x$ to estimate the zero at $x = \pi$ by typing the following:
Define f(x) = sin(x) (enter)
3 STO> x (enter)
x − y0 ÷ yp STO> x (enter)(enter)(enter)

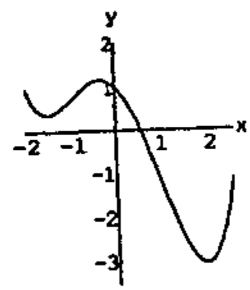
After executing the last command two times the value, $x = 3.14159$, does not change in the fifth decimal place thereafter. The zeros (sin(x), x) command gives 3.14159 · @n1, which means any integer multiple of 3.14159. This matches the above result when @n1 = 1.

14. (a) $f(x) = x^3 - 3x - 1 \Rightarrow f'(x) = 3x^2 - 3 \Rightarrow x_{n+1} = x_n - \frac{x_n^3 - 3x_n - 1}{3x_n^2 - 3} \Rightarrow$ the two negative zeros are -1.53209 and -0.34730

(b) The estimated solutions of $x^3 - 3x - 1 = 0$ are -1.53209, -0.34730, 1.87939.



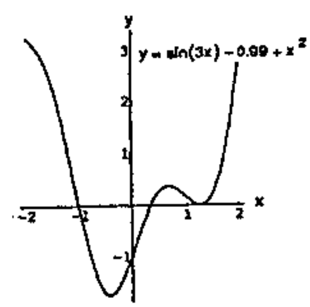
(c) The estimated x-values where $g(x) = 0.25x^4 - 1.5x^2 - x + 5$ has horizontal tangents are the roots of $g'(x) = x^3 - 3x - 1$, and these are -1.53209, -0.34730, 1.87939.



15. $f(x) = \tan x - 2x \Rightarrow f'(x) = \sec^2 x - 2 \Rightarrow x_{n+1} = x_n - \dfrac{\tan(x_n) - 2x_n}{\sec^2(x_n)}$; $x_0 = 1 \Rightarrow x_1 = 1.31047803$ $\Rightarrow x_2 = 1.223929097 \Rightarrow x_6 = x_7 = x_8 = 1.16556119$

16. $f(x) = x^4 - 2x^3 - x^2 - 2x + 2 \Rightarrow f'(x) = 4x^3 - 6x^2 - 2x - 2 \Rightarrow x_{n+1} = x_n - \dfrac{x_n^4 - 2x_n^3 - x_n^2 - 2x_n + 2}{4x_n^3 - 6x_n^2 - 2x_n - 2}$; if $x_0 = 0.5$, then $x_4 = 0.630115396$; if $x_0 = 2.5$, then $x_4 = 2.57327196$

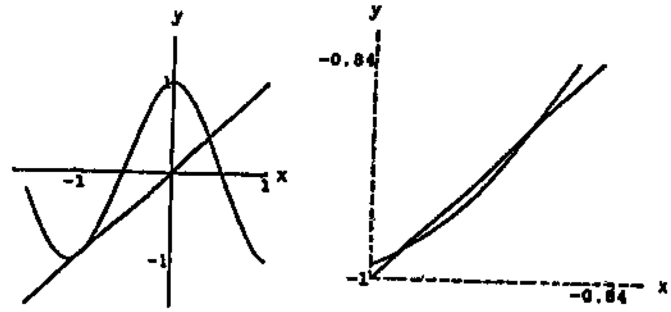
17. (a) The graph of $f(x) = \sin 3x - 0.99 + x^2$ in the window $-2 \le x \le 2$, $-2 \le y \le 3$ suggests three roots. However, when you zoom in on the x-axis near $x = 1.2$, you can see that the graph lies above the axis there. There are only two roots, one near $x = -1$, the other near $x = 0.4$.



(b) $f(x) = \sin 3x - 0.99 + x^2 \Rightarrow f'(x) = 3 \cos 3x + 2x$ $\Rightarrow x_{n+1} = x_n - \dfrac{\sin(3x_n) - 0.99 + x_n^2}{3\cos(3x_n) + 2x_n}$ and the solutions are approximately 0.35003501505249 and -1.0261731615301

18. (a) Yes, three times as indicated by the graphs

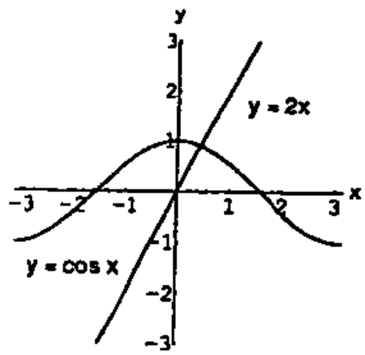
(b) $f(x) = \cos 3x - x \Rightarrow f'(x) = -3 \sin 3x - 1 \Rightarrow x_{n+1} = x_n - \frac{\cos(3x_n) - x_n}{-3 \sin(3x_n) - 1}$; at approximately -0.979367, -0.887726, and 0.39004 we have $\cos 3x = x$



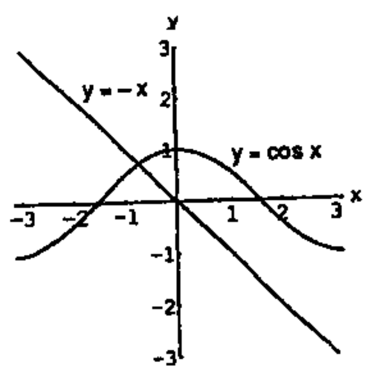
19. $f(x) = 2x^4 - 4x^2 + 1 \Rightarrow f'(x) = 8x^3 - 8x \Rightarrow x_{n+1} = x_n - \frac{2x_n^4 - 4x_n^2 + 1}{8x_n^3 - 8x_n}$; if $x_0 = -2$, then $x_6 = -1.30656296$; if $x_0 = -0.5$, then $x_3 = -0.5411961$; the roots are approximately ± 0.5411961 and ± 1.30656296 because f(x) is an even function.

20. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow x_{n+1} = x_n - \frac{\tan(x_n)}{\sec^2(x_n)}$; $x_0 = 3 \Rightarrow x_1 = 3.13971 \Rightarrow x_2 = 3.14159$ and we approximate π to be 3.14159.

21. From the graph we let $x_0 = 0.5$ and $f(x) = \cos x - 2x$
$\Rightarrow x_{n+1} = x_n - \frac{\cos(x_n) - 2x_n}{-\sin(x_n) - 2} \Rightarrow x_1 = .45063$
$\Rightarrow x_2 = .45018 \Rightarrow$ at $x \approx 0.45$ we have $\cos x = 2x$.



22. From the graph we let $x_0 = -0.7$ and $f(x) = \cos x + x$
$\Rightarrow x_{n+1} = x_n - \frac{x_n + \cos(x_n)}{1 - \sin(x_n)} \Rightarrow x_1 = -.73944$
$\Rightarrow x_2 = -.73908 \Rightarrow$ at $x \approx -0.74$ we have $\cos x = -x$.



23. If $f(x) = x^3 + 2x - 4$, then $f(1) = -1 < 0$ and $f(2) = 8 > 0 \Rightarrow$ by the Intermediate Value Theorem the equation $x^3 + 2x - 4 = 0$ has a solution between 1 and 2. Consequently, $f'(x) = 3x^2 + 2$ and $x_{n+1} = x_n - \frac{x_n^3 + 2x_n - 4}{3x_n^2 + 2}$. Then $x_0 = 1 \Rightarrow x_1 = 1.2 \Rightarrow x_2 = 1.17975 \Rightarrow x_3 = 1.179509 \Rightarrow x_4 = 1.1795090 \Rightarrow$ the root is approximately 1.17951.

24. We wish to solve $8x^4 - 14x^3 - 9x^2 + 11x - 1 = 0$. Let $f(x) = 8x^4 - 14x^3 - 9x^2 + 11x - 1$, then

$f'(x) = 32x^3 - 42x^2 - 18x + 11 \Rightarrow x_{n+1} = x_n - \frac{8x_n^4 - 14x_n^3 - 9x_n^2 + 11x_n - 1}{32x_n^3 - 42x_n^2 - 18x_n + 11}$.

x_0	approximation of corresponding root
−1.0	−0.976823589
0.1	0.100363332
0.6	0.642746671
2.0	1.983713587

25. $f(x) = 4x^4 - 4x^2 \Rightarrow f'(x) = 16x^3 - 8x \Rightarrow x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^3 - x_i}{4x_i^2 - 2}$. Iterations are performed using the procedure in problem 13 in this section.

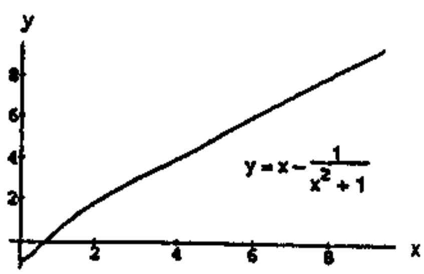
(a) For $x_0 = -2$ or $x_0 = -0.8$, $x_i \to -1$ as i gets large.

(b) For $x_0 = -0.5$ or $x_0 = 0.25$, $x_i \to 0$ as i gets large.

(c) For $x_0 = 0.8$ or $x_0 = 2$, $x_i \to 1$ as i gets large.

(d) (If your calculator has a CAS, put it in exact mode, otherwise approximate the radicals with a decimal value.) For $x_0 = -\frac{\sqrt{21}}{7}$ or $x_0 = \frac{\sqrt{21}}{7}$, Newton's method does not converge. The values of x_i alternate between $-\frac{\sqrt{21}}{7}$ and $\frac{\sqrt{21}}{7}$ as i increases.

26. (a) The distance can be represented by $D(x) = \sqrt{(x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2}$, where $x \geq 0$. The distance D(x) is minimized when $f(x) = (x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2$ is minimized. If $f(x) = (x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2$, then $f'(x) = 4\left(x^3 + x - 1\right)$ and $f''(x) = 4\left(3x^2 + 1\right) > 0$. Now $f'(x) = 0 \Rightarrow x^3 + x - 1 = 0 \Rightarrow x\left(x^2 + 1\right) = 1 \Rightarrow x = \frac{1}{x^2 + 1}$.



(b) Let $g(x) = \frac{1}{x^2 + 1} - x = \left(x^2 + 1\right)^{-1} - x \Rightarrow g'(x) = -\left(x^2 + 1\right)^{-2}(2x) - 1 = \frac{-2x}{\left(x^2 + 1\right)^2} - 1$

$$\Rightarrow x_{n+1} = x_n - \frac{\left(\frac{1}{x_n^2+1} - x_n\right)}{\left(\frac{-2x_n}{\left(x_n^2+1\right)^2} - 1\right)};\ x_0 = 1 \Rightarrow x_4 = 0.68233 \text{ to five decimal places.}$$

27. $f(x) = (x-1)^{40} \Rightarrow f'(x) = 40(x-1)^{39} \Rightarrow x_{n+1} = x_n - \frac{(x_n-1)^{40}}{40(x_n-1)^{39}} = \frac{39x_n+1}{40}$. With $x_0 = 2$, our computer gave $x_{87} = x_{88} = x_{89} = \cdots = x_{200} = 1.11051$, coming within 0.11051 of the root $x = 1$.

28. $f(x) = 4x^4 - 4x^2 \Rightarrow f'(x) = 16x^3 - 8x = 8x\left(2x^2-1\right) \Rightarrow x_{n+1} = x_n - \frac{x_n\left(x_n^2-1\right)}{2\left(2x_n^2-1\right)}$; if $x_0 = .65$, then $x_{12} \approx -.000004$, if $x_0 = .7$, then $x_{12} = -1.000004$; if $x_0 = .8$, then $x_6 = 1.000000$. NOTE: $\frac{\sqrt{21}}{7} \approx .654654$

29. $f(x) = x^3 + 3.6x^2 - 36.4 \Rightarrow f'(x) = 3x^2 + 7.2x \Rightarrow x_{n+1} = x_n - \frac{x_n^3 + 3.6x_n^2 - 36.4}{3x_n^2 + 7.2x_n}$; $x_0 = 2 \Rightarrow x_1 = 2.53\overline{03}$ $\Rightarrow x_2 = 2.45418225 \Rightarrow x_3 = 2.45238021 \Rightarrow x_4 = 2.45237921$ which is 2.45 to two decimal places. Recall that $x = 10^4[H_3O^+] \Rightarrow [H_3O^+] = (x)\left(10^{-4}\right) = (2.45)\left(10^{-4}\right) = 0.000245$

30. Newton's method yields the following:

the initial value	2	i	$\sqrt{3}+i$
the approached value	1	$-5.55931i$	$-29.5815 - 17.0789i$

CHAPTER 3 PRACTICE EXERCISES

1. The global minimum value of $\frac{1}{2}$ occurs at $x = 2$.

2. (a) The values of y' and y'' are both negative where the graph is decreasing and concave down, at T.
   (b) The value of y' is negative and the value of y'' is positive where the graph is decreasing and concave up, at P.

3. (a) The function is increasing on the intervals $[-3,-2]$ and $[1,2]$.
   (b) The function is decreasing on the intervals $[-2,0)$ and $(0,1]$.
   (c) Local maximum values occur only at $x = -2$ and at $x = 2$; local minimum values occur at $x = -3$ and at $x = 1$ provided f is continuous at $x = 0$.

4. The 24th day

5. No, since $f(x) = x^3 + 2x + \tan x \Rightarrow f'(x) = 3x^2 + 2 + \sec^2 x > 0 \Rightarrow f(x)$ is always increasing on its domain.

6. No, since $g(x) = \csc x + 2 \cot x \Rightarrow g'(x) = -\csc x \cot x - 2 \csc^2 x = -\frac{\cos x}{\sin^2 x} - \frac{2}{\sin^2 x} = -\frac{1}{\sin^2 x}(\cos x + 2) < 0$ $\Rightarrow g(x)$ is always decreasing on its domain

7. No absolute minimum because $\lim\limits_{x\to\infty} (7+x)(11-3x)^{1/3} = -\infty$. Next $f'(x) =$

$$(11-3x)^{1/3} - (7+x)(11-3x)^{-2/3} = \frac{(11-3x)-(7+x)}{(11-3x)^{2/3}} = \frac{4(1-x)}{(11-3x)^{2/3}} \Rightarrow x = 1 \text{ and } x = \frac{11}{3} \text{ are critical points.}$$

Since $f' > 0$ if $x < 1$ and $f' < 0$ if $x > 1$, $f(1) = 16$ is the absolute maximum.

8. $f(x) = \frac{ax+b}{x^2-1} \Rightarrow f'(x) = \frac{a\left(x^2-1\right) - 2x(ax+b)}{\left(x^2-1\right)^2} = -\frac{ax^2+2bx+a}{\left(x^2-1\right)^2}$; $f(3) = 1 \Rightarrow \frac{1}{8}(3a+b) = 1$ and $f'(3) = 0$

$\Rightarrow -\frac{1}{64}(9a+6b+a) = 0$. Solving simultaneously, $a = 6$ and $b = -10$. These values mean

$f'(x) = -\frac{6x^2-20x+6}{\left(x^2-1\right)^2} \Rightarrow f' > 0$ if $2 \le x < 3$ and $f' < 0$ if $3 < x \le 4 \Rightarrow$ local maximum value of $f(3) = 1$.

9. Yes, because at each point of $[0, 1)$ except $x = 0$, the function's value is a local minimum value as well as a local maximum value. At $x = 0$ the function's value, 0, is not a local minimum value because each open interval around $x = 0$ on the x-axis contains points to the left of 0 where f equals -1.

10. (a) The first derivative of the function $f(x) = x^3$ is zero at $x = 0$ even though f has no local extreme value at $x = 0$.
    (b) Theorem 2 says only that if f is differentiable and f has a local extreme at $x = c$ then $f'(c) = 0$. It does not assert the (false) reverse implication $f'(c) = 0 \Rightarrow$ f has a local extreme at $x = c$.

11. No, because the interval $0 < x < 1$ fails to be closed. The Extreme Value Theorem says that if the function is continuous throughout a finite closed interval $a \le x \le b$ then the existence of absolute extrema is guaranteed on that interval.

12. The absolute maximum is $|-1| = 1$ and the absolute minimum is $|0| = 0$. The result is consistent because the Extreme Value Theorem for Continuous Functions does not <u>require</u> the interval be closed. However, if it is <u>not</u> closed, absolute extrema may not exist, as Exercise 11 shows. That the interval be closed is a sufficient condition (together with continuity of the function), but it is not necessary for absolute extrema to exist.

13. (a) $g(t) = \sin^2 t - 3t \Rightarrow g'(t) = 2 \sin t \cos t - 3 = \sin(2t) - 3 \Rightarrow g' < 0 \Rightarrow g(t)$ is always falling and hence must decrease on every interval in its domain.
    (b) One, since $\sin^2 t - 3t - 5 = 0$ and $\sin^2 t - 3t = 5$ have the same solutions: $f(t) = \sin^2 t - 3t - 5$ has the same derivative as $g(t)$ in part (a) and is always decreasing with $f(-3) > 0$ and $f(0) < 0$. The Intermediate Value Theorem guarantees the continuous function f has a root in $[-3, 0]$.

14. (a) $y = \tan \theta \Rightarrow \frac{dy}{d\theta} = \sec^2 \theta > 0 \Rightarrow y = \tan \theta$ is always rising on its domain $\Rightarrow y = \tan \theta$ increases on every interval in its domain
    (b) The interval $\left[\frac{\pi}{4}, \pi\right]$ is not in the tangent's domain because $\tan \theta$ is undefined at $\theta = \frac{\pi}{2}$. Thus the tangent need not increase on this interval.

15. (a) $f(x) = x^4 + 2x^2 - 2 \Rightarrow f'(x) = 4x^3 + 4x$. Since $f(0) = -2 < 0$, $f(1) = 1 > 0$ and $f'(x) \ge 0$ for $0 \le x \le 1$, we may conclude from the Intermediate Value Theorem that $f(x)$ has exactly one solution when $0 \le x \le 1$.

(b) $x^2 = \frac{-2 \pm \sqrt{4+8}}{2} > 0 \Rightarrow x^2 = \sqrt{3} - 1$ and $x \geq 0 \Rightarrow x \approx \sqrt{.7320508076} \approx .8555996772$

16. (a) $y = \frac{x}{x+1} \Rightarrow y' = \frac{1}{(x+1)^2} > 0$, for all x in the domain of $\frac{x}{x+1} \Rightarrow y = \frac{x}{x+1}$ is increasing in every interval in its domain

(b) $y = x^3 + 2x \Rightarrow y' = 3x^2 + 2 > 0$ for all $x \Rightarrow$ the graph of $y = x^3 + 2x$ is always increasing and can never have a local maximum or minimum

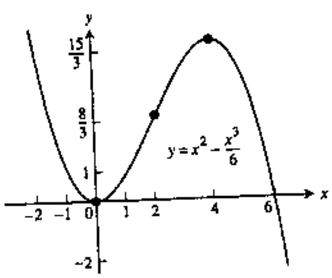
17. Let V(t) represent the volume of the water in the reservoir at time t, in minutes, let $V(0) = a_0$ be the initial amount and $V(1440) = a_0 + (1400)(43{,}560)(7.48)$ gallons be the amount of water contained in the reservoir after the rain, where 24 hr = 1440 min. Assume that V(t) is continuous on [0, 1440] and differentiable on (0, 1440). The Mean Value Theorem says that for some t_0 in (0, 1440) we have $V'(t_0) = \frac{V(1440) - V(0)}{1440 - 0}$ $= \frac{a_0 + (1400)(43{,}560)(7.48) - a_0}{1440} = \frac{456{,}160{,}320 \text{ gal}}{1440 \text{ min}} = 316{,}778$ gal/min. Therefore at t_0 the reservoir's volume was increasing at a rate in excess of 225,000 gal/min.

18. Yes, all differentiable functions g(x) having 3 as a derivative differ by only a constant. Consequently, the difference $3x - g(x)$ is a constant K because $g'(x) = 3 = \frac{d}{dx}(3x)$. Thus $g(x) = 3x + K$, the same form as F(x).

19. No, $\frac{x}{x+1} = 1 + \frac{-1}{x+1} \Rightarrow \frac{x}{x+1}$ differs from $\frac{-1}{x+1}$ by the constant 1. Both functions have the same derivative $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2} = \frac{d}{dx}\left(\frac{-1}{x+1}\right)$.

20. $f'(x) = g'(x) = \frac{2x}{\left(x^2+1\right)^2} \Rightarrow f(x) - g(x) = C$ for some constant $C \Rightarrow$ the graphs differ by a vertical shift.

21.



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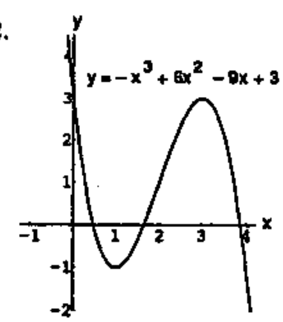

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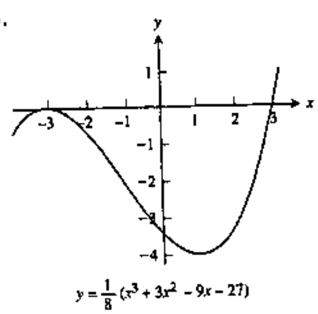

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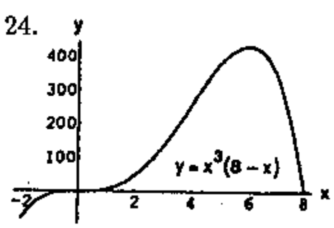
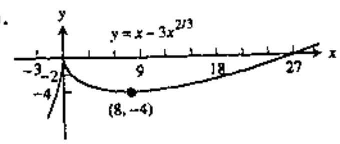
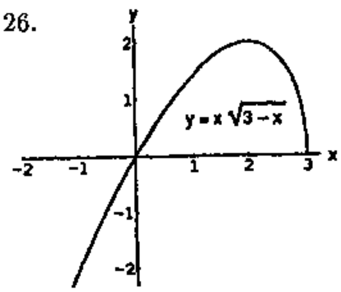
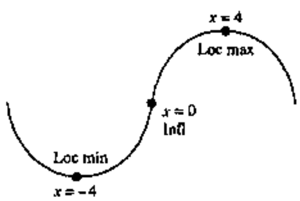


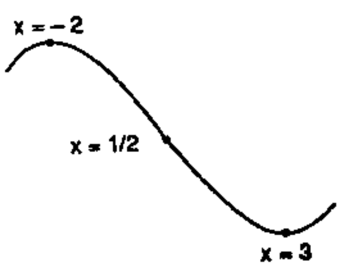
25. 

26. 

27. (a) $y' = 16 - x^2 \Rightarrow y' = \underset{-4}{---|}\underset{4}{+++|}---$ $\Rightarrow$ the curve is rising on $(-4, 4)$, falling on $(-\infty, -4)$ and $(4, \infty)$ $\Rightarrow$ a local maximum at $x = 4$ and a local minimum at $x = -4$; $y'' = -2x \Rightarrow y'' = \underset{0}{+++|}---$ $\Rightarrow$ the curve is concave up on $(-\infty, 0)$, concave down on $(0, \infty)$ $\Rightarrow$ a point of inflection at $x = 0$

(b) 

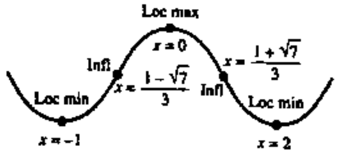
28. (a) $y' = x^2 - x - 6 = (x-3)(x+2) \Rightarrow y' = \underset{-2}{+++|}\underset{3}{---|}+++$ $\Rightarrow$ the curve is rising on $(-\infty, -2)$ and $(3, \infty)$, falling on $(-2, 3)$ $\Rightarrow$ local maximum at $x = -2$ and a local minimum at $x = 3$; $y'' = 2x - 1$ $\Rightarrow y'' = \underset{1/2}{---|}+++$ $\Rightarrow$ concave up on $\left(\frac{1}{2}, \infty\right)$, concave down on $\left(-\infty, \frac{1}{2}\right)$ $\Rightarrow$ a point of inflection at $x = \frac{1}{2}$

(b) 

29. (a) $y' = 6x(x+1)(x-2) = 6x^3 - 6x^2 - 12x \Rightarrow y' = \underset{-1}{---|}\underset{0}{+++|}\underset{2}{---|}+++$ $\Rightarrow$ the graph is rising on $(-1, 0)$ and $(2, \infty)$, falling on $(-\infty, -1)$ and $(0, 2)$ $\Rightarrow$ a local maximum at $x = 0$, local minima at $x = -1$ and $x = 2$; $y'' = 18x^2 - 12x - 12 = 6(3x^2 - 2x - 2) = 6\left(x - \frac{1-\sqrt{7}}{3}\right)\left(x - \frac{1+\sqrt{7}}{3}\right) \Rightarrow$ $y'' = \underset{\frac{1-\sqrt{7}}{3}}{+++|}\underset{\frac{1+\sqrt{7}}{3}}{---|}+++$ $\Rightarrow$ the curve is concave up on $\left(-\infty, \frac{1-\sqrt{7}}{3}\right)$ and $\left(\frac{1+\sqrt{7}}{3}, \infty\right)$, concave down

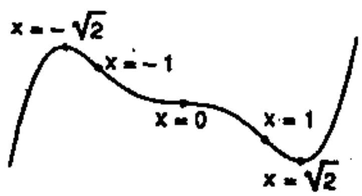
on $\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right) \Rightarrow$ points of inflection at $x = \frac{1 \pm \sqrt{7}}{3}$

(b)



30. (a) $y' = x^4 - 2x^2 = x^2(x^2-2) \Rightarrow y' = +++ \underset{-\sqrt{2}}{|} --- \underset{0}{|} --- \underset{\sqrt{2}}{|} +++ \Rightarrow$ the curve is rising on $(-\infty, -\sqrt{2})$ and $(\sqrt{2}, \infty)$, falling on $(-\sqrt{2}, \sqrt{2}) \Rightarrow$ a local maximum at $x = -\sqrt{2}$ and a local minimum at $x = \sqrt{2}$; $y'' = 4x^3 - 4x = 4x(x-1)(x+1) \Rightarrow y'' = --- \underset{-1}{|} +++ \underset{0}{|} --- \underset{1}{|} +++ \Rightarrow$ concave up on $(-1, 0)$ and $(1, \infty)$, concave down on $(-\infty, -1)$ and $(0, 1) \Rightarrow$ points of inflection at $x = 0$ and $x = \pm 1$

(b)



31. (a) $t = 0, 6, 12$ (b) $t = 3, 9$ (c) $6 < t < 12$ (d) $0 < t < 6$, $12 < t < 14$

32. (a) $t = 4$ (b) at no time (c) $0 < t < 4$ (d) $4 < t < 8$

33. (a) $v(t) = s'(t) = 4 - 6t - 3t^2$
(b) $a(t) = v'(t) = -6 - 6t$
(c) The particle starts at position 3 moving in the positive direction, but decelerating. At approximately $t = 0.528$, it reaches a position 4.128 and changes direction, beginning to move in the negative direction. After that, it continues to accelerate while moving in the negative direction.

34. $s(t) = \frac{1}{2}t^4 - 4t^3 + 6t^2$, $t \ge 0 \Rightarrow v(t) = 2t^3 - 12t^2 + 12t$; $v(t) = 0 \Rightarrow t(t^2 - 6t + 6) = 0 \Rightarrow t = 0$, $t = 3 - \sqrt{3} \approx 1.268$, and $t = 3 + \sqrt{3} \approx 4.732$. For $0 < t < 3 - \sqrt{3}$, $v(t) > 0$, for $3 - \sqrt{3} < t < 3 + \sqrt{3}$, $v(t) < 0$, and for $t > 3 + \sqrt{3}$, $v(t) > 0$, therefore, the particle moves forward during the time intervals, $(0, 3 - \sqrt{3})$ and $(3 + \sqrt{3}, \infty)$.

35. Since $\frac{d}{dx}\left(-\frac{1}{4}x^{-4} - \frac{1}{2}\cos 2x\right) = x^{-5} + \sin 2x$, $f(x) = -\frac{1}{4}x^{-4} - \frac{1}{2}\cos 2x + C$.

36. Since $\frac{d}{dx}\sec x = \sec x \tan x$, $f(x) = \sec x + C$.

37. Since $\frac{d}{dx}\left(-\frac{2}{x} + \frac{1}{3}x^3 + x\right) = \frac{2}{x^2} + x^2 + 1$, $f(x) = -\frac{2}{x} + \frac{1}{3}x^3 + x + C$ for $x > 0$.

38. Since $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + 2x^{1/2}\right) = \sqrt{x} + \frac{1}{\sqrt{x}}$, $f(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$.

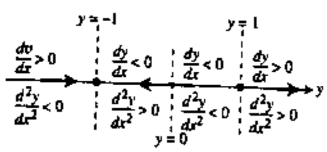
39. $v(t) = s'(t) = 9.8t + 5 \Rightarrow s(t) = 4.9t^2 + 5t + C$; $s(0) = 10 \Rightarrow C = 10 \Rightarrow s(t) = 4.9t^2 + 5t + 10$

40. $a(t) = v'(t) = 32 \Rightarrow v(t) = 32t + C_1$; $v(0) = 20 \Rightarrow C_1 = 20 \Rightarrow v(t) = s'(t) = 32t + 20$
$s(t) = 16t^2 + 20t + C_2$; $s(0) = 5 \Rightarrow C_2 = 5 \Rightarrow s(t) = 16t^2 + 20t + 5$

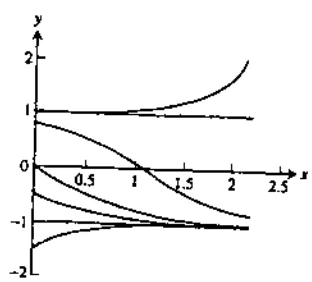
41. $\frac{dy}{dx} = y^2 - 1$

(a) $\frac{dy}{dx} = y^2 - 1 = 0 \Rightarrow y = \pm 1$; $y < -1 \Rightarrow \frac{dy}{dx} > 0$, $-1 < y < 1 \Rightarrow \frac{dy}{dx} < 0$, $y > 1 \Rightarrow \frac{dy}{dx} > 0$. Therefore, $y = -1$ is stable and $y = 1$ is unstable.

(b) $\frac{d^2y}{dx^2} = 2y\,\frac{dy}{dx} = 2y\left(y^2 - 1\right)$

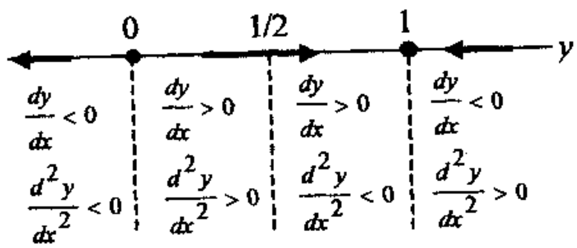


(c)

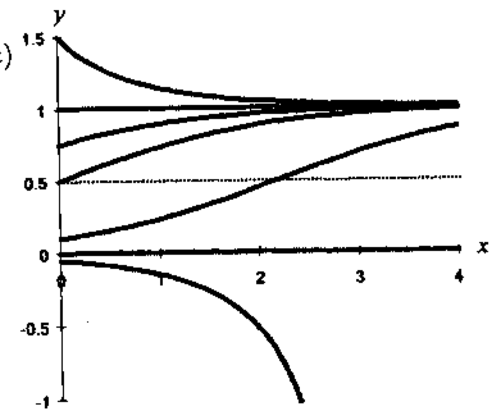


42. (a) $\frac{dy}{dx} = y - y^2 = 0 \Rightarrow y = 0$ or 1; $y < 0 \Rightarrow \frac{dy}{dx} < 0$, $0 < y < 1 \Rightarrow \frac{dy}{dx} > 0$, $y > 1 \Rightarrow \frac{dy}{dx} < 0$.
Therefore, $y = 0$ is unstable and $y = 1$ is stable.

(b) $\frac{d^2y}{dx^2} = (1-2y)\frac{dy}{dx} = (1-2y)(y-y^2) = y(1-2y)(1-y)$

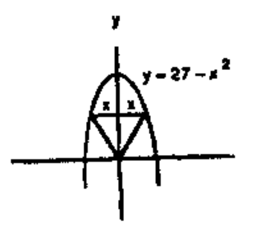


(c)



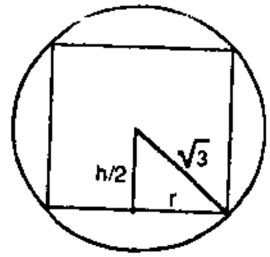
43. Note that $s = 100 - 2r$ and the sector area is given by $A = \pi r^2\left(\frac{s}{2\pi r}\right) = \frac{1}{2}rs = \frac{1}{2}r(100-2r) = 50r - r^2$. To find the domain of $A(r) = 50r - r^2$, note that $r > 0$ and $0 < s < 2\pi r$, which gives $12.1 \approx \frac{50}{\pi+1} < r < 50$. Since $A'(r) = 50 - 2r$, the critical point occurs at $r = 25$. This value is in the domain and corresponds to the maximum area because $A''(r) = -2$, which is negative for all r. The greatest area is attained when $r = 25$ ft and $s = 50$ ft.

44. $A(x) = \frac{1}{2}(2x)(27 - x^2)$ for $0 \le x \le \sqrt{27}$
$\Rightarrow A'(x) = 3(3+x)(3-x)$ and $A''(x) = -6x$.
The critical points are -3 and 3, but -3 is not in the domain. Since $A''(3) = -18 < 0$ and $A(\sqrt{27}) = 0$, the maximum occurs at $x = 3 \Rightarrow$ the largest area is $A(3) = 54$ sq units.

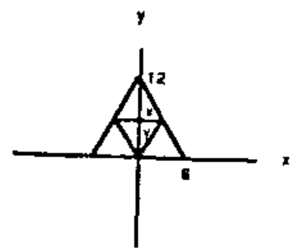


45. From the diagram we have $\left(\frac{h}{2}\right)^2 + r^2 = \left(\sqrt{3}\right)^2$
$\Rightarrow r^2 = \frac{12 - h^2}{4}$. The volume of the cylinder is

$V = \pi r^2 h = \pi\left(\frac{12-h^2}{4}\right)h = \frac{\pi}{4}(12h - h^3)$, where $0 \le h \le 2\sqrt{3}$. Then $V'(h) = \frac{3\pi}{4}(2+h)(2-h)$ $\Rightarrow$ the critical points are -2 and 2, but -2 is not in the domain. At $h = 2$ there is a maximum since $V''(2) = -3\pi < 0$. The dimensions of the largest cylinder are radius $= \sqrt{2}$ and height $= 2$.



46. From the diagram we have $y = 12 - 2x$ and $V(x) = \frac{1}{3}\pi x^2(12 - 2x)$, where $0 \le x \le 6$ $\Rightarrow V'(x) = 2\pi x(4 - x)$ and $V''(4) = -8\pi$. The critical points are 0 and 4; $V(0) = V(6) = 0 \Rightarrow x = 4$ gives the maximum. Thus the values of $r = 4$ and $h = 4$ yield the largest volume for the smaller cone.



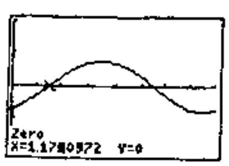
47. The profit $P = 2px + py = 2px + p\left(\frac{40-10x}{5-x}\right)$, where p is the profit on grade B tires and $0 \le x \le 4$. Thus $P'(x) = \frac{2p}{(5-x)^2}(x^2 - 10x + 20) \Rightarrow$ the critical points are $(5-\sqrt{5})$, 5, and $(5+\sqrt{5})$, but only $(5-\sqrt{5})$ is in the domain. Now $P'(x) > 0$ for $0 < x < (5-\sqrt{5})$ and $P'(x) < 0$ for $(5-\sqrt{5}) < x < 4 \Rightarrow$ at $x = (5-\sqrt{5})$ there is a local maximum. Also $P(0) = 8p$, $P(5-\sqrt{5}) = 4p(5-\sqrt{5}) \approx 11p$, and $P(4) = 8p \Rightarrow$ at $x = (5-\sqrt{5})$ there is an absolute maximum. The maximum occurs when $x = (5-\sqrt{5})$ and $y = 2(5-\sqrt{5})$, the units are hundreds of tires, i.e., $x \approx 276$ tires and $y \approx 553$ tires.

48. (a) The distance between the particles is $|f(t)|$ where

$f(t) = -\cos t + \cos\left(t + \frac{\pi}{4}\right)$.

Then $f'(t) = \sin t - \sin\left(t + \frac{\pi}{4}\right)$

Solving $f'(t) = 0$ graphically, we obtain $t \approx 1.178$, $t \approx 4.320$, and so on.



$[0, 2\pi]$ by $[-2, 2]$

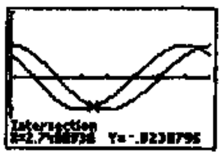
Alternatively, $f'(t) = 0$ may be solved analytically as follows.

$$f'(t) = \sin\left[\left(t+\frac{\pi}{8}\right)-\frac{\pi}{8}\right]-\sin\left[\left(t+\frac{\pi}{8}\right)+\frac{\pi}{8}\right]$$
$$= \left[\sin\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8}-\cos\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8}\right]-\left[\sin\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8}+\cos\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8}\right]$$
$$= -2\sin\frac{\pi}{8}\cos\left(t+\frac{\pi}{8}\right),$$

so the critical points occur when

$\cos\left(t+\frac{\pi}{8}\right)=0$, or $t=\frac{3\pi}{8}+k\pi$. At each of these values, $f(t) = \pm 2\cos\frac{3\pi}{8} \approx \pm 0.765$ units, so the maximum distance between the particles is 0.765 units.

(b) Solving $\cos t = \cos\left(t+\frac{\pi}{4}\right)$ graphically, we obtain $t \approx 2.749$, $t \approx 5.890$, and so on.



[0, 2π] by [−2, 2]

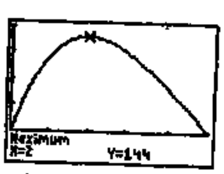
Alternatively, this problem can be solved analytically as follows.

$$\cos t = \cos\left(t+\frac{\pi}{4}\right)$$
$$\cos\left[\left(t+\frac{\pi}{8}\right)-\frac{\pi}{8}\right] = \cos\left[\left(t+\frac{\pi}{8}\right)+\frac{\pi}{8}\right]$$
$$\cos\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8}+\sin\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8} = \cos\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8}-\sin\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8}$$
$$2\sin\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8}=0$$
$$\sin\left(t+\frac{\pi}{8}\right)=0$$
$$t=\frac{7\pi}{8}+k\pi$$

The particles collide when $t=\frac{7\pi}{8}\approx 2.749$ (plus multiples of π if they keep going.)

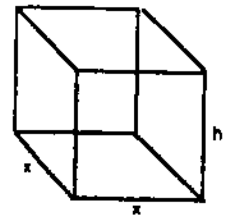
49. The dimensions will be x in. by $10-2x$ in. by $16-2x$ in., so $V(x) = x(10-2x)(16-2x) = 4x^3-52x^2+160x$ for $0<x<5$. Then $V'(x) = 12x^2-104x+160 = 4(x-2)(3x-20)$, so the critical point in the correct domain is $x=2$. This critical point corresponds to the maximum possible volume because $V'(x)>0$ for $0<x<2$ and $V'(x)<0$ for $2<x<5$. The box of largest volume has a height of 2 in. and a base measuring 6 in. by 12 in., and its volume is 144 in.^{3}

Graphical support:

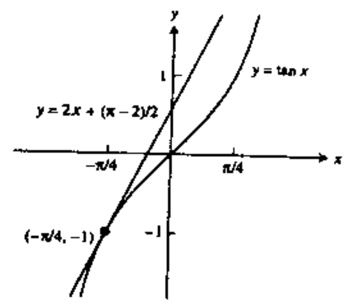


[0, 5] by [−40, 160]

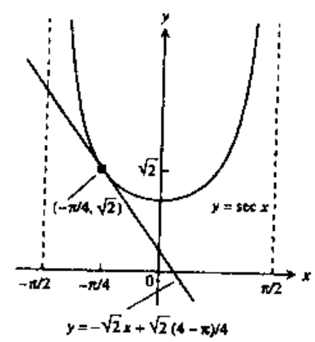
50. The volume is $V = x^2 h = 32 \Rightarrow h = \frac{32}{x^2}$. The surface area is $S(x) = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + \frac{128}{x}$, where $x > 0 \Rightarrow S'(x) = \frac{2x^3 - 128}{x^2} \Rightarrow$ the critical points are 0 and 4, but 0 is not in the domain. Now $S''(4) = 2 + \frac{256}{4^3} > 0 \Rightarrow$ at $x = 4$ there is a minimum. The dimensions 4 ft by 4 ft by 2 ft minimize the surface area.



51. (a) If $f(x) = \tan x$ and $x = -\frac{\pi}{4}$, then $f'(x) = \sec^2 x$, $f\left(-\frac{\pi}{4}\right) = -1$ and $f'\left(-\frac{\pi}{4}\right) = 2$. The linearization of $f(x)$ is $L(x) = 2\left(x + \frac{\pi}{4}\right) + (-1) = 2x + \frac{\pi - 2}{2}$.



(b) If $f(x) = \sec x$ and $x = -\frac{\pi}{4}$, then $f'(x) = \sec x \tan x$, $f\left(-\frac{\pi}{4}\right) = \sqrt{2}$ and $f'\left(-\frac{\pi}{4}\right) = -\sqrt{2}$. The linearization of $f(x)$ is $L(x) = -\sqrt{2}\left(x + \frac{\pi}{4}\right) + \sqrt{2} = -\sqrt{2}x + \frac{\sqrt{2}(4 - \pi)}{4}$.



52. $f(x) = \frac{1}{1 + \tan x} \Rightarrow f'(x) = \frac{-\sec^2 x}{(1 + \tan x)^2}$. The linearization at $x = 0$ is $L(x) = f'(0)(x - 0) + f(0) = 1 - x$.

53. $f(x) = \sqrt{x + 1} + \sin x - 0.5 = (x + 1)^{1/2} + \sin x - 0.5 \Rightarrow f'(x) = \left(\frac{1}{2}\right)(x + 1)^{-1/2} + \cos x$
$\Rightarrow L(x) = f'(0)(x - 0) + f(0) = 1.5(x - 0) + 0.5 \Rightarrow L(x) = 1.5x + 0.5$, the linearization of $f(x)$.

54. $f(x) = \frac{2}{1 - x} + \sqrt{1 + x} - 3.1 = 2(1 - x)^{-1} + (1 + x)^{1/2} - 3.1 \Rightarrow f'(x) = -2(1 - x)^{-2}(-1) + \frac{1}{2}(1 + x)^{-1/2}$
$= \frac{2}{(1 - x)^2} + \frac{1}{2\sqrt{1 + x}} \Rightarrow L(x) = f'(0)(x - 0) + f(0) = 2.5x - 0.1$, the linearization of $f(x)$.

55. When the volume is $V = \frac{1}{3}\pi r^2 h$, then $dV = \frac{2}{3}\pi r_0 h\ dr$ estimates the change in the volume for fixed h.

56. (a) $S = 6r^2 \Rightarrow dS = 12r\ dr$. We want $|dS| \le (2\%)S \Rightarrow |12r\ dr| \le \frac{12r^2}{100} \Rightarrow |dr| \le \frac{r}{100}$. The measurement of the edge r must have an error less than 1%.

(b) When $V = r^3$, then $dV = 3r^2\ dr$. The accuracy of the volume is $\left(\frac{dV}{V}\right)(100\%) = \left(\frac{3r^2\ dr}{r^3}\right)(100\%)$
$= \left(\frac{3}{r}\right)(dr)(100\%) = \left(\frac{3}{r}\right)\left(\frac{r}{100}\right)(100\%) = 3\%$

57. $C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$, $S = 4\pi r^2 = \frac{C^2}{\pi}$, and $V = \frac{4}{3}\pi r^3 = \frac{C^3}{6\pi^2}$. It also follows that $dr = \frac{1}{2\pi}\ dC$, $dS = \frac{2C}{\pi}\ dC$ and $dV = \frac{C^2}{2\pi^2}\ dC$. Recall that $C = 10$ cm and $dC = 0.4$ cm.

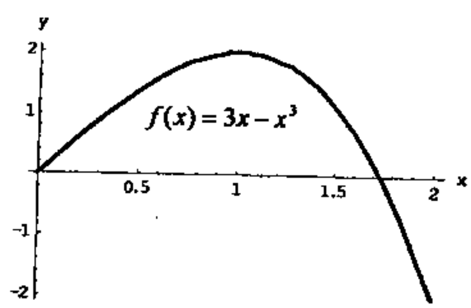
(a) $dr = \frac{0.4}{2\pi} = \frac{0.2}{\pi}$ cm $\Rightarrow \left(\frac{dr}{r}\right)(100\%) = \left(\frac{0.2}{\pi}\right)\left(\frac{2\pi}{10}\right)(100\%) = (.04)(100\%) = 4\%$

(b) $dS = \frac{20}{\pi}(0.4) = \frac{8}{\pi}$ cm $\Rightarrow \left(\frac{dS}{S}\right)(100\%) = \left(\frac{8}{\pi}\right)\left(\frac{\pi}{100}\right)(100\%) = 8\%$

(c) $dV = \frac{10^2}{2\pi^2}(0.4) = \frac{20}{\pi^2}$ cm $\Rightarrow \left(\frac{dV}{V}\right)(100\%) = \left(\frac{20}{\pi^2}\right)\left(\frac{6\pi^2}{1000}\right)(100\%) = 12\%$

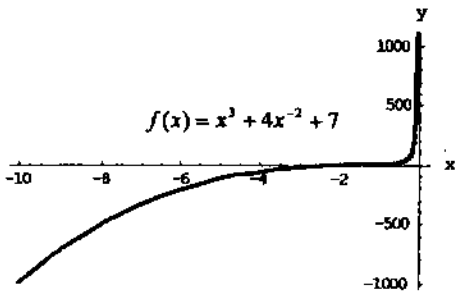
58. Similar triangles yield $\frac{35}{h} = \frac{15}{6} \Rightarrow h = 14$ ft. The same triangles imply that $\frac{20+a}{h} = \frac{a}{6} \Rightarrow h = 120a^{-1} + 6$
$\Rightarrow dh = -120a^{-2}\ da = -\frac{120}{a^2}\ da = \left(-\frac{120}{a^2}\right)\left(\pm\frac{1}{12}\right) = \pm\frac{2}{45} \approx \pm 0.0444$ ft $= \pm 0.53$ inches.

59. The graph of f(x) shows that for $1 \le x \le 2$, $f(x) = 0$ has one solution near $x = 1.7$. (Note: The exact solution is $x = \sqrt{3} \approx 1.732051$. Nonetheless, we use Newton's method to find an estimate for this solution.)



$f(x) = 3x - x^3 \Rightarrow f'(x) = 3 - 3x^2 \Rightarrow x_{n+1} = x_n - \frac{3x_n - x_n^3}{3 - 3x_n^2} \Rightarrow x_0 = 1.7$, $x_1 = 1.732981$, $x_2 = 1.732052$, $x_3 = 1.732051$, $x_4 = 1.732051$. Solution: $x \approx 1.732051$.

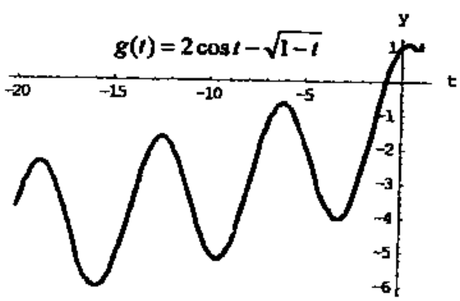
60. The graph of f(x) shows that for $x < 0$, $f(x) = 0$ has one solution near $x = -2$.



$$f(x) = x^3 + 4x^{-2} + 7 \Rightarrow f'(x) = 3x^2 - 8x^{-3} \Rightarrow x_{n+1} = x_n - \frac{x_x^3 + 4x_n^{-2} + 7}{3x_n^2 - 8x_n^{-3}} = x_n - \frac{x_x^6 + 4x_n + 7x_n^3}{3x_n^5 - 8} \Rightarrow x_0 = -2,$$

$x_1 = -2$. This is because $x = -2$ is a root, the one we are looking for.

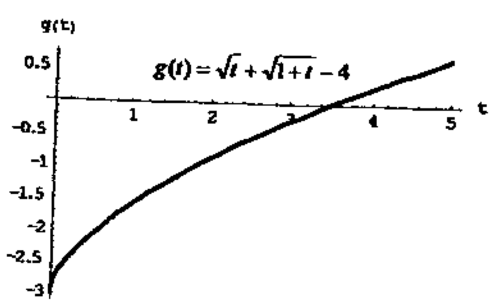
61. The domain of g(t) is $(-\infty, 1]$, and the graph of g(t) shows that $g(t) = 0$ has one solution near $t = -1$.



$$g(t) = 2\cos t - \sqrt{1-t} \Rightarrow g'(t) = -2\sin t + \frac{1}{2\sqrt{1-t}} \Rightarrow t_{n+1} = t_n - \frac{2\cos t_n - \sqrt{1-t_n}}{-2\sin t_n + \frac{1}{2\sqrt{1-t_n}}} \Rightarrow t_0 = -1,$$

$t_1 = -0.836185$, $t_2 = -0.828381$, $t_3 = -0.828361$, $t_4 = -0.828361$. Solution: $t \approx -0.828361$.

62. The graph of g(t) shows that for $t > 0$, $g(t) = 0$ has one solution between $t = 3$ and $t = 4$.



$g(t) = \sqrt{t} + \sqrt{1+t} - 4 \Rightarrow g'(t) = \frac{1}{2\sqrt{t}} + \frac{1}{2\sqrt{1+t}} \Rightarrow t_{n+1} = t_n - \frac{\sqrt{t_n} + \sqrt{1+t_n} - 4}{\frac{1}{2\sqrt{t_n}} + \frac{1}{2\sqrt{1+t_n}}} \Rightarrow x_1 = 3,\ x_2 \approx 3.497423,$

$x_2 \approx 3.515604,\ x_3 \approx 3.515625,\ x_4 \approx 3.515625$

Solution: $x \approx 3.515625$

CHAPTER 3 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. If M and m are the maximum and minimum values, respectively, then $m \le f(x) \le M$ for all $x \in I$. If $m = M$ then f is constant on I.

2. No, the function $f(x) = \begin{cases} 3x+6, & -2 \le x < 0 \\ 9 - x^2, & 0 \le x \le 2 \end{cases}$ has an absolute minimum value of 0 at $x = -2$ and an absolute maximum value of 9 at $x = 0$, but it is discontinuous at $x = 0$.

3. On an open interval the extreme values of a continuous function (if any) must occur at an interior critical point. On a half-open interval the extreme values of a continuous function may be at a critical point or at the closed endpoint. Extreme values occur only where $f' = 0$, f' does not exist, or at the endpoints of the interval. Thus the extreme points will not be at the open ends of an open interval.

4. The pattern $f' = +++\underset{1}{|}----\underset{2}{|}----\underset{3}{|}++++\underset{4}{|}+++$ indicates a local maximum at $x = 1$ and a local minimum at $x = 3$.

5. (a) If $y' = 6(x+1)(x-2)^2$, then $y' < 0$ for $x < -1$ and $y' > 0$ for $x > -1$. The sign pattern is $f' = ---\underset{-1}{|}+++\underset{2}{|}+++ \Rightarrow$ f has a local minimum at $x = -1$. Also $y'' = 6(x-2)^2 + 12(x+1)(x-2)$ $= 6(x-2)(3x) \Rightarrow y'' > 0$ for $x < 0$ or $x > 2$, while $y'' < 0$ for $0 < x < 2$. Therefore f has points of inflection at $x = 0$ and $x = 2$.

   (b) If $y' = 6x(x+1)(x-2)$, then $y' < 0$ for $x < -1$ and $0 < x < 2$; $y' > 0$ for $-1 < x < 0$ and $x > 2$. The sign sign pattern is $y' = ---\underset{-1}{|}+++\underset{0}{|}---\underset{2}{|}+++$. Therefore f has a local maximum at $x = 0$ and local minima at $x = -1$ and $x = 2$. Also, $y'' = 6\left[x - \left(\frac{1-\sqrt{7}}{3}\right)\right]\left[x - \left(\frac{1+\sqrt{7}}{3}\right)\right]$, so $y'' < 0$ for $\frac{1-\sqrt{7}}{3} < x < \frac{1+\sqrt{7}}{3}$ and $y'' > 0$ for all other $x \Rightarrow$ f has points of inflection at $x = \frac{1 \pm \sqrt{7}}{3}$.

6. The Mean Value Theorem indicates that $\frac{f(6)-f(0)}{6-0} = f'(c) \le 2$ for some c in $(0,6)$. Then $f(6) - f(0) \le 12$ indicates the most that f can increase is 12.

7. If f is continuous on $[a,c)$ and $f'(x) \le 0$ on $[a,c)$, then by the Mean Value Theorem for all $x \in [a,c)$ we have $\frac{f(c)-f(x)}{c-x} \le 0 \Rightarrow f(c) - f(x) \le 0 \Rightarrow f(x) \ge f(c)$. Also if f is continuous on $(c,b]$ and $f'(x) \ge 0$ on $(c,b]$, then for all $x \in (c,b]$ we have $\frac{f(x)-f(c)}{x-c} \ge 0 \Rightarrow f(x) - f(c) \ge 0 \Rightarrow f(x) \ge f(c)$. Therefore $f(x) \ge f(c)$ for all $x \in [a,b]$.

8. (a) For all x, $-(x+1)^2 \le 0 \le (x-1)^2 \Rightarrow -(1+x^2) \le 2x \le (1+x^2) \Rightarrow -\frac{1}{2} \le \frac{x}{1+x^2} \le \frac{1}{2}$.

(b) There exists $c \in (a, b)$ such that $\frac{c}{1+c^2} = \frac{f(b)-f(a)}{b-a} \Rightarrow \left|\frac{f(b)-f(a)}{b-a}\right| = \left|\frac{c}{1+c^2}\right| \le \frac{1}{2}$, from part (a) $\Rightarrow |f(b)-f(a)| \le \frac{1}{2}|b-a|$.

9. No. Corollary 1 requires that $f'(x) = 0$ for <u>all</u> x in some interval I, not $f'(x) = 0$ at a single point in I.

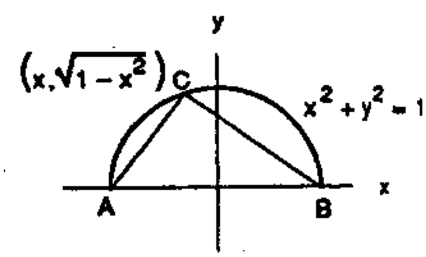
10. (a) $h(x) = f(x)g(x) \Rightarrow h'(x) = f'(x)g(x) + f(x)g'(x)$ which changes signs at $x = a$ since $f'(x), g'(x) > 0$ when $x < a$, $f'(x), g'(x) < 0$ when $x > a$ and $f(x), g(x) > 0$ for all x. Therefore h(x) does have a local maximum at $x = a$.

(b) No, let $f(x) = g(x) = x^3$ which have points of inflection at $x = 0$, but $h(x) = x^6$ has no point of inflection (it has a local minimum at $x = 0$).

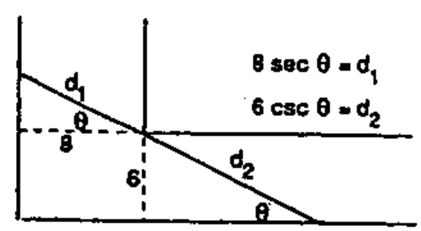
11. From (ii), $f(-1) = \frac{-1+a}{b-c+2} = 0 \Rightarrow a = 1$; from (iii), $1 = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+1}{bx^2+cx+2} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{bx+c+\frac{2}{x}}$ $\Rightarrow b = 0$ (because $b = 1 \Rightarrow \lim_{x\to\infty} f(x) = 0$). Also, if $c = 0$ then $\lim_{x\to\infty} f(x) = \infty$ so we must have $c = 1$. In summary, $a = 1$, $b = 0$, and $c = 1$.

12. $\frac{dy}{dx} = 3x^2 + 2kx + 3 = 0 \Rightarrow x = \frac{-2k \pm \sqrt{4k^2-36}}{6} \Rightarrow$ x has only one value when $4k^2 - 36 = 0 \Rightarrow k^2 = 9$ or $k = \pm 3$.

13. The area of the ΔABC is $A(x) = \frac{1}{2}(2)\sqrt{1-x^2} = (1-x^2)^{1/2}$, where $0 \le x \le 1$. Thus $A'(x) = \frac{-x}{\sqrt{1-x^2}} \Rightarrow 0$ and ± 1 are critical points. Also $A(\pm 1) = 0$ so $A(0) = 1$ is the maximum. When $x = 0$ the ΔABC is isosceles since $AC = BC = \sqrt{2}$.



14. The length of the ladder is $d_1 + d_2 = 8 \sec\theta + 6 \csc\theta$. We wish to maximize $I(\theta) = 8\sec\theta + 6\csc\theta \Rightarrow I'(\theta)$ $= 8\sec\theta\tan\theta - 6\csc\theta\cot\theta$. Then $I'(\theta) = 0$ $\Rightarrow 8\sin^3\theta - 6\cos^3\theta = 0 \Rightarrow \tan\theta = \frac{\sqrt[3]{6}}{2} \Rightarrow$ $d_1 = 4\sqrt{4+\sqrt[3]{36}}$ and $d_2 = \sqrt[3]{36}\sqrt{4+\sqrt[3]{36}}$ $\Rightarrow$ the length of the ladder is about $(4+\sqrt[3]{36})\sqrt{4+\sqrt[3]{36}} = (4+\sqrt[3]{36})^{3/2} \approx 19$ ft (rounded down so that the ladder will make the corner).

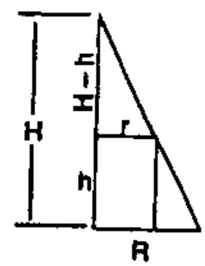


15. The time it would take the water to hit the ground from height y is $\sqrt{\frac{2y}{g}}$, where g is the acceleration of gravity. The product of time and exit velocity (rate) yields the distance the water travels:

$D(y) = \sqrt{\frac{2y}{g}}\sqrt{64(h-y)} = 8\sqrt{\frac{2}{g}}\left(hy - y^2\right)^{1/2}$, $0 \le y \le h \Rightarrow D'(y) = 4\sqrt{\frac{2}{g}}\left(hy - y^2\right)^{-1/2}(h - 2y) \Rightarrow 0$, $\frac{h}{2}$ and h are critical points. Now $D(0) = 0$, $D\left(\frac{h}{2}\right) = \frac{8h}{\sqrt{g}}$ and $D(h) = 0 \Rightarrow$ the best place to drill the hole is at $y = \frac{h}{2}$.

16. From the figure in the text, $\tan(\beta + \theta) = \frac{b+a}{h}$; $\tan(\beta + \theta) = \frac{\tan\beta + \tan\theta}{1 - \tan\beta\tan\theta}$; and $\tan\theta = \frac{a}{h}$. These equations give $\frac{b+a}{h} = \frac{\tan\beta + \frac{a}{h}}{1 - \frac{a}{h}\tan\beta} = \frac{h\tan\beta + a}{h - a\tan\beta}$. Solving for $\tan\beta$ gives $\tan\beta = \frac{bh}{h^2 + a(b+a)}$ or $\left(h^2 + a(b+a)\right)\tan\beta = bh$. Differentiating both sides with respect to h gives $2h\tan\beta + \left(h^2 + a(b+a)\right)\sec^2\beta\,\frac{d\beta}{dh} = b$. Then $\frac{d\beta}{dh} = 0 \Rightarrow 2h\tan\beta = b \Rightarrow 2h\left(\frac{bh}{h^2 + a(b+a)}\right) = b$ $\Rightarrow 2bh^2 = bh^2 + ab(b+a) \Rightarrow h^2 = a(b+a) \Rightarrow h = \sqrt{a(a+b)}$.

17. The surface area of the cylinder is $S = 2\pi r^2 + 2\pi rh$. From the diagram we have $\frac{r}{R} = \frac{H-h}{H} \Rightarrow h = \frac{RH - rH}{R}$ and $S(r) = 2\pi r(r+h) = 2\pi r\left(r + H - r\frac{H}{R}\right) = 2\pi\left(1 - \frac{H}{R}\right)r^2 + 2\pi Hr$, where $0 \le r \le R$.



Case 1: $H < R \Rightarrow S(r)$ is a quadratic equation containing the origin and concave upward $\Rightarrow S(r)$ is maximum at $r = R$.

Case 2: $H = R \Rightarrow S(r)$ is a linear equation containing the origin with a positive slope $\Rightarrow S(r)$ is maximum at $r = R$.

Case 3: $H > R \Rightarrow S(r)$ is a quadratic equation containing the origin and concave downward. Then $\frac{dS}{dr} = 4\pi\left(1 - \frac{H}{R}\right)r + 2\pi H$ and $\frac{dS}{dr} = 0 \Rightarrow 4\pi\left(1 - \frac{H}{R}\right)r + 2\pi H = 0 \Rightarrow r = \frac{RH}{2(H-R)}$. For simplification we let $r^* = \frac{RH}{2(H-R)}$.

(a) If $R < H < 2R$, then $0 \ge H - 2R \Rightarrow H \ge 2(H-R) \Rightarrow \frac{RH}{2(H-R)} \ge R$ which is impossible.

(b) If $H = 2R$, then $r^* = \frac{2R^2}{2R} = R \Rightarrow S(r)$ is maximum at $r = R$.

(c) If $H > 2R$, then $2R + H \le 2H \Rightarrow H \le 2(H-R) \Rightarrow \frac{H}{2(H-R)} \le 1 \Rightarrow \frac{RH}{2(H-R)} \le R \Rightarrow r^* \le R$. Therefore, $S(r)$ is a maximum at $r = r^* = \frac{RH}{2(H-R)}$.

Conclusion: If $H \in (0, R]$ or $H = 2R$, then the maximum surface area is at $r = R$. If $H \in (R, 2R)$, then $r > R$ which is not possible. If $H \in (2R, \infty)$, then the maximum is at $r = r^* = \frac{RH}{2(H-R)}$.

18. $f(x) = mx - 1 + \frac{1}{x} \Rightarrow f'(x) = m - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3} > 0$ when $x > 0$. Then $f'(x) = 0 \Rightarrow x = \frac{1}{\sqrt{m}}$ yields a minimum. If $f\left(\frac{1}{\sqrt{m}}\right) \ge 0$, then $\sqrt{m} - 1 + \sqrt{m} = 2\sqrt{m} - 1 \ge 0 \Rightarrow m \ge \frac{1}{4}$. Thus the smallest acceptable value for m is $\frac{1}{4}$.

19. $\lim\limits_{h\to 0} \frac{f'(c+h)-f'(c)}{h} = f''(c) \Leftrightarrow$ for $\epsilon = \frac{1}{2}|f''(c)| > 0$ there exists a $\delta > 0$ such that $0 < |h| < \delta$
$\Rightarrow \left|\frac{f'(c+h)-f'(c)}{h} - f''(c)\right| < \frac{1}{2}|f''(c)|$. Then $f'(c) = 0 \Rightarrow -\frac{1}{2}|f''(c)| < \frac{f'(c+h)}{h} - f''(c) < \frac{1}{2}|f''(c)|$
$\Rightarrow f''(c) - \frac{1}{2}|f''(c)| < \frac{f'(c+h)}{h} < f''(c) + \frac{1}{2}|f''(c)|$. If $f''(c) < 0$, then $|f''(c)| = -f''(c)$
$\Rightarrow \frac{3}{2}f''(c) < \frac{f'(c+h)}{h} < \frac{1}{2}f''(c) < 0$; likewise if $f''(c) > 0$, then $0 < \frac{1}{2}f''(c) < \frac{f'(c+h)}{h} < \frac{3}{2}f''(c)$.

(a) If $f''(c) < 0$, then $-\delta < h < 0 \Rightarrow f'(c+h) > 0$ and $0 < h < \delta \Rightarrow f'(c+h) < 0$. Therefore, f(c) is a local maximum.

(b) If $f''(c) > 0$, then $-\delta < h < 0 \Rightarrow f'(c+h) < 0$ and $0 < h < \delta \Rightarrow f'(c+h) > 0$. Therefore, f(c) is a local minimum.

20. (a) By completing the square we have $f(x) = a\left(x + \frac{b}{a}\right)^2 + \frac{ac-b^2}{a} \ge 0$. If $a > 0$ and $f(x) \ge 0$, then $\frac{ac-b^2}{a} \ge 0$ $\Rightarrow ac - b^2 > 0 \Rightarrow ac > b^2$. If $ac > b^2$ and $a > 0$, then $\frac{ac-b^2}{a} > 0 \Rightarrow f(x) \ge 0$.

(b) If $f(x) = (a_1x + b_1)^2 + \ldots + (a_nx + b_n)^2$, then let $g(x) = Ax^2 + 2Bx + C$, where $A = \sum\limits_{i=1}^{n} a_i^2$, $B = \sum\limits_{i=1}^{n} a_ib_i$ and $C = \sum\limits_{i=1}^{n} b_i^2$. Part (a) $\Rightarrow B^2 \le AC$ or $\left(\sum\limits_{i=1}^{n} a_ib_i\right)^2 \le \left(\sum\limits_{i=1}^{n} a_i^2\right)\left(\sum\limits_{i=1}^{n} b_i^2\right)$.

(c) $B^2 = AC \Rightarrow$there is a unique $x = x_0$ such that $g(x_0) = A\left(x_0 - \frac{B}{A}\right)^2 + \frac{AC - B^2}{A} = 0$, from part (b). Therefore $f(x_0) = 0 \Rightarrow$ that each $a_ix_0 + b_i = 0 \Rightarrow a_ix_0 = -b_i$ for $i = 1, 2, \ldots, n$.

21. (a) $(1)^2 = \frac{4\pi^2 L}{32.2} \Rightarrow L \approx 0.8156$ ft

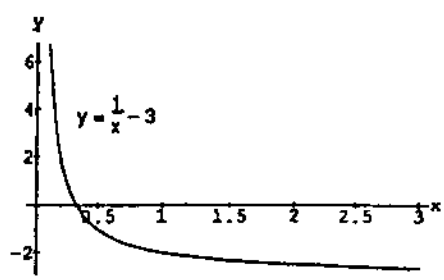
(b) $2T\,dT = \frac{4\pi^2}{g}\,dL \Rightarrow dT = \frac{2\pi^2}{Tg}\,dL = \frac{2\pi^2}{\left(\frac{2\pi\sqrt{L}}{\sqrt{g}}\right)g}\,dL = \frac{\pi}{\sqrt{gL}}\,dL \approx \left(\frac{\pi}{\sqrt{32.2}\,\sqrt{0.8156}}\right)(0.01) = 0.00613$ sec.

(c) The original clock completes 1 swing every second or $(24)(60)(60) = 86{,}400$ swings per day. The new clock completes 1 swing every 1.00613 seconds. Therefore it takes $(86{,}400)(1.00613) = 86{,}929.632$ seconds for the new clock to complete the same number of swings. Thus the new clock loses $\frac{529.632}{60} \approx 8.83$ min/day.

22. (a) If $\frac{1}{x} - 3 = 0$, then $\frac{1-3x}{x} = 0 \Rightarrow x = \frac{1}{3}$.

(b) $f(x) = \frac{1}{x} - 3$ and $f'(x) = -\frac{1}{x^2}$

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{x_n} - 3}{-\frac{1}{x_n^2}}$$
$$= 2x_n - 3x_n^2 = x_n(2 - 3x_n)$$



NOTES.

CHAPTER 4 INTEGRATION

4.1 INDEFINITE INTEGRALS

1. (a) $3x^2$ (b) $\frac{x^8}{8}$ (c) $\frac{x^8}{8} - 3x^2 + 8x$

2. (a) x^{-3} (b) $-\frac{x^{-3}}{3}$ (c) $-\frac{x^{-3}}{3} + x^2 + 3x$

3. (a) $\frac{1}{x^2}$ (b) $\frac{-1}{4x^2}$ (c) $\frac{x^4}{4} + \frac{1}{2x^2}$

4. (a) $\sqrt{x^3}$ (b) $\sqrt{x}$ (c) $\frac{2}{3}\sqrt{x^3} + 2\sqrt{x}$

5. (a) $x^{2/3}$ (b) $x^{1/3}$ (c) $x^{-1/3}$

6. (a) $\cos(\pi x)$ (b) $-3\cos x$ (c) $\frac{-\cos(\pi x)}{\pi} + \cos(3x)$

7. (a) $\tan x$ (b) $2\tan\left(\frac{x}{3}\right)$ (c) $-\frac{2}{3}\tan\left(\frac{3x}{2}\right)$

8. (a) $\sec x$ (b) $\frac{4}{3}\sec(3x)$ (c) $\frac{2}{\pi}\sec\left(\frac{\pi x}{2}\right)$

9. $\int (x+1)\,dx = \frac{x^2}{2} + x + C$

10. $\int \left(3t^2 + \frac{t}{2}\right) dt = t^3 + \frac{t^2}{4} + C$

11. $\int (2x^3 - 5x + 7)\,dx = \frac{1}{2}x^4 - \frac{5}{2}x^2 + 7x + C$

12. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx = \int \left(x^{-2} - x^2 - \frac{1}{3}\right) dx = \frac{x^{-1}}{-1} - \frac{x^3}{3} - \frac{1}{3}x + C = -\frac{1}{x} - \frac{x^3}{3} - \frac{x}{3} + C$

13. $\int x^{-1/3}\,dx = \frac{x^{2/3}}{\frac{2}{3}} + C = \frac{3}{2}x^{2/3} + C$

14. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx = \int \left(x^{1/2} + x^{1/3}\right) dx = \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{4/3}}{\frac{4}{3}} + C = \frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + C$

15. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy = \int \left(8y - 2y^{-1/4}\right) dy = \frac{8y^2}{2} - 2\left(\frac{y^{3/4}}{\frac{3}{4}}\right) + C = 4y^2 - \frac{8}{3}y^{3/4} + C$

16. $\int\left(\frac{\sqrt{x}}{2}+\frac{2}{\sqrt{x}}\right)dx=\int\left(\frac{1}{2}x^{1/2}+2x^{-1/2}\right)dx=\frac{1}{2}\left(\frac{x^{3/2}}{\frac{3}{2}}\right)+2\left(\frac{x^{1/2}}{\frac{1}{2}}\right)+C=\frac{1}{3}x^{3/2}+4x^{1/2}+C$

17. $\int\left(\frac{1}{7}-\frac{1}{y^{5/4}}\right)dy=\int\left(\frac{1}{7}-y^{-5/4}\right)dy=\frac{1}{7}y-\left(\frac{y^{-1/4}}{-\frac{1}{4}}\right)+C=\frac{y}{7}+\frac{4}{y^{1/4}}+C$

18. $\int 2x\left(1-x^{-3}\right)dx=\int\left(2x-2x^{-2}\right)dx=\frac{2x^2}{2}-2\left(\frac{x^{-1}}{-1}\right)+C=x^2+\frac{2}{x}+C$

19. $\int\frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt=\int\left(\frac{t^{3/2}}{t^2}+\frac{t^{1/2}}{t^2}\right)dt=\int\left(t^{-1/2}+t^{-3/2}\right)dt=\frac{t^{1/2}}{\frac{1}{2}}+\left(\frac{t^{-1/2}}{-\frac{1}{2}}\right)+C=2\sqrt{t}-\frac{2}{\sqrt{t}}+C$

20. $\int -2\cos t\,dt=-2\sin t+C$

21. $\int 7\sin\frac{\theta}{3}\,d\theta=-21\cos\frac{\theta}{3}+C$

22. $\int -3\csc^2 x\,dx=3\cot x+C$

23. $\int\left(1+\tan^2\theta\right)d\theta=\int\sec^2\theta\,d\theta=\tan\theta+C$

24. $\int\cot^2 x\,dx=\int\left(\csc^2 x-1\right)dx=-\cot x-x+C$

25. $\int\cos\theta\,(\tan\theta+\sec\theta)\,d\theta=\int(\sin\theta+1)\,d\theta=-\cos\theta+\theta+C$

26. $\int\frac{\csc\theta}{\csc\theta-\sin\theta}\,d\theta=\int\left(\frac{\csc\theta}{\csc\theta-\sin\theta}\right)\left(\frac{\sin\theta}{\sin\theta}\right)d\theta=\int\frac{1}{1-\sin^2\theta}\,d\theta=\int\frac{1}{\cos^2\theta}\,d\theta=\int\sec^2\theta\,d\theta=\tan\theta+C$

27. $\frac{d}{dx}\left(\frac{(7x-2)^4}{28}+C\right)=\frac{4(7x-2)^3(7)}{28}=(7x-2)^3$

28. $\frac{d}{dx}\left(-\frac{(3x+5)^{-1}}{3}+C\right)=-\left(-\frac{(3x+5)^{-2}(3)}{3}\right)=(3x+5)^{-2}$

29. $\frac{d}{dx}\left(-3\cot\left(\frac{x-1}{3}\right)+C\right)=-3\left(-\csc^2\left(\frac{x-1}{3}\right)\right)\left(\frac{1}{3}\right)=\csc^2\left(\frac{x-1}{3}\right)$

30. $\frac{d}{dx}\left(\frac{-1}{x+1}+C\right)=(-1)(-1)(x+1)^{-2}=\frac{1}{(x+1)^2}$

31. (a) Wrong: $\frac{d}{dx}\left(\frac{x^2}{2}\sin x+C\right)=\frac{2x}{2}\sin x+\frac{x^2}{2}\cos x=x\sin x+\frac{x^2}{2}\cos x$

(b) Wrong: $\frac{d}{dx}(-x\cos x+C)=-\cos x+x\sin x$

(c) Right: $\frac{d}{dx}(-x\cos x+\sin x+C)=-\cos x+x\sin x+\cos x=x\sin x$

32. (a) Wrong: $\frac{d}{dx}\left(\frac{(2x+1)^3}{3}+C\right)=\frac{3(2x+1)^2(2)}{3}=2(2x+1)^2$

(b) Wrong: $\frac{d}{dx}\left((2x+1)^3+C\right)=3(2x+1)^2(2)=6(2x+1)^2$

(c) Right: $\frac{d}{dx}\left((2x+1)^3+C\right)=6(2x+1)^2$

33. Graph (b), because $\frac{dy}{dx}=2x \Rightarrow y=x^2+C$. Then $y(1)=4 \Rightarrow C=3$.

34. Graph (b), because $\frac{dy}{dx}=-x \Rightarrow y=-\frac{1}{2}x^2+C$. Then $y(-1)=1 \Rightarrow C=\frac{3}{2}$.

35. $\frac{dy}{dx}=2x-7 \Rightarrow y=x^2-7x+C$; at $x=2$ and $y=0$ we have $0=2^2-7(2)+C \Rightarrow C=10 \Rightarrow y=x^2-7x+10$

36. $\frac{dy}{dx}=\frac{1}{x^2}+x=x^{-2}+x \Rightarrow y=-x^{-1}+\frac{x^2}{2}+C$; at $x=2$ and $y=1$ we have $1=-2^{-1}+\frac{2^2}{2}+C \Rightarrow C=-\frac{1}{2}$
$\Rightarrow y=-x^{-1}+\frac{x^2}{2}-\frac{1}{2}$ or $y=-\frac{1}{x}+\frac{x^2}{2}-\frac{1}{2}$

37. $\frac{dy}{dx}=3x^{-2/3} \Rightarrow y=9x^{1/3}+C$; at $x=-1$ and $y=-5$ we have $-5=9(-1)+C \Rightarrow C=4 \Rightarrow y=9x^{1/3}+4$

38. $\frac{dy}{dx}=\frac{1}{2\sqrt{x}}=\frac{1}{2}x^{-1/2} \Rightarrow y=x^{1/2}+C$; at $x=4$ and $y=0$ we have $0=4^{1/2}+C \Rightarrow C=-2 \Rightarrow y=x^{1/2}-2$

39. $\frac{ds}{dt}=\cos t+\sin t \Rightarrow s=\sin t-\cos t+C$; at $t=\pi$ and $s=1$ we have $1=\sin \pi-\cos \pi+C \Rightarrow C=0$
$\Rightarrow s=\sin t-\cos t$

40. $\frac{dr}{d\theta}=-\pi \sin \pi\theta \Rightarrow r=\cos(\pi\theta)+C$; at $r=0$ and $\theta=0$ we have $0=\cos(\pi 0)+C \Rightarrow C=-1 \Rightarrow r=\cos(\pi\theta)-1$

41. $\frac{dv}{dt}=\frac{1}{2}\sec t \tan t \Rightarrow v=\frac{1}{2}\sec t+C$; at $v=1$ and $t=0$ we have $1=\frac{1}{2}\sec(0)+C \Rightarrow C=\frac{1}{2} \Rightarrow v=\frac{1}{2}\sec t+\frac{1}{2}$

42. $\frac{dv}{dt}=8t+\csc^2 t \Rightarrow v=4t^2-\cot t+C$; at $v=-7$ and $t=\frac{\pi}{2}$ we have $-7=4\left(\frac{\pi}{2}\right)^2-\cot\left(\frac{\pi}{2}\right)+C \Rightarrow C=-7 \ -\pi^2$
$\Rightarrow v=4t^2-\cot t-7-\pi^2$

43. $\frac{d^2y}{dx^2}=2-6x \Rightarrow \frac{dy}{dx}=2x-3x^2+C_1$; at $\frac{dy}{dx}=4$ and $x=0$ we have $4=2(0)-3(0)^2+C_1 \Rightarrow C_1=4$
$\Rightarrow \frac{dy}{dx}=2x-3x^2+4 \Rightarrow y=x^2-x^3+4x+C_2$; at $y=1$ and $x=0$ we have $1=0^2-0^3+4(0)+C_2 \Rightarrow C_2=1$
$\Rightarrow y=x^2-x^3+4x+1$

44. $\frac{d^2r}{dt^2}=\frac{2}{t^3}=2t^{-3} \Rightarrow \frac{dr}{dt}=-t^{-2}+C_1$; at $\frac{dr}{dt}=1$ and $t=1$ we have $1=-(1)^{-2}+C_1 \Rightarrow C_1=2 \Rightarrow \frac{dr}{dt}=-t^{-2}+2$
$\Rightarrow r=t^{-1}+2t+C_2$; at $r=1$ and $t=1$ we have $1=1^{-1}+2(1)+C_2 \Rightarrow C_2=-2 \Rightarrow r=t^{-1}+2t-2$ or
$r=\frac{1}{t}+2t-2$

45. $\frac{d^3y}{dx^3} = 6 \Rightarrow \frac{d^2y}{dx^2} = 6x + C_1$; at $\frac{d^2y}{dx^2} = -8$ and $x = 0$ we have $-8 = 6(0) + C_1 \Rightarrow C_1 = -8 \Rightarrow \frac{d^2y}{dx^2} = 6x - 8$
$\Rightarrow \frac{dy}{dx} = 3x^2 - 8x + C_2$; at $\frac{dy}{dx} = 0$ and $x = 0$ we have $0 = 3(0)^2 - 8(0) + C_2 \Rightarrow C_2 = 0 \Rightarrow \frac{dy}{dx} = 3x^2 - 8x$
$\Rightarrow y = x^3 - 4x^2 + C_3$; at $y = 5$ and $x = 0$ we have $5 = 0^3 - 4(0)^2 + C_3 \Rightarrow C_3 = 5 \Rightarrow y = x^3 - 4x^2 + 5$

46. $y^{(4)} = -\sin t + \cos t \Rightarrow y''' = \cos t + \sin t + C_1$; at $y''' = 7$ and $t = 0$ we have $7 = \cos(0) + \sin(0) + C_1$
$\Rightarrow C_1 = 6 \Rightarrow y''' = \cos t + \sin t + 6 \Rightarrow y'' = \sin t - \cos t + 6t + C_2$; at $y'' = -1$ and $t = 0$ we have
$-1 = \sin(0) - \cos(0) + 6(0) + C_2 \Rightarrow C_2 = 0 \Rightarrow y'' = \sin t - \cos t + 6t \Rightarrow y' = -\cos t - \sin t + 3t^2 + C_3$;
at $y' = -1$ and $t = 0$ we have $-1 = -\cos(0) - \sin(0) + 3(0)^2 + C_3 \Rightarrow C_3 = 0 \Rightarrow y' = -\cos t - \sin t + 3t^2$
$\Rightarrow y = -\sin t + \cos t + t^3 + C_4$; at $y = 0$ and $t = 0$ we have $0 = -\sin(0) + \cos(0) + 0^3 + C_4 \Rightarrow C_4 = -1$
$\Rightarrow y = -\sin t + \cos t + t^3 - 1$

47. $v = \frac{ds}{dt} = 9.8t + 5 \Rightarrow s = 4.9t^2 + 5t + C$; at $s = 10$ and $t = 0$ we have $C = 10 \Rightarrow s = 4.9t^2 + 5t + 10$

48. $v = \frac{ds}{dt} = \frac{2}{\pi}\cos\left(\frac{2t}{\pi}\right) \Rightarrow s = \sin\left(\frac{2t}{\pi}\right) + C$; at $s = 1$ and $t = \pi^2$ we have $C = 1 \Rightarrow s = \sin\left(\frac{2t}{\pi}\right) + 1$

49. $a = 32 \Rightarrow v = 32t + C_1$; at $v = 20$ and $t = 0$ we have $C_1 = 20 \Rightarrow v = 32t + 20 \Rightarrow s = 16t^2 + 20t + C_2$; at $s = 5$ and $t = 0$ we have $C_2 = 5 \Rightarrow s = 16t^2 + 20t + 5$

50. $a = -4\sin(2t) \Rightarrow v = 2\cos(2t) + C_1$; at $v = 2$ and $t = 0$ we have $C_1 = 0 \Rightarrow v = 2\cos(2t)$
$\Rightarrow s = \sin(2t) + C_2$; at $s = -3$ and $t = 0$ we have $C_2 = -3 \Rightarrow s = \sin(2t) - 3$

51. $m = y' = 3\sqrt{x} = 3x^{1/2} \Rightarrow y = 2x^{3/2} + C$; at $(9,4)$ we have $4 = 2(9)^{3/2} + C \Rightarrow C = -50 \Rightarrow y = 2x^{3/2} - 50$

52. (a) $\frac{d^2y}{dx^2} = 6x \Rightarrow \frac{dy}{dx} = 3x^2 + C_1$; at $y' = 0$ and $x = 0$ we have $0 = 3(0)^2 + C_1 \Rightarrow C_1 = 0 \Rightarrow \frac{dy}{dx} = 3x^2$
$\Rightarrow y = x^3 + C_2$; at $y = 1$ and $x = 0$ we have $C_2 = 1 \Rightarrow y = x^3 + 1$

(b) One, because any other possible function would differ from $x^3 + 1$ by a constant that must be zero because of the initial conditions

53. $\frac{dy}{dx} = 1 - \frac{4}{3}x^{1/3} \Rightarrow y = \int\left(1 - \frac{4}{3}x^{1/3}\right)dx = x - x^{4/3} + C$; at $(1, 0.5)$ on the curve we have $0.5 = 1 - 1^{4/3} + C$
$\Rightarrow C = 0.5 \Rightarrow y = x - x^{4/3} + \frac{1}{2}$

54. $\frac{dy}{dx} = x - 1 \Rightarrow y = \int (x+1)\,dx = \frac{x^2}{2} - x + C$; at $(-1,1)$ on the curve we have $1 = \frac{(-1)^2}{2} - (-1) + C$
$\Rightarrow C = -\frac{1}{2} \Rightarrow y = \frac{x^2}{2} - x - \frac{1}{2}$

55. $\frac{dy}{dx} = \sin x - \cos x \Rightarrow y = \int (\sin x - \cos x)\,dx = -\cos x - \sin x + C$; at $(-\pi, -1)$ on the curve we have
$-1 = -\cos(-\pi) - \sin(-\pi) + C \Rightarrow C = -2 \Rightarrow y = -\cos x - \sin x - 2$

56. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \pi \sin \pi x = \frac{1}{2}x^{-1/2} + \pi \sin \pi x \Rightarrow y = \int \left(\frac{1}{2}x^{-1/2} + \pi \sin \pi x\right) dx = x^{1/2} - \cos \pi x + C$; at $(1,2)$ on the curve we have $2 = 1^{1/2} - \cos \pi(1) + C \Rightarrow C = 0 \Rightarrow y = \sqrt{x} - \cos \pi x$

57. $a(t) = v'(t) = 1.6 \Rightarrow v(t) = 1.6t + C$; at $(0,0)$ we have $C = 0 \Rightarrow v(t) = 1.6t$. When $t = 30$, then $v(30) = 48$ m/sec.

58. $a(t) = v'(t) = 20 \Rightarrow v(t) = 20t + C$; at $(0,0)$ we have $C = 0 \Rightarrow v(t) = 20t$. When $t = 60$, then $v(60) = 20(60) = 1200$ m/sec.

59. Step 1: $\frac{d^2s}{dt^2} = -k \Rightarrow \frac{ds}{dt} = -kt + C_1$; at $\frac{ds}{dt} = 88$ and $t = 0$ we have $C_1 = 88 \Rightarrow \frac{ds}{dt} = -kt + 88 \Rightarrow$

$s = -k\left(\frac{t^2}{2}\right) + 88t + C_2$; at $s = 0$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = -\frac{kt^2}{2} + 88t$

Step 2: $\frac{ds}{dt} = 0 \Rightarrow 0 = -kt + 88 \Rightarrow t = \frac{88}{k}$

Step 3: $242 = \frac{-k\left(\frac{88}{k}\right)^2}{2} + 88\left(\frac{88}{k}\right) \Rightarrow 242 = -\frac{(88)^2}{2k} + \frac{(88)^2}{k} \Rightarrow 242 = \frac{(88)^2}{2k} \Rightarrow k = 16$

60. $\frac{d^2s}{dt^2} = -k \Rightarrow \frac{ds}{dt} = \int -k\,dt = -kt + C$; at $\frac{ds}{dt} = 44$ when $t = 0$ we have $44 = -k(0) + C \Rightarrow C = 44$

$\Rightarrow \frac{ds}{dt} = -kt + 44 \Rightarrow s = -\frac{kt^2}{2} + 44t + C_1$; at $s = 0$ when $t = 0$ we have $0 = -\frac{k(0)^2}{2} + 44(0) + C_1 \Rightarrow C_1 = 0$

$\Rightarrow s = -\frac{kt^2}{2} + 44t$. Then $\frac{ds}{dt} = 0 \Rightarrow -kt + 44 = 0 \Rightarrow t = \frac{44}{k}$ and $s\left(\frac{44}{k}\right) = -\frac{k\left(\frac{44}{k}\right)^2}{2} + 44\left(\frac{44}{k}\right) = 45$

$\Rightarrow -\frac{968}{k} + \frac{1936}{k} = 45 \Rightarrow \frac{968}{k} = 45 \Rightarrow k = \frac{968}{45} \approx 21.5$ ft/sec^2.

61. (a) $v = \int a\,dt = \int \left(15t^{1/2} - 3t^{-1/2}\right) dt = 10t^{3/2} - 6t^{1/2} + C$; $\frac{ds}{dt}(1) = 4 \Rightarrow 4 = 10(1)^{3/2} - 6(1)^{1/2} + C \Rightarrow C = 0$

$\Rightarrow v = 10t^{3/2} - 6t^{1/2}$

(b) $s = \int v\,dt = \int \left(10t^{3/2} - 6t^{1/2}\right) dt = 4t^{5/2} - 4t^{3/2} + C$; $s(1) = 0 \Rightarrow 0 = 4(1)^{5/2} - 4(1)^{3/2} + C \Rightarrow C = 0$

$\Rightarrow s = 4t^{5/2} - 4t^{3/2}$

62. $\frac{d^2s}{dt^2} = -5.2 \Rightarrow \frac{ds}{dt} = -5.2t + C_1$; at $\frac{ds}{dt} = 0$ and $t = 0$ we have $C_1 = 0 \Rightarrow \frac{ds}{dt} = -5.2t \Rightarrow s = -2.6t^2 + C_2$; at $s = 4$ and $t = 0$ we have $C_2 = 4 \Rightarrow s = -2.6t^2 + 4$. Then $s = 0 \Rightarrow 0 = -2.6t^2 + 4 \Rightarrow t = \sqrt{\frac{4}{2.6}} \approx 1.24$ sec, since $t > 0$

63. $\frac{d^2s}{dt^2} = a \Rightarrow \frac{ds}{dt} = \int a\,dt = at + C$; $\frac{ds}{dt} = v_0$ when $t = 0 \Rightarrow C = v_0 \Rightarrow \frac{ds}{dt} = at + v_0 \Rightarrow s = \frac{at^2}{2} + v_0t + C_1$; $s = s_0$ when $t = 0 \Rightarrow s_0 = \frac{a(0)^2}{2} + v_0(0) + C_1 \Rightarrow C_1 = s_0 \Rightarrow s = \frac{at^2}{2} + v_0t + s_0$

64. The appropriate initial value problem is: Differential Equation: $\frac{d^2s}{dt^2} = -g$ with Initial Conditions: $\frac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$. Thus, $\frac{ds}{dt} = \int -g\,dt = -gt + C_1$; $\frac{ds}{dt}(0) = v_0 \Rightarrow v_0 = (-g)(0) + C_1 \Rightarrow C_1 = v_0$

$\Rightarrow v = \frac{ds}{dt} = -gt + v_0 \Rightarrow s = -\frac{1}{2}gt^2 + v_0t + C_2$; $s(0) = s_0 \Rightarrow s_0 = -\frac{1}{2}(g)(0)^2 + v_0(0) + C_2 \Rightarrow C_2 = s_0$

$\Rightarrow s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$

65. (a) $\frac{ds}{dt} = 9.8t - 3 \Rightarrow s = 4.9t^2 - 3t + C$; (i) at $s = 5$ and $t = 0$ we have $C = 5 \Rightarrow s = 4.9t^2 - 3t + 5$; displacement $= s(3) - s(1) = ((4.9)(9) - 9 + 5) - (4.9 - 3 + 5) = 33.2$ units; (ii) at $s = -2$ and $t = 0$ we have $C = -2 \Rightarrow s = 4.9t^2 - 3t - 2$; displacement $= s(3) - s(1) = ((4.9)(9) - 9 - 2) - (4.9 - 3 - 2) = 33.2$ units; (iii) at $s = s_0$ and $t = 0$ we have $C = s_0 \Rightarrow s = 4.9t^2 - 3t + s_0$; displacement $= s(3) - s(1)$ $= ((4.9)(9) - 9 + s_0) - (4.9 - 3 + s_0) = 33.2$ units

(b) True. Given an antiderivative $f(t)$ of the velocity function, we know that the body's position function is $s = f(t) + C$ for some constant C. Therefore, the displacement from $t = a$ to $t = b$ is $(f(b) + C) - (f(a) + C)$ $= f(b) - f(a)$. Thus we can find the displacement from any antiderivative f as the numerical difference $f(b) - f(a)$ without knowing the exact values of C and s.

66. Yes. If $F(x)$ and $G(x)$ both solve the initial value problem on an interval I then they both have the same first derivative. Therefore, by Corollary 2 of the Mean Value Theorem there is a constant C such that $F(x) = G(x) + C$ for *all* x. In particular, $F(x_0) = G(x_0) + C$, so $C = F(x_0) - G(x_0) = 0$. Hence $F(x) = G(x)$ for all x.

67-70. Example CAS commands with text inserts:

Maple:

The following commands use the definite integral and the Fundamental Theorem of calculus to construct the solution of the initial value problem.

```
>restart;
>f:=x->(cos(x))^2+sin(x);
>initialx:=Pi;
>initialy:=1;
>y:=x->int(f(t),t=initialx..x)+initialy;
>y(x);
```

Verify that the solution is correct.

```
>diff(y(x),x);
>y(Pi);
```

Plot the solution curve.

```
>plot(y(x),x=-2*Pi..8*Pi);
```

Mathematica:

The following commands use the definite integral and the Fundamental Theorem of calculus to construct the solution of the initial value problem.

```
Clear[y,yprime];
yprime[x_] = Cos[x]^2+Sin[x];
initxval = Pi;
inityval = 1;
y[x_] = ∫_initxval^x yprime[t] dt + inityval
```

Verify that the solution is correct.

```
∂_x y[x]
y[π]
```

Plot the solution curve.
Plot[y[x], {x,-2π,8π}, PlotStyle→ {RGBColor[1,0,0]}];

71-72. Example CAS commands with text inserts:

Maple:

The following commands use a definite integral and the Fundamental Theorem of calculus to construct the solution of the initial value problem.

```
>restart;
>f:=x->3*exp(x/2)+1;
>initialx:=0;
>initialy:=-1;
>initialyprime:=4;
>yprime:=x->int(f(t),t=initialx..x)+initialprime;
>yprime(x);
>y:=x->int(yprime(u),u=initialx..x)+initialy;
>y(x);
>
```

Verify that the solution is correct.

```
>diff(y(x),x,x);
>yprime(0);
>y(0);
```

Plot the solution curve.

```
>plot({yprime(x),y(x)},x=-6..4);
```

Mathematica:

The following commands use a definite integral and the Fundamental Theorem of calculus to construct the solution of the initial value problem.

Clear[y,yprime,y2prime,initxval,inityval,inityprimeval];
y2prime[x_] = $3E^{x/2}$+1;
initxval = 0;
inityprimeval = -1
inityval = 4;
yprime[x_] = $\int_{\text{initxval}}^{x}$ y2prime[t] dt + inityprimeval
y[x_] = $\int_{\text{initxval}}^{x}$ yprime[t] + inityval
Verify that the solution is correct.
D[y[x], {x,2}] // Simplify
yprime[0]
y[0]
Plot the solution curve
Plot[{yprime[x],y[x]},{x,-6,6},
  PlotStyle→{{RGBColor[1,0,0]}, {RGBColor[0,0,1]}}];

4.2 INTEGRAL RULES; INTEGRATION BY SUBSTITUTION

1. Let $u = 2x^2 \Rightarrow du = 4x\,dx \Rightarrow \frac{1}{4}\,du = x\,dx$

$$\int x \sin\left(2x^2\right) dx = \int \frac{1}{4} \sin u \, du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos 2x^2 + C$$

2. Let $u = 1 - \cos\frac{t}{2} \Rightarrow du = \frac{1}{2}\sin\frac{t}{2}\,dt \Rightarrow 2\,du = \sin\frac{t}{2}\,dt$

$$\int \left(1 - \cos\frac{t}{2}\right)^2\left(\sin\frac{t}{2}\right)dt = \int 2u^2\,du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(1 - \cos\frac{t}{2}\right)^3 + C$$

3. Let $u = 7x - 2 \Rightarrow du = 7\,dx \Rightarrow \frac{1}{7}\,du = dx$

$$\int 28(7x-2)^{-5}\,dx = \int \frac{1}{7}(28)u^{-5}\,du = \int 4u^{-5}\,du = -u^{-4} + C = -(7x-2)^{-4} + C$$

4. Let $u = x^4 - 1 \Rightarrow du = 4x^3\,dx \Rightarrow \frac{1}{4}\,du = x^3\,dx$

$$\int x^3\left(x^4 - 1\right)^2 dx = \int \frac{1}{4}u^2\,du = \frac{u^3}{12} + C = \frac{1}{12}\left(x^4 - 1\right)^3 + C$$

5. Let $u = 1 - r^3 \Rightarrow du = -3r^2\,dr \Rightarrow -3\,du = 9r^2\,dr$

$$\int \frac{9r^2\,dr}{\sqrt{1 - r^3}} = \int -3u^{-1/2}\,du = -3(2)u^{1/2} + C = -6\left(1 - r^3\right)^{1/2} + C$$

6. Let $u = y^4 + 4y^2 + 1 \Rightarrow du = \left(4y^3 + 8y\right)dy \Rightarrow 3\,du = 12\left(y^3 + 2y\right)dy$

$$\int 12\left(y^4 + 4y^2 + 1\right)^2\left(y^3 + 2y\right)dy = \int 3u^2\,du = u^3 + C = \left(y^4 + 4y^2 + 1\right)^3 + C$$

7. Let $u = x^{3/2} - 1 \Rightarrow du = \frac{3}{2}x^{1/2}\,dx \Rightarrow \frac{2}{3}\,du = \sqrt{x}\,dx$

$$\int \sqrt{x}\sin^2\left(x^{3/2} - 1\right)dx = \int \frac{2}{3}\sin^2 u\,du = \frac{2}{3}\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right) + C = \frac{1}{3}\left(x^{3/2} - 1\right) - \frac{1}{6}\sin\left(2x^{3/2} - 2\right) + C$$

8. Let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}\,dx$

$$\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx = -\int \cos^2 u\,du = -\left(\frac{u}{2} + \frac{1}{4}\sin 2u\right) + C = -\frac{1}{2x} - \frac{1}{4}\sin\left(\frac{2}{x}\right) + C$$

9. (a) Let $u = \cot 2\theta \Rightarrow du = -2\csc^2 2\theta\,d\theta \Rightarrow -\frac{1}{2}\,du = \csc^2 2\theta\,d\theta$

$$\int \csc^2 2\theta\cot 2\theta\,d\theta = -\int \frac{1}{2}u\,du = -\frac{1}{2}\left(\frac{u^2}{2}\right) + C = -\frac{u^2}{4} + C = -\frac{1}{4}\cot^2 2\theta + C$$

(b) Let $u = \csc 2\theta \Rightarrow du = -2\csc 2\theta\cot 2\theta\,d\theta \Rightarrow -\frac{1}{2}\,du = \csc 2\theta\cot 2\theta\,d\theta$

$$\int \csc^2 2\theta\cot 2\theta\,d\theta = \int -\frac{1}{2}u\,du = -\frac{1}{2}\left(\frac{u^2}{2}\right) + C = -\frac{u^2}{4} + C = -\frac{1}{4}\csc^2 2\theta + C$$

10. (a) Let $u = 5x + 8 \Rightarrow du = 5\,dx \Rightarrow \frac{1}{5}\,du = dx$

$$\int \frac{dx}{\sqrt{5x+8}} = \int \frac{1}{5}\left(\frac{1}{\sqrt{u}}\right)du = \frac{1}{5}\int u^{-1/2}\,du = \frac{1}{5}\left(2u^{1/2}\right) + C = \frac{2}{5}u^{1/2} + C = \frac{2}{5}\sqrt{5x+8} + C$$

(b) Let $u = \sqrt{5x+8} \Rightarrow du = \frac{1}{2}(5x+8)^{-1/2}(5)\ dx \Rightarrow \frac{2}{5}\ du = \frac{dx}{\sqrt{5x+8}}$

$$\int \frac{dx}{\sqrt{5x+8}} = \int \frac{2}{5}\ du = \frac{2}{5}u + C = \frac{2}{5}\sqrt{5x+8} + C$$

11. Let $u = 3 - 2s \Rightarrow du = -2\ ds \Rightarrow -\frac{1}{2}\ du = ds$

$$\int \sqrt{3-2s}\ ds = \int \sqrt{u}\left(-\frac{1}{2}\ du\right) = -\frac{1}{2}\int u^{1/2}\ du = \left(-\frac{1}{2}\right)\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{3}(3-2s)^{3/2} + C$$

12. Let $u = 5s + 4 \Rightarrow du = 5\ ds \Rightarrow \frac{1}{5}\ du = ds$

$$\int \frac{1}{\sqrt{5s+4}}\ ds = \int \frac{1}{\sqrt{u}}\left(\frac{1}{5}\ du\right) = \frac{1}{5}\int u^{-1/2}\ du = \left(\frac{1}{5}\right)\left(2u^{1/2}\right) + C = \frac{2}{5}\sqrt{5s+4} + C$$

13. Let $u = 2 - x \Rightarrow du = -dx \Rightarrow -du = dx$

$$\int \frac{3}{(2-x)^2}\ dx = \int \frac{3(-du)}{u^2} = -3\int u^{-2}\ du = -3\left(\frac{u^{-1}}{-1}\right) + C = \frac{3}{2-x} + C$$

14. Let $u = 1 - \theta^2 \Rightarrow du = -2\theta\ d\theta \Rightarrow -\frac{1}{2}\ du = \theta\ d\theta$

$$\int \theta\sqrt[4]{1-\theta^2}\ d\theta = \int \sqrt[4]{u}\left(-\frac{1}{2}\ du\right) = -\frac{1}{2}\int u^{1/4}\ du = \left(-\frac{1}{2}\right)\left(\frac{4}{5}u^{5/4}\right) + C = -\frac{2}{5}\left(1-\theta^2\right)^{5/4} + C$$

15. Let $u = 7 - 3y^2 \Rightarrow du = -6y\ dy \Rightarrow -\frac{1}{2}\ du = 3y\ dy$

$$\int 3y\sqrt{7-3y^2}\ dy = \int \sqrt{u}\left(-\frac{1}{2}\ du\right) = -\frac{1}{2}\int u^{1/2}\ du = \left(-\frac{1}{2}\right)\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{3}\left(7-3y^2\right)^{3/2} + C$$

16. Let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\ dx \Rightarrow 2\ du = \frac{1}{\sqrt{x}}\ dx$

$$\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\ dx = \int \frac{2\ du}{u^2} = -\frac{2}{u} + C = \frac{-2}{1+\sqrt{x}} + C$$

17. Let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\ dx \Rightarrow 2\ du = \frac{1}{\sqrt{x}}\ dx$

$$\int \frac{\left(1+\sqrt{x}\right)^3}{\sqrt{x}}\ dx = \int u^3\,(2\ du) = 2\left(\frac{1}{4}u^4\right) + C = \frac{1}{2}\left(1+\sqrt{x}\right)^4 + C$$

18. Let $u = 3z + 4 \Rightarrow du = 3\ dz \Rightarrow \frac{1}{3}du = dz$

$$\int \cos(3z+4)\ dz = \int (\cos u)\left(\frac{1}{3}\ du\right) = \frac{1}{3}\int \cos u\ du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3z+4) + C$$

19. Let $u = 3x + 2 \Rightarrow du = 3\ dx \Rightarrow \frac{1}{3}\ du = dx$

$$\int \sec^2 (3x+2)\ dx = \int \left(\sec^2 u\right)\left(\frac{1}{3}\ du\right) = \frac{1}{3}\int \sec^2 u\ du = \frac{1}{3}\tan u + C = \frac{1}{3}\tan (3x+2) + C$$

20. Let $u = \sin\left(\frac{x}{3}\right) \Rightarrow du = \frac{1}{3}\cos\left(\frac{x}{3}\right) dx \Rightarrow 3\ du = \cos\left(\frac{x}{3}\right) dx$

$$\int \sin^5\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right) dx = \int u^5\,(3\ du) = 3\left(\frac{1}{6}u^6\right) + C = \frac{1}{2}\sin^6\left(\frac{x}{3}\right) + C$$

21. Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx \Rightarrow 2\ du = \sec^2\left(\frac{x}{2}\right) dx$

$$\int \tan^7\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right) dx = \int u^7\,(2\ du) = 2\left(\frac{1}{8}u^8\right) + C = \frac{1}{4}\tan^8\left(\frac{x}{2}\right) + C$$

22. Let $u = \frac{r^3}{18} - 1 \Rightarrow du = \frac{r^2}{6}\ dr \Rightarrow 6\ du = r^2\ dr$

$$\int r^2\left(\frac{r^3}{18} - 1\right)^5 dr = \int u^5\,(6\ du) = 6\int u^5\ du = 6\left(\frac{u^6}{6}\right) + C = \left(\frac{r^3}{18} - 1\right)^6 + C$$

23. Let $u = x^{3/2} + 1 \Rightarrow du = \frac{3}{2}x^{1/2}\ dx \Rightarrow \frac{2}{3}\ du = x^{1/2}\ dx$

$$\int x^{1/2}\sin\left(x^{3/2} + 1\right) dx = \int (\sin u)\left(\frac{2}{3}\ du\right) = \frac{2}{3}\int \sin u\ du = \frac{2}{3}(-\cos u) + C = -\frac{2}{3}\cos\left(x^{3/2} + 1\right) + C$$

24. Let $u = \sec\left(v + \frac{\pi}{2}\right) \Rightarrow du = \sec\left(v + \frac{\pi}{2}\right)\tan\left(v + \frac{\pi}{2}\right) dv$

$$\int \sec\left(v + \frac{\pi}{2}\right)\tan\left(v + \frac{\pi}{2}\right) dv = \int du = u + C = \sec\left(v + \frac{\pi}{2}\right) + C$$

25. Let $u = \cos (2t+1) \Rightarrow du = -2\sin (2t+1) \Rightarrow -\frac{1}{2}\ du = \sin (2t+1)$

$$\int \frac{\sin (2t+1)}{\cos^2 (2t+1)}\ dt = \int -\frac{1}{2}\frac{du}{u^2} = \frac{1}{2u} + C = \frac{1}{2\cos (2t+1)} + C$$

26. Let $u = 2 + \sin t \Rightarrow du = \cos t\ dt$

$$\int \frac{6\cos t}{(2+\sin t)^3}\ dt = \int \frac{6}{u^3}\ du = 6\int u^{-3}\ du = 6\left(\frac{u^{-2}}{-2}\right) + C = -3(2+\sin t)^{-2} + C$$

27. Let $u = \cot y \Rightarrow du = -\csc^2 y\ dy \Rightarrow -du = \csc^2 y\ dy$

$$\int \sqrt{\cot y}\,\csc^2 y\ dy = \int \sqrt{u}\,(-du) = -\int u^{1/2}\ du = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(\cot y)^{3/2} + C = -\frac{2}{3}\left(\cot^3 y\right)^{1/2} + C$$

28. Let $u = \frac{1}{t} - 1 = t^{-1} - 1 \Rightarrow du = -t^{-2}\,dt \Rightarrow -du = \frac{1}{t^2}\,dt$

$$\int \frac{1}{t^2}\cos\left(\frac{1}{t} - 1\right) dt = \int (\cos u)(-du) = -\int \cos u\,du = -\sin u + C = -\sin\left(\frac{1}{t} - 1\right) + C$$

29. Let $u = \sqrt{t} + 3 = t^{1/2} + 3 \Rightarrow du = \frac{1}{2}t^{-1/2}\,dt \Rightarrow 2\,du = \frac{1}{\sqrt{t}}\,dt$

$$\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t} + 3\right) dt = \int (\cos u)(2\,du) = 2\int \cos u\,du = 2\sin u + C = 2\sin\left(\sqrt{t} + 3\right) + C$$

30. Let $u = \sin\frac{1}{\theta} \Rightarrow du = \left(\cos\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right) d\theta \Rightarrow -du = \frac{1}{\theta^2}\cos\frac{1}{\theta}\,d\theta$

$$\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta = \int -u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\sin^2\frac{1}{\theta} + C$$

31. Let $u = \csc\sqrt{\theta} \Rightarrow du = \left(-\csc\sqrt{\theta}\cot\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right) d\theta \Rightarrow -2\,du = \frac{1}{\sqrt{\theta}}\cot\sqrt{\theta}\csc\sqrt{\theta}\,d\theta$

$$\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta = \int \frac{1}{\sqrt{\theta}}\cot\sqrt{\theta}\csc\sqrt{\theta}\,d\theta = \int -2\,du = -2u + C = -2\csc\sqrt{\theta} + C = -\frac{2}{\sin\sqrt{\theta}} + C$$

32. Let $u = 1 - \frac{1}{x} \Rightarrow du = \frac{1}{x^2}\,dx$

$$\int \sqrt{\frac{x-1}{x^5}}\,dx = \int \frac{1}{x^2}\sqrt{\frac{x-1}{x}}\,dx = \int \frac{1}{x^2}\sqrt{1 - \frac{1}{x}}\,dx = \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(1 - \frac{1}{x}\right)^{3/2} + C$$

33. (a) Let $u = \tan x \Rightarrow du = \sec^2 x\,dx$; $v = u^3 \Rightarrow dv = 3u^2\,du \Rightarrow 6\,dv = 18u^2\,du$; $w = 2 + v \Rightarrow dw = dv$

$$\int \frac{18\tan^2 x\sec^2 x}{\left(2 + \tan^3 x\right)^2}\,dx = \int \frac{18u^2}{\left(2 + u^3\right)^2}\,du = \int \frac{6\,dv}{(2+v)^2} = \int \frac{6\,dw}{w^2} = 6\int w^{-2}\,dw = -6w^{-1} + C = -\frac{6}{2+v} + C$$

$$= -\frac{6}{2+u^3} + C = -\frac{6}{2+\tan^3 x} + C$$

(b) Let $u = \tan^3 x \Rightarrow du = 3\tan^2 x\sec^2 x\,dx \Rightarrow 6\,du = 18\tan^2 x\sec^2 x\,dx$; $v = 2 + u \Rightarrow dv = du$

$$\int \frac{18\tan^2 x\sec^2 x}{\left(2 + \tan^3 x\right)^2}\,dx = \int \frac{6\,du}{(2+u)^2} = \int \frac{6\,dv}{v^2} = -\frac{6}{v} + C = -\frac{6}{2+u} + C = -\frac{6}{2+\tan^3 x} + C$$

(c) Let $u = 2 + \tan^3 x \Rightarrow du = 3\tan^2 x\sec^2 x\,dx \Rightarrow 6\,du = 18\tan^2 x\sec^2 x\,dx$

$$\int \frac{18\tan^2 x\sec^2 x}{\left(2 + \tan^3 x\right)^2}\,dx = \int \frac{6\,du}{u^2} = -\frac{6}{u} + C = -\frac{6}{2+\tan^3 x} + C$$

34. (a) Let $u = x - 1 \Rightarrow du = dx$; $v = \sin u \Rightarrow dv = \cos u\ du$; $w = 1 + v^2 \Rightarrow dw = 2v\ dv \Rightarrow \frac{1}{2}\ dw = v\ dv$

$$\int \sqrt{1+\sin^2(x-1)}\ \sin(x-1)\ \cos(x-1)\ dx = \int \sqrt{1+\sin^2 u}\ \sin u \cos u\ du = \int v\sqrt{1+v^2}\ dv$$

$$= \int \frac{1}{2}\sqrt{w}\ dw = \frac{1}{3}w^{3/2} + C = \frac{1}{3}\left(1+v^2\right)^{3/2} + C = \frac{1}{3}\left(1+\sin^2 u\right)^{3/2} + C = \frac{1}{3}\left(2+\sin^2(x-1)\right)^{3/2} + C$$

(b) Let $u = \sin(x-1) \Rightarrow du = \cos(x-1)\ dx$; $v = 1 + u^2 \Rightarrow dv = 2u\ du \Rightarrow \frac{1}{2}\ dv = u\ du$

$$\int \sqrt{1+\sin^2(x-1)}\ \sin(x-1)\ \cos(x-1)\ dx = \int u\sqrt{1+u^2}\ du = \int \frac{1}{2}\sqrt{v}\ dv = \int \frac{1}{2}v^{1/2}\ dv$$

$$= \left(\frac{1}{2}\left(\frac{2}{3}\right)v^{3/2}\right) + C = \frac{1}{3}v^{3/2} + C = \frac{1}{3}\left(1+u^2\right)^{3/2} + C = \frac{1}{3}\left(1+\sin^2(x-1)\right)^{3/2} + C$$

(c) Let $u = 1 + \sin^2(x-1) \Rightarrow du = 2\ \sin(x-1)\ \cos(x-1)\ dx \Rightarrow \frac{1}{2}\ du = \sin(x-1)\ \cos(x-1)\ dx$

$$\int \sqrt{1+\sin^2(x-1)}\ \sin(x-1)\ \cos(x-1)\ dx = \int \frac{1}{2}\sqrt{u}\ du = \int \frac{1}{2}u^{1/2}\ du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{3}\left(1+\sin^2(x-1)\right)^{3/2} + C$$

35. Let $u = 3(2r-1)^2 + 6 \Rightarrow du = 6(2r-1)(2)\ dr \Rightarrow \frac{1}{12}\ du = (2r-1)\ dr$; $v = \sqrt{u} \Rightarrow dv = \frac{1}{2\sqrt{u}}\ du \Rightarrow \frac{1}{6}\ dv$ $= \frac{1}{12\sqrt{u}}\ du$

$$\int \frac{(2r-1)\ \cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\ dr = \int \left(\frac{\cos\sqrt{u}}{\sqrt{u}}\right)\left(\frac{1}{12}\ du\right) = \int (\cos v)\left(\frac{1}{6}\ dv\right) = \frac{1}{6}\sin v + C = \frac{1}{6}\sin\sqrt{u} + C$$

$$= \frac{1}{6}\sin\sqrt{3(2r-1)^2+6} + C$$

36. Let $u = \cos\sqrt{\theta} \Rightarrow du = \left(-\sin\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)\ d\theta \Rightarrow -2\ du = \frac{\sin\sqrt{\theta}}{\sqrt{\theta}}\ d\theta$

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\ d\theta = \int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\sqrt{\cos^3\sqrt{\theta}}}\ d\theta = \int \frac{-2\ du}{u^{3/2}} = -2\int u^{-3/2}\ du = -2\left(-2u^{-1/2}\right) + C = \frac{4}{\sqrt{u}} + C$$

$$= \frac{4}{\sqrt{\cos\sqrt{\theta}}} + C$$

37. Let $u = 3t^2 - 1 \Rightarrow du = 6t\ dt \Rightarrow 2\ du = 12t\ dt$

$s = \int 12t\left(3t^2-1\right)^3\ dt = \int u^3\ (2\ du) = 2\left(\frac{1}{4}u^4\right) + C = \frac{1}{2}u^4 + C = \frac{1}{2}\left(3t^2-1\right)^4 + C$;

$s = 3$ when $t = 1 \Rightarrow 3 = \frac{1}{2}(3-1)^4 + C \Rightarrow 3 = 8 + C \Rightarrow C = -5 \Rightarrow s = \frac{1}{2}\left(3t^2-1\right)^4 - 5$

38. Let $u = x^2 + 8 \Rightarrow du = 2x\ dx \Rightarrow 2\ du = 4x\ dx$

$y = \int 4x\left(x^2+8\right)^{-1/3}\ dx = \int u^{-1/3}\ (2\ du) = 2\left(\frac{3}{2}u^{2/3}\right) + C = 3u^{2/3} + C = 3\left(x^2+8\right)^{2/3} + C$;

$y = 0$ when $x = 0 \Rightarrow 0 = 3(8)^{2/3} + C \Rightarrow C = -12 \Rightarrow y = 3\left(x^2+8\right)^{2/3} - 12$

39. Let $u = t + \frac{\pi}{12} \Rightarrow du = dt$

$s = \int 8 \sin^2\left(t + \frac{\pi}{12}\right) dt = \int 8 \sin^2 u \, du = 8\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right) + C = 4\left(t + \frac{\pi}{12}\right) - 2\sin\left(2t + \frac{\pi}{6}\right) + C;$

$s = 8$ when $t = 0 \Rightarrow 8 = 4\left(\frac{\pi}{12}\right) - 2\sin\left(\frac{\pi}{6}\right) + C \Rightarrow C = 8 - \frac{\pi}{3} + 1 = 9 - \frac{\pi}{3} \Rightarrow s = 4t - 2\sin\left(2t + \frac{\pi}{6}\right) + 9$

40. Let $u = \frac{\pi}{4} - \theta \Rightarrow -du = d\theta$

$r = \int 3\cos^2\left(\frac{\pi}{4} - \theta\right) d\theta = -\int 3\cos^2 u \, du = -3\left(\frac{u}{2} + \frac{1}{4}\sin 2u\right) + C = -\frac{3}{2}\left(\frac{\pi}{4} - \theta\right) - \frac{3}{4}\sin\left(\frac{\pi}{2} - 2\theta\right) + C;$

$r = \frac{\pi}{8}$ when $\theta = 0 \Rightarrow \frac{\pi}{8} = -\frac{3\pi}{8} - \frac{3}{4}\sin\frac{\pi}{2} + C \Rightarrow C = \frac{\pi}{2} + \frac{3}{4} \Rightarrow r = -\frac{3}{2}\left(\frac{\pi}{4} - \theta\right) - \frac{3}{4}\sin\left(\frac{\pi}{2} - 2\theta\right) + \frac{\pi}{2} + \frac{3}{4}$

$\Rightarrow r = \frac{3}{2}\theta - \frac{3}{4}\sin\left(\frac{\pi}{2} - 2\theta\right) + \frac{\pi}{8} + \frac{3}{4} \Rightarrow r = \frac{3}{2}\theta - \frac{3}{4}\cos 2\theta + \frac{\pi}{8} + \frac{3}{4}$

41. Let $u = 2t - \frac{\pi}{2} \Rightarrow du = 2\,dt \Rightarrow -2\,du = -4\,dt$

$\frac{ds}{dt} = \int -4\sin\left(2t - \frac{\pi}{2}\right) dt = \int (\sin u)(-2\,du) = 2\cos u + C_1 = 2\cos\left(2t - \frac{\pi}{2}\right) + C_1;$

at $t = 0$ and $\frac{ds}{dt} = 100$ we have $100 = 2\cos\left(-\frac{\pi}{2}\right) + C_1 \Rightarrow C_1 = 100 \Rightarrow \frac{ds}{dt} = 2\cos\left(2t - \frac{\pi}{2}\right) + 100$

$\Rightarrow s = \int \left(2\cos\left(2t - \frac{\pi}{2}\right) + 100\right) dt = \int (\cos u + 50)\,du = \sin u + 50u + C_2 = \sin\left(2t - \frac{\pi}{2}\right) + 50\left(2t - \frac{\pi}{2}\right) + C_2;$

at $t = 0$ and $s = 0$ we have $0 = \sin\left(-\frac{\pi}{2}\right) + 50\left(-\frac{\pi}{2}\right) + C_2 \Rightarrow C_2 = 1 + 25\pi$

$\Rightarrow s = \sin\left(2t - \frac{\pi}{2}\right) + 100t - 25\pi + (1 + 25\pi) \Rightarrow s = \sin\left(2t - \frac{\pi}{2}\right) + 100t + 1$

42. Let $u = \tan 2x \Rightarrow du = 2\sec^2 2x\,dx \Rightarrow 2\,du = 4\sec^2 2x\,dx$; $v = 2x \Rightarrow dv = 2\,dx \Rightarrow \frac{1}{2}\,dv = dx$

$\frac{dy}{dx} = \int 4\sec^2 2x \tan 2x\,dx = \int u(2\,du) = u^2 + C_1 = \tan^2 2x + C_1;$

at $x = 0$ and $\frac{dy}{dx} = 4$ we have $4 = 0 + C_1 \Rightarrow C_1 = 4 \Rightarrow \frac{dy}{dx} = \tan^2 2x + 4 = \left(\sec^2 2x - 1\right) + 4 = \sec^2 2x + 3$

$\Rightarrow y = \int \left(\sec^2 2x + 3\right) dx = \int \left(\sec^2 v + 3\right)\left(\frac{1}{2}\,dv\right) = \frac{1}{2}\tan v + \frac{3}{2}v + C_2 = \frac{1}{2}\tan 2x + 3x + C_2;$

at $x = 0$ and $y = -1$ we have $-1 = \frac{1}{2}(0) + 0 + C_2 \Rightarrow C_2 = -1 \Rightarrow y = \frac{1}{2}\tan 2x + 3x - 1$

43. Let $u = 2t \Rightarrow du = 2\,dt \Rightarrow 3\,du = 6\,dt$

$s = \int 6\sin 2t\,dt = \int (\sin u)(3\,du) = -3\cos u + C = -3\cos 2t + C;$

at $t = 0$ and $s = 0$ we have $0 = -3\cos 0 + C \Rightarrow C = 3 \Rightarrow s = 3 - 3\cos 2t \Rightarrow s\left(\frac{\pi}{2}\right) = 3 - 3\cos(\pi) = 6$ m

44. Let $u = \pi t \Rightarrow du = \pi\ dt \Rightarrow \pi\ du = \pi^2\ dt$

$$v = \int \pi^2 \cos \pi t\ dt = \int (\cos u)(\pi\ du) = \pi \sin u + C_1 = \pi \sin(\pi t) + C_1;$$

at $t = 0$ and $v = 8$ we have $8 = \pi(0) + C_1 \Rightarrow C_1 = 8 \Rightarrow v = \frac{ds}{dt} = \pi \sin(\pi t) + 8 \Rightarrow s = \int (\pi \sin(\pi t) + 8)\ dt$

$= \int \sin u\ du + 8t + C_2 = -\cos(\pi t) + 8t + C_2$; at $t = 0$ and $s = 0$ we have $0 = -1 + C_2 \Rightarrow C_2 = 1$

$\Rightarrow s = 8t - \cos(\pi t) + 1 \Rightarrow s(1) = 8 - \cos \pi + 1 = 10$ m

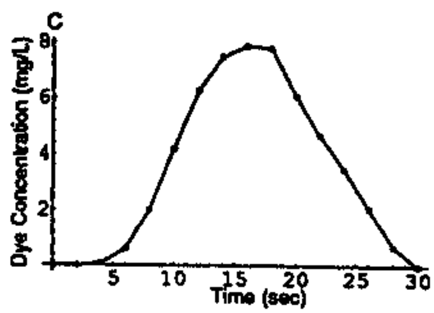
45. All three integrations are correct. In each case, the derivative of the function on the right is the integrand on the left, and each formula has an arbitrary constant for generating the remaining antiderivatives. Moreover, $\sin^2 x + C_1 = 1 - \cos^2 x + C_1 \Rightarrow C_2 = 1 + C_1$; also $-\cos^2 x + C_2 = -\frac{\cos 2x}{2} - \frac{1}{2} + C_2 \Rightarrow C_3 = C_2 - \frac{1}{2} = C_1 + \frac{1}{2}$.

46. Both integrations are correct. In each case, the derivative of the function on the right is the integrand on the left, and each formula has an arbitrary constant for generating the remaining antiderivatives. Moreover, $\frac{\tan^2 x}{2} + C = \frac{\sec^2 x - 1}{2} + C = \frac{\sec^2 x}{2} + \underbrace{\left(C - \frac{1}{2}\right)}_{\text{a constant}}$

4.3 ESTIMATING WITH FINITE SUMS

1. Using values of the function taken from the graph at the midpoints of the intervals, Area $\approx (0.25)(2) + (1.0)(2) + (2.0)(2) + (3.25)(2) + (4.0)(2) + (4.0)(2) + (3.35)(2) + (2.25)(2) + (1.3)(2) + (0.75)(2) + (0.25)(2) = 44.8$ mg·sec/L. Cardiac output $= \frac{\text{amount of dye}}{\text{area under curve}} \times 60 \approx \frac{5\ \text{mg}}{44.5\ \text{mg}\cdot\text{sec/L}} \times 60\ \frac{\text{sec}}{\text{min}} \approx 6.7$ L/min.

2. Using values of the function taken from the graph at the midpoints of the intervals,
Area $\approx 0(2) + (0.1)(2) + (0.4)(2) + (1.2)(2) + (3.2)(2) + (5.3)(2) + (6.8)(2) + (7.6)(2) + (7.7)(2) + (6.9)(2) + (5.6)(2) + (4.0)(2) + (2.8)(2) + (1.6)(2) + (0.2)(2) = 107.0$ mg · sec/L.
Cardiac output $= \frac{\text{dye concentration}}{\text{area estimate}} \times 60 = \frac{10\ \text{mg}}{107.0\ \text{mg}\cdot\text{sec/L}} \times 60\ \frac{\text{sec}}{\text{min}} = 5.61$ L/min.



3. (a) $D \approx (0)(1) + (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) = 87$ inches
(b) $D \approx (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) + (0)(1) = 87$ inches

4. (a) $D \approx (1)(300) + (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300) + (1.0)(300) + (1.8)(300) + (1.5)(300) + (1.2)(300) = 5200$ meters (NOTE: 5 minutes = 300 seconds)
(b) $D \approx (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300) + (1.0)(300) + (1.8)(300) + (1.5)(300) + (1.2)(300) + (0)(300) = 4920$ meters (NOTE: 5 minutes = 300 seconds)

5. (a) $D \approx (0)(10)+(44)(10)+(15)(10)+(35)(10)+(30)(10)+(44)(10)+(35)(10)+(15)(10)+(22)(10)$
$+(35)(10)+(44)(10)+(30)(10) = 3490$ feet ≈ 0.66 miles

(b) $D \approx (44)(10)+(15)(10)+(35)(10)+(30)(10)+(44)(10)+(35)(10)+(15)(10)+(22)(10)+(35)(10)$
$+(44)(10)+(30)(10)+(35)(10) = 3840$ feet ≈ 0.73 miles

6. (a) The distance traveled will be the area under the curve. We will use the approximate velocities at the midpoints of each time interval to approximate this area using rectangles. Thus,
$D \approx (20)(0.001)+(51)(0.001)+(72)(0.001)+(89)(0.001)+(102)(0.001)+(112)(0.001)+(120.5)(0.001)$
$+(128.5)(0.001)+(134.5)(0.001)+(139.5)(0.001) \approx 0.969$ miles

(b) Roughly, after 0.0063 hours, the car would have gone 0.485 miles, where 0.0060 hours = 22.7 sec. At 22.7 sec, the velocity was approximately 120 mi/hr.

7. (a) $S_4 = \pi\left[\sqrt{16-(-2)^2}\right]^2(2)+\pi\left[\sqrt{16-0^2}\right]^2(2)+\pi\left[\sqrt{16-(2)^2}\right]^2(2) = \pi[(16-4)+(16-0)+(16-4)](2)$
$= 80\pi$

(b) $\dfrac{|V-S_4|}{V} = \dfrac{\left|\left(\frac{256}{3}\right)\pi - 80\pi\right|}{\left(\frac{256}{3}\right)\pi} = \dfrac{16}{256} \approx 6\%$

8. (a) $S_5 = \pi\left[(25-(-3)^2)+(25-(-1)^2)+(25-(1)^2)+(25-(3)^2)\right](2) = \pi(16+24+24+16)(2) = 160\pi$

(b) $V = \frac{4}{3}\pi r^3 = \frac{500\pi}{3} \Rightarrow \dfrac{|V-S_5|}{V} = \dfrac{\left|\left(\frac{500}{3}\right)\pi - 160\pi\right|}{\left(\frac{500}{3}\right)\pi} = \dfrac{20}{500} = 4\%$

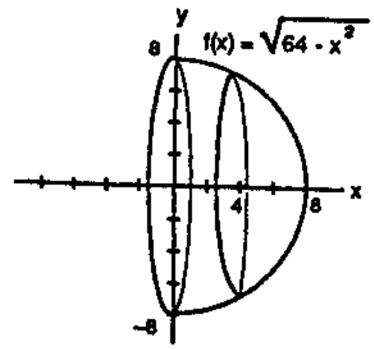
9. (a) $S_8 = \pi\left[(16-0^2)+\left(16-\left(\frac{1}{2}\right)^2\right)+(16-(1)^2)+\left(16-\left(\frac{3}{2}\right)^2\right)+(16-(2)^2)+\left(16-\left(\frac{5}{2}\right)^2\right)\right.$
$\left.+(16-(3)^2)+\left(16-\left(\frac{7}{2}\right)\right)^2\right]\left(\frac{1}{2}\right) = \frac{\pi}{2}\left[128-\frac{1}{4}-1-\frac{9}{4}-4-\frac{25}{4}-9-\frac{49}{4}\right] = \frac{372\pi}{8} = \frac{93\pi}{2}$, overestimates

(b) $V = \frac{2}{3}\pi r^3 = \frac{128\pi}{3} \Rightarrow \dfrac{|V-S_8|}{V} = \dfrac{\left|\left(\frac{128}{3}\right)\pi - \left(\frac{93}{2}\right)\pi\right|}{\left(\frac{128}{3}\right)\pi} = \dfrac{23}{256} \approx 9\%$

10. (a) $S_8 = \pi\left[\left(16-\left(\frac{1}{2}\right)^2\right)+(16-(1)^2)+\left(16-\left(\frac{3}{2}\right)^2\right)+(16-(2)^2)+\left(16-\left(\frac{5}{2}\right)^2\right)+(16-(3)^2)\right.$
$\left.+\left(16-\left(\frac{7}{2}\right)^2\right)\right]\left(\frac{1}{2}\right) = \frac{\pi}{2}\left[112-\frac{1}{4}-1-\frac{9}{4}-4-\frac{25}{4}-9-\frac{49}{4}\right] = \frac{308\pi}{8} = \frac{77\pi}{2}$, underestimates

(b) $\dfrac{|V-S_8|}{V} = \dfrac{\left|\left(\frac{128}{3}\right)\pi - \left(\frac{77}{2}\right)\pi\right|}{\left(\frac{128}{3}\right)\pi} = \dfrac{25}{256} \approx 10\%$

11. (a) To have the same orientation as the hemisphere in Exercise 10, tip the bowl sideways (assume the water is ice). The water covers the interval $[4,8]$. The function which will give us the values of the radii of the approximating cylinders is the equation of the upper semicircle formed by intersecting the hemisphere with the xy-plane, $f(x) = \sqrt{64-x^2}$. Using $\Delta x = \frac{1}{2}$ and left-endpoints for each interval $\Rightarrow S_8 = \pi\Big[(64-(4)^2) + \left(64-\left(\frac{9}{2}\right)^2\right)$



$+ (64-(5)^2) + \left(64-\left(\frac{11}{2}\right)^2\right) + (64-(6)^2)$

$+ \left(64-\left(\frac{13}{2}\right)^2\right) + (64-(7)^2) + \left(64-\left(\frac{15}{2}\right)^2\right)\Big]\left(\frac{1}{2}\right) = \frac{\pi}{2}\left(512 - 16 - \frac{81}{4} - 25 - \frac{121}{4} - 36 - \frac{169}{4} - 49 - \frac{225}{4}\right)$

$= \frac{\pi}{2}\left(386 - \frac{596}{4}\right) = \frac{\pi}{8}(1544 - 596) = \frac{948}{8}\pi = 118.5\pi;$

(b) $\frac{|V - S_8|}{V} = \frac{\left|\left(\frac{320}{3}\right)\pi - \left(\frac{948}{8}\right)\pi\right|}{\left(\frac{320}{3}\right)\pi} = \frac{2844-2560}{2560} \approx 11\%$

12. We are using boxes (rectangular parallelepipeds) that are 30 feet wide, 5 feet long, and h(x) feet deep to approximate the volume of water in the pool.

(a) Using left-hand endpoints in the table: $S = (30)(5)(6.0) + (30)(5)(8.2) + (30)(5)(9.1) + (30)(5)(9.9)$ $+ (30)(5)(10.5) + (30)(5)(11.0) + (30)(500(11.5) + (30)(5)(11.9) + (30)(5)(12.3) + (30)(5)(12.7) = 15{,}465 \text{ ft}^3$.

(b) Using right-hand endpoints in the table: $S = (30)(5)(8.2) + (30)(5)(9.1) + (30)(5)(9.9) + (30)(5)(10.5)$ $+ (30)(5)(11.0) + (30)(5)(11.5) + (30)(5)(11.9) + (30)(5)(12.3) + (30)(5)(12.7) + (30)(5)(13) = 16{,}515 \text{ ft}^3$.

13. (a) $S_5 = \pi\left[(\sqrt{0})^2 + (\sqrt{1})^2 + (\sqrt{2})^2 + (\sqrt{3})^2 + (\sqrt{4})^2\right](1) = 10\pi$, underestimates the volume.

(b) $\frac{|V - S_5|}{V} = \frac{\left(\frac{25}{2}\right)\pi - 10\pi}{\left(\frac{25}{2}\right)\pi} = \frac{5}{25} = 20\%$

14. (a) $S_5 = \pi\left[(\sqrt{1})^2 + (\sqrt{2})^2 + (\sqrt{3})^2 + (\sqrt{4})^2 + (\sqrt{5})^2\right](1) = 15\pi$, overestimates the volume.

(b) $\frac{|V - S_5|}{V} = \frac{15\pi - \left(\frac{25}{2}\right)\pi}{\left(\frac{25}{2}\right)\pi} = \frac{5}{25} = 20\%$

15. (a) Because the acceleration is decreasing, an upper estimate is obtained using left end-points in summing acceleration $\cdot\, \Delta t$. Thus, $\Delta t = 1$ and speed $\approx [32.00 + 19.41 + 11.77 + 7.14 + 4.33](1) = 74.65$ ft/sec

(b) Using right end-points we obtain a lower estimate: speed $\approx [19.41 + 11.77 + 7.14 + 4.33 + 2.63](1)$ $= 45.28$ ft/sec

(c) Upper estimates for the speed at each second are:

t	0	1	2	3	4	5
v	0	32.00	51.41	63.18	70.32	74.65

Thus, the distance fallen when $t = 3$ seconds is $s \approx [32.00 + 51.41 + 63.18](1) = 146.59$ ft.

16. (a) The speed is a decreasing function of time $\Rightarrow$ left end-points give an upper estimate for the height (distance) attained. Also

t	0	1	2	3	4	5
v	400	368	336	304	272	240

gives the time-velocity table by subtracting the constant $g = 32$ from the speed at each time increment $\Delta t = 1$ sec. Thus, the speed ≈ 240 ft/sec after 5 seconds.

(b) A lower estimate for height attained is $h \approx [368 + 336 + 304 + 272 + 240](1) = 1520$ ft.

17. Partition $[0,2]$ into the four subintervals $\left[0,\frac{1}{2}\right], \left[\frac{1}{2},1\right], \left[1,\frac{3}{2}\right]$, and $\left[\frac{3}{2},2\right]$. The midpoints of these subintervals are $m_1 = \frac{1}{4}$, $m_2 = \frac{3}{4}$, $m_3 = \frac{5}{4}$, and $m_4 = \frac{7}{4}$. The heights of the four approximating rectangles are $f(m_1) = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$, $f(m_2) = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$, $f(m_3) = \left(\frac{5}{4}\right)^3 = \frac{125}{64}$, and $f(m_4) = \left(\frac{7}{4}\right)^3 = \frac{343}{64}$

$\Rightarrow$ Average value $\approx \dfrac{\frac{1}{64}+\frac{27}{64}+\frac{125}{64}+\frac{343}{64}}{4} = \dfrac{1+27+125+343}{4 \cdot 64} = \dfrac{496}{256} = \dfrac{31}{16}$. Notice that the average value is approximated by $\dfrac{\left(\frac{1}{4}\right)^3+\left(\frac{3}{4}\right)^3+\left(\frac{5}{4}\right)^3+\left(\frac{7}{4}\right)^3}{4} = \frac{1}{2}\left[\left(\frac{1}{4}\right)^3\left(\frac{1}{2}\right)+\left(\frac{3}{4}\right)^3\left(\frac{1}{2}\right)+\left(\frac{5}{4}\right)^3\left(\frac{1}{2}\right)+\left(\frac{7}{4}\right)^3\left(\frac{1}{2}\right)\right]$

$= \dfrac{1}{\text{length of } [0,2]} \cdot \begin{bmatrix}\text{approximate area under}\\ \text{curve } f(x) = x^3\end{bmatrix}$. We use this observation in solving the next several exercises.

18. Partition $[1,9]$ into the four subintervals $[1,3]$, $[3,5]$, $[5,7]$, and $[7,9]$. The midpoints of these subintervals are $m_1 = 2$, $m_2 = 4$, $m_3 = 6$, and $m_4 = 8$. The heights of the four approximating rectangles are $f(m_1) = \frac{1}{2}$, $f(m_2) = \frac{1}{4}$, $f(m_3) = \frac{1}{6}$, and $f(m_4) = \frac{1}{8}$. The width of each rectangle is $\Delta x = 2$. Thus,

Area $\approx 2\left(\frac{1}{2}\right)+2\left(\frac{1}{4}\right)+2\left(\frac{1}{6}\right)+2\left(\frac{1}{8}\right) = \frac{25}{12} \Rightarrow$ average value $\approx \dfrac{\text{area}}{\text{length of } [1,9]} = \dfrac{\left(\frac{25}{12}\right)}{8} = \dfrac{25}{96}$.

19. Partition $[0,2]$ into the four subintervals $[0,0.5]$, $[0.5,1]$, $[1,1.5]$, and $[1.5,2]$. The midpoints of the subintervals are $m_1 = 0.25$, $m_2 = 0.75$, $m_3 = 1.25$, and $m_4 = 1.75$. The heights of the four approximating rectangles are $f(m_1) = \frac{1}{2} + \sin^2\frac{\pi}{4} = \frac{1}{2}+\frac{1}{2} = 1$, $f(m_2) = \frac{1}{2}+\sin^2\frac{3\pi}{4} = \frac{1}{2}+\frac{1}{2} = 1$, $f(m_3) = \frac{1}{2}+\sin^2\frac{5\pi}{4} = \frac{1}{2}+\left(-\frac{1}{\sqrt{2}}\right)^2$ $= \frac{1}{2}+\frac{1}{2} = 1$, and $f(m_4) = \frac{1}{2}+\sin^2\frac{7\pi}{4} = \frac{1}{2}+\left(-\frac{1}{\sqrt{2}}\right)^2 = 1$. The width of each rectangle is $\Delta x = \frac{1}{2}$. Thus,

Area $\approx (1+1+1+1)\left(\frac{1}{2}\right) = 2 \Rightarrow$ average value $\approx \dfrac{\text{area}}{\text{length of } [0,2]} = \dfrac{2}{2} = 1$.

20. Partition [0, 4] into the four subintervals [0, 1], [1,2,], [2, 3], and [3, 4]. The midpoints of the subintervals are $m_1 = \frac{1}{2}$, $m_2 = \frac{3}{2}$, $m_3 = \frac{5}{2}$, and $m_4 = \frac{7}{2}$. The heights of the four approximating rectangles are

$f(m_1) = 1 - \left(\cos\left(\frac{\pi\left(\frac{1}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{\pi}{8}\right)\right)^4 = 0.27145$ (to 5 decimal places),

$f(m_2) = 1 - \left(\cos\left(\frac{\pi\left(\frac{3}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{3\pi}{8}\right)\right)^4 = 0.97855$, $f(m_3) = 1 - \left(\cos\left(\frac{\pi\left(\frac{5}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{5\pi}{8}\right)\right)^4$

$= 0.97855$, and $f(m_4) = 1 - \left(\cos\left(\frac{\pi\left(\frac{7}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{7\pi}{8}\right)\right)^4 = 0.27145$. The width of each rectangle is

$\Delta x = 1$. Thus, Area $\approx (0.27145)(1) + (0.97855)(1) + (0.97855)(1) + (0.27145)(1) = 2.5 \Rightarrow$ average

value $\approx \frac{\text{area}}{\text{length of } [0,4]} = \frac{2.5}{4} = \frac{5}{8}$.

21. Since the leakage is increasing, an upper estimate uses right end-points and a lower estimate uses left end-points:

(a) upper estimate $= (70)(1) + (97)(1) + (136)(1) + (190)(1) + (265)(1) = 758$ gal,

lower estimate $= (50)(1) + (70)(1) + (97)(1) + (136)(1) + (190)(1) = 543$ gal.

(b) upper estimate $= (70 + 97 + 136 + 190 + 265 + 369 + 516 + 720) = 2363$ gal,

lower estimate $= (50 + 70 + 97 + 136 + 190 + 265 + 369 + 516) = 1693$ gal.

(c) worst case: $2363 + 720t = 25{,}000 \Rightarrow t \approx 31.4$ hrs;

best case: $1693 + 720t = 25{,}000 \Rightarrow t \approx 32.4$ hrs

22. Since the pollutant release increases over time, an upper estimate uses right end-points and a lower estimate uses left end-points:

(a) upper estimate $= (0.2)(30) + (0.25)(30) + (0.27)(30) + (0.34)(30) + (0.45)(30) + (0.52)(30) = 60.9$ tons

lower estimate $= (0.05)(30) + (0.2)(30) + (0.25)(30) + (0.27)(30) + (0.34)(30) + (0.45)(30) = 46.8$ tons

(b) Using the lower (best case) estimate: $46.8 + (0.52)(30) + (0.63)(30) + (0.70)(30) + (0.81)(30) = 126.6$ tons, so near the end of September 125 tons of pollutants will have been released.

23. (a) The diagonal of the square has length 2, so the side length is $\sqrt{2}$. Area $= \left(\sqrt{2}\right)^2 = 2$

(b) Think of the octagon as a collection of 16 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{16} = \frac{\pi}{8}$.

Area $= 16\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = 4\sin\frac{\pi}{4} = 2\sqrt{2} \approx 2.828$

(c) Think of the 16-gon as a collection of 32 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{32} = \frac{\pi}{16}$.

Area $= 32\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{16}\right)\left(\cos\frac{\pi}{16}\right) = 8\sin\frac{\pi}{8} \approx 3.061$

(d) Each area is less than the area of the circle, π. As n increases, the area approaches π.

24. (a) Each of the isosceles triangles is made up of two right triangles having hypotenuse 1 and an acute angle measuring $\frac{2\pi}{2n} = \frac{\pi}{n}$. The area of each isosceles triangle is $A_T = 2\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{n}\right)\left(\cos\frac{\pi}{n}\right) = \frac{1}{2}\sin\frac{2\pi}{n}$.

(b) The area of the polygon is $A_P = nA_T = \frac{n}{2}\sin\frac{2\pi}{n}$, so $\lim\limits_{n\to\infty} A_P = \lim\limits_{n\to\infty}\frac{n}{2}\sin\frac{2\pi}{n} = \lim\limits_{n\to\infty}\pi\cdot\frac{\sin\frac{2\pi}{n}}{\left(\frac{2\pi}{n}\right)} = \pi$

(c) Multiply each area by r^2.

$A_T = \frac{1}{2}r^2\sin\frac{2\pi}{n}$

$A_P = \frac{n}{2}r^2\sin\frac{2\pi}{n}$

$\lim\limits_{n\to\infty} A_P = \pi r^2$

25-28. Example CAS commands:

Maple:

```
with(student):
f:=x -> sin(x); a:= 0; b:= Pi;
plot(f(x),x=a..b);
n:= 1000;
middlebox(f(x),x=a..b,n);
middlesum(f(x),x=a..b,n);
average:= evalf(%)/(b-a);
fsolve(f(x)=average,x);
```

Mathematica:

```
Clear[x]
f[x_] = Sin[x]
{a,b} = {0,Pi};
Plot[ f[x], {x,a,b} ]
n = 100; dx = (b-a)/n;
Table[ N[f[x]], {x,a+dx/2,b,dx} ];
fave = (Plus @@ %) /n
n = 200; dx = (b-a)/n;
Table[ N[f[x]], {x,a+dx/2,b,dx} ];
fave = (Plus @@ %)/n
n = 1000; dx = (b-a)/n;
Table[ N[f[x]], {x,a+dx/2,b,dx} ];
fave = (Plus @@ %)/n
FindRoot[ f[x] - fave, {x,a} ]
```

4.4 RIEMANN SUMS AND DEFINITE INTEGRALS

1. $\sum\limits_{k=1}^{2}\frac{6k}{k+1} = \frac{6(1)}{1+1} + \frac{6(2)}{2+1} = \frac{6}{2} + \frac{12}{3} = 7$

2. $\sum\limits_{k=1}^{3}\frac{k-1}{k} = \frac{1-1}{1} + \frac{2-1}{2} + \frac{3-1}{3} = 0 + \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

3. $\sum\limits_{k=1}^{4}\cos k\pi = \cos(1\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) = -1 + 1 - 1 + 1 = 0$

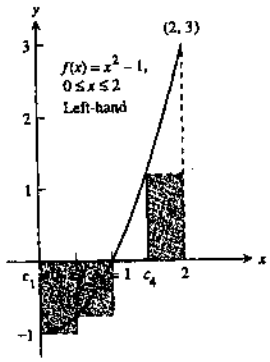
4. $\sum_{k=1}^{5} \sin k\pi = \sin(1\pi) + \sin(2\pi) + \sin(3\pi) + \sin(4\pi) + \sin(5\pi) = 0 + 0 + 0 + 0 + 0 = 0$

5. $\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k} = (-1)^{1+1} \sin \frac{\pi}{1} + (-1)^{2+1} \sin \frac{\pi}{2} + (-1)^{3+1} \sin \frac{\pi}{3} = 0 - 1 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - 2}{2}$

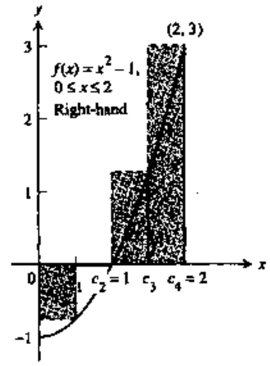
6. $\sum_{k=1}^{4} (-1)^{k} \cos k\pi = (-1)^{1} \cos(1\pi) + (-1)^{2} \cos(2\pi) + (-1)^{3} \cos(3\pi) + (-1)^{4} \cos(4\pi)$

$= -(-1) + 1 - (-1) + 1 = 4$

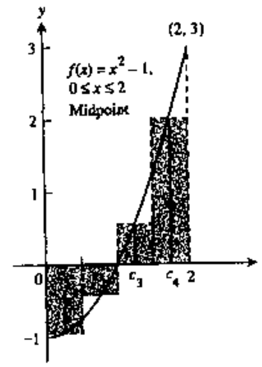
7. (a)



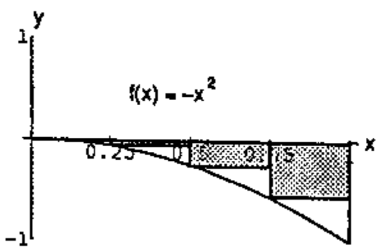
(b)



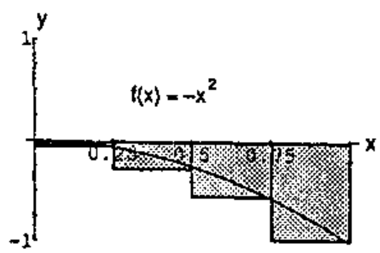
(c)



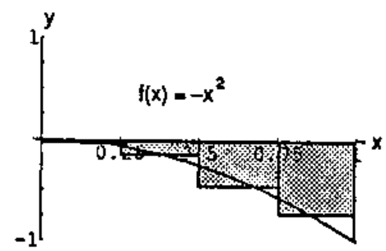
8. (a)



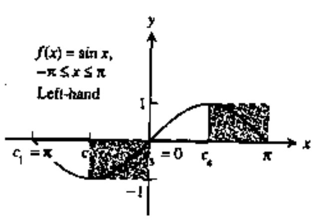
(b)



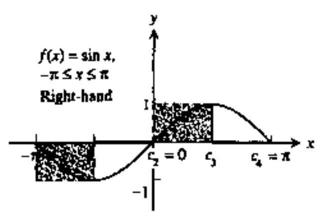
(c)



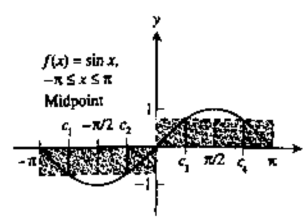
9. (a)



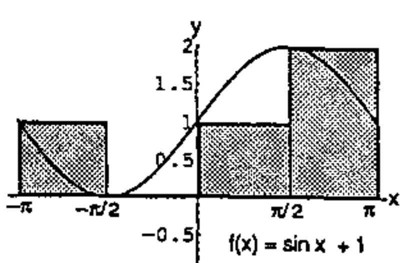
(b)



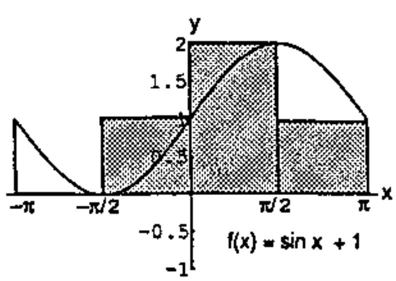
(c)



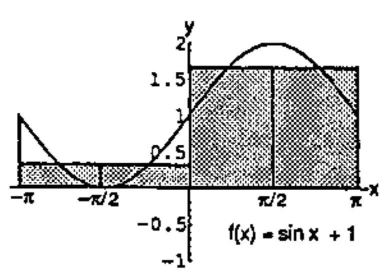
10. (a)



(b)



(c)



11. $\int_0^2 x^2\,dx$

12. $\int_{-1}^0 2x^3\,dx$

13. $\int_{-7}^5 \left(x^2 - 3x\right) dx$

14. $\int_2^3 \frac{1}{1-x}\,dx$

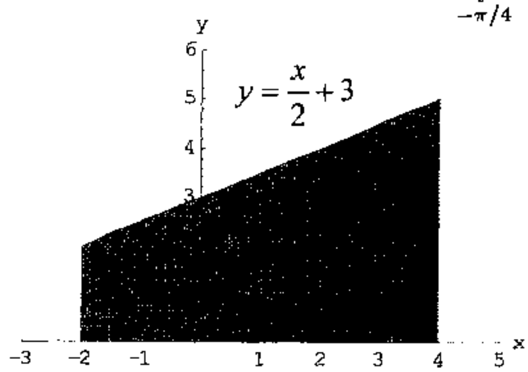
15. $\int_0^1 \sqrt{4-x^2}\,dx$

16. $\int_{-\pi/4}^0 (\sec x)\,dx$

17. The area of the trapezoid is $A = \frac{1}{2}(B+b)h$

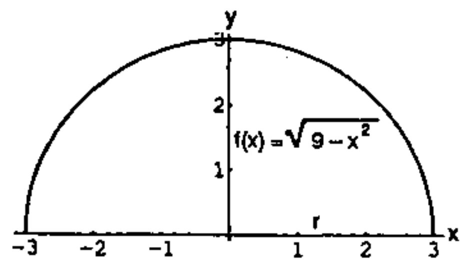
$= \frac{1}{2}(5+2)(6) = 21 \Rightarrow \int_{-2}^4 \left(\frac{x}{2}+3\right) dx$

$= 21$ square units



18. The area of the semicircle is $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$

$= \frac{9}{2}\pi \Rightarrow \int_{-3}^3 \sqrt{9-x^2}\,dx = \frac{9}{2}\pi$ square units

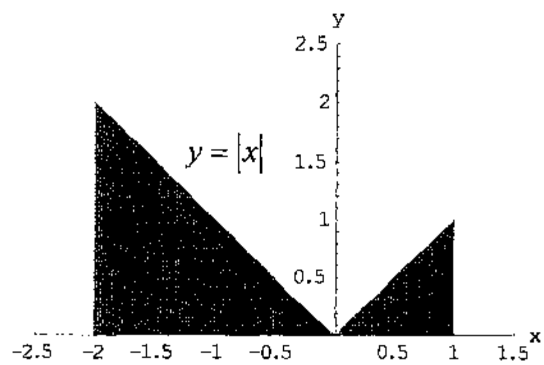


19. The area of the triangle on the left is $A = \frac{1}{2}bh = \frac{1}{2}(2)(2)$

$= 2$. The area of the triangle on the right is $A = \frac{1}{2}bh$

$= \frac{1}{2}(1)(1) = \frac{1}{2}$. Then, the total area is $2.5 \Rightarrow \int_{-2}^1 |x|\,dx$

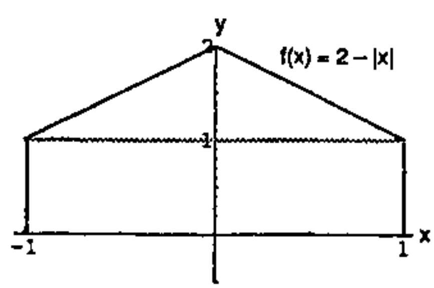
$= 2.5$ square units



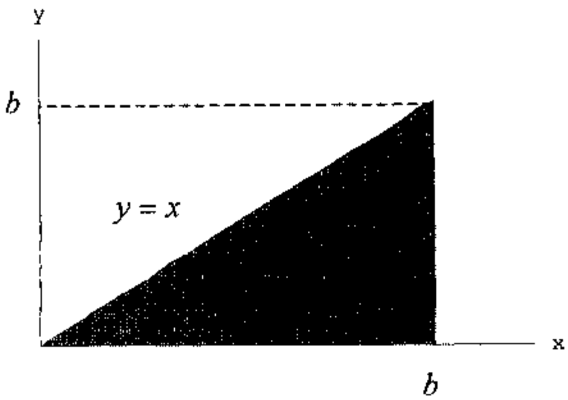
20. The area of the triangular peak is $A = \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1$.

The area of the rectangular base is $S = \ell w = (2)(1) = 2$.

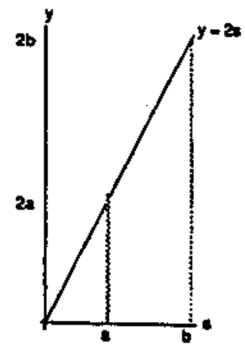
Then the total area is $3 \Rightarrow \int_{-1}^1 \left(2 - |x|\right) dx = 3$ square units



21. $\int_0^b x\, dx = \frac{1}{2}(b)(b) = \frac{b^2}{2}$ square units



22. $\int_a^b 2s\, ds = \frac{1}{2}b(2b) - \frac{1}{2}a(2a) = b^2 - a^2$ square units



23. The graph of $f(x) = 1 - x$ on the interval $[0,1]$ forms a right isosceles triangle in the first quadrant with its two legs, each of length one, lying on the coordinate axes. The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$, which is also the value of the integral $\int_0^1 (1-x)\, dx = \frac{1}{2}$, therefore, $av(f) = \frac{1}{1-0}\int_0^1 (1-x)\, dx = (1)\left(\frac{1}{2}\right) = \frac{1}{2}$.

24. The graph of $f(x) = |x|$ on the interval $[-1,1]$ forms two congruent isosceles right triangles one in the first and the other in the second quadrant. The total area of these two triangles is $A = 2\left(\frac{1}{2}bh\right) = 2\left(\frac{1}{2}\cdot 1\cdot 1\right) = 1$, which is also the value of the integral $\int_{-1}^1 |x|\, dx$, therefore, $av(f) = \frac{1}{1-(-1)}\int_{-1}^1 |x|\, dx = \frac{1}{2}(1) = \frac{1}{2}$.

25. The function $f(x) = \sqrt{1-x^2}$ on the interval $[0,1]$ forms a quarter-circular area of radius 1 lying in the first quadrant with its center on the origin. The area of this quarter-circle is $A = \frac{\pi}{4}r^2 = \frac{\pi}{4}$, which is also the value of the integral $\int_0^1 \sqrt{1-x^2}\, dx$, therefore, $av(f) = \frac{1}{1-0}\int_0^1 \sqrt{1-x^2}\, dx = (1)\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$.

26. The function $f(x) = \sqrt{1-(x-2)^2}$ on the interval $[1,2]$ forms a quarter-circular area of radius 1, lying in the first quadrant with the center of the circle on the point $(2,0)$. The area of this quarter circle is $A = \frac{\pi}{4}r^2 = \frac{\pi}{4}$, which is also the value of the integral $\int_1^2 \sqrt{1-(x-2)^2}\, dx$, therefore,

$$av(f) = \frac{1}{2-1}\int_1^2 \sqrt{1-(x-2)^2} = (1)\left(\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

27. (a) $\int_2^2 g(x)\, dx = 0$ (b) $\int_5^1 g(x)\, dx = -\int_1^5 g(x)\, dx = -8$

(c) $\int_1^2 3f(x)\,dx = 3\int_1^2 f(x)\,dx = 3(-4) = -12$

(d) $\int_2^5 f(x)\,dx = \int_1^5 f(x)\,dx - \int_1^2 f(x)\,dx = 6 - (-4) = 10$

(e) $\int_1^5 [f(x) - g(x)]\,dx = \int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 6 - 8 = -2$

(f) $\int_1^5 [4f(x) - g(x)]\,dx = 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 4(6) - 8 = 16$

28. (a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2(-1) = 2$

(b) $\int_7^9 [f(x) + h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 5 + 4 = 9$

(c) $\int_7^9 [2f(x) - 3h(x)]\,dx = 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx = 2(5) - 3(4) = -2$

(d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = -(-1) = 1$

(e) $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx = -1 - 5 = -6$

(f) $\int_9^7 [h(x) - f(x)]\,dx = \int_7^9 [f(x) - h(x)]\,dx = \int_7^9 f(x)\,dx - \int_7^9 h(x)\,dx = 5 - 4 = 1$

29. (a) $\int_1^2 f(u)\,du = \int_1^2 f(x)\,dx = 5$

(b) $\int_1^2 \sqrt{3}f(z)\,dz = \sqrt{3}\int_1^2 f(z)\,dz = 5\sqrt{3}$

(c) $\int_2^1 f(t)\,dt = -\int_1^2 f(t)\,dt = -5$

(d) $\int_1^2 [-f(x)]\,dx = -\int_1^2 f(x)\,dx = -5$

30. (a) $\int_0^{-3} g(t)\,dt = -\int_{-3}^0 g(t)\,dt = -\sqrt{2}$

(b) $\int_{-3}^0 g(u)\,du = \int_{-3}^0 g(t)\,dt = \sqrt{2}$

(c) $\int_{-3}^0 [-g(x)]\,dx = -\int_{-3}^0 g(x)\,dx = -\sqrt{2}$

(d) $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr = \frac{1}{\sqrt{2}}\int_{-3}^0 g(t)\,dt = \left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}) = 1$

31. (a) $\int_3^4 f(z)\,dz = \int_0^4 f(z)\,dz - \int_0^3 f(z)\,dz = 7 - 3 = 4$

(b) $\int_4^3 f(t)\,dt = -\int_3^4 f(t)\,dt = -4$

32. (a) $\int_1^3 h(r)\,dr = \int_{-1}^3 h(r)\,dr - \int_{-1}^1 h(r)\,dr = 6 - 0 = 6$

(b) $-\int_3^1 h(u)\,du = -\left(-\int_1^3 h(u)\,du\right) = \int_1^3 h(u)\,du = 6$

33. To find where $x - x^2 \ge 0$, let $x - x^2 = 0 \Rightarrow x(1-x) = 0 \Rightarrow x = 0$ or $x = 1$. If $0 < x < 1$, then $x^2 < x$ $\Rightarrow 0 < x - x^2 \Rightarrow a = 0$ and $b = 1$ maximize the integral.

34. To find where $x^4 - 2x^2 \le 0$, let $x^4 - 2x^2 = 0 \Rightarrow x^2(x^2-2) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{2}$. By the sign graph, $+ + + + + + \underset{-\sqrt{2}}{0} - - - \underset{0}{0} - - - \underset{\sqrt{2}}{0} + + + + + + + +$, we can see that $x^4 - 2x^2 \le 0$ on $[-\sqrt{2}, \sqrt{2}] \Rightarrow a = -\sqrt{2}$ and $b = \sqrt{2}$ minimize the integral.

35. By the constant multiple rule, $\int_a^b k\,dx = k\int_a^b 1\,dx$. The Riemann sums definition of the definite integral gives $\int_a^b 1\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^n \Delta x_k$, and if $\Delta x_k = \frac{b-a}{n}$, then $\lim_{\|P\|\to 0} \sum_{k=1}^n \Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^n \frac{b-a}{n}$

$= \lim_{\|P\|\to 0}\left(\frac{b-a}{n} \sum_{k=1}^n 1\right) = \lim_{\|P\|\to 0}\left(\frac{b-a}{n}\cdot n\right) = \lim_{\|P\|\to 0} (b-a) = b - a$. Therefore, $\int_a^b k\,dx = k(b-a)$, for any k.

36. If $f(x) \ge 0$ on $[a,b]$, then $\min f \ge 0$ and $\max f \ge 0$ on $[a,b]$. Now, $(b-a)\min f \le \int_a^b f(x)\,dx \le (b-a)\max f$. Then $b \ge a \Rightarrow b - a \ge 0 \Rightarrow (b-a)\min f \ge 0 \Rightarrow \int_a^b f(x)\,dx \ge 0$.

37. $f(x) = \frac{1}{1+x^2}$ is decreasing on $[0,1] \Rightarrow$ maximum value of f occurs at $0 \Rightarrow \max f = f(0) = 1$; minimum value of f occurs at $1 \Rightarrow \min f = f(1) = \frac{1}{1+1^2} = \frac{1}{2}$. Therefore, $(1-0)\min f \le \int_0^1 \frac{1}{1+x^2}\,dx \le (1-0)\max f$ $\Rightarrow \frac{1}{2} \le \int_0^1 \frac{1}{1+x^2}\,dx \le 1$. That is, an upper bound $= 1$ and a lower bound $= \frac{1}{2}$.

38. See Exercise 37 above. On $[0, 0.5]$, $\max f = \frac{1}{1+0^2} = 1$, $\min f = \frac{1}{1+(0.5)^2} = 0.8$. Therefore $(0.5-0)\min f \le \int_0^{0.5} f(x)\,dx \le (0.5-0)\max f \Rightarrow 0.4 \le \int_0^{0.5} \frac{1}{1+x^2}\,dx \le 0.5$. On $[0.5, 1]$, $\max f = \frac{1}{1+(0.5)^2} = 0.8$

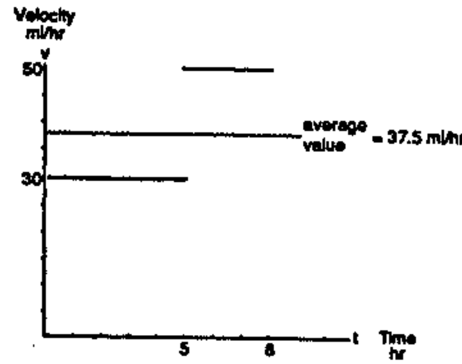
and min $f = \frac{1}{1+1^2} = 0.5$. Thus $(1-0.5)\min f \le \int_{0.5}^{1} \frac{1}{1+x^2}\,dx \le (1-0.5)\max f \Rightarrow 0.25 \le \int_{0.5}^{1} \frac{1}{1+x^2}\,dx \le 0.4$.

Then $0.25 + 0.4 \le \int_{0}^{0.5} \frac{1}{1+x^2}\,dx + \int_{0.5}^{1} \frac{1}{1+x^2}\,dx \le 0.5 + 0.4 \Rightarrow 0.65 \le \int_{0}^{1} \frac{1}{1+x^2}\,dx \le 0.9$.

39. The car drove the first 150 miles in 5 hours and the second 150 miles in 3 hours, which means it drove 300 miles in 8 hours, for an average of $\frac{300}{8}$ mi/hr $= 37.5$ mi/hr. In terms of average values of functions, the function whose average value we seek is

$$v(t) = \begin{cases} 30, & 0 \le t \le 5 \\ 50, & 5 < t \le 8 \end{cases}$$, and the average value is

$\frac{(30)(5)+(50)(3)}{8} = 37.5$ mph. It does not help to consider

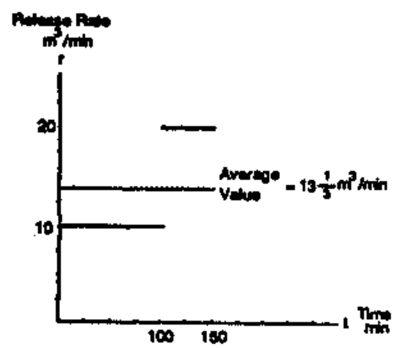


$$v(s) = \begin{cases} 30, & 0 \le s \le 150 \\ 50, & 150 < s \le 300 \end{cases}$$ whose average value is $\frac{(30)(150)+(50)(150)}{300} = 40$ (mph)/mi because we want the average speed with respect to time, not distance.

40. The dam released 1000 m^3 of water in 100 min and then released another 1000 m^3 of water in 50 min, for a total of 2000 m^3 in 150 min, which averages to $\frac{2000}{150}$ $= \frac{40}{3}$ m^3/min. In terms of average values of functions, the function whose average value we seek is

$$r(t) = \begin{cases} 10, & 0 \le t \le 100 \\ 20, & 100 < 1 \le 150 \end{cases}$$, and the average value is

$\frac{(10)(100)+(20)(50)}{150} = \frac{40}{3}$ m^3/min.



41-46. Example CAS commands:

Maple:

```
with(student):
f:= x -> x ∧ 2 + 1; a:= 0; b:= 1;
n:=20;
leftbox(f(x),x=a..b,n);
leftsum(f(x),x=a..b,n);
evalf(%);
rightbox(f(x),x=a..b,n);
rightsum(f(x),x=a..b,n);
evalf(%);
middlebox(f(x),x=a..b,n);
```

```
middlesum(f(x),x=a..b,n);
evalf(%);
```

Mathematica:

This CAS does not have the leftbox, leftsum, etc. commands. Here are definitions of 3 functions that plot the boxes and also return the Riemann sum, using either left endpoints, right endpoints, or midpoints of each subinterval for the values of the function. The arguments to each are:

f: a pure function of one variable
{a,b}: the interval
n: the (positive integer) number of subintervals
plotopts: (optional) options for the plot

```
LeftSum[f_, {a_,b_},n_,plotopts___] := Module[
{x, dx = (b-a)/n, xvals, yvals, boxes},
xvals = Table[ N[x], {x,a,b-dx,dx} ];
yvals = Map[ f, xvals ] // N;
boxes = MapThread[
 Line[{{#1,0},{#1,#3},{#2,#3},{#2,0}}]&,
 {xvals,xvals+dx,yvals} ];
Plot[ f[x], {x,a,b}, Epilog -> boxes, plotopts ];
(Plus @@ yvals)*dx // N
]
RightSum[f_,{a_,b_},n_,plotopts___]  := Module[
{x, dx = (b-a)/n, xvals, yvals, boxes},
xvals = Table[ N[x], {x,a+dx,b,dx} ];
yvals = Map[ f, xvals ] // N;
boxes = MapThread[
 Line[{{#1,0},{#1,#3},{#2,#3},{#2,0}}]&,
 {xvals-dx,xvals,yvals} ];
Plot[ f[x], {x,a,b}, Epilog -> boxes, plotopts ];
(Plus @@ yvals)*dx // N
]
MiddleSum[f_,{a_,b_},n_,plotopts___] := Module[
{x, dx = (b-a)/n, xvals, yvals, boxes},
xvals = Table[ N[x], {x,a+dx/2,b,dx} ];
yvals = Map[ f, xvals ] // N;
boxes = MapThread[
 Line[{{#1,0},{#1,#3},{#2,#3},{#2,0}}]&,
 {xvals-dx/2,xvals+dx/2,yvals} ];
Plot[ f[x], {x,a,b}, Epilog -> boxes, plotopts ];
(Plus @@ yvals)*dx // N
]
Clear[x]
f[x_] = x ∧ 2 + 1
{a,b} = {0,1};
n = 20;
LeftSum[ f, {a,b}, n ]
RightSum[ f, {a,b}, n ]
MiddleSum[ f, {a,b}, n ]
```

4.5 THE MEAN VALUE AND FUNDAMENTAL THEOREMS

1. $\int_{-2}^{0} (2x+5)\,dx = \left[x^2+5x\right]_{-2}^{0} = \left(0^2+5(0)\right)-\left((-2)^2+5(-2)\right) = 6$

2. $\int_{0}^{4} \left(3x-\frac{x^3}{4}\right) dx = \left[\frac{3x^2}{2}-\frac{x^4}{16}\right]_0^4 = \left(\frac{3(4)^2}{2}-\frac{4^4}{16}\right)-\left(\frac{3(0)^2}{2}-\frac{(0)^4}{16}\right) = 8$

3. $\int_{0}^{1} \left(x^2+\sqrt{x}\right) dx = \left[\frac{x^3}{3}+\frac{2}{3}x^{3/2}\right]_0^1 = \left(\frac{1}{3}+\frac{2}{3}\right)-0 = 1$

4. $\int_{-2}^{-1} \frac{2}{x^2}\,dx = \int_{-2}^{-1} 2x^{-2}\,dx = \left[-2x^{-1}\right]_{-2}^{-1} = \left(\frac{-2}{-1}\right)-\left(\frac{-2}{-2}\right) = 1$

5. $\int_{0}^{\pi} (1+\cos x)\,dx = [x+\sin x]_0^{\pi} = (\pi+\sin \pi)-(0+\sin 0) = \pi$

6. $\int_{0}^{\pi/3} 2\sec^2 x\,dx = [2\tan x]_0^{\pi/3} = \left(2\tan\left(\frac{\pi}{3}\right)\right)-(2\tan 0) = 2\sqrt{3}-0 = 2\sqrt{3}$

7. $\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\,d\theta = [-\csc\theta]_{\pi/4}^{3\pi/4} = \left(-\csc\left(\frac{3\pi}{4}\right)\right)-\left(-\csc\left(\frac{\pi}{4}\right)\right) = -\sqrt{2}-\left(-\sqrt{2}\right) = 0$

8. $\int_{0}^{\pi/2} \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)\Big|_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\sin 2(\pi/2)}{2}\right)-\left(0-\frac{\sin 2(0)}{2}\right)\right] = \frac{\pi}{4}$

9. $\int_{-\pi/2}^{\pi/2} \left(8y^2+\sin y\right) dy = \left[\frac{8y^3}{3}-\cos y\right]_{-\pi/2}^{\pi/2} = \left(\frac{8\left(\frac{\pi}{2}\right)^3}{3}-\cos\frac{\pi}{2}\right)-\left(\frac{8\left(-\frac{\pi}{2}\right)^3}{3}-\cos\left(-\frac{\pi}{2}\right)\right) = \frac{2\pi^3}{3}$

10. $\int_{-1}^{1} (r+1)^2\,dr = \int_{-1}^{1} \left(r^2+2r+1\right) dr = \left[\frac{r^3}{3}+r^2+r\right]_{-1}^{1} = \left(\frac{1^3}{3}+1^2+1\right)-\left(\frac{(-1)^3}{3}+(-1)^2+(-1)\right) = \frac{8}{3}$

11. $\int_{1}^{\sqrt{2}} \left(\frac{u^2}{2}-\frac{1}{u^5}\right) du = \int_{1}^{\sqrt{2}} \left(\frac{u^2}{2}-u^{-5}\right) du = \left(\frac{u^3}{6}+\frac{u^{-4}}{4}\right)\Big|_1^{\sqrt{2}} = \left[\left(\frac{\left(\sqrt{2}\right)^3}{6}+\frac{1}{4\left(\sqrt{2}\right)^4}\right)-\left(\frac{(1)^3}{6}+\frac{1}{4(1)^4}\right)\right]$

$= \frac{\sqrt{2}}{3}+\frac{1}{16}-\frac{1}{6}-\frac{1}{4} = \frac{16\sqrt{2}+3-8-12}{48} = \frac{16\sqrt{2}-17}{48}$

12. $\int_4^9 \frac{1-\sqrt{u}}{\sqrt{u}}\,du = \int_4^9 \left(u^{-1/2}-1\right) du = \left(2u^{1/2}-u\right)\Big|_4^9 = \left[\left(2\sqrt{9}-9\right)-\left(2\sqrt{4}-4\right)\right] = -3-(0) = -3$

13. $\int_{-4}^4 |x|\,dx = \int_{-4}^0 |x|\,dx + \int_0^4 |x|\,dx = -\int_{-4}^0 x\,dx + \int_0^4 x\,dx = \left[-\frac{x^2}{2}\right]_{-4}^0 + \left[\frac{x^2}{2}\right]_0^4 = \left(-\frac{0^2}{2}+\frac{(-4)^2}{2}\right)+\left(\frac{4^2}{2}-\frac{0^2}{2}\right)$
$= 16$

14. $\int_0^\pi \frac{1}{2}(\cos x + |\cos x|)\,dx = \int_0^{\pi/2} \frac{1}{2}(\cos x + \cos x)\,dx + \int_{\pi/2}^{\pi} \frac{1}{2}(\cos x - \cos x)\,dx = \int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2}$
$= \sin\frac{\pi}{2} - \sin 0 = 1$

15. (a) $\int_0^{\sqrt{x}} \cos t\,dt = [\sin t]_0^{\sqrt{x}} = \sin\sqrt{x} - \sin 0 = \sin\sqrt{x} \Rightarrow \frac{d}{dx}\left(\int_0^{\sqrt{x}} \cos t\,dt\right) = \frac{d}{dx}\left(\sin\sqrt{x}\right) = \cos\sqrt{x}\left(\frac{1}{2}x^{-1/2}\right)$
$= \frac{\cos\sqrt{x}}{2\sqrt{x}}$

(b) $\frac{d}{dx}\left(\int_0^{\sqrt{x}} \cos t\,dt\right) = \left(\cos\sqrt{x}\right)\left(\frac{d}{dx}\left(\sqrt{x}\right)\right) = \left(\cos\sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$

16. (a) $\int_1^{\sin x} 3t^2\,dt = \left[t^3\right]_1^{\sin x} = \sin^3 x - 1 \Rightarrow \frac{d}{dx}\left(\int_1^{\sin x} 3t^2\,dt\right) = \frac{d}{dx}\left(\sin^3 x - 1\right) = 3\sin^2 x\cos x$

(b) $\frac{d}{dx}\left(\int_1^{\sin x} 3t^2\,dt\right) = \left(3\sin^2 x\right)\left(\frac{d}{dx}(\sin x)\right) = 3\sin^2 x\cos x$

17. (a) $\int_0^{t^4} \sqrt{u}\,du = \int_0^{t^4} u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_0^{t^4} = \frac{2}{3}\left(t^4\right)^{3/2} - 0 = \frac{2}{3}t^6 \Rightarrow \frac{d}{dt}\left(\int_0^{t^4} \sqrt{u}\,du\right) = \frac{d}{dt}\left(\frac{2}{3}t^6\right) = 4t^5$

(b) $\frac{d}{dt}\left(\int_0^{t^4} \sqrt{u}\,du\right) = \sqrt{t^4}\left(\frac{d}{dt}\left(t^4\right)\right) = t^2\left(4t^3\right) = 4t^5$

18. (a) $\int_0^{\tan\theta} \sec^2 y\,dy = [\tan y]_0^{\tan\theta} = \tan(\tan\theta) - 0 = \tan(\tan\theta) \Rightarrow \frac{d}{d\theta}\left(\int_0^{\tan\theta} \sec^2 y\,dy\right) = \frac{d}{d\theta}(\tan(\tan\theta))$
$= \left(\sec^2(\tan\theta)\right)\sec^2\theta$

(b) $\frac{d}{d\theta}\left(\int_0^{\tan\theta} \sec^2 y\ dy\right) = \left(\sec^2(\tan\theta)\right)\left(\frac{d}{d\theta}(\tan\theta)\right) = \left(\sec^2(\tan\theta)\right)\sec^2\theta$

19. $y = \int_0^x \sqrt{1+t^2}\ dt \Rightarrow \frac{dy}{dx} = \sqrt{1+x^2}$

20. $y = \int_1^x \frac{1}{t}\ dt \Rightarrow \frac{dy}{dx} = \frac{1}{x},\ x > 0$

21. $y = \int_{\sqrt{x}}^0 \sin t^2\ dt \rightarrow y = -\int_0^{\sqrt{x}} \sin t^2\ dt \Rightarrow \frac{dy}{dx} = -\left(\sin\left(\sqrt{x}\right)^2\right)\left(\frac{d}{dx}\left(\sqrt{x}\right)\right) = -(\sin x)\left(\frac{1}{2}x^{-1/2}\right) = -\frac{\sin x}{2\sqrt{x}}$

22. $y = \int_0^{x^2} \cos\sqrt{t}\ dt \Rightarrow \frac{dy}{dx} = \left(\cos\sqrt{x^2}\right)\left(\frac{d}{dx}(x^2)\right) = 2x\cos x$

23. $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}},\ |x| < \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 x}}\left(\frac{d}{dx}(\sin x)\right) = \frac{1}{\sqrt{\cos^2 x}}(\cos x) = \frac{\cos x}{|\cos x|} = \frac{\cos x}{\cos x} = 1$ since $|x| < \frac{\pi}{2}$

24. $y = \int_{\tan x}^0 \frac{dt}{1+t^2} \rightarrow y = -\int_0^{\tan x} \frac{dt}{1+t^2} \Rightarrow \frac{dy}{dx} = \left(\frac{-1}{1+\tan^2 x}\right)\left(\frac{d}{dx}(\tan x)\right) = \left(\frac{-1}{\sec^2 x}\right)(\sec^2 x) = -1$

25. Let $u = 1-2x \Rightarrow du = -2\ dx$

$$\int (1-2x)^3\ dx = \int -\frac{1}{2}u^3\ du = -\frac{1}{8}u^4 + C \Rightarrow \int_0^1 (1-2x)^3\ dx = \left[-\frac{1}{8}(1-2x)^4\right]_0^1 = -\frac{1}{8}(-1)^4 - \left(-\frac{1}{8}\right)(1)^4 = 0$$

26. Let $u = t^2+1 \Rightarrow du = 2t\ dt$

$$\int t\sqrt{t^2+1}\ dt = \int \frac{1}{2}u^{1/2}\ du = \frac{1}{3}u^{3/2} + C \Rightarrow \int_0^1 t\sqrt{t^2+1}\ dt = \left[\frac{1}{3}\left(t^2+1\right)^{3/2}\right]_0^1 = \frac{1}{3}(2)^{3/2} - \frac{1}{3}(1)^{3/2}$$
$$= \frac{1}{3}\left(2\sqrt{2}-1\right)$$

27. Let $u = 1+\frac{\theta}{2} \Rightarrow du = \frac{1}{2}\ d\theta$

$$\int \sin^2\left(1+\frac{\theta}{2}\right) d\theta = \int 2\sin^2 u\ du = 2\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right) + C \Rightarrow \int_0^{\pi} \sin^2\left(1+\frac{\theta}{2}\right) d\theta = \left[\left(1+\frac{\theta}{2}\right) - \frac{1}{2}\sin(2+\theta)\right]_0^{\pi}$$
$$= \left[\left(1+\frac{\pi}{2}\right) - \frac{1}{2}\sin(2+\pi)\right] - \left(1 - \frac{1}{2}\sin 2\right) = \frac{\pi}{2} + \sin 2$$

28. Let $u = \sin\frac{x}{4} \Rightarrow du = \frac{1}{4}\cos\frac{x}{4}\ dx$

$$\int \sin^2\frac{x}{4}\cos\frac{x}{4}\ dx = \int 4u^2\ du = \frac{4}{3}u^3 + C \Rightarrow \int_0^{\pi} \sin^2\frac{x}{4}\cos\frac{x}{4}\ dx = \left[\frac{4}{3}\sin^3\frac{x}{4}\right]_0^{\pi} = \frac{4}{3}\sin^3\frac{\pi}{4} - \frac{4}{3}\sin 0$$

$= \frac{4}{3}\left(\frac{\sqrt{2}}{2}\right)^3 = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$

29. $y = \int_2^x \sec t\, dt + 3$

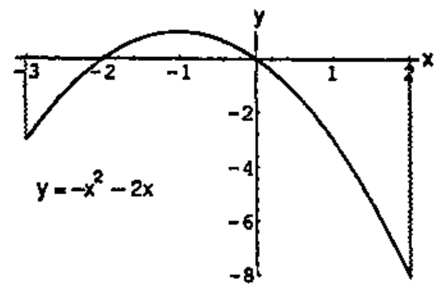
30. $y = \int_1^x t\sqrt{1+t^2}\, dt - 2$

31. $y = \int_0^x \cos^2 t \sin t\, dt - 1$

32. $y = \int_{\frac{\pi}{2}-1}^x \frac{1}{\sqrt{t+1}} \cos \sqrt{t+1}\, dt + 1$

33. $-x^2 - 2x = 0 \Rightarrow -x(x+2) = 0 \Rightarrow x = 0$ or $x = -2$; Area

$= -\int_{-3}^{-2} \left(-x^2 - 2x\right) dx + \int_{-2}^{0} \left(-x^2 - 2x\right) dx - \int_{0}^{2} \left(-x^2 - 2x\right) dx$



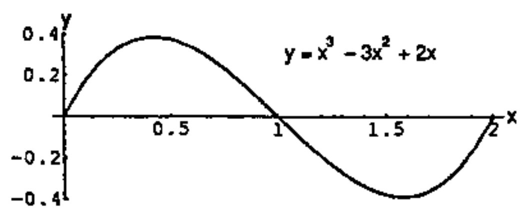
$= -\left[-\frac{x^3}{3} - x^2\right]_{-3}^{-2} + \left[-\frac{x^3}{3} - x^2\right]_{-2}^{0} - \left[-\frac{x^3}{3} - x^2\right]_{0}^{2}$

$= -\left(\left(-\frac{(-2)^3}{3} - (-2)^2\right) - \left(-\frac{(-3)^3}{3} - (-3)^2\right)\right)$

$+\left(\left(-\frac{0^3}{3} - 0^2\right) - \left(-\frac{(-2)^3}{3} - (-2)^2\right)\right) - \left(\left(-\frac{2^3}{3} - 2^2\right) - \left(-\frac{0^3}{3} - 0^2\right)\right) = \frac{28}{3}$

34. $x^3 - 3x^2 + 2x = 0 \Rightarrow x\left(x^2 - 3x + 2\right) = 0$

$\Rightarrow x(x-2)(x-1) = 0$, $x = 0$, 1, or 2;



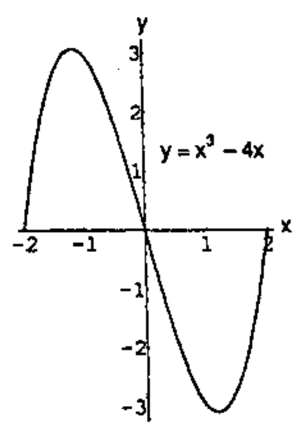
Area $= \int_0^1 \left(x^3 - 3x^2 + 2x\right) dx - \int_1^2 \left(x^3 - 3x^2 + 2x\right) dx$

$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2$

$= \left(\frac{1^4}{4} - 1^3 + 1^2\right) - \left(\frac{0^4}{4} - 0^3 + 0^2\right) - \left[\left(\frac{2^4}{4} - 2^3 + 2^2\right) - \left(\frac{1^4}{4} - 1^3 + 1^2\right)\right] = \frac{1}{2}$

35. $x^3 - 4x = 0 \Rightarrow x\left(x^2 - 4\right) = 0 \Rightarrow x(x-2)(x+2) = 0$

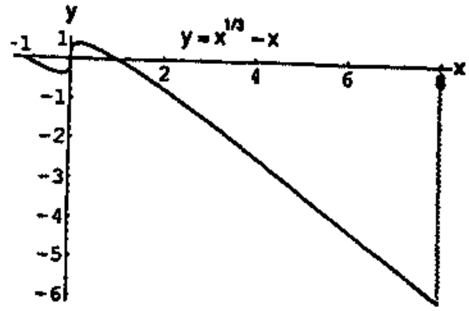
$\Rightarrow x = 0$, 2, or -2; Area $= \int_{-2}^{0} \left(x^3 - 4x\right) dx - \int_0^2 \left(x^3 - 4x\right) dx$



$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - 2x^2\right]_0^2 = \left(\frac{0^4}{4} - 2(0)^2\right)$

$-\left(\frac{(-2)^4}{4} - 2(-2)^2\right) - \left[\left(\frac{2^4}{4} - 2(2)^2\right) - \left(\frac{0^4}{4} - 2(0)^2\right)\right] = 8$

36. $x^{1/3} - x = 0 \Rightarrow x^{1/3}\left(1 - x^{2/3}\right) = 0 \Rightarrow x^{1/3} = 0$ or $1 - x^{2/3} = 0 \Rightarrow x = 0$ or $1 = x^{2/3} \Rightarrow x = 0$ or $1 = x^2 \Rightarrow x = 0$ or ± 1;



$$\text{Area} = -\int_{-1}^{0} \left(x^{1/3} - x\right) dx + \int_{0}^{1} \left(x^{1/3} - x\right) dx - \int_{1}^{8} \left(x^{1/3} - x\right) dx$$

$$= -\left[\frac{3}{4}x^{4/3} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{3}{4}x^{4/3} - \frac{x^2}{2}\right]_{0}^{1} - \left[\frac{3}{4}x^{4/3} - \frac{x^2}{2}\right]_{1}^{8}$$

$$= -\left[\left(\frac{3}{4}(0)^{4/3} - \frac{0^2}{2}\right) - \left(\frac{3}{4}(-1)^{4/3} - \frac{(-1)^2}{2}\right)\right] + \left[\left(\frac{3}{4}(1)^{4/3} - \frac{1^2}{2}\right) - \left(\frac{3}{4}(0)^{4/3} - \frac{0^2}{2}\right)\right]$$

$$-\left[\left(\frac{3}{4}(8)^{4/3} - \frac{8^2}{2}\right) - \left(\frac{3}{4}(1)^{4/3} - \frac{1^2}{2}\right)\right] = \frac{1}{4} + \frac{1}{4} - \left(12 - \frac{64}{2}\right) + \frac{1}{4} = \frac{83}{4}$$

37. The area of the rectangle bounded by the lines $y = 2$, $y = 0$, $x = \pi$, and $x = 0$ is 2π. The area under the curve $y = 1 + \cos x$ on $[0, \pi]$ is $\int_0^{\pi} (1 + \cos x)\, dx = [x + \sin x]_0^{\pi} = (\pi + \sin \pi) - (0 + \sin 0) = \pi$. Therefore the area of the shaded region is $2\pi - \pi = \pi$.

38. The area of the rectangle bounded by the lines $y = 2$, $y = 0$, $t = -\frac{\pi}{4}$, and $t = 1$ is $2\left(1 - \left(-\frac{\pi}{4}\right)\right) = 2 + \frac{\pi}{2}$. The area under the curve $y = \sec^2 t$ on $\left[-\frac{\pi}{4}, 0\right]$ is $\int_{-\pi/4}^{0} \sec^2 t\, dt = [\tan t]_{-\pi/4}^{0} = \tan 0 - \tan\left(-\frac{\pi}{4}\right) = 1$. The area under the curve $y = 1 - t^2$ on $[0, 1]$ is $\int_0^1 \left(1 - t^2\right) dt = \left[t - \frac{t^3}{3}\right]_0^1 = \left(1 - \frac{1^3}{3}\right) - \left(0 - \frac{0^3}{3}\right) = \frac{2}{3}$. Thus, the total area under the curves on $\left[-\frac{\pi}{4}, 1\right]$ is $1 + \frac{2}{3} = \frac{5}{3}$. Therefore the area of the shaded region is $\left(2 + \frac{\pi}{2}\right) - \frac{5}{3} = \frac{1}{3} + \frac{\pi}{2}$.

39. $\frac{dc}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2} \Rightarrow c = \int_0^x \frac{1}{2}t^{-1/2} dt = \left[t^{1/2}\right]_0^x = \sqrt{x}$

(a) $c(100) - c(1) = \sqrt{100} - \sqrt{1} = \$9.00$ (b) $c(400) - c(100) = \sqrt{400} - \sqrt{100} = \$10.00$

40. $r = \int_0^3 \left(2 - \frac{2}{(x+1)^2}\right) dx = 2\int_0^3 \left(1 - \frac{1}{(x+1)^2}\right) dx = 2\left[x - \left(\frac{-1}{x+1}\right)\right]_0^3 = 2\left[\left(3 + \frac{1}{(3+1)}\right) - \left(0 + \frac{1}{(0+1)}\right)\right]$

$= 2\left[3\frac{1}{4} - 1\right] = 2\left(2\frac{1}{4}\right) = 4.5$ or $\$4500$

41. (a) $v = \frac{ds}{dt} = \frac{d}{dt}\int_0^t f(x)\,dx = f(t) \Rightarrow v(5) = f(5) = 2$ m/sec

(b) $a = \frac{df}{dt}$ is negative since the slope of the tangent line at $t = 5$ is negative

(c) $s = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(3) = \frac{9}{2}$ m since the integral is the area of the triangle formed by $y = f(x)$, the x-axis, and $x = 3$

(d) $t = 6$ since after $t = 6$ to $t = 9$, the region lies below the x-axis

(e) At $t = 4$ and $t = 7$, since there are horizontal tangents there

(f) Toward the origin between $t = 6$ and $t = 9$ since the velocity is negative on this interval. Away from the origin between $t = 0$ and $t = 6$ since the velocity is positive there.

(g) Right or positive side, because the integral of f from 0 to 9 is positive, there being more area above the x-axis than below it.

42. (a) $v = \frac{ds}{dt} = \frac{d}{dt}\int_0^t f(x)\,dx = f(t) \Rightarrow v(3) = f(3) = 0$ m/sec.

(b) $a = \frac{df}{dt}$ is positive, since the slope of the tangent line at $t = 3$ is positive

(c) At $t = 3$, the particle's position is $\int_0^3 f(x)\,dx = \frac{1}{2}(3)(-6) = -9$

(d) The particle passes through the origin at $t = 6$ because $s(6) = \int_0^6 f(x)\,dx = 0$

(e) At $t = 7$, since there is a horizontal tangent there

(f) The particle starts at the origin and moves away to the left for $0 < t < 3$. It moves back toward the origin for $3 < t < 6$, passes through the origin at $t = 6$, and moves away to the right for $t > 6$.

(g) Right side, since its position at $t = 9$ is positive, there being more area above the x-axis than below it.

43. $\int_4^8 \pi\left(64 - x^2\right)dx = \pi\left[64x - \frac{x^3}{3}\right]_4^8 = \pi\left[\left(512 - \frac{512}{3}\right) - \left(256 - \frac{64}{3}\right)\right] = \pi\left(256 - \frac{448}{3}\right) = \frac{320\pi}{3}$

44. $\int_0^5 \pi\left(\sqrt{x}\right)^2 dx = \pi\int_0^5 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^5 = \pi\left(\frac{25}{2} - \frac{0}{2}\right) = \frac{25\pi}{2}$

45. $\int_1^x f(t)\,dt = x^2 - 2x + 1 \Rightarrow f(x) = \frac{d}{dx}\int_1^x f(t)\,dt = \frac{d}{dx}\left(x^2 - 2x + 1\right) = 2x - 2$

46. $\int_0^x f(t)\,dt = x\cos\pi x \Rightarrow f(x) = \frac{d}{dx}\int_1^x f(t)\,dt = \cos\pi x - \pi x\sin\pi x \Rightarrow f(4) = \cos\pi(4) - \pi(4)\sin\pi(4) = 1$

47. $f(x) = 2 - \int_2^{x+1} \frac{9}{1+t}\,dt \Rightarrow f'(x) = -\frac{9}{1+(x+1)} = \frac{-9}{x+2} \Rightarrow f'(1) = -3;\ f(1) = 2 - \int_2^{1+1} \frac{9}{1+t}\,dt = 2 - 0 = 2;$

$L(x) = f'(1)(x-1) + f(1) = -3(x-1) + 2 = -3x + 5$

48. $g(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt \Rightarrow g'(x) = \left(\sec\left(x^2-1\right)\right)(2x) = 2x\sec\left(x^2-1\right) \Rightarrow g'(-1) = 2(-1)\sec\left((-1)^2-1\right)$

$= -2;\ g(-1) = 3 + \int_1^{(-1)^2} \sec(t-1)\,dt = 3 + \int_1^1 \sec(t-1)\,dt = 3 + 0 = 3;\ L(x) = g'(-1)(x-(-1)) + g(-1)$

$= -2(x+1) + 3 = -2x + 1$

49. (a) True: since f is continuous, g is differentiable by Part 1 of the Fundamental Theorem of Calculus.
 (b) True: g is continuous because it is differentiable.
 (c) True, since $g'(1) = f(1) = 0$.
 (d) False, since $g''(1) = f'(1) > 0$.
 (e) True, since $g'(1) = 0$ and $g''(1) = f'(1) > 0$.
 (f) False: $g''(x) = f'(x) > 0$, so g'' never changes sign.
 (g) True, since $g'(1) = f(1) = 0$ and $g'(x) = f(x)$ is an increasing function of x (because $f'(x) > 0$).

50. (a) True: by Part 1 of the Fundamental Theorem of Calculus, $h'(x) = f(x)$. Since f is differentiable for all x, h has a second derivative for all x.
 (b) True: they are continuous because they are differentiable.
 (c) True, since $h'(1) = f(1) = 0$.
 (d) True, since $h'(1) = 0$ and $h''(1) = f'(1) < 0$.
 (e) False, since $h''(1) = f'(1) < 0$.
 (f) False, since $h''(x) = f'(x) < 0$ never changes sign.
 (g) True, since $h'(1) = f(1) = 0$ and $h'(x) = f(x)$ is a decreasing function of x (because $f'(x) < 0$).

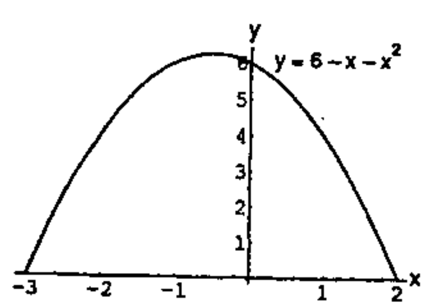
51. (a) $6 - x - x^2 = 0 \Rightarrow x^2 + x - 6 = 0$

$\Rightarrow (x+3)(x-2) = 0 \Rightarrow x = -3$ or $x = 2$;

Area $= \int_{-3}^{2} (6 - x - x^2)\,dx = \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$

$= \left(6(2) - \frac{2^2}{2} - \frac{2^3}{3}\right) - \left(6(-3) - \frac{(-3)^2}{2} - \frac{(-3)^3}{3}\right)$

$= \frac{125}{6}$



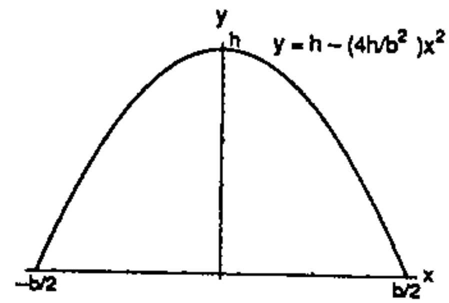
(b) $y' = -1 - 2x = 0 \Rightarrow x = -\frac{1}{2}$; $y' > 0$ for $x < -\frac{1}{2}$ and $y' < 0$ for $x > -\frac{1}{2} \Rightarrow x = -\frac{1}{2}$ yields a local maximum $\Rightarrow \text{height} = y\left(-\frac{1}{2}\right) = \frac{25}{4}$

(c) $\text{Base} = 2 - (-3) = 5$, $\text{height} = y\left(-\frac{1}{2}\right) = \frac{25}{4} \Rightarrow \text{Area} = \frac{2}{3}(\text{Base})(\text{Height}) = \frac{2}{3}(5)\left(\frac{25}{4}\right) = \frac{125}{6}$

(d) $$\text{Area} = \int_{-b/2}^{b/2} \left(h - \left(\frac{4h}{b^2}\right)x^2\right) dx = \left[hx - \frac{4hx^3}{3b^2}\right]_{-b/2}^{b/2}$$

$$= \left(h\left(\frac{b}{2}\right) - \frac{4h\left(\frac{b}{2}\right)^3}{3b^2}\right) - \left(h\left(-\frac{b}{2}\right) - \frac{4h\left(-\frac{b}{2}\right)^3}{3b^2}\right)$$

$$= \left(\frac{bh}{2} - \frac{bh}{6}\right) - \left(-\frac{bh}{2} + \frac{bh}{6}\right) = bh - \frac{bh}{3} = \frac{2}{3}bh$$



52. (a) $$\left(\frac{1}{\frac{1}{60} - 0}\right) \int_0^{1/60} V_{max} \sin 120\pi t \, dt = 60\left[-V_{max}\left(\frac{1}{120\pi}\right)\cos(120\pi t)\right]_0^{1/60} = -\frac{V_{max}}{2\pi}[\cos 2\pi - \cos 0]$$

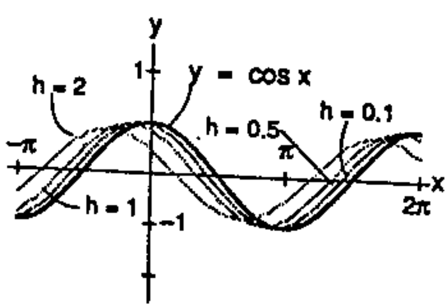
$$= -\frac{V_{max}}{2\pi}[1 - 1] = 0$$

(b) $V_{max} = \sqrt{2}\, V_{rms} = \sqrt{2}(240) \approx 339$ volts

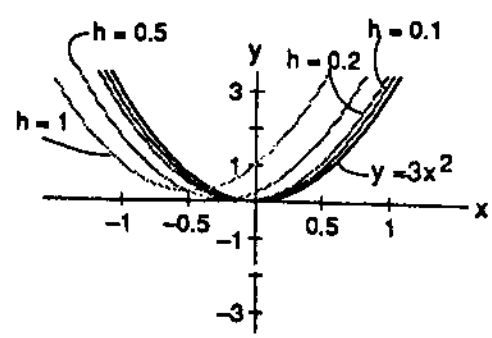
(c) $$\int_0^{1/60} (V_{max})^2 \sin^2 120\pi t \, dt = (V_{max})^2 \int_0^{1/60} \left(\frac{1 - \cos 240\pi t}{2}\right) dt = \frac{(V_{max})^2}{2} \int_0^{1/60} (1 - \cos 240\pi t) \, dt$$

$$= \frac{(V_{max})^2}{2}\left[t - \left(\frac{1}{240\pi}\right)\sin 240\pi t\right]_0^{1/60} = \frac{(V_{max})^2}{2}\left[\left(\frac{1}{60} - \left(\frac{1}{240\pi}\right)\sin(2\pi)\right) - \left(0 - \left(\frac{1}{240\pi}\right)\sin(0)\right)\right] = \frac{(V_{max})^2}{120}$$

53.



54. The limit is $3x^2$



55-58. Example CAS commands:

Maple:

```
f:=x -> x ∧ 3 − 4*x ∧ 2 + 3*x;
F:=x -> int(f(t),t=0..x);
plot({F(x),f(x)},x=0..3.75);
solve(diff(F(x),x),x);
```

```
plot({diff(f(x),x),F(x)}, x=0..3);
map(evalf,[solve(diff(f(x),x)=0)]);
```

Mathematica:

```
Clear[x]
{a,b} = {0,2Pi}; f[x_] = Sin[2x] Cos[x/3]
F[x_] = Integrate[ f[t], {t,a,x} ]
Plot[ {f[x],F[x]}, {x,a,b} ]
x /. Map[
   FindRoot[ F' [x] == 0, {x,#} ]&,
   {2,3,5,6} ]
x /. Map[
   FindRoot[ f'[x] == 0, {x,#} ]&,
   {1,2,4,5,6} ]
```

59-62. Example CAS commands:

Maple:

```
f:=x -> sqrt(1 -x ∧ 2);
u:=x -> x ∧ 2;
F:=x -> int(f(t),t=1..u(x));
dFx:=diff(F(x),x);
simplify(%);
solve(dFx=0,x);
dFxx:=diff(F(x),x$2);
simplify(%);
solve(dFxx=0,x);
evalf(%);
plot (F(x),x=-1..1);
```

Mathematica:

```
a = 1; u[x_] = x ∧ 2; f[x_] = Sqrt[ 1 - x ∧ 2 ]
F[x_] = Integrate[ f[t], {t,a,u[x]} ]
F'[x]
x /. NSolve[ F'[x] == 0, x ]
F''[x]
x /. NSolve [ F''[x] == 0, x]
Plot[ F[x], {x,-1,1} ]
```

63. In Maple type diff(int(f(x),x=a..u(x)),x); or, in Mathematica type $\partial_x \int_a^{u[x]} f[t]\, dt$

64. In Maple type diff(int(f(x),x=a..u(x)),x,x); or in Mathematica type $\partial_{x,x}\left(\int_a^{u[x]} f[t]\, dt\right)$

4.6 SUBSTITUTION IN DEFINITE INTEGRALS

1. (a) Let $u = y + 1 \Rightarrow du = dy$; $y = 0 \Rightarrow u = 1$, $y = 3 \Rightarrow u = 4$

$$\int_0^3 \sqrt{y+1}\, dy = \int_1^4 u^{1/2}\, du = \left[\frac{2}{3}u^{3/2}\right]_1^4 = \left(\frac{2}{3}\right)(4)^{3/2} - \left(\frac{2}{3}\right)(1)^{3/2} = \left(\frac{2}{3}\right)(8) - \left(\frac{2}{3}\right)(1) = \frac{14}{3}$$

(b) Use the same substitution for u as in part (a); $y = -1 \Rightarrow u = 0$, $y = 0 \Rightarrow u = 1$

$$\int_{-1}^{0} \sqrt{y+1}\, dy = \int_{0}^{1} u^{1/2}\, du = \left[\frac{2}{3}u^{3/2}\right]_0^1 = \left(\frac{2}{3}\right)(1)^{3/2} - 0 = \frac{2}{3}$$

2. (a) Let $u = \tan x \Rightarrow du = \sec^2 x\, dx$; $x = 0 \Rightarrow u = 0$, $x = \frac{\pi}{4} \Rightarrow u = 1$

$$\int_{0}^{\pi/4} \tan x \sec^2 x\, dx = \int_{0}^{1} u\, du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1^2}{2} - 0 = \frac{1}{2}$$

(b) Use the same substitution as in part (a); $x = -\frac{\pi}{4} \Rightarrow u = -1$, $x = 0 \Rightarrow u = 0$

$$\int_{-\pi/4}^{0} \tan x \sec^2 x\, dx = \int_{-1}^{0} u\, du = \left[\frac{u^2}{2}\right]_{-1}^{0} = 0 - \frac{1}{2} = -\frac{1}{2}$$

3. (a) Let $u = \cos x \Rightarrow du = -\sin x\, dx \Rightarrow -du = \sin x\, dx$; $x = 0 \Rightarrow u = 1$, $x = \pi \Rightarrow u = -1$

$$\int_{0}^{\pi} 3\cos^2 x \sin x\, dx = \int_{1}^{-1} -3u^2\, du = \left[-u^3\right]_1^{-1} = -(-1)^3 - \left(-(1)^3\right) = 2$$

(b) Use the same substitution as in part (a); $x = 2\pi \Rightarrow u = 1$, $x = 3\pi \Rightarrow u = -1$

$$\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\, dx = \int_{1}^{-1} -3u^2\, du = 2$$

4. (a) Let $u = t^2 + 1 \Rightarrow du = 2t\, dt \Rightarrow \frac{1}{2}\, du = t\, dt$; $t = 0 \Rightarrow u = 1$, $t = \sqrt{7} \Rightarrow u = 8$

$$\int_{0}^{\sqrt{7}} t\left(t^2+1\right)^{1/3} dt = \int_{1}^{8} \frac{1}{2}u^{1/3}\, du = \left[\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)u^{4/3}\right]_1^8 = \left(\frac{3}{8}\right)(8)^{4/3} - \left(\frac{3}{8}\right)(1)^{4/3} = \frac{45}{8}$$

(b) Use the same substitution as in part (a); $t = -\sqrt{7} \Rightarrow u = 8$, $t = 0 \Rightarrow u = 1$

$$\int_{-\sqrt{7}}^{0} t\left(t^2+1\right)^{1/3} dt = \int_{8}^{1} \frac{1}{2}u^{1/3}\, du = -\int_{1}^{8} \frac{1}{2}u^{1/3}\, du = -\frac{45}{8}$$

5. (a) Let $u = 4 + r^2 \Rightarrow du = 2r\, dr \Rightarrow \frac{1}{2}\, du = r\, dr$; $r = -1 \Rightarrow u = 5$, $r = 1 \Rightarrow u = 5$

$$\int_{-1}^{1} \frac{5r}{\left(4+r^2\right)^2}\, dr = 5\int_{5}^{5} \frac{1}{2}u^{-2}\, du = 0$$

(b) Use the same substitution as in part (a); $r = 0 \Rightarrow u = 4$, $r = 1 \Rightarrow u = 5$

$$\int_{0}^{1} \frac{5r}{\left(4+r^2\right)^2}\, dr = 5\int_{4}^{5} \frac{1}{2}u^{-2}\, du = 5\left[-\frac{1}{2}u^{-1}\right]_4^5 = 5\left(-\frac{1}{2}(5)^{-1}\right) - 5\left(-\frac{1}{2}(4)^{-1}\right) = \frac{1}{8}$$

6. (a) Let $u = x^2 + 1 \Rightarrow du = 2x\,dx \Rightarrow 2\,du = 4x\,dx$; $x = 0 \Rightarrow u = 1$, $x = \sqrt{3} \Rightarrow u = 4$

$$\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx = \int_1^4 \frac{2}{\sqrt{u}}\,du = \int_1^4 2u^{-1/2}\,du = \left[4u^{1/2}\right]_1^4 = 4(4)^{1/2} - 4(1)^{1/2} = 4$$

(b) Use the same substitution as in part (a); $x = -\sqrt{3} \Rightarrow u = 4$, $x = \sqrt{3} \Rightarrow u = 4$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx = \int_4^4 \frac{2}{\sqrt{u}}\,du = 0$$

7. (a) Let $u = 1 - \cos 3t \Rightarrow du = 3\sin 3t\,dt \Rightarrow \frac{1}{3}\,du = \sin 3t\,dt$; $t = 0 \Rightarrow u = 0$, $t = \frac{\pi}{6} \Rightarrow u = 1 - \cos\frac{\pi}{2} = 1$

$$\int_0^{\pi/6} (1 - \cos 3t)\sin 3t\,dt = \int_0^1 \frac{1}{3}u\,du = \left[\frac{1}{3}\left(\frac{u^2}{2}\right)\right]_0^1 = \frac{1}{6}(1)^2 - \frac{1}{6}(0)^2 = \frac{1}{6}$$

(b) Use the same substitution as in part (a); $t = \frac{\pi}{6} \Rightarrow u = 1$, $t = \frac{\pi}{3} \Rightarrow u = 1 - \cos\pi = 2$

$$\int_{\pi/6}^{\pi/3} (1 - \cos 3t)\sin 3t\,dt = \int_1^2 \frac{1}{3}u\,du = \left[\frac{1}{3}\left(\frac{u^2}{2}\right)\right]_1^2 = \frac{1}{6}(2)^2 - \frac{1}{6}(1)^2 = \frac{1}{2}$$

8. (a) Let $u = 2 + \tan\frac{t}{2} \Rightarrow du = \frac{1}{2}\sec^2\frac{t}{2}\,dt \Rightarrow 2\,du = \sec^2\frac{t}{2}\,dt$; $t = \frac{-\pi}{2} \Rightarrow u = 2 + \tan\left(\frac{-\pi}{4}\right) = 1$, $t = 0 \Rightarrow u = 2$

$$\int_{-\pi/2}^{0} \left(2 + \tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt = \int_1^2 u\,(2\,du) = \left[u^2\right]_1^2 = 2^2 - 1^2 = 3$$

(b) Use the same substitution as in part (a); $t = \frac{-\pi}{2} \Rightarrow u = 1$, $t = \frac{\pi}{2} \Rightarrow u = 3$

$$\int_{-\pi/2}^{\pi/2} \left(2 + \tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt = 2\int_1^3 u\,du = \left[u^2\right]_1^3 = 3^2 - 1^2 = 8$$

9. (a) Let $u = 4 + 3\sin z \Rightarrow du = 3\cos z\,dz \Rightarrow \frac{1}{3}\,du = \cos z\,dz$; $z = 0 \Rightarrow u = 4$, $z = 2\pi \Rightarrow u = 4$

$$\int_0^{2\pi} \frac{\cos z}{\sqrt{4 + 3\sin z}}\,dz = \int_4^4 \frac{1}{\sqrt{u}}\left(\frac{1}{3}\,du\right) = 0$$

(b) Use the same substitution as in part (a); $z = -\pi \Rightarrow u = 4 + 3\sin(-\pi) = 4$, $z = \pi \Rightarrow u = 4$

$$\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4 + 3\sin z}}\,dz = \int_4^4 \frac{1}{\sqrt{u}}\left(\frac{1}{3}\,du\right) = 0$$

10. (a) Let $u = 3 + 2\cos w \Rightarrow du = -2\sin w\,dw \Rightarrow -\frac{1}{2}\,du = \sin w\,dw$; $w = -\frac{\pi}{2} \Rightarrow u = 3$, $w = 0 \Rightarrow u = 5$

$$\int_{-\pi/2}^{0} \frac{\sin w}{(3 + 2\cos w)^2}\,dw = \int_3^5 u^{-2}\left(-\frac{1}{2}\,du\right) = \frac{1}{2}\left[u^{-1}\right]_3^5 = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{3}\right) = -\frac{1}{15}$$

(b) Use the same substitution as in part (a); $w = 0 \Rightarrow u = 5$, $w = \frac{\pi}{2} \Rightarrow u = 3$

$$\int_0^{\pi/2} \frac{\sin w}{(3 + 2\cos w)^2}\, dw = \int_5^3 u^{-2}\left(-\tfrac{1}{2}\, du\right) = \frac{1}{2}\int_3^5 u^{-2}\, du = \frac{1}{15}$$

11. Let $u = t^5 + 2t \Rightarrow du = \left(5t^4 + 2\right) dt$; $t = 0 \Rightarrow u = 0$, $t = 1 \Rightarrow u = 3$

$$\int_0^1 \sqrt{t^5 + 2t}\left(5t^4 + 2\right) dt = \int_0^3 u^{1/2}\, du = \left[\tfrac{2}{3}u^{3/2}\right]_0^3 = \tfrac{2}{3}(3)^{3/2} - \tfrac{2}{3}(0)^{3/2} = 2\sqrt{3}$$

12. Let $u = 1 + \sqrt{y} \Rightarrow du = \frac{dy}{2\sqrt{y}}$; $y = 1 \Rightarrow u = 2$, $y = 4 \Rightarrow u = 3$

$$\int_1^4 \frac{dy}{2\sqrt{y}\left(1 + \sqrt{y}\right)^2} = \int_2^3 \frac{1}{u^2}\, du = \int_2^3 u^{-2}\, du = \left[-u^{-1}\right]_2^3 = \left(-\tfrac{1}{3}\right) - \left(-\tfrac{1}{2}\right) = \tfrac{1}{6}$$

13. Let $u = \cos 2\theta \Rightarrow du = -2\sin 2\theta\, d\theta \Rightarrow -\frac{1}{2}\, du = \sin 2\theta\, d\theta$; $\theta = 0 \Rightarrow u = 1$, $\theta = \frac{\pi}{6} \Rightarrow u = \cos 2\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\, d\theta = \int_1^{1/2} u^{-3}\left(-\tfrac{1}{2}\, du\right) = -\tfrac{1}{2}\int_1^{1/2} u^{-3}\, du = \left[-\tfrac{1}{2}\left(\tfrac{u^{-2}}{-2}\right)\right]_1^{1/2} = \frac{1}{4\left(\frac{1}{2}\right)^2} - \frac{1}{4(1)^2} = \frac{3}{4}$$

14. Let $u = \tan\left(\frac{\theta}{6}\right) \Rightarrow du = \frac{1}{6}\sec^2\left(\frac{\pi}{6}\right) d\theta \Rightarrow 6\, du = \sec^2\left(\frac{\theta}{6}\right) d\theta$; $\theta = \pi \Rightarrow u = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, $\theta = \frac{3\pi}{2} \Rightarrow$ $u = \tan\frac{\pi}{4} = 1$

$$\int_\pi^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right) d\theta = \int_{1/\sqrt{3}}^1 u^{-5}\,(6\, du) = \left[6\left(\frac{u^{-4}}{-4}\right)\right]_{1/\sqrt{3}}^1 = \left[-\frac{3}{2u^4}\right]_{1/\sqrt{3}}^1 = -\frac{3}{2(1)^4} - \left(-\frac{3}{2\left(\frac{1}{\sqrt{3}}\right)^4}\right) = 12$$

15. Let $u = 1 - \sin 2t \Rightarrow du = -2\cos 2t\, dt \Rightarrow -\frac{1}{2}\, du = \cos 2t\, dt$; $t = 0 \Rightarrow u = 1$, $t = \frac{\pi}{4} \Rightarrow u = 0$

$$\int_0^{\pi/4} (1 - \sin 2t)^{3/2} \cos 2t\, dt = \int_1^0 -\tfrac{1}{2}u^{3/2}\, du = \left[-\tfrac{1}{2}\left(\tfrac{2}{5}u^{5/2}\right)\right]_1^0 = \left(-\tfrac{1}{5}(0)^{5/2}\right) - \left(-\tfrac{1}{5}(1)^{5/2}\right) = \tfrac{1}{5}$$

16. Let $u = 4y - y^2 + 4y^3 + 1 \Rightarrow du = \left(4 - 2y + 12y^2\right) dy$; $y = 0 \Rightarrow u = 1$, $y = 1 \Rightarrow u = 4(1) - (1)^2 + 4(1)^3 + 1 = 8$

$$\int_0^1 \left(4y - y^2 + 4y^3 + 1\right)^{-2/3}\left(12y^2 - 2y + 4\right) dy = \int_1^8 u^{-2/3}\, du = \left[3u^{1/3}\right]_1^8 = 3(8)^{1/3} - 3(1)^{1/3} = 3$$

17. $y(t) = \int \frac{1}{t^2} \sec^2 \frac{\pi}{t}\, dt$ with $y(4) = \frac{2}{\pi}$. Let $u = \frac{\pi}{t} \Rightarrow du = -\frac{\pi}{t^2}\, dt \Rightarrow \frac{1}{t^2}\, dt = -\frac{du}{\pi}$

$\Rightarrow y(t) = \int \frac{1}{t^2} \sec^2 \frac{\pi}{t}\, dt = -\frac{1}{\pi} \int \sec^2 u\, du = -\frac{1}{\pi} \tan u + C = -\frac{1}{\pi} \tan \frac{\pi}{t} + C;$

$y(4) = \frac{2}{\pi} = -\frac{1}{\pi} \tan \frac{\pi}{4} + C \Rightarrow C = \frac{3}{\pi} \Rightarrow y(t) = \frac{1}{\pi}\left(3 - \tan \frac{\pi}{t}\right)$

18. $y(t) = \int \frac{1}{\sqrt{t}} \sin^2 \sqrt{t} \cos \sqrt{t}\, dt$ with $y\left(\frac{\pi^2}{16}\right) = 0$. Let $u = \sin \sqrt{t} \Rightarrow du = \frac{1}{2\sqrt{t}} \cos \sqrt{t}\, dt$

$\Rightarrow 2\, du = \frac{1}{\sqrt{t}} \cos \sqrt{t}\, dt \Rightarrow y(t) = \int 2u^2\, du = \frac{2u^3}{3} + C = \frac{2}{3} \sin^3 \sqrt{t} + C;$

$y\left(\frac{\pi^2}{16}\right) = 0 = \frac{2}{3} \sin^3 \sqrt{\frac{\pi^2}{16}} + C \Rightarrow C = -\frac{\sqrt{2}}{6} \Rightarrow y(t) = \frac{\sqrt{2}}{6}\left(2\sqrt{2} \sin^3 \sqrt{t} - 1\right)$

19. For the sketch given, $a = 0$, $b = \pi$; $f(x) - g(x) = 1 - \cos^2 x = \sin^2 x = \frac{1 - \cos 2x}{2}$;

$$A = \int_0^{\pi} \frac{(1 - \cos 2x)}{2}\, dx = \frac{1}{2} \int_0^{\pi} (1 - \cos 2x)\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}[(\pi - 0) - (0 - 0)] = \frac{\pi}{2}$$

20. For the sketch given, $a = -\frac{\pi}{3}$, $b = \frac{\pi}{3}$; $f(t) - g(t) = \frac{1}{2} \sec^2 t - \left(-4 \sin^2 t\right) = \frac{1}{2} \sec^2 t + 4 \sin^2 t$;

$$A = \int_{-\pi/3}^{\pi/3} \left(\frac{1}{2} \sec^2 t + 4 \sin^2 t\right) dt = \frac{1}{2} \int_{-\pi/3}^{\pi/3} \sec^2 t\, dt + 4 \int_{-\pi/3}^{\pi/3} \sin^2 t\, dt = \frac{1}{2} \int_{-\pi/3}^{\pi/3} \sec^2 t\, dt + 4 \int_{-\pi/3}^{\pi/3} \frac{(1 - \cos 2t)}{2}\, dt$$

$$= \frac{1}{2} \int_{-\pi/3}^{\pi/3} \sec^2 t\, dt + 2 \int_{-\pi/3}^{\pi/3} (1 - \cos 2t)\, dt = \frac{1}{2}[\tan t]_{-\pi/3}^{\pi/3} + 2\left[t - \frac{1}{2} \sin 2t\right]_{-\pi/3}^{\pi/3} = \sqrt{3} + 4 \cdot \frac{\pi}{3} - \sqrt{3} = \frac{4\pi}{3}$$

21. For the sketch given, $a = -2$, $b = 2$; $f(x) - g(x) = 2x^2 - \left(x^4 - 2x^2\right) = 4x^2 - x^4$;

$$A = \int_{-2}^{2} \left(4x^2 - x^4\right) dx = \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} = \left(\frac{32}{3} - \frac{32}{5}\right) - \left[-\frac{32}{3} - \left(-\frac{32}{5}\right)\right] = \frac{64}{3} - \frac{64}{5} = \frac{320 - 192}{15} = \frac{128}{15}$$

22. For the sketch given, $a = -1$, $b = 1$; $f(x) - g(x) = x^2 - \left(-2x^4\right) = x^2 + 2x^4$;

$$A = \int_{-1}^{1} \left(x^2 + 2x^4\right) dx = \left[\frac{x^3}{3} + \frac{2x^5}{5}\right]_{-1}^{1} = \left(\frac{1}{3} + \frac{2}{5}\right) - \left[-\frac{1}{3} + \left(-\frac{2}{5}\right)\right] = \frac{2}{3} + \frac{4}{5} = \frac{10 + 12}{15} = \frac{22}{15}$$

23. AREA = A1 + A2

A1: For the sketch given, $a = -3$ and we find b by solving the equations $y = x^2 - 4$ and $y = -x^2 - 2x$ simultaneously for x: $x^2 - 4 = -x^2 - 2x \Rightarrow 2x^2 + 2x - 4 = 0 \Rightarrow 2(x + 2)(x - 1) \Rightarrow x = -2$ or $x = 1$ so

$b = -2$: $f(x) - g(x) = \left(x^2 - 4\right) - \left(-x^2 - 2x\right) = 2x^2 + 2x - 4 \Rightarrow A1 = \int_{-3}^{-2} \left(2x^2 + 2x - 4\right) dx$

$$= \left[\frac{2x^3}{3} + \frac{2x^2}{2} - 4x\right]_{-3}^{-2} = \left(-\frac{16}{3} + 4 + 8\right) - (-18 + 9 + 12) = 9 - \frac{16}{3} = \frac{11}{3};$$

A2: For the sketch given, $a = -2$ and $b = 1$: $f(x) - g(x) = (-x^2 - 2x) - (x^2 - 4) = -2x^2 - 2x + 4$

$$\Rightarrow A2 = -\int_{-2}^{1} (2x^2 + 2x - 4)\,dx = -\left[\frac{2x^3}{3} + x^2 - 4x\right]_{-2}^{1} = -\left(\frac{2}{3} + 1 - 4\right) + \left(-\frac{16}{3} + 4 + 8\right)$$

$$= -\frac{2}{3} - 1 + 4 - \frac{16}{3} + 4 + 8 = 9;$$

Therefore, AREA $= A1 + A2 = \frac{11}{3} + 9 = \frac{38}{3}$

24. AREA $= A1 + A2 + A3$

A1: For the sketch given, $a = -2$ and $b = -1$: $f(x) - g(x) = (-x + 2) - (4 - x^2) = x^2 - x - 2$

$$\Rightarrow A1 = \int_{-2}^{-1} (x^2 - x - 2)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^{-1} = \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(-\frac{8}{3} - \frac{4}{2} + 4\right) = \frac{7}{3} - \frac{1}{2} = \frac{14 - 3}{6} = \frac{11}{6};$$

A2: For the sketch given, $a = -1$ and $b = 2$: $f(x) - g(x) = (4 - x^2) - (-x + 2) = -(x^2 - x - 2)$

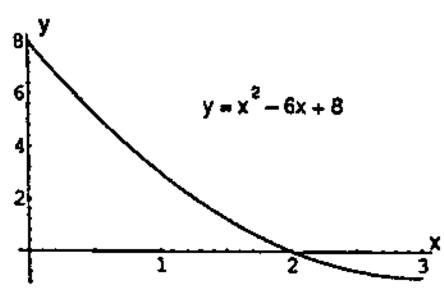
$$\Rightarrow A2 = -\int_{-1}^{2} (x^2 - x - 2)\,dx = -\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2} = -\left(\frac{8}{3} - \frac{4}{2} - 4\right) + \left(-\frac{1}{3} - \frac{1}{2} + 2\right) = -3 + 8 - \frac{1}{2} = \frac{9}{2};$$

A3: For the sketch given, $a = 2$ and $b = 3$: $f(x) - g(x) = (-x + 2) - (4 - x^2) = x^2 - x - 2$

$$\Rightarrow A3 = \int_{2}^{3} (x^2 - x - 2)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{2}^{3} = \left(\frac{27}{3} - \frac{9}{2} - 6\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right) = 9 - \frac{9}{2} - \frac{8}{3};$$

Therefore, AREA $= A1 + A2 + A3 = \frac{11}{6} + \frac{9}{2} + \left(9 - \frac{9}{2} - \frac{8}{3}\right) = 9 - \frac{5}{6} = \frac{49}{6}$

25. $x^2 - 6x + 8 = 0 \Rightarrow (x - 4)(x - 2) = 0 \Rightarrow x = 4$ or $x = 2$, the x-intercepts.



(a) $\int_0^3 (x^2 - 6x + 8)\,dx = \int_0^3 x^2\,dx - 6\int_0^3 x\,dx + \int_0^3 8\,dx$

$$= \frac{3^3}{3} - 6\left(\frac{3^2}{2} - \frac{0^2}{2}\right) + 8(3 - 0) = 6$$

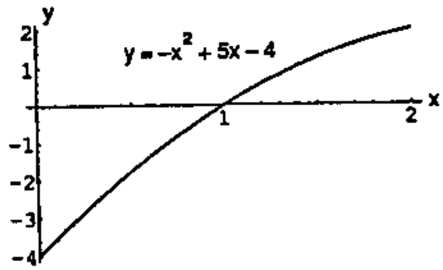
(b) Area $= \int_0^2 (x^2 - 6x + 8)\,dx + \left(-\int_2^3 (x^2 - 6x + 8)\,dx\right)$

$$= \left(\int_0^2 x^2\,dx - 6\int_0^2 x\,dx + \int_0^2 8\,dx\right) - \left(\int_2^3 x^2\,dx - 6\int_2^3 x\,dx + \int_2^3 8\,dx\right)$$

$$= \left[\frac{2^3}{3} - 6\left(\frac{2^2}{2} - \frac{0^2}{2}\right) + 8(2 - 0)\right] - \left(\int_0^3 x^2\,dx - \int_0^2 x^2\,dx - 6\left(\frac{3^2}{2} - \frac{2^2}{2}\right) + 8(3 - 2)\right)$$

$$= \left(\frac{8}{3} - 12 + 16\right) - \left(\frac{3^3}{3} - \frac{2^3}{3} - 15 + 8\right) = \frac{22}{3} = 7\frac{1}{3}$$

26. $-x^2+5x-4=0 \Rightarrow x^2-5x+4=0 \Rightarrow (x-4)(x-1)=0$
$\Rightarrow x=4$ or $x=1$, the x-intercepts.



(a) $\int_0^2 \left(-x^2+5x-4\right) dx = -\int_0^2 x^2\,dx + 5\int_0^2 x\,dx - \int_0^2 4\,dx$

$= -\frac{2^3}{3} + 5\left(\frac{2^2}{2}-\frac{0^2}{2}\right) - 4(2-0) = -\frac{2}{3}$

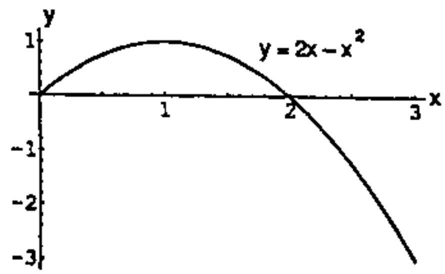
(b) Area $= -\int_0^1 \left(-x^2+5x-4\right) dx + \int_1^2 \left(-x^2+5x-4\right) dx$

$= \int_0^1 x^2\,dx - 5\int_0^1 x\,dx + \int_0^1 4\,dx + \int_1^2 -x^2\,dx + 5\int_1^2 x\,dx - \int_1^2 4\,dx$

$= \frac{1^3}{3} - 5\left(\frac{1^2}{2}-\frac{0^2}{2}\right) + 4(1-0) + \left(\int_0^2 -x^2\,dx - \int_0^1 -x^2\,dx\right) + 5\left(\frac{2^2}{2}-\frac{1^2}{2}\right) - 4(2-1)$

$= \frac{1}{3} - \frac{5}{2} + 4 - \frac{2^3}{3} + \frac{1^3}{3} + \frac{15}{2} - 4 = 3$

27. $2x-x^2=0 \Rightarrow x(2-x)=0 \Rightarrow x=0$ or $x=2$,
the x-intercepts.



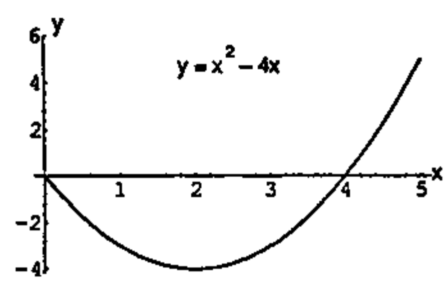
(a) $\int_0^3 \left(2x-x^2\right) dx = 2\int_0^3 x\,dx - \int_0^3 x^2\,dx$

$= 2\left(\frac{3^2}{2}-\frac{0^2}{2}\right) - \frac{3^3}{3} = 0$

(b) Area $= \int_0^2 \left(2x-x^2\right) dx - \int_2^3 \left(2x-x^2\right) dx = 2\int_0^2 x\,dx - \int_0^2 x^2\,dx - \left(2\int_2^3 x\,dx - \int_2^3 x^2\,dx\right)$

$= 2\left(\frac{2^2}{2}-\frac{0^2}{2}\right) - \frac{2^3}{3} - 2\left(\frac{3^2}{2}-\frac{2^2}{2}\right) + \left(\int_0^3 x^2\,dx - \int_0^2 x^2\,dx\right) = 4 - \frac{8}{3} - 5 + \frac{3^3}{3} - \frac{2^3}{3} = \frac{8}{3}$

28. $x^2-4x=0 \Rightarrow x(x-4)=0 \Rightarrow x=0$ or $x=4$,
the x-intercepts.



(a) $\int_0^5 \left(x^2-4x\right) dx = \int_0^5 x^2\,dx - 4\int_0^5 x\,dx$

$= \frac{5^3}{3} - 4\left(\frac{5^2}{2}-\frac{0^2}{2}\right) = -\frac{25}{3}$

(b) $\text{Area} = -\int_0^4 (x^2-4x)\,dx + \int_4^5 (x^2-4x)\,dx$

$$= -\int_0^4 x^2\,dx + 4\int_0^4 x\,dx + \int_4^5 x^2\,dx - 4\int_4^5 x\,dx = -\frac{4^3}{3} + 4\left(\frac{4^2}{2}-\frac{0^2}{2}\right) + \left(\int_0^5 x^2\,dx - \int_0^4 x^2\,dx\right) - 4\left(\frac{5^2}{2}-\frac{4^2}{2}\right)$$

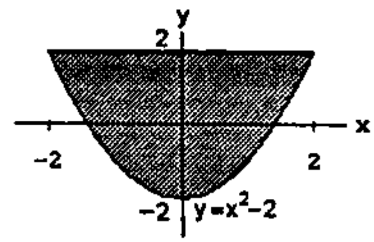
$$= -\frac{64}{3} + 32 + \frac{5^3}{3} - \frac{4^3}{3} - 18 = 13$$

29. $a = -2$, $b = 2$;

$f(x) - g(x) = 2 - (x^2-2) = 4 - x^2$

$$\Rightarrow A = \int_{-2}^{2} (4-x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left(8-\frac{8}{3}\right) - \left(-8+\frac{8}{3}\right)$$

$$= 2\cdot\left(\frac{24}{3}-\frac{8}{3}\right) = \frac{32}{3}$$



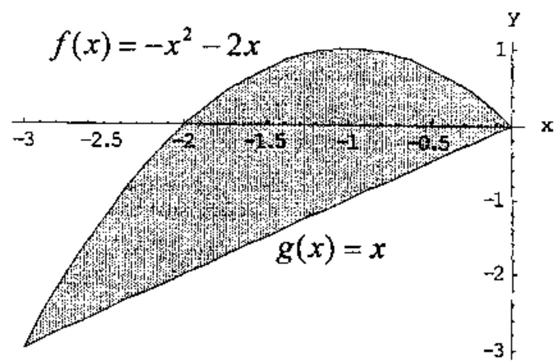
30. Limits of integration: $-x^2-2x = x \Rightarrow x^2 = -3x$

$\Rightarrow x(x+3) = 0 \Rightarrow a = -3$ and $b = 0$

$f(x) - g(x) = (-x^2-2x) - x = -x^2 - 3x$

$$\Rightarrow A = \int_{-3}^{0} (-x^2-3x)\,dx = \left[\frac{-3x^2}{2} - \frac{x^3}{3}\right]_{-3}^{0}$$

$$= 0 - \left(-\frac{27}{2}+\frac{27}{3}\right) = \frac{9}{2}$$



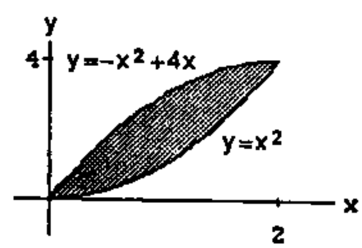
31. Limits of integration: $x^2 = -x^2+4x \Rightarrow 2x^2 - 4x = 0$

$\Rightarrow 2x(x-2) = 0 \Rightarrow a = 0$ and $b = 2$;

$f(x) - g(x) = (-x^2+4x) - x^2 = -2x^2 + 4x$

$$\Rightarrow A = \int_0^2 (-2x^2+4x)\,dx = \left[\frac{-2x^3}{3} + \frac{4x^2}{2}\right]_0^2$$

$$= -\frac{16}{3} + \frac{16}{2} = \frac{-32+48}{6} = \frac{8}{3}$$



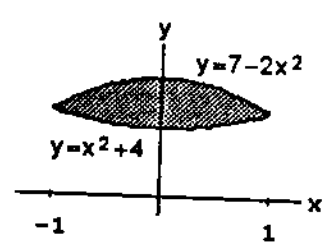
32. Limits of integration: $7 - 2x^2 = x^2 + 4 \Rightarrow 3x^2 - 3 = 0$

$\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow a = -1$ and $b = 1$;

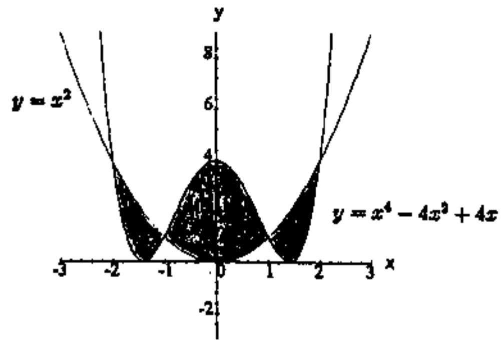
$f(x) - g(x) = (7-2x^2) - (x^2+4) = 3 - 3x^2$

$$\Rightarrow A = \int_{-1}^{1} (3-3x^2)\,dx = 3\left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= 3\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right] = 6\left(\frac{2}{3}\right) = 4$$



33. Limits of integration: $x^4 - 4x^2 + 4 = x^2$
$\Rightarrow x^4 - 5x^2 + 4 = 0 \Rightarrow (x^2 - 4)(x^2 - 1) = 0$
$\Rightarrow (x + 2)(x - 2)(x + 1)(x - 1) = 0 \Rightarrow x = -2, -1, 1, 2;$
$f(x) - g(x) = (x^4 - 4x^2 + 4) - x^2 = x^4 - 5x^2 + 4$ and
$g(x) - f(x) = x^2 - (x^4 - 4x^2 + 4) = -x^4 + 5x^2 - 4$



$$\Rightarrow A = \int_{-2}^{-1} (-x^4 + 5x^2 - 4)\,dx + \int_{-1}^{1} (x^4 - 5x^2 + 4)\,dx + \int_{1}^{2} (-x^4 + 5x^2 - 4)\,dx$$

$$= \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_{-2}^{-1} + \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_{-1}^{1} + \left[\frac{-x^5}{5} + \frac{5x^3}{3} - 4x\right]_{1}^{2} = \left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(\frac{32}{5} - \frac{40}{3} + 8\right) + \left(\frac{1}{5} - \frac{5}{3} + 4\right)$$

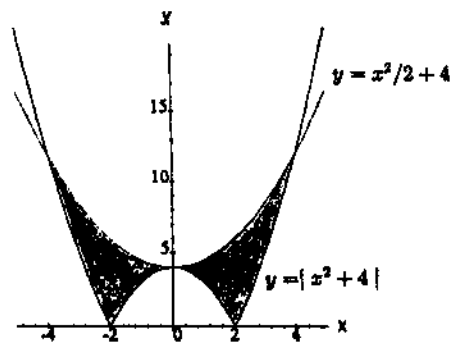
$$- \left(-\frac{1}{5} + \frac{5}{3} - 4\right) + \left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right) = -\frac{60}{5} + \frac{60}{3} = \frac{300 - 180}{15} = 8$$

34. Limits of integration: $y = |x^2 - 4| = \begin{cases} x^2 - 4, & x \le -2 \text{ or } x \ge 2 \\ 4 - x^2, & -2 \le x \le 2 \end{cases}$

for $x \le -2$ and $x \ge 2$: $x^2 - 4 = \frac{x^2}{2} + 4$
$\Rightarrow 2x^2 - 8 = x^2 + 8 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4;$
for $-2 \le x \le 2$: $4 - x^2 = \frac{x^2}{2} + 4 \Rightarrow 8 - 2x^2 = x^2 + 8$
$\Rightarrow x^2 = 0 \Rightarrow x = 0$; by symmetry of the graph,

$$A = 2\int_0^2 \left[\left(\frac{x^2}{2} + 4\right) - (4 - x^2)\right] dx + 2\int_2^4 \left[\left(\frac{x^2}{2} + 4\right) - (x^2 - 4)\right] dx$$

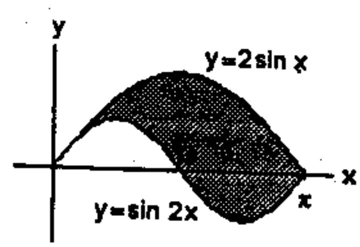
$$= 2\left[\frac{x^3}{2}\right]_0^2 + 2\left[8x - \frac{x^3}{6}\right]_2^4 = 2\left(\frac{8}{2} - 0\right) + 2\left(32 - \frac{64}{6} - 16 + \frac{8}{6}\right) = 40 - \frac{56}{3} = \frac{64}{3}$$



35. $a = 0$, $b = \pi$; $f(x) - g(x) = 2 \sin x - \sin 2x$

$$\Rightarrow A = \int_0^{\pi} (2 \sin x - \sin 2x)\,dx = \left[-2 \cos x + \frac{\cos 2x}{2}\right]_0^{\pi}$$

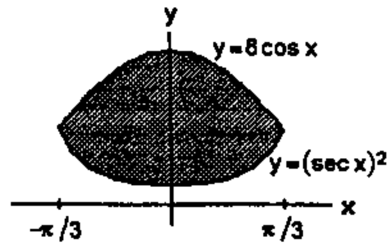
$$= \left[-2(-1) + \frac{1}{2}\right] - \left(-2 \cdot 1 + \frac{1}{2}\right) = 4$$



36. $a = -\frac{\pi}{3}$, $b = \frac{\pi}{3}$; $f(x) - g(x) = 8\cos x - \sec^2 x$

$$\Rightarrow A = \int_{-\pi/3}^{\pi/3} \left(8\cos x - \sec^2 x\right) dx = [8\sin x - \tan x]_{-\pi/3}^{\pi/3}$$

$$= \left(8 \cdot \frac{\sqrt{3}}{2} - \sqrt{3}\right) - \left(-8 \cdot \frac{\sqrt{3}}{2} + \sqrt{3}\right) = 6\sqrt{3}$$



37. $A = A1 + A2$

$a_1 = -1$, $b_1 = 0$ and $a_2 = 0$, $b_2 = 1$;

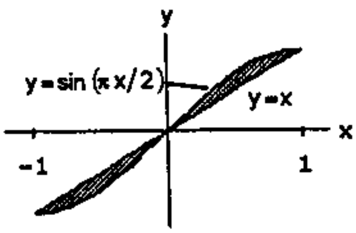
$f_1(x) - g_1(x) = x - \sin\left(\frac{\pi x}{2}\right)$ and $f_2(x) - g_2(x) = \sin\left(\frac{\pi x}{2}\right) - x$

$\Rightarrow$ by symmetry about the origin,

$$A_1 + A_2 = 2A_1 \Rightarrow A = 2\int_0^1 \left[\sin\left(\frac{\pi x}{2}\right) - x\right] dx$$

$$= 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}\right]_0^1 = 2\left[\left(-\frac{2}{\pi} \cdot 0 - \frac{1}{2}\right) - \left(-\frac{2}{\pi} \cdot 1 - 0\right)\right]$$

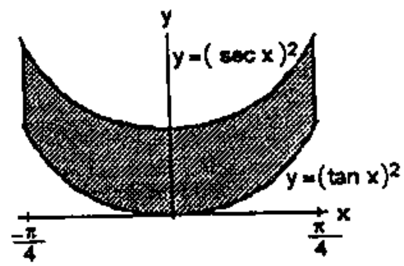
$$= 2\left(\frac{2}{\pi} - \frac{1}{2}\right) = 2\left(\frac{4-\pi}{2\pi}\right) = \frac{4-\pi}{\pi}$$



38. $a = -\frac{\pi}{4}$, $b = \frac{\pi}{4}$; $f(x) - g(x) = \sec^2 x - \tan^2 x$

$$\Rightarrow A = \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \tan^2 x\right) dx = \int_{-\pi/4}^{\pi/4} \left[\sec^2 x - \left(\sec^2 x - 1\right)\right] dx$$

$$= \int_{-\pi/4}^{\pi/4} 1 \cdot dx = [x]_{-\pi/4}^{\pi/4} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$



39. $A = A_1 + A_2$

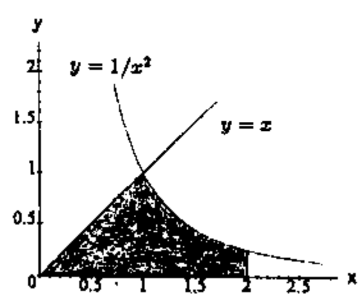
Limits of integration: $y = x$ and $y = \frac{1}{x^2} \Rightarrow x = \frac{1}{x^2}$, $x \neq 0$

$\Rightarrow x^3 = 1 \Rightarrow x = 1$, $f_1(x) - g_1(x) = x - 0 = x$

$$\Rightarrow A_1 = \int_0^1 x\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2};\ f_2(x) - g_2(x) = \frac{1}{x^2} - 0$$

$$= x^{-2} \Rightarrow A_2 = \int_1^2 x^{-2}\, dx = \left[\frac{-1}{x}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2};$$

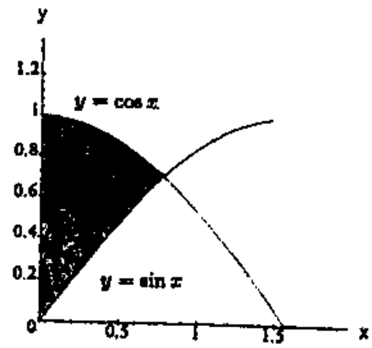
$A = A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = 1$



40. Limits of integration: $\sin x = \cos x \Rightarrow x = \frac{\pi}{4} \Rightarrow a = 0$ and $b = \frac{\pi}{4}$; $f(x) - g(x) = \cos x - \sin x$

$$\Rightarrow A = \int_0^{\pi/4} (\cos x - \sin x)\, dx = [\sin x + \cos x]_0^{\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \sqrt{2} - 1$$



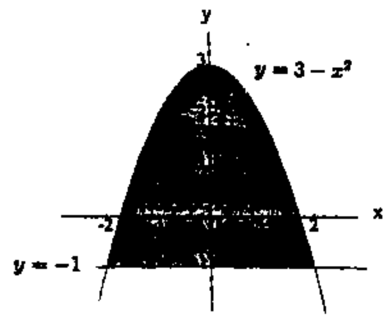
41. Limits of integration: $y = 3 - x^2$ and $y = -1$

$\Rightarrow 3 - x^2 = -1 \Rightarrow x^2 = 4 \Rightarrow a = -2$ and $b = 2$;

$f(x) - g(x) = \left(3 - x^2\right) - (-1) = 4 - x^2$

$$\Rightarrow A = \int_{-2}^{2} \left(4 - x^2\right) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3} = \frac{32}{3}$$



42. Limits of integration: $y = 1 + \sqrt{x}$ and $y = \frac{2}{\sqrt{x}}$

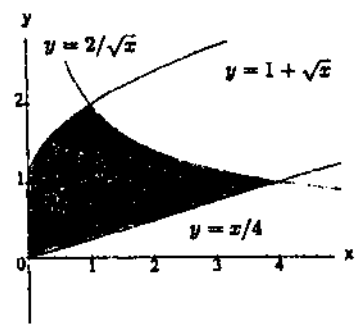
$\Rightarrow 1 + \sqrt{x} = \frac{2}{\sqrt{x}},\ x \neq 0 \Rightarrow \sqrt{x} + x = 2 \Rightarrow x = (2 - x)^2$

$\Rightarrow x = 4 - 4x + x^2 \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x - 4)(x - 1) = 0 \Rightarrow x = 1,\ 4$ (but $x = 4$ does not satisfy the equation); $y = \frac{2}{\sqrt{x}}$ and $y = \frac{x}{4} \Rightarrow \frac{2}{\sqrt{x}} = \frac{x}{4}$

$\Rightarrow 8 = x\sqrt{x} \Rightarrow 64 = x^3 \Rightarrow x = 4$ since $x > 0$;

Therefore, AREA $= A_1 + A_2$: $f_1(x) - g_1(x) = \left(1 + x^{1/2}\right) - \frac{x}{4}$



$$\Rightarrow A_1 = \int_0^1 \left(1 + x^{1/2} - \frac{x}{4}\right) dx = \left[x + \frac{2}{3}x^{3/2} - \frac{x^2}{8}\right]_0^1 = \left(1 + \frac{2}{3} - \frac{1}{8}\right) - 0 = \frac{37}{24};\ f_2(x) - g_2(x) = 2x^{-1/2} - \frac{x}{4}$$

$$\Rightarrow A_2 = \int_1^4 \left(2x^{-1/2} - \frac{x}{4}\right) dx = \left[4x^{1/2} - \frac{x^2}{8}\right]_1^4 = \left(4 \cdot 2 - \frac{16}{8}\right) - \left(4 - \frac{1}{8}\right) = 4 - \frac{15}{8} = \frac{17}{8};$$

Therefore, AREA $= A_1 + A_2 = \frac{37}{24} + \frac{17}{8} = \frac{37 + 51}{24} = \frac{88}{24} = \frac{11}{3}$

43-46. Example CAS commands:

Maple:

```
p:=x^2*cos(x);
q:=x^3 - x;
plot({p,q}, x=-2..2,-2..2);
intpt1:=fsolve(p=q,x=-2..0);
```

```
intpt2:=fsolve(p=q,x=0..2);
intone:=Int(q-p,x=intpt1..0);
inttwo:=Int(p-q,x=0..intpt2);
evalf(intone+inttwo);
```

Mathematica:

```
Clear[x]
f[x_] = x^2 Cos[x]
g[x_] = x^3 - x
Plot[ {f[x],g[x]}, {x,-4,4} ]

Here, need to use FindRoot for each crossing; can do all together using Map
over initial guesses.

pts = x /. Map[
    FindRoot[ f[x] == g[x], {x,#} ]&,
    {-1,0,1} ]
i1 = NIntegrate[ f[x] - g[x], {x,pts[[1]],pts[[2]]}]
i2 = NIntegrate[ g[x] - f[x], {x,pts[[2]],pts[[3]]}]
i1 + i2
```

4.7 NUMERICAL INTEGRATION

1. $\int_1^2 x\,dx$

I. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{h}{2} = \frac{1}{8}$;

$\sum mf(x_i) = 12 \Rightarrow T = \frac{1}{8}(12) = \frac{3}{2}$;

$f(x) = x \Rightarrow f'(x) = 1 \Rightarrow f'' = 0 \Rightarrow M = 0$

$\Rightarrow |E_T| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	2	5/2
x_2	3/2	3/2	2	3
x_3	7/4	7/4	2	7/2
x_4	2	2	1	2

(b) $\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow |E_T| = \int_1^2 x\,dx - T = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{h}{3} = \frac{1}{12}$;

$\sum mf(x_i) = 18 \Rightarrow S = \frac{1}{12}(18) = \frac{3}{2}$;

$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	4	5
x_2	3/2	3/2	2	3
x_3	7/4	7/4	4	7
x_4	2	2	1	2

(b) $\int_1^2 x\,dx = \frac{3}{2} \Rightarrow |E_S| = \int_1^2 x\,dx - S = \frac{3}{2} - \frac{3}{2} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

2. $\int_1^3 (2x-1)\,dx$

I. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4}$;

$\sum mf(x_i) = 24 \Rightarrow T = \frac{1}{4}(24) = 6$;

$f(x) = 2x - 1 \Rightarrow f'(x) = 2 \Rightarrow f'' = 0 \Rightarrow M = 0$

$\Rightarrow |E_T| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	2	4
x_2	2	3	2	6
x_3	5/2	4	2	8
x_4	3	5	1	5

(b) $\int_1^3 (2x-1)\,dx = \left[x^2 - x\right]_1^3 = (9-3) - (1-1) = 6 \Rightarrow |E_T| = \int_1^3 (2x-1)\,dx - T = 6 - 6 = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6}$;

$\sum mf(x_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6$;

$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	4	8
x_2	2	3	2	6
x_3	5/2	4	4	16
x_4	3	5	1	5

(b) $\int_1^3 (2x-1)\,dx = 6 \Rightarrow |E_S| = \int_1^3 (2x-1)\,dx - S = 6 - 6 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

3. $\int_{-1}^1 (x^2+1)\,dx$

I. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4}$;

$\sum mf(x_i) = 11 \Rightarrow T = \frac{1}{4}(11) = 2.75$;

$f(x) = x^2 + 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2 \Rightarrow M = 2$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−1	2	1	2
x_1	−1/2	5/4	2	5/2
x_2	0	1	2	2
x_3	1/2	5/4	2	5/2
x_4	1	2	1	2

$$\Rightarrow |E_T| \le \frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(2) = \frac{1}{12} \text{ or } 0.08333$$

(b) $\int_{-1}^{1} (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_{-1}^{1} = \left(\frac{1}{3}+1\right)-\left(-\frac{1}{3}-1\right) = \frac{8}{3} \Rightarrow E_T = \int_{-1}^{1} (x^2+1)\,dx - T = \frac{8}{3}-\frac{11}{4} = -\frac{1}{12}$

$$\Rightarrow |E_T| = \left|-\frac{1}{12}\right| \approx 0.08333$$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{8}{3}}\right) \times 100 \approx 3\%$

II. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6}$;

$\sum mf(x_i) = 16 \Rightarrow S = \frac{1}{6}(16) = \frac{8}{3} = 2.66667$;

$f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−1	2	1	2
x_1	−1/2	5/4	4	5
x_2	0	1	2	2
x_3	1/2	5/4	4	5
x_4	1	2	1	2

(b) $\int_{-1}^{1} (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_{-1}^{1} = \frac{8}{3} \Rightarrow |E_S| = \int_{-1}^{1} (x^2+1)\,dx - S = \frac{8}{3}-\frac{8}{3} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

4. $\int_{-2}^{0} (x^2-1)\,dx$

I. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4}$;

$\sum mf(x_i) = 3 \Rightarrow T = \frac{1}{4}(3) = \frac{3}{4}$;

$f(x) = x^2 - 1 \Rightarrow f'(x) = 2x + 1 \Rightarrow f''(x) = 2 \Rightarrow M = 2$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−2	3	1	3
x_1	−3/2	5/4	2	5/2
x_2	−1	0	2	0
x_3	− 1/2	−3/4	2	−3/2
x_4	0	−1	1	−1

$$\Rightarrow |E_T| \le \frac{0-(-2)}{12}\left(\frac{1}{2}\right)^2(2) = \frac{1}{12} = 0.08333$$

(b) $\int_{-2}^{0} (x^2-1)\,dx = \left[\frac{x^3}{3}-x\right]_{-2}^{0} = 0-\left(-\frac{8}{3}+2\right) = \frac{2}{3} \Rightarrow E_T = \int_{-2}^{0} (x^2-1)\,dx - T = \frac{2}{3}-\frac{2}{4} = -\frac{1}{12}$

$$\Rightarrow |E_T| = \frac{1}{12}$$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{2}{3}}\right) \times 100 \approx 13\%$

II. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6}$;

$\sum mf(x_i) = 4 \Rightarrow S = \frac{1}{6}(4) = \frac{2}{3}$;

$f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−2	3	1	3
x_1	−3/2	5/4	4	5
x_2	−1	0	2	0
x_3	−1/2	−3/4	4	−3
x_4	0	−1	1	−1

(b) $\int_{-2}^{0} (x^2-1)\,dx = \frac{2}{3} \Rightarrow |E_S| = \int_{-2}^{0} (x^2-1)\,dx - S = \frac{2}{3} - \frac{2}{3} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

5. $\int_0^2 (t^3+t)\,dt$

I. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4}$;

$\sum mf(t_i) = 25 \Rightarrow T = \frac{1}{4}(25) = \frac{25}{4}$;

$f(t) = t^3 + t \Rightarrow f'(t) = 3t^2 + 1 \Rightarrow f''(t) = 6t \Rightarrow M = 12$

$= f''(2) \Rightarrow |E_T| \le \frac{2-0}{12}\left(\frac{1}{2}\right)^2(12) = \frac{1}{2}$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	2	5/4
t_2	1	2	2	4
t_3	3/2	39/8	2	39/4
t_4	2	10	1	10

(b) $\int_0^2 (t^3+t)\,dt = \left[\frac{t^4}{4} + \frac{t^2}{2}\right]_0^2 = \left(\frac{2^4}{4} + \frac{2^2}{2}\right) - 0 = 6 \Rightarrow |E_T| = \int_0^2 (t^3+t)\,dt - T = 6 - \frac{25}{4} = -\frac{1}{4} \Rightarrow |E_T| = \frac{1}{4}$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{\left|-\frac{1}{4}\right|}{6} \times 100 \approx 4\%$

II. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6}$;

$\sum mf(t_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6$;

$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	4	5/2
t_2	1	2	2	4
t_3	3/2	39/8	4	39/2
t_4	2	10	1	10

(b) $\int_0^2 (t^3+t)\,dt = 6 \Rightarrow |E_S| = \int_0^2 (t^3+t)\,dt - S = 6 - 6 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

6. $\int_{-1}^{1} (t^3+1)\,dt$

I. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4}$;

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	−1	0	1	0
t_1	−1/2	7/8	2	7/4
t_2	0	1	2	2
t_3	1/2	9/8	2	9/4
t_4	1	2	1	2

$\sum mf(t_i) = 8 \Rightarrow T = \frac{1}{4}(8) = 2$;

$f(t) = t^3 + 1 \Rightarrow f'(t) = 3t^2 \Rightarrow f''(t) = 6t \Rightarrow M = 6$

$= f''(1) \Rightarrow |E_T| \le \frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(6) = \frac{1}{4}$

(b) $\int_{-1}^{1} (t^3+1)\,dt = \left[\frac{t^4}{4} + t\right]_{-1}^{1} = \left(\frac{1^4}{4} + 1\right) - \left(\frac{(-1)^4}{4} + (-1)\right) = 2 \Rightarrow |E_T| = \int_{-1}^{1} (t^3+1)\,dt - T = 2 - 2 = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6}$;

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	−1	0	1	0
t_1	−1/2	7/8	4	7/2
t_2	0	1	2	2
t_3	1/2	9/8	4	9/2
t_4	1	2	1	2

$\sum mf(t_i) = 12 \Rightarrow S = \frac{1}{6}(12) = 2$;

$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

(b) $\int_{-1}^{1} (t^3+1)\,dt = 2 \Rightarrow |E_S| = \int_{-1}^{1} (t^3+1)\,dt - S = 2 - 2 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

7. $\int_1^2 \frac{1}{s^2}\, ds$

I. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{h}{2} = \frac{1}{8}$;

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	2	32/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	2	32/49
s_4	2	1/4	1	1/4

$\sum mf(s_i) = \frac{179{,}573}{44{,}100} \Rightarrow T = \frac{1}{8}\left(\frac{179{,}573}{44{,}100}\right) = \frac{179{,}573}{352{,}800}$

≈ 0.50899; $f(s) = \frac{1}{s^2} \Rightarrow f'(s) = -\frac{2}{s^3} \Rightarrow f''(s) = \frac{6}{s^4}$

$\Rightarrow M = 6 = f''(1) \Rightarrow |E_T| \le \frac{2-1}{12}\left(\frac{1}{4}\right)^2(6) = \frac{1}{32} = 0.03125$

(b) $\int_1^2 \frac{1}{s^2}\, ds = \int_1^2 s^{-2}\, ds = \left[-\frac{1}{s}\right]_1^2 = -\frac{1}{2} - \left(-\frac{1}{1}\right) = \frac{1}{2} \Rightarrow E_T = \int_1^2 \frac{1}{s^2}\, ds - T = \frac{1}{2} - 0.50899 = -0.00899$

$\Rightarrow |E_T| = 0.00899$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.00899}{0.5} \times 100 \approx 2\%$

II. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{h}{3} = \frac{1}{12}$;

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	4	64/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	4	64/49
s_4	2	1/4	1	1/4

$\sum mf(s_i) = \frac{264{,}821}{44{,}100} \Rightarrow S = \frac{1}{12}\left(\frac{264{,}821}{44{,}100}\right) = \frac{264{,}821}{529{,}200}$

≈ 0.50042; $f^{(3)}(s) = -\frac{24}{s^5} \Rightarrow f^{(4)}(s) = \frac{120}{s^6} \Rightarrow M$

$= 120 = f^{(4)}(1) \Rightarrow |E_S| \le \frac{2-1}{180}\left(\frac{1}{4}\right)^4(120) = \frac{1}{384} \approx 0.002604$

(b) $\int_1^2 \frac{1}{s^2}\, ds = \frac{1}{2} \Rightarrow E_S = \int_1^2 \frac{1}{s^2}\, ds - S = \frac{1}{2} - 0.050041 = -0.00041 \Rightarrow |E_S| = 0.00041$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.0004}{0.5} \times 100 \approx 0.08\%$

8. $\int_2^4 \frac{1}{(s-1)^2}\, ds$

I. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4}$;

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	2	1	1	1
s_1	5/2	4/9	2	8/9
s_2	3	1/4	2	1/2
s_3	7/2	4/25	2	8/25
s_4	4	1/9	1	1/9

$\sum mf(s_i) = \frac{1269}{450} \Rightarrow T = \frac{1}{4}\left(\frac{1269}{450}\right) = \frac{1269}{1800} = 0.70500$;

$f(s) = (s-1)^{-2} \Rightarrow f'(s) = -\frac{2}{(s-1)^3}$

$\Rightarrow f''(s) = \frac{6}{(s-1)^4} \Rightarrow M = 6 = f''(2) \Rightarrow |E_T| \le \frac{4-2}{12}\left(\frac{1}{2}\right)^2(6)$

$= \frac{1}{4} = 0.25$

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds = \left[\frac{-1}{(s-1)}\right]_2^4 = \left(\frac{-1}{4-1}\right) - \left(\frac{-1}{2-1}\right) = \frac{2}{3} \Rightarrow E_T = \int_2^4 \frac{1}{(s-1)^2}\,ds - T = \frac{2}{3} - 0.705 \approx -0.03833$

$\Rightarrow |E_T| \approx 0.03833$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03833}{\left(\frac{2}{3}\right)} \times 100 \approx 6\%$

II. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6}$;

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	2	1	1	1
s_1	5/2	4/9	4	16/9
s_2	3	1/4	2	1/2
s_3	7/2	4/25	4	16/25
s_4	4	1/9	1	1/9

$\sum mf(s_i) = \frac{1813}{450} \Rightarrow S = \frac{1}{6}\left(\frac{1813}{450}\right) = \frac{1813}{2700} \approx 0.67148$;

$f^{(3)}(s) = \frac{-24}{(s-1)^5} \Rightarrow f^{(4)}(s) = \frac{120}{(s-1)^6} \Rightarrow M = 120 = f^{(4)}(2)$

$\Rightarrow |E_S| \le \frac{4-2}{180}\left(\frac{1}{2}\right)^4(120) = \frac{1}{12} \approx 0.08333$

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds = \frac{2}{3} \Rightarrow E_S = \int_2^4 \frac{1}{(s-1)^2}\,ds - S \approx \frac{2}{3} - 0.67148 = -0.00481 \Rightarrow |E_S| \approx 0.00481$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00481}{\left(\frac{2}{3}\right)} \times 100 \approx 1\%$

9. $\int_0^\pi \sin t\,dt$

I. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{h}{2} = \frac{\pi}{8}$;

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	$\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
t_2	$\pi/2$	1	2	2
t_3	$3\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
t_4	π	0	1	0

$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.8284 \Rightarrow T = \frac{\pi}{8}(2 + 2\sqrt{2})$

≈ 1.89612; $f(t) = \sin t \Rightarrow f'(t) = \cos t \Rightarrow f''(t) = -\sin t$

$\Rightarrow M = 1 = |f''(0)| \Rightarrow |E_T| \le \frac{\pi-0}{12}\left(\frac{\pi}{4}\right)^2(1) = \frac{\pi^3}{192} \approx 0.16149$

(b) $\int_0^\pi \sin t\,dt = [-\cos t]_0^\pi = (-\cos \pi) - (-\cos 0) = 2 \Rightarrow |E_T| = \int_0^\pi \sin t\,dt - T \approx 2 - 1.89612 = 0.10388$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.10388}{2} \times 100 \approx 5\%$

II. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4} \Rightarrow \frac{h}{3} = \frac{\pi}{12}$;

$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.6569 \Rightarrow S = \frac{\pi}{12}(2 + 4\sqrt{2})$

≈ 2.00456; $f^{(3)}(t) = -\cos t \Rightarrow f^{(4)}(t) = \sin t$

$\Rightarrow M = 1 = f^{(4)}(0) \Rightarrow |E_S| \le \frac{\pi - 0}{180}\left(\frac{\pi}{4}\right)^4(1) \approx 0.00664$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	$\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_2	$\pi/2$	1	2	2
t_3	$3\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_4	π	0	1	0

(b) $\int_0^{\pi} \sin t\, dt = 2 \Rightarrow E_S = \int_0^{\pi} \sin t\, dt - S \approx 2 - 2.00456 = -0.00456 \Rightarrow |E_S| \approx 0.00456$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00456}{2} \times 100 \approx 0.23\%$

10. $\int_0^1 \sin \pi t\, dt$

I. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{h}{2} = \frac{1}{8}$;

$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.828 \Rightarrow T = \frac{1}{8}(2 + 2\sqrt{2})$

≈ 0.60355; $f(t) = \sin \pi t \Rightarrow f'(t) = \pi \cos \pi t$

$\Rightarrow f''(t) = -\pi^2 \sin \pi t \Rightarrow M = \pi^2 = |f''(0)|$

$\Rightarrow |E_T| \le \frac{1-0}{12}\left(\frac{1}{4}\right)^2(\pi^2) \approx 0.05140$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/4	$\sqrt{2}/2$	2	$\sqrt{2}$
t_2	1/2	1	2	2
t_3	3/4	$\sqrt{2}/2$	2	$\sqrt{2}$
t_4	1	0	1	0

(b) $\int_0^1 \sin \pi t\, dt = \left[-\frac{1}{\pi}\cos \pi t\right]_0^1 = \left(-\frac{1}{\pi}\cos \pi\right) - \left(-\frac{1}{\pi}\cos 0\right) = \frac{2}{\pi} \approx 0.63662 \Rightarrow |E_T| = \int_0^1 \sin \pi t\, dt - T$

$\approx \frac{2}{\pi} - 0.60355 = 0.03307$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03307}{\left(\frac{2}{\pi}\right)} \times 100 \approx 5\%$

II. (a) For $n = 4$, $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{h}{3} = \frac{1}{12}$;

$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.65685 \Rightarrow S = \frac{1}{12}(2 + 4\sqrt{2})$

≈ 0.63807; $f^{(3)}(t) = -\pi^3 \cos \pi t \Rightarrow f^{(4)}(t) = \pi^4 \sin \pi t$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_2	1/2	1	2	2
t_3	3/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_4	1	0	1	0

$\Rightarrow M = \pi^4 = f^{(4)}(0) \Rightarrow |E_S| \le \frac{1-0}{180}\left(\frac{1}{4}\right)^4(\pi^4) \approx 0.00211$

(b) $\int_0^1 \sin \pi t \, dt = \frac{2}{\pi} \approx 0.63662 \Rightarrow E_S = \int_0^1 \sin \pi t \, dt - S \approx \frac{2}{\pi} - 0.63807 = -0.00145 \Rightarrow |E_S| \approx 0.00145$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00145}{\left(\frac{2}{\pi}\right)} \times 100 \approx 0\%$

11. (a) $n = 8 \Rightarrow h = \frac{1}{8} \Rightarrow \frac{h}{2} = \frac{1}{16}$;

$\sum mf(x_i) = 1(0.0) + 2(0.12402) + 2(0.24206) + 2(0.34763) + 2(0.43301) + 2(0.48789) + 2(0.49608)$
$+ 2(0.42361) + 1(0) = 5.1086 \Rightarrow T = \frac{1}{16}(5.1086) = 0.31929$

(b) $n = 8 \Rightarrow h = \frac{1}{8} \Rightarrow \frac{h}{3} = \frac{1}{24}$;

$\sum mf(x_i) = 1(0.0) + 4(0.12402) + 2(0.24206) + 4(0.34763) + 2(0.43301) + 4(0.48789) + 2(0.49608)$
$+ 4(0.42361) + 1(0) = 7.8749 \Rightarrow S = \frac{1}{24}(7.8749) = 0.32812$

(c) Let $u = 1 - x^2 \Rightarrow du = -2x \, dx \Rightarrow -\frac{1}{2} du = x \, dx$; $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 0$

$\int_0^1 x\sqrt{1-x^2} \, dx = \int_1^0 \sqrt{u}\left(-\frac{1}{2} du\right) = \frac{1}{2}\int_0^1 u^{1/2} \, du = \left[\frac{1}{2}\left(\frac{u^{3/2}}{\frac{3}{2}}\right)\right]_0^1 = \left[\frac{1}{3}u^{3/2}\right]_0^1 = \frac{1}{3}(\sqrt{1})^3 - \frac{1}{3}(\sqrt{0})^3 = \frac{1}{3}$;

$E_T = \int_0^1 x\sqrt{1-x^2} \, dx - T \approx \frac{1}{3} - 0.31929 = 0.01404$; $E_S = \int_0^1 x\sqrt{1-x^2} \, dx - S \approx \frac{1}{3} - 0.32812 = 0.00521$

12. (a) $n = 8 \Rightarrow h = \frac{3}{8} \Rightarrow \frac{h}{2} = \frac{3}{16}$;

$\sum mf(\theta_i) = 1(0) + 2(0.09334) + 2(0.18429) + 2(0.27075) + 2(0.35112) + 2(0.42443) + 2(0.49026)$
$+ 2(0.58466) + 1(0.6) = 5.3977 \Rightarrow T = \frac{3}{16}(5.3977) = 1.01207$

(b) $n = 8 \Rightarrow h = \frac{3}{8} \Rightarrow \frac{h}{3} = \frac{1}{8}$;

$\sum mf(\theta_i) = 1(0) + 4(0.09334) + 2(0.18429) + 4(0.27075) + 2(0.35112) + 4(0.42443) + 2(0.49026)$
$+ 4(0.58466) + 1(0.6) = 8.14406 \Rightarrow S = \frac{1}{8}(8.14406) = 1.01801$

(c) Let $u = 16 + \theta^2 \Rightarrow du = 2\theta \, d\theta \Rightarrow \frac{1}{2} du = \theta \, d\theta$; $\theta = 0 \Rightarrow u = 16$, $\theta = 3 \Rightarrow u = 16 + 3^2 = 25$

$\int_0^3 \frac{\theta}{\sqrt{16+\theta^2}} \, d\theta = \int_{16}^{25} \frac{1}{\sqrt{u}}\left(\frac{1}{2} du\right) = \frac{1}{2}\int_{16}^{25} u^{-1/2} \, du = \left[\frac{1}{2}\left(\frac{u^{1/2}}{\frac{1}{2}}\right)\right]_{16}^{25} = \sqrt{25} - \sqrt{16} = 1$;

$E_T = \int_0^3 \frac{\theta}{\sqrt{16+\theta^2}} \, d\theta - T \approx 1 - 1.01207 = -0.01207$; $E_S = \int_0^3 \frac{\theta}{\sqrt{16+\theta^2}} \, d\theta - S \approx 1 - 0.01801 = -0.01801$

13. (a) $n = 8 \Rightarrow h = \frac{\pi}{8} \Rightarrow \frac{h}{2} = \frac{\pi}{16}$;

$\sum mf(t_i) = 1(0.0) + 2(0.99138) + 2(1.26906) + 2(1.05961) + 2(0.75) + 2(0.48821) + 2(0.28946) + 2(0.13429)$
$+ 1(0) = 9.96402 \Rightarrow T = \frac{\pi}{16}(9.96402) \approx 1.95643$

(b) $n = 8 \Rightarrow h = \frac{\pi}{8} \Rightarrow \frac{h}{3} = \frac{\pi}{24}$;

$\sum mf(t_i) = 1(0.0) + 4(0.99138) + 2(1.26906) + 4(1.05961) + 2(0.75) + 4(0.48821) + 2(0.28946) + 4(0.13429)$
$+ 1(0) = 15.311 \Rightarrow S \approx \frac{\pi}{24}(15.311) \approx 2.00421$

(c) Let $u = 2 + \sin t \Rightarrow du = \cos t\, dt$; $t = -\frac{\pi}{2} \Rightarrow u = 2 + \sin\left(-\frac{\pi}{2}\right) = 1$, $t = \frac{\pi}{2} \Rightarrow u = 2 + \sin\frac{\pi}{2} = 3$

$$\int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2+\sin t)^2}\, dt = \int_1^3 \frac{3}{u^2}\, du = 3\int_1^3 u^{-2}\, du = \left[3\left(\frac{u^{-1}}{-1}\right)\right]_1^3 = 3\left(-\frac{1}{3}\right) - 3\left(-\frac{1}{1}\right) = 2;$$

$$E_T = \int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2+\sin t)^2}\, dt - T \approx 2 - 1.95643 = 0.04357;\ E_S = \int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2+\sin t)^2}\, dt - S$$

$\approx 2 - 2.00421 = -0.00421$

14. (a) $n = 8 \Rightarrow h = \frac{\pi}{32} \Rightarrow \frac{h}{2} = \frac{\pi}{64}$;

$\sum mf(y_i) = 1(2.0) + 2(1.51606) + 2(1.18237) + 2(0.93998) + 2(0.75402) + 2(0.60145) + 2(0.46364)$
$+ 2(0.31688) + 1(0) = 13.5488 \Rightarrow T \approx \frac{\pi}{64}(13.5488) = 0.66508$

(b) $n = 8 \Rightarrow h = \frac{\pi}{32} \Rightarrow \frac{h}{3} = \frac{\pi}{96}$;

$\sum mf(y_i) = 1(2.0) + 4(1.51606) + 2(1.18237) + 4(0.93988) + 2(0.75402) + 4(0.60145) + 2(0.46364)$
$+ 4(0.31688) + 1(0) = 20.29754 \Rightarrow S \approx \frac{\pi}{96}(20.29754) = 0.66424$

(c) Let $u = \cot y \Rightarrow du = -\csc^2 y\, dy$; $y = \frac{\pi}{4} \Rightarrow u = 1$, $y = \frac{\pi}{2} \Rightarrow u = 0$

$$\int_{\pi/4}^{\pi/2} \left(\csc^2 y\right)\sqrt{\cot y}\, dy = \int_1^0 \sqrt{u}\,(-du) = \int_0^1 u^{1/2}\, du = \left[\frac{u^{3/2}}{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left(\sqrt{1}\right)^3 - \frac{2}{3}\left(\sqrt{0}\right)^3 = \frac{2}{3};$$

$$E_T = \int_{\pi/4}^{\pi/2} \left(\csc^2 y\right)\sqrt{\cot y}\, dy - T \approx \frac{2}{3} - 0.66508 = 0.00159;\ E_S = \int_{\pi/4}^{\pi/2} \left(\csc^2 y\right)\sqrt{\cot y}\, dy - S$$

$\approx \frac{2}{3} - 0.66424 = 0.00243$

15. $\frac{5}{2}(6.0 + 2(8.2) + 2(9.1 + \ldots + 2(12.7) + 13.0)(30) = 15{,}990\ \text{ft}^3$

16. (a) Using the Trapezoid Rule, $h = 200 \Rightarrow \frac{h}{2} = \frac{200}{2} = 100$;

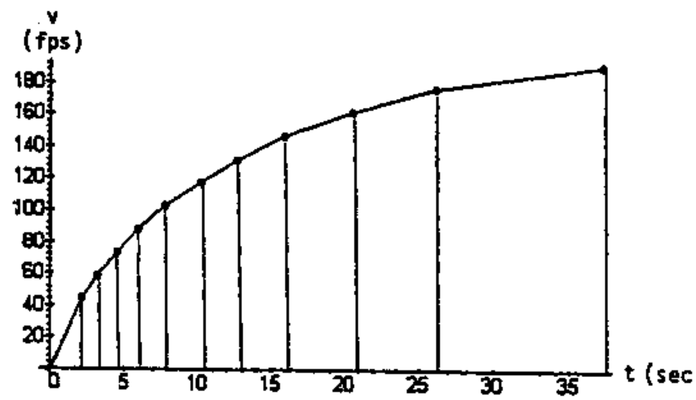
$\sum mf(x_i) = 13{,}180 \Rightarrow \text{Area} \approx 100(13{,}180)$

$= 1{,}318{,}000\ \text{ft}^2$. Since the average depth = 20 ft we obtain Volume $\approx 20\,(\text{Area}) \approx 26{,}360{,}000\ \text{ft}^3$.

(b) The number of fish $= \frac{\text{Volume}}{1000} = 26{,}360$ (to the nearest fish) $\Rightarrow$ Maximum to be caught = 75% of 26,360 = 19,770 $\Rightarrow$ Number of licenses $= \frac{19{,}770}{20} = 988$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	200	520	2	1040
x_2	400	800	2	1600
x_3	600	1000	2	2000
x_4	800	1140	2	2280
x_4	1000	1160	2	2320
x_6	1200	1110	2	2220
x_7	1400	860	2	1720
x_8	1600	0	1	0

17. Use the conversion 30 mph = 44 fps (ft per sec) since time is measured in seconds. The distance traveled as the car accelerates from, say, 40 mph = 58.67 fps to 50 mph = 73.33 fps in $(4.5 - 3.2) = 1.3$ sec is the area of the trapezoid (see figure) associated with that time interval: $\frac{1}{2}(58.67 + 73.33)(1.3) = 85.8$ ft. The total distance traveled by the Ford Mustang Cobra is the sum of all these eleven trapezoids (using $\frac{\Delta t}{2}$ and the table below):



$$s = (44)(1.1) + (102.67)(0.5) + (132)(0.65) + (161.33)(0.7) + (190.67)(0.95) + (220)(1.2) + (249.33)(1.25)$$
$$+ (278.67)(1.65) + (308)(2.3) + (337.33)(2.8) + (366.67)(5.45) = 5166.35\ \text{ft} \approx 0.980\ \text{mi}$$

v (mph)	0	30	40	50	60	70	80	90	100	110	120	130
v (fps)	0	44	58.67	73.33	88	102.67	117.33	132	146.67	161.33	176	190.67
t (sec)	0	2.2	3.2	4.5	5.9	7.8	10.2	12.7	16	20.6	26.2	37.1
$\Delta t/2$	0	1.1	0.5	0.65	0.7	0.95	1.2	1.25	1.65	2.3	2.8	5.45

18. Using Simpson's Rule, $h = \frac{b-a}{n} = \frac{24-0}{6} = \frac{24}{6} = 4$;

$\sum my_i = 350 \Rightarrow S = \frac{4}{3}(350) = \frac{1400}{3} \approx 466.7\ \text{in.}^2$

	x_i	y_i	m	my_i
x_0	0	0	1	0
x_1	4	18.75	4	75
x_2	8	24	2	48
x_3	12	26	4	104
x_4	16	24	2	48
x_5	20	18.75	4	75
x_6	24	0	1	0

19. Using Simpson's Rule, $h = 1 \Rightarrow \frac{h}{3} = \frac{1}{3}$;

$\sum my_i = 33.6 \Rightarrow$ Cross Section Area $\approx \frac{1}{3}(33.6) = 11.2\ \text{ft}^2$.
Let x be the length of the tank. Then the Volume V
$= (\text{Cross Sectional Area})\,x = 11.2x$. Now 5000 lb of
gasoline at $42\ \text{lb/ft}^3 \Rightarrow V = \frac{5000}{42} = 119.05\ \text{ft}^3$
$\Rightarrow 119.05 = 11.2x \Rightarrow x \approx 10.63$ ft

	x_i	y_i	m	my_i
x_0	0	1.5	1	1.5
x_1	1	1.6	4	6.4
x_2	2	1.8	2	3.6
x_3	3	1.9	4	7.6
x_4	4	2.0	2	4.0
x_5	5	2.1	4	8.4
x_6	6	2.1	1	2.1

20. $\frac{24}{2}[0.019 + 2(0.020) + 2(0.021) + \ldots + 2(0.031) + 0.035] = 4.2$ L

21. $n = 2 \Rightarrow h = \frac{2-0}{2} = 1 \Rightarrow \frac{h}{3} = \frac{1}{3}$;

$\sum mf(x_i) = 12 \Rightarrow S = \frac{1}{3}(12) = 4$;

$$\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = \frac{2^4}{4} - \frac{0^4}{4} = 4$$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	1	1	4	4
x_2	2	8	1	8

22. (a) $|E_S| \le \frac{b-a}{180}(h^4)M$; $n = 4 \Rightarrow h = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$; $|f^{(4)}| \le 1 \Rightarrow M = 1 \Rightarrow |E_S| \le \frac{\left(\frac{\pi}{2}-0\right)}{180}\left(\frac{\pi}{8}\right)^4(1) \approx 0.00021$

(b) $h = \frac{\pi}{8} \Rightarrow \frac{h}{3} = \frac{\pi}{24}$;

$\sum mf(x_i) = 10.472087048$

$\Rightarrow S = \frac{\pi}{24}(10.472087048) \approx 1.37079$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	1	1	1
x_1	$\pi/8$	0.974495358	4	3.897981432
x_2	$\pi/4$	0.900316316	2	1.800632632
x_3	$3\pi/8$	0.784213303	4	3.136853212
x_4	$\pi/2$	0.636619772	1	0.636619772

(c) $\approx \left(\frac{0.00021}{1.37079}\right) \times 100 \approx 0.015\%$

23. (a) $h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1 \Rightarrow \text{erf}(1) = \frac{2}{\sqrt{\pi}}\left(\frac{0.1}{3}\right)(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_9 + y_{10})$

$$= \frac{2}{30\sqrt{\pi}}\left(e^0 + 4e^{-0.01} + 2e^{-0.04} + 4e^{-0.09} + \ldots + 4e^{-0.81} + e^{-1}\right) \approx 0.843$$

(b) $|E_s| \le \frac{1-0}{180}(0.1)^4(12) \approx 6.7 \times 10^{-6}$

24. The average of the 13 discrete temperatures gives equal weight to the low values at the end.

Exercises 25-28 were done using a graphing calculator.

25. 3.1415927 26. 1.0894294 27. 1.3707622 28. 0.82811633

29. (a) $T_{10} \approx 1.983523538$

$T_{100} \approx 1.999835504$

$T_{1000} \approx 1.999998355$

(b)

n	$\lvert E_T \rvert = 2 - T_n$
10	$0.016476462 = 1.6476462 \times 10^{-2}$
100	1.64496×10^{-4}
1000	1.645×10^{-6}

(c) $\left|E_{T_{10n}}\right| \approx 10^{-2}\left|E_{T_n}\right|$

(d) $b - a = \pi$, $h^2 = \frac{\pi^2}{n^2}$, $M = 1$

$$\left|E_{T_n}\right| \le \frac{\pi}{12}\left(\frac{\pi^2}{n^2}\right) = \frac{\pi^3}{12n^2}$$

$$\left|E_{T_{10n}}\right| \le \frac{\pi^3}{12(10n)^2} = 10^{-2}\left|E_{T_n}\right|$$

30. (a) $S_{10} \approx 2.000109517$

$S_{100} \approx 2.000000011$

$S_{1000} \approx 2.000000000$

(b)

n	$\lvert E_S \rvert = 2 - S_n$
10	1.09517×10^{-4}
100	1.1×10^{-8}
1000	0

(c) $\left|E_{S_{10n}}\right| \approx 10^{-4}\left|E_{S_n}\right|$

(d) $b - a = \pi$, $h^4 = \frac{\pi^4}{n^4}$, $M = 1$

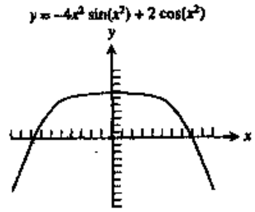
$$\left|E_{S_n}\right| \le \frac{\pi}{180}\left(\frac{\pi^4}{n^4}\right) = \frac{\pi^5}{180n^4}$$

$$\left|E_{S_{10n}}\right| \le \frac{\pi^5}{180(10n)^4} = 10^{-4}\left|E_{S_n}\right|$$

31. (a) $f'(x) = 2x\cos(x^2)$

$f''(x) = 2x \cdot -2x\sin(x^2) + 2\cos(x^2) = -4x^2\sin(x^2) + 2\cos(x^2)$

(b)



(c) The graph shows that $-3 \le f''(x) \le 2$ so $|f''(x)| \le 3$ for $-1 \le x \le 1$.

(d) $|E_T| \le \frac{1-(-1)}{12}(h^2)(3) = \frac{h^2}{2}$

(e) For $0 < h \le 0.1$, $|E_T| \le \frac{h^2}{2} \le \frac{0.1^2}{2} = 0.005 < 0.01$

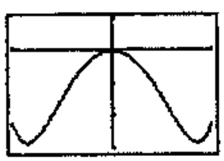
(f) $n \ge \frac{1-(-1)}{h} \ge \frac{2}{0.1} = 20$

32. (a) $f'''(x) = -4x^2 \cdot 2x\cos(x^2) - 8x\sin(x^2) - 4x\sin(x^2) = -8x^3\cos(x^2) - 12x\sin(x^2)$

$f^{(4)}(x) = -8x^3 \cdot -2x\sin(x^2) - 24x^2\cos(x^2) - 12x \cdot 2x\cos(x^2) - 12\sin(x^2)$

$= (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$

(b)



[−1, 1] by [−30, 10]

(c) The graph shows that $-30 \le f^{(4)}(x) \le 0$ so $\left|f^{(4)}(x)\right| \le 30$ for $-1 \le x \le 1$.

(d) $|E_T| \le \frac{1-(-1)}{180}(h^4)(30) = \frac{h^4}{3}$

(e) For $0 < h \le 0.4$, $|E_S| \le \frac{h^4}{3} \le \frac{0.4^4}{3} \approx 0.00853 < 0.01$

(f) $n \ge \frac{1-(-1)}{h} \ge \frac{2}{0.4} = 5$

CHAPTER 4 PRACTICE EXERCISES

1. $\int (x^3 + 5x - 7)\,dx = \frac{x^4}{4} + \frac{5x^2}{2} - 7x + C$

2. $\int \left(8t^3 - \frac{t^2}{2} + t\right) dt = \frac{8t^4}{4} - \frac{t^3}{6} + \frac{t^2}{2} + C = 2t^4 - \frac{t^3}{6} + \frac{t^2}{2} + C$

3. $\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt = \int \left(3t^{1/2} + 4t^{-2}\right) dt = \frac{3t^{3/2}}{\left(\frac{3}{2}\right)} + \frac{4t^{-1}}{-1} + C = 2t^{3/2} - \frac{4}{t} + C$

4. $\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4}\right) dt = \int \left(\frac{1}{2}t^{-1/2} - 3t^{-4}\right) dt = \frac{1}{2}\left(\frac{t^{1/2}}{\frac{1}{2}}\right) - \frac{3t^{-3}}{(-3)} + C = \sqrt{t} + \frac{1}{t^3} + C$

5. Let $u = r^2 + 5 \Rightarrow du = 2r\ dr \Rightarrow \frac{1}{2}\ du = r\ dr$

$$\int \frac{r\ dr}{\left(r^2+5\right)^2} = \int \frac{\left(\frac{1}{2}\right) du}{u^2} = \frac{1}{2} \int u^{-2}\ du = \frac{1}{2}\left(\frac{u^{-1}}{-1}\right) + C = -\frac{1}{2}u^{-1} + C = -\frac{1}{2\left(r^2+5\right)} + C$$

6. Let $u = r^3 - \sqrt{2} \Rightarrow du = 3r^2\ dr \Rightarrow 2\ du = 6r^2\ dr$

$$\int \frac{6r^2\ dr}{\left(r^3 - \sqrt{2}\right)^3} = \int \frac{2\ du}{u^3} = 2\int u^{-3}\ du = 2\left(\frac{u^{-2}}{-2}\right) + C = -u^{-2} + C = -\frac{1}{\left(r^3 - \sqrt{2}\right)^2} + C$$

7. Let $u = 2 - \theta^2 \Rightarrow du = -2\theta\ d\theta \Rightarrow -\frac{1}{2}\ du = \theta\ d\theta$

$$\int 3\theta\sqrt{2-\theta^2}\ d\theta = \int \sqrt{u}\left(-\frac{3}{2}\ du\right) = -\frac{3}{2}\int u^{1/2}\ du = -\frac{3}{2}\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + C = -u^{3/2} + C = -\left(2-\theta^2\right)^{3/2} + C$$

8. Let $u = 73 + \theta^3 \Rightarrow du = 3\theta^2\ d\theta \Rightarrow \frac{1}{27}\ du = \frac{\theta^2}{9}\ d\theta$

$$\int \frac{\theta^2}{9\sqrt{73+\theta^3}}\ d\theta = \int \frac{1}{\sqrt{u}}\left(\frac{1}{27}\ du\right) = \frac{1}{27}\int u^{-1/2}\ du = \frac{1}{27}\left(\frac{u^{1/2}}{\frac{1}{2}}\right) + C = \frac{2}{27}u^{1/2} + C = \frac{2}{27}\sqrt{73+\theta^3} + C$$

9. Let $u = 1 + x^4 \Rightarrow du = 4x^3\ dx \Rightarrow \frac{1}{4}\ du = x^3\ dx$

$$\int x^3\left(1+x^4\right)^{-1/4} dx = \int u^{-1/4}\left(\frac{1}{4}\ du\right) = \frac{1}{4}\int u^{-1/4}\ du = \frac{1}{4}\left(\frac{u^{3/4}}{\frac{3}{4}}\right) + C = \frac{1}{3}u^{3/4} + C = \frac{1}{3}\left(1+x^4\right)^{3/4} + C$$

10. Let $u = 2 - x \Rightarrow du = -dx \Rightarrow -du = dx$

$$\int (2-x)^{3/5}\ dx = \int u^{3/5}(-du) = -\int u^{3/5}\ du = -\frac{u^{8/5}}{\left(\frac{8}{5}\right)} + C = -\frac{5}{8}u^{8/5} + C = -\frac{5}{8}(2-x)^{8/5} + C$$

11. Let $u = \frac{s}{10} \Rightarrow du = \frac{1}{10}\ ds \Rightarrow 10\ du = ds$

$$\int \sec^2 \frac{s}{10}\ ds = \int \left(\sec^2 u\right)(10\ du) = 10\int \sec^2 u\ du = 10\tan u + C = 10\tan\frac{s}{10} + C$$

12. Let $u = \pi s \Rightarrow du = \pi\ ds \Rightarrow \frac{1}{\pi}\, du = ds$

$$\int \csc^2 \pi s\ ds = \int \left(\csc^2 u\right)\left(\frac{1}{\pi}\, du\right) = \frac{1}{\pi}\int \csc^2 u\ du = -\frac{1}{\pi}\cot u + C = -\frac{1}{\pi}\cot \pi s + C$$

13. Let $u = \sqrt{2}\,\theta \Rightarrow du = \sqrt{2}\ d\theta \Rightarrow \frac{1}{\sqrt{2}}\, du = d\theta$

$$\int \csc \sqrt{2}\theta\ \cot \sqrt{2}\theta\ d\theta = \int (\csc u\ \cot u)\left(\frac{1}{\sqrt{2}}\, du\right) = \frac{1}{\sqrt{2}}(-\csc u) + C = -\frac{1}{\sqrt{2}}\csc \sqrt{2}\theta + C$$

14. Let $u = \frac{\theta}{3} \Rightarrow du = \frac{1}{3}\, d\theta \Rightarrow 3\ du = d\theta$

$$\int \sec \frac{\theta}{3}\tan \frac{\theta}{3}\, d\theta = \int (\sec u\ \tan u)(3\ du) = 3\sec u + C = 3\sec \frac{\theta}{3} + C$$

15. Let $u = \frac{x}{4} \Rightarrow du = \frac{1}{4}\, dx \Rightarrow 4\ du = dx$

$$\int \sin^2 \frac{x}{4}\, dx = \int \left(\sin^2 u\right)(4\ du) = \int 4\left(\frac{1-\cos 2u}{2}\right) du = 2\int (1-\cos 2u)\ du = 2\left(u - \frac{\sin 2u}{2}\right) + C$$

$$= 2u - \sin 2u + C = 2\left(\frac{x}{4}\right) - \sin 2\left(\frac{x}{4}\right) + C = \frac{x}{2} - \sin \frac{x}{2} + C$$

16. Let $u = \frac{x}{2} \Rightarrow du = \frac{1}{2}\, dx \Rightarrow 2\ du = dx$

$$\int \cos^2 \frac{x}{2}\, dx = \int \left(\cos^2 u\right)(2\ du) = \int 2\left(\frac{1+\cos 2u}{2}\right) du = \int (1+\cos 2u)\ du = u + \frac{\sin 2u}{2} + C$$

$$= \frac{x}{2} + \frac{1}{2}\sin x + C$$

17. Let $u = \cos x \Rightarrow du = -\sin x\ dx \Rightarrow -du = \sin x\ dx$

$$\int 2(\cos x)^{-1/2}\sin x\ dx = \int 2u^{-1/2}(-du) = -2\int u^{-1/2}\ du = -2\left(\frac{u^{1/2}}{\frac{1}{2}}\right) + C = -4u^{1/2} + C$$

$$= -4(\cos x)^{1/2} + C$$

18. Let $u = \tan x \Rightarrow du = \sec^2 x\ dx$

$$\int (\tan x)^{-3/2}\sec^2 x\ dx = \int u^{-3/2}\ du = \frac{u^{-1/2}}{\left(-\frac{1}{2}\right)} + C = -2u^{-1/2} + C = \frac{-2}{(\tan x)^{1/2}} + C$$

19. $$\int \left(t - \frac{2}{t}\right)\left(t + \frac{2}{t}\right) dt = \int \left(t^2 - \frac{4}{t^2}\right) dt = \int \left(t^2 - 4t^{-2}\right) dt = \frac{t^3}{3} - 4\left(\frac{t^{-1}}{-1}\right) + C = \frac{t^3}{3} + \frac{4}{t} + C$$

20. $$\int \frac{(t+1)^2 - 1}{t^4}\, dt = \int \frac{t^2 + 2t}{t^4}\, dt = \int \left(\frac{1}{t^2} + \frac{2}{t^3}\right) dt = \int \left(t^{-2} + 2t^{-3}\right) dt = \frac{t^{-1}}{(-1)} + 2\left(\frac{t^{-2}}{-2}\right) + C = -\frac{1}{t} - \frac{1}{t^2} + C$$

21. (a) Each time subinterval is of length $\Delta t = 0.4$ sec. The distance traveled over each subinterval, using the midpoint rule, is $\Delta h = \frac{1}{2}(v_i + v_{i+1})\Delta t$, where v_i is the velocity at the left, and v_{i+1} the velocity at the right, endpoint of the subinterval. We then add Δh to the height attained so far at the left endpoint v_i to arrive at the height associated with velocity v_{i+1} at the right endpoint. Using this methodology we build the following table based on the figure in the text:

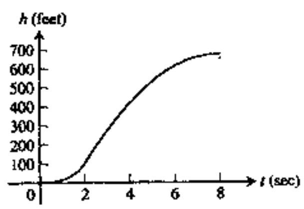
t (sec)	0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	4.4	4.8	5.2	5.6	6.0
v (fps)	0	10	25	55	100	190	180	170	155	140	130	120	105	90	80	65
h (ft)	0	2	9	25	56	114	188	258	323	382	436	486	531	570	604	633

t (sec)	6.4	6.8	7.2	7.6	8.0
v (fps)	52	40	30	15	0
h (ft)	656	674	688	697	700

NOTE: Your table values may vary slightly from ours depending on the v-values you read from the graph. Remember that some shifting of the graph occurs in the printing process.

The total height attained is about 700 ft.

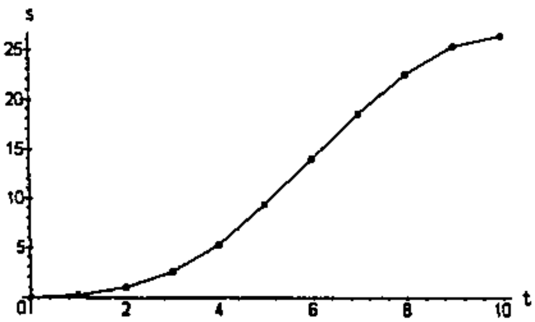
(b) The graph is based on the table in part (a).



22. (a) Each time subinterval is of length $\Delta t = 1$ sec. The distance traveled over each subinterval, using the midpoint rule, is $\Delta s = \frac{1}{2}(v_i + v_{i+1})\Delta t$, where v_i is the velocity at the left, and v_{i+1} the velocity at the right, endpoint of the subinterval. We then add Δs to the distance attained so far at the left endpoint v_i to arrive at the distance associated with velocity v_{i+1} at the right endpoint. Using this methodology we build the table given below based on the figure in the text, obtaining approximately 26 m for the total distance traveled:

t (sec)	0	1	2	3	4	5	6	7	8	9	10
v (m/sec)	0	0.5	1.0	2	3.5	4.5	4.8	4.5	3.5	2	0
s (m)	0	0.25	1.00	2.5	5.25	9.25	13.9	18.55	22.55	25.3	26.3

(b) The graph shows the distance traveled by the moving body as a function of time for $0 \le t \le 10$.



23. Let $u = 2x - 1 \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}\,du = dx$; $x = 1 \Rightarrow u = 1$, $x = 5 \Rightarrow u = 9$

$$\int_1^5 (2x-1)^{-1/2}\,dx = \int_1^9 u^{-1/2}\left(\tfrac{1}{2}\,du\right) = \left[u^{1/2}\right]_1^9 = 3 - 1 = 2$$

24. Let $u = x^2 - 1 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$; $x = 1 \Rightarrow u = 0$, $x = 3 \Rightarrow u = 8$

$$\int_1^3 x\left(x^2-1\right)^{1/3}\,dx = \int_0^8 u^{1/3}\left(\tfrac{1}{2}\,du\right) = \left[\tfrac{3}{8}u^{4/3}\right]_0^8 = \tfrac{3}{8}(16-0) = 6$$

25. Let $u = \frac{x}{2} \Rightarrow 2\,du = dx$; $x = -\pi \Rightarrow u = -\frac{\pi}{2}$, $x = 0 \Rightarrow u = 0$

$$\int_{-\pi}^0 \cos\left(\tfrac{x}{2}\right)dx = \int_{-\pi/2}^0 (\cos u)(2\,du) = [2\sin u]_{-\pi/2}^0 = 2\sin 0 - 2\sin\left(-\tfrac{\pi}{2}\right) = 2(0-(-1)) = 2$$

26. Let $u = \sin x \Rightarrow du = \cos x\,dx$; $x = 0 \Rightarrow u = 0$, $x = \frac{\pi}{2} \Rightarrow u = 1$

$$\int_0^{\pi/2} (\sin x)(\cos x)\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$$

27. (a) $\int_{-2}^2 f(x)\,dx = \frac{1}{3}\int_{-2}^2 3\,f(x)\,dx = \frac{1}{3}(12) = 4$ (b) $\int_2^5 f(x)\,dx = \int_{-2}^5 f(x)\,dx - \int_{-2}^2 f(x)\,dx = 6 - 4 = 2$

(c) $\int_5^{-2} g(x)\,dx = -\int_{-2}^5 g(x)\,dx = -2$ (d) $\int_{-2}^5 (-\pi\,g(x))\,dx = -\pi\int_{-2}^5 g(x)\,dx = -\pi(2) = -2\pi$

(e) $\int_{-2}^5 \left(\frac{f(x)+g(x)}{5}\right)dx = \frac{1}{5}\int_{-2}^5 f(x)\,dx + \frac{1}{5}\int_{-2}^5 g(x)\,dx = \frac{1}{5}(6) + \frac{1}{5}(2) = \frac{8}{5}$

28. (a) $\int_0^2 g(x)\,dx = \frac{1}{7}\int_0^2 7\,g(x)\,dx = \frac{1}{7}(7) = 1$ (b) $\int_1^2 g(x)\,dx = \int_0^2 g(x)\,dx - \int_0^1 g(x)\,dx = 1 - 2 = -1$

(c) $\int_2^0 f(x)\,dx = -\int_0^2 f(x)\,dx = -\pi$ (d) $\int_0^2 \sqrt{2}\,f(x)\,dx = \sqrt{2}\int_0^2 f(x)\,dx = \sqrt{2}\,(\pi) = \pi\sqrt{2}$

(e) $\int_0^2 [g(x) - 3\,f(x)]\,dx = \int_0^2 g(x)\,dx - 3\int_0^2 f(x)\,dx = 1 - 3\pi$

29. $\int_{-1}^{1} (3x^2 - 4x + 7)\,dx = \left[x^3 - 2x^2 + 7x\right]_{-1}^{1} = \left[1^3 - 2(1)^2 + 7(1)\right] - \left[(-1)^3 - 2(-1)^2 + 7(-1)\right] = 6 - (-10) = 16$

30. $\int_0^1 (8s^3 - 12s^2 + 5)\,ds = \left[2s^4 - 4s^3 + 5s\right]_0^1 = \left[2(1)^4 - 4(1)^3 + 5(1)\right] - 0 = 3$

31. $\int_1^2 \frac{4}{v^2}\,dv = \int_1^2 4v^{-2}\,dv = \left[-4v^{-1}\right]_1^2 = \left(\frac{-4}{2}\right) - \left(\frac{-4}{1}\right) = 2$

32. $\int_1^{27} x^{-4/3}\,dx = \left[-3x^{-1/3}\right]_1^{27} = -3(27)^{-1/3} - \left(-3(1)^{-1/3}\right) = -3\left(\frac{1}{3}\right) + 3(1) = 2$

33. $\int_1^4 \frac{dt}{t\sqrt{t}} = \int_1^4 \frac{dt}{t^{3/2}} = \int_1^4 t^{-3/2}\,dt = \left[-2t^{-1/2}\right]_1^4 = \frac{-2}{\sqrt{4}} - \frac{(-2)}{\sqrt{1}} = 1$

34. Let $x = 1 + \sqrt{u} \Rightarrow dx = \frac{1}{2}u^{-1/2}\,du \Rightarrow 2\,dx = \frac{du}{\sqrt{u}}$; $u = 1 \Rightarrow x = 2$, $u = 4 \Rightarrow x = 3$

$$\int_1^4 \frac{(1+\sqrt{u})^{1/2}}{\sqrt{u}}\,du = \int_2^3 x^{1/2}(2\,dx) = \left[2\left(\frac{2}{3}\right)x^{3/2}\right]_2^3 = \frac{4}{3}(3^{3/2}) - \frac{4}{3}(2^{3/2}) = 4\sqrt{3} - \frac{8}{3}\sqrt{2} = \frac{4}{3}(3\sqrt{3} - 2\sqrt{2})$$

35. Let $u = 2x + 1 \Rightarrow du = 2\,dx \Rightarrow 18\,du = 36\,dx$; $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 3$

$$\int_0^1 \frac{36\,dx}{(2x+1)^3} = \int_1^3 18u^{-3}\,du = \left[\frac{18u^{-2}}{-2}\right]_1^3 = \left[\frac{-9}{u^2}\right]_1^3 = \left(\frac{-9}{3^2}\right) - \left(\frac{-9}{1^2}\right) = 8$$

36. Let $u = 7 - 5r \Rightarrow du = -5\,dr \Rightarrow -\frac{1}{5}\,du = dr$; $r = 0 \Rightarrow u = 7$, $r = 1 \Rightarrow u = 2$

$$\int_0^1 \frac{dr}{\sqrt[3]{(7-5r)^2}} = \int_0^1 (7-5r)^{-2/3}\,dr = \int_7^2 u^{-2/3}\left(-\frac{1}{5}\,du\right) = -\frac{1}{5}\left[3u^{1/3}\right]_7^2 = \frac{3}{5}\left(\sqrt[3]{7} - \sqrt[3]{2}\right)$$

37. Let $u = 1 - x^{2/3} \Rightarrow du = -\frac{2}{3}x^{-1/3}\,dx \Rightarrow -\frac{3}{2}\,du = x^{-1/3}\,dx$; $x = \frac{1}{8} \Rightarrow u = 1 - \left(\frac{1}{8}\right)^{2/3} = \frac{3}{4}$,
$x = 1 \Rightarrow u = 1 - 1^{2/3} = 0$

$$\int_{1/8}^{1} x^{-1/3}\left(1-x^{2/3}\right)^{3/2} dx = \int_{3/4}^{0} u^{3/2}\left(-\frac{3}{2}\,du\right) = \left[\left(-\frac{3}{2}\right)\left(\frac{u^{5/2}}{\frac{5}{2}}\right)\right]_{3/4}^{0} = \left[-\frac{3}{5}u^{5/2}\right]_{3/4}^{0} = -\frac{3}{5}(0)^{5/2} - \left(-\frac{3}{5}\right)\left(\frac{3}{4}\right)^{5/2}$$
$$= \frac{27\sqrt{3}}{160}$$

38. Let $u = 1 + 9x^4 \Rightarrow du = 36x^3\,dx \Rightarrow \frac{1}{36}\,du = x^3\,dx$; $x = 0 \Rightarrow u = 1$, $x = \frac{1}{2} \Rightarrow u = 1 + 9\left(\frac{1}{2}\right)^4 = \frac{25}{16}$

$$\int_{0}^{1/2} x^3\left(1+9x^4\right)^{-3/2} dx = \int_{1}^{25/16} u^{-3/2}\left(\frac{1}{36}\,du\right) = \left[\frac{1}{36}\left(\frac{u^{-1/2}}{-\frac{1}{2}}\right)\right]_{1}^{25/16} = \left[-\frac{1}{18}u^{-1/2}\right]_{1}^{25/16}$$
$$= -\frac{1}{18}\left(\frac{25}{16}\right)^{-1/2} - \left(-\frac{1}{18}(1)^{-1/2}\right) = \frac{1}{90}$$

39. Let $u = 5r \Rightarrow du = 5\,dr \Rightarrow \frac{1}{5}\,du = dr$; $r = 0 \Rightarrow u = 0$, $r = \pi \Rightarrow u = 5\pi$

$$\int_{0}^{\pi} \sin^2 5r\,dr = \int_{0}^{5\pi} \left(\sin^2 u\right)\left(\frac{1}{5}\,du\right) = \frac{1}{5}\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_{0}^{5\pi} = \left(\frac{\pi}{2} - \frac{\sin 10\pi}{20}\right) - \left(0 - \frac{\sin 0}{20}\right) = \frac{\pi}{2}$$

40. Let $u = 4t - \frac{\pi}{4} \Rightarrow du = 4\,dt \Rightarrow \frac{1}{4}\,du = dt$; $t = 0 \Rightarrow u = -\frac{\pi}{4}$, $t = \frac{\pi}{4} \Rightarrow u = \frac{3\pi}{4}$

$$\int_{0}^{\pi/4} \cos^2\left(4t - \frac{\pi}{4}\right) dt = \int_{-\pi/4}^{3\pi/4} \left(\cos^2 u\right)\left(\frac{1}{4}\,du\right) = \frac{1}{4}\left[\frac{u}{2} + \frac{\sin 2u}{4}\right]_{-\pi/4}^{3\pi/4} = \frac{1}{4}\left(\frac{3\pi}{8} + \frac{\sin\left(\frac{3\pi}{2}\right)}{4}\right) - \frac{1}{4}\left(-\frac{\pi}{8} + \frac{\sin\left(-\frac{\pi}{2}\right)}{4}\right)$$
$$= \frac{\pi}{8} - \frac{1}{16} + \frac{1}{16} = \frac{\pi}{8}$$

41. $$\int_{0}^{\pi/3} \sec^2\theta\,d\theta = [\tan\theta]_0^{\pi/3} = \tan\frac{\pi}{3} - \tan 0 = \sqrt{3}$$

42. $$\int_{\pi/4}^{3\pi/4} \csc^2 x\,dx = [-\cot x]_{\pi/4}^{3\pi/4} = \left(-\cot\frac{3\pi}{4}\right) - \left(-\cot\frac{\pi}{4}\right) = 2$$

43. Let $u = \frac{x}{6} \Rightarrow du = \frac{1}{6}\,dx \Rightarrow 6\,du = dx$; $x = \pi \Rightarrow u = \frac{\pi}{6}$, $x = 3\pi \Rightarrow u = \frac{\pi}{2}$

$$\int_{\pi}^{3\pi} \cot^2\frac{x}{6}\,dx = \int_{\pi/6}^{\pi/2} 6\cot^2 u\,du = 6\int_{\pi/6}^{\pi/2} \left(\csc^2 u - 1\right) du = [6(-\cot u - u)]_{\pi/6}^{\pi/2} = 6\left(-\cot\frac{\pi}{2} - \frac{\pi}{2}\right) - 6\left(-\cot\frac{\pi}{6} - \frac{\pi}{6}\right)$$
$$= 6\sqrt{3} - 2\pi$$

44. Let $u = \frac{\theta}{3} \Rightarrow du = \frac{1}{3}\,d\theta \Rightarrow 3\,du = d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \pi \Rightarrow u = \frac{\pi}{3}$

$$\int_{0}^{\pi} \tan^2\frac{\theta}{3}\,d\theta = \int_{0}^{\pi} \left(\sec^2\frac{\theta}{3} - 1\right) d\theta = \int_{0}^{\pi/3} 3\left(\sec^2 u - 1\right) du = [3\tan u - 3u]_0^{\pi/3}$$

$$= \left[3 \tan \frac{\pi}{3} - 3\left(\frac{\pi}{3}\right)\right] - (3 \tan 0 - 0) = 3\sqrt{3} - \pi$$

45. $\displaystyle\int_{-\pi/3}^{0} \sec x \tan x \, dx = [\sec x]_{-\pi/3}^{0} = \sec 0 - \sec\left(-\frac{\pi}{3}\right) = 1 - 2 = -1$

46. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc z \cot z \, dz = [-\csc z]_{\pi/4}^{3\pi/4} = \left(-\csc \frac{3\pi}{4}\right) - \left(-\csc \frac{\pi}{4}\right) = -\sqrt{2} + \sqrt{2} = 0$

47. Let $u = \sin x \Rightarrow du = \cos x \, dx$; $x = 0 \Rightarrow u = 0$, $x = \frac{\pi}{2} \Rightarrow u = 1$

$$\int_{0}^{\pi/2} 5(\sin x)^{3/2} \cos x \, dx = \int_{0}^{1} 5u^{3/2} \, du = \left[5\left(\frac{2}{5}\right)u^{5/2}\right]_{0}^{1} = \left[2u^{5/2}\right]_{0}^{1} = 2(1)^{5/2} - 2(0)^{5/2} = 2$$

48. Let $u = 1 - x^2 \Rightarrow du = -2x \, dx \Rightarrow -du = 2x \, dx$; $x = -1 \Rightarrow u = 0$, $x = 1 \Rightarrow u = 0$

$$\int_{-1}^{1} 2x \sin\left(1 - x^2\right) dx = \int_{0}^{0} -\sin u \, du = 0$$

49. Let $u = 1 + 3 \sin^2 x \Rightarrow du = 6 \sin x \cos x \, dx \Rightarrow \frac{1}{2} du = 3 \sin x \cos x \, dx$; $x = 0 \Rightarrow u = 1$, $x = \frac{\pi}{2}$
$\Rightarrow u = 1 + 3 \sin^2 \frac{\pi}{2} = 4$

$$\int_{0}^{\pi/2} \frac{3 \sin x \cos x}{\sqrt{1 + 3 \sin^2 x}} \, dx = \int_{1}^{4} \frac{1}{\sqrt{u}}\left(\frac{1}{2} du\right) = \int_{1}^{4} \frac{1}{2}u^{-1/2} \, du = \left[\frac{1}{2}\left(\frac{u^{1/2}}{\frac{1}{2}}\right)\right]_{1}^{4} = \left[u^{1/2}\right]_{1}^{4} = 4^{1/2} - 1^{1/2} = 1$$

50. Let $u = 1 + 7 \tan x \Rightarrow du = 7 \sec^2 x \, dx \Rightarrow \frac{1}{7} du = \sec^2 x \, dx$; $x = 0 \Rightarrow u = 1 + 7 \tan 0 = 1$,
$x = \frac{\pi}{4} \Rightarrow u = 1 + 7 \tan \frac{\pi}{4} = 8$

$$\int_{0}^{\pi/4} \frac{\sec^2 x}{(1 + 7 \tan x)^{2/3}} \, dx = \int_{1}^{8} \frac{1}{u^{2/3}}\left(\frac{1}{7} du\right) = \int_{1}^{8} \frac{1}{7}u^{-2/3} \, du = \left[\frac{1}{7}\left(\frac{u^{1/3}}{\frac{1}{3}}\right)\right]_{1}^{8} = \left[\frac{3}{7}u^{1/3}\right]_{1}^{8} = \frac{3}{7}(8)^{1/3} - \frac{3}{7}(1)^{1/3} = \frac{3}{7}$$

51. Let $u = \sec \theta \Rightarrow du = \sec \theta \tan \theta \, d\theta$; $\theta = 0 \Rightarrow u = \sec 0 = 1$, $\theta = \frac{\pi}{3} \Rightarrow u = \sec \frac{\pi}{3} = 2$

$$\int_{0}^{\pi/3} \frac{\tan \theta}{\sqrt{2 \sec \theta}} \, d\theta = \int_{0}^{\pi/3} \frac{\sec \theta \tan \theta}{\sec \theta \sqrt{2 \sec \theta}} \, d\theta = \int_{0}^{\pi/3} \frac{\sec \theta \tan \theta}{\sqrt{2}\,(\sec \theta)^{3/2}} \, d\theta = \int_{1}^{2} \frac{1}{\sqrt{2}\, u^{3/2}} \, du = \frac{1}{\sqrt{2}} \int_{1}^{2} u^{-3/2} \, du$$

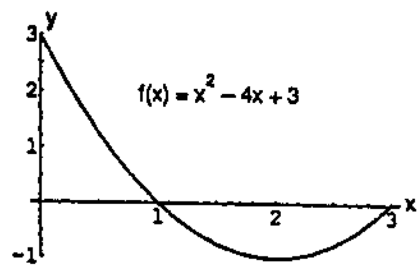
$$= \frac{1}{\sqrt{2}}\left[\frac{u^{-1/2}}{\left(-\frac{1}{2}\right)}\right]_{1}^{2} = \left[-\frac{2}{\sqrt{2u}}\right]_{1}^{2} = -\frac{2}{\sqrt{2(2)}} - \left(-\frac{2}{\sqrt{2(1)}}\right) = \sqrt{2} - 1$$

52. Let $u = \sin\sqrt{t} \Rightarrow du = (\cos\sqrt{t})\left(\frac{1}{2}t^{-1/2}\right)dt = \frac{\cos\sqrt{t}}{2\sqrt{t}}\,dt \Rightarrow 2\,du = \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$; $t = \frac{\pi^2}{36} \Rightarrow u = \sin\frac{\pi}{6} = \frac{1}{2}$,

$t = \frac{\pi^2}{4} \Rightarrow u = \sin\frac{\pi}{2} = 1$

$$\int_{\pi^2/36}^{\pi^2/4} \frac{\cos\sqrt{t}}{\sqrt{t\sin\sqrt{t}}}\,dt = \int_{1/2}^{1} \frac{1}{\sqrt{u}}(2\,du) = 2\int_{1/2}^{1} u^{-1/2}\,du = \left[4\sqrt{u}\right]_{1/2}^{1} = 4\sqrt{1} - 4\sqrt{\tfrac{1}{2}} = 2\left(2-\sqrt{2}\right)$$

53. $x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 3$ or $x = 1$;



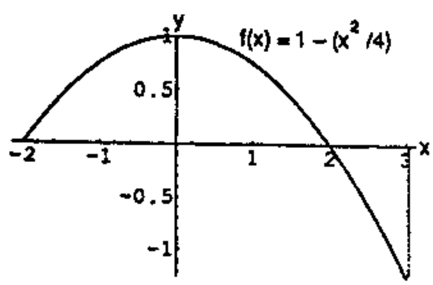
$$\text{Area} = \int_0^1 \left(x^2 - 4x + 3\right)dx - \int_1^3 \left(x^2 - 4x + 3\right)dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3$$

$$= \left[\left(\frac{1^3}{3} - 2(1)^2 + 3(1)\right) - 0\right]$$

$$- \left[\left(\frac{3^3}{3} - 2(3)^2 + 3(3)\right) - \left(\frac{1^3}{3} - 2(1)^2 + 3(1)\right)\right] = \left(\frac{1}{3} + 1\right) - \left[0 - \left(\frac{1}{3} + 1\right)\right] = \frac{8}{3}$$

54. $1 - \frac{x^2}{4} = 0 \Rightarrow 4 - x^2 - 0 \Rightarrow x = \pm 2$;



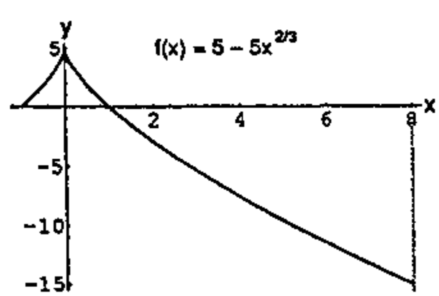
$$\text{Area} = \int_{-2}^{2} \left(1 - \frac{x^2}{4}\right)dx - \int_2^3 \left(1 - \frac{x^2}{4}\right)dx$$

$$= \left[x - \frac{x^3}{12}\right]_{-2}^{2} - \left[x - \frac{x^3}{12}\right]_2^3$$

$$= \left[\left(2 - \frac{2^3}{12}\right) - \left(-2 - \frac{(-2)^3}{12}\right)\right]$$

$$- \left[\left(3 - \frac{3^3}{12}\right) - \left(2 - \frac{2^3}{12}\right)\right] = \left[\frac{4}{3} - \left(-\frac{4}{3}\right)\right] - \left(\frac{3}{4} - \frac{4}{3}\right) = \frac{13}{4}$$

55. $5 - 5x^{2/3} = 0 \Rightarrow 1 - x^{2/3} = 0 \Rightarrow x = \pm 1$;



$$\text{Area} = \int_{-1}^{1} \left(5 - 5x^{2/3}\right)dx - \int_1^8 \left(5 - 5x^{2/3}\right)dx$$

$$= \left[5x - 3x^{5/3}\right]_{-1}^{1} - \left[5x - 3x^{5/3}\right]_1^8$$

$$= \left[\left(5(1) - 3(1)^{5/3}\right) - \left(5(-1) - 3(-1)^{5/3}\right)\right]$$

$$- \left[\left(5(8) - 3(8)^{5/3}\right) - \left(5(1) - 3(1)^{5/3}\right)\right] = [2 - (-2)] - [(40 - 96) - 2] = 62$$

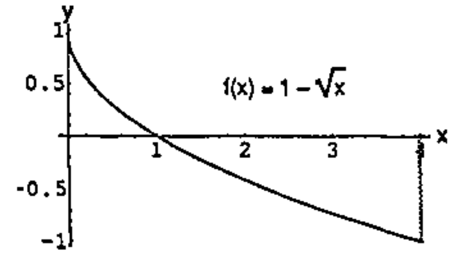
56. $1-\sqrt{x}=0 \Rightarrow x=1$;

$$\text{Area} = \int_0^1 \left(1-\sqrt{x}\right)dx - \int_1^4 \left(1-\sqrt{x}\right)dx$$

$$= \left[x-\tfrac{2}{3}x^{3/2}\right]_0^1 - \left[x-\tfrac{2}{3}x^{3/2}\right]_1^4$$

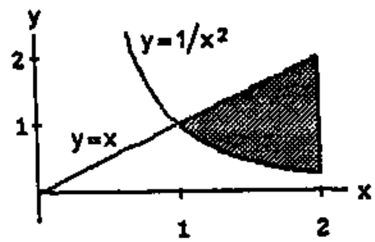
$$= \left[\left(1-\tfrac{2}{3}(1)^{3/2}\right)-0\right] - \left[\left(4-\tfrac{2}{3}(4)^{3/2}\right)-\left(1-\tfrac{2}{3}(1)^{3/2}\right)\right]$$

$$= \tfrac{1}{3} - \left[\left(4-\tfrac{16}{3}\right)-\tfrac{1}{3}\right] = 2$$



57. $f(x)=x,\ g(x)=\frac{1}{x^2},\ a=1,\ b=2 \Rightarrow A = \int_a^b [f(x)-g(x)]\,dx$

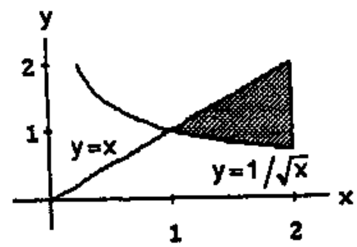
$$= \int_1^2 \left(x-\tfrac{1}{x^2}\right)dx = \left[\tfrac{x^2}{2}+\tfrac{1}{x}\right]_1^2 = \left(\tfrac{4}{2}+\tfrac{1}{2}\right)-\left(\tfrac{1}{2}+1\right)=1$$



58. $f(x)=x,\ g(x)=\frac{1}{\sqrt{x}},\ a=1,\ b=2 \Rightarrow A = \int_a^b [f(x)-g(x)]\,dx$

$$= \int_1^2 \left(x-\tfrac{1}{\sqrt{x}}\right)dx = \left[\tfrac{x^2}{2}-2\sqrt{x}\right]_1^2 = \left(\tfrac{4}{2}-2\sqrt{2}\right)-\left(\tfrac{1}{2}-2\right)$$

$$= \frac{7-4\sqrt{2}}{2}$$



59. $f(x)=\left(1-\sqrt{x}\right)^2,\ g(x)=0,\ a=0,\ b=1 \Rightarrow A = \int_a^b [f(x)-g(x)]\,dx = \int_0^1 \left(1-\sqrt{x}\right)^2 dx = \int_0^1 \left(1-2\sqrt{x}+x\right)dx$

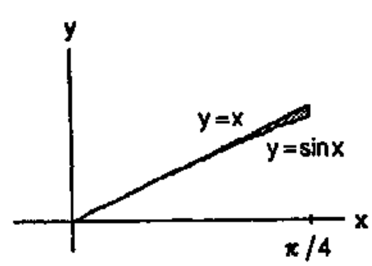
$$= \int_0^1 \left(1-2x^{1/2}+x\right)dx = \left[x-\tfrac{4}{3}x^{3/2}+\tfrac{x^2}{2}\right]_0^1 = 1-\tfrac{4}{3}+\tfrac{1}{2} = \tfrac{1}{6}(6-8+3) = \tfrac{1}{6}$$

60. $f(x)=\left(1-x^3\right)^2,\ g(x)=0,\ a=0,\ b=1 \Rightarrow A = \int_a^b [f(x)-g(x)]\,dx = \int_0^1 \left(1-x^3\right)^2 dx = \int_0^1 \left(1-2x^3+x^6\right)dx$

$$= \left[x-\tfrac{x^4}{2}+\tfrac{x^7}{7}\right]_0^1 = 1-\tfrac{1}{2}+\tfrac{1}{7} = \tfrac{9}{14}$$

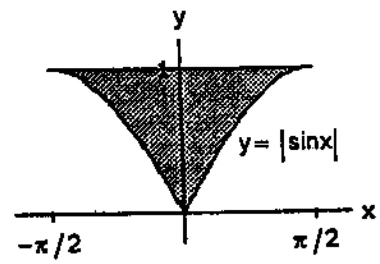
61. $f(x)=x,\ g(x)=\sin x,\ a=0,\ b=\frac{\pi}{4}$

$$\Rightarrow A = \int_a^b [f(x)-g(x)]\,dx = \int_0^{\pi/4} (x-\sin x)\,dx$$



$= \left[\frac{x^2}{2} + \cos x\right]_0^{\pi/4} = \left(\frac{\pi^2}{32} + \frac{\sqrt{2}}{2}\right) - 1$

62. $f(x) = 1$, $g(x) = |\sin x|$, $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$

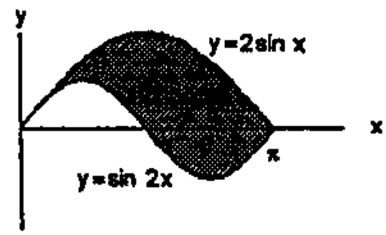


$\Rightarrow A = \int_a^b [f(x) - g(x)]\,dx = \int_{-\pi/2}^{\pi/2} (1 - |\sin x|)\,dx$

$= \int_{-\pi/2}^{0} (1 + \sin x)\,dx + \int_0^{\pi/2} (1 - \sin x)\,dx$

$= 2\int_0^{\pi/2} (1 - \sin x)\,dx = 2[x + \cos x]_0^{\pi/2} = 2\left(\frac{\pi}{2} - 1\right) = \pi - 2$

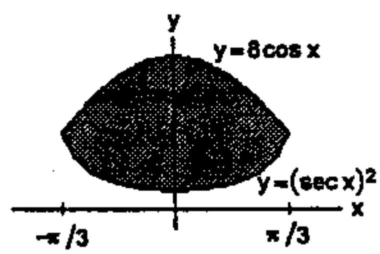
63. $a = 0$, $b = \pi$, $f(x) - g(x) = 2\sin x - \sin 2x$



$\Rightarrow A = \int_0^{\pi} (2\sin x - \sin 2x)\,dx = \left[-2\cos x + \frac{\cos 2x}{2}\right]_0^{\pi}$

$= \left[-2\cdot(-1) + \frac{1}{2}\right] - \left(-2\cdot 1 + \frac{1}{2}\right) = 4$

64. $a = -\frac{\pi}{3}$, $b = \frac{\pi}{3}$, $f(x) - g(x) = 8\cos x - \sec^2 x$



$\Rightarrow A = \int_{-\pi/3}^{\pi/3} (8\cos x - \sec^2 x)\,dx = [8\sin x - \tan x]_{-\pi/3}^{\pi/3}$

$= \left(8\cdot\frac{\sqrt{3}}{2} - \sqrt{3}\right) - \left(-8\cdot\frac{\sqrt{3}}{2} + \sqrt{3}\right) = 6\sqrt{3}$

65. $f(x) = x^3 - 3x^2 = x^2(x-3) \Rightarrow f'(x) = 3x^2 - 6x = 3x(x-2) \Rightarrow f' = +++\underset{0}{|}----\underset{2}{|}+++$

$\Rightarrow f(0) = 0$ is a maximum and $f(2) = -4$ is a minimum. Then $A = -\int_0^3 (x^3 - 3x^2)\,dx = -\left[\frac{x^4}{4} - x^3\right]_0^3$

$= -\left(\frac{81}{4} - 27\right) = \frac{27}{4}$

66. $A = \int_0^1 \left(1 - x^{1/3}\right)^3 dx = \int_0^1 \left(1 - 3x^{1/3} + 3x^{2/3} - x\right) dx = \left[x - \frac{9}{4}x^{4/3} + \frac{9}{5}x^{5/3} - \frac{x^2}{2}\right]_0^1 = \frac{1}{20}$

67. $y = \int \frac{x^2+1}{x^2}\,dx = \int \left(1 + x^{-2}\right) dx = x - x^{-1} + C = x - \frac{1}{x} + C$; $y = -1$ when $x = 1 \Rightarrow 1 - \frac{1}{1} + C = -1$

$\Rightarrow C = -1 \Rightarrow y = x - \frac{1}{x} - 1$

68. $y = \int \left(x + \frac{1}{x}\right)^2 dx = \int \left(x^2 + 2 + \frac{1}{x^2}\right) dx = \int \left(x^2 + 2 + x^{-2}\right) dx = \frac{x^3}{3} + 2x - x^{-1} + C = \frac{x^3}{3} + 2x - \frac{1}{x} + C;$

$y = 1$ when $x = 1 \Rightarrow \frac{1}{3} + 2 - \frac{1}{1} + C = 1 \Rightarrow C = -\frac{1}{3} \Rightarrow y = \frac{x^3}{3} + 2x - \frac{1}{x} - \frac{1}{3}$

69. $\frac{dr}{dt} = \int \left(15\sqrt{t} + \frac{3}{\sqrt{t}}\right) dt = \int \left(15t^{1/2} + 3t^{-1/2}\right) dt = 10t^{3/2} + 6t^{1/2} + C;\ \frac{dr}{dt} = 8$ when $t = 1$

$\Rightarrow 10(1)^{3/2} + 6(1)^{1/2} + C = 8 \Rightarrow C = -8.$ Thus $\frac{dr}{dt} = 10t^{3/2} + 6t^{1/2} - 8 \Rightarrow r = \int \left(10t^{3/2} + 6t^{1/2} - 8\right) dt$

$= 4t^{5/2} + 4t^{3/2} - 8t + C;\ r = 0$ when $t = 1 \Rightarrow 4(1)^{5/2} + 4(1)^{3/2} - 8(1) + C_1 = 0 \Rightarrow C_1 = 0.$ Therefore, $r = 4t^{5/2} + 4t^{3/2} - 8t$

70. $\frac{d^2r}{dt^2} = \int -\cos t\, dt = -\sin t + C;\ r'' = 0$ when $t = 0 \Rightarrow -\sin 0 + C = 0 \Rightarrow C = 0.$ Thus, $\frac{d^2r}{dt^2} = -\sin t$

$\Rightarrow \frac{dr}{dt} = \int -\sin t\, dt = \cos t + C_1;\ r' = 0$ when $t = 0 \Rightarrow 1 + C_1 = 0 \Rightarrow C_1 = -1.$ Then $\frac{dr}{dt} = \cos t - 1$

$\Rightarrow r = \int (\cos t - 1)\, dt = \sin t - t + C_2;\ r = -1$ when $t = 0 \Rightarrow 0 - 0 + C_2 = -1.$ Therefore, $r = \sin t - t - 1$

71. $y = x^2 + \int_1^x \frac{1}{t}\, dt \Rightarrow \frac{dy}{dx} = 2x + \frac{1}{x} \Rightarrow \frac{d^2y}{dx^2} = 2 - \frac{1}{x^2};\ y(1) = 1 + \int_1^1 \frac{1}{t}\, dt = 1$ and $y'(1) = 2 + 1 = 3$

72. $y = \int_0^x \left(1 + 2\sqrt{\sec t}\right) dt \Rightarrow \frac{dy}{dx} = 1 + 2\sqrt{\sec x} \Rightarrow \frac{d^2y}{dx^2} = 2\left(\frac{1}{2}\right)(\sec x)^{-1/2}(\sec x \tan x) = \sqrt{\sec x}\,(\tan x);$

$x = 0 \Rightarrow y = \int_0^0 \left(1 + 2\sqrt{\sec t}\right) dt = 0$ and $x = 0 \Rightarrow \frac{dy}{dx} = 1 + 2\sqrt{\sec 0} = 3$

73. $y = \int_5^x \frac{\sin t}{t}\, dt - 3 \Rightarrow \frac{dy}{dx} = \frac{\sin x}{x};\ x = 5 \Rightarrow y = \int_5^5 \frac{\sin t}{t}\, dt - 3 = -3$

74. $y = \int_{-1}^x \sqrt{2 - \sin^2 t}\, dt + 2$ so that $\frac{dy}{dx} = \sqrt{2 - \sin^2 x};\ x = -1 \Rightarrow y = \int_{-1}^{-1} \sqrt{2 - \sin^2 t}\, dt + 2 = 2$

75. (a) $\text{av}(f) = \frac{1}{1-(-1)} \int_{-1}^1 (mx + b)\, dx = \frac{1}{2}\left[\frac{mx^2}{2} + bx\right]_{-1}^1 = \frac{1}{2}\left[\left(\frac{m(1)^2}{2} + b(1)\right) - \left(\frac{m(-1)^2}{2} + b(-1)\right)\right] = \frac{1}{2}(2b) = b$

(b) $\text{av}(f) = \frac{1}{k-(-k)} \int_{-k}^k (mx + b)\, dx = \frac{1}{2k}\left[\frac{mx^2}{2} + bx\right]_{-k}^k = \frac{1}{2k}\left[\left(\frac{m(k)^2}{2} + b(k)\right) - \left(\frac{m(-k)^2}{2} + b(-k)\right)\right]$

$= \frac{1}{2k}(2bk) = b$

76. (a) $y_{av} = \frac{1}{3-0}\int_0^3 \sqrt{3x}\,dx = \frac{1}{3}\int_0^3 \sqrt{3}\,x^{1/2}\,dx = \frac{\sqrt{3}}{3}\left[\frac{2}{3}x^{3/2}\right]_0^3 = \frac{\sqrt{3}}{3}\left[\frac{2}{3}(3)^{3/2} - \frac{2}{3}(0)^{3/2}\right] = \frac{\sqrt{3}}{3}(2\sqrt{3}) = 2$

(b) $y_{av} = \frac{1}{a-0}\int_0^a \sqrt{ax}\,dx = \frac{1}{a}\int_0^a \sqrt{a}\,x^{1/2}\,dx = \frac{\sqrt{a}}{a}\left[\frac{2}{3}x^{3/2}\right]_0^a = \frac{\sqrt{a}}{a}\left(\frac{2}{3}(a)^{3/2} - \frac{2}{3}(0)^{3/2}\right) = \frac{\sqrt{a}}{a}\left(\frac{2}{3}a\sqrt{a}\right) = \frac{2}{3}a$

77. $f'_{av} = \frac{1}{b-a}\int_a^b f'(x)\,dx = \frac{1}{b-a}[f(x)]_a^b = \frac{1}{b-a}[f(b)-f(a)] = \frac{f(b)-f(a)}{b-a}$ so the average value of f' over $[a,b]$ is the slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$.

78. Yes, because the average value of f on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$. If the length of the interval is 2, then $b - a = 2$ and the average value of the function is $\frac{1}{2}\int_a^b f(x)\,dx$.

79. $\frac{dy}{dx} = \sqrt{2+\cos^3 x}$

80. $\frac{dy}{dx} = \sqrt{2+\cos^3(7x^2)} \cdot \frac{d}{dx}(7x^2) = 14x\sqrt{2+\cos^3(7x^2)}$

81. $\frac{dy}{dx} = \frac{d}{dx}\left(-\int_1^x \frac{6}{3+t^4}\,dt\right) = -\frac{6}{3+x^4}$

82. $\frac{dy}{dx} = \frac{d}{dx}\int_{\sec x}^2 \frac{1}{t^2+1}\,dt = -\frac{d}{dx}\int_2^{\sec x} \frac{1}{t^2+1}\,dt = -\frac{1}{1+\sec^2 x}\frac{d}{dx}(\sec x) = -\frac{\sec x \tan x}{1+\sec^2 x}$

83. $h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6} \Rightarrow \frac{h}{2} = \frac{\pi}{12}$;

$\sum_{i=0}^{6} mf(x_i) = 12 \Rightarrow T = \left(\frac{\pi}{12}\right)(12) = \pi$;

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	$\pi/6$	1/2	2	1
x_2	$\pi/3$	3/2	2	3
x_3	$\pi/2$	2	2	4
x_4	$2\pi/3$	3/2	2	3
x_5	$5\pi/6$	1/2	2	1
x_6	π	0	1	0

$\sum_{i=0}^{6} mf(x_i) = 18$ and $\frac{h}{3} = \frac{\pi}{18} \Rightarrow S = \left(\frac{\pi}{18}\right)(18) = \pi$.

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	$\pi/6$	1/2	4	2
x_2	$\pi/3$	3/2	2	3
x_3	$\pi/2$	2	4	8
x_4	$2\pi/3$	3/2	2	3
x_5	$5\pi/6$	1/2	4	2
x_6	π	0	1	0

84. (a) Each interval is 5 min $= \frac{1}{12}$ hour.

$$\frac{1}{24}[2.5 + 2(2.4) + 2(2.3) + \ldots + 2(2.4) + 2.3] = \frac{29}{12} \approx 2.42 \text{ gal}$$

(b) $(60 \text{ mph})\left(\frac{12}{29} \text{ hours/gal}\right) \approx 24.83 \text{ mi/gal}$

85. $$y_{av} = \frac{1}{365-0}\int_0^{365}\left[37\sin\left(\frac{2\pi}{365}(x-101)\right)+25\right]dx = \frac{1}{365}\left[-37\left(\frac{365}{2\pi}\cos\left(\frac{2\pi}{365}(x-101)\right)+25x\right)\right]_0^{365}$$

$$= \frac{1}{365}\left[\left(-37\left(\frac{365}{2\pi}\right)\cos\left[\frac{2\pi}{365}(365-101)\right]+25(365)\right)-\left(-37\left(\frac{365}{2\pi}\right)\cos\left[\frac{2\pi}{365}(0-101)\right]+25(0)\right)\right]$$

$$= -\frac{37}{2\pi}\cos\left(\frac{2\pi}{365}(264)\right)+25+\frac{37}{2\pi}\cos\left(\frac{2\pi}{365}(-101)\right) = -\frac{37}{2\pi}\left(\cos\left(\frac{2\pi}{365}(264)\right)-\cos\left(\frac{2\pi}{365}(-101)\right)\right)+25$$

$$\approx -\frac{37}{2\pi}(0.16705 - 0.16705) + 25 = 25^\circ\text{F}$$

86. $$av(C_v) = \frac{1}{675-20}\int_{20}^{675}\left[8.27 + 10^{-5}\left(26T - 1.87T^2\right)\right]dT = \frac{1}{655}\left[8.27T + \frac{13}{10^5}T^2 - \frac{0.62333}{10^5}T^3\right]_{20}^{675}$$

$$\approx \frac{1}{655}[(5582.25 + 59.23125 - 1917.03194) - (165.4 + 0.052 - 0.04987)] \approx 5.434;$$

$$8.27 + 10^{-5}\left(26T - 1.87T^2\right) = 5.434 \Rightarrow 1.87T^2 - 26T - 283{,}600 = 0 \Rightarrow T \approx \frac{26 + \sqrt{676 + 4(1.87)(283{,}600)}}{2(1.87)}$$

$$\approx 396.45^\circ\text{C}$$

87. Using the trapezoidal rule, $h = 15 \Rightarrow \frac{h}{2} = 7.5$;

$\sum mf(x_i) = 794.8 \Rightarrow \text{Area} \approx (794.8)(7.5) = 5961 \text{ ft}^2$;

The cost is Area $\cdot$ ($\$2.10/\text{ft}^2$) $\approx (5961 \text{ ft}^2)(\$2.10/\text{ft}^2)$
$= \$12{,}518.10 \Rightarrow$ the job cannot be done for \$11,000.

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	15	36	2	72
x_2	30	54	2	108
x_3	45	51	2	102
x_4	60	49.5	2	99
x_5	75	54	2	108
x_6	90	64.4	2	128.8
x_7	105	67.5	2	135
x_8	120	42	1	42

88. (a) Upper estimate:
$3(5.30 + 5.25 + 5.04 + \ldots + 1.11) = 103.05$ ft
Lower estimate:
$3(5.25 + 5.04 + 4.71 + \ldots + 0) = 87.15$ ft

(b) $\frac{3}{2}[5.30 + 2(5.25) + 2(5.04) + \ldots + 2(1.11) + 0] = 95.1$ ft

89. Yes. The function f, being differentiable on [a, b], is then continuous on [a, b]. The Fundamental Theorem of Calculus says that every continuous function on [a, b] is the derivative of a function on [a, b].

90. The second part of the Fundamental Theorem of Calculus states that if F(x) is an antiderivative of f(x) on [a, b], then $\int_a^b f(x)\,dx = F(b) - F(a)$. In particular, if F(x) is an antiderivative of $\sqrt{1+x^4}$ on [0, 1], then

$$\int_0^1 \sqrt{1+x^4}\,dx = F(1) - F(0).$$

91. $y(x) = \int_5^x \frac{\sin t}{t}\,dt + 3$

92. $y' = \cos x - \frac{d}{dx}\int_\pi^x \cos 2t\,dt = \cos x - \cos 2x$

$y'' = -\sin x + 2\sin 2x$

Thus, it satisfies condition **i.**

$y(\pi) = \sin \pi + \int_\pi^\pi \cos 2t\,dt + 1 = 1$

$y'(\pi) = \cos \pi - \cos 2\pi = -2$

Thus, it satisfies condition **ii.**

93. (a) $g(1) = \int_1^1 f(t)\,dt = 0$

(b) $g(3) = \int_1^3 f(t)\,dt = -\frac{1}{2}(2)(1) = -1$

(c) $g(-1) = \int_1^{-1} f(t)\,dt = -\int_{-1}^1 f(t)\,dt = -\frac{1}{4}\pi(2)^2 = -\pi$

(d) $g'(x) = f(x)$; Since $f(x) > 0$ for $-3 < x < 1$ and $f(x) < 0$ for $1 < x < 3$, g(x) has a relative maximum at $x = 1$.

(e) $g'(-1) = f(-1) = 2$
The equation of the tangent line is $y - (-\pi) = 2(x+1)$ or $y = 2x + 2 - \pi$

(f) $g''(x) = f'(x)$, $f'(x) = 0$ at $x = -1$ and f' is not defined at $x = 2$. The inflection points are at $x = -1$ and $x = 2$. Note that $g''(x) = f'(x)$ is undefined at $x = 1$ as well, but since $g''(x) = f'(x)$ is negative on both sides of $x = 1$, $x = 1$ is not an inflection point.

(g) Note that the absolute maximum is $g(1) = 0$ and the absolute minimum is

$$g(-3) = \int_1^{-3} f(t)\,dt = -\int_{-3}^{1} f(t)\,dt = -\tfrac{1}{2}\pi(2)^2 = -2\pi.$$

The range of g is $[-2\pi, 0]$.

94. (a) Before the chute opens for A, $a = -32$ ft/sec^2. Since the helicopter is hovering, $v_0 = 0$ ft/sec $\Rightarrow v = \int -32\,dt = -32t + v_0 = -32t$. Then $s_0 = 6400$ ft $\Rightarrow s = \int -32t\,dt = -16t^2 + s_0 = -16t^2 + 6400$. At $t = 4$ sec, $s = -16(4)^2 + 6400 = 6144$ ft when A's chute opens;

(b) For B, $s_0 = 7000$ ft, $v_0 = 0$, $a = -32$ ft/sec^2 $\Rightarrow v = \int -32\,dt = -32t + v_0 = -32t \Rightarrow s = \int -32t\,dt$ $= -16t^2 + s_0 = -16t^2 + 7000$. At $t = 13$ sec, $s = -16(13)^2 + 7000 = 4296$ ft when B's chute opens;

(c) After the chutes open, $v = -16$ ft/sec $\Rightarrow s = \int -16\,dt = -16t + s_0$. For A, $s_0 = 6144$ ft and for B, $s_0 = 4296$ ft. Therefore, for A, $s = -16t + 6144$ and for B, $s = -16t + 4296$. When they hit the ground, $s = 0 \Rightarrow$ for A, $0 = -16t + 6144 \Rightarrow t = \frac{6144}{16} = 384$ seconds, and for B, $0 = -16t + 4296 \Rightarrow t = \frac{4296}{16}$ $= 268.5$ seconds to hit the ground after the chutes open. Since B's chute opens 54 seconds after A's opens $\Rightarrow$ B hits the ground first.

95. $av(I) = \frac{1}{30}\int_0^{30} (1200 - 40t)\,dt = \frac{1}{30}\left[1200t - 20t^2\right]_0^{30} = \frac{1}{30}\left[\left((1200(30) - 20(30)^2\right) - \left(1200(0) - 20(0)^2\right)\right]$ $= \frac{1}{30}(18{,}000) = 600$; Average Daily Holding Cost $= (600)(\$0.03) = \$18$

96. $av(I) = \frac{1}{14}\int_0^{14} (600 + 600t)\,dt = \frac{1}{14}\left[600t + 300t^2\right]_0^{14} = \frac{1}{14}\left[600(14) + 300(14)^2 - 0\right] = 4800$; Average Daily Holding Cost $= (4800)(\$0.04) = \$192$

97. $av(I) = \frac{1}{30}\int_0^{30} \left(450 - \frac{t^2}{2}\right) dt = \frac{1}{30}\left[450t - \frac{t^3}{6}\right]_0^{30} = \frac{1}{30}\left[450(30) - \frac{30^3}{6} - 0\right] = 300$; Average Daily Holding Cost $= (300)(\$0.02) = \$6$

98. $av(I) = \frac{1}{60}\int_0^{60} \left(600 - 20\sqrt{15t}\right) dt = \frac{1}{60}\int_0^{60} \left(600 - 20\sqrt{15}\,t^{1/2}\right) dt = \frac{1}{60}\left[600t - 20\sqrt{15}\left(\frac{2}{3}\right)t^{3/2}\right]_0^{60}$

$= \frac{1}{60}\left[600(60) - \frac{40\sqrt{15}}{3}(60)^{3/2} - 0\right] = \frac{1}{60}\left(36{,}000 - \left(\frac{320}{3}\right)15^2\right) = 200$; Average Daily Holding Cost

$= (200)(\$0.005) = \$1.00$

CHAPTER 4 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. (a) Yes, because $\int_0^1 f(x)\,dx = \frac{1}{7}\int_0^1 7f(x)\,dx = \frac{1}{7}(7) = 1$

   (b) No. For example, $\int_0^1 8x\,dx = \left[4x^2\right]_0^1 = 4$, but $\int_0^1 \sqrt{8x}\,dx = \left[2\sqrt{2}\left(\frac{x^{3/2}}{\frac{3}{2}}\right)\right]_0^1 = \frac{4\sqrt{2}}{3}\left(1^{3/2} - 0^{3/2}\right)$

   $= \frac{4\sqrt{2}}{3} \ne \sqrt{4}$

2. (a) True: $\int_5^2 f(x)\,dx = -\int_2^5 f(x)\,dx = -3$

   (b) True: $\int_{-2}^5 [f(x) + g(x)]\,dx = \int_{-2}^5 f(x)\,dx + \int_{-2}^5 g(x)\,dx = \int_{-2}^2 f(x)\,dx + \int_2^5 f(x)\,dx + \int_{-2}^5 g(x)\,dx$

   $= 4 + 3 + 2 = 9$

   (c) False: $\int_{-2}^5 f(x)\,dx = 4 + 3 = 7 > 2 = \int_{-2}^5 g(x)\,dx \Rightarrow \int_{-2}^5 [f(x) - g(x)]\,dx > 0 \Rightarrow \int_{-2}^5 [g(x) - f(x)]\,dx < 0.$

   On the other hand, $f(x) \le g(x) \Rightarrow [g(x) - f(x)] \ge 0 \Rightarrow \int_{-2}^5 [g(x) - f(x)]\,dx \ge 0$ which would be a contradiction.

3. $y = \frac{1}{a}\int_0^x f(t)\sin a(x-t)\,dt = \frac{1}{a}\int_0^x f(t)\sin ax\cos at\,dt - \frac{1}{a}\int_0^x f(t)\cos ax\sin at\,dt$

   $= \frac{\sin ax}{a}\int_0^x f(t)\cos at\,dt - \frac{\cos ax}{a}\int_0^x f(t)\sin at\,dt \Rightarrow \frac{dy}{dx} = \cos ax\int_0^x f(t)\cos at\,dt$

   $+ \frac{\sin ax}{a}\left(\frac{d}{dx}\int_0^x f(t)\cos at\,dt\right) + \sin ax\int_0^x f(t)\sin at\,dt - \frac{\cos ax}{a}\left(\frac{d}{dx}\int_0^x f(t)\sin at\,dt\right)$

   $= \cos ax\int_0^x f(t)\cos at\,dt + \frac{\sin ax}{a}(f(x)\cos ax) + \sin ax\int_0^x f(t)\sin at\,dt - \frac{\cos ax}{a}(f(x)\sin ax)$

   $\Rightarrow \frac{dy}{dx} = \cos ax\int_0^x f(t)\cos at\,dt + \sin ax\int_0^x f(t)\sin at\,dt$. Next,

$$\frac{d^2y}{dx^2} = -a \sin ax \int_0^x f(t) \cos at\, dt + (\cos ax)\left(\frac{d}{dx}\int_0^x f(t) \cos at\, dt\right) + a \cos ax \int_0^x f(t) \sin at\, dt$$

$$+ (\sin ax)\left(\frac{d}{dx}\int_0^x f(t) \sin at\, dt\right) = -a \sin ax \int_0^x f(t) \cos at\, dt + (\cos ax)f(x) \cos ax$$

$$+ a \cos ax \int_0^x f(t) \sin at\, dt + (\sin ax)f(x) \sin ax = -a \sin ax \int_0^x f(t) \cos at\, dt + a \cos ax \int_0^x f(t) \sin at\, dt + f(x).$$

Therefore, $y'' + a^2y = a \cos ax \int_0^x f(t) \sin at\, dt - a \sin ax \int_0^x f(t) \cos at\, dt + f(x)$

$+ a^2\left(\frac{\sin ax}{a}\int_0^x f(t) \cos at\, dt - \frac{\cos ax}{a}\int_0^x f(t) \sin at\, dt\right) = f(x)$. Note also that $y'(0) = y(0) = 0$.

4. $x = \int_0^y \frac{1}{\sqrt{1+4t^2}}\, dt \Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}\int_0^y \frac{1}{\sqrt{1+4t^2}}\, dt = \frac{d}{dy}\left[\int_0^y \frac{1}{\sqrt{1+4t^2}}\, dt\right]\left(\frac{dy}{dx}\right)$ from the chain rule

$\Rightarrow 1 = \frac{1}{\sqrt{1+4y^2}}\left(\frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx} = \sqrt{1+4y^2}$. Then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\sqrt{1+4y^2}\right) = \frac{d}{dy}\left(\sqrt{1+4y^2}\right)\left(\frac{dy}{dx}\right)$

$= \frac{1}{2}\left(1+4y^2\right)^{-1/2}(8y)\left(\frac{dy}{dx}\right) = \frac{4y\left(\frac{dy}{dx}\right)}{\sqrt{1+4y^2}} = \frac{4y\left(\sqrt{1+4y^2}\right)}{\sqrt{1+4y^2}} = 4y$. Thus $\frac{d^2y}{dx^2} = 4y$, and the constant of proportionality is 4.

5. (a) $\int_0^{x^2} f(t)\, dt = x \cos \pi x \Rightarrow \frac{d}{dx}\int_0^{x^2} f(t)\, dt = \cos \pi x - \pi x \sin \pi x \Rightarrow f\left(x^2\right)(2x) = \cos \pi x - \pi x \sin \pi x$

$\Rightarrow f\left(x^2\right) = \frac{\cos \pi x - \pi x \sin \pi x}{2x}$. Thus, $x = 2 \Rightarrow f(4) = \frac{\cos 2\pi - 2\pi \sin 2\pi}{4} = \frac{1}{4}$

(b) $\int_0^{f(x)} t^2\, dt = \left[\frac{t^3}{3}\right]_0^{f(x)} = \frac{1}{3}(f(x))^3 \Rightarrow \frac{1}{3}(f(x))^3 = x \cos \pi x \Rightarrow (f(x))^3 = 3x \cos \pi x \Rightarrow f(x) = \sqrt[3]{3x \cos \pi x}$

$\Rightarrow f(4) = \sqrt[3]{3(4) \cos 4\pi} = \sqrt[3]{12}$

6. $\int_0^a f(x)\, dx = \frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a$ and let $F(a) = \int_0^a f(t)\, dt \Rightarrow f(a) = F'(a)$. Now $F(a) = \frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a$

$\Rightarrow f(a) = F'(a) = a + \frac{1}{2} \sin a + \frac{a}{2} \cos a - \frac{\pi}{2} \sin a \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{1}{2} \sin \frac{\pi}{2} + \frac{\left(\frac{\pi}{2}\right)}{2} \cos \frac{\pi}{2} - \frac{\pi}{2} \sin \frac{\pi}{2} = \frac{\pi}{2} + \frac{1}{2} - \frac{\pi}{2} = \frac{1}{2}$

7. $\int_1^b f(x)\,dx = \sqrt{b^2+1} - \sqrt{2} \Rightarrow f(b) = \frac{d}{db}\int_1^b f(x)\,dx = \frac{1}{2}\left(b^2+1\right)^{-1/2}(2b) = \frac{b}{\sqrt{b^2+1}} \Rightarrow f(x) = \frac{x}{\sqrt{x^2+1}}$

8. The derivative of the left side of the equation is: $\frac{d}{dx}\left[\int_0^x \left[\int_0^u f(t)\,dt\right] du\right] = \int_0^x f(t)\,dt$; the derivative of the right side of the equation is: $\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] = \frac{d}{dx}\int_0^x f(u)\,x\,du - \frac{d}{dx}\int_0^x u\,f(u)\,du$

$= \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\int_0^x u\,f(u)\,du = \int_0^x f(u)\,du + x\left[\frac{d}{dx}\int_0^x f(u)\,du\right] - xf(x) = \int_0^x f(u)\,du + xf(x) - xf(x)$

$= \int_0^x f(u)\,du$. Since each side has the same derivative, they differ by a constant, and since both sides equal 0 when $x = 0$, the constant must be 0. Therefore, $\int_0^x \left[\int_0^u f(t)\,dt\right] du = \int_0^x f(u)(x-u)\,du$.

9. $\frac{dy}{dx} = 3x^2 + 2 \Rightarrow y = \int \left(3x^2+2\right) dx = x^3 + 2x + C$. Then $(1,-1)$ on the curve $\Rightarrow 1^3 + 2(1) + C = -1 \Rightarrow C = -4$ $\Rightarrow y = x^3 + 2x - 4$

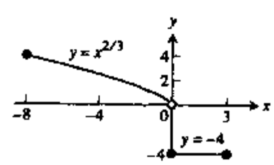
10. The acceleration due to gravity downward is $-32\ \text{ft/sec}^2 \Rightarrow v = \int -32\,dt = -32t + v_0$, where v_0 is the initial velocity $\Rightarrow v = -32t + 32 \Rightarrow s = \int (-32t+32)\,dt = -16t^2 + 32t + C$. If the release point is $s = 0$, then $C = 0$ $\Rightarrow s = -16t^2 + 32t$. Then $s = 17 \Rightarrow 17 = -16t^2 + 32t \Rightarrow 16t^2 - 32t + 17 = 0$. The discriminant of this quadratic equation is -64 which says there is no real time when $s = 17$ ft. You had better duck.

11. $\int_{-8}^3 f(x)\,dx = \int_{-8}^0 x^{2/3}\,dx + \int_0^3 -4\,dx$

$= \left[\frac{3}{5}x^{5/3}\right]_{-8}^0 + [-4x]_0^3$

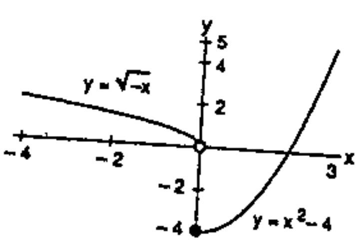
$= \left(0 - \frac{3}{5}(-8)^{5/3}\right) + (-4(3) - 0) = \frac{96}{5} - 12$

$= \frac{36}{5}$



12. $\int_{-4}^3 f(x)\,dx = \int_{-4}^0 \sqrt{-x}\,dx + \int_0^3 \left(x^2-4\right) dx$

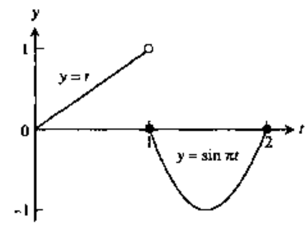
$= \left[-\frac{2}{3}(-x)^{3/2}\right]_{-4}^0 + \left[\frac{x^3}{3} - 4x\right]_0^3$



$$= \left[0 - \left(-\frac{2}{3}(4)^{3/2}\right)\right] + \left(\frac{3^3}{3} - 4(3)\right) - 0$$

$$= \frac{16}{3} - 3 = \frac{7}{3}$$

13. $\displaystyle\int_0^2 f(t)\,dt = \int_0^1 t\,dt + \int_1^2 \sin \pi t\,dt$

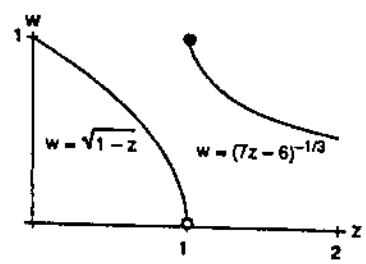


$$= \left[\frac{t^2}{2}\right]_0^1 + \left[-\frac{1}{\pi}\cos \pi t\right]_1^2$$

$$= \left(\frac{1}{2} - 0\right) + \left[-\frac{1}{\pi}\cos 2\pi - \left(-\frac{1}{\pi}\cos \pi\right)\right]$$

$$= \frac{1}{2} - \frac{2}{\pi}$$

14. $\displaystyle\int_0^2 f(z)\,dz = \int_0^1 \sqrt{1-z}\,dz + \int_1^2 (7z-6)^{-1/3}\,dz$



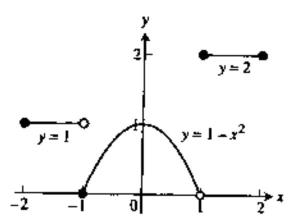
$$= \left[-\frac{2}{3}(1-z)^{3/2}\right]_0^1 + \left[\frac{3}{14}(7z-6)^{2/3}\right]_1^2$$

$$= \left[-\frac{2}{3}(1-1)^{3/2} - \left(-\frac{2}{3}(1-0)^{3/2}\right)\right]$$

$$+ \left[\frac{3}{14}(7(2)-6)^{2/3} - \frac{3}{14}(7(1)-6)^{2/3}\right]$$

$$= \frac{2}{3} + \left(\frac{6}{7} - \frac{3}{14}\right) = \frac{55}{42}$$

15. $\displaystyle\int_{-2}^2 f(x)\,dx = \int_{-2}^{-1} dx + \int_{-1}^1 \left(1-x^2\right)dx + \int_1^2 2\,dx$

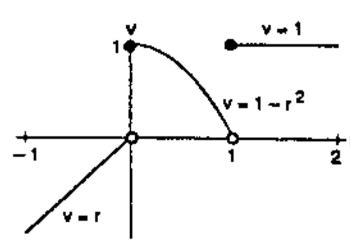


$$= [x]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^1 + [2x]_1^2$$

$$= -1 - (-2) + \left(1 - \frac{1^3}{3}\right) - \left(-1 - \frac{(-1)^3}{3}\right) + 2(2) - 2(1)$$

$$= 1 + \frac{2}{3} - \left(-\frac{2}{3}\right) + 4 - 2 = \frac{13}{3}$$

16. $\displaystyle\int_{-1}^2 h(r)\,dr = \int_{-1}^0 r\,dr + \int_0^1 \left(1-r^2\right)dr + \int_1^2 dr$



$$= \left[\frac{r^2}{2}\right]_{-1}^0 + \left[r - \frac{r^3}{3}\right]_0^1 + [r]_1^2$$

$$= 0 - \frac{(-1)^2}{2} + \left(1 - \frac{1^3}{3}\right) - 0 + 2 - 1$$

$$= -\frac{1}{2} + \frac{2}{3} + 1 = \frac{7}{6}$$

17. Ave. value $= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\left[\int_0^1 x\,dx + \int_1^2 (x-1)\,dx\right] = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^2$

$= \frac{1}{2}\left[\frac{1^2}{2} - 0 + \left(\frac{2^2}{2} - 2\right) - \left(\frac{1^2}{2} - 1\right)\right] = \frac{1}{2}$

18. Ave. value $= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3-0}\int_0^3 f(x)\,dx = \frac{1}{3}\left[\int_0^1 dx + \int_1^2 0\,dx + \int_2^3 dx\right] = \frac{1}{3}[1 - 0 + 0 + 3 - 2] = \frac{2}{3}$

19. $f(x) = \int_{1/x}^{x} \frac{1}{t}\,dt \Rightarrow f'(x) = \frac{1}{x}\left(\frac{dx}{dx}\right) - \left(\frac{1}{\frac{1}{x}}\right)\left(\frac{d}{dx}\left(\frac{1}{x}\right)\right) = \frac{1}{x} - x\left(-\frac{1}{x^2}\right) = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}$

20. $f(x) = \int_{\cos x}^{\sin x} \frac{1}{1-t^2}\,dt \Rightarrow f'(x) = \left(\frac{1}{1-\sin^2 x}\right)\left(\frac{d}{dx}(\sin x)\right) - \left(\frac{1}{1-\cos^2 x}\right)\left(\frac{d}{dx}(\cos x)\right) = \frac{\cos x}{\cos^2 x} + \frac{\sin x}{\sin^2 x}$

$= \frac{1}{\cos x} + \frac{1}{\sin x}$

21. $g(y) = \int_{\sqrt{y}}^{2\sqrt{y}} \sin t^2\,dt \Rightarrow g'(y) = \left(\sin\left(2\sqrt{y}\right)^2\right)\left(\frac{d}{dy}\left(2\sqrt{y}\right)\right) - \left(\sin\left(\sqrt{y}\right)^2\right)\left(\frac{d}{dy}\left(\sqrt{y}\right)\right) = \frac{\sin 4y}{\sqrt{y}} - \frac{\sin y}{2\sqrt{y}}$

22. $f(x) = \int_{x}^{x+3} t(5-t)\,dt \Rightarrow f'(x) = (x+3)(5-(x+3))\left(\frac{d}{dx}(x+3)\right) - x(5-x)\left(\frac{dx}{dx}\right) = (x+3)(2-x) - x(5-x)$

$= 6 - x - x^2 - 5x + x^2 = 6 - 6x$. Thus $f'(x) = 0 \Rightarrow 6 - 6x = 0 \Rightarrow x = 1$. Also, $f''(x) = -6 < 0 \Rightarrow x = 1$ gives a maximum.

NOTES:

CHAPTER 5 APPLICATIONS OF INTEGRALS

5.1 VOLUMES BY SLICING AND ROTATION ABOUT AN AXIS

1. (a) $A = \pi(\text{radius})^2$ and radius $= \sqrt{1-x^2} \Rightarrow A(x) = \pi\left(1-x^2\right)$

   (b) $A = \text{width} \cdot \text{height}$, width $=$ height $= 2\sqrt{1-x^2} \Rightarrow A(x) = 4\left(1-x^2\right)$

   (c) $A = (\text{side})^2$ and diagonal $= \sqrt{2}(\text{side}) \Rightarrow A = \frac{(\text{diagonal})^2}{2}$; diagonal $= 2\sqrt{1-x^2} \Rightarrow A(x) = 2\left(1-x^2\right)$

   (d) $A = \frac{\sqrt{3}}{4}(\text{side})^2$ and side $= 2\sqrt{1-x^2} \Rightarrow A(x) = \sqrt{3}\left(1-x^2\right)$

2. (a) $A = \pi(\text{radius})^2$ and radius $= \sqrt{x} \Rightarrow A(x) = \pi x$

   (b) $A = \text{width} \cdot \text{height}$, width $=$ height $= 2\sqrt{x} \Rightarrow A(x) = 4x$

   (c) $A = (\text{side})^2$ and diagonal $= \sqrt{2}(\text{side}) \Rightarrow A = \frac{(\text{diagonal})^2}{2}$; diagonal $= 2\sqrt{x} \Rightarrow A(x) = 2x$

   (d) $A = \frac{\sqrt{3}}{4}(\text{side})^2$ and side $= 2\sqrt{x} \Rightarrow A(x) = \sqrt{3}x$

3. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left(\sqrt{x}-\left(-\sqrt{x}\right)\right)^2}{2} = 2x$ (see Exercise 1c); $a = 0$, $b = 4$;

   $$V = \int_a^b A(x)\,dx = \int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16$$

4. $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi\left[\left(2-x^2\right)-x^2\right]^2}{4} = \frac{\pi\left[2\left(1-x^2\right)\right]^2}{4} = \pi\left(1-2x^2+x^4\right)$; $a = -1$, $b = 1$;

   $$V = \int_a^b A(x)\,dx = \int_{-1}^1 \pi\left(1-2x^2+x^4\right)dx = \pi\left[x-\frac{2}{3}x^3+\frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{16\pi}{15}$$

5. $A(x) = (\text{edge})^2 = \left[\sqrt{1-x^2}-\left(-\sqrt{1-x^2}\right)\right]^2 = \left(2\sqrt{1-x^2}\right)^2 = 4\left(1-x^2\right)$; $a = -1$, $b = 1$;

   $$V = \int_a^b A(x)\,dx = \int_{-1}^1 4\left(1-x^2\right)dx = 4\left[x-\frac{x^3}{3}\right]_{-1}^1 = 8\left(1-\frac{1}{3}\right) = \frac{16}{3}$$

6. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left[\sqrt{1-x^2}-\left(-\sqrt{1-x^2}\right)\right]^2}{2} = \frac{\left(2\sqrt{1-x^2}\right)^2}{2} = 2\left(1-x^2\right)$ (see Exercise 1c); $a = -1$, $b = 1$;

   $$V = \int_a^b A(x)\,dx = 2\int_{-1}^1 \left(1-x^2\right)dx = 2\left[x-\frac{x^3}{3}\right]_{-1}^1 = 4\left(1-\frac{1}{3}\right) = \frac{8}{3}$$

7. (a) STEP 1) $A(x) = \frac{1}{2}(\text{side}) \cdot (\text{side}) \cdot \left(\sin \frac{\pi}{3}\right) = \frac{1}{2} \cdot \left(2\sqrt{\sin x}\right) \cdot \left(2\sqrt{\sin x}\right)\left(\sin \frac{\pi}{3}\right) = \sqrt{3} \sin x$

STEP 2) $a = 0,\ b = \pi$

STEP 3) $V = \int_a^b A(x)\,dx = \sqrt{3} \int_0^{\pi} \sin x\,dx = \left[-\sqrt{3} \cos x\right]_0^{\pi} = \sqrt{3}(1+1) = 2\sqrt{3}$

(b) STEP 1) $A(x) = (\text{side})^2 = \left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right) = 4 \sin x$

STEP 2) $a = 0,\ b = \pi$

STEP 3) $V = \int_a^b A(x)\,dx = \int_0^{\pi} 4 \sin x\,dx = [-4 \cos x]_0^{\pi} = 8$

8. (a) STEP 1) $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi}{4}(\sec x - \tan x)^2 = \frac{\pi}{4}\left(\sec^2 x + \tan^2 x - 2 \sec x \tan x\right)$

$$= \frac{\pi}{4}\left[\sec^2 x + \left(\sec^2 x - 1\right) - 2\,\frac{\sin x}{\cos^2 x}\right]$$

STEP 2) $a = -\frac{\pi}{3},\ b = \frac{\pi}{3}$

STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3} \frac{\pi}{4}\left(2 \sec^2 x - 1 - \frac{2 \sin x}{\cos^2 x}\right) dx = \frac{\pi}{4}\left[2 \tan x - x + 2\left(-\frac{1}{\cos x}\right)\right]_{-\pi/3}^{\pi/3}$

$$= \frac{\pi}{4}\left[2\sqrt{3} - \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right) - \left(-2\sqrt{3} + \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right)\right)\right] = \frac{\pi}{4}\left(4\sqrt{3} - \frac{2\pi}{3}\right)$$

(b) STEP 1) $A(x) = (\text{edge})^2 = (\sec x - \tan x)^2 = \left(2 \sec^2 x - 1 - 2\,\frac{\sin x}{\cos^2 x}\right)$

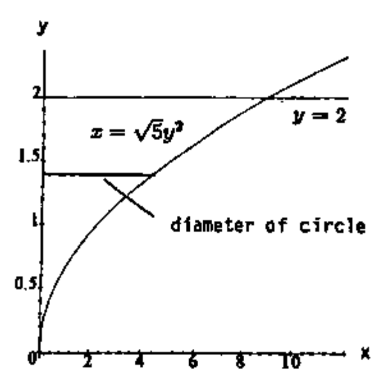
STEP 2) $a = -\frac{\pi}{3},\ b = \frac{\pi}{3}$

STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3} \left(2 \sec^2 x - 1 - \frac{2 \sin x}{\cos^2 x}\right) dx = 2\left(2\sqrt{3} - \frac{\pi}{3}\right) = 4\sqrt{3} - \frac{2\pi}{3}$

9. $A(y) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{5}y^2 - 0\right)^2 = \frac{5\pi}{4} y^4;$

$c = 0,\ d = 2;\ V = \int_c^d A(y)\,dy = \int_0^2 \frac{5\pi}{4} y^4\,dy$

$$= \left[\left(\frac{5\pi}{4}\right)\left(\frac{y^5}{5}\right)\right]_0^2 = \frac{\pi}{4}\left(2^5 - 0\right) = 8\pi$$



10. $A(y) = \frac{1}{2}(\text{leg})(\text{leg}) = \frac{1}{2}\left[\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\right]^2 = \frac{1}{2}\left(2\sqrt{1-y^2}\right)^2 = 2\left(1-y^2\right)$; $c = -1$, $d = 1$;

$$V = \int_c^d A(y)\,dy = \int_{-1}^{1} 2\left(1-y^2\right) dy = 2\left[y - \frac{y^3}{3}\right]_{-1}^{1} = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$$

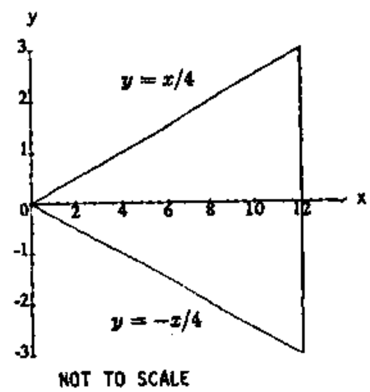
11. (a) It follows from Cavalieri's Theorem that the volume of a column is the same as the volume of a right prism with a square base of side length s and altitude h. Thus, STEP 1) $A(x) = (\text{side length})^2 = s^2$; STEP 2) $a = 0$, $b = h$; STEP 3) $V = \int_a^b A(x)\,dx = \int_0^h s^2\,dx = s^2h$

(b) From Cavalieri's Theorem we conclude that the volume of the column is the same as the volume of the prism described above, regardless of the number of turns $\Rightarrow V = s^2h$

12. 1) The solid and the cone have the same altitude of 12.

2) The cross sections of the solid are disks of diameter $x - \left(\frac{x}{2}\right) = \frac{x}{2}$. If we place the vertex of the cone at the origin of the coordinate system and make its axis of symmetry coincide with the x-axis then the cone's cross sections will be circular disks of diameter $\frac{x}{4} - \left(-\frac{x}{4}\right) = \frac{x}{2}$ (see accompanying figure).

3) The solid and the cone have equal altitudes and identical parallel cross sections. From Cavalieri's Theorem we conclude that the solid and the cone have the same volume.



13. $R(x) = y = 1 - \frac{x}{2} \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 \left(1 - \frac{x}{2}\right)^2 dx = \pi\int_0^2 \left(1 - x + \frac{x^2}{4}\right) dx = \pi\left[x - \frac{x^2}{2} + \frac{x^3}{12}\right]_0^2$

$= \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$

14. $R(y) = x = \frac{3y}{2} \Rightarrow V = \int_0^2 \pi[R(y)]^2\,dy = \pi\int_0^2 \left(\frac{3y}{2}\right)^2 dy = \pi\int_0^2 \frac{9}{4}y^2\,dy = \pi\left[\frac{3}{4}y^3\right]_0^2 = \pi \cdot \frac{3}{4} \cdot 8 = 6\pi$

15. $R(x) = \tan\left(\frac{\pi}{4}y\right)$; $u = \frac{\pi}{4}y \Rightarrow du = \frac{\pi}{4}\,dy \Rightarrow 4\,du = \pi\,dy$; $y = 0 \Rightarrow u = 0$, $y = 1 \Rightarrow u = \frac{\pi}{4}$;

$$V = \int_0^1 \pi[R(y)]^2\,dy = \pi\int_0^1 \left[\tan\left(\frac{\pi}{4}y\right)\right]^2 dy = 4\int_0^{\pi/4} \tan^2 u\,du = 4\int_0^{\pi/4} \left(-1 + \sec^2 u\right) du = 4[-u + \tan u]_0^{\pi/4}$$

$= 4\left(-\frac{\pi}{4} + 1 - 0\right) = 4 - \pi$

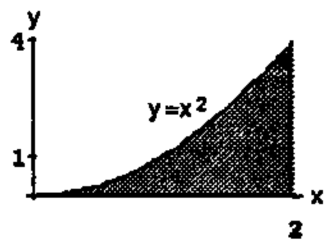
16. $R(x) = \sin x \cos x$; $R(x) = 0 \Rightarrow a = 0$ and $b = \frac{\pi}{2}$ are the limits of integration; $V = \int_0^{\pi/2} \pi[R(x)]^2\,dx$

$= \pi \int_0^{\pi/2} (\sin x \cos x)^2\,dx = \pi \int_0^{\pi/2} \frac{(\sin 2x)^2}{4}\,dx$; $\left[u = 2x \Rightarrow du = 2\,dx \Rightarrow \frac{du}{8} = \frac{dx}{4};\ x = 0 \Rightarrow u = 0,\right.$

$\left. x = \frac{\pi}{2} \Rightarrow u = \pi\right] \to V = \pi \int_0^{\pi} \frac{1}{8}\sin^2 u\,du = \frac{\pi}{8}\left[\frac{u}{2} - \frac{1}{4}\sin 2u\right]_0^{\pi} = \frac{\pi}{8}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{\pi^2}{16}$

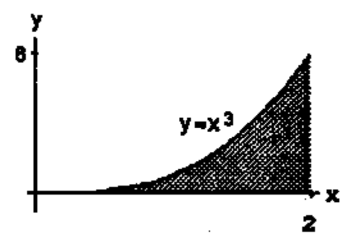
17. $R(x) = x^2 \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi \int_0^2 \left(x^2\right)^2\,dx$

$= \pi \int_0^2 x^4\,dx = \pi\left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$



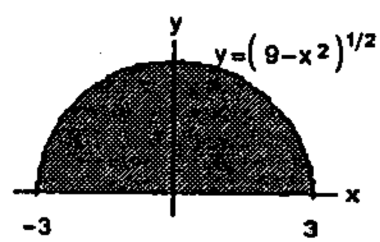
18. $R(x) = x^3 \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi \int_0^2 \left(x^3\right)^2\,dx$

$= \pi \int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$



19. $R(x) = \sqrt{9 - x^2} \Rightarrow V = \int_{-3}^{3} \pi[R(x)]^2\,dx = \pi \int_{-3}^{3} \left(9 - x^2\right)dx$

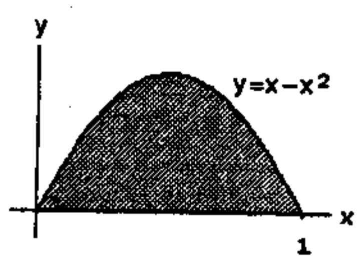
$= \pi\left[9x - \frac{x^3}{3}\right]_{-3}^{3} = 2\pi\left[9(3) - \frac{27}{3}\right] = 2 \cdot \pi \cdot 18 = 36\pi$



20. $R(x) = x - x^2 \Rightarrow V = \int_0^1 \pi[R(x)]^2\,dx = \pi \int_0^1 \left(x - x^2\right)^2\,dx$

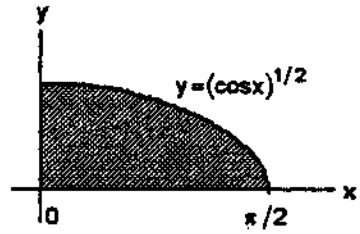
$= \pi \int_0^1 \left(x^2 - 2x^3 + x^4\right)dx = \pi\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1$

$= \pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{30}(10 - 15 + 6) = \frac{\pi}{30}$



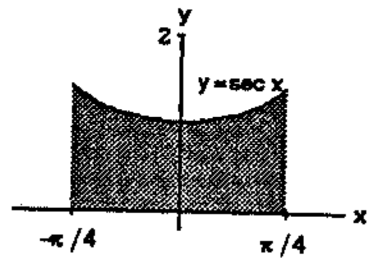
21. $R(x) = \sqrt{\cos x} \Rightarrow V = \int_0^{\pi/2} \pi[R(x)]^2\,dx = \pi \int_0^{\pi/2} \cos x\,dx$

$= \pi[\sin x]_0^{\pi/2} = \pi(1-0) = \pi$



22. $R(x) = \sec x \Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi[R(x)]^2\,dx = \pi \int_{-\pi/4}^{\pi/4} \sec^2 x\,dx$

$= \pi[\tan x]_{-\pi/4}^{\pi/4} = \pi[1-(-1)] = 2\pi$

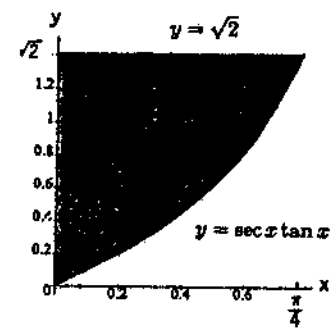


23. $R(x) = \sqrt{2} - \sec x \tan x \Rightarrow V = \int_0^{\pi/4} \pi[R(x)]^2\,dx$

$= \pi \int_0^{\pi/4} \left(\sqrt{2} - \sec x \tan x\right)^2 dx$

$= \pi \int_0^{\pi/4} \left(2 - 2\sqrt{2}\sec x \tan x + \sec^2 x \tan^2 x\right) dx$

$= \pi\left(\int_0^{\pi/4} 2\,dx - 2\sqrt{2}\int_0^{\pi/4} \sec x \tan x\,dx + \int_0^{\pi/4} (\tan x)^2 \sec^2 x\,dx\right)$

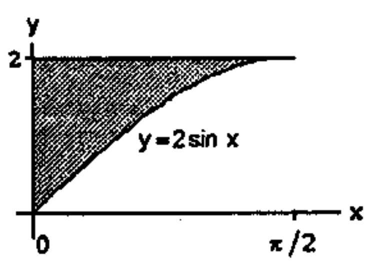


$= \pi\left([2x]_0^{\pi/4} - 2\sqrt{2}[\sec x]_0^{\pi/4} + \left[\frac{\tan^3 x}{3}\right]_0^{\pi/4}\right) = \pi\left[\left(\frac{\pi}{2} - 0\right) - 2\sqrt{2}\left(\sqrt{2}-1\right) + \frac{1}{3}\left(1^3 - 0\right)\right] = \pi\left(\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3}\right)$

24. $R(x) = 2 - 2\sin x = 2(1-\sin x) \Rightarrow V = \int_0^{\pi/2} \pi[R(x)]^2\,dx$

$= \pi \int_0^{\pi/2} 4(1-\sin x)^2\,dx = 4\pi \int_0^{\pi/2} \left(1 + \sin^2 x - 2\sin x\right) dx$

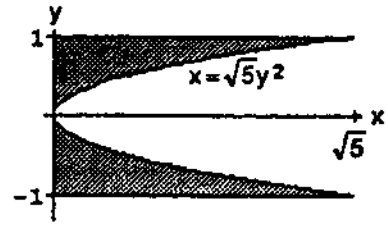
$= 4\pi \int_0^{\pi/2} \left[1 + \frac{1}{2}(1-\cos 2x) - 2\sin x\right] dx$



$= 4\pi \int_0^{\pi/2} \left(\frac{3}{2} - \frac{\cos 2x}{2} - 2\sin x\right) = 4\pi\left[\frac{3}{2}x - \frac{\sin 2x}{4} + 2\cos x\right]_0^{\pi/2} = 4\pi\left[\left(\frac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 2)\right] = \pi(3\pi - 8)$

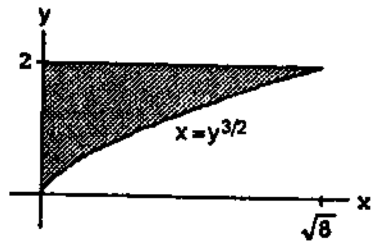
25. $R(y) = \sqrt{5} \cdot y^2 \Rightarrow V = \int_{-1}^{1} \pi[R(y)]^2\, dy = \pi \int_{-1}^{1} 5y^4\, dy$

$= \pi\left[y^5\right]_{-1}^{1} = \pi[1-(-1)] = 2\pi$



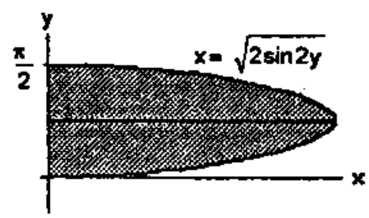
26. $R(y) = y^{3/2} \Rightarrow V = \int_{0}^{2} \pi[R(y)]^2\, dy = \pi \int_{0}^{2} y^3\, dy$

$= \pi\left[\frac{y^4}{4}\right]_0^2 = 4\pi$



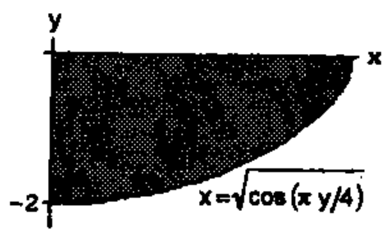
27. $R(y) = \sqrt{2 \sin 2y} \Rightarrow V = \int_{0}^{\pi/2} \pi[R(y)]^2\, dy = \pi \int_{0}^{\pi/2} 2 \sin 2y\, dy$

$= \pi\left[-\cos 2y\right]_0^{\pi/2} = \pi[1-(-1)] = 2\pi$



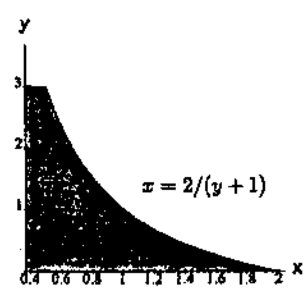
28. $R(y) = \sqrt{\cos \frac{\pi y}{4}} \Rightarrow V = \int_{-2}^{0} \pi[R(y)]^2\, dy = \pi \int_{-2}^{0} \cos\left(\frac{\pi y}{4}\right) dy$

$= 4\left[\sin \frac{\pi y}{4}\right]_{-2}^{0} = 4[0-(-1)] = 4$



29. $R(y) = \frac{2}{y+1} \Rightarrow V = \int_{0}^{3} \pi[R(y)]^2\, dy = 4\pi \int_{0}^{3} \frac{1}{(y+1)^2}\, dy$

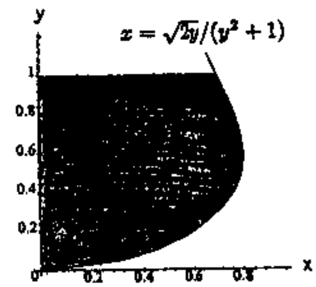
$= 4\pi\left[\frac{-1}{y+1}\right]_0^3 = 4\pi\left[-\frac{1}{4}-(-1)\right] = 3\pi$



30. $R(y) = \frac{\sqrt{2y}}{y^2+1} \Rightarrow V = \int_0^1 \pi[R(y)]^2\, dy = \pi \int_0^1 2y\left(y^2+1\right)^{-2} dy;$

$\left[u = y^2 + 1 \Rightarrow du = 2y\, dy;\ y = 0 \Rightarrow u = 1,\ y = 1 \Rightarrow u = 2\right]$

$\to V = \pi \int_1^2 u^{-2}\, du = \pi\left[-\frac{1}{u}\right]_1^2 = \pi\left[-\frac{1}{2} - (-1)\right] = \frac{\pi}{2}$



31. For the sketch given, $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$; $R(x) = 1$, $r(x) = \sqrt{\cos x}$; $V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

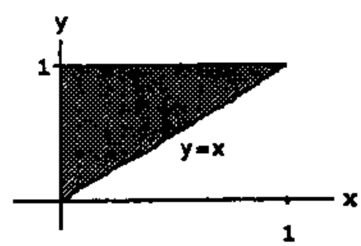
$= \int_{-\pi/2}^{\pi/2} \pi(1 - \cos x)\, dx = 2\pi \int_0^{\pi/2} (1 - \cos x)\, dx = 2\pi[x - \sin x]_0^{\pi/2} = 2\pi\left(\frac{\pi}{2} - 1\right) = \pi^2 - 2\pi$

32. For the sketch given, $c = 0$, $d = \frac{\pi}{4}$; $R(y) = 1$, $r(y) = \tan y$; $V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^{\pi/4} \left(1 - \tan^2 y\right) dy = \pi \int_0^{\pi/4} \left(2 - \sec^2 y\right) dy = \pi[2y - \tan y]_0^{\pi/4} = \pi\left(\frac{\pi}{2} - 1\right) = \frac{\pi^2}{2} - \pi$

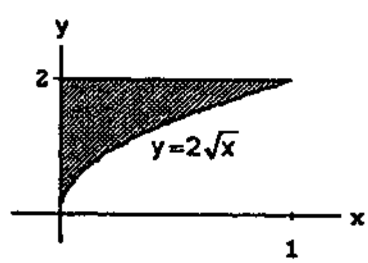
33. $r(x) = x$ and $R(x) = 1 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \int_0^1 \pi\left(1 - x^2\right) dx = \pi\left[x - \frac{x^3}{3}\right]_0^1 = \pi\left[\left(1 - \frac{1}{3}\right) - 0\right] = \frac{2\pi}{3}$

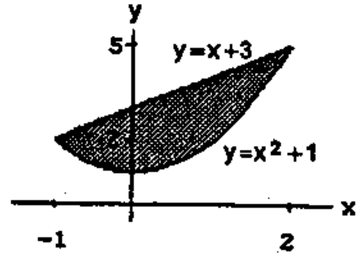


34. $r(x) = 2\sqrt{x}$ and $R(x) = 2 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_0^1 (4 - 4x)\, dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1 = 4\pi\left(1 - \frac{1}{2}\right) = 2\pi$



35. $r(x) = x^2 + 1$ and $R(x) = x + 3 \Rightarrow V = \int_{-1}^{2} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$



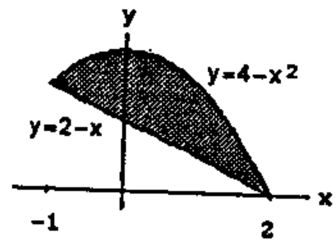
$= \pi \int_{-1}^{2} \left[(x+3)^2 - \left(x^2+1\right)^2\right] dx$

$= \pi \int_{-1}^{2} \left[\left(x^2+6x+9\right) - \left(x^4+2x^2+1\right)\right] dx$

$= \pi \int_{-1}^{2} \left(-x^4 - x^2 + 6x + 8\right) dx = \pi\left[-\frac{x^5}{5} - \frac{x^3}{3} + \frac{6x^2}{2} + 8x\right]_{-1}^{2} = \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + \frac{24}{2} + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + \frac{6}{2} - 8\right)\right]$

$= \pi\left(-\frac{33}{5} - 3 + 28 - 3 + 8\right) = \pi\left(\frac{5 \cdot 30 - 33}{5}\right) = \frac{117\pi}{5}$

36. $r(x) = 2 - x$ and $R(x) = 4 - x^2 \Rightarrow V = \int_{-1}^{2} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$



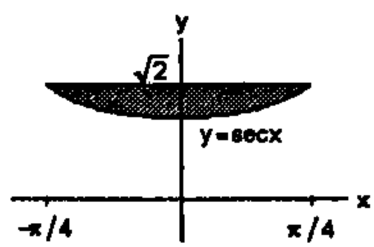
$= \pi \int_{-1}^{2} \left[\left(4-x^2\right)^2 - (2-x)^2\right] dx$

$= \pi \int_{-1}^{2} \left[\left(16 - 8x^2 + x^4\right) - \left(4 - 4x + x^2\right)\right] dx$

$= \pi \int_{-1}^{2} \left(12 + 4x - 9x^2 + x^4\right) dx = \pi\left[12x + 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{-1}^{2} = \pi\left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right]$

$= \pi\left(15 + \frac{33}{5}\right) = \frac{108\pi}{5}$

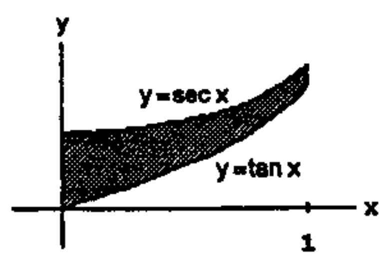
37. $r(x) = \sec x$ and $R(x) = \sqrt{2} \Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$



$= \pi \int_{-\pi/4}^{\pi/4} \left(2 - \sec^2 x\right) dx = \pi[2x - \tan x]_{-\pi/4}^{\pi/4}$

$= \pi\left[\left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)\right] = \pi(\pi - 2)$

38. $R(x) = \sec x$ and $r(x) = \tan x \Rightarrow V = \int_{0}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

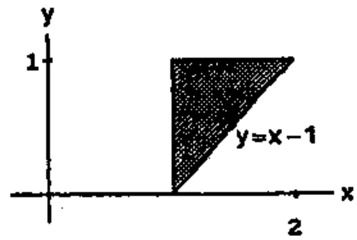


$= \pi \int_{0}^{1} \left(\sec^2 x - \tan^2 x\right) dx = \pi \int_{0}^{1} 1\, dx = \pi[x]_0^1 = \pi$

39. $r(y) = 1$ and $R(y) = 1 + y \Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^1 \left[(1+y)^2 - 1\right] dy = \pi \int_0^1 \left(1 + 2y + y^2 - 1\right) dy$

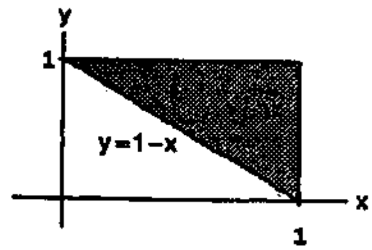
$= \pi \int_0^1 \left(2y + y^2\right) dy = \pi\left[y^2 + \frac{y^3}{3}\right]_0^1 = \pi\left(1 + \frac{1}{3}\right) = \frac{4\pi}{3}$



40. $R(y) = 1$ and $r(y) = 1 - y \Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

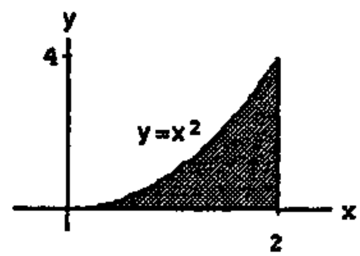
$= \pi \int_0^1 \left[1 - (1-y)^2\right] dy = \pi \int_0^1 \left[1 - \left(1 - 2y + y^2\right)\right] dy$

$= \pi \int_0^1 \left(2y - y^2\right) dy = \pi\left[y^2 - \frac{y^3}{3}\right]_0^1 = \pi\left(1 - \frac{1}{3}\right) = \frac{2\pi}{3}$



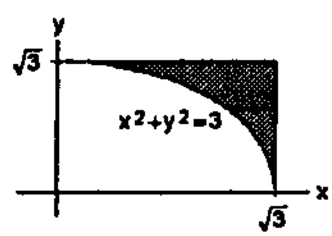
41. $R(y) = 2$ and $r(y) = \sqrt{y} \Rightarrow V = \int_0^4 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^4 (4 - y)\, dy = \pi\left[4y - \frac{y^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$



42. $R(y) = \sqrt{3}$ and $r(y) = \sqrt{3 - y^2} \Rightarrow V = \int_0^{\sqrt{3}} \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^{\sqrt{3}} \left[3 - \left(3 - y^2\right)\right] dy = \pi \int_0^{\sqrt{3}} y^2\, dy = \pi\left[\frac{y^3}{3}\right]_0^{\sqrt{3}} = \pi\sqrt{3}$

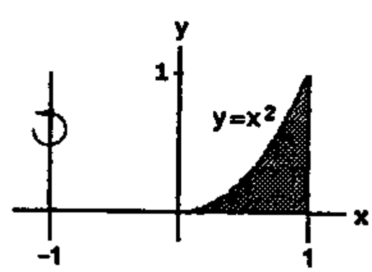


43. $R(y) = 2$ and $r(y) = 1 + \sqrt{y} \Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

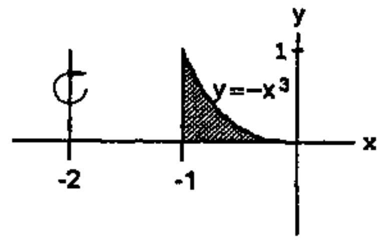
$= \pi \int_0^1 \left[4 - \left(1 + \sqrt{y}\right)^2\right] dy = \pi \int_0^1 \left(4 - 1 - 2\sqrt{y} - y\right) dy$

$= \pi \int_0^1 \left(3 - 2\sqrt{y} - y\right) dy = \pi\left[3y - \frac{4}{3}y^{3/2} - \frac{y^2}{2}\right]_0^1$

$= \pi\left(3 - \frac{4}{3} - \frac{1}{2}\right) = \pi\left(\frac{18 - 8 - 3}{6}\right) = \frac{7\pi}{6}$



44. $R(y) = 2 - y^{1/3}$ and $r(y) = 1 \Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

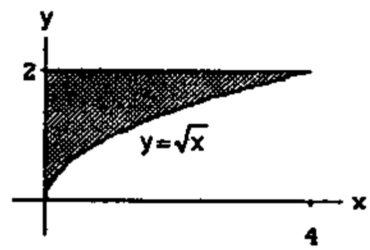


$= \pi \int_0^1 \left[\left(2 - y^{1/3}\right)^2 - 1\right] dy = \pi \int_0^1 \left(4 - 4y^{1/3} + y^{2/3} - 1\right) dy$

$= \pi \int_0^1 \left(3 - 4y^{1/3} + y^{2/3}\right) dy = \pi\left[3y - 3y^{4/3} + \frac{3y^{5/3}}{5}\right]_0^1$

$= \pi\left(3 - 3 + \frac{3}{5}\right) = \frac{3\pi}{5}$

45. (a) $r(x) = \sqrt{x}$ and $R(x) = 2 \Rightarrow V = \int_0^4 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$



$= \pi \int_0^4 (4 - x)\, dx = \pi\left[4x - \frac{x^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$

(b) $r(y) = 0$ and $R(y) = y^2 \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^2 y^4\, dy = \pi\left[\frac{y^5}{5}\right]_0^2 = \frac{32\pi}{5}$

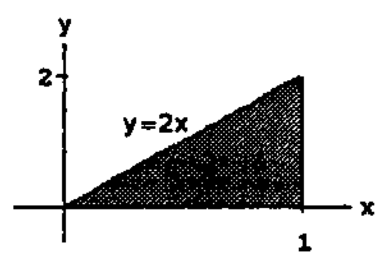
(c) $r(x) = 0$ and $R(x) = 2 - \sqrt{x} \Rightarrow V = \int_0^4 \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_0^4 \left(2 - \sqrt{x}\right)^2 dx$

$= \pi \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx = \pi\left[4x - \frac{8x^{3/2}}{3} + \frac{x^2}{2}\right]_0^4 = \pi\left(16 - \frac{64}{3} + \frac{16}{2}\right) = \frac{8\pi}{3}$

(d) $r(y) = 4 - y^2$ and $R(y) = 4 \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[16 - \left(4 - y^2\right)^2\right] dy$

$= \pi \int_0^2 \left(16 - 16 + 8y^2 - y^4\right) dy = \pi \int_0^2 \left(8y^2 - y^4\right) dy = \pi\left[\frac{8}{3}y^3 - \frac{y^5}{5}\right]_0^2 = \pi\left(\frac{64}{3} - \frac{32}{5}\right) = \frac{224\pi}{15}$

46. (a) $r(y) = 0$ and $R(y) = 1 - \frac{y}{2} \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$



$= \pi \int_0^2 \left(1 - \frac{y}{2}\right)^2 dy = \pi \int_0^2 \left(1 - y + \frac{y^2}{4}\right) dy$

$= \pi\left[y - \frac{y^2}{2} + \frac{y^3}{12}\right]_0^2 = \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$

(b) $r(y) = 1$ and $R(y) = 2 - \frac{y}{2} \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^2 \left[\left(2 - \frac{y}{2}\right)^2 - 1\right] dy = \pi \int_0^2 \left(4 - 2y + \frac{y^2}{4} - 1\right) dy = \pi \int_0^2 \left(3 - 2y + \frac{y^2}{4}\right) dy = \pi\left[3y - y^2 + \frac{y^3}{12}\right]_0^2$

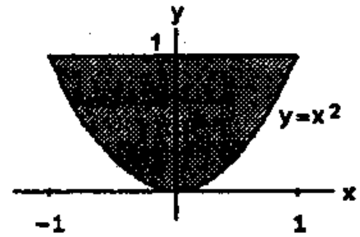
$= \pi\left(6 - 4 + \frac{8}{12}\right) = \pi\left(2 + \frac{2}{3}\right) = \frac{8\pi}{3}$

47. (a) $r(x) = 0$ and $R(x) = 1 - x^2 \Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_{-1}^{1} \left(1 - x^2\right)^2 dx = \pi \int_{-1}^{1} \left(1 - 2x^2 + x^4\right) dx$

$= \pi\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 2\pi\left(\frac{15 - 10 + 3}{15}\right)$

$= \frac{16\pi}{15}$



(b) $r(x) = 1$ and $R(x) = 2 - x^2 \Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^{1} \left[\left(2 - x^2\right)^2 - 1\right] dx$

$= \pi \int_{-1}^{1} \left(4 - 4x^2 + x^4 - 1\right) dx = \pi \int_{-1}^{1} \left(3 - 4x^2 + x^4\right) dx = \pi\left[3x - \frac{4}{3}x^3 + \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(3 - \frac{4}{3} + \frac{1}{5}\right)$

$= \frac{2\pi}{15}(45 - 20 + 3) = \frac{56\pi}{15}$

(c) $r(x) = 1 + x^2$ and $R(x) = 2 \Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^{1} \left[4 - \left(1 + x^2\right)^2\right] dx$

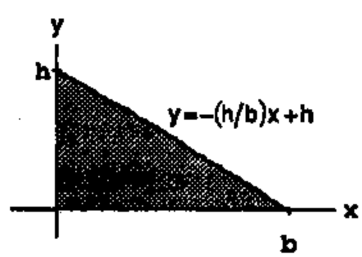
$= \pi \int_{-1}^{1} \left(4 - 1 - 2x^2 - x^4\right) dx = \pi \int_{-1}^{1} \left(3 - 2x^2 - x^4\right) dx = \pi\left[3x - \frac{2}{3}x^3 - \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(3 - \frac{2}{3} - \frac{1}{5}\right)$

$= \frac{2\pi}{15}(45 - 10 - 3) = \frac{64\pi}{15}$

48. (a) $r(x) = 0$ and $R(x) = -\frac{h}{b}x + h \Rightarrow V = \int_0^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_0^b \left(-\frac{h}{b}x + h\right)^2 dx = \pi \int_0^b \left(\frac{h^2}{b^2}x^2 - \frac{2h^2}{b}x + h^2\right) dx$

$= \pi h^2\left[\frac{x^3}{3b^2} - \frac{x^2}{b} + x\right]_0^b = \pi h^2\left(\frac{b}{3} - b + b\right) = \frac{\pi h^2 b}{3}$



(b) $r(y) = 0$ and $R(y) = b\left(1 - \frac{y}{h}\right) \Rightarrow V = \int_0^h \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi b^2 \int_0^h \left(1 - \frac{y}{h}\right)^2 dy$

$= \pi b^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy = \pi b^2\left[y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right]_0^h = \pi b^2\left(h - h + \frac{h}{3}\right) = \frac{\pi b^2 h}{3}$

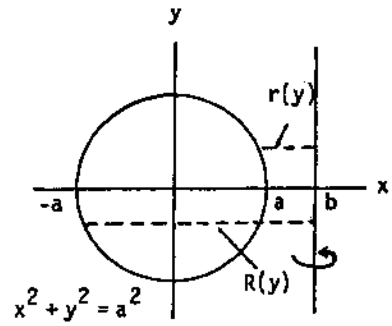
49. $R(y) = b + \sqrt{a^2 - y^2}$ and $r(y) = b - \sqrt{a^2 - y^2}$

$\Rightarrow V = \int_{-a}^{a} \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_{-a}^{a} \left[\left(b + \sqrt{a^2 - y^2}\right)^2 - \left(b - \sqrt{a^2 - y^2}\right)^2\right] dy$

$= \pi \int_{-a}^{a} 4b\sqrt{a^2 - y^2}\, dy = 4b\pi \int_{-a}^{a} \sqrt{a^2 - y^2}\, dy$

$= 4b\pi \cdot$ area of semicircle of radius $a = 4b\pi \cdot \frac{\pi a^2}{2} = 2a^2 b\pi^2$



50. (a) A cross section has radius $r = \sqrt{2y}$ and area $\pi r^2 = 2\pi y$. The volume is $\int_0^5 2\pi y\, dy = \pi\left[y^2\right]_0^5 = 25\pi$.

(b) $V(h) = \int A(h)\, dh$, so $\frac{dV}{dh} = A(h)$. Therefore $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = A(h) \cdot \frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{1}{A(h)} \cdot \frac{dV}{dt}$

For $h = 4$, the area is $2\pi(4) = 8\pi$, so $\frac{dh}{dt} = \frac{1}{8\pi} \cdot 3\ \frac{\text{units}^3}{\text{sec}} = \frac{3}{8\pi} \cdot \frac{\text{units}^3}{\text{sec}}$.

51. (a) $R(y) = \sqrt{a^2 - y^2} \Rightarrow V = \pi \int_{-a}^{h-a} \left(a^2 - y^2\right) dy = \pi\left[a^2 y - \frac{y^3}{3}\right]_{-a}^{h-a} = \pi\left[a^2 h - a^3 - \frac{(h-a)^3}{3} - \left(-a^3 + \frac{a^3}{3}\right)\right]$

$= \pi\left[a^2 h - \frac{1}{3}\left(h^3 - 3h^2 a + 3ha^2 - a^3\right) - \frac{a^3}{3}\right] = \pi\left(a^2 h - \frac{h^3}{3} + h^2 a - ha^2\right) = \frac{\pi h^2(3a - h)}{3}$

(b) Given $\frac{dV}{dt} = 0.2\ \text{m}^3/\text{sec}$ and $a = 5$ m, find $\left.\frac{dh}{dt}\right|_{h=4}$. From part (a), $V(h) = \frac{\pi h^2(15 - h)}{3} = 5\pi h^2 - \frac{\pi h^3}{3}$

$\Rightarrow \frac{dV}{dh} = 10\pi h - \pi h^2 \Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \pi h(10 - h)\, \frac{dh}{dt} \Rightarrow \left.\frac{dh}{dt}\right|_{h=4} = \frac{0.2}{4\pi(10 - 4)} = \frac{1}{(20\pi)(6)} = \frac{1}{120\pi}$ m/sec.

52. Partition the appropriate interval in the axis of revolution and measure the radius $r(x)$ of the shadow region at these points. Then use an approximation such as the trapezoidal rule to estimate the integral $\int_a^b \pi r^2(x)\, dx$.

53. The cross section of a solid right circular cylinder with a cone removed is a disk with radius R from which a disk of radius h has been removed (figure provided). Thus its area is $A_1 = \pi R^2 - \pi h^2 = \pi\left(R^2 - h^2\right)$. The cross section of the hemisphere is a disk of radius $\sqrt{R^2 - h^2}$ (figure provided). Therefore its area is

$A_2 = \pi\left(\sqrt{R^2-h^2}\right)^2 = \pi\left(R^2-h^2\right)$. We can see that $A_1 = A_2$. The altitudes of both solids are R. Applying Cavalieri's Theorem we find Volume of Hemisphere = (Volume of Cylinder) − (Volume of Cone) $= \left(\pi R^2\right)R - \frac{1}{3}\pi\left(R^2\right)R = \frac{2}{3}\pi R^3$.

54. (a) $R(x) = \sqrt{a^2-x^2} \Rightarrow V = \int_{-a}^{a} \pi[R(x)]^2\,dx = \pi\int_{-a}^{a}\left(a^2-x^2\right)dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$

$= \pi\left[\left(a^3-\frac{a^3}{3}\right)-\left(-a^3+\frac{a^3}{3}\right)\right] = 2\pi\left(\frac{2a^3}{3}\right) = \frac{4}{3}\pi a^3$, the volume of a sphere of radius a

(b) $R(x) = \frac{rx}{h} \Rightarrow V = \int_0^h \pi[R(x)]^2\,dx = \pi\int_0^h \frac{r^2x^2}{h^2}\,dx$

$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \left(\frac{\pi r^2}{h^2}\right)\left(\frac{h^3}{3}\right) = \frac{1}{3}\pi r^2 h$, the volume of a cone of radius r and height h

55. $R(y) = \sqrt{256-y^2} \Rightarrow V = \int_{-16}^{-7} \pi[R(y)]^2\,dy = \pi\int_{-16}^{-7}\left(256-y^2\right)dy = \pi\left[256y - \frac{y^3}{3}\right]_{-16}^{-7}$

$= \pi\left[(256)(-7)+\frac{7^3}{3}-\left((256)(-16)+\frac{16^3}{3}\right)\right] = \pi\left(\frac{7^3}{3}+256(16-7)-\frac{16^3}{3}\right) = 1053\pi\ \text{cm}^3 \approx 3308\ \text{cm}^3$

56. $R(x) = \frac{x}{12}\sqrt{36-x^2} \Rightarrow V = \int_0^6 \pi[R(x)]^2\,dx = \pi\int_0^6 \frac{x^2}{144}\left(36-x^2\right)dx = \frac{\pi}{144}\int_0^6\left(36x^2-x^4\right)dx$

$= \frac{\pi}{144}\left[12x^3-\frac{x^5}{5}\right]_0^6 = \frac{\pi}{144}\left(12\cdot 6^3-\frac{6^5}{5}\right) = \frac{\pi\cdot 6^3}{144}\left(12-\frac{36}{5}\right) = \left(\frac{216\pi}{144}\right)\left(\frac{60-36}{5}\right) = \frac{36\pi}{5}\ \text{cm}^3$. The plumb bob will weigh about $W = (8.5)\left(\frac{36\pi}{5}\right) \approx 192$ gm, to the nearest gram

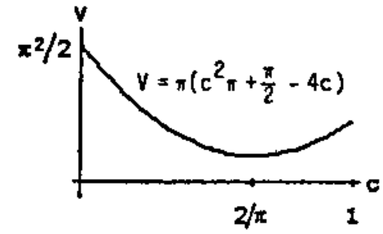
57. (a) $R(x) = |c-\sin x|$, so $V = \pi\int_0^\pi [R(x)]^2\,dx = \pi\int_0^\pi (c-\sin x)^2\,dx = \pi\int_0^\pi\left(c^2-2c\sin x+\sin^2 x\right)dx$

$= \pi\int_0^\pi\left(c^2-2c\sin x+\frac{1-\cos 2x}{2}\right)dx = \pi\int_0^\pi\left(c^2+\frac{1}{2}-2c\sin x-\frac{\cos 2x}{2}\right)dx$

$= \pi\left[\left(c^2+\frac{1}{2}\right)x+2c\cos x-\frac{\sin 2x}{4}\right]_0^\pi = \pi\left[\left(c^2\pi+\frac{\pi}{2}-2c-0\right)-(0+2c-0)\right] = \pi\left(c^2\pi+\frac{\pi}{2}-4c\right)$. Let $V(c) = \pi\left(c^2\pi+\frac{\pi}{2}-4c\right)$. We find the extreme values of $V(c)$: $\frac{dV}{dc} = \pi(2c\pi-4) = 0 \Rightarrow c = \frac{2}{\pi}$ is a critical point, and $V\left(\frac{2}{\pi}\right) = \pi\left(\frac{4}{\pi}+\frac{\pi}{2}-\frac{8}{\pi}\right) = \pi\left(\frac{\pi}{2}-\frac{4}{\pi}\right) = \frac{\pi^2}{2}-4$; Evaluate V at the endpoints: $V(0) = \frac{\pi^2}{2}$ and $V(1) = \pi\left(\frac{3}{2}\pi-4\right) = \frac{\pi^2}{2}-(4-\pi)\pi$. Now we see that the function's absolute minimum value is $\frac{\pi^2}{2}-4$, taken on at the critical point $c = \frac{2}{\pi}$. (See also the accompanying graph.)

(b) From the discussion in part (a) we conclude that the function's absolute maximum value is $\frac{\pi^2}{2}$, taken on at the endpoint $c = 0$.

(c) The graph of the solid's volume as a function of c for $0 \le c \le 1$ is given at the right. As c moves away from $[0, 1]$ the volume of the solid increases without bound. If we approximate the solid as a set of solid disks, we can see that the radius of a typical disk increases without bounds as c moves away from $[0, 1]$.

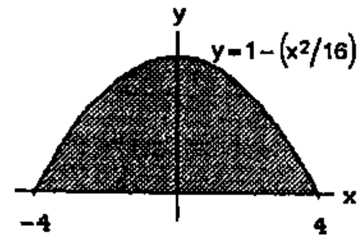


58. (a) $R(x) = 1 - \frac{x^2}{16} \Rightarrow V = \int_{-4}^{4} \pi[R(x)]^2\,dx = \pi \int_{-4}^{4} \left(1 - \frac{x^2}{16}\right)^2 dx$

$= \pi \int_{-4}^{4} \left(1 - \frac{x^2}{8} + \frac{x^4}{16^2}\right) dx = \pi\left[x - \frac{x^3}{24} + \frac{x^5}{5 \cdot 16^2}\right]_{-4}^{4}$

$= 2\pi\left(4 - \frac{4^3}{24} + \frac{4^5}{5 \cdot 16^2}\right) = 2\pi\left(4 - \frac{8}{3} + \frac{4}{5}\right)$

$= \frac{2\pi}{15}(60 - 40 + 12) = \frac{64\pi}{15}\ \text{ft}^3$



(b) The helicopter will be able to fly $\left(\frac{64\pi}{15}\right)(7.481)(2) \approx 201$ additional miles.

59. (a) Using $d = \frac{C}{\pi}$, and $A = \pi\left(\frac{d}{2}\right)^2 = \frac{C^2}{4\pi}$ yields the following areas (in square inches, rounded to the nearest tenth: 2.3, 1.6, 1.5, 2.1, 3.2, 4.8, 7.0, 9.3, 10.7, 10.7, 9.3, 6.4, 3.2.

(b) If C(y) is the circumference as a function of y, then the area of a cross section is

$A(y) = \pi\left(\frac{C(y)/\pi}{2}\right)^2 = \frac{C^2(y)}{4\pi}$, and the volume is $\frac{1}{4\pi}\int_0^6 C^2(y)\,dy$.

(c) $\int_0^6 A(y)\,dy = \frac{1}{4\pi}\int_0^6 C^2(y)\,dy \approx \frac{1}{4\pi}\left(\frac{6-0}{24}\right)[5.4^2 + 2(4.5^2 + 4.4^2 + 5.1^2 + 6.3^2 + 7.8^2 + 9.4^2 + 10.8^2$
$+ 11.6^2 + 11.6^2 + 10.8^2 + 9.0^2) + 6.3^2] \approx 34.7\ \text{in.}^3$

(d) $V = \frac{1}{4\pi}\int_0^6 C^2(y)\,dy \approx \frac{1}{4\pi}\left(\frac{6-0}{36}\right)\Big[5.4^2 + 4(4.5^2) + 2(4.4^2) + 4(5.1^2) + 2(6.3^2) + 4(7.8^2) + 2(9.4^2)$
$+ 4(10.8^2) + 2(11.6^2) + 4(11.6^2) + 2(10.8^2) + 4(9.0^2) + 6.3^2\Big] = 34.792\ \text{in.}^3$

by Simpson's rule. The Simpson's rule estimate should be more accurate than the trapezoid estimate. The error in the Simpson's estimate is proportional to $h^4 = 0.0625$ whereas the error in the trapezoid estimate is proportional to $h^2 = 0.25$, a larger number when $h = 0.5$ in.

60. (a) Displacement Volume $V \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$, $x_0 = 0$, $x_n = 10 - h$,

$h = 2.54$, $n = 10 \Rightarrow V = \int_{x_0}^{x_n} A(x)\,dx \approx \frac{2.54}{3}[0 + 4(1.07) + 2(3.84) + 4(7.82) + 2(12.20) + 4(15.18)$

$+ 2(16.14) + 4(14.00) + 2(9.21) + 4(3.24) + 0] = \frac{2.54}{3}(4.28 + 7.68 + 31.28 + 24.4 + 60.72 + 32.28$

$+ 56 + 18.42 + 12.96) = \frac{2.54}{3}(248.02) = 209.99 \approx 210\ \text{ft}^3$

(b) The weight of water displaced is approximately $64 \cdot 210 = 13{,}440$ lb

(c) The volume of a prism $= (25.4)\cdot(16.14) = 409.96 \approx 410\ \text{ft}^3$. Thus, the prismatic coefficient is $\frac{210\ \text{ft}^3}{410\ \text{ft}^3} \approx 0.51$

5.2 MODELING VOLUME USING CYLINDRICAL SHELLS

1. For the sketch given, $a = 0$, $b = 2$;

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx = \int_0^2 2\pi x\left(1 + \frac{x^2}{4}\right)dx = 2\pi\int_0^2 \left(x + \frac{x^3}{4}\right)dx = 2\pi\left[\frac{x^2}{2} + \frac{x^4}{16}\right]_0^2 = 2\pi\left(\frac{4}{2} + \frac{16}{16}\right)$$

$= 2\pi \cdot 3 = 6\pi$

2. For the sketch given, $a = 0$, $b = 2$;

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx = \int_0^2 2\pi x\left(2 - \frac{x^2}{4}\right)dx = 2\pi\int_0^2 \left(2x - \frac{x^3}{4}\right)dx = 2\pi\left[x^2 - \frac{x^4}{16}\right]_0^2 = 2\pi(4-1) = 6\pi$$

3. For the sketch given, $c = 0$, $d = \sqrt{2}$;

$$V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^{\sqrt{2}} 2\pi y\cdot(y^2)\,dy = 2\pi\int_0^{\sqrt{2}} y^3\,dy = 2\pi\left[\frac{y^4}{4}\right]_0^{\sqrt{2}} = 2\pi$$

4. For the sketch given, $c = 0$, $d = \sqrt{3}$;

$$V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^{\sqrt{3}} 2\pi y\cdot\left[3 - \left(3 - y^2\right)\right]dy = 2\pi\int_0^{\sqrt{3}} y^3\,dy = 2\pi\left[\frac{y^4}{4}\right]_0^{\sqrt{3}} = \frac{9\pi}{2}$$

5. For the sketch given, $a = 0$, $b = \sqrt{3}$;

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx = \int_0^{\sqrt{3}} 2\pi x\cdot\left(\sqrt{x^2+1}\right)dx;$$

$[u = x^2 + 1 \Rightarrow du = 2x\,dx;\ x = 0 \Rightarrow u = 1,\ x = \sqrt{3} \Rightarrow u = 4]$

$$\Rightarrow V = \pi\int_1^4 u^{1/2}\,du = \pi\left[\frac{2}{3}u^{3/2}\right]_1^4 = \frac{2\pi}{3}\left(4^{3/2} - 1\right) = \left(\frac{2\pi}{3}\right)(8-1) = \frac{14\pi}{3}$$

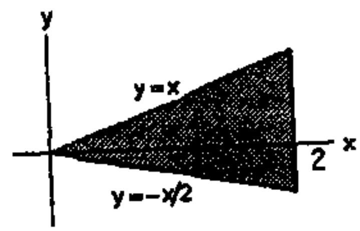
6. For the sketch given, $a = 0$, $b = 3$;

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^3 2\pi x \left(\frac{9x}{\sqrt{x^3+9}} \right) dx;$$

$\left[u = x^3 + 9 \Rightarrow du = 3x^2\, dx \Rightarrow 3\, du = 9x^2\, dx;\ x = 0 \Rightarrow u = 9,\ x = 3 \Rightarrow u = 36\right]$

$$\Rightarrow V = 2\pi \int_9^{36} 3u^{-1/2}\, du = 6\pi \left[2u^{1/2}\right]_9^{36} = 12\pi\left(\sqrt{36} - \sqrt{9}\right) = 36\pi$$

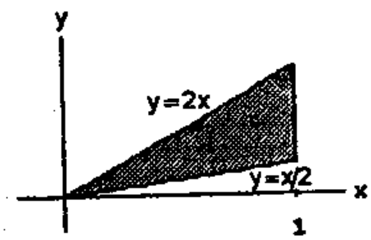
7. $a = 0$, $b = 2$;



$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^2 2\pi x \left[x - \left(-\frac{x}{2}\right)\right] dx$$

$$= \int_0^2 2\pi x^2 \cdot \frac{3}{2}\, dx = \pi \int_0^2 3x^2\, dx = \pi \left[x^3\right]_0^2 = 8\pi$$

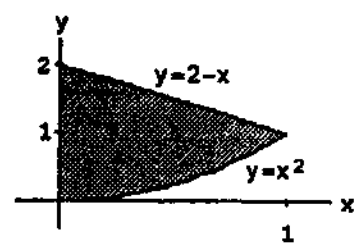
8. $a = 0$, $b = 1$;



$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi x \left(2x - \frac{x}{2}\right) dx$$

$$= \pi \int_0^1 2\left(\frac{3x^2}{2}\right) dx = \pi \int_0^1 3x^2\, dx = \pi\left[x^3\right]_0^1 = \pi$$

9. $a = 0$, $b = 1$;

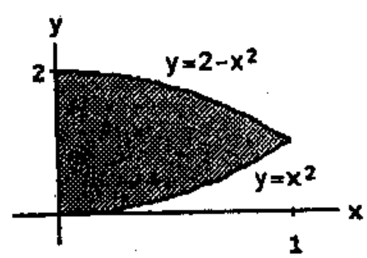


$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi x \left[(2 - x) - x^2\right] dx$$

$$= 2\pi \int_0^1 \left(2x - x^2 - x^3\right) dx = 2\pi \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = 2\pi\left(\frac{12 - 4 - 3}{12}\right) = \frac{10\pi}{12} = \frac{5\pi}{6}$$

10. $a = 0$, $b = 1$;



$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi x \left[\left(2 - x^2\right) - x^2\right] dx$$

$$= 2\pi \int_0^1 x\left(2 - 2x^2\right) dx = 4\pi \int_0^1 \left(x - x^3\right) dx$$

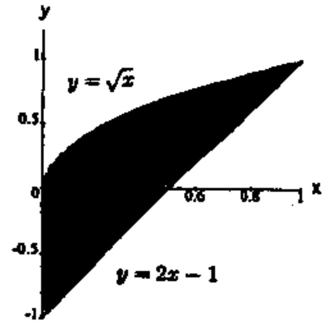
$$= 4\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 4\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \pi$$

11. $a = 0$, $b = 1$;

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi x\left[\sqrt{x} - (2x-1)\right] dx$$

$$= 2\pi \int_0^1 \left(x^{3/2} - 2x^2 + x\right) dx = 2\pi\left[\frac{2}{5}x^{5/2} - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1$$

$$= 2\pi\left(\frac{2}{5} - \frac{2}{3} + \frac{1}{2}\right) = 2\pi\left(\frac{12-20+15}{30}\right) = \frac{7\pi}{15}$$

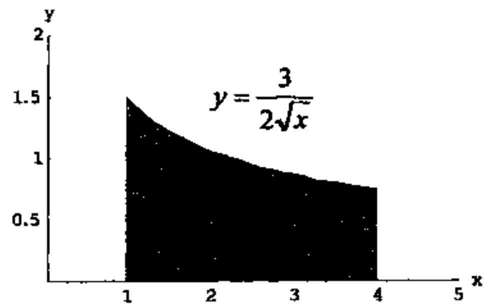


12. $a = 1$, $b = 4$;

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_1^4 2\pi x\left(\frac{3}{2}x^{-1/2}\right) dx$$

$$= 3\pi \int_1^4 x^{1/2}\, dx = 3\pi\left[\frac{2}{3}x^{3/2}\right]_1^4 = 2\pi\left(4^{3/2} - 1\right)$$

$$= 2\pi(8-1) = 14\pi$$



13. (a) $xf(x) = \begin{cases} x \cdot \frac{\sin x}{x}, & 0 < x \le \pi \\ x, & x = 0 \end{cases} \Rightarrow xf(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ 0, & x = 0 \end{cases}$; since $\sin 0 = 0$ we have

$$xf(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ \sin x, & x = 0 \end{cases} \Rightarrow xf(x) = \sin x,\ 0 \le x \le \pi$$

(b) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^\pi 2\pi x \cdot f(x)\, dx$ and $x \cdot f(x) = \sin x$, $0 \le x \le \pi$ by part (a)

$$\Rightarrow V = 2\pi \int_0^\pi \sin x\, dx = 2\pi[-\cos x]_0^\pi = 2\pi(-\cos \pi + \cos 0) = 4\pi$$

14. (a) $xg(x) = \begin{cases} x \cdot \frac{\tan^2 x}{x}, & 0 < x \le \frac{\pi}{4} \\ x \cdot 0, & x = 0 \end{cases} \Rightarrow xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ 0, & x = 0 \end{cases}$; since $\tan 0 = 0$ we have

$$xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ \tan^2 x, & x = 0 \end{cases} \Rightarrow xg(x) = \tan^2 x,\ 0 \le x \le \pi/4$$

(b) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^{\pi/4} 2\pi x \cdot g(x)\, dx$ and $x \cdot g(x) = \tan^2 x$, $0 \le x \le \pi/4$ by part (a)

$$\Rightarrow V = 2\pi \int_0^{\pi/4} \tan^2 x\, dx = 2\pi \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx = 2\pi[\tan x - x]_0^{\pi/4} = 2\pi\left(1 - \frac{\pi}{4}\right) = \frac{4\pi - \pi^2}{2}$$

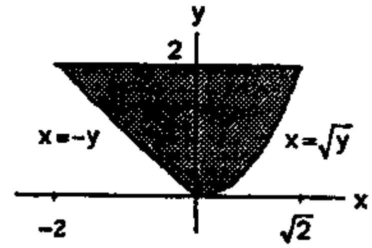
15. $c = 0,\ d = 2;$

$$V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y\left[\sqrt{y} - (-y)\right] dy$$

$$= 2\pi \int_0^2 \left(y^{3/2} + y^2\right) dy = 2\pi\left[\frac{2y^{5/2}}{5} + \frac{y^3}{3}\right]_0^2$$

$$= 2\pi\left[\frac{2}{5}\left(\sqrt{2}\right)^5 + \frac{2^3}{3}\right] = 2\pi\left(\frac{8\sqrt{2}}{5} + \frac{8}{3}\right) = 16\pi\left(\frac{\sqrt{2}}{5} + \frac{1}{3}\right)$$

$$= \frac{16\pi}{15}\left(3\sqrt{2} + 5\right)$$

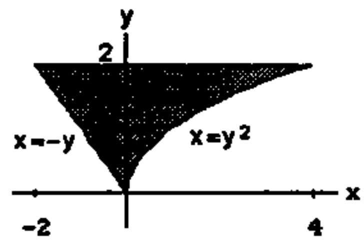


16. $c = 0,\ d = 2;$

$$V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y\left[y^2 - (-y)\right] dy$$

$$= 2\pi \int_0^2 \left(y^3 + y^2\right) dy = 2\pi\left[\frac{y^4}{4} + \frac{y^3}{3}\right]_0^2 = 16\pi\left(\frac{2}{4} + \frac{1}{3}\right)$$

$$= 16\pi\left(\frac{5}{6}\right) = \frac{40\pi}{3}$$

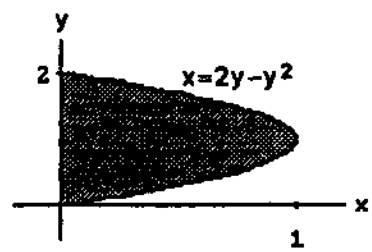


17. $c = 0,\ d = 2;$

$$V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y\left(2y - y^2\right) dy$$

$$= 2\pi \int_0^2 \left(2y^2 - y^3\right) dy = 2\pi\left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = 2\pi\left(\frac{16}{3} - \frac{16}{4}\right)$$

$$= 32\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{32\pi}{12} = \frac{8\pi}{3}$$

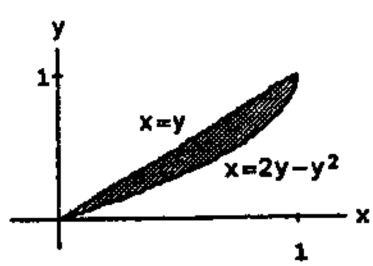


18. $c = 0,\ d = 1;$

$$V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi y\left(2y - y^2 - y\right) dy$$

$$= 2\pi \int_0^1 y\left(y - y^2\right) dy = 2\pi \int_0^1 \left(y^2 - y^3\right) dy$$

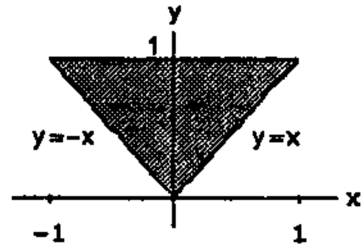
$$= 2\pi\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$$



19. $c = 0$, $d = 1$;

$$V = \int_c^d 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy = 2\pi \int_0^1 y[y - (-y)]\,dy$$

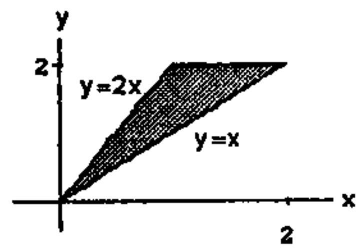
$$= 2\pi \int_0^1 2y^2\,dy = \frac{4\pi}{3}\left[y^3\right]_0^1 = \frac{4\pi}{3}$$



20. $c = 0$, $d = 2$;

$$V = \int_c^d 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy = \int_0^2 2\pi y\left(y - \frac{y}{2}\right) dy$$

$$= 2\pi \int_0^2 \frac{y^2}{2}\,dy = \frac{\pi}{3}\left[y^3\right]_0^2 = \frac{8\pi}{3}$$

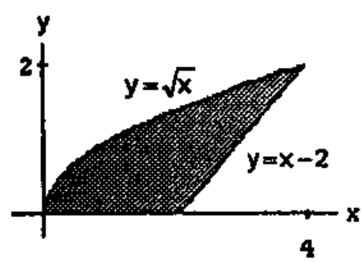


21. $c = 0$, $d = 2$;

$$V = \int_c^d 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy = \int_0^2 2\pi y\left[(2+y) - y^2\right] dy$$

$$= 2\pi \int_0^2 \left(2y + y^2 - y^3\right) dy = 2\pi\left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2$$

$$= 2\pi\left(4 + \frac{8}{3} - \frac{16}{4}\right) = \frac{\pi}{6}(48 + 32 - 48) = \frac{16\pi}{3}$$

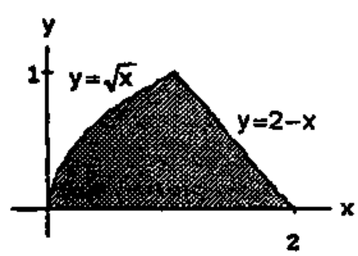


22. $c = 0$, $d = 1$;

$$V = \int_c^d 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy = \int_0^1 2\pi y\left[(2-y) - y^2\right] dy$$

$$= 2\pi \int_0^1 \left(2y - y^2 - y^3\right) dy = 2\pi\left[y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$$

$$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}(12 - 4 - 3) = \frac{5\pi}{6}$$



23. (a) $V = \int_c^d 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy = \int_0^1 2\pi y \cdot 12\left(y^2 - y^3\right) dy = 24\pi \int_0^1 \left(y^3 - y^4\right) dy = 24\pi\left[\frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$

$$= 24\pi\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{24\pi}{20} = \frac{6\pi}{5}$$

(b) $V = \int_c^d 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy = \int_0^1 2\pi(1-y)\left[12\left(y^2 - y^3\right)\right] dy = 24\pi \int_0^1 (1-y)\left(y^2 - y^3\right) dy$

$$= 24\pi \int_0^1 \left(y^2 - 2y^3 + y^4\right) dy = 24\pi\left[\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right]_0^1 = 24\pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = 24\pi\left(\frac{1}{30}\right) = \frac{4\pi}{5}$$

(c) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi\left(\frac{8}{5} - y\right)\left[12\left(y^2 - y^3\right)\right] dy = 24\pi \int_0^1 \left(\frac{8}{5} - y\right)\left(y^2 - y^3\right) dy$

$$= 24\pi \int_0^1 \left(\frac{8}{5}y^2 - \frac{13}{5}y^3 + y^4\right) dy = 24\pi\left[\frac{8}{15}y^3 - \frac{13}{20}y^4 + \frac{y^5}{5}\right]_0^1 = 24\pi\left(\frac{8}{15} - \frac{13}{20} + \frac{1}{5}\right) = \frac{24\pi}{60}(32 - 39 + 12)$$

$$= \frac{24\pi}{12} = 2\pi$$

(d) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi\left(y + \frac{2}{5}\right)\left[12\left(y^2 - y^3\right)\right] dy = 24\pi \int_0^1 \left(y + \frac{2}{5}\right)\left(y^2 - y^3\right) dy$

$$= 24\pi \int_0^1 \left(y^3 - y^4 + \frac{2}{5}y^2 - \frac{2}{5}y^3\right) dy = 24\pi \int_0^1 \left(\frac{2}{5}y^2 + \frac{3}{5}y^3 - y^4\right) dy = 24\pi\left[\frac{2}{15}y^3 + \frac{3}{20}y^4 - \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{2}{15} + \frac{3}{20} - \frac{1}{5}\right) = \frac{24\pi}{60}(8 + 9 - 12) = \frac{24\pi}{12} = 2\pi$$

24\. (a) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right] dy = \int_0^2 2\pi y\left(y^2 - \frac{y^4}{4}\right) dy = 2\pi \int_0^2 \left(y^3 - \frac{y^5}{4}\right) dy$

$$= 2\pi\left[\frac{y^4}{4} - \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{2^4}{4} - \frac{2^6}{24}\right) = 32\pi\left(\frac{1}{4} - \frac{4}{24}\right) = 32\pi\left(\frac{1}{4} - \frac{1}{6}\right) = 32\pi\left(\frac{2}{24}\right) = \frac{8\pi}{3}$$

(b) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi(2 - y)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right] dy = \int_0^2 2\pi(2 - y)\left(y^2 - \frac{y^4}{4}\right) dy$

$$= 2\pi \int_0^2 \left(2y^2 - \frac{y^4}{2} - y^3 + \frac{y^5}{4}\right) dy = 2\pi\left[\frac{2y^3}{3} - \frac{y^5}{10} - \frac{y^4}{4} + \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{16}{3} - \frac{32}{10} - \frac{16}{4} + \frac{64}{24}\right) = \frac{8\pi}{5}$$

(c) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi(5 - y)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right] dy = \int_0^2 2\pi(5 - y)\left(y^2 - \frac{y^4}{4}\right) dy$

$$= 2\pi \int_0^2 \left(5y^2 - \frac{5}{4}y^4 - y^3 + \frac{y^5}{4}\right) dy = 2\pi\left[\frac{5y^3}{3} - \frac{5y^5}{20} - \frac{y^4}{4} + \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{40}{3} - \frac{160}{20} - \frac{16}{4} + \frac{64}{24}\right) = 8\pi$$

(d) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi\left(y + \frac{5}{8}\right)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right] dy = \int_0^2 2\pi\left(y + \frac{5}{8}\right)\left(y^2 - \frac{y^4}{4}\right) dy$

$$= 2\pi \int_0^2 \left(y^3 - \frac{y^5}{4} + \frac{5}{8}y^2 - \frac{5}{32}y^4\right) dy = 2\pi\left[\frac{y^4}{4} - \frac{y^6}{24} + \frac{5y^3}{24} - \frac{5y^5}{160}\right]_0^2 = 2\pi\left(\frac{16}{4} - \frac{64}{24} + \frac{40}{24} - \frac{160}{160}\right) = 4\pi$$

25. (a) About the x-axis: $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\,dy = \int_0^1 2\pi y\left(\sqrt{y} - y\right)dy = 2\pi\int_0^1 \left(y^{3/2} - y^2\right)dy$

$= 2\pi\left(\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right)\Big|_{y=0}^{1} = 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) = \frac{2\pi}{15}$

About the y-axis: $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\,dx = \int_0^1 2\pi x\left(x - x^2\right)dx = 2\pi\int_0^1 \left(x^2 - x^3\right)dx$

$= 2\pi\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_{x=0}^{1} = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$

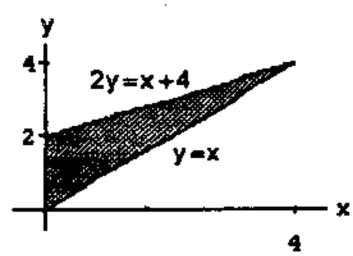
(b) About the x-axis: $R(x) = x$ and $r(x) = x^2 \Rightarrow V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right]dx = \int_0^1 \pi\left[x^2 - x^4\right]dx$

$= \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_{r=0}^{1} = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$

About the y-axis: $R(y) = \sqrt{y}$ and $r(y) = y \Rightarrow V = \int_c^d \pi\left[R(y)^2 - r(y)^2\right]dy = \int_0^1 \pi\left[y - y^2\right]dy$

$= \pi\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{y=0}^{1} = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$

26. (a) $V = \int_a^b \pi\left[R^2(x) - r^2(x)\right]dx = \pi\int_0^4 \left[\left(\frac{x}{2} + 2\right)^2 - x^2\right]dx$



$= \pi\int_0^4 \left(-\frac{3}{4}x^2 + 2x + 4\right)dx = \pi\left[-\frac{x^3}{4} + x^2 + 4x\right]_0^4$

$= \pi(-16 + 16 + 16) = 16\pi$

(b) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\,dx = \int_0^4 2\pi x\left(\frac{x}{2} + 2 - x\right)dx = \int_0^4 2\pi x\left(2 - \frac{x}{2}\right)dx = 2\pi\int_0^4 \left(2x - \frac{x^2}{2}\right)dx$

$= 2\pi\left[x^2 - \frac{x^3}{6}\right]_0^4 = 2\pi\left(16 - \frac{64}{6}\right) = \frac{32\pi}{3}$

(c) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\,dx = \int_0^4 2\pi(4 - x)\left(\frac{x}{2} + 2 - x\right)dx = 2\pi\int_0^4 (4 - x)\left(2 - \frac{x}{2}\right)dx$

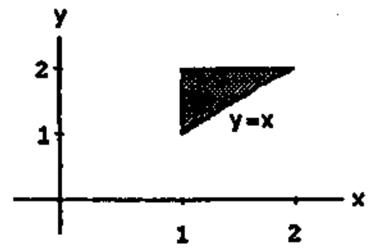
$= 2\pi\int_0^4 \left(8 - 4x + \frac{x^2}{2}\right)dx = 2\pi\left[8x - 2x^2 + \frac{x^3}{6}\right]_0^4 = 2\pi\left(32 - 32 + \frac{64}{6}\right) = \frac{64\pi}{3}$

(d) $V = \int_a^b \pi\left[R^2(x) - r^2(x)\right]dx = \int_0^4 \pi\left[(8 - x)^2 - \left(6 - \frac{x}{2}\right)^2\right]dx$

$= \pi\int_0^4 \left[\left(64 - 16x + x^2\right) - \left(36 - 6x + \frac{x^2}{4}\right)\right]dx = \pi\int_0^4 \left(\frac{3}{4}x^2 - 10x + 28\right)dx = \pi\left[\frac{x^3}{4} - 5x^2 + 28x\right]_0^4$

$$= \pi[16 - (5)(16) + (7)(16)] = \pi(3)(16) = 48\pi$$

27. (a) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_1^2 2\pi y(y-1)\,dy$



$$= 2\pi \int_1^2 (y^2 - y)\,dy = 2\pi\left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2 = 2\pi\left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

$$= 2\pi\left(\frac{7}{3} - 2 + \frac{1}{2}\right) = \frac{\pi}{3}(14 - 12 + 3) = \frac{5\pi}{3}$$

(b) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_1^2 2\pi x(2-x)\,dx = 2\pi \int_1^2 (2x - x^2)\,dx = 2\pi\left[x^2 - \frac{x^3}{3}\right]_1^2$

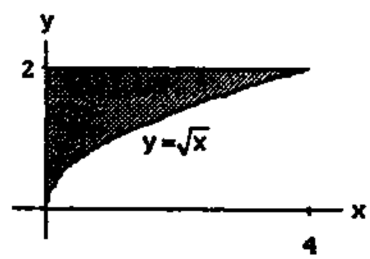
$$= 2\pi\left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = 2\pi\left[\left(\frac{12-8}{3}\right) - \left(\frac{3-1}{3}\right)\right] = 2\pi\left(\frac{4}{3} - \frac{2}{3}\right) = \frac{4\pi}{3}$$

(c) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_1^2 2\pi\left(\frac{10}{3} - x\right)(2-x)\,dx = 2\pi \int_1^2 \left(\frac{20}{3} - \frac{16}{3}x + x^2\right) dx$

$$= 2\pi\left[\frac{20}{3}x - \frac{8}{3}x^2 + \frac{1}{3}x^3\right]_1^2 = 2\pi\left[\left(\frac{40}{3} - \frac{32}{3} + \frac{8}{3}\right) - \left(\frac{20}{3} - \frac{8}{3} + \frac{1}{3}\right)\right] = 2\pi\left(\frac{3}{3}\right) = 2\pi$$

(d) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_1^2 2\pi(y-1)(y-1)\,dy = 2\pi \int_1^2 (y-1)^2 = 2\pi\left[\frac{(y-1)^3}{3}\right]_1^2 = \frac{2\pi}{3}$

28. (a) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y\left(y^2 - 0\right) dy$



$$= 2\pi \int_0^2 y^3\,dy = 2\pi\left[\frac{y^4}{4}\right]_0^2 = 2\pi\left(\frac{2^4}{4}\right) = 8\pi$$

(b) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi x\left(2 - \sqrt{x}\right) dx = 2\pi \int_0^4 \left(2x - x^{3/2}\right) dx = 2\pi\left[x^2 - \frac{2}{5}x^{5/2}\right]_0^4$

$$= 2\pi\left(16 - \frac{2 \cdot 2^5}{5}\right) = 2\pi\left(16 - \frac{64}{5}\right) = \frac{2\pi}{5}(80 - 64) = \frac{32\pi}{5}$$

(c) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi(4-x)\left(2 - \sqrt{x}\right) dx = 2\pi \int_0^4 \left(8 - 4x^{1/2} - 2x + x^{3/2}\right) dx$

$$= 2\pi\left[8x - \frac{8}{3}x^{3/2} - x^2 + \frac{2}{5}x^{5/2}\right]_0^4 = 2\pi\left(32 - \frac{64}{3} - 16 + \frac{64}{5}\right) = \frac{2\pi}{15}(240 - 320 + 192) = \frac{2\pi}{15}(112) = \frac{224\pi}{15}$$

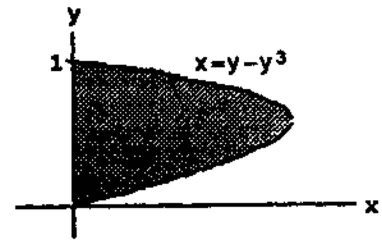
(d) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi(2-y)\left(y^2\right) dy = 2\pi \int_0^2 \left(2y^2 - y^3\right) dy = 2\pi\left[\frac{2}{3}y^3 - \frac{y^4}{4}\right]_0^2$

$$= 2\pi\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{32\pi}{12}(4 - 3) = \frac{8\pi}{3}$$

29. (a) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^1 2\pi y(y - y^3)\,dy$

$= \int_0^1 2\pi(y^2 - y^4)\,dy = 2\pi\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{5}\right)$

$= \frac{4\pi}{15}$



(b) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^1 2\pi(1-y)(y - y^3)\,dy = 2\pi\int_0^1 (y - y^2 - y^3 + y^4)\,dy$

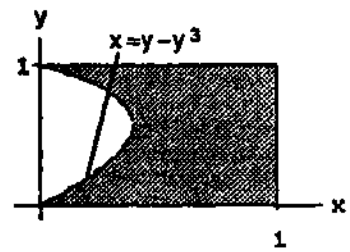
$= 2\pi\left[\frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5}\right]_0^1 = 2\pi\left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right) = \frac{2\pi}{60}(30 - 20 - 15 + 12) = \frac{7\pi}{30}$

30. (a) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^1 2\pi y[1 - (y - y^3)]\,dy$

$= 2\pi\int_0^1 (y - y^2 + y^4)\,dy = 2\pi\left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^5}{5}\right]_0^1$

$= 2\pi\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{5}\right) = \frac{2\pi}{30}(15 - 10 + 6)$

$= \frac{11\pi}{15}$



(b) Use the washer method:

$V = \int_c^d \pi[R^2(y) - r^2(y)]\,dy = \int_0^1 \pi\left[1^2 - (y - y^3)^2\right]dy = \pi\int_0^1 (1 - y^2 - y^6 + 2y^4)\,dy = \pi\left[y - \frac{y^3}{3} - \frac{y^7}{7} + \frac{2y^5}{5}\right]_0^1$

$= \pi\left(1 - \frac{1}{3} - \frac{1}{7} + \frac{2}{5}\right) = \frac{\pi}{105}(105 - 35 - 15 + 42) = \frac{97\pi}{105}$

(c) Use the washer method:

$V = \int_c^d \pi[R^2(y) - r^2(y)]\,dy = \int_0^1 \pi\left[[1 - (y - y^3)]^2 - 0\right]dy = \pi\int_0^1 \left[1 - 2(y - y^3) + (y - y^3)^2\right]dy$

$= \pi\int_0^1 (1 + y^2 + y^6 - 2y + 2y^3 - 2y^4)\,dy = \pi\left[y + \frac{y^3}{3} + \frac{y^7}{7} - y^2 + \frac{y^4}{2} - \frac{2y^5}{5}\right]_0^1 = \pi\left(1 + \frac{1}{3} + \frac{1}{7} - 1 + \frac{1}{2} - \frac{2}{5}\right)$

$= \frac{\pi}{210}(70 + 30 + 105 - 2\cdot 42) = \frac{121\pi}{210}$

(d) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^1 2\pi(1-y)[1 - (y - y^3)]\,dy = 2\pi\int_0^1 (1-y)(1 - y + y^3)\,dy$

$= 2\pi\int_0^1 (1 - y + y^3 - y + y^2 - y^4)\,dy = 2\pi\int_0^1 (1 - 2y + y^2 + y^3 - y^4)\,dy = 2\pi\left[y - y^2 + \frac{y^3}{3} + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$

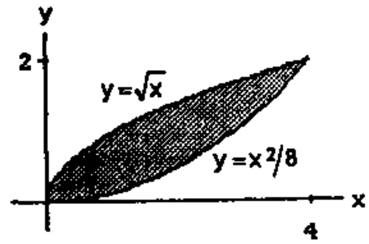
$= 2\pi\left(1 - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{5}\right) = \frac{2\pi}{60}(20 + 15 - 12) = \frac{23\pi}{30}$

31. (a) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y\left(\sqrt{8y} - y^2\right) dy$

$= 2\pi \int_0^2 \left(2\sqrt{2}\, y^{3/2} - y^3\right) dy = 2\pi\left[\frac{4\sqrt{2}}{5} y^{5/2} - \frac{y^4}{4}\right]_0^2$

$= 2\pi\left(\frac{4\sqrt{2}\cdot(\sqrt{2})^5}{5} - \frac{2^4}{4}\right) = 2\pi\left(\frac{4\cdot 2^3}{5} - \frac{4\cdot 4}{4}\right)$

$= 2\pi\cdot 4\left(\frac{8}{5} - 1\right) = \frac{8\pi}{5}(8-5) = \frac{24\pi}{5}$



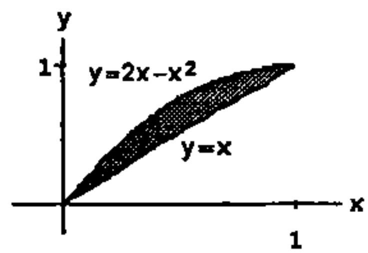
(b) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi x\left(\sqrt{x} - \frac{x^2}{8}\right) dx = 2\pi \int_0^4 \left(x^{3/2} - \frac{x^3}{8}\right) dx = 2\pi\left[\frac{2}{5}x^{5/2} - \frac{x^4}{32}\right]_0^4$

$= 2\pi\left(\frac{2\cdot 2^5}{5} - \frac{4^4}{32}\right) = 2\pi\left(\frac{2^6}{5} - \frac{2^8}{32}\right) = \frac{\pi\cdot 2^7}{160}(32-20) = \frac{\pi\cdot 2^9\cdot 3}{160} = \frac{\pi\cdot 2^4\cdot 3}{5} = \frac{48\pi}{5}$

32. (a) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi x\left[\left(2x - x^2\right) - x\right] dx$

$= 2\pi \int_0^1 x\left(x - x^2\right) dx = 2\pi \int_0^1 \left(x^2 - x^3\right) dx$

$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$



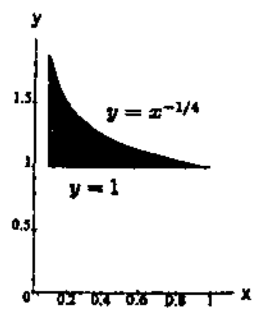
(b) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi(1-x)\left[\left(2x - x^2\right) - x\right] dx = 2\pi \int_0^1 (1-x)\left(x - x^2\right) dx$

$= 2\pi \int_0^1 \left(x - 2x^2 + x^3\right) dx = 2\pi\left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{2\pi}{12}(6 - 8 + 3) = \frac{\pi}{6}$

33. (a) $V = \int_a^b \pi\left[R^2(x) - r^2(x)\right] dx = \pi \int_{1/16}^1 \left(x^{-1/2} - 1\right) dx$

$= \pi\left[2x^{1/2} - x\right]_{1/16}^1 = \pi\left[(2-1) - \left(2\cdot\frac{1}{4} - \frac{1}{16}\right)\right]$

$= \pi\left(1 - \frac{7}{16}\right) = \frac{9\pi}{16}$



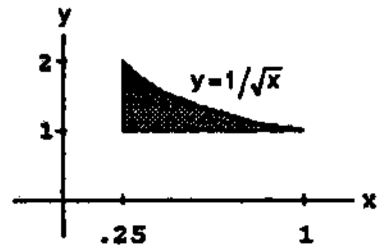
(b) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_1^2 2\pi y\left(\frac{1}{y^4} - \frac{1}{16}\right) dy = 2\pi \int_1^2 \left(y^{-3} - \frac{y}{16}\right) dy = 2\pi\left[-\frac{1}{2}y^{-2} - \frac{y^2}{32}\right]_1^2$

$= 2\pi\left[\left(-\frac{1}{8} - \frac{1}{8}\right) - \left(-\frac{1}{2} - \frac{1}{32}\right)\right] = 2\pi\left(\frac{1}{4} + \frac{1}{32}\right) = \frac{2\pi}{32}(8+1) = \frac{9\pi}{16}$

34. (a) $V = \int_c^d \pi\left[R^2(y) - r^2(y)\right] dy = \int_{1/16}^{1} \pi\left(\frac{1}{y^4} - \frac{1}{16}\right) dy$

$= \pi\left[-\frac{1}{3}y^{-3} - \frac{y}{16}\right]_1^2 = \pi\left[\left(-\frac{1}{24} - \frac{1}{8}\right) - \left(-\frac{1}{3} - \frac{1}{16}\right)\right]$

$= \frac{\pi}{48}(-2 - 6 + 16 + 3) = \frac{11\pi}{48}$



(b) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_{1/4}^{1} 2\pi x\left(\frac{1}{\sqrt{x}} - 1\right) dx = 2\pi \int_{1/4}^{1} \left(x^{1/2} - x\right) dx = 2\pi\left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_{1/4}^{1}$

$= 2\pi\left[\left(\frac{2}{3} - \frac{1}{2}\right) - \left(\frac{2}{3}\cdot\frac{1}{8} - \frac{1}{32}\right)\right] = \pi\left(\frac{4}{3} - 1 - \frac{1}{6} + \frac{1}{16}\right) = \frac{\pi}{48}(4\cdot 16 - 48 - 8 + 3) = \frac{11\pi}{48}$

35. (a) *Disc*: $V = V_1 - V_2$

$V_1 = \int_{a_1}^{b_1} \pi[R_1(x)]^2\,dx$ and $V_2 = \int_{a_2}^{b_2} \pi[R_2(x)]^2$ with $R_1(x) = \sqrt{\frac{x+2}{3}}$ and $R_2(x) = \sqrt{x}$,

$a_1 = -2$, $b_1 = 1$; $a_2 = 0$, $b_2 = 1 \Rightarrow$ two integrals are required

(b) *Washer*: $V = V_1 - V_2$

$V_1 = \int_{a_1}^{b_1} \pi\left([R_1(x)]^2 - [r_1(x)]^2\right) dx$ with $R_1(x) = \sqrt{\frac{x+2}{3}}$ and $r_1(x) = 0$; $a_1 = -2$ and $b_1 = 0$;

$V_2 = \int_{a_2}^{b_2} \pi\left([R_2(x)]^2 - [r_2(x)]^2\right) dx$ with $R_2(x) = \sqrt{\frac{x+2}{3}}$ and $r_2(x) = \sqrt{x}$; $a_2 = 0$ and $b_2 = 1$

$\Rightarrow$ two integrals are required

(c) *Shell*: $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_c^d 2\pi y\binom{\text{shell}}{\text{height}} dy$ where shell height $= y^2 - \left(3y^2 - 2\right) = 2 - 2y^2$;

$c = 0$ and $d = 1$. Only *one* integral is required. It is, therefore preferable to use the *shell* method. However, whichever method you use, you will get $V = \pi$.

36. (a) *Disc*: $V = V_1 - V_2 - V_3$

$V_i = \int_{c_i}^{d_i} \pi[R_i(y)]^2\,dy$, $i = 1, 2, 3$ with $R_1(y) = 1$ and $c_1 = -1$, $d_1 = 1$; $R_2(y) = \sqrt{y}$ and $c_2 = 0$ and $d_2 = 1$;

$R_3(y) = (-y)^{1/4}$ and $c_3 = -1$, $d_3 = 0 \Rightarrow$ three integrals are required

(b) *Washer*: $V = V_1 + V_2$

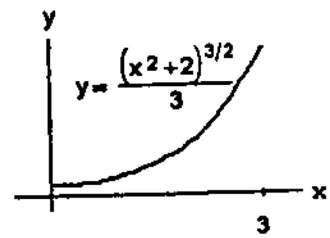
$V_i = \int_{c_i}^{d_i} \pi\left([R_i(y)]^2 - [r_i(y)]^2\right) dy$, $i = 1, 2$ with $R_1(y) = 1$, $r_1(y) = \sqrt{y}$, $c_1 = 0$ and $d_1 = 1$;

$R_2(y) = 1$, $r_2(y) = (-y)^{1/4}$, $c_2 = -1$ and $d_2 = 0 \Rightarrow$ two integrals are required

(c) *Shell*: $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\,dx$ where shell radius $= x$, shell height $= x^2 - (-x^4) = x^2 + x^4$, $a = 0$ and $b = 1 \Rightarrow$ only one integral is required. It is, therefore preferable to use the *shell* method. However, whichever method you use, you will get $V = \frac{5\pi}{6}$.

5.3 LENGTHS OF PLANE CURVES

1. $\frac{dy}{dx} = \frac{1}{3}\cdot\frac{3}{2}(x^2+2)^{1/2}\cdot 2x = \sqrt{(x^2+2)}\cdot x$

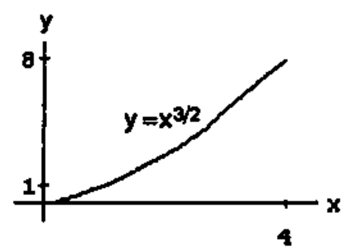


$\Rightarrow L = \int_0^3 \sqrt{1+(x^2+2)x^2}\,dx = \int_0^3 \sqrt{1+2x^2+x^4}\,dx$

$= \int_0^3 \sqrt{(1+x^2)^2}\,dx = \int_0^3 (1+x^2)\,dx = \left[x + \frac{x^3}{3}\right]_0^3$

$= 3 + \frac{27}{3} = 12$

2. $\frac{dy}{dx} = \frac{3}{2}\sqrt{x} \Rightarrow L = \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx;\ \left[u = 1 + \frac{9}{4}x\right.$

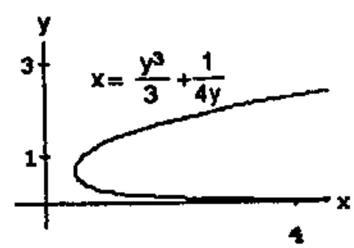


$\Rightarrow du = \frac{9}{4}\,dx \Rightarrow \frac{4}{9}\,du = dx;\ x = 0 \Rightarrow u = 1;\ x = 4$

$\Rightarrow u = 10] \to L = \int_1^{10} u^{1/2}\left(\frac{4}{9}\,du\right) = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10}$

$= \frac{8}{27}(10\sqrt{10} - 1)$

3. $\frac{dx}{dy} = y^2 - \frac{1}{4y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4 - \frac{1}{2} + \frac{1}{16y^4}$

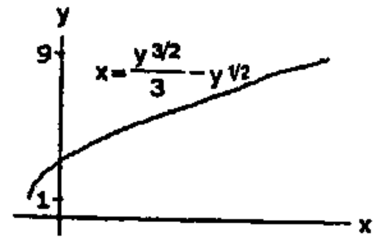


$\Rightarrow L = \int_1^3 \sqrt{1 + y^4 - \frac{1}{2} + \frac{1}{16y^4}}\,dy = \int_1^3 \sqrt{y^4 + \frac{1}{2} + \frac{1}{16y^4}}\,dy$

$= \int_1^3 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2}\,dy = \int_1^3 \left(y^2 + \frac{1}{4y^2}\right)dy = \left[\frac{y^3}{3} - \frac{y^{-1}}{4}\right]_1^3$

$= \left(\frac{27}{3} - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4} = 9 + \frac{(-1-4+3)}{12} = 9 + \frac{(-2)}{12} = \frac{53}{6}$

4. $\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)$

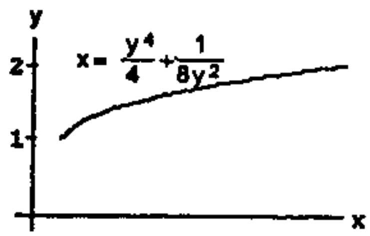


$$\Rightarrow L = \int_1^9 \sqrt{1 + \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}\, dy = \int_1^9 \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}\, dy$$

$$= \int_1^9 \frac{1}{2}\sqrt{\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2}\, dy = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right) dy$$

$$= \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \left[\frac{y^{3/2}}{3} + y^{1/2}\right]_1^9 = \left(\frac{3^3}{3} + 3\right) - \left(\frac{1}{3} + 1\right) = 11 - \frac{1}{3} = \frac{32}{3}$$

5. $\frac{dx}{dy} = y^3 - \frac{1}{4y^3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^6 - \frac{1}{2} + \frac{1}{16y^6}$

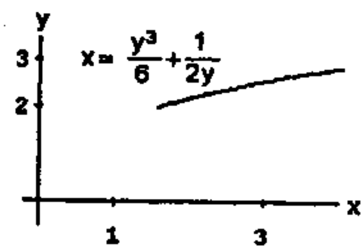


$$\Rightarrow L = \int_1^2 \sqrt{1 + y^6 - \frac{1}{2} + \frac{1}{16y^6}}\, dy = \int_1^2 \sqrt{y^6 + \frac{1}{2} + \frac{1}{16y^6}}\, dy$$

$$= \int_1^2 \sqrt{\left(y^3 + \frac{y^{-3}}{4}\right)^2}\, dy = \int_1^2 \left(y^3 + \frac{y^{-3}}{4}\right) dy$$

$$= \left[\frac{y^4}{4} - \frac{y^{-2}}{8}\right]_1^2 = \left(\frac{16}{4} - \frac{1}{(16)(2)}\right) - \left(\frac{1}{4} - \frac{1}{8}\right) = 4 - \frac{1}{32} - \frac{1}{4} + \frac{1}{8} = \frac{128 - 1 - 8 + 4}{32} = \frac{123}{32}$$

6. $\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y^4 - 2 + y^{-4}\right)$

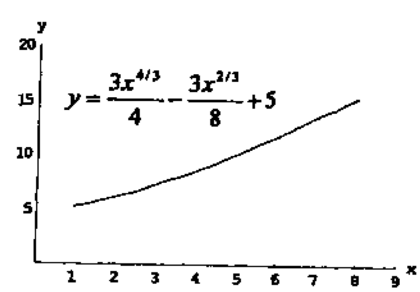


$$\Rightarrow L = \int_2^3 \sqrt{1 + \frac{1}{4}\left(y^4 - 2 + y^{-4}\right)}\, dy = \int_2^3 \sqrt{\frac{1}{4}\left(y^4 + 2 + y^{-4}\right)}\, dy$$

$$= \frac{1}{2}\int_2^3 \sqrt{\left(y^2 + y^{-2}\right)^2}\, dy = \frac{1}{2}\int_2^3 \left(y^2 + y^{-2}\right) dy = \frac{1}{2}\left[\frac{y^3}{3} - y^{-1}\right]_2^3$$

$$= \frac{1}{2}\left[\left(\frac{27}{3} - \frac{1}{3}\right) - \left(\frac{8}{3} - \frac{1}{2}\right)\right] = \frac{1}{2}\left(\frac{26}{3} - \frac{8}{3} + \frac{1}{2}\right) = \frac{1}{2}\left(6 + \frac{1}{2}\right) = \frac{13}{4}$$

7. $\frac{dy}{dx} = x^{1/3} - \frac{1}{4}x^{-1/3} \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^{2/3} - \frac{1}{2} + \frac{x^{-2/3}}{16}$

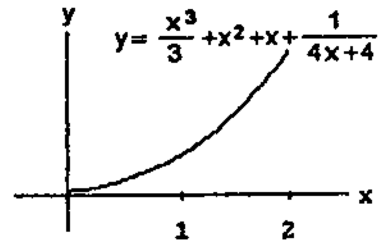


$$\Rightarrow L = \int_1^8 \sqrt{1 + x^{2/3} - \frac{1}{2} + \frac{x^{-2/3}}{16}}\, dx = \int_1^8 \sqrt{x^{2/3} + \frac{1}{2} + \frac{x^{-2/3}}{16}}\, dx$$

$$= \int_1^8 \sqrt{\left(x^{1/3} + \frac{1}{4}x^{-1/3}\right)^2}\, dx = \int_1^8 \left(x^{1/3} + \frac{1}{4}x^{-1/3}\right) dx$$

$$= \left[\frac{3}{4}x^{4/3} + \frac{3}{8}x^{2/3}\right]_1^8 = \frac{3}{8}\left[2x^{4/3} + x^{2/3}\right]_1^8 = \frac{3}{8}\left[\left(2 \cdot 2^4 + 2^2\right) - (2 + 1)\right] = \frac{3}{8}(32 + 4 - 3) = \frac{99}{8}$$

8. $\frac{dy}{dx} = x^2 + 2x + 1 - \frac{4}{(4x+4)^2} = x^2 + 2x + 1 - \frac{1}{4}\frac{1}{(1+x)^2}$



$= (1+x)^2 - \frac{1}{4}\frac{1}{(1+x)^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = (1+x)^4 - \frac{1}{2} + \frac{1}{16(1+x)^4}$

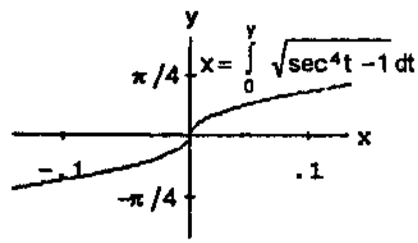
$\Rightarrow L = \int_0^2 \sqrt{1 + (1+x)^4 - \frac{1}{2} + \frac{(1+x)^{-4}}{16}}\, dx$

$= \int_0^2 \sqrt{(1+x)^4 + \frac{1}{2} + \frac{(1+x)^{-4}}{16}}\, dx = \int_0^2 \sqrt{\left[(1+x)^2 + \frac{(1+x)^{-2}}{4}\right]^2}\, dx = \int_0^2 \left[(1+x)^2 + \frac{(1+x)^{-2}}{4}\right] dx;$

$[u = 1 + x \Rightarrow du = dx;\ x = 0 \Rightarrow u = 1,\ x = 2 \Rightarrow u = 3]$

$\rightarrow L = \int_1^3 \left(u^2 + \frac{1}{4}u^{-2}\right) du = \left[\frac{u^3}{3} - \frac{1}{4}u^{-1}\right]_1^3 = \left(9 - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{108 - 1 - 4 + 3}{12} = \frac{106}{12} = \frac{53}{6}$

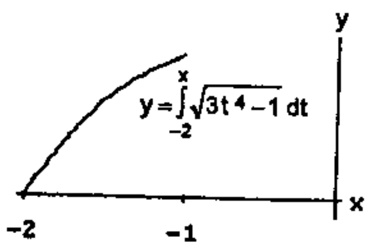
9. $\frac{dx}{dy} = \sqrt{\sec^4 y - 1} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^4 y - 1$



$\Rightarrow L = \int_{-\pi/4}^{\pi/4} \sqrt{1 + \left(\sec^4 y - 1\right)}\, dy = \int_{-\pi/4}^{\pi/4} \sec^2 y\ dy$

$= \left[\tan y\right]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$

10. $\frac{dy}{dx} = \sqrt{3x^4 - 1} \Rightarrow \left(\frac{dy}{dx}\right)^2 = 3x^4 - 1$



$\Rightarrow L = \int_{-2}^{-1} \sqrt{1 + \left(3x^4 - 1\right)}\, dx = \int_{-2}^{-1} \sqrt{3}\, x^2\ dx$

$= \sqrt{3}\left[\frac{x^3}{3}\right]_{-2}^{-1} = \frac{\sqrt{3}}{3}\left[-1 - (-2)^3\right] = \frac{\sqrt{3}}{3}(-1 + 8) = \frac{7\sqrt{3}}{3}$

11. $\frac{dx}{dt} = -a \sin t$ and $\frac{dy}{dt} = a \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-a \sin t)^2 + (a \cos t)^2} = \sqrt{a^2\left(\sin^2 t + \cos^2 t\right)} = |a|$

$\Rightarrow \text{Length} = \int_0^{2\pi} |a|\, dt = |a| \int_0^{2\pi} dt = 2\pi |a|.$

12. $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = 1 + \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (1 + \cos t)^2} = \sqrt{2 + 2 \cos t}$

$\Rightarrow \text{Length} = \int_0^{\pi} \sqrt{2 + 2 \cos t}\, dt = \sqrt{2} \int_0^{\pi} \sqrt{\left(\frac{1 - \cos t}{1 - \cos t}\right)(1 + \cos t)}\, dt = \sqrt{2} \int_0^{\pi} \sqrt{\frac{\sin^2 t}{1 - \cos t}}\, dt$

$= \sqrt{2} \int_0^{\pi} \frac{\sin t}{\sqrt{1-\cos t}}\, dt$ (since $\sin t \geq 0$ on $[0,\pi]$); $[u = 1 - \cos t \Rightarrow du = \sin t\, dt;\ t = 0 \Rightarrow u = 0,$

$t = \pi \Rightarrow u = 2] \to \sqrt{2} \int_0^2 u^{-1/2}\, du = \sqrt{2}\left[2u^{1/2}\right]_0^2 = 4$

13. $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 3t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(3t^2\right)^2 + (3t)^2} = \sqrt{9t^4 + 9t^2} = 3t\sqrt{t^2+1}$ (since $t \geq 0$ on $[0, \sqrt{3}]$)

$\Rightarrow \text{Length} = \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\, dt;\ \left[u = t^2 + 1 \Rightarrow \frac{3}{2}\, du = 3t\, dt;\ t = 0 \Rightarrow u = 1,\ t = \sqrt{3} \Rightarrow u = 4\right]$

$\to \int_1^4 \frac{3}{2} u^{1/2}\, du = \left[u^{3/2}\right]_1^4 = (8-1) = 7$

14. $\frac{dx}{dt} = t$ and $\frac{dy}{dt} = (2t+1)^{1/2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^2 + (2t+1)} = \sqrt{(t+1)^2} = |t+1| = t+1$ since $0 \leq t \leq 4$

$\Rightarrow \text{Length} = \int_0^4 (t+1)\, dt = \left[\frac{t^2}{2} + t\right]_0^4 = (8+4) = 12$

15. $\frac{dx}{dt} = (2t+3)^{1/2}$ and $\frac{dy}{dt} = 1 + t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t+3) + (1+t)^2} = \sqrt{t^2 + 4t + 4} = |t+2| = t+2$

since $0 \leq t \leq 3 \Rightarrow \text{Length} = \int_0^3 (t+2)\, dt = \left[\frac{t^2}{2} + 2t\right]_0^3 = \frac{21}{2}$

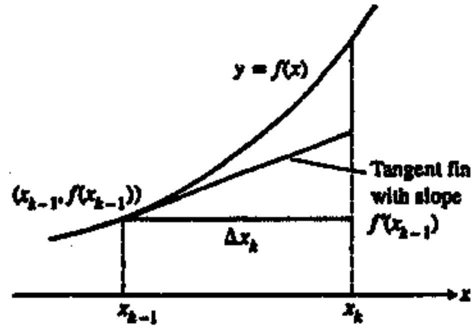
16. $\frac{dx}{dt} = 8t\cos t$ and $\frac{dy}{dt} = 8t\sin t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(8t\cos t)^2 + (8t\sin t)^2} = \sqrt{64t^2\cos^2 t + 64t^2\sin^2 t}$

$= |8t| = 8t$ since $0 \leq t \leq \frac{\pi}{2} \Rightarrow \text{Length} = \int_0^{\pi/2} 8t\, dt = \left[4t^2\right]_0^{\pi/2} = \pi^2$

17. $\sqrt{2}\, a = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,\ a \geq 0 \Rightarrow \sqrt{2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Rightarrow \frac{dy}{dx} = \pm 1 \Rightarrow y = f(x) = \pm x + C$ where C is any

real number.

18. (a) From the accompanying figure and definition of the differential (change along the tangent line) we see that $dy = f'(x_{k-1})\,\Delta x_k \Rightarrow$ length of kth tangent fin is $\sqrt{(\Delta x_k)^2 + (dy)^2} = \sqrt{(\Delta x_k)^2 + [f'(x_{k-1})\,\Delta x_k]^2}$.



(b) Length of curve $= \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n}$ (length of kth tangent fin) $= \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f'(x_{k-1})\,\Delta x_k]^2}$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(x_{k-1})]^2}\,\Delta x_k = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

19. (a) $\left(\frac{dy}{dx}\right)^2$ corresponds to $\frac{1}{4x}$ here, so take $\frac{dy}{dx}$ as $\frac{1}{2\sqrt{x}}$. Then $y = \sqrt{x} + C$, and since $(1,1)$ lies on the curve, $C = 0$. So $y = \sqrt{x}$ from $(1,1)$ to $(4,2)$.

(b) Only one. We know the derivative of the function and the value of the function at one value of x.

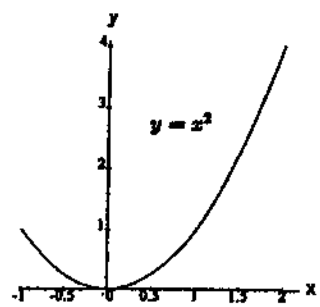
20. (a) $\left(\frac{dx}{dy}\right)^2$ corresponds to $\frac{1}{y^4}$ here, so take $\frac{dx}{dy}$ as $\frac{1}{y^2}$. Then $x = -\frac{1}{y} + C$ and, since $(0,1)$ lies on the curve, $C = 1$. So $y = \frac{1}{1-x}$.

(b) Only one. We know the derivative of the function and the value of the function at one value of x.

21. (a) $\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)^2 = 4x^2 \Rightarrow L = \int_{-1}^{2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= \int_{-1}^{2} \sqrt{1+4x^2}\,dx$$

(b)

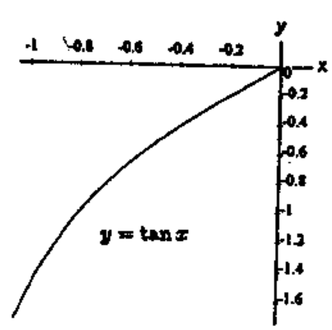


(c) $L \approx 6.13$

22. (a) $\frac{dy}{dx} = \sec^2 x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \sec^4 x$

$$\Rightarrow L = \int_{-\pi/3}^{0} \sqrt{1+\sec^4 x}\,dx$$

(b)

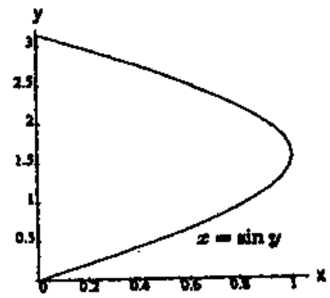


(c) $L \approx 2.06$

23. (a) $\frac{dx}{dy} = \cos y \Rightarrow \left(\frac{dx}{dy}\right)^2 = \cos^2 y$

$$\Rightarrow L = \int_0^{\pi} \sqrt{1 + \cos^2 y}\, dy$$

(b)



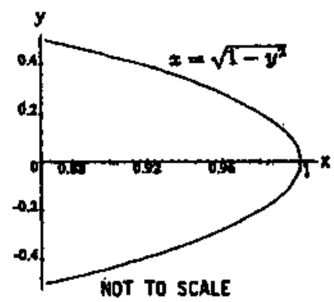
(c) $L \approx 3.82$

24. (a) $\frac{dx}{dy} = -\frac{y}{\sqrt{1-y^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{1-y^2}$

$$\Rightarrow L = \int_{-1/2}^{1/2} \sqrt{1 + \frac{y^2}{(1-y^2)}}\, dy = \int_{-1/2}^{1/2} \sqrt{\frac{1}{1-y^2}}\, dy$$

$$= \int_{-1/2}^{1/2} \left(1-y^2\right)^{-1/2} dy$$

(b)

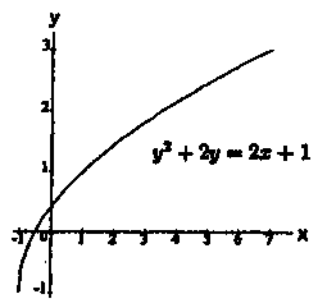


(c) $L \approx 1.05$

25. (a) $2y + 2 = 2\frac{dx}{dy} \Rightarrow \left(\frac{dx}{dy}\right)^2 = (y+1)^2$

$$\Rightarrow L = \int_{-1}^{3} \sqrt{1 + (y+1)^2}\, dy$$

(b)

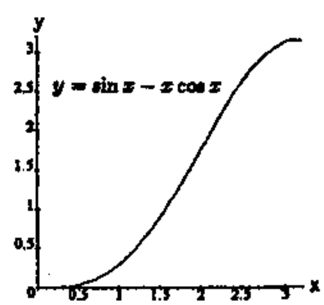


(c) $L \approx 9.29$

26. (a) $\frac{dy}{dx} = \cos x - \cos x + x \sin x \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^2 \sin^2 x$

$$\Rightarrow L = \int_0^{\pi} \sqrt{1 + x^2 \sin^2 x}\, dx$$

(b)



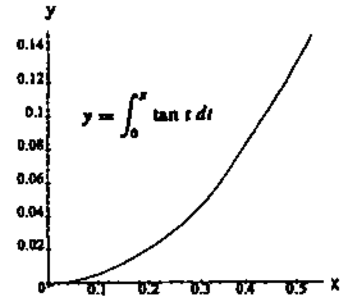
(c) $L \approx 4.70$

27. (a) $\frac{dy}{dx} = \tan x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x$

$$\Rightarrow L = \int_0^{\pi/6} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/6} \sqrt{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}}\, dx$$

$$= \int_0^{\pi/6} \frac{dx}{\cos x} = \int_0^{\pi/6} \sec x\, dx$$

(b)



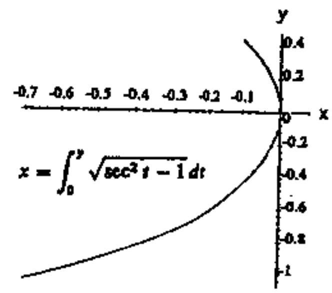
(c) $L \approx 0.55$

28. (a) $\frac{dx}{dy} = \sqrt{\sec^2 y - 1} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^2 y - 1$

$$\Rightarrow L = \int_{-\pi/3}^{\pi/4} \sqrt{1 + \left(\sec^2 y - 1\right)}\, dy$$

$$= \int_{-\pi/3}^{\pi/4} |\sec y|\, dy = \int_{-\pi/3}^{\pi/4} \sec y\, dy$$

(b)



(c) $L \approx 2.20$

29. The length of the curve $y = \sin\left(\frac{3\pi}{20}x\right)$ from 0 to 20 is: $L = \int_0^{20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$; $\frac{dy}{dx} = \frac{3\pi}{20}\cos\left(\frac{3\pi}{20}x\right) \Rightarrow \left(\frac{dy}{dx}\right)^2$ $= \frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right) \Rightarrow L = \int_0^{20} \sqrt{1 + \frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right)}\, dx$. Using numerical integration we find $L \approx 21.07$ in

30. First, we'll find the length of the cosine curve: $L = \int_{-25}^{25} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$; $\frac{dy}{dx} = -\frac{25\pi}{50}\sin\left(\frac{\pi x}{50}\right)$ $\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{50}\right) \Rightarrow L = \int_{-25}^{25} \sqrt{1 + \frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{50}\right)}\, dx$. Using a numerical integrator we find

$L \approx 73.1848$ ft. Surface area is: $A = \text{length} \cdot \text{width} \approx (73.1848)(300) = 21{,}955.44$ ft.
Cost $= 1.75A = (1.75)(21{,}955.44) = \$38{,}422.02$. Answers may vary slightly, depending on the numerical integration used.

31-36. Example CAS commands:

Maple:

```
xi:=(i,n) -> (a + (b−a)*i/n);
digits := 6:
f:= x -> sqrt(1−x^2); a:= −1: b:= 1:
n:=8;
segs := [seq([xi(i,n),f(xi(i,n))], i = 0..n)];i:= 'i':
plot({f(x),segs}, x=a..b);
```

```
approx:= sum(sqrt((xi(j,n) - xi(j-1,n))^2 + (f(xi(j,n))-f(xi(j-1,n)))^2),j=1..n);
evalf(approx);
int(sqrt(1+D(f)(x)^2), x = a..b);
evalf(%);
```

Mathematica:

```
Clear[x]
{a,b} = {-1,1}; f[x_] = Sqrt[ 1 - x^2 ]
p1 = Plot[ f[x], {x,a,b} ]
n = 8;
pts = Table[ {xn,f[xn]}, {xn,a,b,(b-a)/n} ] // N
Show[{ p1, Graphics[{Line[pts]}] }]
Sum[ Sqrt[
    (pts[[i+1,1]] - pts[[i,1]])^2 +
    (pts[[i+1,2]] - pts[[i,2]])^2 ] ,
    {i,1,n} ]
NIntegrate[ Sqrt[1+f' [x]^2], {x,a,b}]
```

5.4 SPRINGS, PUMPING AND LIFTING

1. The force required to lift the water is equal to the water's weight, which varies steadily from 40 lb to 0 lb over the 20-ft lift. When the bucket is x ft off the ground, the water weighs: $F(x) = 40\left(\frac{20-x}{20}\right) = 40\left(1 - \frac{x}{20}\right)$ $= 40 - 2x$ lb. The work done is: $W = \int_a^b F(x)\,dx = \int_0^{20} (40-2x)\,dx = \left[40x - x^2\right]_0^{20} = (40)(20) - 20^2$ $= 800 - 400 = 400$ ft · lb

2. The water's weight varies steadily from 16 lb to 8 lb over the 20-ft lift. When the bucket is x ft off the ground, the water weighs: $F(x) = 16\left(\frac{40-x}{40}\right) = 16\left(1 - \frac{x}{40}\right) = 16 - \frac{2x}{5}$ lb. The work done is: $W = \int_a^b F(x)\,dx$ $= \int_0^{20} \left(16 - \frac{2x}{5}\right) dx = \left[16x - \frac{x^2}{5}\right]_0^{20} = (16)(20) - \frac{20^2}{5} = 320 - \frac{400}{5} = 320 - 80 = 240$ ft · lb

3. The force required to haul up the rope is equal to the rope's weight, which varies steadily and is proportional to x, the length of the rope still hanging: $F(x) = 0.624x$. The work done is: $W = \int_0^{50} F(x)\,dx = \int_0^{50} 0.624x\,dx$ $= 0.624\left[\frac{x^2}{2}\right]_0^{50} = 780$ J

4. The weight of sand varies steadily from 144 lb to 72 lb over the 18 ft length. When the bag is x ft off the ground, the sand weighs: $F(x) = 144\left(\frac{36-x}{36}\right) = 144\left(1 - \frac{x}{36}\right)$. The work done is: $W = \int_a^b F(x)\,dx$

$$= \int_0^{18} 144\left(1 - \frac{x}{36}\right) dx = 144\left[x - \frac{x^2}{72}\right]_0^{18} = 144\left(18 - \frac{18^2}{18 \cdot 4}\right) = 144\left(18 - \frac{18}{4}\right) = \frac{144 \cdot 18 \cdot 3}{4} = 36 \cdot 18 \cdot 3 = 1944 \text{ ft} \cdot \text{lb}$$

5. The force required to lift the cable is equal to the weight of the cable paid out: $F(x) = (4.5)(180 - x)$ where x is the position of the car off the first floor. The work done is: $W = \int_0^{180} F(x)\, dx = 4.5 \int_0^{180} (180 - x)\, dx$

$$= 4.5\left[180x - \frac{x^2}{2}\right]_0^{180} = 4.5\left(180^2 - \frac{180^2}{2}\right) = \frac{4.5 \cdot 180^2}{2} = 72{,}900 \text{ ft} \cdot \text{lb}$$

6. Since the force is acting <u>toward</u> the origin, it acts opposite to the positive x-direction. Thus $F(x) = -\frac{k}{x^2}$. The work done is $W = \int_a^b -\frac{k}{x^2}\, dx = k \int_a^b -\frac{1}{x^2}\, dx = k\left[\frac{1}{x}\right]_a^b = k\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{k(a-b)}{ab}$

7. The force against the piston is $F = pA$. If $V = Ax$, where x is the height of the cylinder, then $dV = A\, dx$

$$\Rightarrow \text{Work} = \int F\, dx = \int pA\, dx = \int_{(p_1, V_1)}^{(p_2, V_2)} p\, dV.$$

8. $pV^{1.4} = c$, a constant $\Rightarrow p = cV^{-1.4}$. If $V_1 = 243 \text{ in}^3$ and $p_1 = 50 \text{ lb/in}^3$, then $c = (50)(243)^{1.4} = 109{,}350$ lb. Thus $W = \int_{243}^{32} 109{,}350V^{-1.4}\, dV = \left[-\frac{109{,}350}{0.4V^{0.4}}\right]_{243}^{32} = -\frac{109{,}350}{0.4}\left(\frac{1}{32^{0.4}} - \frac{1}{243^{0.4}}\right) = -\frac{109{,}350}{0.4}\left(\frac{1}{4} - \frac{1}{9}\right)$

$= -\frac{(109{,}350)(5)}{(0.4)(36)} = -37{,}968.75 \text{ in} \cdot \text{lb}$. Note that when a system is compressed, the work done is negative.

9. The force required to stretch the spring from its natural length of 2 m to a length of 5 m is $F(x) = kx$. The work done by F is $W = \int_0^3 F(x)\, dx = k \int_0^3 x\, dx = \frac{k}{2}\left[x^2\right]_0^3 = \frac{9k}{2}$. This work is equal to 1800 J $\Rightarrow \frac{9}{2}k = 1800$

$\Rightarrow k = 400$ N/m

10. (a) We find the force constant from Hooke's Law: $F = kx \Rightarrow k = \frac{F}{x} \Rightarrow k = \frac{800}{4} = 200$ lb/in

(b) The work done to stretch the spring 2 inches beyond its natural length is $W = \int_0^2 kx\, dx$

$$= 200 \int_0^2 x\, dx = 200\left[\frac{x^2}{2}\right]_0^2 = 200(2 - 0) = 400 \text{ in} \cdot \text{lb} = 33.3 \text{ ft} \cdot \text{lb}$$

(c) We substitute $F = 1600$ into the equation $F = 200x$ to find $1600 = 200x \Rightarrow x = 8$ in

11. We find the force constant from Hooke's law: $F = kx$. A force of 2 N stretches the spring to 0.02 m $\Rightarrow 2 = k \cdot (0.02) \Rightarrow k = 100\,\frac{\text{N}}{\text{m}}$. The force of 4 N will stretch the rubber band y m, where $F = ky \Rightarrow y = \frac{F}{k}$

$\Rightarrow y = \frac{4\text{N}}{100\frac{\text{N}}{\text{m}}} \Rightarrow y = 0.04$ m $= 4$ cm. The work done to stretch the rubber band 0.04 m is $W = \int_0^{0.04} kx\,dx$ $= 100\int_0^{0.04} x\,dx = 100\left[\frac{x^2}{2}\right]_0^{0.04} = \frac{(100)(0.04)^2}{2} = 0.08$ J

12. We find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} \Rightarrow k = \frac{90}{1} \Rightarrow k = 90\,\frac{\text{N}}{\text{m}}$. The work done to stretch the spring 5 m beyond its natural length is $W = \int_0^5 kx\,dx = 90\int_0^5 x\,dx = 90\left[\frac{x^2}{2}\right]_0^5 = (90)\left(\frac{25}{2}\right) = 1125$ J

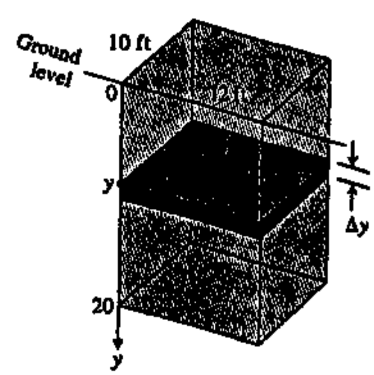
13. (a) We find the spring's constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} = \frac{21{,}714}{8-5} = \frac{21{,}714}{3} \Rightarrow k = 7238\,\frac{\text{lb}}{\text{in}}$

(b) The work done to compress the assembly the first half inch is $W = \int_0^{0.5} kx\,dx = 7238\int_0^{0.5} x\,dx$ $= 7238\left[\frac{x^2}{2}\right]_0^{0.5} = (7238)\frac{(0.5)^2}{2} = \frac{(7238)(0.25)}{2} \approx 905$ in·lb. The work done to compress the assembly the second half inch is: $W = \int_{0.5}^{1.0} kx\,dx = 7238\int_{0.5}^{1.0} x\,dx = 7238\left[\frac{x^2}{2}\right]_{0.5}^{1.0} = \frac{7238}{2}\left[1-(0.5)^2\right] = \frac{(7238)(0.75)}{2}$ ≈ 2714 in·lb

14. First, we find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} = \frac{150}{\left(\frac{1}{16}\right)} = 16\cdot 150 = 2{,}400\,\frac{\text{lb}}{\text{in}}$. If someone compresses the scale $x = \frac{1}{8}$ in, he/she must weigh $F = kx = 2{,}400\left(\frac{1}{8}\right) = 300$ lb. The work done to compress the scale this far is $W = \int_0^{1/8} kx\,dx = 2400\left[\frac{x^2}{2}\right]_0^{1/8} = \frac{2400}{2\cdot 64} = 18.75$ lb·in. $= \frac{25}{16}$ ft·lb

15. We will use the coordinate system given.

(a) The typical slab between the planes at y and $y+\Delta y$ has a volume of $\Delta V = (10)(12)\,\Delta y = 120\,\Delta y$ ft^3. The force F required to lift the slab is equal to its weight: $F = 62.4\,\Delta V = 62.4\cdot 120\,\Delta y$ lb. The distance through which F must act is about y ft, so the work done lifting the slab is about ΔW = force × distance $= 62.4\cdot 120\cdot y\cdot \Delta y$ ft·lb. The work it takes to lift all the water is approximately $W \approx \sum_0^{20} \Delta W$ $= \sum_0^{20} 62.4\cdot 120y\cdot\Delta y$ ft·lb. This is a Riemann sum for the function $62.4\cdot 120y$ over the interval $0 \le y \le 20$.



The work of pumping the tank empty is the limit of these sums: $W = \int_0^{20} 62.4 \cdot 120y\,dy$

$= (62.4)(120)\left[\frac{y^2}{2}\right]_0^{20} = (62.4)(120)\left(\frac{400}{2}\right) = (62.4)(120)(200) = 1{,}497{,}600$ ft · lb

(b) The time t it takes to empty the full tank with $\left(\frac{5}{11}\right)$-hp motor is $t = \frac{W}{250\ \frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = \frac{1{,}497{,}600\ \text{ft}\cdot\text{lb}}{250\ \frac{\text{ft}\cdot\text{lb}}{\text{sec}}}$

$= 5990.4$ sec $= 1.664$ hr $\Rightarrow t \approx 1$ hr and 40 min

(c) Following all the steps of part (a), we find that the work it takes to lower the water level 10 ft is

$W = \int_0^{10} 62.4 \cdot 120y\,dy = (62.4)(120)\left[\frac{y^2}{2}\right]_0^{10} = (62.4)(120)\left(\frac{100}{2}\right) = 374{,}400$ ft · lb and the time is

$t = \frac{W}{250\ \frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = 1497.6$ sec $= 0.416$ hr ≈ 25 min

(d) In a location where water weighs $62.26\ \frac{\text{lb}}{\text{ft}^3}$:

a) $W = (62.26)(24{,}000) = 1{,}494{,}240$ ft · lb.

b) $t = \frac{1{,}494{,}240}{250} = 5976.96$ sec ≈ 1.660 hr $\Rightarrow t \approx 1$ hr and 40 min

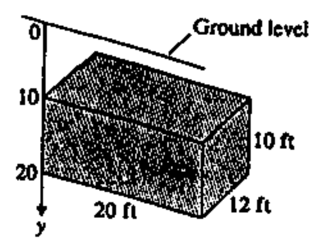
In a location where water weighs $62.59\ \frac{\text{lb}}{\text{ft}^3}$

a) $W = (62.59)(24{,}000) = 1{,}502{,}160$ ft · lb

b) $t = \frac{1{,}502{,}160}{250} = 6008.64$ sec ≈ 1.669 hr $\Rightarrow t \approx 1$ hr and 40.1 min

16. We will use the coordinate system given.

(a) The typical slab between the planes at y and $y + \Delta y$ has a volume of $\Delta V = (20)(12)\,\Delta y = 240\,\Delta y$ ft^3. The force F required to lift the slab is equal to its weight: $F = 62.4\,\Delta V = 62.4 \cdot 240\,\Delta y$ lb. The distance through which F must act is about y ft, so the work done lifting the slab is about ΔW = force × distance $= 62.4 \cdot 240 \cdot y \cdot \Delta y$ ft · lb. The work it takes to lift all the water is approximately $W \approx \sum_{10}^{20} \Delta W$



$= \sum_{10}^{20} 62.4 \cdot 240y \cdot \Delta y$ ft · lb. This is a Riemann sum for the function $62.4 \cdot 240y$ over the interval

$10 \le y \le 20$. The work it takes to empty the cistern is the limit of these sums: $W = \int_{10}^{20} 62.4 \cdot 240y\,dy$

$= (62.4)(240)\left[\frac{y^2}{2}\right]_{10}^{20} = (62.4)(240)(200 - 50) = (62.4)(240)(150) = 2{,}246{,}400$ ft · lb

(b) $t = \frac{W}{275\ \frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = \frac{2{,}246{,}400\ \text{ft}\cdot\text{lb}}{275} \approx 8168.73$ sec ≈ 2.27 hours ≈ 2 hr and 16.1 min

(c) Following all the steps of part (a), we find that the work it takes to empty the tank halfway is

$$W = \int_{10}^{15} 62.4 \cdot 240y \, dy = (62.4)(240)\left[\frac{y^2}{2}\right]_{10}^{15} = (62.4)(240)\left(\frac{225}{2} - \frac{100}{2}\right) = (62.4)(240)\left(\frac{125}{2}\right) = 936{,}000 \text{ ft.}$$

Then the time is $t = \dfrac{W}{275 \frac{\text{ft} \cdot \text{lb}}{\text{sec}}} = \dfrac{936{,}000}{275} \approx 3403.64$ sec ≈ 56.7 min

(d) In a location where water weighs $62.26 \frac{\text{lb}}{\text{ft}^3}$:

a) $W = (62.26)(240)(150) = 2{,}241{,}360$ ft $\cdot$ lb.

b) $t = \dfrac{2{,}241{,}360}{275} = 8150.40$ sec $= 2.264$ hours ≈ 2 hr and 15.8 min

c) $W = (62.26)(240)\left(\frac{125}{2}\right) = 933{,}900$ ft $\cdot$ lb; $t = \dfrac{933{,}900}{275} = 3396$ sec ≈ 0.94 hours ≈ 56.6 min

In a location where water weighs $62.59 \frac{\text{lb}}{\text{ft}^3}$

a) $W = (62.59)(240)(150) = 2{,}253{,}240$ ft $\cdot$ lb.

b) $t = \dfrac{2{,}253{,}240}{275} = 8193.60$ sec $= 2.276$ hours ≈ 2 hr and 16.61 min

c) $W = (62.59)(240)\left(\frac{125}{2}\right) = 938{,}850$ ft $\cdot$ lb; $t = \dfrac{938{,}850}{275} \approx 3414$ sec ≈ 0.95 hours ≈ 56.9 min

17. Using exactly the same procedure as done in Example 6 we change only the distance through which F must act: distance $\approx (10 - y)$ m. Then $\Delta W = 245{,}000\pi(10 - y)\,\Delta y$ J $\Rightarrow W \approx \sum_0^{10} \Delta W = \sum_0^{10} 245{,}000\pi(10 - y)\,\Delta y$

$$\Rightarrow W = \int_0^{10} 245{,}000\pi(10 - y)\,dy = 245{,}000\pi \int_0^{10} (10 - y)\,dy = 245{,}000\pi\left[10y - \frac{y^2}{2}\right]_0^{10} = 245{,}000\pi\left(100 - \frac{100}{2}\right)$$

$\approx (245{,}000\pi)(50) \approx 38{,}484{,}510$ J

18. Exactly as done in Example 6 with the change in the upper limit of the sums and the integral: $W \approx \sum_0^5 \Delta W$

$$= \sum_0^5 245{,}000\pi(14 - y)\,\Delta y \text{ J} \Rightarrow W = \int_0^5 245{,}000\pi(14 - y)\,dy = 245{,}000\pi\left[14y - \frac{y^2}{2}\right]_0^5 = 245{,}000\pi\left(70 - \frac{25}{2}\right)$$

$= (245{,}000\pi)\left(\frac{115}{2}\right) \approx 44{,}257{,}186.5$ J

19. The typical slab between the planes at y and and $y + \Delta y$ has a volume of $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\frac{20}{2}\right)^2 \Delta y = \pi \cdot 100\ \Delta y\ \text{ft}^3$. The force F required to lift the slab is equal to its weight: $F = 51.2\,\Delta V = 51.2 \cdot 100\pi\,\Delta y$ lb $\Rightarrow F = 5120\pi\,\Delta y$ lb. The distance through which F must act is about $(30 - y)$ ft. The work it takes to lift all the kerosene is approximately $W \approx \sum_0^{30} \Delta W$ $= \sum_0^{30} 5120\pi(30 - y)\,\Delta y$ ft $\cdot$ lb which is a Riemann sum. The work to pump the tank dry is the limit of these sums: $W = \int_0^{30} 5120\pi(30 - y)\,dy = 5120\pi\left[30y - \frac{y^2}{2}\right]_0^{30} = 5120\pi\left(\frac{900}{2}\right) = (5120)(450\pi)$

$\approx 7{,}238{,}229.47$ ft $\cdot$ lb

20. For both ways of filling the tank, the typical slab between the planes at y and $y + \Delta y$ has a volume of $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi(2)^2\Delta y$. The force F required to lift this slab is equal to its weight: $F = 62.4\,\Delta V = \pi(4)(62.4)\,\Delta y$. The distance through which F must act *does* depend on the way of filling.

(a) If we pump the water through a hose attached to a valve in the bottom, the distance is $(15 + y)$ so the work done lifting the slab is about $\Delta W_1 = (62.4)(4\pi)(15 + y)\,\Delta y$. The work done lifting all the slabs is

$W_1 \approx \sum_0^6 (62.4)(4\pi)(15 + y)\,\Delta y$ and taking the limit we get $W_1 = \int_0^6 (62.4)(4\pi)(15 + y)\,dy$

$$= (62.4)(4\pi)\left[15y + \frac{y^2}{2}\right]_0^6 = (62.4)(4\pi)\left(15 \cdot 6 + \frac{36}{2}\right) = (62.4)(4\pi)(90 + 18) = (62.4)(4\pi)(108)$$

$\approx 84{,}687.3$ ft · lb

(b) If we attach the hose to the rim of the tank and let the water pour in, the distance is $(15 + 6)$, so the work done by the *pump* on one slab is $\Delta W_2 = (62.4)(4\pi)(15 + 6)\,\Delta y$. The work done lifting all the slabs is:

$W_2 \approx \sum_0^6 (62.4)(4\pi)(15 + 6)\,\Delta y$ and taking the limit we get $W_2 = \int_0^6 (62.4)(4\pi)(15 + 6)\,dy$

$= (62.4)(4\pi)(21) \int_0^6 dy = (62.4)(4\pi)(126) \approx 98{,}801.8$ ft · lb. We see that $W_2 > W_1$ and if we assume that the pump produces a constant amount of work per hour then it takes more time to do work W_2.

21. (a) Follow all the steps of Example 7 but make the substitution of $64.5\,\frac{\text{lb}}{\text{ft}^3}$ for $57\,\frac{\text{lb}}{\text{ft}^3}$. Then,

$$W = \int_0^8 \frac{64.5\pi}{4}(10 - y)y^2\,dy = \frac{64.5\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 = \frac{64.5\pi}{4}\left(\frac{10 \cdot 8^3}{3} - \frac{8^4}{4}\right) = \left(\frac{64.5\pi}{4}\right)(8^3)\left(\frac{10}{3} - 2\right)$$

$$= \frac{64.5\pi \cdot 8^3}{3} = 21.5\pi \cdot 8^3 \approx 34{,}583 \text{ ft} \cdot \text{lb}$$

(b) Exactly as done in Example 7 but change the distance through which F acts to distance $\approx (13 - y)$ ft.

Then $W = \int_0^8 \frac{57\pi}{4}(13 - y)y^2\,dy = \frac{57\pi}{4}\left[\frac{13y^3}{3} - \frac{y^4}{4}\right]_0^8 = \frac{57\pi}{4}\left(\frac{13 \cdot 8^3}{3} - \frac{8^4}{4}\right) = \left(\frac{57\pi}{4}\right)(8^3)\left(\frac{13}{3} - 2\right) = \frac{57\pi \cdot 8^3 \cdot 7}{3 \cdot 4}$

$= (19\pi)(8^2)(7)(2) \approx 53{,}482$ ft · lb

22. The typical slab between the planes of y and $y + \Delta y$ has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\sqrt{y}\right)^2\Delta y = \pi y\,\Delta y\ \text{m}^3$. The force F(y) is equal to the slab's weight: $F(y) = 10{,}000\,\frac{\text{N}}{\text{m}^3} \cdot \Delta V$ $= \pi 10{,}000y\,\Delta y$ N. The height of the tank is $4^2 = 16$ m. The distance through which F(y) must act to lift the slab to the level of the top of the tank is about $(16 - y)$ m, so the work done lifting the slab is about $\Delta W = 10{,}000\pi y(16 - y)\,\Delta y$ N · m. The work done lifting all the slabs from $y = 0$ to $y = 16$ to the top is approximately $W \approx \sum_0^{16} 10{,}000\pi y(16 - y)\Delta y$. Taking the limit of these Riemann sums, we get

$$W = \int_0^{16} 10{,}000\pi y(16-y)\,dy = 10{,}000\pi \int_0^{16} \left(16y - y^2\right) dy = 10{,}000\pi\left[\frac{16y^2}{2} - \frac{y^3}{3}\right]_0^{16} = 10{,}000\pi\left(\frac{16^3}{2} - \frac{16^3}{3}\right)$$

$$= \frac{10{,}000 \cdot \pi \cdot 16^3}{6} \approx 21{,}446{,}605.9 \text{ J}$$

23. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\sqrt{25-y^2}\right)^2 \Delta y \text{ m}^3$. The force F(y) required to lift this slab is equal to its weight: $F(y) = 9800 \cdot \Delta V$ $= 9800\pi\left(\sqrt{25-y^2}\right)^2 \Delta y = 9800\pi\left(25-y^2\right)\Delta y$ N. The distance through which F(y) must act to lift the slab to the level of 4 m above the top of the reservoir is about $(4-y)$ m, so the work done is approximately $\Delta W \approx 9800\pi\left(25-y^2\right)(4-y)\,\Delta y \text{ N}\cdot\text{m}$. The work done lifting all the slabs from $y = -5$ m to $y = 0$ m is approximately $W \approx \sum_{-5}^{0} 9800\pi\left(25-y^2\right)(4-y)\,\Delta y \text{ N}\cdot\text{m}$. Taking the limit of these Riemann sums, we get

$$W = \int_{-5}^{0} 9800\pi\left(25-y^2\right)(4-y)\,dy = 9800\pi \int_{-5}^{0} \left(100 - 25y - 4y^2 + y^3\right) dy = 9800\pi\left[100y - \frac{25}{2}y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_{-5}^{0}$$

$$= -9800\pi\left(-500 - \frac{25 \cdot 25}{2} + \frac{4}{3} \cdot 125 + \frac{625}{4}\right) \approx 15{,}073{,}100 \text{ J}$$

24. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\sqrt{100-y^2}\right)^2 \Delta y = \pi\left(100-y^2\right)\Delta y \text{ ft}^3$. The force is $F(y) = \frac{56 \text{ lb}}{\text{ft}^3} \cdot \Delta V = 56\pi\left(100-y^2\right)\Delta y$ lb. The distance through which F(y) must act to lift the slab to the level of 2 ft above the top of the tank is about $(12-y)$ ft, so the work done is $\Delta W \approx 56\pi\left(100-y^2\right)(12-y)\,\Delta y \text{ lb}\cdot\text{ft}$. The work done lifting all the slabs from $y = 0$ ft to $y = 10$ ft is approximately $W \approx \sum_{0}^{10} 56\pi\left(100-y^2\right)(12-y)\,\Delta y \text{ lb}\cdot\text{ft}$. Taking the limit of these Riemann sums, we get $W = \int_0^{10} 56\pi\left(100-y^2\right)(12-y)\,dy = 56\pi \int_0^{10} \left(100-y^2\right)(12-y)\,dy$

$$= 56\pi \int_0^{10} \left(1200 - 100y - 12y^2 + y^3\right) dy = 56\pi\left[1200y - \frac{100y^2}{2} - \frac{12y^3}{3} + \frac{y^4}{4}\right]_0^{10}$$

$$= 56\pi\left(12{,}000 - \frac{10{,}000}{2} - 4 \cdot 1000 + \frac{10{,}000}{4}\right) = (56\pi)\left(12 - 5 - 4 + \frac{5}{2}\right)(1000) \approx 967{,}611 \text{ ft}\cdot\text{lb}.$$

It would cost $(0.5)(967.611) = 483{,}805¢ = \$4838.05$. Yes, we can afford to hire the firm.

25. $F = m\,\frac{dv}{dt} = m\,\frac{dv}{dx} \cdot \frac{dx}{dt} = mv\,\frac{dv}{dx}$ by the chain rule $\Rightarrow W = \int_{x_1}^{x_2} mv\,\frac{dv}{dx}\,dx = m \int_{x_1}^{x^2} \left(v\,\frac{dv}{dx}\right) dx = m\left[\frac{1}{2}v^2(x)\right]_{x_1}^{x_2}$ $= \frac{1}{2}m\left[v^2(x_2) - v^2(x_1)\right] = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$, as claimed.

26. weight $= 2 \text{ oz} = \frac{2}{16}$ lb; mass $= \frac{\text{weight}}{32} = \frac{\frac{1}{8}}{32} = \frac{1}{256}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{1}{256} \text{ slugs}\right)(160 \text{ ft/sec})^2 \approx 50 \text{ ft}\cdot\text{lb}$

27. 90 mph $= \frac{90 \text{ mi}}{1 \text{ hr}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} = 132$ ft/sec; $m = \frac{0.3125 \text{ lb}}{32 \text{ ft/sec}^2} = \frac{0.3125}{32}$ slugs;

$W = \left(\frac{1}{2}\right)\left(\frac{0.3125 \text{ lb}}{32 \text{ ft/sec}^2}\right)(132 \text{ ft/sec})^2 \approx 85.1 \text{ ft} \cdot \text{lb}$

28. weight $= 1.6$ oz $= 0.1$ lb $\Rightarrow m = \frac{0.1 \text{ lb}}{32 \text{ ft/sec}^2} = \frac{1}{320}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{1}{320} \text{ slugs}\right)(280 \text{ ft/sec})^2 = 122.5 \text{ ft} \cdot \text{lb}$

29. weight $= 2$ oz $= \frac{1}{8}$ lb $\Rightarrow m = \frac{\frac{1}{8}}{32}$ slugs $= \frac{1}{256}$ slugs; 124 mph $= \frac{(124)(5280)}{(60)(60)} \approx 181.87$ ft/sec;

$W = \left(\frac{1}{2}\right)\left(\frac{1}{256} \text{ slugs}\right)(181.87 \text{ ft/sec})^2 \approx 64.6 \text{ ft} \cdot \text{lb}$

30. weight $= 14.5$ oz $= \frac{14.5}{16}$ lb $\Rightarrow m = \frac{14.5}{(16)(32)}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{14.5}{(16)(32)} \text{ slugs}\right)(88 \text{ ft/sec})^2 \approx 109.7 \text{ ft} \cdot \text{lb}$

31. weight $= 6.5$ oz $= \frac{6.5}{16}$ lb $\Rightarrow m = \frac{6.5}{(16)(32)}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{6.5}{(16)(32)} \text{ slugs}\right)(132 \text{ ft/sec})^2 \approx 110.6 \text{ ft} \cdot \text{lb}$

32. $F = (18 \text{ lb/ft})x \Rightarrow W = \int_0^{1/4} 18x \, dx = \left[9x^2\right]_0^{1/4} = \frac{9}{16}$ ft · lb. Now $W = \frac{1}{2}mv^2 - \frac{1}{2}mv_1^2$, where $W = \frac{9}{16}$ ft · lb,

$m = \frac{\frac{1}{8}}{32} = \frac{1}{256}$ slugs and $v_1 = 0$ ft/sec. Thus, $\frac{9}{16}$ ft · lb. $= \left(\frac{1}{2}\right)\left(\frac{1}{256} \text{ slugs}\right)v^2 \Rightarrow v = 12\sqrt{2}$ ft/sec. With $v = 0$ at the top of the bearing's path and $v = 12\sqrt{2} - 32t \Rightarrow t = \frac{3\sqrt{2}}{8}$ sec when the bearing is at the top of its path.

The height the bearing reaches is $s = 12\sqrt{2}\,t - 16t^2 \Rightarrow$ at $t = \frac{3\sqrt{2}}{8}$ the bearing reaches a height of

$(12\sqrt{2})\left(\frac{3\sqrt{2}}{8}\right) - (16)\left(\frac{3\sqrt{2}}{8}\right)^2 = 9 - \frac{16 \cdot 18}{64} = 4\frac{1}{2}$ ft

33. (a) From the diagram,

$r(y) = 60 - x = 60 - \sqrt{50^2 - (y-325)^2}$

for $325 \le y \le 375$ ft.

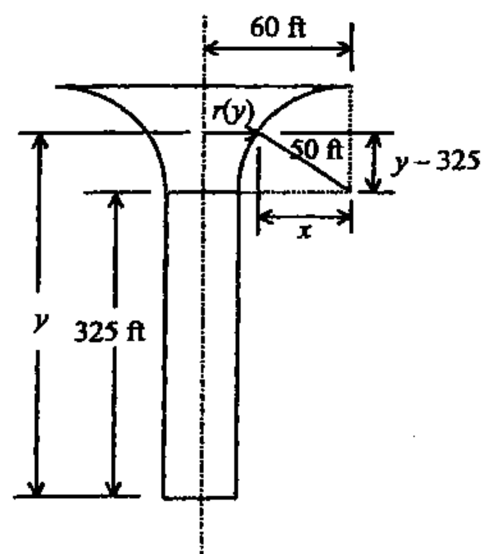
(b) The volume of a horizontal slice of the funnel is $\Delta V \approx \pi[r(y)]^2 \Delta y$

$= \pi\left[60 - \sqrt{2500 - (y-325)^2}\right]^2 \Delta y$

(c) The work required to lift the single slice of water is $\Delta W \approx 62.4 \Delta V(375 - y)$

$= 62.4(375 - y)\pi\left[60 - \sqrt{2500 - (y-325)^2}\right]^2 \Delta y.$

The total work to pump out the funnel is



$$W=\int_{325}^{375}62.4(375-y)\pi\left[60-\sqrt{2500-(y-325)^2}\right]^2 dy=6.3358\cdot 10^7 \text{ ft}\cdot\text{lb}.$$

34. (a) From the result in Example 8, the work to pump out the throat is 1,353,869,354 ft · lb. Therefore, the total work required to pump out the throat and the funnel is 1,353,869,354 + 63,358,000 = 1,417,227,354 ft · lb.

(b) In horsepower-hours, the work required to pump out the glory hole is $\frac{1{,}417{,}227{,}354}{1.98\cdot 10^6}$ = 7158. Therefore, it would take $\frac{715.8 \text{ hp}\cdot\text{h}}{1000 \text{ hp}}=0.7158$ hours ≈ 43 minutes.

35. We imagine the milkshake divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0,7]$. The typical slab between the planes at y and $y+\Delta y$ has a volume of about $\Delta V=\pi(\text{radius})^2(\text{thickness})=\pi\left(\frac{y+17.5}{14}\right)^2\Delta y \text{ in}^3$. The force $F(y)$ required to lift this slab is equal to its weight: $F(y)=\frac{4}{9}\Delta V=\frac{4\pi}{9}\left(\frac{y+17.5}{14}\right)^2\Delta y$ oz. The distance through which $F(y)$ must act to lift this slab to the level of 1 inch above the top is about $(8-y)$ in. The work done lifting the slab is about $\Delta W=\left(\frac{4\pi}{9}\right)\frac{(y+17.5)^2}{14^2}(8-y)\,\Delta y$ in · oz. The work done lifting all the slabs from $y=0$ to $y=7$ is approximately $W=\sum_0^7\frac{4\pi}{9\cdot 14^2}(y+17.5)^2(8-y)\,\Delta y$ in · oz which is a Riemann sum. The work is the limit of these sums as the norm of the partition goes to zero: $W=\int_0^7\frac{4\pi}{9\cdot 14^2}(y+17.5)^2(8-y)\,dy$

$$=\frac{4\pi}{9\cdot 14^2}\int_0^7\left(2450-26.25y-27y^2-y^3\right)dy=\frac{4\pi}{9\cdot 14^2}\left[-\frac{y^4}{4}-9y^3-\frac{26.25}{2}y^2+2450y\right]_0^7$$

$$=\frac{4\pi}{9\cdot 14^2}\left[-\frac{7^4}{4}-9\cdot 7^3-\frac{26.25}{2}\cdot 7^2+2450\cdot 7\right]\approx 91.32 \text{ in}\cdot\text{oz}$$

36. We fill the pipe and the tank.

(a) To find the work required to fill the tank follow Example 6 with radius = 10 ft. Then $\Delta V=\pi\cdot 100\,\Delta y \text{ ft}^3$. The force required will be $F=62.4\cdot\Delta V=62.4\cdot 100\pi\,\Delta y=6240\pi\,\Delta y$ lb. The distance through which F must act is y so the work done lifting the slab is about $\Delta W_1=6240\pi\cdot y\cdot\Delta y$ lb · ft. The work it takes to lift all the water into the tank is: $W_1\approx\sum_{360}^{385}\Delta W_1=\sum_{360}^{385}6240\pi\cdot y\cdot\Delta y$ lb · ft. Taking the limit we end up with $W_1=\int_{360}^{385}6240\pi y\,dy=6240\pi\left[\frac{y^2}{2}\right]_{360}^{385}=\frac{6240\pi}{2}\left[385^2-360^2\right]\approx 182{,}557{,}949$ ft · lb

(b) To find the work required to fill the pipe, do as in part (a), but take the radius to be $\frac{4}{2}$ in $=\frac{1}{6}$ ft. Then $\Delta V=\pi\cdot\frac{1}{36}\Delta y \text{ ft}^3$ and $F=62.4\cdot\Delta V=\frac{62.4\pi}{36}\Delta y$. Also take different limits of summation and integration: $W_2\approx\sum_0^{360}\Delta W_2\Rightarrow W_2=\int_0^{360}\frac{62.4}{36}\pi y\,dy=\frac{62.4\pi}{36}\left[\frac{y^2}{2}\right]_0^{360}=\left(\frac{62.4\pi}{36}\right)\left(\frac{360^2}{2}\right)\approx 352{,}864$ ft · lb.

The total work is $W = W_1 + W_2 \approx 182{,}557{,}949 + 352{,}864 \approx 182{,}910{,}813$ ft·lb. The time it takes to fill the tank and the pipe is Time $= \frac{W}{1650} \approx \frac{182{,}910{,}813}{1650} \approx 110{,}855$ sec ≈ 31 hr

37. Work $= \int_{6,370,000}^{35,780,000} \frac{1000\,MG}{r^2}\,dr = 1000\,MG \int_{6,370,000}^{35,780,000} \frac{dr}{r^2} = 1000\,MG\left[-\frac{1}{r}\right]_{6,370,000}^{35,780,000}$

$= (1000)(5.975 \cdot 10^{24})(6.672 \cdot 10^{-11})\left(\frac{1}{6{,}370{,}000} - \frac{1}{35{,}780{,}000}\right) \approx 5.144 \times 10^{10}$ J

38. (a) Let ρ be the x-coordinate of the second electron. Then $r^2 = (\rho - 1)^2 \Rightarrow W = \int_{-1}^{0} F(\rho)\,d\rho$

$= \int_{-1}^{0} \frac{(23 \times 10^{-29})}{(\rho-1)^2}\,d\rho = -\left[\frac{23 \times 10^{-29}}{\rho - 1}\right]_{-1}^{0} = (23 \times 10^{-29})\left(1 - \frac{1}{2}\right) = 11.5 \times 10^{-29}$

(b) $W = W_1 + W_2$ where W_1 is the work done against the field of the first electron and W_2 is the work done against the field of the second electron. Let ρ be the x-coordinate of the third electron. Then $r_1^2 = (\rho - 1)^2$

and $r_2^2 = (\rho+1)^2 \Rightarrow W_1 = \int_3^5 \frac{23 \times 10^{-29}}{r_1^2}\,d\rho = \int_3^5 \frac{23 \times 10^{-29}}{(\rho-1)^2}\,d\rho = -23 \times 10^{-29}\left[\frac{1}{\rho-1}\right]_3^5$

$= (-23 \times 10^{-29})\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{23}{4} \times 10^{-29}$, and $W_2 = \int_3^5 \frac{23 \times 10^{-29}}{r_2^2}\,d\rho = \int_3^5 \frac{23 \times 10^{-29}}{(\rho+1)^2}\,d\rho$

$= -23 \times 10^{-29}\left[\frac{1}{\rho+1}\right]_3^5 = (-23 \times 10^{-29})\left(\frac{1}{6} - \frac{1}{4}\right) = \frac{23 \times 10^{-29}}{12}(3-2) = \frac{23}{12} \times 10^{-29}$. Therefore

$W = W_1 + W_2 = \left(\frac{23}{4} \times 10^{-29}\right) + \left(\frac{23}{12} \times 10^{-29}\right) = \frac{23}{3} \times 10^{-29} \approx 7.67 \times 10^{-29}$ J

5.5 FLUID FORCES

1. To find the width of the plate at a typical depth y, we first find an equation for the line of the plate's right-hand edge: $y = x - 5$. If we let x denote the width of the right-hand half of the triangle at depth y, then $x = 5 + y$ and the total width is $L(y) = 2x = 2(5 + y)$. The depth of the strip is $(-y)$. The force exerted by the water against one side of the plate is therefore $F = \int_{-5}^{-2} w(-y) \cdot L(y)\,dy = \int_{-5}^{-2} 62.4 \cdot (-y) \cdot 2(5+y)\,dy$

$= 124.8 \int_{-5}^{-2} (-5y - y^2)\,dy = 124.8\left[-\frac{5}{2}y^2 - \frac{1}{3}y^3\right]_{-5}^{-2} = 124.8\left[\left(-\frac{5}{2} \cdot 4 + \frac{1}{3} \cdot 8\right) - \left(-\frac{5}{2} \cdot 25 + \frac{1}{3} \cdot 125\right)\right]$

$= (124.8)\left(\frac{105}{2} - \frac{117}{3}\right) = (124.8)\left(\frac{315 - 234}{6}\right) = 1684.8$ lb

2. An equation for the line of the plate's right-hand edge is $y = x - 3 \Rightarrow x = y + 3$. Thus the total width is $L(y) = 2x = 2(y+3)$. The depth of the strip is $(2-y)$. The force exerted by the water is

$$F = \int_{-3}^{0} w(2-y)L(y)\,dy = \int_{-3}^{0} 62.4 \cdot (2-y) \cdot 2(3+y)\,dy = 124.8 \int_{-3}^{0} \left(6 - y - y^2\right) dy = 124.8\left[6y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-3}^{0}$$

$$= (-124.8)\left(-18 - \frac{9}{2} + 9\right) = (-124.8)\left(-\frac{27}{2}\right) = 1684.8 \text{ lb}$$

3. Using the coordinate system of Exercise 2, we find the equation for the line of the plate's right-hand edge is $y = x - 3 \Rightarrow x = y + 3$. Thus the total width is $L(y) = 2x = 2(y+3)$. The depth of the strip changes to $(4-y)$

$$\Rightarrow F = \int_{-3}^{0} w(4-y)L(y)\,dy = \int_{-3}^{0} 62.4 \cdot (4-y) \cdot 2(y+3)\,dy = 124.8 \int_{-3}^{0} \left(12 + y - y^2\right) dy$$

$$= 124.8\left[12y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-3}^{0} = (-124.8)\left(-36 + \frac{9}{2} + 9\right) = (-124.8)\left(-\frac{45}{2}\right) = 2808 \text{ lb}$$

4. Using the coordinate system of Exercise 2, we see that the equation for the line of the plate's right-hand edge remains the same: $y = x - 3 \Rightarrow x = 3 + y$ and $L(y) = 2x = 2(y+3)$. The depth of the strip changes to $(-y)$

$$\Rightarrow F = \int_{-3}^{0} w(-y)L(y)\,dy = \int_{-3}^{0} 62.4 \cdot (-y) \cdot 2(y+3)\,dy = 124.8 \int_{-3}^{0} \left(-y^2 - 3y\right) dy = 124.8\left[-\frac{y^3}{3} - \frac{3}{2}y^2\right]_{-3}^{0}$$

$$= (-124.8)\left(\frac{27}{3} - \frac{27}{2}\right) = \frac{(-124.8)(27)(2-3)}{6} = 561.6 \text{ lb}$$

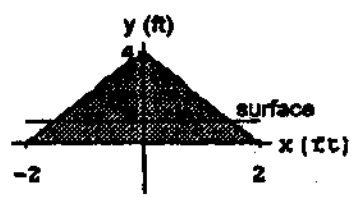
5. Using the coordinate system of Exercise 2, we find the equation for the line of the plate's right-hand edge to be $y = 2x - 4 \Rightarrow x = \frac{y+4}{2}$ and $L(y) = 2x = y + 4$. The depth of the strip is $(1-y)$.

(a) $$F = \int_{-4}^{0} w(1-y)L(y)\,dy = \int_{-4}^{0} 62.4 \cdot (1-y)(y+4)\,dy = 62.4 \int_{-4}^{0} \left(4 - 3y - y^2\right) dy = 62.4\left[4y - \frac{3y^2}{2} - \frac{y^3}{3}\right]_{-4}^{0}$$

$$= (-62.4)\left[(-4)(4) - \frac{(3)(16)}{2} + \frac{64}{3}\right] = (-62.4)\left(-16 - 24 + \frac{64}{3}\right) = \frac{(-62.4)(-120+64)}{3} = 1164.8 \text{ lb}$$

(b) $$F = (-64.0)\left[(-4)(4) - \frac{(3)(16)}{2} + \frac{64}{3}\right] = \frac{(-64.0)(-120+64)}{3} \approx 1194.7 \text{ lb}$$

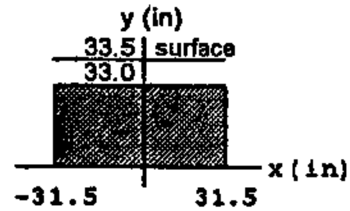
6. Using the coordinate system given, we find an equation for the line of the plate's right-hand edge to be $y = -2x + 4$ $\Rightarrow x = \frac{4-y}{2}$ and $L(y) = 2x = 4 - y$. The depth of the strip is $(1-y) \Rightarrow F = \int_{0}^{1} w(1-y)(4-y)\,dy$



$$= 62.4 \int_{0}^{1} \left(y^2 - 5y + 4\right) dy = 62.4\left[\frac{y^3}{3} - \frac{5y^2}{2} + 4y\right]_{0}^{1} = (62.4)\left(\frac{1}{3} - \frac{5}{2} + 4\right) = (62.4)\left(\frac{2 - 15 + 24}{6}\right)$$

$$= \frac{(62.4)(11)}{6} = 114.4 \text{ lb}$$

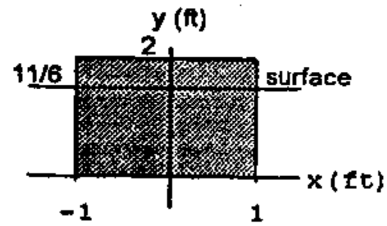
7. Using the coordinate system given in the accompanying figure, we see that the total width is $L(y) = 63$ and the depth of the strip is $(33.5 - y) \Rightarrow F = \int_0^{33} w(33.5 - y)L(y)\,dy$



$$= \int_0^{33} \frac{64}{12^3}\cdot(33.5 - y)\cdot 63\,dy = \left(\frac{64}{12^3}\right)(63)\int_0^{33}(33.5 - y)\,dy$$

$$= \left(\frac{64}{12^3}\right)(63)\left[33.5y - \frac{y^2}{2}\right]_0^{33} = \left(\frac{64\cdot 63}{12^3}\right)\left[(33.5)(33) - \frac{33^2}{2}\right] = \frac{(64)(63)(33)(67-33)}{(2)\left(12^3\right)} = 1309 \text{ lb}$$

8. (a) Use the coordinate system given in the accompanying figure. The depth of the strip is $\left(\frac{11}{6} - y\right)$ ft



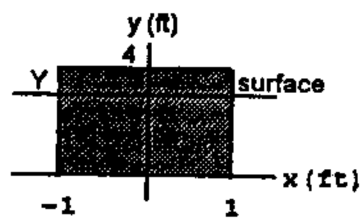
$$\Rightarrow F = \int_0^{11/6} w\left(\frac{11}{6} - y\right)(\text{width})\,dy$$

$$= (62.4)(\text{width})\int_0^{11/6}\left(\frac{11}{6} - y\right)dy$$

$$= (62.4)(\text{width})\left[\frac{11}{6}y - \frac{y^2}{2}\right]_0^{11/6} = (62.4)(\text{width})\left[\left(\frac{11}{6}\right)^2\cdot\frac{1}{2}\right] \Rightarrow F_{\text{end}} = (62.4)(2)\left(\frac{121}{36}\right)\left(\frac{1}{2}\right) \approx 209.73 \text{ lb and}$$

$F_{\text{side}} = (62.4)(4)\left(\frac{121}{36}\right)\left(\frac{1}{2}\right) \approx 419.47$ lb

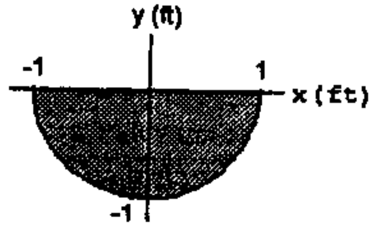
(b) Use the coordinate system given in the accompanying figure. Find Y from the condition that the entire volume of the water is conserved (no spilling): $\frac{11}{6}\cdot 2\cdot 4 = 2\cdot 2\cdot Y$ $\Rightarrow Y = \frac{11}{3}$ ft. The depth of a typical strip is $\left(\frac{11}{3} - y\right)$ ft and the total width is $L(y) = 2$ ft. Thus,



$$F = \int_0^{11/3} w\left(\frac{11}{3} - y\right)L(y)\,dy = \int_0^{11/3}(62.4)\left(\frac{11}{3} - y\right)\cdot 2\,dy = (62.4)(2)\left[\frac{11}{3}y - \frac{y^2}{2}\right]_0^{11/3} = (62.4)(2)\left[\left(\frac{1}{2}\right)\left(\frac{11}{3}\right)^2\right]$$

$$= \frac{(62.4)(121)}{9} \approx 838.93 \text{ lb} \Rightarrow \text{the fluid force doubles.}$$

9. Using the coordinate system given in the accompanying figure, we see that the right-hand edge is $x = \sqrt{1 - y^2}$ for $-1 \le y \le 0$ so the total width is $L(y) = 2x = 2\sqrt{1 - y^2}$ and the depth of the strip is $(-y)$. The force exerted by the water is therefore $F = \int_{-1}^{0} w\cdot(-y)\cdot 2\sqrt{1 - y^2}\,dy$



$$= 62.4 \int_{-1}^{0} \sqrt{1-y^2}\, d(1-y^2) = 62.4\left[\tfrac{2}{3}(1-y^2)^{3/2}\right]_{-1}^{0} = (62.4)\left(\tfrac{2}{3}\right)(1-0) = 41.6 \text{ lb}$$

10. Using the same coordinate system as in Exercise 9, the right-hand edge is $x = \sqrt{3^2 - y^2}$ and the total width is $L(y) = 2x = 2\sqrt{9-y^2}$. The depth of the strip is $(-y)$. The force exerted by the milk is therefore

$$F = \int_{-3}^{0} w\cdot(-y)\cdot 2\sqrt{9-y^2}\, dy = 64.5 \int_{-3}^{0} \sqrt{9-y^2}\, d(9-y^2) = 64.5\left[\tfrac{2}{3}(9-y^2)^{3/2}\right]_{-3}^{0} = (64.5)\left(\tfrac{2}{3}\right)(27-0)$$

$= (64.5)(18) = 1161$ lb

11. The coordinate system is given in the text. The right-hand edge is $x = \sqrt{y}$ and the total width is $L(y) = 2x = 2\sqrt{y}$.

(a) The depth of the strip is $(2-y)$ so the force exerted by the liquid on the gate is $F = \int_0^1 w(2-y)L(y)\, dy$

$$= \int_0^1 50(2-y)\cdot 2\sqrt{y}\, dy = 100 \int_0^1 (2-y)\sqrt{y}\, dy = 100 \int_0^1 \left(2y^{1/2} - y^{3/2}\right) dy = 100\left[\tfrac{4}{3}y^{3/2} - \tfrac{2}{5}y^{5/2}\right]_0^1$$

$= 100\left(\frac{4}{3} - \frac{2}{5}\right) = \left(\frac{100}{15}\right)(20-6) = 93.33$ lb

(b) Suppose that H is the maximum height to which the container can be filled without exceeding its design limitation. The depth of a typical strip is $(H-y)$ and the force is $F = \int_0^1 w(H-y)L(y)\, dy = F_{max}$, where $F_{max} = 160$ lb. Therefore, $F_{max} = w \int_0^1 (H-y)\cdot 2\sqrt{y}\, dy = 100 \int_0^1 (H-y)\sqrt{y}\, dy$

$= 100 \int_0^1 \left(Hy^{1/2} - y^{3/2}\right) dy = 100\left[\frac{2}{3}Hy^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1 = 100\left(\frac{2H}{3} - \frac{2}{5}\right) = \left(\frac{100}{15}\right)(10H-6)$. When $F_{max} = 160$ lb we have $160 = \left(\frac{100}{15}\right)(10H-6) \Rightarrow 10H - 6 = 24 \Rightarrow H = 3$ ft

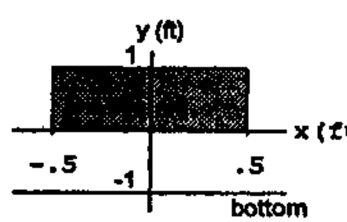
12. Use the coordinate system given in the accompanying figure. The total width is $L(y) = 1$.

(a) The depth of the strip is $(3-1) - y = (2-y)$ ft. The force exerted by the fluid in the window is

$$F = \int_0^1 w(2-y)L(y)\, dy = 62.4 \int_0^1 (2-y)\cdot 1\, dy = (62.4)\left[2y - \frac{y^2}{2}\right]_0^1 = (62.4)\left(2 - \frac{1}{2}\right) = \frac{(62.4)(3)}{2} = 93.6 \text{ lb}$$

(b) Suppose that H is the maximum height to which the tank can be filled without exceeding its design limitation. This means that the depth of a typical strip is $(H-1) - y$ and the force is

$F = \int_0^1 w[(H-1) - y]L(y)\, dy = F_{max}$, where

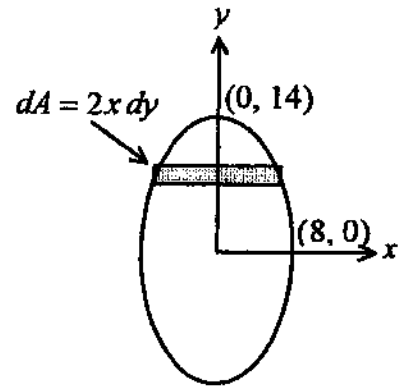


$F_{max} = 312$ lb. Thus, $F_{max} = w \int_0^1 [(H-1)-y]\cdot 1\ dy = (62.4)\left[(H-1)y - \frac{y^2}{2}\right]_0^1 = (62.4)\left(H - \frac{3}{2}\right)$

$= \left(\frac{62.4}{2}\right)(2H-3) = -93.6 + 62.4H$. Then $F_{max} = -93.6 + 62.4H \Rightarrow 312 = -93.6 + 62.4H \Rightarrow H = \frac{405.6}{62.4}$

$= 6.5$ ft

13. (a) The equation of the ellipse for the penstock gate is $\left(\frac{x}{8}\right)^2 + \left(\frac{y}{14}\right)^2 = 1$ or $49x^2 + 16y^2 = 3136 \Rightarrow x = \frac{\sqrt{3136-16y^2}}{7}$, where y is measured from the center of the ellipse.



(b) $L(y) = 2x = \frac{2}{7}\sqrt{3136-16y^2}$

(c) $\Delta F \approx 62.4[389-(y+115)](2x\,\Delta y) = 124.8(274-y)\,\frac{\sqrt{3136-16y^2}}{7}\,\Delta y$. Therefore,

$F = \int_{-14}^{14} 17.829(274-y)\sqrt{3136-16y^2}\,dy = 6.0159 \cdot 10^6 \text{ lb} = 3008$ tons.

14. (a) After 9 hours of filling there are $V = 1000 \cdot 9 = 9000$ cubic feet of water in the pool. The level of the water is $h = \frac{V}{\text{Area}}$, where Area $= 50 \cdot 30 = 1500 \Rightarrow h = \frac{9000}{1500} = 6$ ft. The depth of the typical horizontal strip at level y is then $(6-y)$ where y is measured up from the bottom of the pool. An equation for the drain plate's right-hand edge is $y = x \Rightarrow$ total width is $L(y) = 2x = 2y$. Thus the force against the drain plate is

$F = \int_0^1 w(6-y)L(y)\,dy = 62.4\int_0^1 (6-y)\cdot 2y\,dy = (62.4)(2)\int_0^1 \left(6y - y^2\right) = (62.4)(2)\left[\frac{6y^2}{2} - \frac{y^3}{3}\right]_0^1$

$= (124.8)\left(3 - \frac{1}{3}\right) = (124.8)\left(\frac{8}{3}\right) = 332.8$ lb

(b) Suppose that h is the maximum height. Then, the depth of a typical strip is $(h-y)$ and the force

$F = \int_0^1 w(h-y)L(y)\,dy = F_{max}$, where $F_{max} = 520$ lb. Hence, $F_{max} = (62.4)\int_0^1 (h-y)\cdot 2y\,dy$

$= 124.8\int_0^1 \left(hy - y^2\right)dy = (124.8)\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^1 = (124.8)\left(\frac{h}{2} - \frac{1}{3}\right) = (20.8)(3h-2) \Rightarrow \frac{520}{20.8} = 3h - 2$

$\Rightarrow h = \frac{27}{3} = 9$ ft

15. (a) The pressure at level y is $p(y) = w \cdot y \Rightarrow$ the average pressure is $\bar{p} = \frac{1}{b}\int_0^b p(y)\,dy = \frac{1}{b}\int_0^b w \cdot y\,dy = \frac{1}{b}w\left[\frac{y^2}{2}\right]_0^b$ $= \left(\frac{w}{b}\right)\left(\frac{b^2}{2}\right) = \frac{wb}{2}$. This is the pressure at level $\frac{b}{2}$, which is the pressure at the middle of the plate.

(b) The force exerted by the fluid is $F = \int_0^b w(\text{depth})(\text{length})\,dy = \int_0^b w \cdot y \cdot a\,dy$ $= (w \cdot a)\int_0^b y\,dy = (w \cdot a)\left[\frac{y^2}{2}\right]_0^b = w\left(\frac{ab^2}{2}\right) = \left(\frac{wb}{2}\right)(ab) = \bar{p} \cdot \text{Area}$, where $\bar{p}$ is the average value of the pressure (see part (a)).

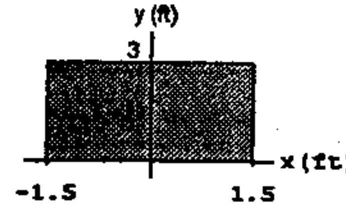
16. When the water reaches the top of the tank the force on the movable side is $\int_{-2}^{0} (62.4)\left(2\sqrt{4-y^2}\right)(-y)\,dy$ $= (62.4)\int_{-2}^{0}\left(4-y^2\right)^{1/2}(-2y)\,dy = (62.4)\left[\frac{2}{3}\left(4-y^2\right)^{3/2}\right]_{-2}^{0} = (62.4)\left(\frac{2}{3}\right)\left(4^{3/2}\right) = 332.8 \text{ ft} \cdot \text{lb}$. The force compressing the spring is $F = 100x$, so when the tank is full we have $332.8 = 100x \Rightarrow x \approx 3.33$ ft. Therefore the movable end does not reach the required 5 ft to allow drainage $\Rightarrow$ the tank will overflow.

17. (a) An equation of the right-hand edge is $y = \frac{3}{2}x \Rightarrow x = \frac{2}{3}y$ and $L(y) = 2x = \frac{4y}{3}$. The depth of the strip is $(3-y) \Rightarrow F = \int_0^3 w(3-y)L(y)\,dy = \int_0^3 (62.4)(3-y)\left(\frac{4}{3}y\right)dy = (62.4)\cdot\left(\frac{4}{3}\right)\int_0^3\left(3y-y^2\right)dy$ $= (62.4)\left(\frac{4}{3}\right)\left[\frac{3}{2}y^2 - \frac{y^3}{3}\right]_0^3 = (62.4)\left(\frac{4}{3}\right)\left[\frac{27}{2} - \frac{27}{3}\right] = (62.4)\left(\frac{4}{3}\right)\left(\frac{27}{6}\right) = 374.4$ lb

(b) We want to find a new water level Y such that $F_Y = \frac{1}{2}(374.4) = 187.2$ lb. The new depth of the strip is $(Y-y)$, and Y is the new upper limit of integration. Thus, $F_Y = \int_0^Y w(Y-y)L(y)\,dy$ $= 62.4\int_0^Y (Y-y)\left(\frac{4}{3}y\right)dy = (62.4)\left(\frac{4}{3}\right)\int_0^Y\left(Yy-y^2\right)dy = (62.4)\left(\frac{4}{3}\right)\left[Y\cdot\frac{y^2}{2} - \frac{y^3}{3}\right]_0^Y = (62.4)\left(\frac{4}{3}\right)\left(\frac{Y^3}{2} - \frac{Y^3}{3}\right)$ $= (62.4)\left(\frac{2}{9}\right)Y^3$. Therefore $Y^3 = \frac{9F_Y}{2\cdot(62.4)} = \frac{(9)(187.2)}{124.8} \Rightarrow Y = \sqrt[3]{\frac{(9)(187.2)}{124.8}} = \sqrt[3]{13.5} \approx 2.3811$ ft. So, $\Delta Y = 3 - Y \approx 3 - 2.3811 \approx 0.6189$ ft ≈ 7.5 in to the nearest half inch

(c) No, it does not matter how long the trough is. The fluid pressure and the resulting force depend only on depth of the water.

18. (a) Using the given coordinate system we see that the total width is $L(y) = 3$ and the depth of the strip is $(3 - y)$.



Thus, $F = \int_0^3 w(3-y)L(y)\,dy = \int_0^3 (62.4)(3-y)\cdot 3\,dy$

$= (62.4)(3)\int_0^3 (3-y)\,dy = (62.4)(3)\left[3y - \frac{y^2}{2}\right]_0^3$

$= (62.4)(3)\left(9 - \frac{9}{2}\right) = (62.4)(3)\left(\frac{9}{2}\right) = 842.4$ lb

(b) Find a new water level Y such that $F_Y = (0.75)(842.4 \text{ lb}) = 631.8$ lb. The new depth of the strip is $(Y - y)$ and Y is the new upper limit of integration. Thus, $F_Y = \int_0^Y w(Y-y)L(y)\,dy$

$= 62.4\int_0^Y (Y-y)\cdot 3\,dy = (62.4)(3)\int_0^Y (Y-y)\,dy = (62.4)(3)\left[Yy - \frac{y^2}{2}\right]_0^Y = (62.4)(3)\left(Y^2 - \frac{Y^2}{2}\right)$

$= (62.4)(3)\left(\frac{Y^2}{2}\right)$. Therefore, $Y = \sqrt{\frac{2F_Y}{(62.4)(3)}} = \sqrt{\frac{1263.6}{187.2}} = \sqrt{6.75} \approx 2.598$ ft. So, $\Delta Y = 3 - Y$

$\approx 3 - 2.598 \approx 0.402 \text{ ft} \approx 4.8$ in

19. Use the same coordinate system as in Exercise 20 with $L(y) = 3.75$ and the depth of a typical strip being $(7.75 - y)$. Then $F = \int_0^{7.75} w(7.75-y)L(y)\,dy = \left(\frac{64.5}{12^3}\right)(3.75)\int_0^{7.75} (7.75-y)\,dy = \left(\frac{64.5}{12^3}\right)(3.75)\left[7.75y - \frac{y^2}{2}\right]_0^{7.75}$

$= \left(\frac{64.5}{12^3}\right)(3.75)\frac{(7.75)^2}{2} \approx 4.2$ lb

20. The force against the base is $F_{\text{base}} = pA = whA = w\cdot h\cdot(\text{length})(\text{width}) = \left(\frac{57}{12^3}\right)(10)(5.75)(3.5) \approx 6.64$ lb.

To find the fluid force against each side, use the coordinate system described in Exercise 10 with the depth of a typical strip being $(10 - y)$: $F = \int_0^{10} w(10-y)\binom{\text{width of}}{\text{the side}}dy = \left(\frac{57}{12^3}\right)\binom{\text{width of}}{\text{the side}}\left[10y - \frac{y^2}{2}\right]_0^{10}$

$= \left(\frac{57}{12^3}\right)\binom{\text{width of}}{\text{the side}}\left(\frac{100}{2}\right) \Rightarrow F_{\text{end}} = \left(\frac{57}{12^3}\right)(50)(3.5) \approx 5.773$ lb and $F_{\text{side}} = \left(\frac{57}{12^3}\right)(50)(5.75) \approx 9.484$ lb

21. Suppose that h is the maximum height. Using the coordinate system given in the text, we find an equation for the line of the end plate's right-hand edge is $y = \frac{5}{2}x \Rightarrow x = \frac{2}{5}y$. The total width is $L(y) = 2x = \frac{4}{5}y$ and the depth of the typical horizontal strip at level y is $(h - y)$. Then the force is $F = \int_0^h w(h-y)L(y)\,dy = F_{\max}$,

where $F_{max} = 6667$ lb. Hence, $F_{max} = w \int_0^h (h-y)\cdot\frac{4}{5}y\,dy = (62.4)\left(\frac{4}{5}\right)\int_0^h (hy - y^2)\,dy$

$= (62.4)\left(\frac{4}{5}\right)\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h = (62.4)\left(\frac{4}{5}\right)\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = (62.4)\left(\frac{4}{5}\right)\left(\frac{1}{6}\right)h^3 = (10.4)\left(\frac{4}{5}\right)h^3 \Rightarrow h = \sqrt[3]{\left(\frac{5}{4}\right)\left(\frac{F_{max}}{10.4}\right)}$

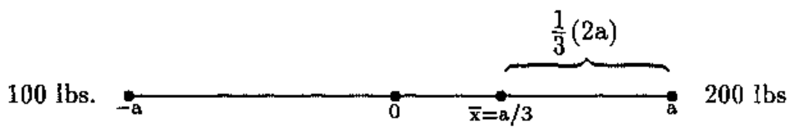
$= \sqrt[3]{\left(\frac{5}{4}\right)\left(\frac{6667}{10.4}\right)} \approx 9.288$ ft. The volume of water which the tank can hold is $V = \frac{1}{2}(\text{Base})(\text{Height})\cdot 30$, where

Height $= h$ and $\frac{1}{2}(\text{Base}) = \frac{2}{5}h \Rightarrow V = \left(\frac{2}{5}h^2\right)(30) = 12h^2 \approx 12(9.288)^2 \approx 1035\ \text{ft}^3$.

5.6 MOMENTS AND CENTERS OF MASS

1. Because the children are balanced, the moment of the system about the origin must be equal to zero: $5\cdot 80 = x\cdot 100 \Rightarrow x = 4$ ft, the distance of the 100-lb child from the fulcrum.

2. Suppose the log has length 2a. Align the log along the x-axis so the 100-lb end is placed at $x = -a$ and the 200-lb end at $x = a$. Then the center of mass $\bar{x}$ satisfies $\bar{x} = \frac{100(-a) + 200(a)}{300} \Rightarrow \bar{x} = \frac{a}{3}$. That is, $\bar{x}$ is located at a distance $a - \frac{a}{3} = \frac{2a}{3} = \frac{1}{3}(2a)$ which is $\frac{1}{3}$ of the length of the log from the 200-lb (heavier) end (see figure) or $\frac{2}{3}$ of the way from the lighter end toward the heavier end.



3. The center of mass of each rod is in its center (see Example 1). The rod system is equivalent to two point masses located at the centers of the rods at coordinates $\left(\frac{L}{2}, 0\right)$ and $\left(0, \frac{L}{2}\right)$. Therefore $\bar{x} = \frac{m_y}{m}$

$= \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2} = \frac{\frac{L}{2}\cdot m + 0}{m + m} = \frac{L}{4}$ and $\bar{y} = \frac{m_x}{m} = \frac{y_1 m_2 + y_2 m_2}{m_1 + m_2} = \frac{0 + \frac{L}{2}\cdot m}{m + m} = \frac{L}{4} \Rightarrow \left(\frac{L}{4}, \frac{L}{4}\right)$ is the center of mass location.

4. Let the rods have lengths $x = L$ and $y = 2L$. The center of mass of each rod is in its center (see Example 1). The rod system is equivalent to two point masses located at the centers of the rods at coordinates $\left(\frac{L}{2}, 0\right)$ and $(0, L)$. Therefore $\bar{x} = \frac{\frac{L}{2}\cdot m + 0\cdot 2m}{m + 2m} = \frac{L}{6}$ and $\bar{y} = \frac{0\cdot m + L\cdot 2m}{m + 2m} = \frac{2L}{3} \Rightarrow \left(\frac{L}{6}, \frac{2L}{3}\right)$ is the center of mass location.

5. $M_0 = \int_0^2 x\cdot 4\,dx = \left[4\frac{x^2}{2}\right]_0^2 = 4\cdot\frac{4}{2} = 8$; $M = \int_0^2 4\,dx = [4x]_0^2 = 4\cdot 2 = 8 \Rightarrow \bar{x} = \frac{M_0}{M} = 1$

6. $M_0 = \int_1^3 x\cdot 4\,dx = \left[4\frac{x^2}{2}\right]_1^3 = \frac{4}{2}(9-1) = 16$; $M = \int_1^3 4\,dx = [4x]_1^3 = 12 - 4 = 8 \Rightarrow \bar{x} = \frac{M_0}{M} = \frac{16}{8} = 2$

7. $M_0 = \int_0^3 x\left(1+\frac{x}{3}\right) dx = \int_0^3 \left(x+\frac{x^2}{3}\right) dx = \left[\frac{x^2}{2}+\frac{x^3}{9}\right]_0^3 = \left(\frac{9}{2}+\frac{27}{9}\right) = \frac{15}{2}$; $M = \int_0^3 \left(1+\frac{x}{3}\right) dx = \left[x+\frac{x^2}{6}\right]_0^3$

$= 3+\frac{9}{6} = \frac{9}{2} \Rightarrow \bar{x} = \frac{M_0}{M} = \frac{\left(\frac{15}{2}\right)}{\left(\frac{9}{2}\right)} = \frac{15}{9} = \frac{5}{3}$

8. $M_0 = \int_0^4 x\left(2-\frac{x}{4}\right) dx = \int_0^4 \left(2x-\frac{x^2}{4}\right) dx = \left[x^2-\frac{x^3}{12}\right]_0^4 = \left(16-\frac{64}{12}\right) = 16-\frac{16}{3} = 16\cdot\frac{2}{3} = \frac{32}{3}$;

$M = \int_0^4 \left(2-\frac{x}{4}\right) dx = \left[2x-\frac{x^2}{8}\right]_0^4 = 8-\frac{16}{8} = 6 \Rightarrow \bar{x} = \frac{M_0}{M} = \frac{32}{3\cdot 6} = \frac{16}{9}$

9. $M_0 = \int_1^4 x\left(1+\frac{1}{\sqrt{x}}\right) dx = \int_1^4 \left(x+x^{1/2}\right) dx = \left[\frac{x^2}{2}+\frac{2x^{3/2}}{3}\right]_1^4 = \left(8+\frac{16}{3}\right)-\left(\frac{1}{2}+\frac{2}{3}\right) = \frac{15}{2}+\frac{14}{3} = \frac{45+28}{6} = \frac{73}{6}$;

$M = \int_1^4 \left(1+x^{-1/2}\right) dx = \left[x+2x^{1/2}\right]_1^4 = (4+4)-(1+2) = 5 \Rightarrow \bar{x} = \frac{M_0}{M} = \frac{\left(\frac{73}{6}\right)}{5} = \frac{73}{30}$

10. $M_0 = \int_{1/4}^1 x\cdot 3\left(x^{-3/2}+x^{-5/2}\right) dx = 3\int_{1/4}^1 \left(x^{-1/2}+x^{-3/2}\right) dx = 3\left[2x^{1/2}-\frac{2}{x^{1/2}}\right]_{1/4}^1 = 3\left[(2-2)-\left(2\cdot\frac{1}{2}-\frac{2}{\left(\frac{1}{2}\right)}\right)\right]$

$= 3(4-1) = 9$; $M = 3\int_{1/4}^1 \left(x^{-3/2}+x^{-5/2}\right) dx = 3\left[\frac{-2}{x^{1/2}}-\frac{2}{3x^{3/2}}\right]_{1/4}^1 = 3\left[\left(-2-\frac{2}{3}\right)-\left(-4-\frac{16}{3}\right)\right] = 3\left(2+\frac{14}{3}\right)$

$= 6+14 = 20 \Rightarrow \bar{x} = \frac{M_0}{M} = \frac{9}{20}$

11. $M_0 = \int_0^1 x(2-x)\, dx + \int_1^2 x\cdot x\, dx = \int_0^1 \left(2x-x^2\right) dx + \int_1^2 x^2\, dx = \left[\frac{2x^2}{2}-\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}\right]_1^2 = \left(1-\frac{1}{3}\right)+\left(\frac{8}{3}-\frac{1}{3}\right)$

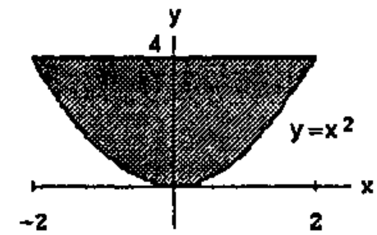
$= \frac{9}{3} = 3$; $M = \int_0^1 (2-x)\, dx + \int_1^2 x\, dx = \left[2x-\frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2}\right]_1^2 = \left(2-\frac{1}{2}\right)+\left(\frac{4}{2}-\frac{1}{2}\right) = 3 \Rightarrow \bar{x} = \frac{M_0}{M} = 1$

12. $M_0 = \int_0^1 x(x+1)\, dx + \int_1^2 2x\, dx = \int_0^1 \left(x^2+x\right) dx + \int_1^2 2x\, dx = \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1 + \left[x^2\right]_1^2 = \left(\frac{1}{3}+\frac{1}{2}\right)+(4-1)$

$= 3+\frac{5}{6} = \frac{23}{6}$; $M = \int_0^1 (x+1)\, dx + \int_1^2 2\, dx = \left[\frac{x^2}{2}+x\right]_0^1 + [2x]_1^2 = \left(\frac{1}{2}+1\right)+(4-2) = 2+\frac{3}{2} = \frac{7}{2}$

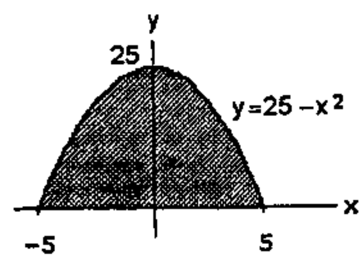
$\Rightarrow \bar{x} = \frac{M_0}{M} = \left(\frac{23}{6}\right)\left(\frac{2}{7}\right) = \frac{23}{21}$

13. Since the plate is symmetric about the y-axis and its density is constant, the distribution of mass is symmetric about the y-axis and the center of mass lies on the y-axis. This means that $\overline{x} = 0$. It remains to find $\overline{y} = \frac{M_x}{M}$. We model the distribution of mass with *vertical* strips. The typical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{x^2+4}{2}\right)$, length: $4 - x^2$, width: dx, area: $dA = (4 - x^2)\,dx$, mass: $dm = \delta\,dA = \delta(4 - x^2)\,dx$. The moment of the strip about the x-axis is $\tilde{y}\ dm = \left(\frac{x^2+4}{2}\right)\delta(4 - x^2)\,dx = \frac{\delta}{2}(16 - x^4)\,dx$. The moment of the plate about the x-axis is $M_x = \int \tilde{y}\,dm = \int_{-2}^{2} \frac{\delta}{2}(16 - x^4)\,dx = \frac{\delta}{2}\left[16x - \frac{x^5}{5}\right]_{-2}^{2} = \frac{\delta}{2}\left[\left(16 \cdot 2 - \frac{2^5}{5}\right) - \left(-16 \cdot 2 + \frac{2^5}{5}\right)\right]$ $= \frac{\delta \cdot 2}{2}\left(32 - \frac{32}{5}\right) = \frac{128\delta}{5}$. The mass of the plate is $M = \int \delta(4 - x^2)\,dx = \delta\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 2\delta\left(8 - \frac{8}{3}\right) = \frac{32\delta}{3}$.



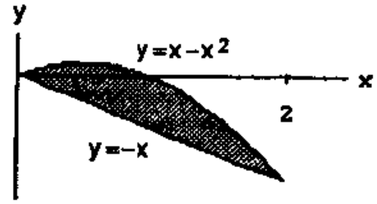
Therefore $\overline{y} = \frac{M_x}{M} = \frac{\left(\frac{128\delta}{5}\right)}{\left(\frac{32\delta}{3}\right)} = \frac{12}{5}$. The plate's center of mass is the point $(\overline{x}, \overline{y}) = \left(0, \frac{12}{5}\right)$.

14. Applying the symmetry argument analogous to the one in Exercise 13, we find $\overline{x} = 0$. To find $\overline{y} = \frac{M_x}{M}$, we use the *vertical* strips technique. The typical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{25 - x^2}{2}\right)$, length: $25 - x^2$, width: dx, area: $dA = (25 - x^2)\,dx$, mass: $dm = \delta\,dA = \delta(25 - x^2)\,dx$.



The moment of the strip about the x-axis is $\tilde{y}\,dm = \left(\frac{25 - x^2}{2}\right)\delta(25 - x^2)\,dx = \frac{\delta}{2}(25 - x^2)^2\,dx$. The moment of the plate about the x-axis is $M_x = \int \tilde{y}\,dm = \int_{-5}^{5} \frac{\delta}{2}(25 - x^2)^2\,dx = \frac{\delta}{2}\int_{-5}^{5} (625 - 50x^2 + x^4)\,dx = \frac{\delta}{2}\left[625x - \frac{50}{3}x^3 + \frac{x^5}{5}\right]_{-5}^{5}$ $= 2 \cdot \frac{\delta}{2}\left(625 \cdot 5 - \frac{50}{3} \cdot 5^3 + \frac{5^5}{5}\right) = \delta \cdot 625\left(5 - \frac{10}{3} + 1\right) = \delta \cdot 625 \cdot \left(\frac{8}{3}\right)$. The mass of the plate is $M = \int dm = \int_{-5}^{5} \delta(25 - x^2)\,dx = \delta\left[25x - \frac{x^3}{3}\right]_{-5}^{5} = 2\delta\left(5^3 - \frac{5^3}{3}\right) = \frac{4}{3}\delta \cdot 5^3$. Therefore $\overline{y} = \frac{M_x}{M}$ $= \frac{\delta \cdot 5^4 \cdot \left(\frac{8}{3}\right)}{\delta \cdot 5^3 \cdot \left(\frac{4}{3}\right)} = 10$. The plate's center of mass is the point $(\overline{x}, \overline{y}) = (0, 10)$.

15. Intersection points: $x - x^2 = -x \Rightarrow 2x - x^2 = 0$ $\Rightarrow x(2-x) = 0 \Rightarrow x = 0$ or $x = 2$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{(x-x^2)+(-x)}{2}\right)$ $= \left(x, -\frac{x^2}{2}\right)$, length: $(x-x^2)-(-x) = 2x - x^2$, width: dx, area: $dA = (2x-x^2)\,dx$, mass: $dm = \delta\, dA = \delta(2x-x^2)\,dx$.



The moment of the strip about the x-axis is $\tilde{y}\,dm = \left(-\frac{x^2}{2}\right)\delta(2x-x^2)\,dx$; about the y-axis it is $\tilde{x}\,dm = x\cdot\delta(2x-x^2)\,dx$. Thus, $M_x = \int \tilde{y}\,dm = -\int_0^2 \left(\frac{\delta}{2}x^2\right)(2x-x^2)\,dx = -\frac{\delta}{2}\int_0^2 (2x^3-x^4)\,dx$

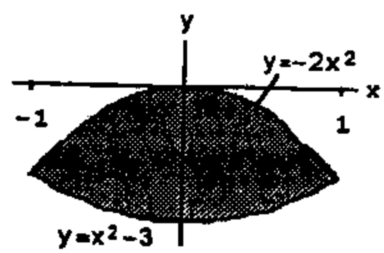
$= -\frac{\delta}{2}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = -\frac{\delta}{2}\left(2^3 - \frac{2^5}{5}\right) = -\frac{\delta}{2}\cdot 2^3\left(1-\frac{4}{5}\right) = -\frac{4\delta}{5}$; $M_y = \int \tilde{x}\,dm = \int_0^2 x\cdot\delta(2x-x^2)\,dx$

$= \delta\int_0^2 (2x^2-x^3) = \delta\left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_0^2 = \delta\left(2\cdot\frac{2^3}{3} - \frac{2^4}{4}\right) = \frac{\delta\cdot 2^4}{12} = \frac{4\delta}{3}$; $M = \int dm = \int_0^2 \delta(2x-x^2)\,dx$

$= \delta\int_0^2 (2x-x^2)\,dx = \delta\left[x^2 - \frac{x^3}{3}\right]_0^2 = \delta\left(4-\frac{8}{3}\right) = \frac{4\delta}{3}$. Therefore, $\bar{x} = \frac{M_y}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{3}{4\delta}\right) = 1$ and $\bar{y} = \frac{M_x}{M}$

$= \left(-\frac{4\delta}{5}\right)\left(\frac{3}{4\delta}\right) = -\frac{3}{5} \Rightarrow (\bar{x},\bar{y}) = \left(1, -\frac{3}{5}\right)$ is the center of mass.

16. Intersection points: $x^2 - 3 = -2x^2 \Rightarrow 3x^2 - 3 = 0$ $\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow x = -1$ or $x = 1$. Applying the symmetry argument analogous to the one in Exercise 13, we find $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{-2x^2+(x^2-3)}{2}\right) = \left(x, \frac{-x^2-3}{2}\right)$, length: $-2x^2 - (x^2-3) = 3(1-x^2)$, width: dx,



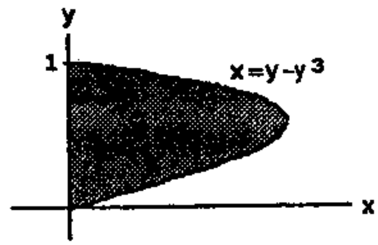
area: $dA = 3(1-x^2)\,dx$, mass: $dm = \delta\,dA = 3\delta(1-x^2)\,dx$. The moment of the strip about the x-axis is $\tilde{y}\,dm = \frac{3}{2}\delta(-x^2-3)(1-x^2)\,dx = \frac{3}{2}\delta(x^4+3x^2-x^2-3)\,dx = \frac{3}{2}\delta(x^4+2x^2-3)\,dx$; $M_x = \int \tilde{y}\,dm$

$= \frac{3}{2}\delta\int_{-1}^{1}(x^4+2x^2-3)\,dx = \frac{3}{2}\delta\left[\frac{x^5}{5} + \frac{2x^3}{3} - 3x\right]_{-1}^{1} = \frac{3}{2}\cdot\delta\cdot 2\left(\frac{1}{5}+\frac{2}{3}-3\right) = 3\delta\left(\frac{3+10-45}{15}\right) = -\frac{32\delta}{5}$;

$M = \int dm = 3\delta\int_{-1}^{1}(1-x^2)\,dx = 3\delta\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 3\delta\cdot 2\left(1-\frac{1}{3}\right) = 4\delta$. Therefore, $\bar{y} = \frac{M_x}{M} = -\frac{\delta\cdot 32}{5\cdot\delta\cdot 4} = -\frac{8}{5}$

$\Rightarrow (\bar{x},\bar{y}) = \left(0, -\frac{8}{5}\right)$ is the center of mass.

17. The typical *horizontal* strip has center of mass:



$(\tilde{x},\tilde{y}) = \left(\frac{y-y^3}{2}, y\right)$, length: $y-y^3$, width: dy,

area: $dA = (y-y^3)\,dy$, mass: $dm = \delta\,dA = \delta(y-y^3)\,dy$.

The moment of the strip about the y-axis is

$\tilde{x}\,dm = \delta\left(\frac{y-y^3}{2}\right)(y-y^3)\,dy = \frac{\delta}{2}(y-y^3)^2\,dy$

$= \frac{\delta}{2}(y^2-2y^4+y^6)\,dy$; the moment about the x-axis is $\tilde{y}\,dm = \delta y(y-y^3)\,dy = \delta(y^2-y^4)\,dy$. Thus,

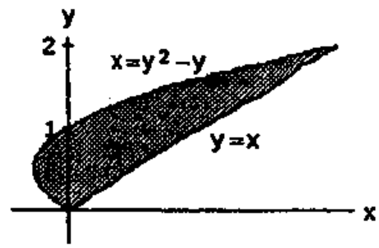
$M_x = \int \tilde{y}\,dm = \delta\int_0^1 (y^2-y^4)\,dy = \delta\left[\frac{y^3}{3}-\frac{y^5}{5}\right]_0^1 = \delta\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{2\delta}{15}$; $M_y = \int \tilde{x}\,dm = \frac{\delta}{2}\int_0^1 (y^2-2y^4+y^6)\,dy$

$= \frac{\delta}{2}\left[\frac{y^3}{3}-\frac{2y^5}{5}+\frac{y^7}{7}\right]_0^1 = \frac{\delta}{2}\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right) = \frac{\delta}{2}\left(\frac{35-42+15}{3\cdot5\cdot7}\right) = \frac{4\delta}{105}$; $M = \int dm = \delta\int_0^1 (y-y^3)\,dy$

$= \delta\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1 = \delta\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\delta}{4}$. Therefore, $\bar{x} = \frac{M_y}{M} = \left(\frac{4\delta}{105}\right)\left(\frac{4}{\delta}\right) = \frac{16}{105}$ and $\bar{y} = \frac{M_x}{M} = \left(\frac{2\delta}{15}\right)\left(\frac{4}{\delta}\right) = \frac{8}{15}$

$\Rightarrow (\bar{x},\bar{y}) = \left(\frac{16}{105}, \frac{8}{15}\right)$ is the center of mass.

18. Intersection points: $y = y^2 - y \Rightarrow y^2-2y = 0 \Rightarrow y(y-2)$ $= 0 \Rightarrow y = 0$ or $y = 2$. The typical *horizontal* strip has center of mass: $(\tilde{x},\tilde{y}) = \left(\frac{(y^2-y)+y}{2}, y\right) = \left(\frac{y^2}{2}, y\right)$,



length: $y-(y^2-y) = 2y-y^2$, width: dy,

area: $dA = (2y-y^2)\,dy$, mass: $dm = \delta\,dA = \delta(2y-y^2)\,dy$.

The moment about the y-axis is $\tilde{x}\,dm = \frac{\delta}{2}\cdot y^2(2y-y^2)\,dy$

$= \frac{\delta}{2}(2y^3-y^4)\,dy$; the moment about the x-axis is $\tilde{y}\,dm = \delta y(2y-y^2)\,dy = \delta(2y^2-y^3)\,dy$. Thus,

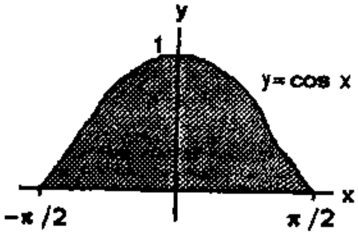
$M_x = \int \tilde{y}\,dm = \int_0^2 \delta(2y^2-y^3)\,dy = \delta\left[\frac{2y^3}{3}-\frac{y^4}{4}\right]_0^2 = \delta\left(\frac{16}{3}-\frac{16}{4}\right) = \frac{16\delta}{12}(4-3) = \frac{4\delta}{3}$; $M_y = \int \tilde{x}\,dm$

$= \int_0^2 \frac{\delta}{2}(2y^3-y^4)\,dy = \frac{\delta}{2}\left[\frac{y^4}{2}-\frac{y^5}{5}\right]_0^2 = \frac{\delta}{2}\left(8-\frac{32}{5}\right) = \frac{\delta}{2}\left(\frac{40-32}{5}\right) = \frac{4\delta}{5}$; $M = \int dm = \int_0^2 \delta(2y-y^2)\,dy$

$= \delta\left[y^2-\frac{y^3}{3}\right]_0^2 = \delta\left(4-\frac{8}{3}\right) = \frac{4\delta}{3}$. Therefore, $\bar{x} = \frac{M_y}{M} = \left(\frac{4\delta}{5}\right)\left(\frac{3}{4\delta}\right) = \frac{3}{5}$ and $\bar{y} = \frac{M_x}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{3}{4\delta}\right) = 1$

$\Rightarrow (\bar{x},\bar{y}) = \left(\frac{3}{5}, 1\right)$ is the center of mass.

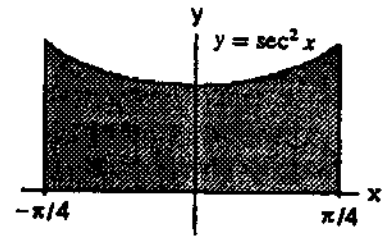
19. Applying the symmetry argument analogous to the one used in Exercise 13, we find $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\cos x}{2}\right)$, length: $\cos x$, width: dx, area: $dA = \cos x\, dx$, mass: $dm = \delta\, dA = \delta \cos x\, dx$. The moment of the strip about the x-axis is $\tilde{y}\, dm = \delta \cdot \frac{\cos x}{2} \cdot \cos x\, dx$ $= \frac{\delta}{2} \cos^2 x\, dx = \frac{\delta}{2}\left(\frac{1+\cos 2x}{2}\right) dx = \frac{\delta}{4}(1+\cos 2x)\, dx$; thus,



$M_x = \int \tilde{y}\, dm = \int_{-\pi/2}^{\pi/2} \frac{\delta}{4}(1+\cos 2x)\, dx = \frac{\delta}{4}\left[x + \frac{\sin 2x}{2}\right]_{-\pi/2}^{\pi/2} = \frac{\delta}{4}\left[\left(\frac{\pi}{2}+0\right) - \left(-\frac{\pi}{2}\right)\right] = \frac{\delta\pi}{4}$; $M = \int dm$

$= \delta \int_{-\pi/2}^{\pi/2} \cos x\, dx = \delta[\sin x]_{-\pi/2}^{\pi/2} = 2\delta$. Therefore, $\bar{y} = \frac{M_x}{M} = \frac{\delta\pi}{4 \cdot 2\delta} = \frac{\pi}{8} \Rightarrow (\bar{x}, \bar{y}) = \left(0, \frac{\pi}{8}\right)$ is the center of mass.

20. Applying the symmetry argument analogous to the one used in Exercise 13, we find $\bar{x} = 0$. The typical vertical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sec^2 x}{2}\right)$, length: $\sec^2 x$, width: dx, area: $dA = \sec^2 x\, dx$, mass: $dm = \delta\, dA = \delta \sec^2 x\, dx$. The moment about the x-axis is $\tilde{y}\, dm = \left(\frac{\sec^2 x}{2}\right)\left(\delta \sec^2 x\right) dx$



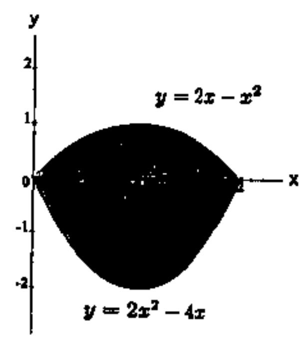
$= \frac{\delta}{2} \sec^4 x\, dx$. $M_x = \int_{-\pi/4}^{\pi/4} \tilde{y}\, dm = \frac{\delta}{2} \int_{-\pi/4}^{\pi/4} \sec^4 x\, dx = \frac{\delta}{2} \int_{-\pi/4}^{\pi/4} \left(\tan^2 x + 1\right)\left(\sec^2 x\right) dx$

$= \frac{\delta}{2} \int_{-\pi/4}^{\pi/4} (\tan x)^2 \left(\sec^2 x\right) dx + \frac{\delta}{2} \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \frac{\delta}{2}\left[\frac{(\tan x)^3}{3}\right]_{-\pi/4}^{\pi/4} + \frac{\delta}{2}[\tan x]_{-\pi/4}^{\pi/4}$

$= \frac{\delta}{2}\left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] + \frac{\delta}{2}[1 - (-1)] = \frac{\delta}{3} + \delta = \frac{4\delta}{3}$; $M = \int dm = \delta \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \delta[\tan x]_{-\pi/4}^{\pi/4}$

$= \delta[1 - (-1)] = 2\delta$. Therefore, $\bar{y} = \frac{M_x}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{1}{2\delta}\right) = \frac{2}{3} \Rightarrow (\bar{x}, \bar{y}) = \left(0, \frac{2}{3}\right)$ is the center of mass.

21. Since the plate is symmetric about the line $x = 1$ and its density is constant, the distribution of mass is symmetric about this line and the center of mass lies on it. This means that $\bar{x} = 1$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\left(2x - x^2\right) + \left(2x^2 - 4x\right)}{2}\right) = \left(x, \frac{x^2 - 2x}{2}\right)$, length: $\left(2x - x^2\right) - \left(2x^2 - 4x\right) = -3x^2 + 6x = 3\left(2x - x^2\right)$, width: dx, area: $dA = 3\left(2x - x^2\right) dx$, mass: $dm = \delta\, dA$ $= 3\delta\left(2x - x^2\right) dx$. The moment about the x-axis is



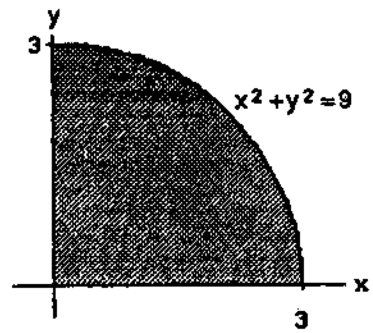
$\tilde{y}\, dm = \frac{3}{2}\delta\left(x^2 - 2x\right)\left(2x - x^2\right) dx = -\frac{3}{2}\delta\left(x^2 - 2x\right)^2 dx = -\frac{3}{2}\delta\left(x^4 - 4x^3 + 4x^2\right) dx$. Thus, $M_x = \int \tilde{y}\, dm$

$$= -\int_0^2 \frac{3}{2}\delta\left(x^4 - 4x^3 + 4x^2\right) dx = -\frac{3}{2}\delta\left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_0^2 = -\frac{3}{2}\delta\left(\frac{2^5}{5} - 2^4 + \frac{4}{3}\cdot 2^3\right) = -\frac{3}{2}\delta\cdot 2^4\left(\frac{2}{5} - 1 + \frac{2}{3}\right)$$

$$= -\frac{3}{2}\delta\cdot 2^4\left(\frac{6-15+10}{15}\right) = -\frac{8\delta}{5};\ M = \int dm = \int_0^2 3\delta\left(2x - x^2\right) dx = 3\delta\left[x^2 - \frac{x^3}{3}\right]_0^2 = 3\delta\left(4 - \frac{8}{3}\right) = 4\delta.$$

Therefore, $\overline{y} = \frac{M_x}{M} = \left(-\frac{8\delta}{5}\right)\left(\frac{1}{4\delta}\right) = -\frac{2}{5} \Rightarrow (\overline{x}, \overline{y}) = \left(1, -\frac{2}{5}\right)$ is the center of mass.

22. (a) Since the plate is symmetric about the line $x = y$ and its density is constant, the distribution of mass is symmetric about this line. This means that $\overline{x} = \overline{y}$. The typical *vertical* strip has center of mass:



$(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{9-x^2}}{2}\right)$, length: $\sqrt{9-x^2}$, width: dx,

area: $dA = \sqrt{9-x^2}\,dx$, mass: $dm = \delta\,dA = \delta\sqrt{9-x^2}\,dx$.

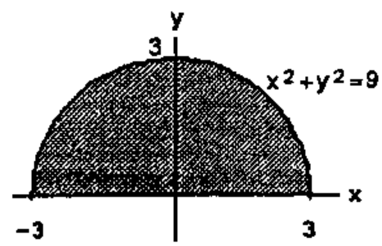
The moment about the x-axis is $\tilde{y}\,dm = \delta\left(\frac{\sqrt{9-x^2}}{2}\right)\sqrt{9-x^2}\,dx$

$= \frac{\delta}{2}\left(9 - x^2\right) dx$. Thus, $M_x = \int \tilde{y}\,dm = \int_0^3 \frac{\delta}{2}\left(9 - x^2\right) dx = \frac{\delta}{2}\left[9x - \frac{x^3}{3}\right]_0^3 = \frac{\delta}{2}(27 - 9) = 9\delta$;

$M = \int dm = \int \delta\,dA = \delta \int dA = \delta(\text{Area of a quarter of a circle of radius } 3) = \delta\left(\frac{9\pi}{4}\right) = \frac{9\pi\delta}{4}$. Therefore,

$\overline{y} = \frac{M_x}{M} = (9\delta)\left(\frac{4}{9\pi\delta}\right) = \frac{4}{\pi} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{4}{\pi}, \frac{4}{\pi}\right)$ is the center of mass.

(b) Applying the symmetry argument analogous to the one used in Exercise 13, we find that $\overline{x} = 0$. The typical vertical strip has the same parameters as in part (a). Thus,



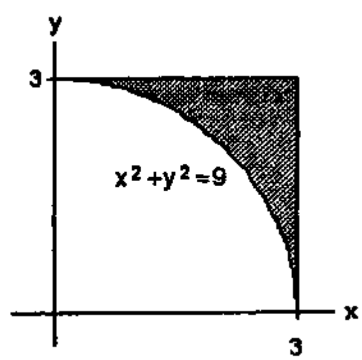
$M_x = \int \tilde{y}\,dm = \int_{-3}^3 \frac{\delta}{2}\left(9 - x^2\right) dx = 2\int_0^3 \frac{\delta}{2}\left(9 - x^2\right) dx$

$= 2(9\delta) = 18\delta$; $M = \int dm = \int \delta\,dA = \delta \int dA$

$= \delta(\text{Area of a semi-circle of radius } 3) = \delta\left(\frac{9\pi}{2}\right) = \frac{9\pi\delta}{2}$. Therefore, $\overline{y} = \frac{M_x}{M} = (18\delta)\left(\frac{2}{9\pi\delta}\right) = \frac{4}{\pi}$, the same $\overline{y}$

as in part (a) $\Rightarrow (\overline{x}, \overline{y}) = \left(0, \frac{4}{\pi}\right)$ is the center of mass.

23. Since the plate is symmetric about the line $x = y$ and its density is constant, the distribution of mass is symmetric about this line. This means that $\overline{x} = \overline{y}$. The typical *vertical* strip has



center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{3 + \sqrt{9-x^2}}{2}\right)$,

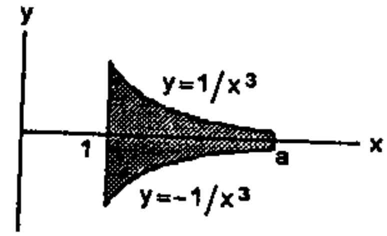
length: $3 - \sqrt{9-x^2}$, width: dx, area: $dA = \left(3 - \sqrt{9-x^2}\right) dx$,

mass: $dm = \delta\,dA = \delta\left(3 - \sqrt{9-x^2}\right) dx$. The moment about the

x-axis is $\tilde{y}\,dm = \delta\frac{\left(3 + \sqrt{9-x^2}\right)\left(3 - \sqrt{9-x^2}\right)}{2}\,dx$

$= \frac{\delta}{2}[9 - (9 - x^2)]\,dx = \frac{\delta x^2}{2}\,dx$. Thus, $M_x = \int_0^3 \frac{\delta x^2}{2}\,dx = \frac{\delta}{6}[x^3]_0^3 = \frac{9\delta}{2}$. The area equals the area of a square with side length 3 minus one quarter the area of a disk with radius 3 $\Rightarrow A = 3^2 - \frac{\pi 9}{4} = \frac{9}{4}(4 - \pi) \Rightarrow M = \delta A = \frac{9\delta}{4}(4 - \pi)$. Therefore, $\overline{y} = \frac{M_x}{M} = \left(\frac{9\delta}{2}\right)\left[\frac{4}{9\delta(4-\pi)}\right] = \frac{2}{4-\pi} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{2}{4-\pi}, \frac{2}{4-\pi}\right)$ is the center of mass.

24. Applying the symmetry argument analogous to the one used in Exercise 13, we find that $\overline{y} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\frac{1}{x^3} - \frac{1}{x^3}}{2}\right) = (x, 0)$, length: $\frac{1}{x^3} - \left(-\frac{1}{x^3}\right) = \frac{2}{x^3}$, width: dx, area: $dA = \frac{2}{x^3}\,dx$, mass: $dm = \delta\,dA = \frac{2\delta}{x^3}\,dx$. The moment about the y-axis is

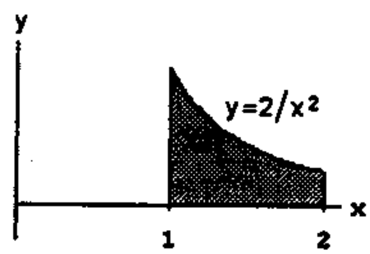


$\tilde{x}\,dm = x \cdot \frac{2\delta}{x^3}\,dx = \frac{2\delta}{x^2}\,dx$. Thus, $M_y = \int \tilde{x}\,dm = \int_1^a \frac{2\delta}{x^2}\,dx = 2\delta\left[-\frac{1}{x}\right]_1^a = 2\delta\left(-\frac{1}{a} + 1\right) = \frac{2\delta(a-1)}{a}$; $M = \int dm = \int_1^a \frac{2\delta}{x^3}\,dx = \delta\left[-\frac{1}{x^2}\right]_1^a = \delta\left(-\frac{1}{a^2} + 1\right) = \frac{\delta(a^2 - 1)}{a^2}$. Therefore, $\overline{x} = \frac{M_y}{M} = \left[\frac{2\delta(a-1)}{a}\right]\left[\frac{a^2}{\delta(a^2-1)}\right] = \frac{2a}{a+1} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{2a}{a+1}, 0\right)$. Also, $\lim_{a\to\infty} \overline{x} = 2$.

25. $M_x = \int \tilde{y}\,dm = \int_1^2 \frac{\left(\frac{2}{x^2}\right)}{2} \cdot \delta \cdot \left(\frac{2}{x^2}\right) dx$

$= \int_1^2 \left(\frac{1}{x^2}\right)(x^2)\left(\frac{2}{x^2}\right) dx = \int_1^2 \frac{2}{x^2}\,dx = 2\int_1^2 x^{-2}\,dx$

$= 2\left[-x^{-1}\right]_1^2 = 2\left[\left(-\frac{1}{2}\right) - (-1\right] = 2\left(\frac{1}{2}\right) = 1$;

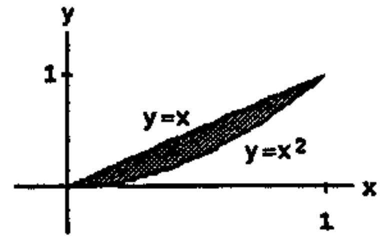


$M_y = \int \tilde{x}\,dm = \int_1^2 x \cdot \delta \cdot \left(\frac{2}{x^2}\right) dx$

$= \int_1^2 x(x^2)\left(\frac{2}{x^2}\right) dx = 2\int_1^2 x\,dx = \;= 2\left[\frac{x^2}{2}\right]_1^2 = 2\left(2 - \frac{1}{2}\right) = 4 - 1 = 3$; $M = \int dm = \int_1^2 \delta\left(\frac{2}{x^2}\right) dx$

$= \int_1^2 x^2\left(\frac{2}{x^2}\right) dx = 2\int_1^2 dx = 2[x]_1^2 = 2(2-1) = 2$. So $\overline{x} = \frac{M_y}{M} = \frac{3}{2}$ and $\overline{y} = \frac{M_x}{M} = \frac{1}{2} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{3}{2}, \frac{1}{2}\right)$ is the center of mass.

26. We use the *vertical* strip approach:



$$M_x = \int \tilde{y}\,dm = \int_0^1 \frac{(x+x^2)}{2}(x-x^2)\cdot\delta\,dx = \frac{1}{2}\int_0^1 (x^2-x^4)\cdot 12x\,dx$$

$$= 6\int_0^1 (x^3-x^5)\,dx = 6\left[\frac{x^4}{4}-\frac{x^6}{6}\right]_0^1 = 6\left(\frac{1}{4}-\frac{1}{6}\right) = \frac{6}{4}-1 = \frac{1}{2};$$

$$M_y = \int \tilde{x}\,dm = \int_0^1 x(x-x^2)\cdot\delta\,dx = \int_0^1 (x^2-x^3)\cdot 12x\,dx = 12\int_0^1 (x^3-x^4)\,dx = 12\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_0^1 = 12\left(\frac{1}{4}-\frac{1}{5}\right)$$

$$= \frac{12}{20} = \frac{3}{5};\ M = \int dm = \int_0^1 (x-x^2)\cdot\delta\,dx = 12\int_0^1 (x^2-x^3)\,dx = 12\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = 12\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{12}{12} = 1.\ \text{So}$$

$\overline{x} = \frac{M_y}{M} = \frac{3}{5}$ and $\overline{y} = \frac{M_x}{M} = \frac{1}{2} \Rightarrow \left(\frac{3}{5},\frac{1}{2}\right)$ is the center of mass.

27. (a) We use the shell method:

$$V = \int_a^b 2\pi\begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}dx = \int_1^4 2\pi x\left[\frac{4}{\sqrt{x}}-\left(-\frac{4}{\sqrt{x}}\right)\right]dx$$

$$= 16\pi\int_1^4 \frac{x}{\sqrt{x}}\,dx = 16\pi\int_1^4 x^{1/2}\,dx = 16\pi\left[\frac{2}{3}x^{3/2}\right]_1^4$$

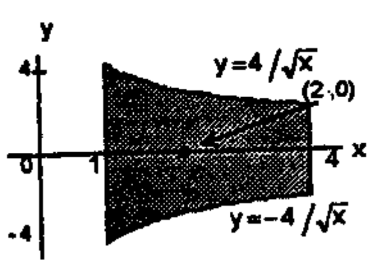
$$= 16\pi\left(\frac{2}{3}\cdot 8-\frac{2}{3}\right) = \frac{32\pi}{3}(8-1) = \frac{224\pi}{3}$$

(b) Since the plate is symmetric about the x-axis and its density $\delta(x) = \frac{1}{x}$ is a function of x alone, the distribution of its mass is symmetric about the x-axis. This means that $\overline{y} = 0$. We use the vertical strip approach to find $\overline{x}$: $M_y = \int \tilde{x}\,dm = \int_1^4 x\cdot\left[\frac{4}{\sqrt{x}}-\left(-\frac{4}{\sqrt{x}}\right)\right]\cdot\delta\,dx = \int_1^4 x\cdot\frac{8}{\sqrt{x}}\cdot\frac{1}{x}\,dx = 8\int_1^4 x^{-1/2}\,dx$

$$= 8\left[2x^{1/2}\right]_1^4 = 8(2\cdot 2-2) = 16;\ M = \int dm = \int_1^4 \left[\frac{4}{\sqrt{x}}-\left(\frac{-4}{\sqrt{x}}\right)\right]\cdot\delta\,dx = 8\int_1^4 \left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{x}\right)dx = 8\int_1^4 x^{-3/2}\,dx$$

$$= 8\left[-2x^{-1/2}\right]_1^4 = 8[-1-(-2)] = 8.\ \text{So } \overline{x} = \frac{M_y}{M} = \frac{16}{8} = 2 \Rightarrow (\overline{x},\overline{y}) = (2,0) \text{ is the center of mass.}$$

(c)



28. (a) We use the disk method:

$$V = \int_a^b \pi R^2(x)\,dx = \int_1^4 \pi\left(\frac{4}{x^2}\right)dx = 4\pi\int_1^4 x^{-2}\,dx$$

$$= 4\pi\left[-\frac{1}{x}\right]_1^4 = 4\pi\left[\frac{-1}{4}-(-1)\right] = \pi[-1+4] = 3\pi$$

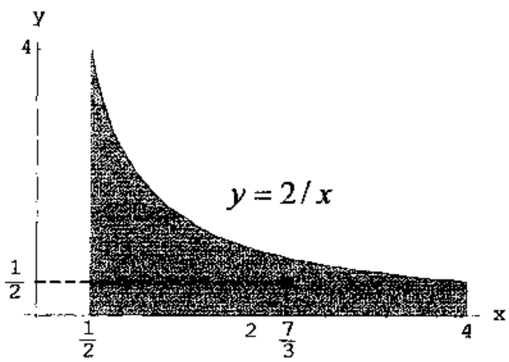
(b) We model the distribution of mass with vertical strips: $M_x = \int \tilde{y}\,dm = \int_1^4 \frac{\left(\frac{2}{x}\right)}{2}\cdot\left(\frac{2}{x}\right)\cdot\delta\,dx = \int_1^4 \frac{2}{x^2}\cdot\sqrt{x}\,dx$

$$= 2\int_1^4 x^{-3/2}\,dx = 2\left[\frac{-2}{\sqrt{x}}\right]_1^4 = 2[-1-(-2)] = 2;\ M_y = \int \tilde{x}\,dm = \int_1^4 x\cdot\tfrac{2}{x}\cdot\delta\,dx = 2\int_1^4 x^{1/2}\,dx$$

$$= 2\left[\frac{2x^{3/2}}{3}\right]_1^4 = 2\left[\frac{16}{3}-\frac{2}{3}\right] = \frac{28}{3};\ M = \int dm = \int_1^4 \frac{2}{x}\cdot\delta\,dx = 2\int_1^4 \frac{\sqrt{x}}{x}\,dx = 2\int_1^4 x^{-1/2}\,dx = 2\left[2x^{1/2}\right]_1^4$$

$= 2(4-2) = 4$. So $\overline{x} = \frac{M_y}{M} = \frac{\left(\frac{28}{3}\right)}{4} = \frac{7}{3}$ and $\overline{y} = \frac{M_x}{M} = \frac{2}{4} = \frac{1}{2} \Rightarrow (\overline{x},\overline{y}) = \left(\frac{7}{3},\frac{1}{2}\right)$ is the center of mass.

(c)



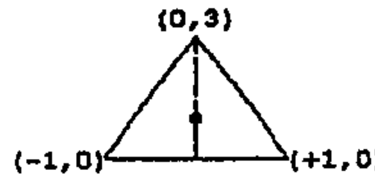
29. The mass of a horizontal strip is $dm = \delta\,dA = \delta L$, where L is the width of the triangle at a distance of y above its base on the x-axis as shown in the figure in the text. Also, by similar triangles we have $\frac{L}{b} = \frac{h-y}{h}$

$$\Rightarrow L = \frac{b}{h}(h-y).\ \text{Thus, } M_x = \int \tilde{y}\,dm = \int_0^h \delta y\left(\frac{b}{h}\right)(h-y)\,dy = \frac{\delta b}{h}\int_0^h \left(hy-y^2\right)dy = \frac{\delta b}{h}\left[\frac{hy^2}{2}-\frac{y^3}{3}\right]_0^h$$

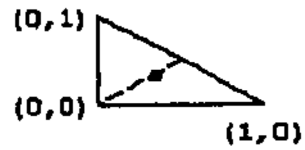
$$= \frac{\delta b}{h}\left(\frac{h^3}{2}-\frac{h^3}{3}\right) = \delta bh^2\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{\delta bh^2}{6};\ M = \int dm = \int_0^h \delta\left(\frac{b}{h}\right)(h-y)\,dy = \frac{\delta b}{h}\int_0^h (h-y)\,dy = \frac{\delta b}{h}\left[hy-\frac{y^2}{2}\right]_0^h$$

$= \frac{\delta b}{h}\left(h^2-\frac{h^2}{2}\right) = \frac{\delta bh}{2}$. So $\overline{y} = \frac{M_x}{M} = \left(\frac{\delta bh^2}{6}\right)\left(\frac{2}{\delta bh}\right) = \frac{h}{3} \Rightarrow$ the center of mass lies above the base of the triangle one-third of the way toward the opposite vertex. Similarly the other two sides of the triangle can be placed on the x-axis and the same results will occur. Therefore the centroid does lie at the intersection of the medians, as claimed.

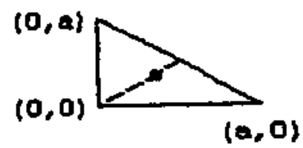
30. From the symmetry about the y-axis it follows that $\overline{x} = 0$. It also follows that the line through the points $(0,0)$ and $(0,3)$ is a median $\Rightarrow \overline{y} = \frac{1}{3}(3-0) = 1 \Rightarrow (\overline{x},\overline{y}) = (0,1)$.



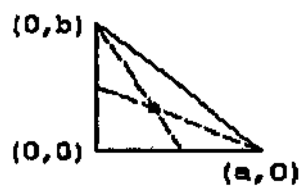
31. From the symmetry about the line $x = y$ it follows that $\overline{x} = \overline{y}$. It also follows that the line through the points $(0,0)$ and $\left(\frac{1}{2},\frac{1}{2}\right)$ is a median $\Rightarrow \overline{y} = \overline{x} = \frac{2}{3}\cdot\left(\frac{1}{2}-0\right) = \frac{1}{3}$ $\Rightarrow (\overline{x},\overline{y}) = \left(\frac{1}{3},\frac{1}{3}\right)$.



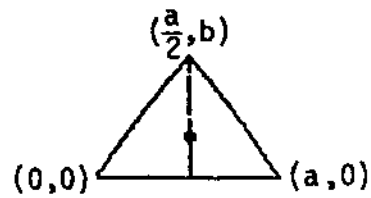
32. From the symmetry about the line $x = y$ it follows that $\overline{x} = \overline{y}$. It also follows that the line through the point $(0,0)$ and $\left(\frac{a}{2},\frac{a}{2}\right)$ is a median $\Rightarrow \overline{y} = \overline{x} = \frac{2}{3}\left(\frac{a}{2}-0\right) = \frac{1}{3}a$ $\Rightarrow (\overline{x},\overline{y}) = \left(\frac{a}{3},\frac{a}{3}\right)$.



33. The point of intersection of the median from the vertex $(0,b)$ to the opposite side has coordinates $\left(0,\frac{a}{2}\right) \Rightarrow \overline{y} = (b-0)\cdot\frac{1}{3}$ $= \frac{b}{3}$ and $\overline{x} = \left(\frac{a}{2}-0\right)\cdot\frac{2}{3} = \frac{a}{3} \Rightarrow (\overline{x},\overline{y}) = \left(\frac{a}{3},\frac{b}{3}\right)$.



34. From the symmetry about the line $x = \frac{a}{2}$ it follows that $\overline{x} = \frac{a}{2}$. It also follows that the line through the points $\left(\frac{a}{2},0\right)$ and $\left(\frac{a}{2},b\right)$ is a median $\Rightarrow \overline{y} = \frac{1}{3}(b-0) = \frac{b}{3}$ $\Rightarrow (\overline{x},\overline{y}) = \left(\frac{a}{2},\frac{b}{3}\right)$.



35. $y = x^{1/2} \Rightarrow dy = \frac{1}{2}x^{-1/2}\,dx \Rightarrow ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+\frac{1}{4x}}\,dx$; $M_x = \delta\int_0^2 x\sqrt{1+\frac{1}{4x}}\,dx$

$$= \delta\int_0^2 x+\frac{1}{4}\,dx = \frac{2\delta}{3}\left[\left(x+\frac{1}{4}\right)^{3/2}\right]_0^2 = \frac{2\delta}{3}\left[\left(2+\frac{1}{4}\right)^{3/2}-\left(\frac{1}{4}\right)^{3/2}\right] = \frac{2\delta}{3}\left[\left(\frac{9}{4}\right)^{3/2}-\left(\frac{1}{4}\right)^{3/2}\right] = \frac{2\delta}{3}\left(\frac{27}{8}-\frac{1}{8}\right) = \frac{13\delta}{6}$$

36. $y = x^3 \Rightarrow dy = 3x^2\,dx \Rightarrow dx = \sqrt{(dx)^2+\left(3x^2\,dx\right)^2} = \sqrt{1+9x^4}\,dx$; $M_x = \delta\int_0^1 x^3\sqrt{1+9x^4}\,dx$;

$\left[u = 1+9x^4 \Rightarrow du = 36x^3\,dx \Rightarrow \frac{1}{36}\,du = x^3\,dx;\ x = 0 \Rightarrow u = 1,\ x = 1 \Rightarrow u = 10\right]$

$$\to M_x = \delta \int_1^{10} \frac{1}{36} u^{1/2}\, du = \frac{\delta}{36}\left[\frac{2}{3} u^{3/2}\right]_1^{10} = \frac{\delta}{54}\left(10^{3/2} - 1\right)$$

37. From Example 6 we have $M_x = \int_0^{\pi} a(a \sin \theta)(k \sin \theta)\, d\theta = a^2 k \int_0^{\pi} \sin^2 \theta\, d\theta = \frac{a^2 k}{2} \int_0^{\pi} (1 - \cos 2\theta)\, d\theta$

$= \frac{a^2 k}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{a^2 k \pi}{2}$; $M_y = \int_0^{\pi} a(a \cos \theta)(k \sin \theta)\, d\theta = a^2 k \int_0^{\pi} \sin \theta \cos \theta\, d\theta = \frac{a^2 k}{2}\left[\sin^2 \theta\right]_0^{\pi} = 0$;

$M = \int_0^{\pi} ak \sin \theta\, d\theta = ak[-\cos \theta]_0^{\pi} = 2ak$. Therefore, $\overline{x} = \frac{M_y}{M} = 0$ and $\overline{y} = \frac{M_x}{M} = \left(\frac{a^2 k \pi}{2}\right)\left(\frac{1}{2ak}\right) = \frac{a\pi}{4} \Rightarrow \left(0, \frac{a\pi}{4}\right)$

is the center of mass.

38. $M_x = \int \widetilde{y}\, dm = \int_0^{\pi} (a \sin \theta) \cdot \delta \cdot a\, d\theta = \int_0^{\pi} \left(a^2 \sin \theta\right)\left(1 + k|\cos \theta|\right) d\theta$

$= a^2 \int_0^{\pi/2} (\sin \theta)(1 + k \cos \theta)\, d\theta + a^2 \int_{\pi/2}^{\pi} (\sin \theta)(1 - k \cos \theta)\, d\theta$

$= a^2 \int_0^{\pi/2} \sin \theta\, d\theta + a^2 k \int_0^{\pi/2} \sin \theta \cos \theta\, d\theta + a^2 \int_{\pi/2}^{\pi} \sin \theta\, d\theta - a^2 k \int_{\pi/2}^{\pi} \sin \theta \cos \theta\, d\theta$

$= a^2 [-\cos \theta]_0^{\pi/2} + a^2 k\left[\frac{\sin^2 \theta}{2}\right]_0^{\pi/2} + a^2[-\cos \theta]_{\pi/2}^{\pi} - a^2 k\left[\frac{\sin^2 \theta}{2}\right]_{\pi/2}^{\pi}$

$= a^2[0 - (-1)] + a^2 k\left(\frac{1}{2} - 0\right)^{\pi/2} + a^2[-(-1) - 0] - a^2 k\left(0 - \frac{1}{2}\right) = a^2 + \frac{a^2 k}{2} + a^2 + \frac{a^2 k}{2} = 2a^2 + a^2 k = a^2(2 + k)$;

$M_y = \int \widetilde{x}\, dm = \int_0^{\pi} (a \cos \theta) \cdot \delta \cdot a\, d\theta = \int_0^{\pi} \left(a^2 \cos \theta\right)\left(1 + k|\cos \theta|\right) d\theta$

$= a^2 \int_0^{\pi/2} (\cos \theta)(1 + k \cos \theta)\, d\theta + a^2 \int_{\pi/2}^{\pi} (\cos \theta)(1 - k \cos \theta)\, d\theta$

$= a^2 \int_0^{\pi/2} \cos \theta\, d\theta + a^2 k \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta + a^2 \int_{\pi/2}^{\pi} \cos \theta\, d\theta - a^2 k \int_{\pi/2}^{\pi} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$

$= a^2[\sin \theta]_0^{\pi/2} + \frac{a^2 k}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} + a^2[\sin \theta]_{\pi/2}^{\pi} - \frac{a^2 k}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{\pi}$

$= a^2(1 - 0) + \frac{a^2 k}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0 + 0)\right] + a^2(0 - 1) - \frac{a^2 k}{2}\left[(\pi + 0) - \left(\frac{\pi}{2} + 0\right)\right] = a^2 + \frac{a^2 k \pi}{4} - a^2 - \frac{a^2 k \pi}{4} = 0$;

$M = \int_0^{\pi} \delta \cdot a\, d\theta = a \int_0^{\pi} (1 + k\, |\cos \theta|\, d\theta = a \int_0^{\pi/2} (1 + k \cos \theta)\, d\theta + a \int_{\pi/2}^{\pi} (1 - k \cos \theta)\, d\theta$

$= a[\theta + k \sin \theta]_0^{\pi/2} + a[\theta - k \sin \theta]_{\pi/2}^{\pi} = a\left[\left(\frac{\pi}{2} + k\right) - 0\right] + a\left[(\pi + 0) - \left(\frac{\pi}{2} - k\right)\right]$

$= \frac{a\pi}{2} + ak + a\left(\frac{\pi}{2} + k\right) = a\pi + 2ak = a(\pi + 2k)$. So $\overline{x} = \frac{M_y}{M} = 0$ and $\overline{y} = \frac{M_x}{M} = \frac{a^2(2+k)}{a(\pi + 2k)} = \frac{a(2+k)}{\pi + 2k}$

$= \left(0, \frac{2a + ka}{\pi + 2k}\right)$ is the center of mass.

39. Consider the curve as an infinite number of line segments joined together. From the derivation of arc length we have that the length of a particular segment is $ds = \sqrt{(dx)^2 + (dy)^2}$. This implies that $M_x = \int \delta y \, ds$, $M_y = \int \delta x \, ds$ and $M = \int \delta \, ds$. If δ is constant, then $\overline{x} = \frac{M_y}{M} = \frac{\int x \, ds}{\int ds} = \frac{\int x \, ds}{\text{length}}$ and $\overline{y} = \frac{M_x}{M} = \frac{\int y \, ds}{\int ds} = \frac{\int y \, ds}{\text{length}}$.

40. Applying the symmetry argument analogous to the one used in Exercise 13, we find that $\overline{x} = 0$. The typical vertical strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{a + \frac{x^2}{4p}}{2}\right)$, length: $a - \frac{x^2}{4p}$, width: dx, area: $dA = \left(a - \frac{x^2}{4p}\right) dx$, mass: $dm = \delta \, dA = \delta\left(a - \frac{x^2}{4p}\right) dx$. Thus, $M_x = \int \widetilde{y} \, dm = \int_{-2\sqrt{pa}}^{2\sqrt{pa}} \frac{1}{2}\left(a + \frac{x^2}{4p}\right)\left(a - \frac{x^2}{4p}\right)\delta \, dx$

$= \frac{\delta}{2} \int_{-2\sqrt{pa}}^{2\sqrt{pa}} \left(a^2 - \frac{x^4}{16p^2}\right) dx = \frac{\delta}{2}\left[a^2 x - \frac{x^5}{80p^2}\right]_{-2\sqrt{pa}}^{2\sqrt{pa}} = 2 \cdot \frac{\delta}{2}\left[a^2 x - \frac{x^5}{80p^2}\right]_0^{2\sqrt{pa}} = \delta\left(2a^2\sqrt{pa} - \frac{2^5 p^2 a^2 \sqrt{pa}}{80p^2}\right)$

$= 2a^2\delta\sqrt{pa}\left(1 - \frac{16}{80}\right) = 2a^2\delta\sqrt{pa}\left(\frac{80-16}{80}\right) = 2a^2\delta\sqrt{pa}\left(\frac{64}{80}\right) = \frac{8a^2\delta\sqrt{pa}}{5}$; $M = \int dm = \delta \int_{-2\sqrt{pa}}^{2\sqrt{pa}} \left(a - \frac{x^2}{4p}\right) dx$

$= \delta\left[ax - \frac{x^3}{12p}\right]_{-2\sqrt{pa}}^{2\sqrt{pa}} = 2 \cdot \delta\left[ax - \frac{x^3}{12p}\right]_0^{2\sqrt{pa}} = 2\delta\left(2a\sqrt{pa} - \frac{2^3 pa\sqrt{pa}}{12p}\right) = 4a\delta\sqrt{pa}\left(1 - \frac{4}{12}\right) = 4a\delta\sqrt{pa}\left(\frac{12-4}{12}\right)$

$= \frac{8a\delta\sqrt{pa}}{3}$. So $\overline{y} = \frac{M_x}{M} = \left(\frac{8a^2\delta\sqrt{pa}}{5}\right)\left(\frac{3}{8a\delta\sqrt{pa}}\right) = \frac{3}{5}a$, as claimed.

41. A generalization of Example 6 yields $M_x = \int \widetilde{y} \, dm = \int_{\pi/2 - \alpha}^{\pi/2 + \alpha} a^2 \sin \theta \, d\theta = a^2[-\cos \theta]_{\pi/2 - \alpha}^{\pi/2 + \alpha}$

$= a^2\left[-\cos\left(\frac{\pi}{2} + \alpha\right) + \cos\left(\frac{\pi}{2} - \alpha\right)\right] = a^2(\sin \alpha + \sin \alpha) = 2a^2 \sin \alpha$; $M = \int dm = \int_{\pi/2 - \alpha}^{\pi/2 + \alpha} a \, d\theta = a[\theta]_{\pi/2 - \alpha}^{\pi/2 + \alpha}$

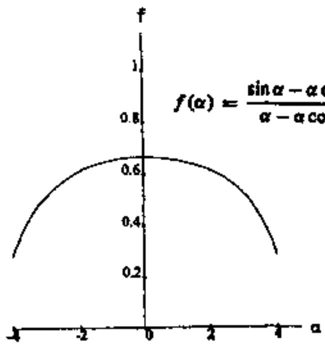
$= a\left[\left(\frac{\pi}{2} + \alpha\right) - \left(\frac{\pi}{2} - \alpha\right)\right] = 2a\alpha$. Thus, $\overline{y} = \frac{M_x}{M} = \frac{2a^2 \sin \alpha}{2a\alpha} = \frac{a \sin \alpha}{\alpha}$. Now $s = a(2\alpha)$ and $a \sin \alpha = \frac{c}{2}$

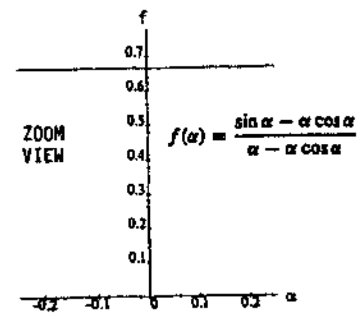
$\Rightarrow c = 2a \sin \alpha$. Then $\bar{y} = \frac{a(2a \sin \alpha)}{2a\alpha} = \frac{ac}{s}$, as claimed.

42. (a) First, we note that $\bar{y} = (\text{distance from origin to } \overline{AB}) + d \Rightarrow \frac{a \sin \alpha}{\alpha} = a \cos \alpha + d \Rightarrow d = \frac{a(\sin \alpha - \alpha \cos \alpha)}{\alpha}$.

Moreover, $h = a - a \cos \alpha \Rightarrow \frac{d}{h} = \frac{a(\sin \alpha - \alpha \cos \alpha)}{a(\alpha - \alpha \cos \alpha)} = \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha}$. The graphs below suggest that

$\lim_{\alpha \to 0^+} \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha} \approx \frac{2}{3}$.



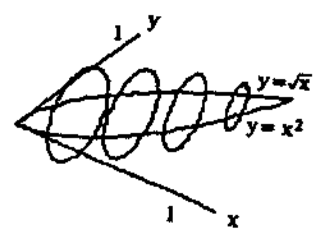


(b) Equation (9): $\frac{d}{h} = \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha}$

α	0.2	0.4	0.6	0.8	1.0
$f(\alpha)$	0.666222	0.664879	0.662615	0.659389	0.655145

CHAPTER 5 PRACTICE EXERCISES

1. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{x} - x^2\right)^2 = \frac{\pi}{4}\left(x - 2\sqrt{x} \cdot x^2 + x^4\right)$;



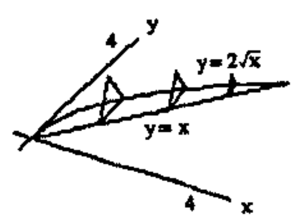
$a = 0,\ b = 1 \Rightarrow V = \int_a^b A(x)\,dx = \frac{\pi}{4} \int_0^1 \left(x - 2x^{5/2} + x^4\right) dx$

$= \frac{\pi}{4}\left[\frac{x^2}{2} - \frac{4}{7}x^{7/2} + \frac{x^5}{5}\right]_0^1 = \frac{\pi}{4}\left(\frac{1}{2} - \frac{4}{7} + \frac{1}{5}\right) = \frac{\pi}{4 \cdot 70}(35 - 40 + 14)$

$= \frac{9\pi}{280}$

2. $A(x) = \frac{1}{2}(\text{side})^2\left(\sin \frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}(2\sqrt{x} - x)^2 = \frac{\sqrt{3}}{4}\left(4x - 4x\sqrt{x} + x^2\right)$;

$a = 0,\ b = 4 \Rightarrow V = \int_a^b A(x)\,dx = \frac{\sqrt{3}}{4} \int_0^4 \left(4x - 4x^{3/2} + x^2\right) dx$



$= \frac{\sqrt{3}}{4}\left[2x^2 - \frac{8}{5}x^{5/2} + \frac{x^3}{3}\right]_0^4 = \frac{\sqrt{3}}{4}\left(32 - \frac{8 \cdot 32}{5} + \frac{64}{3}\right)$

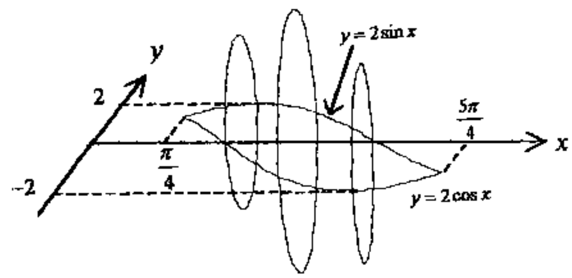
$= \frac{32\sqrt{3}}{4}\left(1 - \frac{8}{5} + \frac{2}{3}\right) = \frac{8\sqrt{3}}{15}(15 - 24 + 10) = \frac{8\sqrt{3}}{15}$

3. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}(2 \sin x - 2 \cos x)^2$

$= \frac{\pi}{4} \cdot 4(\sin^2 x - 2 \sin x \cos x + \cos^2 x) = \pi(1 - \sin 2x)$; $a = \frac{\pi}{4}$,

$b = \frac{5\pi}{4} \Rightarrow V = \int_a^b A(x)\, dx = \pi \int_{\pi/4}^{5\pi/4} (1 - \sin 2x)\, dx$

$= \pi\left[x + \frac{\cos 2x}{2}\right]_{\pi/4}^{5\pi/4} = \pi\left[\left(\frac{5\pi}{4} + \frac{\cos \frac{5\pi}{2}}{2}\right) - \left(\frac{\pi}{4} - \frac{\cos \frac{\pi}{2}}{2}\right)\right] = \pi^2$



4. $A(x) = (\text{edge})^2 = \left(\left(\sqrt{6} - \sqrt{x}\right)^2 - 0\right)^2 = \left(\sqrt{6} - \sqrt{x}\right)^4 = 36 - 24\sqrt{6}\,\sqrt{x} + 36x - 4\sqrt{6}\,x^{3/2} + x^2$;

$a = 0,\ b = 6 \Rightarrow V = \int_a^b A(x)\, dx = \int_0^6 \left(36 - 24\sqrt{6}\,\sqrt{x} + 36x - 4\sqrt{6}\,x^{3/2} + x^2\right) dx$

$= \left[36x - 24\sqrt{6} \cdot \frac{2}{3}x^{3/2} + 18x^2 - 4\sqrt{6} \cdot \frac{2}{5}x^{5/2} + \frac{x^3}{3}\right]_0^6 = 216 - 16 \cdot \sqrt{6}\,\sqrt{6} \cdot 6 + 18 \cdot 6^2 - \frac{8}{5}\sqrt{6}\,\sqrt{6} \cdot 6^2 + \frac{6^3}{3}$

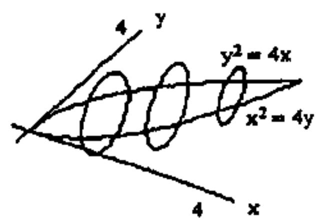
$= 216 - 576 + 648 - \frac{1728}{5} + 72 = 360 - \frac{1728}{5} = \frac{1800 - 1728}{5} = \frac{72}{5}$

5. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(2\sqrt{x} - \frac{x^2}{4}\right)^2 = \frac{\pi}{4}\left(4x - x^{5/2} + \frac{x^4}{16}\right)$;

$a = 0,\ b = 4 \Rightarrow V = \int_a^b A(x)\, dx = \frac{\pi}{4} \int_0^4 \left(4x - x^{5/2} + \frac{x^4}{16}\right) dx$

$= \frac{\pi}{4}\left[2x^2 - \frac{2}{7}x^{7/2} + \frac{x^5}{5 \cdot 16}\right]_0^4 = \frac{\pi}{4}\left(32 - 32 \cdot \frac{8}{7} + \frac{2}{5} \cdot 32\right)$

$= \frac{32\pi}{4}\left(1 - \frac{8}{7} + \frac{2}{5}\right) = \frac{8\pi}{35}(35 - 40 + 14) = \frac{72\pi}{35}$

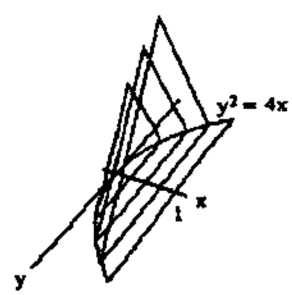


6. $A(x) = \frac{1}{2}(\text{edge})^2 \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}\left[2\sqrt{x} - \left(-2\sqrt{x}\right)\right]^2$

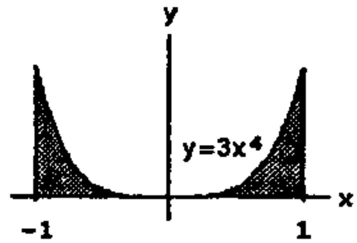
$= \frac{\sqrt{3}}{4}\left(4\sqrt{x}\right)^2 = 4\sqrt{3}\,x$; $a = 0$, $b = 1$

$\Rightarrow V = \int_a^b A(x)\, dx = \int_0^1 4\sqrt{3}\,x\, dx = \left[2\sqrt{3}\,x^2\right]_0^1$

$= 2\sqrt{3}$



7. (a) *disk method*:



$$V = \int_a^b \pi R^2(x)\, dx = \int_{-1}^{1} \pi\left(3x^4\right)^2 dx = \pi \int_{-1}^{1} 9x^8\, dx$$

$$= \pi\left[x^9\right]_{-1}^{1} = 2\pi$$

(b) *shell method*:

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi x\left(3x^4\right) dx = 2\pi \cdot 3 \int_0^1 x^5\, dx = 2\pi \cdot 3\left[\frac{x^6}{6}\right]_0^1 = \pi$$

Note: The lower limit of integration is 0 rather than -1.

(c) *shell method*:

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = 2\pi \int_{-1}^{1} (1-x)\left(3x^4\right) dx = 2\pi\left[\frac{3x^5}{5} - \frac{x^6}{2}\right]_{-1}^{1} = 2\pi\left[\left(\frac{3}{5} - \frac{1}{2}\right) - \left(-\frac{3}{5} - \frac{1}{2}\right)\right] = \frac{12\pi}{5}$$

(d) *washer method*:

$$R(x) = 3,\ r(x) = 3 - 3x^4 = 3\left(1 - x^4\right) \Rightarrow V = \int_a^b \pi\left[R^2(x) - r^2(x)\right] dx = \int_{-1}^{1} \pi\left[9 - 9\left(1 - x^4\right)^2\right] dx$$

$$= 9\pi \int_{-1}^{1} \left[1 - \left(1 - 2x^4 + x^8\right)\right] dx = 9\pi \int_{-1}^{1} \left(2x^4 - x^8\right) dx = 9\pi\left[\frac{2x^5}{5} - \frac{x^9}{9}\right]_{-1}^{1} = 18\pi\left[\frac{2}{5} - \frac{1}{9}\right] = \frac{2\pi \cdot 13}{5} = \frac{26\pi}{5}$$

8. (a) *washer method*:

$$R(x) = \frac{4}{x^3},\ r(x) = \frac{1}{2} \Rightarrow V = \int_a^b \left[R^2(x) - r^2(x)\right] dx = \int_1^2 \pi\left[\left(\frac{4}{x^3}\right)^2 - \left(\frac{1}{2}\right)^2\right] dx = \pi\left[-\frac{16}{5}x^{-5} - \frac{x}{4}\right]_1^2$$

$$= \pi\left[\left(\frac{-16}{5 \cdot 32} - \frac{1}{2}\right) - \left(-\frac{16}{5} - \frac{1}{4}\right)\right] = \pi\left(-\frac{1}{10} - \frac{1}{2} + \frac{16}{5} + \frac{1}{4}\right) = \frac{\pi}{20}(-2 - 10 + 64 + 5) = \frac{57\pi}{20}$$

(b) *shell method*:

$$V = 2\pi \int_1^2 x\left(\frac{4}{x^3} - \frac{1}{2}\right) dx = 2\pi\left[-4x^{-1} - \frac{x^2}{4}\right]_1^2 = 2\pi\left[\left(-\frac{4}{2} - 1\right) - \left(-4 - \frac{1}{4}\right)\right] = 2\pi\left(\frac{5}{4}\right) = \frac{5\pi}{2}$$

(c) *shell method*:

$$V = 2\pi \int_a^b \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = 2\pi \int_1^2 (2-x)\left(\frac{4}{x^3} - \frac{1}{2}\right) dx = 2\pi \int_1^2 \left(\frac{8}{x^3} - \frac{4}{x^2} - 1 + \frac{x}{2}\right) dx$$

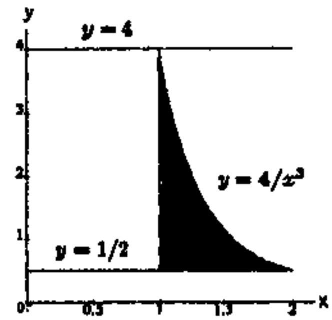
$$= 2\pi\left[-\frac{4}{x^2} + \frac{4}{x} - x + \frac{x^2}{4}\right]_1^2 = 2\pi\left[(-1 + 2 - 2 + 1) - \left(-4 + 4 - 1 + \frac{1}{4}\right)\right] = \frac{3\pi}{2}$$

(d) *disk method*:

$$V = \int_a^b \left[R^2(x) - r^2(x)\right] dx = \pi \int_1^2 \left[\left(\frac{7}{2}\right)^2 - \left(4 - \frac{4}{x^3}\right)^2\right] dx = \frac{49\pi}{4} - 16\pi \int_1^2 \left(1 - 2x^{-3} + x^{-6}\right) dx$$

$$= \frac{49\pi}{4} - 16\pi\left[x + x^{-2} - \frac{x^{-5}}{5}\right]_1^2 = \frac{49\pi}{4} - 16\pi\left[\left(2 + \frac{1}{4} - \frac{1}{5 \cdot 32}\right) - \left(1 + 1 - \frac{1}{5}\right)\right]$$

$$= \frac{49\pi}{4} - 16\pi\left(\frac{1}{4} - \frac{1}{160} + \frac{1}{5}\right) = \frac{49\pi}{4} - \frac{16\pi}{160}(40 - 1 + 32) = \frac{49\pi}{4} - \frac{71\pi}{10} = \frac{103\pi}{20}$$



9. (a) *disk method*:

$$V = \pi \int_1^5 \left(\sqrt{x-1}\right)^2 dx = \int_1^5 (x-1)\, dx = \pi\left[\frac{x^2}{2} - x\right]_1^5$$

$$= \pi\left[\left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right)\right] = \pi\left(\frac{24}{2} - 4\right) = 8\pi$$

(b) *washer method*:

$$R(y) = 5,\ r(y) = y^2 + 1 \Rightarrow V = \int_c^d \pi\left[R^2(y) - r^2(y)\right] dy = \pi \int_{-2}^{2} \left[25 - \left(y^2+1\right)^2\right] dy$$

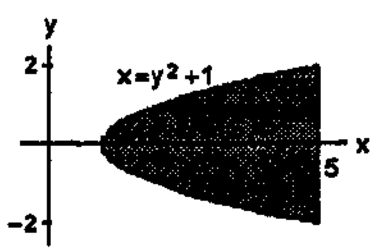
$$= \pi \int_{-2}^{2} \left(25 - y^4 - 2y^2 - 1\right) dy = \pi \int_{-2}^{2} \left(24 - y^4 - 2y^2\right) dy = \pi\left[24y - \frac{y^5}{5} - \frac{2}{3}y^3\right]_{-2}^{2} = 2\pi\left(24 \cdot 2 - \frac{32}{5} - \frac{2}{3} \cdot 8\right)$$

$$= 32\pi\left(3 - \frac{2}{5} - \frac{1}{3}\right) = \frac{32\pi}{15}(45 - 6 - 5) = \frac{1088\pi}{15}$$

(c) *disk method*:

$$R(y) = 5 - \left(y^2 + 1\right) = 4 - y^2 \Rightarrow V = \int_c^d \pi R^2(y)\, dy = \int_{-2}^{2} \pi\left(4 - y^2\right)^2 dy = \pi \int_{-2}^{2} \left(16 - 8y^2 + y^4\right) dy$$

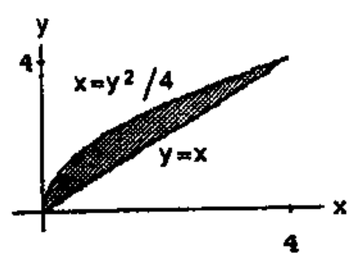
$$= \pi\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_{-2}^{2} = 2\pi\left(32 - \frac{64}{3} + \frac{32}{5}\right) = 64\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{64\pi}{15}(15 - 10 + 3) = \frac{512\pi}{15}$$



10. (a) *shell method*:

$$V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^4 2\pi y\left(y - \frac{y^2}{4}\right) dy$$

$$= 2\pi \int_0^4 \left(y^2 - \frac{y^3}{4}\right) dy = 2\pi\left[\frac{y^3}{3} - \frac{y^4}{16}\right]_0^4 = 2\pi\left(\frac{64}{3} - \frac{64}{4}\right)$$



$= \frac{2\pi}{12} \cdot 64 = \frac{32\pi}{3}$

(b) *shell method*:

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi x \left(2\sqrt{x} - x\right) dx = 2\pi \int_0^4 \left(2x^{3/2} - x^2\right) dx = 2\pi \left[\frac{4}{5}x^{5/2} - \frac{x^3}{3}\right]_0^4$$

$$= 2\pi\left(\frac{4}{5} \cdot 32 - \frac{64}{3}\right) = \frac{128\pi}{15}$$

(c) *shell method*:

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi(4-x)\left(2\sqrt{x} - x\right) dx = 2\pi \int_0^4 \left(8x^{1/2} - 4x - 2x^{3/2} + x^2\right) dx$$

$$= 2\pi\left[\frac{16}{3}x^{3/2} - 2x^2 - \frac{4}{5}x^{5/2} + \frac{x^3}{3}\right]_0^4 = 2\pi\left(\frac{16}{3} \cdot 8 - 32 - \frac{4}{5} \cdot 32 + \frac{64}{3}\right) = 64\pi\left(\frac{4}{3} - 1 - \frac{4}{5} + \frac{2}{3}\right) = 64\pi\left(1 - \frac{4}{5}\right) = \frac{64\pi}{5}$$

(d) *shell method*:

$$V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^4 2\pi(4-y)\left(y - \frac{y^2}{4}\right) dy = 2\pi \int_0^4 \left(4y - y^2 - y^2 + \frac{y^3}{4}\right) dy$$

$$= 2\pi \int_0^4 \left(4y - 2y^2 + \frac{y^3}{4}\right) dy = 2\pi\left[2y^2 - \frac{2}{3}y^3 + \frac{y^4}{16}\right]_0^4 = 2\pi\left(32 - \frac{2}{3} \cdot 64 + 16\right) = 32\pi\left(2 - \frac{8}{3} + 1\right) = \frac{32\pi}{3}$$

11. *disk method*:

$$R(x) = \tan x,\ a = 0,\ b = \frac{\pi}{3} \Rightarrow V = \pi \int_0^{\pi/3} \tan^2 x\, dx = \pi \int_0^{\pi/3} \left(\sec^2 x - 1\right) dx = \pi[\tan x - x]_0^{\pi/3} = \frac{\pi\left(3\sqrt{3} - \pi\right)}{3}$$

12. *disk method*:

$$V = \pi \int_0^\pi (2 - \sin x)^2\, dx = \pi \int_0^\pi \left(4 - 4\sin x + \sin^2 x\right) dx = \pi \int_0^\pi \left(4 - 4\sin x + \frac{1 - \cos 2x}{2}\right) dx$$

$$= \pi\left[4x + 4\cos x + \frac{x}{2} - \frac{\sin 2x}{4}\right]_0^\pi = \pi\left[\left(4\pi - 4 + \frac{\pi}{2} - 0\right) - (0 + 4 + 0 - 0)\right] = \pi\left(\frac{9\pi}{2} - 8\right) = \frac{\pi}{2}(9\pi - 16)$$

13. (a) *disk method*:

$$V = \pi \int_0^2 \left(x^2 - 2x\right)^2 dx = \pi \int_0^2 \left(x^4 - 4x^3 + 4x^2\right) dx = \pi\left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_0^2 = \pi\left(\frac{32}{5} - 16 + \frac{32}{3}\right)$$

$$= \frac{16\pi}{15}(6 - 15 + 10) = \frac{16\pi}{15}$$

(b) *disk method*:

$$V = 2\pi - \pi \int_0^2 \left[1 + \left(x^2 - 2x\right)\right]^2 dx = 2\pi - \pi \int_0^2 \left[1 + 2\left(x^2 - 2x\right) + \left(x^2 - 2x\right)^2\right] dx$$

$$= 2\pi - \pi \int_0^2 \left(1 + 2x^2 - 4x + x^4 - 4x^3 + 4x^2\right) dx = 2\pi - \pi \int_0^2 \left(x^4 - 4x^3 + 6x^2 - 4x + 1\right) dx$$

$$= 2\pi - \pi\left[\frac{x^5}{5} - x^4 + 2x^3 - 2x^2 + x\right]_0^2 = 2\pi - \pi\left(\frac{32}{5} - 16 + 16 - 8 + 2\right) = 2\pi - \frac{\pi}{5}(32 - 30) = 2\pi - \frac{2\pi}{5} = \frac{8\pi}{5}$$

(c) *shell method:*

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = 2\pi \int_0^2 (2-x)\left[-\left(x^2 - 2x\right)\right] dx = 2\pi \int_0^2 (2-x)\left(2x - x^2\right) dx$$

$$= 2\pi \int_0^2 \left(4x - 2x^2 - 2x^2 + x^3\right) dx = 2\pi \int_0^2 \left(x^3 - 4x^2 + 4x\right) dx = 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2 = 2\pi\left(4 - \frac{32}{3} + 8\right)$$

$$= \frac{2\pi}{3}(36 - 32) = \frac{8\pi}{3}$$

(d) *disk method:*

$$V = \pi \int_0^2 \left[2 - \left(x^2 - 2x\right)\right]^2 dx - \pi \int_0^2 2^2\, dx = \pi \int_0^2 \left[4 - 4\left(x^2 - 2x\right) + \left(x^2 - 2x\right)^2\right] dx - 8\pi$$

$$= \pi \int_0^2 \left(4 - 4x^2 + 8x + x^4 - 4x^3 + 4x^2\right) dx - 8\pi = \pi \int_0^2 \left(x^4 - 4x^3 + 8x + 4\right) dx - 8\pi$$

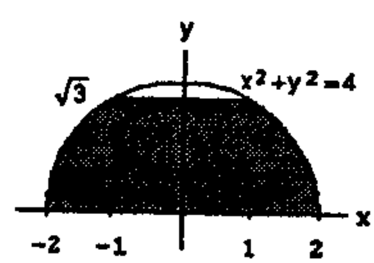
$$= \pi\left[\frac{x^5}{5} - x^4 + 4x^2 + 4x\right]_0^2 - 8\pi = \pi\left(\frac{32}{5} - 16 + 16 + 8\right) - 8\pi = \frac{\pi}{5}(32 + 40) - 8\pi = \frac{72\pi}{5} - \frac{40\pi}{5} = \frac{32\pi}{5}$$

14. *disk method:*

$$V = 2\pi \int_0^{\pi/4} 4\tan^2 x\, dx = 8\pi \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx = 8\pi[\tan x - x]_0^{\pi/4} = 2\pi(4 - \pi)$$

15. The volume cut out is equivalent to the volume of the solid generated by revolving the region shown here about the x-axis. Using the *shell* method:

$$V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^{\sqrt{3}} 2\pi y\left[\sqrt{4 - y^2} - \left(-\sqrt{4 - y^2}\right)\right] dy$$



$$= 2\pi \int_0^{\sqrt{3}} 2y\sqrt{4 - y^2}\, dy = -2\pi \int_0^{\sqrt{3}} \sqrt{4 - y^2}\, d\left(4 - y^2\right)$$

$$= (-2\pi)\left(\frac{2}{3}\right)\left[\left(4 - y^2\right)^{3/2}\right]_0^{\sqrt{3}} = -\frac{4\pi}{3}(1 - 8) = \frac{28\pi}{3}$$

16. We rotate the region enclosed by the curve $y = \sqrt{12\left(1 - \frac{4x^2}{121}\right)}$ and the x-axis around the x-axis. To find the volume we use the *disk* method: $V = \int_a^b \pi R^2(x)\,dx = \int_{-11/2}^{11/2} \pi\left(\sqrt{12\left(1 - \frac{4x^2}{121}\right)}\right)^2 dx = \pi \int_{-11/2}^{11/2} 12\left(1 - \frac{4x^2}{121}\right) dx$

$= 12\pi \int_{-11/2}^{11/2} \left(1 - \frac{4x^2}{121}\right) dx = 12\pi\left[x - \frac{4x^3}{363}\right]_{-11/2}^{11/2} = 24\pi\left[\frac{11}{2} - \left(\frac{4}{363}\right)\left(\frac{11}{2}\right)^3\right] = 132\pi\left[1 - \left(\frac{4}{363}\right)\left(\frac{11^2}{4}\right)\right]$

$= 132\pi\left(1 - \frac{1}{3}\right) = \frac{264\pi}{3} = 88\pi \approx 276 \text{ in}^3$

17. $y = x^{1/2} - \frac{x^{3/2}}{3} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(\frac{1}{x} - 2 + x\right) \Rightarrow L = \int_1^4 \sqrt{1 + \frac{1}{4}\left(\frac{1}{x} - 2 + x\right)}\,dx$

$\Rightarrow L = \int_1^4 \sqrt{\frac{1}{4}\left(\frac{1}{x} + 2 + x\right)}\,dx = \int_1^4 \sqrt{\frac{1}{4}\left(x^{-1/2} + x^{1/2}\right)^2}\,dx = \int_1^4 \frac{1}{2}\left(x^{-1/2} + x^{1/2}\right) dx = \frac{1}{2}\left[2x^{1/2} + \frac{2}{3}x^{3/2}\right]_1^4$

$= \frac{1}{2}\left[\left(4 + \frac{2}{3}\cdot 8\right) - \left(2 + \frac{2}{3}\right)\right] = \frac{1}{2}\left(2 + \frac{14}{3}\right) = \frac{10}{3}$

18. $x = y^{2/3} \Rightarrow \frac{dx}{dy} = \frac{2}{3}x^{-1/3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4x^{-2/3}}{9} \Rightarrow L = \int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\,dy$

$= \int_1^8 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\,dx = \frac{1}{3}\int_1^8 \sqrt{9x^{2/3} + 4}\,\left(x^{-1/3}\right) dx$; $\left[u = 9x^{2/3} + 4 \Rightarrow du = 6y^{-1/3}\,dy;\ x = 1 \Rightarrow u = 13,\right.$

$\left. x = 8 \Rightarrow u = 40\right] \to L = \frac{1}{18}\int_{13}^{40} u^{1/2}\,du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \approx 7.634$

19. $y = \frac{5}{12}x^{6/5} - \frac{5}{8}x^{4/5} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{1/5} - \frac{1}{2}x^{-1/5} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)$

$\Rightarrow L = \int_1^{32} \sqrt{1 + \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)}\,dx \Rightarrow L = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{2/5} + 2 + x^{-2/5}\right)}\,dx = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{1/5} + x^{-1/5}\right)^2}\,dx$

$= \int_1^{32} \frac{1}{2}\left(x^{1/5} + x^{-1/5}\right) dx = \frac{1}{2}\left[\frac{5}{6}x^{6/5} + \frac{5}{4}x^{4/5}\right]_1^{32} = \frac{1}{2}\left[\left(\frac{5}{6}\cdot 2^6 + \frac{5}{4}\cdot 2^4\right) - \left(\frac{5}{6} + \frac{5}{4}\right)\right] = \frac{1}{2}\left(\frac{315}{6} + \frac{75}{4}\right)$

$= \frac{1}{48}(1260 + 450) = \frac{1710}{48} = \frac{285}{8}$

20. $x = \frac{1}{12}y^3 + \frac{1}{y} \Rightarrow \frac{dx}{dy} = \frac{1}{4}y^2 - \frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4} \Rightarrow L = \int_1^2 \sqrt{1 + \left(\frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}\right)}\, dy$

$= \int_1^2 \sqrt{\frac{1}{16}y^4 + \frac{1}{2} + \frac{1}{y^4}}\, dy = \int_1^2 \sqrt{\left(\frac{1}{4}y^2 + \frac{1}{y^2}\right)^2}\, dy = \int_1^2 \left(\frac{1}{4}y^2 + \frac{1}{y^2}\right) dy = \left[\frac{1}{12}y^3 - \frac{1}{y}\right]_1^2$

$= \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right) = \frac{7}{12} + \frac{1}{2} = \frac{13}{12}$

21. $\frac{dx}{dt} = -5 \sin t + 5 \sin 5t$ and $\frac{dy}{dt} = 5 \cos t - 5 \cos 5t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{(5 \sin 5t - 5 \sin t)^2 + (5 \cos t - 5 \cos 5t)^2}$

$= 5\sqrt{\sin^2 5t - 2 \sin t \sin 5t + \sin^2 t + \cos^2 t - 2 \cos t \cos 5t + \cos^2 5t} = 5\sqrt{2 - 2(\sin t \sin 5t + \cos t \cos 5t)}$

$= 5\sqrt{2(1 - \cos 4t)} = 5\sqrt{4\left(\frac{1}{2}\right)(1 - \cos 4t)} = 10\sqrt{\sin^2 2t} = 10|\sin 2t| = 10 \sin 2t$ (since $0 \le t \le \pi/2$)

$\Rightarrow \text{Length} = \int_0^{\pi/2} 10 \sin 2t\, dt = (-5 \cos 2t)\Big|_{t=0}^{\pi/2} = (-5)(-1) - (-5)(1) = 10$

22. $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 2 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t)^2 + (2)^2} = 2\sqrt{t^2 + 1}$

$\Rightarrow \text{Length} = \int_0^1 2\sqrt{t^2 + 1}\, dt = \sqrt{2} + \ln\left(\sqrt{2} + 1\right) \approx 2.29559$ (Integral evaluated on TI-92 Plus calculator.)

23. $\frac{dx}{d\theta} = -3 \sin \theta$ and $\frac{dy}{d\theta} = 3 \cos \theta \Rightarrow \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{(-3 \sin \theta)^2 + (3 \cos \theta)^2} = \sqrt{3^2\left(\sin^2 \theta + \cos^2 \theta\right)} = 3$

$\Rightarrow \text{Length} = \int_0^{3\pi/2} 3\, d\theta = 3 \int_0^{3\pi/2} d\theta = 3\left(\frac{3\pi}{2} - 0\right) = \frac{9\pi}{2}$

24. $x = t^2$ and $y = \frac{t^3}{3} - t$, $-\sqrt{3} \le t \le \sqrt{3} \Rightarrow \frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = t^2 - 1 \Rightarrow \text{Length} = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(2t)^2 + \left(t^2 - 1\right)^2}\, dt$

$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{\left(t^2 + 1\right)^2}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} \left(t^2 + 1\right) dt = \left[\frac{t^3}{3} + t\right]_{-\sqrt{3}}^{\sqrt{3}} = 4\sqrt{3}$

25. The equipment alone: the force required to lift the equipment is equal to its weight $\Rightarrow F_1(x) = 100$ N.

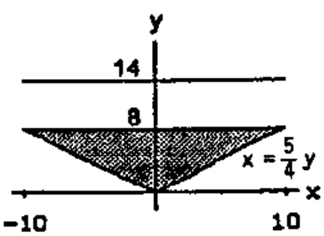
The work done is $W_1 = \int_a^b F_1(x)\, dx = \int_0^{40} 100\, dx = [100x]_0^{40} = 4000$ J; the rope alone: the force required to lift the rope is equal to the weight of the rope paid out at elevation x $\Rightarrow F_2(x) = 0.8(40-x)$. The work done is $W_2 = \int_a^b F_2(x)\,dx = \int_0^{40} 0.8(40-x)\,dx = 0.8\left[40x - \frac{x^2}{2}\right]_0^{40} = 0.8\left(40^2 - \frac{40^2}{2}\right) = \frac{(0.8)(1600)}{2} = 640$ J; the total work is $W = W_1 + W_2 = 4000 + 640 = 4640$ J

26. The force required to lift the water is equal to the water's weight, which varies steadily from $8 \cdot 800$ lb to $8 \cdot 400$ lb over the 4750 ft elevation. When the truck is x ft off the base of Mt. Washington, the water weight is $F(x) = 8 \cdot 800 \cdot \left(\frac{2 \cdot 4750 - x}{2 \cdot 4750}\right) = (6400)\left(1 - \frac{x}{9500}\right)$ lb. The work done is $W = \int_a^b F(x)\,dx$ $= \int_0^{4750} 6400\left(1 - \frac{x}{9500}\right) dx = 6400\left[x - \frac{x^2}{2 \cdot 9500}\right]_0^{4750} = 6400\left(4750 - \frac{4750^2}{4 \cdot 4750}\right) = \left(\frac{3}{4}\right)(6400)(4750)$ $= 22{,}800{,}000$ ft $\cdot$ lb

27. Force constant: $F = kx \Rightarrow 20 = k \cdot 1 \Rightarrow k = 20$ lb/ft; the work to stretch the spring 1 ft is $W = \int_0^1 kx\,dx = k\int_0^1 x\,dx = \left[20\frac{x^2}{2}\right]_0^1 = 10$ ft $\cdot$ lb; the work to stretch the spring an additional foot is $W = \int_1^2 kx\,dx = k\int_1^2 x\,dx = 20\left[\frac{x^2}{2}\right]_1^2 = 20\left(\frac{4}{2} - \frac{1}{2}\right) = 20\left(\frac{3}{2}\right) = 30$ ft $\cdot$ lb

28. Force constant: $F = kx \Rightarrow 200 = k(0.8) \Rightarrow k = 250$ N/m; the 300 N force stretches the spring $x = \frac{F}{k}$ $= \frac{300}{250} = 1.2$ m; the work required to stretch the spring that far is then $W = \int_0^{1.2} F(x)\,dx = \int_0^{1.2} 250x\,dx$ $= [125x^2]_0^{1.2} = 125(1.2)^2 = 180$ J

29. We imagine the water divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0, 8]$. The typical slab between the planes at y and $y + \Delta y$ has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\frac{5}{4}y\right)^2 = \frac{25\pi}{16}y^2\,\Delta y$ ft^3. The force F(y) required to lift this slab is equal to its weight: $F(y) = 62.4\,\Delta V$ $= \frac{(62.4)(25)}{16}\pi y^2\,\Delta y$ lb. The distance through which F(y) must act to lift this slab to the level 6 ft above the top is about $(6 + 8 - y)$ ft, so the work done lifting the slab is about $\Delta W = \frac{(62.4)(25)}{16}\pi y^2(14 - y)\,\Delta y$ ft $\cdot$ lb. The work done lifting all the slabs from $y = 0$ to $y = 8$ to the level 6 ft above the top is approximately $W \approx \sum_0^8 \frac{(62.4)(25)}{16}\pi y^2\,(14 - y)\,\Delta y$ ft $\cdot$ lb so the work to pump the water is the limit of these Riemann sums as the norm of the partition goes to zero:



Reservoir's Cross Section

$$W = \int_0^8 \frac{(62.4)(25)}{(16)}\pi y^2(14-y)\,dy = \frac{(62.4)(25)\pi}{16}\int_0^8 (14y^2 - y^3)\,dy = (62.4)\left(\frac{25\pi}{16}\right)\left[\frac{14}{3}y^3 - \frac{y^4}{4}\right]_0^8$$

$$= (62.4)\left(\frac{25\pi}{16}\right)\left(\frac{14}{3}\cdot 8^3 - \frac{8^4}{4}\right) \approx 418{,}208.81 \text{ ft}\cdot\text{lb}$$

30. The same as in Exercise 29, but change the distance through which $F(y)$ must act to $(8-y)$ rather than $(6+8-y)$. Also change the upper limit of integration from 8 to 5. The integral is:

$$W = \int_0^5 \frac{(62.4)(25)\pi}{16}y^2(8-y)\,dy = (62.4)\left(\frac{25\pi}{16}\right)\int_0^5 (8y^2 - y^3)\,dy = (62.4)\left(\frac{25\pi}{16}\right)\left[\frac{8}{3}y^3 - \frac{y^4}{4}\right]_0^5$$

$$= (62.4)\left(\frac{25\pi}{16}\right)\left(\frac{8}{3}\cdot 5^3 - \frac{5^4}{4}\right) \approx 54{,}241.56 \text{ ft}\cdot\text{lb}$$

31. The tank's cross section looks like the figure in Exercise 29 with right edge given by $x = \frac{5}{10}y = \frac{y}{2}$. A typical horizontal slab has volume $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{y}{2}\right)^2\Delta y = \frac{\pi}{4}y^2\,\Delta y$. The force required to lift this slab is its weight: $F(y) = 60\cdot\frac{\pi}{4}y^2\,\Delta y$. The distance through which $F(y)$ must act is $(2+10-y)$ ft, so the work to pump the liquid is $W = 60\int_0^{10}\pi(12-y)\left(\frac{y^2}{4}\right)dy = 15\pi\left[\frac{12y^3}{3} - \frac{y^4}{4}\right]_0^{10} = 22{,}500\pi$ ft · lb; the time needed to empty the tank is $\frac{22{,}500\pi \text{ ft}\cdot\text{lb}}{275 \text{ ft}\cdot\text{lb/sec}} \approx 257$ sec

32. A typical horizontal slab has volume about $\Delta V = (20)(2x)\Delta y = (20)\left(2\sqrt{16-y^2}\right)\Delta y$ and the force required to lift this slab is its weight $F(y) = (57)(20)\left(2\sqrt{16-y^2}\right)\Delta y$. The distance through which $F(y)$ must act is $(6+4-y)$ ft, so the work to pump the olive oil from the half-full tank is

$$W = 57\int_{-4}^0 (10-y)(20)\left(2\sqrt{16-y^2}\right)dy = 2880\int_{-4}^0 10\sqrt{16-y^2}\,dy + 1140\int_{-4}^0 \left(16-y^2\right)^{1/2}(-2y)\,dy$$

$$= 22{,}800\cdot(\text{area of a quarter circle having radius }4) + \frac{2}{3}(1140)\left[\left(16-y^2\right)^{3/2}\right]_{-4}^0 = (22{,}800)(4\pi) + 48{,}640$$

$$= 334{,}153.25 \text{ ft}\cdot\text{lb}$$

33. $F = \int_a^b W\cdot\begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix}\cdot L(y)\,dy \Rightarrow F = 2\int_0^2 (62.4)(2-y)(2y)\,dy = 249.6\int_0^2 \left(2y-y^2\right)dy = 249.6\left[y^2 - \frac{y^3}{3}\right]_0^2$

$$= (249.6)\left(4-\frac{8}{3}\right) = (249.6)\left(\frac{4}{3}\right) = 332.8 \text{ lb}$$

34. $F = \int_a^b W\cdot\begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix}\cdot L(y)\,dy \Rightarrow F = \int_0^{5/6} 75\left(\frac{5}{6}-y\right)(2y+4)\,dy = 75\int_0^{5/6}\left(\frac{5}{3}y + \frac{10}{3} - 2y^2 - 4y\right)dy$

$$= 75\int_0^{5/6}\left(\frac{10}{3} - \frac{7}{3}y - 2y^2\right)dy = 75\left[\frac{10}{3}y - \frac{7}{6}y^2 - \frac{2}{3}y^3\right]_0^{5/6} = (75)\left[\left(\frac{50}{18}\right) - \left(\frac{7}{6}\right)\left(\frac{25}{36}\right) - \left(\frac{2}{3}\right)\left(\frac{125}{216}\right)\right]$$

$= (75)\left(\frac{25}{9} - \frac{175}{216} - \frac{250}{3\cdot 216}\right) = \left(\frac{75}{9\cdot 216}\right)(25\cdot 216 - 175\cdot 9 - 250\cdot 3) = \frac{(75)(3075)}{9\cdot 216} \approx 118.63 \text{ lb.}$

35. $F = \int_a^b W\cdot\binom{\text{strip}}{\text{depth}}\cdot L(y)\,dy \Rightarrow F = 62.4\int_0^4 (9-y)\left(2\cdot\frac{\sqrt{y}}{2}\right)dy = 62.4\int_0^4 \left(9y^{1/2} - y^{3/2}\right)dy$

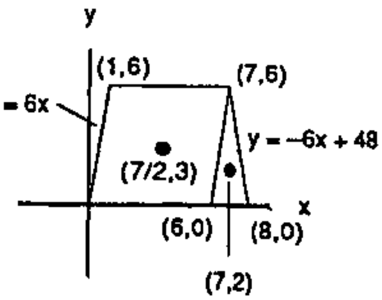
$= 62.4\left[6y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^4 = (62.4)\left(6\cdot 8 - \frac{2}{5}\cdot 32\right) = \left(\frac{62.4}{5}\right)(48\cdot 5 - 64) = \frac{(62.4)(176)}{5} = 2196.48 \text{ lb}$

36. $F = 62.4\int_0^6 (10-y)\left[\left(8-\frac{y}{6}\right) - \left(\frac{y}{6}\right)\right]dy$

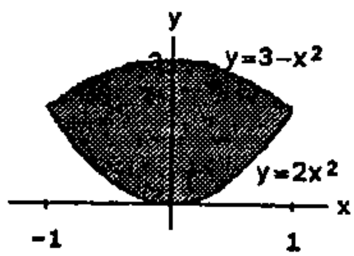
$= \frac{62.4}{3}\int_0^6 \left(240 - 34y + y^2\right)dy$

$= \frac{62.4}{3}\left[240y - 17y^2 + \frac{y^3}{3}\right]_0^6 = \frac{62.4}{3}(1440 - 612 + 72)$

$= 18{,}720$ lb.



37. Intersection points: $3 - x^2 = 2x^2 \Rightarrow 3x^2 - 3 = 0$ $\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow x = -1$ or $x = 1$. Applying the symmetry argument analogous to the one used in Exercise 5.6.13, we find that $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{2x^2 + (3-x^2)}{2}\right) = \left(x, \frac{x^2+3}{2}\right)$, length: $(3-x^2) - 2x^2 = 3(1-x^2)$, width: dx,



area: $dA = 3(1-x^2)\,dx$, and mass: $dm = \delta\cdot dA = 3\delta(1-x^2)\,dx$

$\Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \frac{3}{2}\delta(x^2+3)(1-x^2)\,dx = \frac{3}{2}\delta(-x^4 - 2x^2 + 3)\,dx$

$\Rightarrow M_x = \int \tilde{y}\,dm = \frac{3}{2}\delta\int_{-1}^{1}(-x^4 - 2x^2 + 3)\,dx = \frac{3}{2}\delta\left[-\frac{x^5}{5} - \frac{2x^3}{3} + 3x\right]_{-1}^{1} = 3\delta\left(-\frac{1}{5} - \frac{2}{3} + 3\right)$

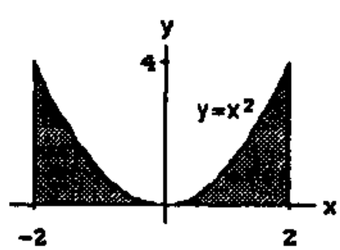
$= \frac{3\delta}{15}(-3 - 10 + 45) = \frac{32\delta}{5}$; $M = \int dm = 3\delta\int_{-1}^{1}(1-x^2)\,dx = 3\delta\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 6\delta\left(1 - \frac{1}{3}\right) = 4\delta$

$\Rightarrow \bar{y} = \frac{M_x}{M} = \frac{32\delta}{5\cdot 4\delta} = \frac{8}{5}$. Therefore, the centroid is $(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right)$.

38. Applying the symmetry argument analogous to the one used in Exercise 5.6.13, we find that $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{x^2}{2}\right)$, length: x^2, width: dx, area: $dA = x^2\,dx$, mass: $dm = \delta\cdot dA = \delta x^2\,dx$

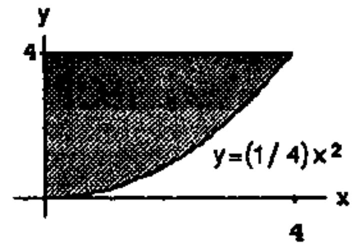
$\Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \frac{\delta}{2}x^2\cdot x^2\,dx = \frac{\delta}{2}x^4\,dx$

$\Rightarrow M_x = \int \tilde{y}\,dm = \frac{\delta}{2}\int_{-2}^{2} x^4\,dx = \frac{\delta}{10}\left[x^5\right]_{-2}^{2} = \frac{2\delta}{10}(2^5) = \frac{32\delta}{5}$;



$M = \int dm = \delta \int_{-2}^{2} x^2\,dx = \delta\left[\frac{x^3}{3}\right]_{-2}^{2} = \frac{2\delta}{3}(2^3) = \frac{16\delta}{3} \Rightarrow \bar{y} = \frac{M_x}{M} = \frac{32\cdot\delta\cdot 3}{5\cdot 16\cdot\delta} = \frac{6}{5}$. Therefore, the centroid is $(\bar{x},\bar{y}) = \left(0,\frac{6}{5}\right)$.

39. The typical *vertical* strip has: center of mass: $(\tilde{x},\tilde{y}) = \left(x, \frac{4+\frac{x^2}{4}}{2}\right)$, length: $4-\frac{x^2}{4}$, width: dx, area: $dA = \left(4-\frac{x^2}{4}\right)dx$, mass: $dm = \delta\cdot dA = \delta\left(4-\frac{x^2}{4}\right)dx \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \delta\cdot\frac{\left(4+\frac{x^2}{4}\right)}{2}\left(4-\frac{x^2}{4}\right)dx = \frac{\delta}{2}\left(16-\frac{x^4}{16}\right)dx$; the moment about



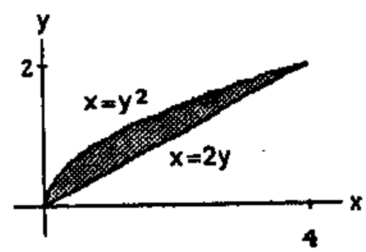
the y-axis is $\tilde{x}\,dm = \delta\left(4-\frac{x^2}{4}\right)\cdot x\,dx = \delta\left(4x-\frac{x^3}{4}\right)dx$. Thus, $M_x = \int \tilde{y}\,dm = \frac{\delta}{2}\int_0^4 \left(16-\frac{x^4}{16}\right)dx$

$= \frac{\delta}{2}\left[16x - \frac{x^5}{5\cdot 16}\right]_0^4 = \frac{\delta}{2}\left[64-\frac{64}{5}\right] = \frac{128\delta}{5}$; $M_y = \int \tilde{x}\,dm = \delta\int_0^4 \left(4x-\frac{x^3}{4}\right)dx = \delta\left[2x^2-\frac{x^4}{16}\right]_0^4$

$= \delta(32-16) = 16\delta$; $M = \int dm = \delta\int_{-0}^{4}\left(4-\frac{x^2}{4}\right)dx = \delta\left[4x-\frac{x^3}{12}\right]_0^4 = \delta\left(16-\frac{64}{12}\right) = \frac{32\delta}{3}$

$\Rightarrow \bar{x} = \frac{M_y}{M} = \frac{16\cdot\delta\cdot 3}{32\cdot\delta} = \frac{3}{2}$ and $\bar{y} = \frac{M_x}{M} = \frac{128\cdot\delta\cdot 3}{5\cdot 32\cdot\delta} = \frac{12}{5}$. Therefore, the centroid is $(\bar{x},\bar{y}) = \left(\frac{3}{2},\frac{12}{5}\right)$.

40. A typical *horizontal* strip has:
center of mass: $(\tilde{x},\tilde{y}) = \left(\frac{y^2+2y}{2}, y\right)$, length: $2y-y^2$, width: dy, area: $dA = (2y-y^2)\,dy$, mass: $dm = \delta\cdot dA = \delta(2y-y^2)\,dy$; the moment about the x-axis is $\tilde{y}\,dm = \delta\cdot y\cdot(2y-y^2)\,dy = \delta(2y^2-y^3)$; the moment about the y-axis is $\tilde{x}\,dm = \delta\cdot\frac{(y^2+2y)}{2}\cdot(2y-y^2)\,dy = \frac{\delta}{2}(4y^2-y^4)\,dy$

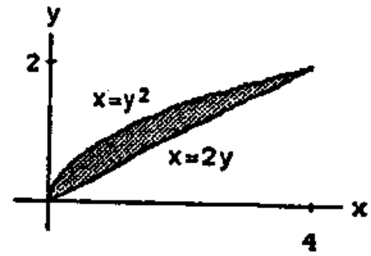


$\Rightarrow M_x = \int \tilde{y}\,dm = \delta\int_0^2 (2y^2-y^3)\,dy = \delta\left[\frac{2}{3}y^3-\frac{y^4}{4}\right]_0^2 = \delta\left(\frac{2}{3}\cdot 8 - \frac{16}{4}\right) = \delta\left(\frac{16}{3}-\frac{16}{4}\right) = \frac{\delta\cdot 16}{12} = \frac{4\delta}{3}$;

$M_y = \int \tilde{x}\,dm = \frac{\delta}{2}\int_0^2 (4y^2-y^4)\,dy = \frac{\delta}{2}\left[\frac{4}{3}y^3-\frac{y^5}{5}\right]_0^2 = \frac{\delta}{2}\left(\frac{4\cdot 8}{3}-\frac{32}{5}\right) = \frac{32\delta}{15}$; $M = \int dm = \delta\int_0^2 (2y-y^2)\,dy$

$= \delta\left[y^2-\frac{y^3}{3}\right]_0^2 = \delta\left(4-\frac{8}{3}\right) = \frac{4\delta}{3} \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{\delta\cdot 32\cdot 3}{15\cdot\delta\cdot 4} = \frac{8}{5}$ and $\bar{y} = \frac{M_x}{M} = \frac{4\cdot\delta\cdot 3}{3\cdot 4\cdot\delta} = 1$. Therefore, the centroid is $(\bar{x},\bar{y}) = \left(\frac{8}{5},1\right)$.

41. A typical horizontal strip has: center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{y^2+2y}{2}, y\right)$, length: $2y - y^2$, width: dy, area: $dA = \left(2y - y^2\right) dy$, mass: $dm = \delta \cdot dA = (1+y)\left(2y-y^2\right) dy \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = y(1+y)\left(2y-y^2\right) dy = \left(2y^2+2y^3-y^3-y^4\right) dy$ $= \left(2y^2+y^3-y^4\right) dy$; the moment about the y-axis is



$\tilde{x}\,dm = \left(\frac{y^2+2y}{2}\right)(1+y)\left(2y-y^2\right) dy = \frac{1}{2}\left(4y^2-y^4\right)(1+y)\,dy$

$= \frac{1}{2}\left(4y^2+4y^3-y^4-y^5\right) dy \Rightarrow M_x = \int \tilde{y}\,dm = \int_0^2 \left(2y^2+y^3-y^4\right) dy = \left[\frac{2}{3}y^3 + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^2$

$= \left(\frac{16}{3}+\frac{16}{4}-\frac{32}{5}\right) = 16\left(\frac{1}{3}+\frac{1}{4}-\frac{2}{5}\right) = \frac{16}{60}(20+15-24) = \frac{4}{15}(11) = \frac{44}{15}$; $M_y = \int \tilde{x}\,dm$

$= \int_0^2 \frac{1}{2}\left(4y^2+4y^3-y^4-y^5\right) dy = \frac{1}{2}\left[\frac{4}{3}y^3+y^4-\frac{y^5}{5}-\frac{y^6}{6}\right]_0^2 = \frac{1}{2}\left(\frac{4\cdot 2^3}{3}+2^4-\frac{2^5}{5}-\frac{2^6}{6}\right)$

$= 4\left(\frac{4}{3}+2-\frac{4}{5}-\frac{8}{6}\right) = 4\left(2-\frac{4}{5}\right) = \frac{24}{5}$; $M = \int dm = \int_0^2 (1+y)\left(2y-y^2\right) dy = \int_0^2 \left(2y+y^2-y^3\right) dy$

$= \left[y^2+\frac{y^3}{3}-\frac{y^4}{4}\right]_0^2 = \left(4+\frac{8}{3}-\frac{16}{4}\right) = \frac{8}{3} \Rightarrow \overline{x} = \frac{M_y}{M} = \left(\frac{24}{5}\right)\left(\frac{3}{8}\right) = \frac{9}{5}$ and $\overline{y} = \frac{M_x}{M} = \left(\frac{44}{15}\right)\left(\frac{3}{8}\right) = \frac{44}{40} = \frac{11}{10}$. Therefore, the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{9}{5}, \frac{11}{10}\right)$.

42. A typical vertical strip has: center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{3}{2x^{3/2}}\right)$, length: $\frac{3}{x^{3/2}}$, width: dx, area: $dA = \frac{3}{x^{3/2}}\,dx$, mass: $dm = \delta \cdot dA = \delta \cdot \frac{3}{x^{3/2}}\,dx \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \frac{3}{2x^{3/2}} \cdot \frac{3}{x^{3/2}}\,dx = \frac{9}{2x^3}\,dx$; the moment about the y-axis is $\tilde{x}\,dm = x \cdot \frac{3}{x^{3/2}}\,dx = \frac{3}{x^{1/2}}\,dx$.

(a) $M_x = \delta \int_1^9 \frac{1}{2}\left(\frac{9}{x^3}\right) dx = \frac{9\delta}{2}\left[-\frac{x^{-2}}{2}\right]_1^9 = \frac{20\delta}{9}$; $M_y = \delta \int_1^9 x\left(\frac{3}{x^{3/2}}\right) dx = 3\delta\left[2x^{1/2}\right]_1^9 = 12\delta$;

$M = \delta \int_1^9 \frac{3}{x^{3/2}}\,dx = -6\delta\left[x^{-1/2}\right]_1^9 = 4\delta \Rightarrow \overline{x} = \frac{M_y}{M} = \frac{12\delta}{4\delta} = 3$ and $\overline{y} = \frac{M_x}{M} = \frac{\left(\frac{20\delta}{9}\right)}{4\delta} = \frac{5}{9}$

(b) $M_x = \int_1^9 \frac{x}{2}\left(\frac{9}{x^3}\right) dx = \frac{9}{2}\left[-\frac{1}{x}\right]_1^9 = 4$; $M_y = \int_1^9 x^2\left(\frac{3}{x^{3/2}}\right) dx = \left[2x^{3/2}\right]_1^9 = 52$; $M = \int_1^9 x\left(\frac{3}{x^{3/2}}\right) dx$

$= 6\left[x^{1/2}\right]_1^9 = 12 \Rightarrow \overline{x} = \frac{M_y}{M} = \frac{13}{3}$ and $\overline{y} = \frac{M_x}{M} = \frac{1}{3}$

CHAPTER 5 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. $V = \pi \int_0^a [f(x)]^2\, dx = a^2 + a \Rightarrow \pi \int_0^x [f(t)]^2\, dt = x^2 + x$ for all $x > a \Rightarrow \pi[f(x)]^2 = 2x + 1 \Rightarrow f(x) = \sqrt{\frac{2x+1}{\pi}}$

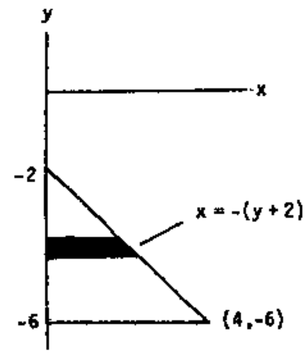
2. By the shell method we have $2\pi b^3 = 2\pi \int_0^b xf(x)\, dx \Rightarrow x^3 = \int_0^x tf(t)\, dt$, where $x > 0$. By the Fundamental Theorem of Calculus we have $3x^2 = xf(x) \Rightarrow f(x) = 3x$.

3. $s(x) = Cx \Rightarrow \int_0^x \sqrt{1 + [f'(t)]^2}\, dt = Cx \Rightarrow \sqrt{1 + [f'(x)]^2} = C \Rightarrow f'(x) = \sqrt{C^2 - 1}$ for $C \ge 1$ $\Rightarrow f(x) = \int_0^x \sqrt{C^2 - 1}\, dt + k$. Then $f(0) = a \Rightarrow a = 0 + k \Rightarrow f(x) = \int_0^x \sqrt{C^2 - 1}\, dt + a \Rightarrow f(x) = x\sqrt{C^2 - 1} + a$, where $C \ge 1$.

4. (a) The graph of $f(x) = \sin x$ traces out a path from $(0,0)$ to $(\alpha, \sin \alpha)$ whose length is $L = \int_0^\alpha \sqrt{1 + \cos^2\theta}\, d\theta$. The line segment from $(0,0)$ to $(\alpha, \sin \alpha)$ has length $\sqrt{(\alpha - 0)^2 + (\sin \alpha - 0)^2} = \sqrt{\alpha^2 + \sin^2\alpha}$. Since the shortest distance between two points is the length of the straight line segment joining them, we have immediately that $\int_0^\alpha \sqrt{1 + \cos^2\theta}\, d\theta > \sqrt{\alpha^2 + \sin^2\alpha}$ if $0 < \alpha \le \frac{\pi}{2}$.
   (b) In general, if $y = f(x)$ is continuously differentiable and $f(0) = 0$, then $\int_0^\alpha \sqrt{1 + [f'(t)]^2}\, dt > \sqrt{\alpha^2 + f^2(\alpha)}$ for $\alpha > 0$.

5. Converting to pounds and feet, $2 \text{ lb/in} = \frac{2 \text{ lb}}{1 \text{ in}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} = 24 \text{ lb/ft}$. Thus, $F = 24x \Rightarrow W = \int_0^{1/2} 24x\, dx$ $= \left[12x^2\right]_0^{1/2} = 3 \text{ ft}\cdot\text{lb}$. Since $W = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_1^2$, where $W = 3 \text{ ft}\cdot\text{lb}$, $m = \left(\frac{1}{10} \text{ lb}\right)\left(\frac{1}{32 \text{ ft/sec}^2}\right)$ $= \frac{1}{320}$ slugs, and $v_1 = 0$ ft/sec, we have $3 = \left(\frac{1}{2}\right)\left(\frac{1}{320} v_0^2\right) \Rightarrow v_0^2 = 3 \cdot 640$. For the projectile height, $s = -16t^2 + v_0 t$ (since $s = 0$ at $t = 0$) $\Rightarrow \frac{ds}{dt} = v = -32t + v_0$. At the top of the ball's path, $v = 0 \Rightarrow t = \frac{v_0}{32}$ and the height is $s = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = \frac{v_0^2}{64} = \frac{3 \cdot 640}{64} = 30$ ft.

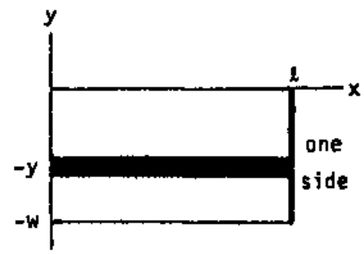
6. The submerged triangular plate is depicted in the figure at the right. The hypotenuse of the triangle has slope -1 $\Rightarrow y-(-2)=-(x-0) \Rightarrow x=-(y+2)$ is an equation of the hypotenuse. Using a typical horizontal strip, the fluid pressure is $F=\int (62.4)\cdot\begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix}\cdot\begin{pmatrix}\text{strip}\\ \text{length}\end{pmatrix}dy$



$$= \int_{-6}^{-2} (62.4)(-y)[-(y+2)]\,dy = 62.4\int_{-6}^{-2}\left(y^2+2y\right)dy$$

$$= 62.4\left[\frac{y^3}{3}+y^2\right]_{-6}^{-2} = (62.4)\left[\left(-\frac{8}{3}+4\right)-\left(-\frac{216}{3}+36\right)\right] = (62.4)\left(\frac{208}{3}-32\right) = \frac{(62.4)(112)}{3} \approx 2329.6 \text{ lb}$$

7. Consider a rectangular plate of length ℓ and width w. The length is parallel with the surface of the fluid of weight density ω. The force on one side of the plate is $F=\omega\int_{-w}^{0}(-y)(\ell)\,dy = -\omega\ell\left[\frac{y^2}{2}\right]_{-w}^{0} = \frac{\omega\ell w^2}{2}$. The average force on one side of the plate is $F_{av}=\frac{\omega}{w}\int_{-w}^{0}(-y)\,dy$



$= \frac{\omega}{w}\left[-\frac{y^2}{2}\right]_{-w}^{0} = \frac{\omega w}{2}$. Therefore the force $\frac{\omega\ell w^2}{2} = \left(\frac{\omega w}{2}\right)(\ell w)$

$=$ (the average pressure up and down) $\cdot$ (the area of the plate).

8. For y measured downward from the fluid's surface the width of a horizontal strip is $2(y-2)$ when $2\le y\le 8$ and it is $12-2(y-8)$ when $8\le y\le 14$. Using the hint given in the Exercise, the fluid force on the plate is: $F = 2\int_{2}^{8} w_1 y(y-2)\,dy + 2\int_{8}^{14}[8w_1+w_2(y-8)](14-y)\,dy = (216w_1)+(288w_1+72w_2)$

$= 504w_1 + 72w_2$.

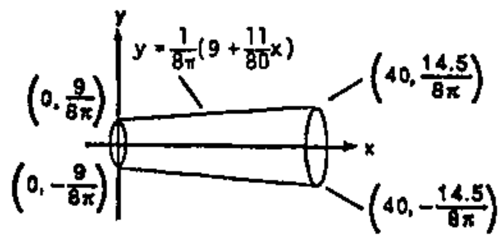
9. From the symmetry of $y=1-x^n$, n even, about the y-axis for $-1\le x\le 1$, we have $\bar{x}=0$. To find $\bar{y}=\frac{M_x}{M}$, we use the vertical strips technique. The typical strip has center of mass: $(\tilde{x},\tilde{y})=\left(x,\frac{1-x^n}{2}\right)$, length: $1-x^n$, width: dx, area: $dA=\left(1-x^n\right)dx$, mass: $dm=1\cdot dA=\left(1-x^n\right)dx$. The moment of the strip about the x-axis is $\tilde{y}\,dm=\frac{\left(1-x^n\right)^2}{2}dx \Rightarrow M_x=\int_{-1}^{1}\frac{\left(1-x^n\right)^2}{2}dx = 2\int_{0}^{1}\frac{1}{2}\left(1-2x^n+x^{2n}\right)dx = \left[x-\frac{2x^{n+1}}{n+1}+\frac{x^{2n+1}}{2n+1}\right]_0^1$

$$= 1-\frac{2}{n+1}+\frac{1}{2n+1} = \frac{(n+1)(2n+1)-2(2n+1)+(n+1)}{(n+1)(2n+1)} = \frac{2n^2+3n+1-4n-2+n+1}{(n+1)(2n+1)} = \frac{2n^2}{(n+1)(2n+1)}.$$

Also, $M = \int_{-1}^{1} dA = \int_{-1}^{1} (1-x^n)\,dx = 2\int_{0}^{1} (1-x^n)\,dx = 2\left[x - \frac{x^{n+1}}{n+1}\right]_0^1 = 2\left(1-\frac{1}{n+1}\right) = \frac{2n}{n+1}$. Therefore,

$\overline{y} = \frac{M_x}{M} = \frac{2n^2}{(n+1)(2n+1)} \cdot \frac{(n+1)}{2n} = \frac{n}{2n+1} \Rightarrow \left(0, \frac{n}{2n+1}\right)$ is the location of the centroid. As $n \to \infty$, $\overline{y} \to \frac{1}{2}$ so the limiting position of the centroid is $\left(0, \frac{1}{2}\right)$.

10. Align the telephone pole along the x-axis as shown in the accompanying figure. The slope of the top length of pole is $\frac{\left(\frac{14.5}{8\pi} - \frac{9}{8\pi}\right)}{40} = \frac{1}{8\pi} \cdot \frac{1}{40} \cdot (14.5-9) = \frac{5.5}{8\pi \cdot 40} = \frac{11}{8\pi \cdot 80}$. Thus, $y = \frac{9}{8\pi} + \frac{11}{8\pi \cdot 80}x = \frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)$ is an equation of the



line representing the top of the pole. Then, $M_y = \int_a^b x \cdot \pi y^2\,dx = \pi \int_0^{40} x\left[\frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)\right]^2 dx$

$= \frac{1}{64\pi} \int_0^{40} x\left(9 + \frac{11}{80}x\right)^2 dx$; $M = \int_a^b \pi y^2\,dx = \pi \int_0^{40} \left[\frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)\right]^2 dx = \frac{1}{64\pi}\int_0^{40} \left(9 + \frac{11}{80}x\right)^2 dx$. Thus,

$\overline{x} = \frac{M_y}{M} \approx \frac{129{,}700}{5623.3} \approx 23.06$ (using a calculator to compute the integrals). By symmetry about the x-axis, $\overline{y} = 0$ so the center of mass is about 23 ft from the top of the pole.

11. (a) Consider a single vertical strip with center of mass $(\widetilde{x}, \widetilde{y})$. If the plate lies to the right of the line, then the moment of this strip about the line $x = b$ is $(\widetilde{x} - b)\,dm = (\widetilde{x} - b)\,\delta\,dA \Rightarrow$ the plate's first moment about $x = b$ is the integral $\int (x-b)\delta\,dA = \int \delta x\,dA - \int \delta b\,dA = M_y - b\delta A$.

(b) If the plate lies to the left of the line, the moment of a vertical strip about the line $x = b$ is $(b - \widetilde{x})\,dm = (b - \widetilde{x})\delta\,dA \Rightarrow$ the plate's first moment about $x = b$ is $\int (b-x)\delta\,dA = \int b\delta\,dA - \int \delta x\,dA = b\delta A - M_y$.

12. (a) By symmetry of the plate about the x-axis, $\overline{y} = 0$. A typical vertical strip has center of mass: $(\widetilde{x}, \widetilde{y}) = (x, 0)$, length: $4\sqrt{ax}$, width: dx, area: $4\sqrt{ax}\,dx$, mass: $dm = \delta\,dA = kx \cdot 4\sqrt{ax}\,dx$, for some proportionality constant k. The moment of the strip about the y-axis is $M_y = \int \widetilde{x}\,dm = \int_0^a 4kx^2\sqrt{ax}\,dx$

$= 4k\sqrt{a}\int_0^a x^{5/2}\,dx = 4k\sqrt{a}\left[\frac{2}{7}x^{7/2}\right]_0^a = 4ka^{1/2} \cdot \frac{2}{7}a^{7/2} = \frac{8ka^4}{7}$. Also, $M = \int dm = \int_0^a 4kx\sqrt{ax}\,dx$

$= 4k\sqrt{a}\int_0^a x^{3/2}\,dx = 4k\sqrt{a}\left[\frac{2}{5}x^{5/2}\right]_0^a = 4ka^{1/2} \cdot \frac{2}{5}a^{5/2} = \frac{8ka^3}{5}$. Thus, $\overline{x} = \frac{M_y}{M} = \frac{8ka^4}{7} \cdot \frac{5}{8ka^3} = \frac{5}{7}a$

$\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{5a}{7}, 0\right)$ is the center of mass.

(b) A typical horizontal strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{\frac{y^2}{4a} + a}{2}, 0\right) = \left(\frac{y^2 + 4a^2}{8a}, 0\right)$, length: $a - \frac{y^2}{4a}$, width: dy, area: $\left(a - \frac{y^2}{4a}\right) dy$, mass: $dm = \delta\, dA = |y| \left(a - \frac{y^2}{4a}\right) dy$. Thus, $M_x = \int \tilde{y}\, dm$

$$= \int_{-2a}^{2a} y|y| \left(a - \frac{y^2}{4a}\right) dy = \int_{-2a}^{0} -y^2\left(a - \frac{y^2}{4a}\right) dy + \int_{0}^{2a} y^2\left(a - \frac{y^2}{4a}\right) dy$$

$$= \int_{-2a}^{0} \left(-ay^2 + \frac{y^4}{4a}\right) dy + \int_{0}^{2a} \left(ay^2 - \frac{y^4}{4a}\right) dy = \left[-\frac{a}{3}y^3 + \frac{y^5}{20a}\right]_{-2a}^{0} + \left[\frac{a}{3}y^3 - \frac{y^5}{20a}\right]_{0}^{2a}$$

$$= -\frac{8a^4}{3} + \frac{32a^5}{20a} + \frac{8a^4}{3} - \frac{32a^5}{20a} = 0;\ M_y = \int \tilde{x}\, dm = \int_{-2a}^{2a} \left(\frac{y^2 + 4a^2}{8a}\right) |y| \left(a - \frac{y^2}{4a}\right) dy$$

$$= \frac{1}{8a} \int_{-2a}^{2a} |y|\left(y^2 + 4a^2\right)\left(\frac{4a^2 - y^2}{4a}\right) dy = \frac{1}{32a^2} \int_{-2a}^{2a} |y|\left(16a^4 - y^4\right) dy$$

$$= \frac{1}{32a^2} \int_{-2a}^{0} \left(-16a^4 y + y^5\right) dy + \frac{1}{32a^2} \int_{0}^{2a} \left(16a^4 - y^5\right) dy = \frac{1}{32a^2}\left[-8a^4y^2 + \frac{y^6}{6}\right]_{-2a}^{0} + \frac{1}{32a^2}\left[8a^4y^2 - \frac{y^6}{6}\right]_{0}^{2a}$$

$$= \frac{1}{32a^2}\left[8a^4 \cdot 4a^2 - \frac{64a^6}{6}\right] + \frac{1}{32a^2}\left[8a^4 \cdot 4a^2 - \frac{64a^6}{6}\right] = \frac{1}{16a^2}\left(32a^6 - \frac{32a^6}{3}\right) = \frac{1}{16a^2} \cdot \frac{2}{3}\left(32a^6\right) = \frac{4}{3}a^4;$$

$$M = \int dm = \int_{-2a}^{2a} |y|\left(\frac{4a^2 - y^2}{4a}\right) dy = \frac{1}{4a} \int_{-2a}^{2a} |y|\left(4a^2 - y^2\right) dy$$

$$= \frac{1}{4a} \int_{-2a}^{0} \left(-4a^2y + y^3\right) dy + \frac{1}{4a} \int_{0}^{2a} \left(4a^2y - y^3\right) dy = \frac{1}{4a}\left[-2a^2 + \frac{y^4}{4}\right]_{-2a}^{0} + \frac{1}{4a}\left[2a^2y^2 - \frac{y^4}{4}\right]_{0}^{2a}$$

$$= 2 \cdot \frac{1}{4a}\left(2a^2 \cdot 4a^2 - \frac{16a^4}{4}\right) = \frac{1}{2a}\left(8a^4 - 4a^4\right) = 2a^3.$$ Therefore, $\bar{x} = \frac{M_y}{M} = \left(\frac{4}{3}a^4\right)\left(\frac{1}{2a^3}\right) = \frac{2a}{3}$ and $\bar{y} = \frac{M_x}{M} = 0$ is the center of mass.

13. (a) On $[0, a]$ a typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{b^2 - x^2} + \sqrt{a^2 - x^2}}{2}\right)$, length: $\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}$, width: dx, area: $dA = \left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx$, mass: $dm = \delta\, dA = \delta\left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx$. On $[a, b]$ a typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{b^2 - x^2}}{2}\right)$, length: $\sqrt{b^2 - x^2}$, width: dx, area: $dA = \sqrt{b^2 - x^2}\, dx$, mass: $dm = \delta\, dA = \delta \sqrt{b^2 - x^2}\, dx$. Thus, $M_x = \int \tilde{y}\, dm$

$$= \int_0^a \frac{1}{2}\left(\sqrt{b^2-x^2}+\sqrt{a^2-x^2}\right)\delta\left(\sqrt{b^2-x^2}-\sqrt{a^2-x^2}\right)dx + \int_a^b \frac{1}{2}\sqrt{b^2-x^2}\,\delta\sqrt{b^2-x^2}\,dx$$

$$= \frac{\delta}{2}\int_0^a \left[\left(b^2-x^2\right)-\left(a^2-x^2\right)\right]dx + \frac{\delta}{2}\int_a^b \left(b^2-x^2\right)dx = \frac{\delta}{2}\int_0^a \left(b^2-a^2\right)dx + \frac{\delta}{2}\int_a^b \left(b^2-x^2\right)dx$$

$$= \frac{\delta}{2}\left[\left(b^2-a^2\right)x\right]_0^a + \frac{\delta}{2}\left[b^2x-\frac{x^3}{3}\right]_a^b = \frac{\delta}{2}\left[\left(b^2-a^2\right)a\right] + \frac{\delta}{2}\left[\left(b^3-\frac{b^3}{3}\right)-\left(b^2a-\frac{a^3}{3}\right)\right]$$

$$= \frac{\delta}{2}\left(ab^2-a^3\right) + \frac{\delta}{2}\left(\frac{2}{3}b^3-ab^2+\frac{a^3}{3}\right) = \frac{\delta b^3}{3}-\frac{\delta a^3}{3} = \delta\left(\frac{b^3-a^3}{3}\right);\ M_y = \int \tilde{x}\,dm$$

$$= \int_0^a x\delta\left(\sqrt{b^2-x^2}-\sqrt{a^2-x^2}\right)dx + \int_a^b x\delta\sqrt{b^2-x^2}\,dx$$

$$= \delta\int_0^a x\left(b^2-x^2\right)^{1/2}dx - \delta\int_0^a x\left(a^2-x^2\right)^{1/2}dx + \delta\int_a^b x\left(b^2-x^2\right)^{1/2}dx$$

$$= \frac{-\delta}{2}\left[\frac{2\left(b^2-x^2\right)^{3/2}}{3}\right]_0^a + \frac{\delta}{2}\left[\frac{2\left(a^2-x^2\right)^{3/2}}{3}\right]_0^a - \frac{\delta}{2}\left[\frac{2\left(b^2-x^2\right)^{3/2}}{3}\right]_a^b$$

$$= -\frac{\delta}{3}\left[\left(b^2-a^2\right)^{3/2}-\left(b^2\right)^{3/2}\right] + \frac{\delta}{3}\left[0-\left(a^2\right)^{3/2}\right] - \frac{\delta}{3}\left[0-\left(b^2-a^2\right)^{3/2}\right] = \frac{\delta b^3}{3}-\frac{\delta a^3}{3} = \frac{\delta\left(b^3-a^3\right)}{3} = M_x;$$

We calculate the mass geometrically: $M = \delta A = \delta\left(\frac{\pi b^2}{4}\right) - \delta\left(\frac{\pi a^2}{4}\right) = \frac{\delta\pi}{4}\left(b^2-a^2\right)$. Thus, $\overline{x} = \frac{M_y}{M}$

$$= \frac{\delta\left(b^3-a^3\right)}{3}\cdot\frac{4}{\delta\pi\left(b^2-a^2\right)} = \frac{4}{3\pi}\left(\frac{b^3-a^3}{b^2-a^2}\right) = \frac{4}{3\pi}\,\frac{(b-a)\left(a^2+ab+b^2\right)}{(b-a)(b+a)} = \frac{4\left(a^2+ab+b^2\right)}{3\pi(a+b)};\text{ likewise}$$

$$\overline{y} = \frac{M_x}{M} = \frac{4\left(a^2+ab+b^2\right)}{3\pi(a+b)}.$$

(b) $\lim\limits_{a\to b} \frac{4}{3\pi}\left(\frac{a^2+ab+b^2}{a+b}\right) = \left(\frac{4}{3\pi}\right)\left(\frac{b^2+b^2+b^2}{b+b}\right) = \left(\frac{4}{3\pi}\right)\left(\frac{3b^2}{2b}\right) = \frac{2b}{\pi} \Rightarrow (\overline{x},\overline{y}) = \left(\frac{2b}{\pi},\frac{2b}{\pi}\right)$ is the limiting position of the centroid as $a \to b$. This is the centroid of a circle of radius a (and we note the two circles coincide when $a = b$).

14. Assume that the x and y intercepts of the triangular corner are a and b, respectively. Then the equation of the sloped edge of the triangle is $y = b - \left(\frac{b}{a}\right)x$. The x-coordinate of the centroid must be greater than 6 in. because the triangular cutout will shift the centroid to the right of the center of the square. Therefore, we assume that $\overline{x} = 7$ in. Using vertical strips of area and noting that $\frac{1}{2}ab = 36\ \text{in}^2$, we calculate $\overline{x}$ as follows:

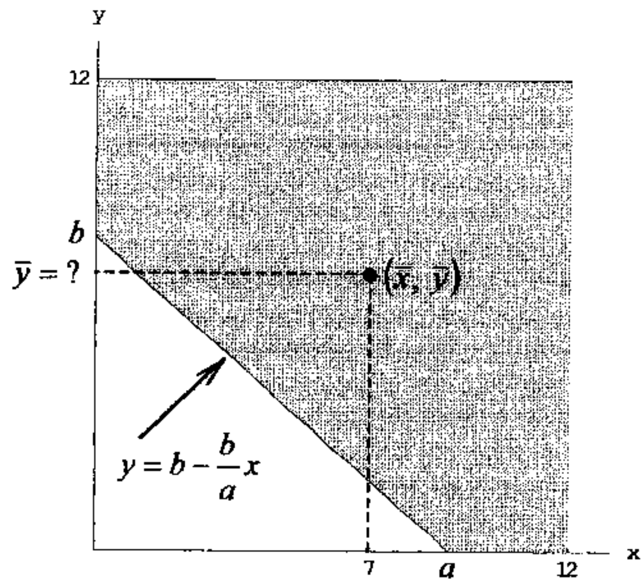
$$\overline{x} = 7\text{ in.} = \frac{\int_0^a x\left(12-\left(b-\frac{b}{a}x\right)\right)dx + \int_a^{12} 12x\,dx}{144-36} = \frac{6a^2-\frac{ba^2}{2}+\frac{ba^3}{3a}+12\left(72-\frac{a^2}{2}\right)}{108}$$

$$= \frac{6a^2-\left(\frac{1}{2}ab\right)a+\frac{2}{3}\left(\frac{1}{2}ab\right)a+864-6a^2}{108} = \frac{-36a+24a+864}{108}$$

Solving for a and b gives a = 9 in. and b = 8 in. Next we calculate $\overline{y}$ using horizontal strips of area, but first we express the equation of the sloped edge in terms of y as $x = 9 - \left(\frac{9}{8}\right)y$.

$$\overline{y} = \frac{\int_0^8 y\left(12 - \left(9 - \frac{9}{8}y\right)\right) dy + \int_8^{12} 12y\ dy}{144 - 36} = \frac{6\left(8^2\right) - \frac{9\left(8^2\right)}{2} + \frac{9\left(8^3\right)}{3(8)} + 12\left(72 - \frac{8^2}{2}\right)}{108} = \frac{64}{9} \text{ in.} \approx 7.11 \text{ in.}$$

Therefore, the centroid is $\frac{64}{9}$ in. from the bottom of the square.



NOTES:

CHAPTER 6 TRANSCENDENTAL FUNCTIONS AND DIFFERENTIAL EQUATIONS

6.1 LOGARITHMS

1. $y = \ln 3x \Rightarrow y' = \left(\frac{1}{3x}\right)(3) = \frac{1}{x}$

2. $y = \ln\left(t^2\right) \Rightarrow \frac{dy}{dt} = \left(\frac{1}{t^2}\right)(2t) = \frac{2}{t}$

3. $y = \ln \frac{3}{x} = \ln 3x^{-1} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{3x^{-1}}\right)\left(-3x^{-2}\right) = -\frac{1}{x}$

4. $y = \ln(\theta + 1) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\theta + 1}\right)(1) = \frac{1}{\theta + 1}$

5. $y = \ln x^3 \Rightarrow \frac{dy}{dx} = \left(\frac{1}{x^3}\right)\left(3x^2\right) = \frac{3}{x}$

6. $y = (\ln x)^3 \Rightarrow \frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{d}{dx}(\ln x) = \frac{3(\ln x)^2}{x}$

7. $y = t(\ln t)^2 \Rightarrow \frac{dy}{dt} = (\ln t)^2 + 2t(\ln t) \cdot \frac{d}{dt}(\ln t) = (\ln t)^2 + \frac{2t \ln t}{t} = (\ln t)^2 + 2 \ln t$

8. $y = t\sqrt{\ln t} = t(\ln t)^{1/2} \Rightarrow \frac{dy}{dt} = (\ln t)^{1/2} + \frac{1}{2}t(\ln t)^{-1/2} \cdot \frac{d}{dt}(\ln t) = (\ln t)^{1/2} + \frac{t(\ln t)^{-1/2}}{2t}$
$= (\ln t)^{1/2} + \frac{1}{2(\ln t)^{1/2}}$

9. $y = \frac{x^4}{4} \ln x - \frac{x^4}{16} \Rightarrow \frac{dy}{dx} = x^3 \ln x + \frac{x^4}{4} \cdot \frac{1}{x} - \frac{4x^3}{16} = x^3 \ln x$

10. $y = \frac{\ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (\ln t)(1)}{t^2} = \frac{1 - \ln t}{t^2}$

11. $y = \frac{1 + \ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (1 + \ln t)(1)}{t^2} = \frac{1 - 1 - \ln t}{t^2} = -\frac{\ln t}{t^2}$

12. $y = \frac{\ln x}{1 + \ln x} \Rightarrow y' = \frac{(1 + \ln x)\left(\frac{1}{x}\right) - (\ln x)\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \frac{\frac{1}{x} + \frac{\ln x}{x} - \frac{\ln x}{x}}{(1 + \ln x)^2} = \frac{1}{x(1 + \ln x)^2}$

13. $y = \frac{x \ln x}{1 + \ln x} \Rightarrow y' = \frac{(1 + \ln x)\left(\ln x + x \cdot \frac{1}{x}\right) - (x \ln x)\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \frac{(1 + \ln x)^2 - \ln x}{(1 + \ln x)^2} = 1 - \frac{\ln x}{(1 + \ln x)^2}$

14. $y = \ln(\ln x) \Rightarrow y' = \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) = \frac{1}{x \ln x}$

15. $y = \ln(\ln(\ln x)) \Rightarrow y' = \frac{1}{\ln(\ln x)} \cdot \frac{d}{dx}(\ln(\ln x)) = \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{d}{dx}(\ln x) = \frac{1}{x(\ln x)\ln(\ln x)}$

16. $y = \theta[\sin(\ln\theta) + \cos(\ln\theta)] \Rightarrow \frac{dy}{d\theta} = [\sin(\ln\theta) + \cos(\ln\theta)] + \theta\left[\cos(\ln\theta)\cdot\frac{1}{\theta} - \sin(\ln\theta)\cdot\frac{1}{\theta}\right]$

$= \sin(\ln\theta) + \cos(\ln\theta) + \cos(\ln\theta) - \sin(\ln\theta) = 2\cos(\ln\theta)$

17. $y = \ln(\sec\theta + \tan\theta) \Rightarrow \frac{dy}{d\theta} = \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta} = \frac{\sec\theta(\tan\theta + \sec\theta)}{\tan\theta + \sec\theta} = \sec\theta$

18. $y = \ln\frac{1}{x\sqrt{x+1}} = -\ln x - \frac{1}{2}\ln(x+1) \Rightarrow y' = -\frac{1}{x} - \frac{1}{2}\left(\frac{1}{x+1}\right) = -\frac{2(x+1)+x}{2x(x+1)} = -\frac{3x+2}{2x(x+1)}$

19. $y = \frac{1+\ln t}{1-\ln t} \Rightarrow \frac{dy}{dt} = \frac{(1-\ln t)\left(\frac{1}{t}\right) - (1+\ln t)\left(\frac{-1}{t}\right)}{(1-\ln t)^2} = \frac{\frac{1}{t} - \frac{\ln t}{t} + \frac{1}{t} + \frac{\ln t}{t}}{(1-\ln t)^2} = \frac{2}{t(1-\ln t)^2}$

20. $y = \sqrt{\ln\sqrt{t}} = \left(\ln t^{1/2}\right)^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2}\cdot\frac{d}{dt}\left(\ln t^{1/2}\right) = \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2}\cdot\frac{1}{t^{1/2}}\cdot\frac{d}{dt}\left(t^{1/2}\right)$

$= \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2}\cdot\frac{1}{t^{1/2}}\cdot\frac{1}{2}t^{-1/2} = \frac{1}{4t\sqrt{\ln\sqrt{t}}}$

21. $y = \ln(\sec(\ln\theta)) \Rightarrow \frac{dy}{d\theta} = \frac{1}{\sec(\ln\theta)}\cdot\frac{d}{d\theta}(\sec(\ln\theta)) = \frac{\sec(\ln\theta)\tan(\ln\theta)}{\sec(\ln\theta)}\cdot\frac{d}{d\theta}(\ln\theta) = \frac{\tan(\ln\theta)}{\theta}$

22. $y = \ln\left(\frac{\left(x^2+1\right)^5}{\sqrt{1-x}}\right) = 5\ln\left(x^2+1\right) - \frac{1}{2}\ln(1-x) \Rightarrow y' = \frac{5\cdot 2x}{x^2+1} - \frac{1}{2}\left(\frac{1}{1-x}\right)(-1) = \frac{10x}{x^2+1} + \frac{1}{2(1-x)}$

23. $y = \int_{x^2/2}^{x^2} \ln\sqrt{t}\,dt \Rightarrow \frac{dy}{dx} = \left(\ln\sqrt{x^2}\right)\cdot\frac{d}{dx}\left(x^2\right) - \left(\ln\sqrt{\frac{x^2}{2}}\right)\cdot\frac{d}{dx}\left(\frac{x^2}{2}\right) = 2x\ln|x| - x\ln\frac{|x|}{\sqrt{2}}$

24. $y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\,dt \Rightarrow \frac{dy}{dx} = \left(\ln\sqrt[3]{x}\right)\cdot\frac{d}{dx}\left(\sqrt[3]{x}\right) - \left(\ln\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right) = \left(\ln\sqrt[3]{x}\right)\left(\frac{1}{3}x^{-2/3}\right) - \left(\ln\sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right)$

$= \frac{\ln\sqrt[3]{x}}{3\sqrt[3]{x^2}} - \frac{\ln\sqrt{x}}{2\sqrt{x}}$

25. $\int_{-3}^{-2} \frac{1}{x}\,dx = [\ln|x|]_{-3}^{-2} = \ln 2 - \ln 3 = \ln\frac{2}{3}$

26. $\int_{-1}^{0} \frac{3}{3x-2}\,dx = [\ln|3x-2|]_{-1}^{0} = \ln 2 - \ln 5 = \ln\frac{2}{5}$

27. $\int \frac{2y}{y^2-25}\,dy = \ln\left|y^2-25\right| + C$

28. $\int \frac{8r}{4r^2-5}\,dr = \ln\left|4r^2-5\right| + C$

29. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\,dt = [\ln|2-\cos t|]_0^{\pi} = \ln 3 - \ln 1 = \ln 3$; or let $u = 2 - \cos t \Rightarrow du = \sin t\,dt$ with $t = 0$ $\Rightarrow u = 1$ and $t = \pi \Rightarrow u = 3 \Rightarrow \int_0^{\pi} \frac{\sin t}{2-\cos t}\,dt = \int_1^3 \frac{1}{u}\,du = [\ln|u|]_1^3 = \ln 3 - \ln 1 = \ln 3$

30. $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta = [\ln|1-4\cos\theta|]_0^{\pi/3} = \ln|1-2| = -\ln 3 = \ln\frac{1}{3}$; or let $u = 1-4\cos\theta \Rightarrow du = 4\sin\theta\,d\theta$ with $\theta = 0 \Rightarrow u = -3$ and $\theta = \frac{\pi}{3} \Rightarrow u = -1 \Rightarrow \int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta = \int_{-3}^{-1} \frac{1}{u}\,du = [\ln|u|]_{-3}^{-1} = -\ln 3 = \ln\frac{1}{3}$

31. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 1 \Rightarrow u = 0$ and $x = 2 \Rightarrow u = \ln 2$;

$$\int_1^2 \frac{2\ln x}{x}\,dx = \int_0^{\ln 2} 2u\,du = [u^2]_0^{\ln 2} = (\ln 2)^2$$

32. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 4 \Rightarrow u = \ln 4$;

$$\int_2^4 \frac{dx}{x\ln x} = \int_{\ln 2}^{\ln 4} \frac{1}{u}\,du = [\ln u]_{\ln 2}^{\ln 4} = \ln(\ln 4) - \ln(\ln 2) = \ln\left(\frac{\ln 4}{\ln 2}\right) = \ln\left(\frac{\ln 2^2}{\ln 2}\right) = \ln\left(\frac{2\ln 2}{\ln 2}\right) = \ln 2$$

33. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 4 \Rightarrow u = \ln 4$;

$$\int_2^4 \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\ln 4} u^{-2}\,du = \left[-\frac{1}{u}\right]_{\ln 2}^{\ln 4} = -\frac{1}{\ln 4} + \frac{1}{\ln 2} = -\frac{1}{\ln 2^2} + \frac{1}{\ln 2} = -\frac{1}{2\ln 2} + \frac{1}{\ln 2} = \frac{1}{2\ln 2} = \frac{1}{\ln 4}$$

34. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 16 \Rightarrow u = \ln 16$;

$$\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} = \frac{1}{2}\int_{\ln 2}^{\ln 16} u^{-1/2}\,du = [u^{1/2}]_{\ln 2}^{\ln 16} = \sqrt{\ln 16} - \sqrt{\ln 2} = \sqrt{4\ln 2} - \sqrt{\ln 2} = 2\sqrt{\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}$$

35. Let $u = 6 + 3\tan t \Rightarrow du = 3\sec^2 t\,dt$;

$$\int \frac{3\sec^2 t}{6+3\tan t}\,dt = \int \frac{du}{u} = \ln|u| + C = \ln|6+3\tan t| + C$$

36. Let $u = 2 + \sec y \Rightarrow du = \sec y\tan y\,dy$;

$$\int \frac{\sec y\tan y}{2+\sec y}\,dy = \int \frac{du}{u} = \ln|u| + C = \ln|2+\sec y| + C$$

37. Let $u = \cos\frac{x}{2} \Rightarrow du = -\frac{1}{2}\sin\frac{x}{2}\,dx \Rightarrow -2\,du = \sin\frac{x}{2}\,dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{2} \Rightarrow u = \frac{1}{\sqrt{2}}$;

$$\int_0^{\pi/2} \tan\frac{x}{2}\,dx = \int_0^{\pi/2} \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\,dx = -2\int_1^{1/\sqrt{2}} \frac{du}{u} = \left[-2\ln|u|\right]_1^{1/\sqrt{2}} = -2\ln\frac{1}{\sqrt{2}} = 2\ln\sqrt{2} = \ln 2$$

38. Let $u = \sin t \Rightarrow du = \cos t\,dt$; $t = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$ and $t = \frac{\pi}{2} \Rightarrow u = 1$;

$$\int_{\pi/4}^{\pi/2} \cot t\,dt = \int_{\pi/4}^{\pi/2} \frac{\cos t}{\sin t}\,dt = \int_{1/\sqrt{2}}^{1} \frac{du}{u} = \left[\ln|u|\right]_{1/\sqrt{2}}^{1} = -\ln\frac{1}{\sqrt{2}} = \ln\sqrt{2}$$

39. Let $u = \sin\frac{\theta}{3} \Rightarrow du = \frac{1}{3}\cos\frac{\theta}{3}\,d\theta \Rightarrow 6\,du = 2\cos\frac{\theta}{3}\,d\theta$; $\theta = \frac{\pi}{2} \Rightarrow u = \frac{1}{2}$ and $\theta = \pi \Rightarrow u = \frac{\sqrt{3}}{2}$;

$$\int_{\pi/2}^{\pi} 2\cot\frac{\theta}{3}\,d\theta = \int_{\pi/2}^{\pi} \frac{2\cos\frac{\theta}{3}}{\sin\frac{\theta}{3}}\,d\theta = 6\int_{1/2}^{\sqrt{3}/2} \frac{du}{u} = 6\left[\ln|u|\right]_{1/2}^{\sqrt{3}/2} = 6\left(\ln\frac{\sqrt{3}}{2} - \ln\frac{1}{2}\right) = 6\ln\sqrt{3} = \ln 27$$

40. Let $u = \cos 3x \Rightarrow du = -3\sin 3x\,dx \Rightarrow -2\,du = 6\sin 3x\,dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{12} \Rightarrow u = \frac{1}{\sqrt{2}}$;

$$\int_0^{\pi/12} 6\tan 3x\,dx = \int_0^{\pi/12} \frac{6\sin 3x}{\cos 3x}\,dx = -2\int_1^{1/\sqrt{2}} \frac{du}{u} = -2\left[\ln|u|\right]_1^{1/\sqrt{2}} = -2\ln\frac{1}{\sqrt{2}} - \ln 1 = 2\ln\sqrt{2} = \ln 2$$

41. $\int \frac{dx}{2\sqrt{x} + 2x} = \int \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)}$; let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$; $\int \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)} = \int \frac{du}{u} = \ln|u| + C$

$= \ln\left|1+\sqrt{x}\right| + C = \ln\left(1+\sqrt{x}\right) + C$

42. Let $u = \sec x + \tan x \Rightarrow du = \left(\sec x\tan x + \sec^2 x\right)dx = (\sec x)(\tan x + \sec x)\,dx \Rightarrow \sec x\,dx = \frac{du}{u}$;

$$\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}} = \int \frac{du}{u\sqrt{\ln u}} = \int (\ln u)^{-1/2}\cdot\frac{1}{u}\,du = 2(\ln u)^{1/2} + C = 2\sqrt{\ln(\sec x + \tan x)} + C$$

43. $y = \sqrt{x(x+1)} = (x(x+1))^{1/2} \Rightarrow \ln y = \frac{1}{2}\ln(x(x+1)) \Rightarrow 2\ln y = \ln(x) + \ln(x+1) \Rightarrow \frac{2y'}{y} = \frac{1}{x} + \frac{1}{x+1}$

$$\Rightarrow y' = \left(\frac{1}{2}\right)\sqrt{x(x+1)}\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{\sqrt{x(x+1)}\,(2x+1)}{2x(x+1)} = \frac{2x+1}{2\sqrt{x(x+1)}}$$

44. $y = \sqrt{\frac{t}{t+1}} = \left(\frac{t}{t+1}\right)^{1/2} \Rightarrow \ln y = \frac{1}{2}[\ln t - \ln(t+1)] \Rightarrow \frac{1}{y}\frac{dy}{dt} = \frac{1}{2}\left(\frac{1}{t} - \frac{1}{t+1}\right)$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{2}\sqrt{\frac{t}{t+1}}\left(\frac{1}{t} - \frac{1}{t+1}\right) = \frac{1}{2}\sqrt{\frac{t}{t+1}}\left[\frac{1}{t(t+1)}\right] = \frac{1}{2\sqrt{t}\,(t+1)^{3/2}}$$

45. $y = \sqrt{\theta+3}\,(\sin\theta) = (\theta+3)^{1/2}\sin\theta \Rightarrow \ln y = \frac{1}{2}\ln(\theta+3) + \ln(\sin\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{1}{2(\theta+3)} + \frac{\cos\theta}{\sin\theta}$

$\Rightarrow \frac{dy}{d\theta} = \sqrt{\theta+3}\,(\sin\theta)\left[\frac{1}{2(\theta+3)} + \cot\theta\right]$

46. $y = (\tan\theta)\sqrt{2\theta+1} = (\tan\theta)(2\theta+1)^{1/2} \Rightarrow \ln y = \ln(\tan\theta) + \frac{1}{2}\ln(2\theta+1) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{\sec^2\theta}{\tan\theta} + \left(\frac{1}{2}\right)\left(\frac{2}{2\theta+1}\right)$

$\Rightarrow \frac{dy}{d\theta} = (\tan\theta)\sqrt{2\theta+1}\left(\frac{\sec^2\theta}{\tan\theta} + \frac{1}{2\theta+1}\right) = \left(\sec^2\theta\right)\sqrt{2\theta+1} + \frac{\tan\theta}{\sqrt{2\theta+1}}$

47. $y = t(t+1)(t+2) \Rightarrow \ln y = \ln t + \ln(t+1) + \ln(t+2) \Rightarrow \frac{1}{y}\frac{dy}{dt} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}$

$\Rightarrow \frac{dy}{dt} = t(t+1)(t+2)\left(\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}\right) = t(t+1)(t+2)\left[\frac{(t+1)(t+2) + t(t+2) + t(t+1)}{t(t+1)(t+2)}\right] = 3t^2 + 6t + 2$

48. $y = \frac{\theta+5}{\theta\cos\theta} \Rightarrow \ln y = \ln(\theta+5) - \ln\theta - \ln(\cos\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin\theta}{\cos\theta}$

$\Rightarrow \frac{dy}{d\theta} = \left(\frac{\theta+5}{\theta\cos\theta}\right)\left(\frac{1}{\theta+5} - \frac{1}{\theta} + \tan\theta\right)$

49. $y = \frac{\theta\sin\theta}{\sqrt{\sec\theta}} \Rightarrow \ln y = \ln\theta + \ln(\sin\theta) - \frac{1}{2}\ln(\sec\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \left[\frac{1}{\theta} + \frac{\cos\theta}{\sin\theta} - \frac{(\sec\theta)(\tan\theta)}{2\sec\theta}\right]$

$\Rightarrow \frac{dy}{d\theta} = \frac{\theta\sin\theta}{\sqrt{\sec\theta}}\left(\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right)$

50. $y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} \Rightarrow \ln y = \ln x + \frac{1}{2}\ln\left(x^2+1\right) - \frac{2}{3}\ln(x+1) \Rightarrow \frac{y'}{y} = \frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}$

$\Rightarrow y' = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right]$

51. $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}} \Rightarrow \ln y = \frac{1}{3}\left[\ln x + \ln(x-2) - \ln\left(x^2+1\right)\right] \Rightarrow \frac{y'}{y} = \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$

$\Rightarrow y' = \frac{1}{3}\sqrt[3]{\frac{x(x-2)}{x^2+1}}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$

52. $y = \sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}} \Rightarrow \ln y = \frac{1}{3}\left[\ln x + \ln(x+1) + \ln(x-2) - \ln\left(x^2+1\right) - \ln(2x+3)\right]$

$\Rightarrow y' = \frac{1}{3}\sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right)$

53. $\frac{dy}{dx} = 1 + \frac{1}{x}$ at $(1,3) \Rightarrow y = x + \ln|x| + C$; $y = 3$ at $x = 1 \Rightarrow C = 2 \Rightarrow y = x + \ln|x| + 2$

54. $\frac{d^2y}{dx^2} = \sec^2 x \Rightarrow \frac{dy}{dx} = \tan x + C$ and $1 = \tan 0 + C \Rightarrow \frac{dy}{dx} = \tan x + 1 \Rightarrow y = \int (\tan x + 1)\, dx$
$= \ln|\sec x| + x + C_1$ and $0 = \ln|\sec 0| + 0 + C_1 \Rightarrow C_1 = 0 \Rightarrow y = \ln|\sec x| + x$

55. $\int \frac{\log_{10} x}{x}\, dx = \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx;\ \left[u = \ln x \Rightarrow du = \frac{1}{x}\, dx\right]$
$\to \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx = \frac{1}{\ln 10} \int u\, du = \left(\frac{1}{\ln 10}\right)\left(\frac{1}{2}u^2\right) + C = \frac{(\ln x)^2}{2 \ln 10} + C$

56. $\int_1^4 \frac{\ln 2 \log_2 x}{x}\, dx = \int_1^4 \left(\frac{\ln 2}{x}\right)\left(\frac{\ln x}{\ln 2}\right) dx = \int_1^4 \frac{\ln x}{x}\, dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}\left[(\ln 4)^2 - (\ln 1)^2\right] = \frac{1}{2}(\ln 4)^2$
$= \frac{1}{2}(2 \ln 2)^2 = 2(\ln 2)^2$

57. $\int_0^2 \frac{\log_2 (x+2)}{x+2}\, dx = \frac{1}{\ln 2} \int_0^2 [\ln (x+2)]\left(\frac{1}{x+2}\right) dx = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln (x+2))^2}{2}\right]_0^2 = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln 4)^2}{2} - \frac{(\ln 2)^2}{2}\right]$
$= \left(\frac{1}{\ln 2}\right)\left[\frac{4(\ln 2)^2}{2} - \frac{(\ln 2)^2}{2}\right] = \frac{3}{2} \ln 2$

58. $\int_0^9 \frac{2 \log_{10} (x+1)}{x+1}\, dx = \frac{2}{\ln 10} \int_0^9 \ln (x+1)\left(\frac{1}{x+1}\right) dx = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln (x+1))^2}{2}\right]_0^9 = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln 10)^2}{2} - \frac{(\ln 1)^2}{2}\right]$
$= \ln 10$

59. $\int \frac{dx}{x \log_{10} x} = \int \left(\frac{\ln 10}{\ln x}\right)\left(\frac{1}{x}\right) dx = (\ln 10) \int \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) dx;\ \left[u = \ln x \Rightarrow du = \frac{1}{x}\, dx\right]$
$\to (\ln 10) \int \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) dx = (\ln 10) \int \frac{1}{u}\, du = (\ln 10) \ln|u| + C = (\ln 10) \ln|\ln x| + C$

60. $\int \frac{dx}{x (\log_8 x)^2} = \int \frac{dx}{x\left(\frac{\ln x}{\ln 8}\right)^2} = (\ln 8)^2 \int \frac{(\ln x)^{-2}}{x}\, dx = (\ln 8)^2 \frac{(\ln x)^{-1}}{-1} + C = -\frac{(\ln 8)^2}{\ln x} + C$

61. $y = \log_2 5\theta = \frac{\ln 5\theta}{\ln 2} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 2}\right)\left(\frac{1}{5\theta}\right)(5) = \frac{1}{\theta \ln 2}$

62. $y = \frac{\ln x}{\ln 4} + \frac{\ln x^2}{\ln 4} = \frac{\ln x}{\ln 4} + 2\frac{\ln x}{\ln 4} = 3\frac{\ln x}{\ln 4} \Rightarrow y' = \frac{3}{x \ln 4}$

63. $y = \log_2 r \cdot \log_4 r = \left(\frac{\ln r}{\ln 2}\right)\left(\frac{\ln r}{\ln 4}\right) = \frac{\ln^2 r}{(\ln 2)(\ln 4)} \Rightarrow \frac{dy}{dr} = \left[\frac{1}{(\ln 2)(\ln 4)}\right](2 \ln r)\left(\frac{1}{r}\right) = \frac{2 \ln r}{r(\ln 2)(\ln 4)}$

64. $y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)^{\ln 3}}{\ln 3} = \frac{(\ln 3)\,\ln\left(\frac{x+1}{x-1}\right)}{\ln 3} = \ln\left(\frac{x+1}{x-1}\right) = \ln(x+1) - \ln(x-1)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$

65. $y = \theta \sin(\log_7 \theta) = \theta \sin\left(\frac{\ln\theta}{\ln 7}\right) \Rightarrow \frac{dy}{d\theta} = \sin\left(\frac{\ln\theta}{\ln 7}\right) + \theta\left[\cos\left(\frac{\ln\theta}{\ln 7}\right)\right]\left(\frac{1}{\theta\ln 7}\right) = \sin(\log_7\theta) + \frac{1}{\ln 7}\cos(\log_7\theta)$

66. $y = 3\log_8(\log_2 t) = \frac{3\ln(\log_2 t)}{\ln 8} = \frac{3\ln\left(\frac{\ln t}{\ln 2}\right)}{\ln 8} \Rightarrow \frac{dy}{dt} = \left(\frac{3}{\ln 8}\right)\left[\frac{1}{(\ln t)/(\ln 2)}\right]\left(\frac{1}{t\ln 2}\right) = \frac{3}{t(\ln t)(\ln 8)}$
$= \frac{1}{t(\ln t)(\ln 2)}$

67. (a) $f(x) = \ln(\cos x) \Rightarrow f'(x) = -\frac{\sin x}{\cos x} = -\tan x = 0 \Rightarrow x = 0$; $f'(x) > 0$ for $-\frac{\pi}{4} \le x < 0$ and $f'(x) < 0$ for $0 < x \le \frac{\pi}{3} \Rightarrow$ there is a relative maximum at $x = 0$ with $f(0) = \ln(\cos 0) = \ln 1 = 0$; $f\left(-\frac{\pi}{4}\right) = \ln\left(\cos\left(-\frac{\pi}{4}\right)\right)$ $= \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2$ and $f\left(\frac{\pi}{3}\right) = \ln\left(\cos\left(\frac{\pi}{3}\right)\right) = \ln\frac{1}{2} = -\ln 2$. Therefore, the absolute minimum occurs at $x = \frac{\pi}{3}$ with $f\left(\frac{\pi}{3}\right) = -\ln 2$ and the absolute maximum occurs at $x = 0$ with $f(0) = 0$.

(b) $f(x) = \cos(\ln x) \Rightarrow f'(x) = \frac{-\sin(\ln x)}{x} = 0 \Rightarrow x = 1$; $f'(x) > 0$ for $\frac{1}{2} \le x < 1$ and $f'(x) < 0$ for $1 < x \le 2$ $\Rightarrow$ there is a relative maximum at $x = 1$ with $f(1) = \cos(\ln 1) = \cos 0 = 1$; $f\left(\frac{1}{2}\right) = \cos\left(\ln\left(\frac{1}{2}\right)\right)$ $= \cos(-\ln 2) = \cos(\ln 2)$ and $f(2) = \cos(\ln 2)$. Therefore, the absolute minimum occurs at $x = \frac{1}{2}$ and $x = 2$ with $f\left(\frac{1}{2}\right) = f(2) = \cos(\ln 2)$, and the absolute maximum occurs at $x = 1$ with $f(1) = 1$.

68. (a) $f(x) = x - \ln x \Rightarrow f'(x) = 1 - \frac{1}{x}$; if $x > 1$, then $f'(x) > 0$ which means that $f(x)$ is increasing

(b) $f(1) = 1 - \ln 1 = 1 \Rightarrow f(x) = x - \ln x > 0$, if $x > 1$ by part (a) $\Rightarrow x > \ln x$ if $x > 1$

69. $\int_1^5 (\ln 2x - \ln x)\,dx = \int_1^5 (-\ln x + \ln 2 + \ln x)\,dx = (\ln 2)\int_1^5 dx = (\ln 2)(5-1) = \ln 2^4 = \ln 16$

70. $A = \int_{-\pi/4}^{0} -\tan x\,dx + \int_0^{\pi/3} \tan x\,dx = \int_{-\pi/4}^{0} \frac{-\sin x}{\cos x}\,dx - \int_0^{\pi/3} \frac{-\sin x}{\cos x}\,dx = \left[\ln|\cos x|\right]_{-\pi/4}^{0} - \left[\ln|\cos x|\right]_0^{\pi/3}$
$= \left(\ln 1 - \ln\frac{1}{\sqrt{2}}\right) - \left(\ln\frac{1}{2} - \ln 1\right) = \ln\sqrt{2} + \ln 2 = \frac{3}{2}\ln 2$

71. $V = \pi\int_0^3 \left(\frac{2}{\sqrt{y+1}}\right)^2 dy = 4\pi\int_0^3 \frac{1}{y+1}\,dy = 4\pi\left[\ln|y+1|\right]_0^3 = 4\pi(\ln 4 - \ln 1) = 4\pi\ln 4$

72. $V = \pi \int_{\pi/6}^{\pi/2} \cot x \, dx = \pi \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x} \, dx = \pi [\ln (\sin x)]_{\pi/6}^{\pi/2} = \pi\left(\ln 1 - \ln \frac{1}{2}\right) = \pi \ln 2$

73. $V = 2\pi \int_{1/2}^{2} x\left(\frac{1}{x^2}\right) dx = 2\pi \int_{1/2}^{2} \frac{1}{x} \, dx = 2\pi[\ln |x|]_{1/2}^{2} = 2\pi\left(\ln 2 - \ln \frac{1}{2}\right) = 2\pi(2 \ln 2) = \pi \ln 2^4 = \pi \ln 16$

74. (a) $y = \frac{x^2}{8} - \ln x \Rightarrow 1 + (y')^2 = 1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2 = 1 + \left(\frac{x^2-4}{4x}\right)^2 = \left(\frac{x^2+4}{4x}\right)^2 \Rightarrow L = \int_4^8 \sqrt{1+(y')^2} \, dx$

$= \int_4^8 \frac{x^2+4}{4x} \, dx = \int_4^8 \left(\frac{x}{4} + \frac{1}{x}\right) dx = \left[\frac{x^2}{8} + \ln |x|\right]_4^8 = (8 + \ln 8) - (2 + \ln 4) = 6 + \ln 2$

(b) $x = \left(\frac{y}{4}\right)^2 - 2 \ln\left(\frac{y}{4}\right) \Rightarrow \frac{dx}{dy} = \frac{y}{8} - \frac{2}{y} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{y}{8} - \frac{2}{y}\right)^2 = 1 + \left(\frac{y^2-16}{8y}\right)^2 = \left(\frac{y^2+16}{8y}\right)^2$

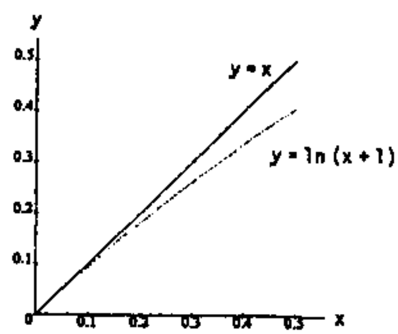
$\Rightarrow L = \int_4^{12} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_4^{12} \frac{y^2+16}{8y} \, dy = \int_4^{12} \left(\frac{y}{8} + \frac{2}{y}\right) dy = \left[\frac{y^2}{16} + 2 \ln y\right]_4^{12} = (9 + 2 \ln 12) - (1 + 2 \ln 4)$

$= 8 + 2 \ln 3 = 8 + \ln 9$

75. (a) $L(x) = f(0) + f'(0) \cdot x$, and $f(x) = \ln (1 + x) \Rightarrow f'(x)\big|_{x=0} = \frac{1}{1+x}\Big|_{x=0} = 1 \Rightarrow L(x) = \ln 1 + 1 \cdot x \Rightarrow L(x) = x$

(b) On $[0, 0.1]$, $f(x)$ and $L(x)$are both increasing because $f'(x) = \frac{1}{1+x} > 0$ and $L'(x) = 1 > 0$ for $0 \le x \le 0.1$. In addition $0 \le x \le 0.1 \Rightarrow 1 \le 1 + x \le 1.1 \Rightarrow \frac{1}{1.1} \le \frac{1}{x+1} \le 1 \Rightarrow L'(x) \ge f'(x) \Rightarrow E(x) = f(x) - L(x)$ is non-increasing on $[0, 0.1]$ because $E'(x) = f'(x) - L' \le 0$ on the interval. Therefore, the largest error is $|f(0.1) - L(0.1)| = |\ln (1.1) - 1.1| \approx 0.00469$.

(c) The approximation $y = x$ for $\ln (1 + x)$ is best for smaller positive values of x on the interval $[0, 0.1]$ as seen on the graph. As x increases so does the magnitude of the error $|\ln (x + 1) - x|$. From the graph, an upper bound for the magnitude of the error is $|\ln (1.1) - 0.1| \approx 0.00469$ which is consistent with the analytical result obtained in part (b).



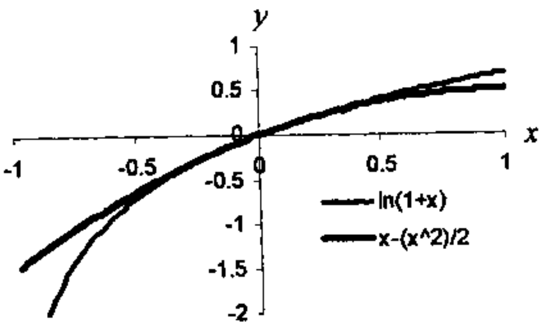
76. (a) $Q(x) = b_2(x - a)^2 + b_1(x - a) + b_0$. In this case, $a = 0$ and $f(x) = \ln (1 + x)$.

Require: (i) $Q(0) = f(0) \Rightarrow b_0 = \ln (1) = 0$. (ii) $Q'(0) = f'(0) \Rightarrow b_1 = \frac{1}{1+0} = 1$.

(iii) $Q''(0) = f''(0) \Rightarrow 2b_2 = -\frac{1}{(1+0)^2} \Rightarrow b_2 = -\frac{1}{2}$

Therefore, $Q(x) = -\frac{x^2}{2} + x$.

(b)



The approximation is best (exact) at x = 0 and least good at x = 1. The closer x is to 0, the better the approximation.

(c) $\ln 1.1 \approx Q(0.1) = -\frac{0.1^2}{2} + 0.1 = 0.095$ compared to the calculator value of 0.0953102.

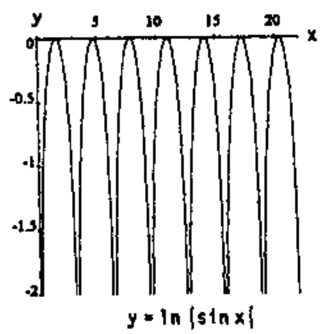
$\ln 2 \approx Q(1) = -\frac{1^2}{2} + 1 = 0.5$ compared to the calculator value of 0.6931472.

77. $\ln(1.2) = \ln(1 + .2) \approx 0.2$, $\ln(.8) = \ln(1 + (-0.2)) \approx -0.2$; with Simpson's rule for n = 2, $\ln(1.2) = \int_1^{1.2} \frac{1}{t}\,dt$

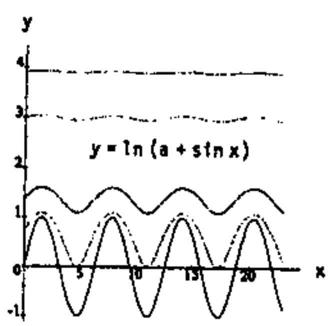
≈ 0.182323232 and $\ln(0.8) = \int_1^{0.8} \frac{1}{t}\,dt \approx -0.223148148$; alternatively, $\ln(1.2) = \ln(1 + 0.2) = \int_0^{0.2} \frac{1}{1+t}\,dt$

≈ 0.182323232 and $\ln(0.8) = \int_0^{-0.2} \frac{1}{1+t}\,dt \approx -0.223148148$.

78. Since $-1 \le \sin x \le 1$, $y = \ln|\sin x|$ must be nonpositive; y = 0 when $\sin x = \pm 1$ $\Rightarrow$ x = odd multiples of $\frac{\pi}{2}$; $y \to -\infty$ when $\sin x \to 0 \Rightarrow x \to$ even multiples of $\frac{\pi}{2}$.
To turn the arches upside down we would use the formula $y = -\ln|\sin x| = \ln\frac{1}{|\sin x|}$.

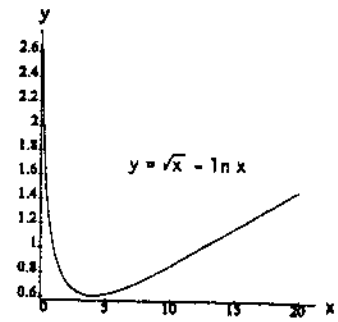


79. (a)



(b) As a increases, the value of $a + \sin x$ gets closer and closer to $a \pm 1$. Thus, $\ln(a + \sin x)$ looks more and more like the constant value $\ln a$ for larger and larger values of $a \Rightarrow$ the curves flatten as a increases.

80. (a) The graph of $y = \sqrt{x} - \ln x$ <u>appears</u> to be concave upward for all $x > 0$. However, by graphing y'' it appears as though there is a value of x at which y'' changes from being positive to being negative. Zooming in shows this situation occurs at about $x = 16$. Using trace or the "zero" function verifies this expectation.



(b) $y = \sqrt{x} - \ln x \Rightarrow y' = \frac{1}{2\sqrt{x}} - \frac{1}{x} \Rightarrow y'' = -\frac{1}{4x^{3/2}} + \frac{1}{x^2} = \frac{1}{x^2}\left(-\frac{\sqrt{x}}{4} + 1\right) = 0 \Rightarrow \sqrt{x} = 4 \Rightarrow x = 16.$

Thus, $y'' > 0$ if $0 < x < 16$ and $y'' < 0$ if $x > 16$ so a point of inflection exists at $x = 16$. Even though the graph of $y = \sqrt{x} - \ln x$ closely resembles a straight line for $x \ge 10$, it is possible to find the inflection point by graphing.

6.2 EXPONENTIAL FUNCTIONS

1. $y = e^{-5x} \Rightarrow y' = e^{-5x}\frac{d}{dx}(-5x) \Rightarrow y' = -5e^{-5x}$

2. $y = e^{5-7x} \Rightarrow y' = e^{5-7x}\frac{d}{dx}(5 - 7x) \Rightarrow y' = -7e^{5-7x}$

3. $y = e^{\left(4\sqrt{x}+x^2\right)} \Rightarrow y' = e^{\left(4\sqrt{x}+x^2\right)}\frac{d}{dx}\left(4\sqrt{x} + x^2\right) \Rightarrow y' = \left(\frac{2}{\sqrt{x}} + 2x\right)e^{\left(4\sqrt{x}+x^2\right)}$

4. $y = xe^x - e^x \Rightarrow y' = \left(e^x + xe^x\right) - e^x = xe^x$

5. $y = \left(x^2 - 2x + 2\right)e^x \Rightarrow y' = (2x - 2)e^x + \left(x^2 - 2x + 2\right)e^x = x^2e^x$

6. $y = e^\theta(\sin\theta + \cos\theta) \Rightarrow y' = e^\theta(\sin\theta + \cos\theta) + e^\theta(\cos\theta - \sin\theta) = 2e^\theta\cos\theta$

7. $y = \ln\left(3\theta e^{-\theta}\right) = \ln 3 + \ln\theta + \ln e^{-\theta} = \ln 3 + \ln\theta - \theta \Rightarrow \frac{dy}{d\theta} = \frac{1}{\theta} - 1$

8. $y=\cos\left(e^{-\theta^2}\right)\Rightarrow \frac{dy}{d\theta}=-\sin\left(e^{-\theta^2}\right)\frac{d}{d\theta}\left(e^{-\theta^2}\right)=\left(-\sin\left(e^{-\theta^2}\right)\right)\left(e^{-\theta^2}\right)\frac{d}{d\theta}(-\theta^2)=2\theta e^{-\theta^2}\sin\left(e^{-\theta^2}\right)$

9. $y=\ln\left(2e^{-t}\sin t\right)=\ln 2+\ln e^{-t}+\ln\sin t=\ln 2-t+\ln\sin t\Rightarrow \frac{dy}{dt}=-1+\left(\frac{1}{\sin t}\right)\frac{d}{dt}(\sin t)=-1+\frac{\cos t}{\sin t}$
$=\frac{\cos t-\sin t}{\sin t}$

10. $y=\ln\frac{e^\theta}{1+e^\theta}=\ln e^\theta-\ln\left(1+e^\theta\right)=\theta-\ln\left(1+e^\theta\right)\Rightarrow \frac{dy}{d\theta}=1-\left(\frac{1}{1+e^\theta}\right)\frac{d}{d\theta}\left(1+e^\theta\right)=1-\frac{e^\theta}{1+e^\theta}=\frac{1}{1+e^\theta}$

11. $y=\ln\frac{\sqrt{\theta}}{1+\sqrt{\theta}}=\ln\sqrt{\theta}-\ln\left(1+\sqrt{\theta}\right)\Rightarrow \frac{dy}{d\theta}=\left(\frac{1}{\sqrt{\theta}}\right)\frac{d}{d\theta}\left(\sqrt{\theta}\right)-\left(\frac{1}{1+\sqrt{\theta}}\right)\frac{d}{d\theta}\left(1+\sqrt{\theta}\right)$
$=\left(\frac{1}{\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right)-\left(\frac{1}{1+\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right)=\frac{\left(1+\sqrt{\theta}\right)-\sqrt{\theta}}{2\theta\left(1+\sqrt{\theta}\right)}=\frac{1}{2\theta\left(1+\sqrt{\theta}\right)}=\frac{1}{2\theta\left(1+\theta^{1/2}\right)}$

12. $y=e^{\sin t}\left(\ln t^2+1\right)\Rightarrow \frac{dy}{dt}=e^{\sin t}(\cos t)\left(\ln t^2+1\right)+\frac{2}{t}e^{\sin t}=e^{\sin t}\left[\left(\ln t^2+1\right)(\cos t)+\frac{2}{t}\right]$

13. $\int_0^{\ln x}\sin e^t\,dt\Rightarrow y'=\left(\sin e^{\ln x}\right)\cdot\frac{d}{dx}(\ln x)=\frac{\sin x}{x}$

14. $y=\int_{e^{4\sqrt{x}}}^{e^{2x}}\ln t\,dt\Rightarrow y'=\left(\ln e^{2x}\right)\cdot\frac{d}{dx}\left(e^{2x}\right)-\left(\ln e^{4\sqrt{x}}\right)\cdot\frac{d}{dx}\left(e^{4\sqrt{x}}\right)=(2x)\left(2e^{2x}\right)-\left(4\sqrt{x}\right)\left(e^{4\sqrt{x}}\right)\cdot\frac{d}{dx}\left(4\sqrt{x}\right)$
$=4xe^{2x}-4\sqrt{x}e^{4\sqrt{x}}\left(\frac{2}{\sqrt{x}}\right)=4xe^{2x}-8e^{4\sqrt{x}}$

15. $\ln y=e^y\sin x\Rightarrow\left(\frac{1}{y}\right)y'=\left(y'e^y\right)(\sin x)+e^y\cos x\Rightarrow y'\left(\frac{1}{y}-e^y\sin x\right)=e^y\cos x$
$\Rightarrow y'\left(\frac{1-ye^y\sin x}{y}\right)=e^y\cos x\Rightarrow y'=\frac{ye^y\cos x}{1-ye^y\sin x}$

16. $\ln xy=e^{x+y}\Rightarrow\ln x+\ln y=e^{x+y}\Rightarrow\frac{1}{x}+\left(\frac{1}{y}\right)y'=\left(1+y'\right)e^{x+y}\Rightarrow y'\left(\frac{1}{y}-e^{x+y}\right)=e^{x+y}-\frac{1}{x}$
$\Rightarrow y'\left(\frac{1-ye^{x+y}}{y}\right)=\frac{xe^{x+y}-1}{x}\Rightarrow y'=\frac{y\left(xe^{x+y}-1\right)}{x\left(1-ye^{x+y}\right)}$

17. $e^{2x}=\sin(x+3y)\Rightarrow 2e^{2x}=\left(1+3y'\right)\cos(x+3y)\Rightarrow 1+3y'=\frac{2e^{2x}}{\cos(x+3y)}\Rightarrow 3y'=\frac{2e^{2x}}{\cos(x+3y)}-1$
$\Rightarrow y'=\frac{2e^{2x}-\cos(x+3y)}{3\cos(x+3y)}$

18. $\tan y=e^x+\ln x\Rightarrow\left(\sec^2 y\right)y'=e^x+\frac{1}{x}\Rightarrow y'=\frac{\left(xe^x+1\right)\cos^2 y}{x}$

19. $\int \left(e^{3x} + 5e^{-x}\right) dx = \frac{e^{3x}}{3} - 5e^{-x} + C$

20. $\int_{\ln 2}^{\ln 3} e^{x}\, dx = \left[e^{x}\right]_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2} = 3 - 2 = 1$

21. $\int 8e^{(x+1)}\, dx = 8e^{(x+1)} + C$

22. $\int_{\ln 4}^{\ln 9} e^{x/2}\, dx = \left[2e^{x/2}\right]_{\ln 4}^{\ln 9} = 2\left[e^{(\ln 9)/2} - e^{(\ln 4)/2}\right] = 2\left(e^{\ln 3} - e^{\ln 2}\right) = 2(3-2) = 2$

23. Let $u = -r^{1/2} \Rightarrow du = -\frac{1}{2}r^{-1/2}\, dr \Rightarrow -2\, du = r^{-1/2}\, dr$;

$$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\, dr = \int e^{-r^{1/2}} \cdot r^{-1/2}\, dr = -2 \int e^{u}\, du = -2e^{-r^{1/2}} + C = -2e^{-\sqrt{r}} + C$$

24. Let $u = -t^{2} \Rightarrow du = -2t\, dt \Rightarrow -du = 2t\, dt$;

$$\int 2te^{-t^2}\, dt = -\int e^{u}\, du = -e^{u} + C = -e^{-t^2} + C$$

25. Let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}\, dx \Rightarrow -du = \frac{1}{x^2}\, dx$;

$$\int \frac{e^{1/x}}{x^2}\, dx = \int -e^{u}\, du = -e^{u} + C = -e^{1/x} + C$$

26. Let $u = -x^{-2} \Rightarrow du = 2x^{-3}\, dx \Rightarrow \frac{1}{2}\, du = x^{-3}\, dx$;

$$\int \frac{e^{-1/x^2}}{x^3}\, dx = \int e^{-x^{-2}} \cdot x^{-3}\, dx = \frac{1}{2}\int e^{u}\, du = \frac{1}{2}e^{u} + C = \frac{1}{2}e^{-x^{-2}} + C = \frac{1}{2}e^{-1/x^2} + C$$

27. Let $u = \tan\theta \Rightarrow du = \sec^2\theta\, d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \frac{\pi}{4} \Rightarrow u = 1$;

$$\int_0^{\pi/4} \left(1 + e^{\tan\theta}\right)\sec^2\theta\, d\theta = \int_0^{\pi/4} \sec^2\theta\, d\theta + \int_0^1 e^{u}\, du = [\tan\theta]_0^{\pi/4} + \left[e^{u}\right]_0^1 = \left[\tan\left(\frac{\pi}{4}\right) - \tan(0)\right] + \left(e^{1} - e^{0}\right)$$
$$= (1-0) + (e-1) = e$$

28. Let $u = \cot\theta \Rightarrow du = -\csc^2\theta\, d\theta$; $\theta = \frac{\pi}{4} \Rightarrow u = 1$, $\theta = \frac{\pi}{2} \Rightarrow u = 0$;

$$\int_{\pi/4}^{\pi/2} \left(1 + e^{\cot\theta}\right)\csc^2\theta\, d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta - \int_1^0 e^{u}\, du = [-\cot\theta]_{\pi/4}^{\pi/2} - \left[e^{u}\right]_1^0 = \left[-\cot\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{4}\right)\right] - \left(e^{0} - e^{1}\right)$$
$$= (0+1) - (1-e) = e$$

29. Let $u = \sec \pi t \Rightarrow du = \frac{1}{\pi} \sec \pi t \tan \pi t \, dt \Rightarrow \pi \, du = \sec \pi t \tan \pi t \, dt$;

$$\int e^{\sec(\pi t)} \sec(\pi t) \tan(\pi t) \, dt = \frac{1}{\pi} \int e^u \, du = \frac{e^u}{\pi} + C = \frac{e^{\sec(\pi t)}}{\pi} + C$$

30. Let $u = e^{x^2} \Rightarrow du = 2xe^{x^2} \, dx$; $x = 0 \Rightarrow u = 1$, $x = \sqrt{\ln \pi} \Rightarrow u = e^{\ln \pi} = \pi$;

$$\int_0^{\sqrt{\ln \pi}} 2xe^{x^2} \cos\left(e^{x^2}\right) dx = \int_1^{\pi} \cos u \, du = [\sin u]_1^{\pi} = \sin(\pi) - \sin(1) = -\sin(1) \approx -0.84147$$

31. Let $u = 1 + e^r \Rightarrow du = e^r \, dr$;

$$\int \frac{e^r}{1 + e^r} \, dr = \int \frac{1}{u} \, du = \ln|u| + C = \ln\left(1 + e^r\right) + C$$

32. $\int \frac{1}{1 + e^x} \, dx = \int \frac{e^{-x}}{e^{-x} + 1} \, dx$;

let $u = e^{-x} + 1 \Rightarrow du = -e^{-x} \, dx \Rightarrow -du = e^{-x} \, dx$;

$$\int \frac{e^{-x}}{e^{-x} + 1} \, dx = -\int \frac{1}{u} \, du = -\ln|u| + C = -\ln\left(e^{-x} + 1\right) + C$$

33. $y = 2^x \Rightarrow y' = 2^x \ln 2$

34. $y = 2^{\sqrt{s}} \Rightarrow \frac{dy}{ds} = 2^{\sqrt{s}} (\ln 2)\left(\frac{1}{2} s^{-1/2}\right) = \left(\frac{\ln 2}{2\sqrt{s}}\right) 2^{\sqrt{s}}$

35. $y = x^{\pi} \Rightarrow y' = \pi x^{(\pi - 1)}$

36. $y = (\cos \theta)^{\sqrt{2}} \Rightarrow \frac{dy}{d\theta} = -\sqrt{2} \, (\cos \theta)^{\left(\sqrt{2} - 1\right)} (\sin \theta)$

37. $y = 7^{\sec \theta} \ln 7 \Rightarrow \frac{dy}{d\theta} = \left(7^{\sec \theta} \ln 7\right)(\ln 7)(\sec \theta \tan \theta) = 7^{\sec \theta} (\ln 7)^2 (\sec \theta \tan \theta)$

38. $y = 2^{\sin 3t} \Rightarrow \frac{dy}{dt} = \left(2^{\sin 3t} \ln 2\right)(\cos 3t)(3) = (3 \cos 3t)\left(2^{\sin 3t}\right)(\ln 2)$

39. $y = t^{1-e} \Rightarrow \frac{dy}{dt} = (1 - e) \, t^{-e}$

40. $y = (\ln \theta)^{\pi} \Rightarrow \frac{dy}{d\theta} = \pi (\ln \theta)^{(\pi - 1)} \left(\frac{1}{\theta}\right) = \frac{\pi (\ln \theta)^{(\pi - 1)}}{\theta}$

41. $y = \log_3 \left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)^{\ln 3}}{\ln 3} = \frac{(\ln 3) \ln\left(\frac{x+1}{x-1}\right)}{\ln 3} = \ln\left(\frac{x+1}{x-1}\right) = \ln(x+1) - \ln(x-1)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$$

42. $y = \log_5 \sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}} = \log_5 \left(\frac{7x}{3x+2}\right)^{(\ln 5)/2} = \frac{\ln\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2}}{\ln 5} = \left(\frac{\ln 5}{2}\right)\left[\frac{\ln\left(\frac{7x}{3x+2}\right)}{\ln 5}\right] = \frac{1}{2}\ln\left(\frac{7x}{3x+2}\right)$

$= \frac{1}{2}\ln 7x - \frac{1}{2}\ln(3x+2) \Rightarrow \frac{dy}{dx} = \frac{7}{2 \cdot 7x} - \frac{3}{2 \cdot (3x+2)} = \frac{(3x+2) - 3x}{2x(3x+2)} = \frac{1}{x(3x+2)}$

43. $y = \log_7\left(\frac{\sin\theta\cos\theta}{e^\theta 2^\theta}\right) = \frac{\ln(\sin\theta) + \ln(\cos\theta) - \ln e^\theta - \ln 2^\theta}{\ln 7} = \frac{\ln(\sin\theta) + \ln(\cos\theta) - \theta - \theta\ln 2}{\ln 7}$

$\Rightarrow \frac{dy}{d\theta} = \frac{\cos\theta}{(\sin\theta)(\ln 7)} - \frac{\sin\theta}{(\cos\theta)(\ln 7)} - \frac{1}{\ln 7} - \frac{\ln 2}{\ln 7} = \left(\frac{1}{\ln 7}\right)(\cot\theta - \tan\theta - 1 - \ln 2)$

44. $y = \log_2\left(\frac{x^2e^2}{2\sqrt{x+1}}\right) = \frac{\ln x^2 + \ln e^2 - \ln 2 - \ln\sqrt{x+1}}{\ln 2} = \frac{2\ln x + 2 - \ln 2 - \frac{1}{2}\ln(x+1)}{\ln 2}$

$\Rightarrow y' = \frac{2}{x\ln 2} - \frac{1}{2(\ln 2)(x+1)} = \frac{4(x+1) - x}{2x(x+1)(\ln 2)} = \frac{3x+4}{2x(x+1)\ln 2}$

45. $y = (x+1)^x \Rightarrow \ln y = \ln(x+1)^x = x\ln(x+1) \Rightarrow \frac{y'}{y} = \ln(x+1) + x \cdot \frac{1}{(x+1)} \Rightarrow y' = (x+1)^x\left[\frac{x}{x+1} + \ln(x+1)\right]$

46. $y = t^{\sqrt{t}} = t^{\left(t^{1/2}\right)} \Rightarrow \ln y = \ln t^{\left(t^{1/2}\right)} = \left(t^{1/2}\right)(\ln t) \Rightarrow \frac{1}{y}\frac{dy}{dt} = \left(\frac{1}{2}t^{-1/2}\right)(\ln t) + t^{1/2}\left(\frac{1}{t}\right) = \frac{\ln t + 2}{2\sqrt{t}}$

$\Rightarrow \frac{dy}{dt} = \left(\frac{\ln t + 2}{2\sqrt{t}}\right)t^{\sqrt{t}}$

47. $y = (\sin x)^x \Rightarrow \ln y = \ln(\sin x)^x = x\ln(\sin x) \Rightarrow \frac{y'}{y} = \ln(\sin x) + x\left(\frac{\cos x}{\sin x}\right) \Rightarrow y' = (\sin x)^x[\ln(\sin x) + x\cot x]$

48. $y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = (\sin x)(\ln x) \Rightarrow \frac{y'}{y} = (\cos x)(\ln x) + (\sin x)\left(\frac{1}{x}\right) = \frac{\sin x + x(\ln x)(\cos x)}{x}$

$\Rightarrow y' = x^{\sin x}\left[\frac{\sin x + x(\ln x)(\cos x)}{x}\right]$

49. $y = x^{\ln x},\ x > 0 \Rightarrow \ln y = (\ln x)^2 \Rightarrow \frac{y'}{y} = 2(\ln x)\left(\frac{1}{x}\right) \Rightarrow y' = \left(x^{\ln x}\right)\left(\frac{\ln x^2}{x}\right)$

50. $y = (\ln x)^{\ln x} \Rightarrow \ln y = (\ln x)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{x}\right)\ln(\ln x) + (\ln x)\left(\frac{1}{\ln x}\right)\frac{d}{dx}(\ln x) = \frac{\ln(\ln x)}{x} + \frac{1}{x}$

$\Rightarrow y' = \left(\frac{\ln(\ln x) + 1}{x}\right)(\ln x)^{\ln x}$

51. Let $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx;\ x = 1 \Rightarrow u = 1,\ x = \sqrt{2} \Rightarrow u = 2;$

$$\int_1^{\sqrt{2}} x2^{\left(x^2\right)}\,dx = \int_1^2 \left(\frac{1}{2}\right)2^u\,du = \frac{1}{2}\left[\frac{2^u}{\ln 2}\right]_1^2 = \left(\frac{1}{2\ln 2}\right)\left(2^2 - 2^1\right) = \frac{1}{\ln 2}$$

52. Let $u = \cos t \Rightarrow du = -\sin t\,dt \Rightarrow -du = \sin t\,dt;\ t = 0 \Rightarrow u = 1,\ t = \frac{\pi}{2} \Rightarrow u = 0;$

$$\int_0^{\pi/2} 7^{\cos t} \sin t\,dt = -\int_1^0 7^u\,du = \left[-\frac{7^u}{\ln 7}\right]_1^0 = \left(\frac{-1}{\ln 7}\right)\left(7^0 - 7\right) = \frac{6}{\ln 7}$$

53. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 1 \Rightarrow u = 0$, $x = 2 \Rightarrow u = \ln 2$;

$$\int_1^2 \frac{2^{\ln x}}{x}\,dx = \int_0^{\ln 2} 2^u\,du = \left[\frac{2^u}{\ln 2}\right]_0^{\ln 2} = \left(\frac{1}{\ln 2}\right)\left(2^{\ln 2} - 2^0\right) = \frac{2^{\ln 2} - 1}{\ln 2}$$

54. $\int 3x^{\sqrt{3}}\,dx = \frac{3x^{\left(\sqrt{3}+1\right)}}{\sqrt{3}+1} + C$

55. $\int x^{\left(\sqrt{2}-1\right)}\,dx = \frac{x^{\sqrt{2}}}{\sqrt{2}} + C$

56. $\int_0^3 \left(\sqrt{2}+1\right)x^{\sqrt{2}}\,dx = \left[x^{\left(\sqrt{2}+1\right)}\right]_0^3 = 3^{\left(\sqrt{2}+1\right)}$

57. $\int_1^e x^{(\ln 2)-1}\,dx = \left[\frac{x^{\ln 2}}{\ln 2}\right]_1^e = \frac{e^{\ln 2} - 1^{\ln 2}}{\ln 2} = \frac{2-1}{\ln 2} = \frac{1}{\ln 2}$

58. Let $u = \ln t \Rightarrow du = \frac{dt}{t}$; $u = 0$ when $t = 1$, and $u = x$ when $t = e^x$.

$$\int_1^{e^x} \frac{3^{\ln t}}{t}\,dt = \int_0^x 3^u\,du = \frac{3^u}{\ln 3}\Big|_0^x = \frac{1}{\ln 3}\left(3^x - 1\right)$$

59. $\frac{dy}{dt} = e^t \sin\left(e^t - 2\right) \Rightarrow y = \int e^t \sin\left(e^t - 2\right)dt$;

let $u = e^t - 2 \Rightarrow du = e^t\,dt \Rightarrow y = \int \sin u\,du = -\cos u + C = -\cos\left(e^t - 2\right) + C$; $y(\ln 2) = 0$

$\Rightarrow -\cos\left(e^{\ln 2} - 2\right) + C = 0 \Rightarrow -\cos(2-2) + C = 0 \Rightarrow C = \cos 0 = 1$; thus, $y = 1 - \cos\left(e^t - 2\right)$

60. $\frac{dy}{dt} = e^{-t}\sec^2\left(\pi e^{-t}\right) \Rightarrow y = \int e^{-t}\sec^2\left(\pi e^{-t}\right)dt$;

let $u = \pi e^{-t} \Rightarrow du = -\pi e^{-t}\,dt \Rightarrow -\frac{1}{\pi}\,du = e^{-t}\,dt \Rightarrow y = -\frac{1}{\pi}\int \sec^2 u\,du = -\frac{1}{\pi}\tan u + C$

$= -\frac{1}{\pi}\tan\left(\pi e^{-t}\right) + C$; $y(\ln 4) = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan\left(\pi e^{-\ln 4}\right) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan\left(\pi \cdot \frac{1}{4}\right) + C = \frac{2}{\pi}$

$\Rightarrow -\frac{1}{\pi}(1) + C = \frac{2}{\pi} \Rightarrow C = \frac{3}{\pi}$; thus, $y = \frac{3}{\pi} - \frac{1}{\pi}\tan\left(\pi e^{-t}\right)$

61. $\frac{d^2y}{dx^2} = 2e^{-x} \Rightarrow \frac{dy}{dx} = -2e^{-x} + C$; $x = 0$ and $\frac{dy}{dx} = 0 \Rightarrow 0 = -2e^0 + C \Rightarrow C = 2$; thus $\frac{dy}{dx} = -2e^{-x} + 2$

$\Rightarrow y = 2e^{-x} + 2x + C_1$; $x = 0$ and $y = 1 \Rightarrow 1 = 2e^0 + C_1 \Rightarrow C_1 = -1 \Rightarrow y = 2e^{-x} + 2x - 1 = 2\left(e^{-x} + x\right) - 1$

62. $\frac{d^2y}{dt^2} = 1 - e^{2t} \Rightarrow \frac{dy}{dt} = t - \frac{1}{2}e^{2t} + C$; $t = 1$ and $\frac{dy}{dt} = 0 \Rightarrow 0 = 1 - \frac{1}{2}e^2 + C \Rightarrow C = \frac{1}{2}e^2 - 1$; thus

$\frac{dy}{dt} = t - \frac{1}{2}e^{2t} + \frac{1}{2}e^2 - 1 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t + C_1$; $t = 1$ and $y = -1 \Rightarrow -1 = \frac{1}{2} - \frac{1}{4}e^2 + \frac{1}{2}e^2 - 1 + C_1$

$\Rightarrow C_1 = -\frac{1}{2} - \frac{1}{4}e^2 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t - \left(\frac{1}{2} + \frac{1}{4}e^2\right)$

63. $f(x) = e^x - 2x \Rightarrow f'(x) = e^x - 2$; $f'(x) = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2$; $f(0) = 1$, the absolute maximum; $f(\ln 2) = 2 - 2\ln 2 \approx 0.613706$, the absolute minimum; $f(1) = e - 2 \approx 0.71828$, a relative or local maximum since $f''(x) = e^x$ is always positive

64. The function $f(x) = 2e^{\sin(x/2)}$ has a maximum whenever $\sin\frac{x}{2} = 1$ and a minimum whenever $\sin\frac{x}{2} = -1$. Therefore the maximums occur at $x = \pi + 2k(2\pi)$ and the minimums occur at $x = 3\pi + 2k(2\pi)$, where k is any integer. The maximum is $2e \approx 5.43656$ and the minimum is $\frac{2}{e} \approx 0.73576$

65. $f(x) = x^2 \ln\frac{1}{x} \Rightarrow f'(x) = 2x\ln\frac{1}{x} + x^2\left(\frac{1}{\frac{1}{x}}\right)(-x^{-2}) = 2x\ln\frac{1}{x} - x = -x(2\ln x + 1)$; $f'(x) = 0 \Rightarrow x = 0$ or $\ln x = -\frac{1}{2}$. Since $x = 0$ is not in the domain of f, $x = e^{-1/2} = \frac{1}{\sqrt{e}}$. Also, $f'(x) > 0$ for $0 < x < \frac{1}{\sqrt{e}}$ and $f'(x) < 0$ for $x > \frac{1}{\sqrt{e}}$. Therefore, $f\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e}\ln\sqrt{e} = \frac{1}{e}\ln e^{1/2} = \frac{1}{2e}\ln e = \frac{1}{2e}$ is the absolute maximum value of f assumed at $x = \frac{1}{\sqrt{e}}$.

66. $\int_0^{\ln 3} (e^{2x} - e^x)\,dx = \left[\frac{e^{2x}}{2} - e^x\right]_0^{\ln 3} = \left(\frac{e^{2\ln 3}}{2} - e^{\ln 3}\right) - \left(\frac{e^0}{2} - e^0\right) = \left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right) = \frac{8}{2} - 2 = 2$

67. Let $x = \frac{r}{k} \Rightarrow k = \frac{r}{x}$ and as $k \to \infty$, $x \to 0 \Rightarrow \lim\limits_{k\to\infty}\left(1 + \frac{r}{k}\right)^k = \lim\limits_{x\to 0}(1+x)^{r/x} = \lim\limits_{x\to 0}\left((1+x)^{1/x}\right)^r$ $= \left(\lim\limits_{x\to 0}(1+x)^{1/x}\right)^r$, since u^r is continuous. However, $\lim\limits_{x\to 0}(1+x)^{1/x} = e$ (by Theorem 2), therefore, $\lim\limits_{k\to\infty}\left(1 + \frac{r}{k}\right)^k = e^r$.

68. $L = \int_0^1 \sqrt{1 + \frac{e^x}{4}}\,dx \Rightarrow \frac{dy}{dx} = \pm\frac{e^{x/2}}{2} \Rightarrow y = \pm e^{x/2} + C$; $y(0) = 0 \Rightarrow 0 = \pm e^0 + C \Rightarrow C = \mp 1 \Rightarrow y = e^{x/2} - 1$ $y = -e^{x/2} + 1$

69. $y = mx + b \Rightarrow x = \frac{y}{m} - \frac{b}{m} \Rightarrow f^{-1}(x) = \frac{1}{m}x - \frac{b}{m}$; the graph of $f^{-1}(x)$ is a line with slope $\frac{1}{m}$ and y-intercept $-\frac{b}{m}$.

70. $A = \int_{-1}^{1} 2^{(1-x)}\,dx = 2\int_{-1}^{1}\left(\frac{1}{2}\right)^x dx = 2\left[\frac{\left(\frac{1}{2}\right)^x}{\ln\left(\frac{1}{2}\right)}\right]_{-1}^{1} = -\frac{2}{\ln 2}\left(\frac{1}{2} - 2\right) = \left(-\frac{2}{\ln 2}\right)\left(-\frac{3}{2}\right) = \frac{3}{\ln 2}$

71. Note that $y = \ln x$ and $e^y = x$ are the same curve; $\int_1^a \ln x\,dx$ = area under the curve between 1 and a; $\int_0^{\ln a} e^y\,dy$ = area to the left of the curve. The sum of these areas is equal to the area of the rectangle

$$\Rightarrow \int_1^a \ln x\,dx + \int_0^{\ln a} e^y\,dy = a \ln a.$$

72. (a) $y = e^x \Rightarrow y'' = e^x > 0$ for all $x \Rightarrow$ the graph of $y = e^x$ is always concave upward

(b) area of the trapezoid ABCD $< \int_{\ln a}^{\ln b} e^x\,dx <$ area of the trapezoid AEFD $\Rightarrow \frac{1}{2}(AB + CD)(\ln b - \ln a)$

$< \int_{\ln a}^{\ln b} e^x\,dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$. Now $\frac{1}{2}(AB + CD)$ is the height of the midpoint

$M = e^{(\ln a + \ln b)/2}$ since the curve containing the points B and C is linear $\Rightarrow e^{(\ln a + \ln b)/2}(\ln b - \ln a)$

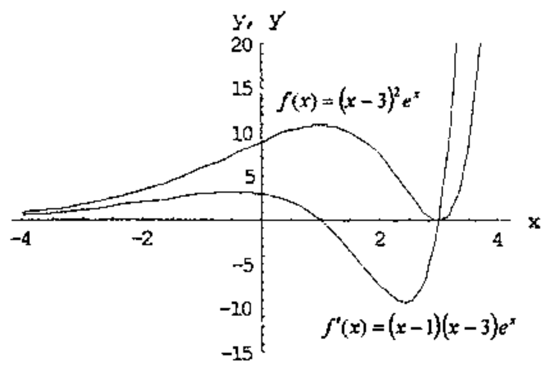
$< \int_{\ln a}^{\ln b} e^x\,dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$

(c) $\int_{\ln a}^{\ln b} e^x\,dx = \left[e^x\right]_{\ln a}^{\ln b} = e^{\ln b} - e^{\ln a} = b - a$, so part (b) implies that

$e^{(\ln a + \ln b)/2}(\ln b - \ln a) < b - a < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a) \Rightarrow e^{(\ln a + \ln b)/2} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$

$\Rightarrow e^{\ln(ab)/2} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2} \Rightarrow e^{\ln (ab)^{1/2}} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2} \Rightarrow \sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$

73. $f(x) = (x-3)^2 e^x \Rightarrow f'(x) = 2(x-3)e^x + (x-3)^2 e^x$
$= (x-3)e^x(2 + x - 3) = (x-1)(x-3)e^x$; thus
$f'(x) > 0$ for $x < 1$ or $x > 3$, and $f'(x) < 0$ for
$1 < x < 3 \Rightarrow f(1) = 4e \approx 10.87$ is a local maximum and
$f(3) = 0$ is a local minimum. Since $f(x) \ge 0$ for all x,
$f(3) = 0$ is also an absolute minimum.



74. $\frac{d}{dx}\left(-\frac{1}{2}x^2 + k\right) = -x$ and $\frac{d}{dx}(\ln x + c) = \frac{1}{x}$.
Since $-x \cdot \frac{1}{x} = -1$ for any $x \neq 0$, these two curves will have perpendicular tangent lines.

75. $e^{\ln x} = x$ for $x > 0$ and $\ln\left(e^x\right) = x$ for all x

76. Using Newton's Method: $f(x) = \ln(x) - 1 \Rightarrow f'(x) = \frac{1}{x} \Rightarrow x_{n+1} = x_n - \frac{\ln(x_n) - 1}{\left(\frac{1}{x_n}\right)} \Rightarrow x_{n+1} = x_n\left[2 - \ln(x_n)\right]$.

Then $x_1 = 2 \Rightarrow x_2 = 2.61370564$, $x_3 = 2.71624393$ and $x_5 = 2.71828183$. Many other methods may be used. For example, graph $y = \ln x - 1$ and determine the zero of y.

77. (a) The point of tangency is $(p, \ln p)$ and $m_{\text{tangent}} = \frac{1}{p}$ since $\frac{dy}{dx} = \frac{1}{x}$. The tangent line passes through $(0,0)$

$\Rightarrow$ the equation of the tangent line is $y = \frac{1}{p}x$. The tangent line also passes through $(p, \ln p) \Rightarrow \ln p = \frac{1}{p}p$

$= 1 \Rightarrow p = e$, and the tangent line equation is $y = \frac{1}{e}x$.

(b) $\frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$ for $x \neq 0 \Rightarrow y = \ln x$ is concave downward over its domain. Therefore, $y = \ln x$ lies below the graph of $y = \frac{1}{e}x$ for all $x > 0$, $x \neq e$ and $\ln x < \frac{x}{e}$ for $x > 0$, $x \neq e$.

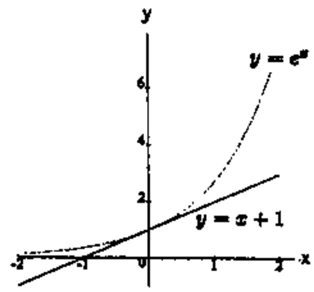
(c) Multiplying by e, $e \ln x < x$ or $\ln x^e < x$.

(d) Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

(e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

78. (a) $f(x) = e^x \Rightarrow f'(x) = e^x$; $L(x) = f(0) + f'(0)(x - 0) \Rightarrow L(x) = 1 + x$

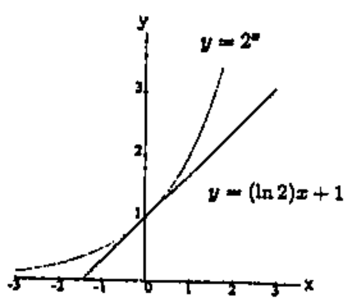
(b) On $[0, 0.2]$, f(x) and L(x) are both increasing because $f'(x) = e^x > 0$ and $L'(x) = 1 > 0$ for $0 \le x \le 0.2$. Also, $f'(x) \ge L'(x)$ on the interval $[0, 0.2]$ since $e^x \ge 1$ on the interval $E(x) = f(x) - L(x)$ is non-decreasing on $[0, 0.2]$ since $E'(x) = f'(x) - L'(x) \ge 0$ on $[0, 0.2]$. Therefore, the largest error is $|E(0.2)| = |f(0.2) - L(0.2)| = |e^{0.2} - 1.2| \approx 0.02141$ on $[0, 0.2]$.

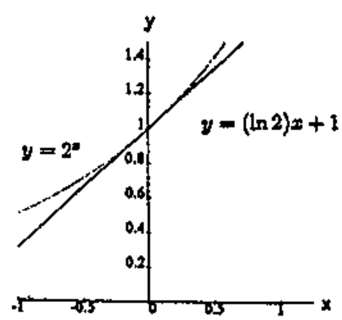


(c) Since f(x) is concave upward for all x, the tangent line lies below the curve $y = e^x$ for all x except at $x = 0$. Consequently, the linear approximation is never an overestimate.

79. (a) $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2$; $L(x) = (2^0 \ln 2)x + 2^0 = x \ln 2 + 1 \approx 0.69x + 1$

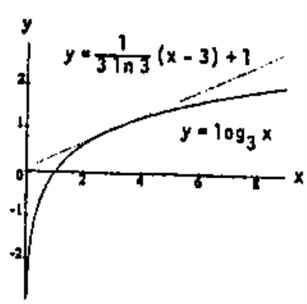
(b)

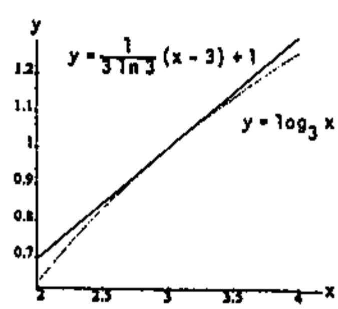




80. (a) $f(x) = \log_3 x \Rightarrow f'(x) = \frac{1}{x \ln 3}$, and $f(3) = \frac{\ln 3}{\ln 3} \Rightarrow L(x) = \frac{1}{3 \ln 3}(x - 3) + \frac{\ln 3}{\ln 3} = \frac{x}{3 \ln 3} - \frac{1}{\ln 3} + 1$ $\approx 0.30x + 0.09$

(b)





81-88. Example CAS commands:

Maple:

```
identity:= z -> z;
eq:= y=(3*x + 2)/(2*x - 11);
solve(eq,y);
simplify(%): f:= unapply(%,x);
diff(f(x),x);
simplify(%): df:= unapply(%,x);
plot({f,df}, -5..5, -5..5);
solve(eq,x);
g:= unapply(%,y);
finv:= y -> g(y);
plot({f,finv,identity}, -1..1, -2..1);
x0:= 1/2; y0:= f(x0):
ftan:= x -> f(x0) + df(x0)*(x - x0);
finvtan:= y -> x0 + (1/df)(x0)*(y - y0);
plot({f,finv,identity,ftan,finvtan,[x0,y0,y0,x0]}, -1..1, -1.5..1, scaling=constrained);
```

Mathematica

```
Clear[x,y]
{a,b} = {-2,2}; x0 = 1/2 ;
f[x_] = (3x + 2)/(2x - 11)
Plot[ {f[x],f'[x]}, {x,a,b} ]
Solve[ y == f[x], x ]
g[y_] = x / . First [%]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x - x0)
gtan[y_] = x0 + (1/f'[x0]) (y - y0)
Plot[{f[x],ftan[x],g[x],gtan[x],Identity[x]}, {x,a,b},
Epilog -> {Line[{{x0,y0},{y0,x0}}]},
PlotRange -> {{a,b}, {a,b}},
AspectRatio -> Automatic]
```

Remark:

Other problems are similar to the example, except for adjusting plot ranges to see both source and inverse points. (Note: functions involving cube roots only show the positive branch.)

89-90. Example CAS commands:

Maple:

```
identity:= z -> z;
eq:= y^(1/3) - 1 = (x + 2)^3;
solve(eq,y);
f:= unapply(%,x);
diff(f(x),x);
df:= unapply(%,x);
plot({f,df}, -2..0,-5..5);
solve(eq,x);
g:= unapply(%[1],y);
finv:= y -> if (1<=y) then g(y) elif (0<=y) then -(1 - y^(1/3))^(1/3) - 2 elif (-1<y) then -(
1+(-y)^(1/3))^(1/3) - 2 else -((-y^(1/3) + 1)^(1/3) - 2 fi;
plot({f,finv,identity}, -2..2, -5..5);
x0:= -3/2: y0:= f(x0):
ftan:= x -> f(x0) + df(x0)*(x - x0);
```

```
finvtan:= y -> x0 + (1/df)(x0)*(y - y0);
plot({f,finv,identity,ftan,finvtan,[x0,y0,y0,x0]}, -5..5,-5..5, scaling = constrained);
```

Mathematica:

```
Clear[x,y]
{a,b} = {-5,5}; x0 = -3/2;
eqn = y^(1/3) - 1 == (x + 2)^3
Solve[ eqn, y ]
f[x_] = y /. First[%]
Plot[ {f[x],f'[x]}, {x,a,b} ]
Solve[ eqn, x ]
g[y_] = x /. First[%]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x - x0)
gtan[y_] = x0 + (1/f'[x0]) (y - y0)
Plot[{f[x],ftan[x],g[x],gtan[x],Identity[x]},{x,a,b},
Epilog -> {Line[{{x0,y0},{y0,x0}}]},
PlotRange -> {{a,b}, {a,b}},
AspectRatio -> Automatic]
```

6.3 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS; INTEGRALS

1. $y = \cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

2. $y = \sin^{-1}(1-t) \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-(1-t)^2}} = \frac{-1}{\sqrt{2t-t^2}}$

3. $y = \sec^{-1}(2s+1) \Rightarrow \frac{dy}{ds} = \frac{2}{|2s+1|\sqrt{(2s+1)^2-1}} = \frac{2}{|2s+1|\sqrt{4s^2+4s}} = \frac{1}{|2s+1|\sqrt{s^2+s}}$

4. $y = \csc^{-1}\left(x^2+1\right) \Rightarrow \frac{dy}{dx} = -\frac{2x}{|x^2+1|\sqrt{\left(x^2+1\right)^2-1}} = \frac{-2x}{\left(x^2+1\right)\sqrt{x^4+2x^2}}$

5. $y = \sec^{-1}\left(\frac{1}{t}\right) = \cos^{-1} t \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$

6. $y = \cot^{-1}\sqrt{t} = \cot^{-1} t^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1+\left(t^{1/2}\right)^2} = \frac{-1}{2\sqrt{t}(1+t)}$

7. $y = \ln\left(\tan^{-1} x\right) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{1+x^2}\right)}{\tan^{-1} x} = \frac{1}{\left(\tan^{-1} x\right)\left(1+x^2\right)}$

8. $y = \tan^{-1}(\ln x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)}{1+(\ln x)^2} = \frac{1}{x\left[1+(\ln x)^2\right]}$

9. $y = \cos^{-1}\left(e^{-t}\right) \Rightarrow \frac{dy}{dt} = -\frac{-e^{-t}}{\sqrt{1-\left(e^{-t}\right)^2}} = \frac{e^{-t}}{\sqrt{1-e^{-2t}}}$

10. $y = s\sqrt{1-s^2} + \cos^{-1} s = s\left(1-s^2\right)^{1/2} + \cos^{-1} s \Rightarrow \frac{dy}{ds} = \left(1-s^2\right)^{1/2} + s\left(\frac{1}{2}\right)\left(1-s^2\right)^{-1/2}(-2s) - \frac{1}{\sqrt{1-s^2}}$

$= \sqrt{1-s^2} - \frac{s^2}{\sqrt{1-s^2}} - \frac{1}{\sqrt{1-s^2}} = \sqrt{1-s^2} - \frac{s^2+1}{\sqrt{1-s^2}} = \frac{1-s^2-s^2-1}{\sqrt{1-s^2}} = \frac{-2s^2}{\sqrt{1-s^2}}$

11. $y = \tan^{-1}\sqrt{x^2-1} + \csc^{-1} x = \tan^{-1}\left(x^2-1\right)^{1/2} + \csc^{-1} x \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)\left(x^2-1\right)^{-1/2}(2x)}{1+\left[\left(x^2-1\right)^{1/2}\right]^2} - \frac{1}{|x|\sqrt{x^2-1}}$

$= \frac{1}{x\sqrt{x^2-1}} - \frac{1}{|x|\sqrt{x^2-1}} = 0$, for $x > 1$

12. $y = \cos^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x = \frac{\pi}{2} - \tan^{-1}\left(x^{-1}\right) - \tan^{-1} x \Rightarrow \frac{dy}{dx} = 0 - \frac{-x^{-2}}{1+\left(x^{-1}\right)^2} - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0$

13. $y = x\sin^{-1} x + \sqrt{1-x^2} = x\sin^{-1} x + \left(1-x^2\right)^{1/2} \Rightarrow \frac{dy}{dx} = \sin^{-1} x + x\left(\frac{1}{\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)$

$= \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x$

14. $y = \ln\left(x^2+4\right) - x\tan^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - x\left[\frac{\left(\frac{1}{2}\right)}{1+\left(\frac{x}{2}\right)^2}\right] = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{4+x^2}$

$= -\tan^{-1}\left(\frac{x}{2}\right)$

15. $\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int \frac{2}{\sqrt{1-(2x)^2}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = 2x$ and $du = 2\,dx$

$= \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1}(2x) + C$

16. $\int_0^{3\sqrt{2}/4} \frac{dx}{9+3x^2} = \frac{1}{3}\int_0^{3\sqrt{2}/4} \frac{1}{\left(\sqrt{3}\right)^2+x^2}\,dx = \frac{\sqrt{3}}{9}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\Bigg|_0^{3\sqrt{2}/4} = \frac{\sqrt{3}}{9}\left[\tan^{-1}\left(\frac{\sqrt{6}}{4}\right) - \tan^{-1} 0\right]$

$= \frac{\sqrt{3}}{9}\tan^{-1}\left(\frac{\sqrt{6}}{4}\right) \approx 0.1057$

17. $\int \frac{dx}{x\sqrt{25x^2-2}} = \int \frac{du}{u\sqrt{u^2-2}}$, where $u = 5x$ and $du = 5\,dx$

$= \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{u}{\sqrt{2}}\right| + C = \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{5x}{\sqrt{2}}\right| + C$

18. $\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int_0^{3\sqrt{2}/2} \frac{du}{\sqrt{9-u^2}}$, where $u = 2x$ and $du = 2\ dx$; $x = 0 \Rightarrow u = 0$, $x = \frac{3\sqrt{2}}{4} \Rightarrow u = \frac{3\sqrt{2}}{2}$

$= \left[\frac{1}{2}\sin^{-1}\frac{u}{3}\right]_0^{3\sqrt{2}/2} = \frac{1}{2}\left(\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1} 0\right) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$

19. $\int_0^2 \frac{dt}{8+2t^2} = \frac{1}{\sqrt{2}}\int_0^{2\sqrt{2}} \frac{du}{8+u^2}$, where $u = \sqrt{2}t$ and $du = \sqrt{2}\ dt$; $t = 0 \Rightarrow u = 0$, $t = 2 \Rightarrow u = 2\sqrt{2}$

$= \left[\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{8}}\tan^{-1}\frac{u}{\sqrt{8}}\right]_0^{2\sqrt{2}} = \frac{1}{4}\left(\tan^{-1}\frac{2\sqrt{2}}{\sqrt{8}} - \tan^{-1} 0\right) = \frac{1}{4}\left(\tan^{-1} 1 - \tan^{-1} 0\right) = \frac{1}{4}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{16}$

20. $\int_{-1}^{-\sqrt{2}} \frac{dy}{y\sqrt{4y^2-1}} = \int_{-2}^{-2\sqrt{2}} \frac{du}{u\sqrt{u^2-1}}$, where $u = 2y$ and $du = 2\ dy$; $y = -1 \Rightarrow u = -2$, $y = -\sqrt{2} \Rightarrow u = -2\sqrt{2}$

$= \left[\sec^{-1}|u|\right]_{-2}^{-2\sqrt{2}} = \sec^{-1}\left|-2\sqrt{2}\right| - \sec^{-1}|-2| = \cos^{-1}\left(\frac{\sqrt{2}}{4}\right) - \frac{\pi}{3}$

21. $\int \frac{3\ dr}{\sqrt{1-4(r-1)^2}} = \frac{3}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = 2(r-1)$ and $du = 2\ dr$

$= \frac{3}{2}\sin^{-1} u + C = \frac{3}{2}\sin^{-1}[2(r-1)] + C$

22. $\int \frac{dx}{1+(3x+1)^2} = \frac{1}{3}\int \frac{du}{1+u^2}$, where $u = 3x+1$ and $du = 3\ dx$

$= \frac{1}{3}\tan^{-1} u + C = \frac{1}{3}\tan^{-1}(3x+1) + C$

23. $\int \frac{y\ dy}{\sqrt{1-y^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = y^2$ and $du = 2y\ dy$

$= \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} y^2 + C$

24. $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\ d\theta}{1+(\sin\theta)^2} = 2\int_{-1}^{1} \frac{du}{1+u^2}$, where $u = \sin\theta$ and $du = \cos\theta\ d\theta$

$= \left[2\tan^{-1} u\right]_{-1}^{1} = 2\left(\tan^{-1} 1 - \tan^{-1}(-1)\right) = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

25. $\int_0^{\ln\sqrt{3}} \frac{e^x\ dx}{1+e^{2x}} = \int_1^{\sqrt{3}} \frac{du}{1+u^2}$, where $u = e^x$ and $du = e^x\ dx$; $x = 0 \Rightarrow u = 1$, $x = \ln\sqrt{3} \Rightarrow u = \sqrt{3}$

$= \left[\tan^{-1} u\right]_1^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

26. $\int\limits_1^{e^{\pi/4}} \frac{4\,dt}{t\left(1+\ln^2 t\right)} = 4 \int\limits_0^{\pi/4} \frac{du}{1+u^2}$, where $u = \ln t$ and $du = \frac{1}{t}\,dt$; $t = 1 \Rightarrow u = 0$, $t = e^{\pi/4} \Rightarrow u = \frac{\pi}{4}$

$= \left[4\tan^{-1} u\right]_0^{\pi/4} = 4\left(\tan^{-1}\frac{\pi}{4} - \tan^{-1} 0\right) = 4\tan^{-1}\frac{\pi}{4}$

27. $\int \frac{dx}{\sqrt{-x^2+4x-3}} = \int \frac{dx}{\sqrt{1-\left(x^2-4x+4\right)}} = \int \frac{dx}{\sqrt{1-(x-2)^2}} = \sin^{-1}(x-2) + C$

28. $\int\limits_{-1}^{0} \frac{6\,dt}{\sqrt{3-2t-t^2}} = 6\int\limits_{-1}^{0} \frac{dt}{\sqrt{4-\left(t^2+2t+1\right)}} = 6\int\limits_{-1}^{0} \frac{dt}{\sqrt{2^2-(t+1)^2}} = 6\left[\sin^{-1}\left(\frac{t+1}{2}\right)\right]_{-1}^{0}$

$= 6\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0\right] = 6\left(\frac{\pi}{6} - 0\right) = \pi$

29. $\int \frac{dy}{y^2-2y+5} = \int \frac{dy}{4+y^2-2y+1} = \int \frac{dy}{2^2+(y-1)^2} = \frac{1}{2}\tan^{-1}\left(\frac{y-1}{2}\right) + C$

30. $\int\limits_1^2 \frac{8\,dx}{x^2-2x+2} = 8\int\limits_1^2 \frac{dx}{1+\left(x^2-2x+1\right)} = 8\int\limits_1^2 \frac{dx}{1+(x-1)^2} = 8\left[\tan^{-1}(x-1)\right]_1^2$

$= 8\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 8\left(\frac{\pi}{4} - 0\right) = 2\pi$

31. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{x^2+2x+1-1}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}}$

$= \int \frac{du}{u\sqrt{u^2-1}}$, where $u = x+1$ and $du = dx$

$= \sec^{-1}|u| + C = \sec^{-1}|x+1| + C$

32. $\int \frac{dx}{(x-2)\sqrt{x^2-4x-3}} = \int \frac{dx}{(x-2)\sqrt{x^2-4x+4-7}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-\sqrt{7}}}$

$= \int \frac{1}{u\sqrt{u^2-\left(\sqrt{7}\right)^2}}\,du$, where $u = x-2$ and $du = dx$ and $a = \sqrt{7}$

$= \frac{1}{\sqrt{7}}\sec^{-1}\left|\frac{u}{a}\right| + C = \frac{1}{\sqrt{7}}\sec^{-1}\left|\frac{x-2}{\sqrt{7}}\right| + C$

33. $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}\,dx = \int e^u\,du$, where $u = \sin^{-1} x$ and $du = \frac{dx}{\sqrt{1-x^2}}$

$= e^u + C = e^{\sin^{-1} x} + C$

34. $\int \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}\,dx = \int u^2\,du$, where $u = \sin^{-1}x$ and $du = \frac{dx}{\sqrt{1-x^2}}$

$= \frac{u^3}{3} + C = \frac{(\sin^{-1}x)^3}{3}$

35. $\int \frac{1}{(\tan^{-1}y)(1+y^2)}\,dy = \int \frac{\left(\frac{1}{1+y^2}\right)}{\tan^{-1}y}\,dy = \int \frac{1}{u}\,du$, where $u = \tan^{-1}y$ and $du = \frac{dy}{1+y^2}$

$= \ln|u| + C = \ln\left|\tan^{-1}y\right| + C$

36. $\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1}x)}{x\sqrt{x^2-1}}\,dx = \int_{\pi/4}^{\pi/3} \sec^2 u\,du$, where $u = \sec^{-1}x$ and $du = \frac{dx}{x\sqrt{x^2-1}}$; $x = \sqrt{2} \Rightarrow u = \frac{\pi}{4}$, $x = 2 \Rightarrow u = \frac{\pi}{3}$

$= [\tan u]_{\pi/4}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{4} = \sqrt{3} - 1$

37. If $y = \ln x - \frac{1}{2}\ln(1+x^2) - \frac{\tan^{-1}x}{x} + C$, then $dy = \left[\frac{1}{x} - \frac{x}{1+x^2} - \frac{\left(\frac{x}{1+x^2}\right) - \tan^{-1}x}{x^2}\right]dx$

$= \left(\frac{1}{x} - \frac{x}{1+x^2} - \frac{1}{x(1+x^2)} + \frac{\tan^{-1}x}{x^2}\right)dx = \frac{x(1+x^2) - x^3 - x + (\tan^{-1}x)(1+x^2)}{x^2(1+x^2)}\,dx = \frac{\tan^{-1}x}{x^2}\,dx,$

which verifies the formula

38. If $y = \frac{x^4}{4}\cos^{-1}5x + \frac{5}{4}\int \frac{x^4}{\sqrt{1-25x^2}}\,dx$, then $dy = \left[x^3\cos^{-1}5x + \left(\frac{x^4}{4}\right)\left(\frac{-5}{\sqrt{1-25x^2}}\right) + \frac{5}{4}\left(\frac{x^4}{\sqrt{1-25x^2}}\right)\right]dx$

$= (x^3\cos^{-1}5x)\,dx$, which verifies the formula

39. If $y = x(\sin^{-1}x)^2 - 2x + 2\sqrt{1-x^2}\,\sin^{-1}x + C$, then

$dy = \left[(\sin^{-1}x)^2 + \frac{2x(\sin^{-1}x)}{\sqrt{1-x^2}} - 2 + \frac{-2x}{\sqrt{1-x^2}}\sin^{-1}x + 2\sqrt{1-x^2}\left(\frac{1}{\sqrt{1-x^2}}\right)\right]dx = (\sin^{-1}x)^2\,dx$, which verifies the formula

40. If $y = x\ln(a^2+x^2) - 2x + 2a\tan^{-1}\left(\frac{x}{a}\right) + C$, then $dy = \left[\ln(a^2+x^2) + \frac{2x^2}{a^2+x^2} - 2 + \frac{2}{1+\left(\frac{x^2}{a^2}\right)}\right]dx$

$= \left[\ln(a^2+x^2) + 2\left(\frac{a^2+x^2}{a^2+x^2}\right) - 2\right]dx = \ln(a^2+x^2)\,dx$, which verifies the formula

41. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow dy = \frac{dx}{\sqrt{1-x^2}} \Rightarrow y = \sin^{-1}x + C$; $x = 0$ and $y = 0 \Rightarrow 0 = \sin^{-1}0 + C \Rightarrow C = 0 \Rightarrow y = \sin^{-1}x$

42. $\frac{dy}{dx} = \frac{1}{x^2+1} - 1 \Rightarrow dy = \left(\frac{1}{1+x^2} - 1\right) dx \Rightarrow y = \tan^{-1}(x) - x + C$; $x = 0$ and $y = 1 \Rightarrow 1 = \tan^{-1} 0 - 0 + C$
$\Rightarrow C = 1 \Rightarrow y = \tan^{-1}(x) - x + 1$

43. $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} \Rightarrow dy = \frac{dx}{x\sqrt{x^2-1}} \Rightarrow y = \sec^{-1}|x| + C$; $x = 2$ and $y = \pi \Rightarrow \pi = \sec^{-1} 2 + C \Rightarrow C = \pi - \sec^{-1} 2$
$= \pi - \frac{\pi}{3} = \frac{2\pi}{3} \Rightarrow y = \sec^{-1}(x) + \frac{2\pi}{3}$, $x > 1$

44. $\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} \Rightarrow dy = \left(\frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}}\right) dx \Rightarrow y = \tan^{-1} x - 2\sin^{-1} x + C$; $x = 0$ and $y = 2$
$\Rightarrow 2 = \tan^{-1} 0 - 2\sin^{-1} 0 + C \Rightarrow C = 2 \Rightarrow y = \tan^{-1} x - 2\sin^{-1} x + 2$

45. $\cos^{-1} u = \frac{\pi}{2} - \sin^{-1} u \Rightarrow \frac{d}{dx}\left(\cos^{-1} u\right) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{\sqrt{1-u^2}}$, $|u| < 1$

46. $\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u \Rightarrow \frac{d}{dx}\left(\cot^{-1} u\right) = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{1+u^2} = -\frac{\frac{du}{dx}}{1+u^2}$

47. $\csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u \Rightarrow \frac{d}{dx}\left(\csc^{-1} u\right) = \frac{d}{dx}\left(\frac{\pi}{2} - \sec^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}} = -\frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}}$, $|u| > 1$

48. From the accompanying figure, $\alpha + \beta + \theta = \pi$, $\cot \alpha = \frac{x}{1}$
and $\cot \beta = \frac{2-x}{1} \Rightarrow \theta = \pi - \cot^{-1} x - \cot^{-1}(2-x)$
$\Rightarrow \frac{d\theta}{dx} = \frac{1}{1+x^2} - \frac{1}{1+(2-x)^2} = \frac{1+(2-x)^2-\left(1+x^2\right)}{\left(1+x^2\right)\left[1+(2-x)^2\right]}$
$= \frac{4-4x}{\left(1+x^2\right)\left[1+(2-x)^2\right]}$; solving $\frac{d\theta}{dx} = 0 \Rightarrow x = 1$; $\frac{d\theta}{dx} > 0$ for $0 < x < 1$ and $\frac{d\theta}{dx} < 0$ for $x > 1$
$\Rightarrow$ at $x = 1$ there is a maximum $\theta = \pi - \cot^{-1} 1 - \cot^{-1}(2-1) = \pi - \frac{\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$

49. $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{(\cos x)_{\sin^{-1} x}} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\cos\left(\sin^{-1} x\right)} = \frac{1}{\sqrt{1-\sin^2\left(\sin^{-1} x\right)}} = \frac{1}{\sqrt{1-x^2}}$

50. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\left(\sec^2 x\right)_{\tan^{-1} x}} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\sec^2\left(\tan^{-1} x\right)} = \frac{1}{1+\tan^2\left(\tan^{-1} x\right)} = \frac{1}{1+x^2}$

51. $V = \pi \int_{-\sqrt{3}/3}^{\sqrt{3}} \left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx = \pi \int_{-\sqrt{3}/3}^{\sqrt{3}} \frac{1}{1+x^2}\, dx = \pi\left[\tan^{-1} x\right]_{-\sqrt{3}/3}^{\sqrt{3}} = \pi\left[\tan^{-1}\sqrt{3} - \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right]$
$= \pi\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi^2}{2}$

52. $y = \sqrt{1-x^2} = (1-x^2)^{1/2} \Rightarrow y' = \left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x) \Rightarrow 1 + (y')^2 = \frac{1}{1-x^2}$; $L = \int_{-1/2}^{1/2} \sqrt{1+(y')^2}\, dx$

$= 2\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx = 2\left[\sin^{-1} x\right]_0^{1/2} = 2\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{3}$

53. (a) $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}}\right)\right]^2 = \frac{\pi}{1+x^2} \Rightarrow V = \int_a^b A(x)\, dx = \int_{-1}^{1} \frac{\pi\, dx}{1+x^2}$

$= \pi\left[\tan^{-1} x\right]_{-1}^{1} = (\pi)(2)\left(\frac{\pi}{4}\right) = \frac{\pi^2}{2}$

(b) $A(x) = (\text{edge})^2 = \left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}}\right)\right]^2 = \frac{\pi}{1+x^2} \Rightarrow V = \int_a^b A(x)\, dx = \int_{-1}^{1} \frac{4\, dx}{1+x^2}$

$= 4\left[\tan^{-1} x\right]_{-1}^{1} = 4\left[\tan^{-1}(1) - \tan^{-1}(-1)\right] = 4\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 2\pi$

54. (a) $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\frac{2}{\sqrt[4]{1-x^2}} - 0\right)^2 = \frac{\pi}{4}\left(\frac{4}{\sqrt{1-x^2}}\right) = \frac{\pi}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x)\, dx$

$= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{\pi}{\sqrt{1-x^2}}\, dx = \pi\left[\sin^{-1} x\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = \pi\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right] = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}$

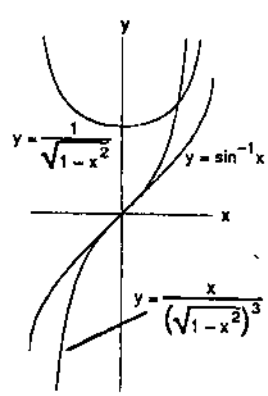
(b) $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{1}{2}\left(\frac{2}{\sqrt[4]{1-x^2}} - 0\right)^2 = \frac{2}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x)\, dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{2}{\sqrt{1-x^2}}\, dx$

$= 2\left[\sin^{-1} x\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = 2\left(\frac{\pi}{4} \cdot 2\right) = \pi$

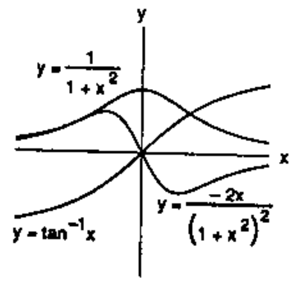
55. A calculator or computer numerical integrator yields $\sin^{-1} 0.6 \approx 0.643517104$.

56. A calculator or computer numerical integrator yields $\pi \approx 3.1415925$.

57. The values of f increase over the interval $[-1, 1]$ because $f' > 0$, and the graph of f steepens as the values of f' increase towards the ends of the interval. The graph of f is concave down to the left of the origin where $f'' < 0$, and concave up to the right of the origin where $f'' > 0$. There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local minimum value.



58. The values of f increase throughout the interval $(-\infty, \infty)$ because $f' > 0$, and they increase most rapidly near the origin where the values of f' are relatively large. The graph of f is concave up to the left of the origin where $f'' > 0$, and concave down to the right of the origin where $f'' < 0$. There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local maximum value.



6.4 FIRST ORDER SEPARABLE DIFFERENTIAL EQUATIONS

1. (a) $y = e^{-x} \Rightarrow y' = -e^{-x} \Rightarrow 2y' + 3y = 2(-e^{-x}) + 3e^{-x} = e^{-x}$

(b) $y = e^{-x} + e^{-3x/2} \Rightarrow y' = -e^{-x} - \frac{3}{2}e^{-3x/2} \Rightarrow 2y' + 3y = 2\left(-e^{-x} - \frac{3}{2}e^{-3x/2}\right) + 3\left(e^{-x} + e^{-3x/2}\right) = e^{-x}$

(c) $y = e^{-x} + Ce^{-3x/2} \Rightarrow y' = -e^{-x} - \frac{3}{2}Ce^{-3x/2} \Rightarrow 2y' + 3y = 2\left(-e^{-x} - \frac{3}{2}Ce^{-3x/2}\right) + 3\left(e^{-x} + Ce^{-3x/2}\right) = e^{-x}$

2. (a) $y = -\frac{1}{x} \Rightarrow y' = \frac{1}{x^2} = \left(-\frac{1}{x}\right)^2 = y^2$

(b) $y = -\frac{1}{x+3} \Rightarrow y' = \frac{1}{(x+3)^2} = \left[-\frac{1}{(x+3)}\right]^2 = y^2$

(c) $y = \frac{1}{x+C} \Rightarrow y' = \frac{1}{(x+C)^2} = \left[-\frac{1}{x+C}\right]^2 = y^2$

3. $y = (x-2)e^{-x^2} \Rightarrow y' = e^{-x^2} + \left(-2xe^{-x^2}\right)(x-2) \Rightarrow y' = e^{-x^2} - 2xy;\ y(2) = (2-2)e^{-2^2} = 0$

4. $y = \frac{\cos x}{x} \Rightarrow y' = \frac{-x \sin x - \cos x}{x^2} \Rightarrow y' = -\frac{\sin x}{x} - \frac{1}{x}\left(\frac{\cos x}{x}\right) \Rightarrow y' = -\frac{\sin x}{x} - \frac{y}{x} \Rightarrow xy' = -\sin x - y$

$\Rightarrow xy' + y = -\sin x;\ y\left(\frac{\pi}{2}\right) = \frac{\cos(\pi/2)}{(\pi/2)} = 0$

5. $2\sqrt{xy}\,\frac{dy}{dx} = 1 \Rightarrow 2x^{1/2}y^{1/2}\,dy = dx \Rightarrow 2y^{1/2}\,dy = x^{-1/2}\,dx \Rightarrow \int 2y^{1/2}\,dy = \int x^{-1/2}\,dx \Rightarrow 2\left(\frac{2}{3}y^{3/2}\right)$

$= 2x^{1/2} + C_1 \Rightarrow \frac{2}{3}y^{3/2} - x^{1/2} = C$, where $C = \frac{1}{2}C_1$

6. $\frac{dy}{dx} = x^2\sqrt{y} \Rightarrow dy = x^2y^{1/2}\,dx \Rightarrow y^{-1/2}\,dy = x^2\,dx \Rightarrow \int y^{-1/2}\,dy = \int x^2\,dx \Rightarrow 2y^{1/2} = \frac{x^3}{3} + C$

$\Rightarrow 2y^{1/2} - \frac{1}{3}x^3 = C$

7. $\frac{dy}{dx} = e^{x-y} \Rightarrow dy = e^x e^{-y}\,dx \Rightarrow e^y\,dy = e^x\,dx \Rightarrow \int e^y\,dy = \int e^x\,dx \Rightarrow e^y = e^x + C \Rightarrow e^y - e^x = C$

8. $\frac{dy}{dx} = 3x^2e^{-y} \Rightarrow dy = 3x^2e^{-y}\,dx \Rightarrow e^y\,dy = 3x^2\,dx \Rightarrow \int e^y\,dy = \int 3x^2\,dx \Rightarrow e^y = x^3 + C \Rightarrow e^y - x^3 = C$

9. $\frac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y} \Rightarrow dy = \left(\sqrt{y}\cos^2\sqrt{y}\right)dx \Rightarrow \frac{\sec^2\sqrt{y}}{\sqrt{y}}\,dy = dx \Rightarrow \int \frac{\sec^2\sqrt{y}}{\sqrt{y}}\,dy = \int dx$. In the integral on the left-hand side, substitute $u = \sqrt{y} \Rightarrow du = \frac{1}{2\sqrt{y}}\,dy \Rightarrow 2\,du = \frac{1}{\sqrt{y}}\,dy$, and we have $2\int \sec^2 u\,du = \int dx \Rightarrow 2\tan u = x + C \Rightarrow -x + 2\tan\sqrt{y} = C$

10. $\sqrt{2xy}\,\frac{dy}{dx} = 1 \Rightarrow dt = \frac{1}{\sqrt{2xy}}\,dx \Rightarrow \sqrt{2}\sqrt{y}\,dy = \frac{1}{\sqrt{x}}\,dx \Rightarrow \sqrt{2}\,y^{1/2}\,dy = x^{-1/2}\,dx$

$\Rightarrow \sqrt{2}\int y^{1/2}\,dy = \int x^{-1/2}\,dx \Rightarrow \sqrt{2}\left(\frac{y^{3/2}}{\frac{1}{2}}\right) = \left(\frac{x^{1/2}}{\frac{1}{2}}\right) + C_1 \Rightarrow \sqrt{2}\,y^{3/2} = 3\sqrt{x} + \frac{3}{2}C_1$

$\Rightarrow \sqrt{2}\left(\sqrt{y}\right)^3 - 3\sqrt{x} = C$, where $C = \frac{3}{2}C_1$.

11. $\sqrt{x}\,\frac{dy}{dx} = e^{y+\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}} \Rightarrow dy = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}}\,dx \Rightarrow e^{-y}\,dy = \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \Rightarrow \int e^{-y}\,dy = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

In the integral on the right-hand side, substitute $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx$. and we have

$\int e^{-y}\,dy = 2\int e^u\,du \Rightarrow -e^{-y} = 2e^u + C_1 \Rightarrow e^{-y} + 2e^{\sqrt{x}} = C$, where $C = -C_1$.

12. $(\sec x)\,\frac{dy}{dx} = e^{y+\sin x} \Rightarrow \frac{dy}{dx} = e^{y+\sin x}\cos x \Rightarrow dy = \left(e^y e^{\sin x}\cos x\right)dx \Rightarrow e^{-y}\,dy = \left(e^{\sin x}\cos x\right)dx$

$\Rightarrow \int e^{-y}\,dy = \int \left(e^{\sin x}\cos x\right)dx \Rightarrow -e^{-y} = e^{\sin x} + C_1 \Rightarrow e^{-y} + e^{\sin x} = C$, where $C = -C_1$

13. $\frac{dy}{dx} = 2x\sqrt{1-y^2} \Rightarrow dy = 2x\sqrt{1-y^2}\,dx \Rightarrow \frac{dy}{\sqrt{1-y^2}} = 2x\,dx \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx \Rightarrow \sin^{-1} y = x^2 + C$ since $|y| < 1 \Rightarrow y = \sin\left(x^2 + C\right)$.

14. $\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}} \Rightarrow dy = \frac{e^{2x-y}}{e^{x+y}}\,dx \Rightarrow dy = \frac{e^{2x}e^{-y}}{e^x e^y}\,dx = \frac{e^x}{e^{2y}}\,dx \Rightarrow e^{2y}\,dy = e^x\,dx \Rightarrow \int e^{2y}\,dy = \int e^x\,dx$

$\Rightarrow \frac{e^{2y}}{2} = e^x + C_1 \Rightarrow e^{2y} - 2e^x = C$ where $C = 2C_1$.

15. (a) $\frac{dp}{dh} = kp \Rightarrow p = p_0 e^{kh}$ where $p_0 = 1013$; $90 = 1013e^{20k} \Rightarrow k = \frac{\ln(90) - \ln(1013)}{20} \approx -0.121$

(b) $p = 1013e^{-6.05} \approx 2.389$ millibars

(c) $900 = 1013e^{(-0.121)h} \Rightarrow -0.121h = \ln\left(\frac{900}{1013}\right) \Rightarrow h = \frac{\ln(1013) - \ln(900)}{0.121} \approx 0.977$ km

16. $\frac{dy}{dt} = -0.6y \Rightarrow y = y_0 e^{-0.6t}$; $y_0 = 100 \Rightarrow y = 100e^{-0.6t} \Rightarrow y = 100e^{-0.6} \approx 54.88$ grams when $t = 1$ hr

17. $A = A_0 e^{kt} \Rightarrow 800 = 1000e^{10k} \Rightarrow k = \frac{\ln(0.8)}{10} \Rightarrow A = 1000e^{((\ln(0.8))/10)t}$, where A represents the amount of sugar that remains after time t. Thus after another 14 hrs, $A = 1000e^{((\ln(0.8)/10)14} \approx 731.688$ kg

18. $L(x) = L_0 e^{-kx} \Rightarrow \frac{L_0}{2} = L_0 e^{-18k} \Rightarrow \ln\frac{1}{2} = -18k \Rightarrow k = \frac{\ln 2}{18} \approx 0.0385 \Rightarrow L(x) = L_0 e^{-0.0385x}$; when the intensity is one-tenth of the surface value, $\frac{L_0}{10} = L_0 e^{-0.0385x} \Rightarrow \ln 10 = 0.0385x \Rightarrow x \approx 59.8$ ft

19. $V(t) = V_0 e^{-t/40} \Rightarrow 0.1V_0 = V_0 e^{-t/40}$ when the voltage is 10% of its original value $\Rightarrow t = -40\ln(0.1)$ ≈ 92.1 sec

20. $0.9P_0 = P_0 e^{k} \Rightarrow k = \ln 0.9$; when the well's output falls to one-fifth of its present value $P = 0.2P_0$ $\Rightarrow 0.2P_0 = P_0 e^{(\ln 0.9)t} \Rightarrow 0.2 = e^{(\ln 0.9)t} \Rightarrow \ln(0.2) = (\ln 0.9)t \Rightarrow t = \frac{\ln 0.2}{\ln 0.9} \approx 15.28$ yr

21. (a) $\frac{dQ}{dt} = -kQ + r$ where k is a positive constant and $Q = Q(t)$.

(b) $dQ = (-kQ + r)\,dt \Rightarrow dQ = -k\left(Q - \frac{r}{k}\right)dt \Rightarrow \frac{dQ}{\left(Q - \frac{r}{k}\right)} = -k\,dt \Rightarrow \int \frac{dQ}{\left(Q - \frac{r}{k}\right)} = -\int k\,dt$

$\Rightarrow \ln\left|Q - \frac{r}{k}\right| = -kt + C_1 \Rightarrow e^{\ln\left|Q - \frac{r}{k}\right|} = e^{-kt + C_1} \Rightarrow \left|Q - \frac{r}{k}\right| = e^{-kt}e^{C_1} \Rightarrow Q(t) = \frac{r}{k} \pm C_2 e^{-kt}$

$\Rightarrow Q(t) = \frac{r}{k} + Ce^{-kt}$ where $C_2 = e^{C_1}$ and $C = \pm C_2$. Apply the initial condition $Q(0) = Q_0 = \frac{r}{k} + Ce^0$

$\Rightarrow C = Q_0 - \frac{r}{k} \Rightarrow Q(t) = \frac{r}{k} + \left(Q_0 - \frac{r}{k}\right)e^{-kt}$

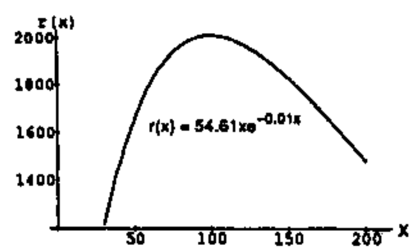
(c) $\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\left(\frac{r}{k} + \left(Q_0 - \frac{r}{k}\right)e^{-kt}\right) = \frac{r}{k} + \left(Q_0 - \frac{r}{k}\right)(0) = \frac{r}{k}$.

22. (a) $\frac{dp}{dx} = -\frac{1}{100}p \Rightarrow \frac{dp}{p} = -\frac{1}{100}dx \Rightarrow \ln p = -\frac{1}{100}x + C \Rightarrow p = e^{(-0.01x + C)} = e^{C}e^{-0.01x} = C_1 e^{-0.01x}$;

$p(100) = 20.09 \Rightarrow 20.09 = C_1 e^{(-0.01)(100)} \Rightarrow C_1 = 20.09e \approx 54.61 \Rightarrow p(x) = 54.61e^{-0.01x}$ (in dollars)

(b) $p(10) = 54.61e^{(-0.01)(10)} = \$49.41$, and $p(90) = 54.61e^{(-0.01)(90)} = \$22.20$

(c) $r(x) = xp(x) \Rightarrow r'(x) = p(x) + xp'(x)$; $p'(x) = -.5461e^{-0.01x} \Rightarrow r'(x)$ $= (54.61 - .5461x)e^{-0.01x}$. Thus, $r'(x) = 0$ $\Rightarrow 54.61 = .5461x \Rightarrow x = 100$. Since $r' > 0$ for any $x < 100$ and $r' < 0$ for $x > 100$, then $r(x)$ must be a maximum at $x = 100$.



23. (a) $A_0 e^{(0.04)5} = A_0 e^{0.2}$

(b) $2A_0 = A_0 e^{(0.04)t} \Rightarrow \ln 2 = (0.04)t \Rightarrow t = \frac{\ln 2}{0.04} \approx 17.33$ years; $3A_0 = A_0 e^{(0.04)t} \Rightarrow \ln 3 = (0.04)t$ $\Rightarrow t = \frac{\ln 3}{0.04} \approx 27.47$ years

24. (a) The amount of money invested A_0 after t years is $A(t) = A_0e^t$

(b) If $A(t) = 3A_0$, then $3A_0 = A_0e^t \Rightarrow \ln 3 = t$ or $t = 1.099$ years

(c) At the beginning of a year the account balance is A_0e^t, while at the end of the year the balance is $A_0e^{(t+1)}$. The amount earned is $A_0e^{(t+1)} - A_0e^t = A_0e^t(e-1) \approx 1.7$ times the beginning amount.

25. $y = y_0e^{-0.18t}$ represents the decay equation; solving $(0.9)y_0 = y_0e^{-0.18t} \Rightarrow t = \frac{\ln(0.9)}{-0.18} \approx 0.585$ days

26. $A = A_0e^{kt}$ and $\frac{1}{2}A_0 = A_0e^{139k} \Rightarrow \frac{1}{2} = e^{139k} \Rightarrow k = \frac{\ln(0.5)}{139} \approx -0.00499$; then $0.05A_0 = A_0e^{-0.00499t}$ $\Rightarrow t = \frac{\ln 0.05}{-0.00499} \approx 600$ days

27. $y = y_0e^{-kt} = y_0e^{-(k)(3/k)} = y_0e^{-3} = \frac{y_0}{e^3} < \frac{y_0}{20} = (0.05)(y_0) \Rightarrow$ after three mean lifetimes less than 5% remains

28. (a) $A = A_0e^{-kt} \Rightarrow \frac{1}{2} = e^{-2.645k} \Rightarrow k = \frac{\ln 2}{2.645} \approx 0.262$

(b) $\frac{1}{k} \approx 3.816$ years

(c) $(0.05)A = A\exp\left(-\frac{\ln 2}{2.645}t\right) \Rightarrow -\ln 20 = \left(-\frac{\ln 2}{2.645}\right)t \Rightarrow t = \frac{2.645 \ln 20}{\ln 2} \approx 11.431$ years

29. $T - T_s = (T_0 - T_s)e^{-kt}$, $T_0 = 90°C$, $T_s = 20°C$, $T = 60°C \Rightarrow 60 - 20 = 70e^{-10k} \Rightarrow \frac{4}{7} = e^{-10k}$ $\Rightarrow k = \frac{\ln\left(\frac{7}{4}\right)}{10} \approx 0.05596$

(a) $35 - 20 = 70e^{-0.05596t} \Rightarrow t \approx 27.5$ min is the total time $\Rightarrow$ it will take $27.5 - 10 = 17.5$ to reach 35°C

(b) $T - T_s = (T_0 - T_s)e^{-kt}$, $T_0 = 90°C$, $T_s = -15°C \Rightarrow 35 + 15 = 105e^{-0.05596t} \Rightarrow t \approx 13.26$ min

30. $T - 65° = (T_0 - 65°)e^{-kt} \Rightarrow 35° - 65° = (T_0 - 65°)e^{-10k}$ and $50° - 65° = (T_0 - 65°)e^{-20k}$. Solving $-30° = (T_0 - 65°)e^{-10k}$ and $-15° = (T_0 - 65°)e^{-20k}$ simultaneously $\Rightarrow (T_0 - 65°)e^{-10k} = 2(T_0 - 65°)e^{-20k} \Rightarrow e^{10k} = 2$ $\Rightarrow k = \frac{\ln 2}{10}$ and $-30° = \frac{T_0 - 65°}{e^{10k}} \Rightarrow -30°\left[e^{10\left(\frac{\ln 2}{10}\right)}\right] = T_0 - 65° \Rightarrow T_0 = 65° - 30°\left(e^{\ln 2}\right) = 65° - 60° = 5°$

31. $T - T_s = (T_0 - T_s)e^{-kt} \Rightarrow 39 - T_s = (46 - T_s)e^{-10k}$ and $33 - T_s = (46 - T_s)e^{-20k} \Rightarrow \frac{39 - T_s}{46 - T_s} = e^{-10k}$ and $\frac{33 - T_s}{46 - T_s} = e^{-20k} = \left(e^{-10k}\right)^2 \Rightarrow \frac{33 - T_s}{46 - T_s} = \left(\frac{39 - T_s}{46 - T_s}\right)^2 \Rightarrow (33 - T_s)(46 - T_s) = (39 - T_s)^2 \Rightarrow 1518 - 79T_s + T_s^2$ $= 1521 - 78T_s + T_s^2 \Rightarrow -T_s = 3 \Rightarrow T_s = -3°C$

32. Let x represent how far above room temperature the silver will be 15 min from now, y how far above room temperature the silver will be 120 min from now, and t_0 the time the silver will be 10°C above room temperature. We then have the following time-temperature table:

time in min.	0	20 (Now)	35	140	t_0
temperature	$T_s + 70°$	$T_s + 60°$	$T_s + x$	$T_s + y$	$T_s + 10°$

$T - T_s = (T_0 - T_s)\,e^{-kt} \Rightarrow (60 + T_s) - T_s = [(70 + T_s) - T_s]e^{-20k} \Rightarrow 60 = 70e^{-20k} \Rightarrow k = \left(-\frac{1}{20}\right)\ln\left(\frac{6}{7}\right)$
≈ 0.00771

(a) $T - T_s = (T_0 - T_s)\,e^{-0.00771t} \Rightarrow (T_s + x) - T_s = [(70 + T_s) - T_s]e^{-(0.00771)(35)} \Rightarrow x = 70e^{-0.26985} \approx 53.44°C$

(b) $T - T_s = (T_0 - T_s)\,e^{-0.00771t} \Rightarrow (T_s + y) - T_s = [(70 + T_s) - T_s]e^{-(0.00771)(140)} \Rightarrow y = 70e^{-1.0794} \approx 23.79°C$

(c) $T - T_s = (T_0 - T_s)\,e^{-0.00771t} \Rightarrow (T_s + 10) - T_s = [(70 + T_s) - T_s]e^{-(0.00771)\,t_0} \Rightarrow 10 = 70e^{-0.00771t_0}$

$\Rightarrow \ln\left(\frac{1}{7}\right) = -0.00771t_0 \Rightarrow t_0 = \left(-\frac{1}{0.00771}\right)\ln\left(\frac{1}{7}\right) = 252.39 \Rightarrow 252.39 - 20 \approx 232$ minutes from now the silver will be 10°C above room temperature

33. From Example 5, the half-life of carbon-14 is 5700 yr $\Rightarrow \frac{1}{2}c_0 = c_0e^{-k(5700)} \Rightarrow k = \frac{\ln 2}{5700} \approx 0.0001216$

$\Rightarrow c = c_0e^{-0.0001216t} \Rightarrow (0.445)c_0 = c_0e^{-0.0001216t} \Rightarrow t = \frac{\ln(0.445)}{-0.0001216} \approx 6658$ years

34. From Exercise 33, $k \approx 0.0001216$ for carbon-14.

(a) $c = c_0e^{-0.0001216t} \Rightarrow (0.17)c_0 = c_0e^{-0.0001216t} \Rightarrow t \approx 14{,}571.44$ years $\Rightarrow$ 12,571 BC

(b) $(0.18)c_0 = c_0e^{-0.0001216t} \Rightarrow t \approx 14{,}101.41$ years $\Rightarrow$ 12,101 BC

(c) $(0.16)c_0 = c_0e^{-0.0001216t} \Rightarrow t \approx 15{,}069.98$ years $\Rightarrow$ 13,070 BC

35. From Exercise 33, $k \approx 0.0001216$ for carbon-14. Thus, $c = c_0e^{-0.0001216t} \Rightarrow (0.995)c_0 = c_0e^{-0.0001216t}$

$\Rightarrow t = \frac{\ln(0.995)}{-0.0001216} \approx 41$ years old

36. (a) Since there are 48 half-hour doubling times in 24 hours, there will be $2^{48} \approx 2.8 \times 10^{14}$ bacteria.

(b) The bacteria reproduce fast enough that even if many are destroyed there are still enough left to make the person sick.

37. Note that the total mass is $66 + 7 = 73$ kg, therefore, $v = v_0e^{-(k/m)t} \Rightarrow v = 9e^{-3.9t/73}$

(a) $s(t) = \int 9e^{-3.9t/73}\,dt = -\frac{2190}{13}e^{-3.9t/73} + C$

Since $s(0) = 0$ we have $C = \frac{2190}{13}$ and $\lim_{t\to\infty} s(t) = \lim_{t\to\infty} \frac{2190}{13}\left(1 - e^{-3.9t/3}\right) = \frac{2190}{13} \approx 168.5$

The cyclist will coast about 168.5 meters.

(b) $1 = 9e^{-3.9t/73} \Rightarrow \frac{3.9t}{73} = \ln 9 \Rightarrow t = \frac{73\ln 9}{3.9} \approx 41.13$ sec

It will take about 41.13 seconds.

38. $v = v_0e^{-(k/m)t} \Rightarrow v = 9e^{-(59{,}000/51{,}000{,}000)t} \Rightarrow v = 9e^{-59t/51{,}000}$

(a) $s(t) = \int 9e^{-59t/51{,}000}\,dt = -\frac{459{,}000}{59}e^{59t/51{,}000} + C$

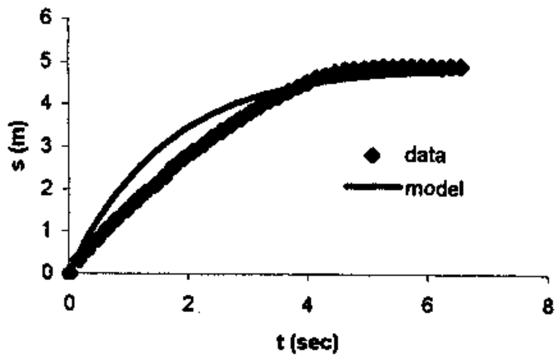
Since $s(0) = 0$, we have $C = \frac{459{,}000}{59}$ and $\lim_{t\to\infty} s(t) = \lim_{t\to\infty} \frac{459{,}000}{59}\left(1 - e^{-59t/51{,}000}\right) = \frac{459{,}000}{59} \approx 7780$ m

The ship will coast about 7780 m, or 7.78 km.

(b) $1 = 9e^{-59t/51,000} \Rightarrow \frac{59t}{51,000} = \ln 9 \Rightarrow t = \frac{51,000 \ln 9}{59} \approx 1899.3$ sec

It will take about 31.65 minutes.

39. The total distance traveled $= \frac{v_0 m}{k} \Rightarrow \frac{(2.75)(39.92)}{k} = 4.91 \Rightarrow k = 22.36$. Therefore, the distance traveled is given by the function $s(t) = 4.91(1 - e^{-(22.36/39.92)t})$. The graphs shows s(t) and the data points.

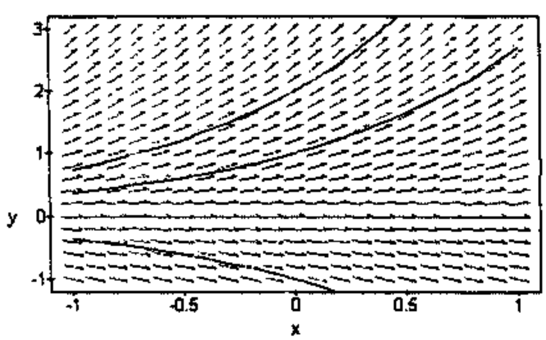


40. $\frac{v_0 m}{k} = \text{coasting distance} \Rightarrow \frac{(0.80)(49.90)}{k} = 1.32 \Rightarrow k = \frac{998}{33}$

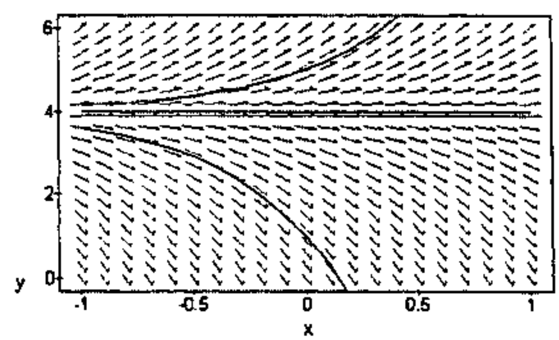
We know that $\frac{v_0 m}{k} = 1.32$ and $\frac{k}{m} = \frac{998}{33(49.9)} = \frac{20}{33}$.

Using Equation 3, we have: $s(t) = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right) = 1.32\left(1 - e^{-20t/33}\right) \approx 1.32\left(1 - e^{-0.606t}\right)$

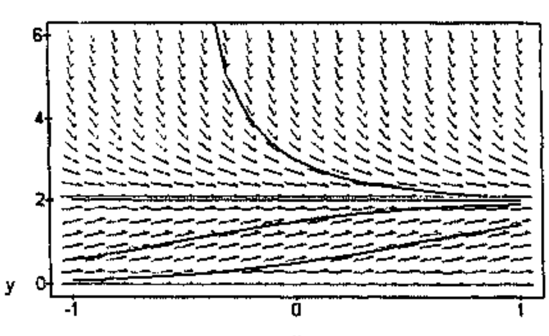
41.



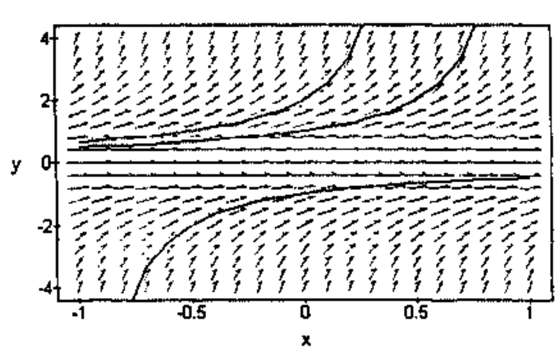
42.



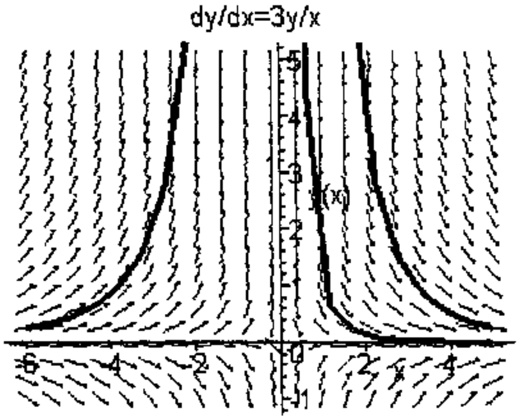
43.



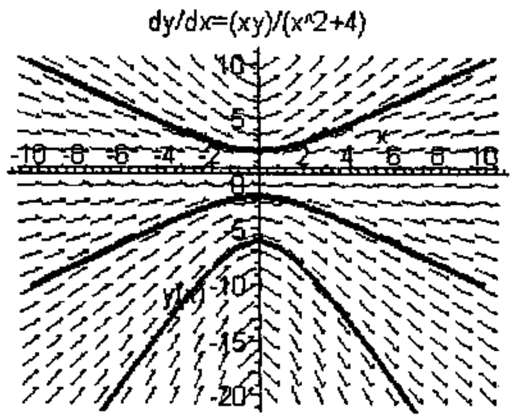
44.



45.



46.



6.5 LINEAR FIRST ORDER DIFFERENTIAL EQUATIONS

1. $x\frac{dy}{dx}+y=e^x$

   Step 1: $\frac{dy}{dx}+\left(\frac{1}{x}\right)y=\frac{e^x}{x}$, $P(x)=\frac{1}{x}$, $Q(x)=\frac{e^x}{x}$

   Step 2: $\int P(x)\,dx=\int \frac{1}{x}\,dx=\ln|x|=\ln x,\ x>0$

   Step 3: $v(x)=e^{\int P(x)\,dx}=e^{\ln x}=x$

   Step 4: $y=\frac{1}{v(x)}\int v(x)\,Q(x)\,dx=\frac{1}{x}\int x\left(\frac{e^x}{x}\right)dx=\frac{1}{x}(e^x+C)=\frac{e^x+C}{x}$

2. $e^x\frac{dy}{dx}+2e^xy=1$

   Step 1: $\frac{dy}{dx}+2y=e^{-x}$, $P(x)=2$, $Q(x)=e^{-x}$

Step 2: $\int P(x)\,dx = \int 2\,dx = 2x$

Step 3: $v(x) = e^{\int P(x)\,dx} = e^{2x}$

Step 4: $y = \frac{1}{e^{2x}} \int e^{2x} \cdot e^{-x}\,dx = \frac{1}{e^{2x}} \int e^{x}\,dx = \frac{1}{e^{2x}}(e^{x} + C) = e^{-x} + Ce^{-2x}$

3. $xy' + 3y = \frac{\sin x}{x^2}$, $x > 0$

Step 1: $\frac{dy}{dx} + \left(\frac{3}{x}\right)y = \frac{\sin x}{x^3}$, $P(x) = \frac{3}{x}$, $Q(x) = \frac{\sin x}{x^3}$

Step 2: $\int \frac{3}{x}\,dx = 3 \ln|x| = \ln x^3$, $x > 0$

Step 3: $v(x) = e^{\ln x^3} = x^3$

Step 4: $y = \frac{1}{x^3} \int x^3\left(\frac{\sin x}{x^3}\right)dx = \frac{1}{x^3} \int \sin x\,dx = \frac{1}{x^3}(-\cos x + C) = \frac{C - \cos x}{x^3}$, $x > 0$

4. $y' + (\tan x)\,y = \cos^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Step 1: $\frac{dy}{dx} + (\tan x)\,y = \cos^2 x$, $P(x) = \tan x$, $Q(x) = \cos^2 x$

Step 2: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| = \ln(\cos x)^{-1}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Step 3: $v(x) = e^{\ln(\cos x)^{-1}} = (\cos x)^{-1}$

Step 4: $y = \frac{1}{(\cos x)^{-1}} \int (\cos x)^{-1} \cdot \cos^2 x\,dx = (\cos x) \int \cos x\,dx = (\cos x)(\sin x + C) = \sin x \cos x + C \cos x$

5. $x\frac{dy}{dx} + 2y = 1 - \frac{1}{x}$, $x > 0$

Step 1: $\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{1}{x} - \frac{1}{x^2}$, $P(x) = \frac{2}{x}$, $Q(x) = \frac{1}{x} - \frac{1}{x^2}$

Step 2: $\int \frac{2}{x}\,dx = 2 \ln|x| = \ln x^2$, $x > 0$

Step 3: $v(x) = e^{\ln x^2} = x^2$

Step 4: $y = \frac{1}{x^2} \int x^2\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \frac{1}{x^2} \int (x - 1)\,dx = \frac{1}{x^2}\left(\frac{x^2}{2} - x + C\right) = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}$, $x > 0$

6. $(1 + x)\,y' + y = \sqrt{x}$

Step 1: $\frac{dy}{dx} + \left(\frac{1}{1 + x}\right)y = \frac{\sqrt{x}}{1 + x}$, $P(x) = \frac{1}{1 + x}$, $Q(x) = \frac{\sqrt{x}}{1 + x}$

Step 2: $\int \frac{1}{1+x}\,dx = \ln(1+x)$, since $x > 0$

Step 3: $v(x) = e^{\ln(1+x)} = 1+x$

Step 4: $y = \frac{1}{1+x}\int (1+x)\left(\frac{\sqrt{x}}{1+x}\right)dx = \frac{1}{1+x}\int \sqrt{x}\,dx = \left(\frac{1}{1+x}\right)\left(\frac{2}{3}x^{3/2}+C\right) = \frac{2x^{3/2}}{3(1+x)} + \frac{C}{1+x}$

7. $\frac{dy}{dx} - \frac{1}{2}y = \frac{1}{2}e^{x/2} \Rightarrow P(x) = -\frac{1}{2},\ Q(x) = \frac{1}{2}e^{x/2} \Rightarrow \int P(x)\,dx = -\frac{1}{2}x \Rightarrow v(x) = e^{-x/2}$

$\Rightarrow y = \frac{1}{e^{-x/2}}\int e^{-x/2}\left(\frac{1}{2}e^{x/2}\right)dx = e^{x/2}\int \frac{1}{2}\,dx = e^{x/2}\left(\frac{1}{2}x + C\right) = \frac{1}{2}xe^{x/2} + Ce^{x/2}$

8. $\frac{dy}{dx} + 2y = 2xe^{-2x} \Rightarrow P(x) = 2,\ Q(x) = 2xe^{-2x} \Rightarrow \int P(x)\,dx = \int 2\,dx = 2x \Rightarrow v(x) = e^{2x}$

$\Rightarrow y = \frac{1}{e^{2x}}\int e^{2x}\left(2xe^{-2x}\right)dx = \frac{1}{e^{2x}}\int 2x\,dx = e^{-2x}\left(x^2+C\right) = x^2e^{-2x} + Ce^{-2x}$

9. $\frac{dy}{dx} - \left(\frac{1}{x}\right)y = 2\ln x \Rightarrow P(x) = -\frac{1}{x},\ Q(x) = 2\ln x \Rightarrow \int P(x)\,dx = -\int \frac{1}{x}\,dx = -\ln x,\ x > 0$

$\Rightarrow v(x) = e^{-\ln x} = \frac{1}{x} \Rightarrow y = x\int \left(\frac{1}{x}\right)(2\ln x)\,dx = x\left[(\ln x)^2 + C\right] = x(\ln x)^2 + Cx$

10. $\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{\cos x}{x^2},\ x > 0 \Rightarrow P(x) = \frac{2}{x},\ Q(x) = \frac{\cos x}{x^2} \Rightarrow \int P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2,\ x > 0$

$\Rightarrow v(x) = e^{\ln x^2} = x^2 \Rightarrow y = \frac{1}{x^2}\int x^2\left(\frac{\cos x}{x^2}\right)dx = \frac{1}{x^2}\int \cos x\,dx = \frac{1}{x^2}(\sin x + C) = \frac{\sin x + C}{x^2}$

11. $\frac{ds}{dt} + \left(\frac{4}{t-1}\right)s = \frac{t+1}{(t-1)^3} \Rightarrow P(t) = \frac{4}{t-1},\ Q(t) = \frac{t+1}{(t-1)^3} \Rightarrow \int P(t)\,dt = \int \frac{4}{t-1}\,dt = 4\ln|t-1| = \ln(t-1)^4$

$\Rightarrow v(t) = e^{\ln(t-1)^4} = (t-1)^4 \Rightarrow s = \frac{1}{(t-1)^4}\int (t-1)^4\left[\frac{t+1}{(t-1)^3}\right]dt = \frac{1}{(t-1)^4}\int \left(t^2-1\right)dt$

$= \frac{1}{(t-1)^4}\left(\frac{t^3}{3} - t + C\right) = \frac{t^3}{3(t-1)^4} - \frac{t}{(t-1)^4} + \frac{C}{(t-1)^4}$

12. $(t+1)\frac{ds}{dt} + 2s = 3(t+1) + \frac{1}{(t+1)^2} \Rightarrow \frac{ds}{dt} + \left(\frac{2}{t+1}\right)s = 3 + \frac{1}{(t+1)^3} \Rightarrow P(t) = \frac{2}{t+1},\ Q(t) = 3 + (t+1)^{-3}$

$\Rightarrow \int P(t)\,dt = \int \frac{2}{t+1}\,dt = 2\ln|t+1| = \ln(t+1)^2 \Rightarrow v(t) = e^{\ln(t+1)^2} = (t+1)^2$

$\Rightarrow s = \frac{1}{(t+1)^2}\int (t+1)^2\left[3 + (t+1)^{-3}\right]dt = \frac{1}{(t+1)^2}\int \left[3(t+1)^2 + (t+1)^{-1}\right]dt$

$= \frac{1}{(t+1)^2}\left[(t+1)^3 + \ln|t+1| + C\right] = (t+1) + (t+1)^{-2}\ln(t+1) + \frac{C}{(t+1)^2},\ t > -1$

13. $\frac{dr}{d\theta} + (\cot\theta)r = \sec\theta \Rightarrow P(\theta) = \cot\theta,\ Q(\theta) = \sec\theta \Rightarrow \int P(\theta)\,d\theta = \int \cot\theta\,d\theta = \ln|\sin\theta| \Rightarrow v(\theta) = e^{\ln|\sin\theta|}$

$= \sin\theta$ because $0 < \theta < \frac{\pi}{2} \Rightarrow r = \frac{1}{\sin\theta}\int(\sin\theta)(\sec\theta)\,d\theta = \frac{1}{\sin\theta}\int\tan\theta\,d\theta = \frac{1}{\sin\theta}\left(\ln|\sec\theta| + C\right)$

$= (\csc\theta)\left(\ln|\sec\theta| + C\right)$

14. $\tan\theta\,\frac{dr}{d\theta} + r = \sin^2\theta \Rightarrow \frac{dr}{d\theta} + \frac{r}{\tan\theta} = \frac{\sin^2\theta}{\tan\theta} \Rightarrow \frac{dr}{d\theta} + (\cot\theta)r = \sin\theta\cos\theta \Rightarrow P(\theta) = \cot\theta,\ Q(\theta) = \sin\theta\cos\theta$

$\Rightarrow \int P(\theta)\,d\theta = \int\cot\theta\,d\theta = \ln|\sin\theta| = \ln(\sin\theta)$ since $0 < \theta < \frac{\pi}{2} \Rightarrow v(\theta) = e^{\ln(\sin\theta)} = \sin\theta$

$\Rightarrow r = \frac{1}{\sin\theta}\int(\sin\theta)(\sin\theta\cos\theta)\,d\theta = \frac{1}{\sin\theta}\int\sin^2\theta\cos\theta\,d\theta = \left(\frac{1}{\sin\theta}\right)\left(\frac{\sin^3\theta}{3} + C\right) = \frac{\sin^2\theta}{3} + \frac{C}{\sin\theta}$

15. $\frac{dy}{dt} + 2y = 3 \Rightarrow P(t) = 2,\ Q(t) = 3 \Rightarrow \int P(t)\,dt = \int 2\,dt = 2t \Rightarrow v(t) = e^{2t} \Rightarrow y = \frac{1}{e^{2t}}\int 3e^{2t}\,dt$

$= \frac{1}{e^{2t}}\left(\frac{3}{2}e^{2t} + C\right);\ y(0) = 1 \Rightarrow \frac{3}{2} + C = 1 \Rightarrow C = -\frac{1}{2} \Rightarrow y = \frac{3}{2} - \frac{1}{2}e^{-2t}$

16. $\frac{dy}{dt} + \frac{2y}{t} = t^2 \Rightarrow P(t) = \frac{2}{t},\ Q(t) = t^2 \Rightarrow \int P(t)\,dt = 2\ln|t| \Rightarrow v(t) = e^{\ln t^2} = t^2 \Rightarrow y = \frac{1}{t^2}\int(t^2)(t^2)\,dt$

$= \frac{1}{t^2}\int t^4\,dt = \frac{1}{t^2}\left(\frac{t^5}{5} + C\right) = \frac{t^3}{5} + \frac{C}{t^2};\ y(2) = 1 \Rightarrow \frac{8}{5} + \frac{C}{4} = 1 \Rightarrow C = -\frac{12}{5} \Rightarrow y = \frac{t^3}{5} - \frac{12}{5t^2}$

17. $\frac{dy}{d\theta} + \left(\frac{1}{\theta}\right)y = \frac{\sin\theta}{\theta} \Rightarrow P(\theta) = \frac{1}{\theta},\ Q(\theta) = \frac{\sin\theta}{\theta} \Rightarrow \int P(\theta)\,d\theta = \ln|\theta| \Rightarrow v(\theta) = e^{\ln|\theta|} = |\theta|$

$\Rightarrow y = \frac{1}{|\theta|}\int|\theta|\left(\frac{\sin\theta}{\theta}\right)d\theta = \frac{1}{\theta}\int\theta\left(\frac{\sin\theta}{\theta}\right)d\theta$ for $\theta \neq 0 \Rightarrow y = \frac{1}{\theta}\int\sin\theta\,d\theta = \frac{1}{\theta}(-\cos\theta + C)$

$= -\frac{1}{\theta}\cos\theta + \frac{C}{\theta};\ y\left(\frac{\pi}{2}\right) = 1 \Rightarrow C = \frac{\pi}{2} \Rightarrow y = -\frac{1}{\theta}\cos\theta + \frac{\pi}{2\theta}$

18. $\frac{dy}{d\theta} - \left(\frac{2}{\theta}\right)y = \theta^2\sec\theta\tan\theta \Rightarrow P(\theta) = -\frac{2}{\theta},\ Q(\theta) = \theta^2\sec\theta\tan\theta \Rightarrow \int P(\theta)\,d\theta = -2\ln|\theta| \Rightarrow v(\theta) = e^{-2\ln|\theta|}$

$= \theta^{-2} \Rightarrow y = \frac{1}{\theta^{-2}}\int(\theta^{-2})(\theta^2\sec\theta\tan\theta)\,d\theta = \theta^2\int\sec\theta\tan\theta\,d\theta = \theta^2(\sec\theta + C) = \theta^2\sec\theta + C\theta^2;$

$y\left(\frac{\pi}{3}\right) = 2 \Rightarrow 2 = \left(\frac{\pi^2}{9}\right)(2) + C\left(\frac{\pi^2}{9}\right) \Rightarrow C = \frac{18}{\pi^2} - 2 \Rightarrow y = \theta^2\sec\theta + \left(\frac{18}{\pi^2} - 2\right)\theta^2$

19. $(x+1)\frac{dy}{dx} - 2(x^2 + x)y = \frac{e^{x^2}}{x+1} \Rightarrow \frac{dy}{dx} - 2\left[\frac{x(x+1)}{x+1}\right]y = \frac{e^{x^2}}{(x+1)^2} \Rightarrow \frac{dy}{dx} - 2xy = \frac{e^{x^2}}{(x+1)^2} \Rightarrow P(x) = -2x,$

$Q(x) = \frac{e^{x^2}}{(x+1)^2} \Rightarrow \int P(x)\,dx = \int -2x\,dx = -x^2 \Rightarrow v(x) = e^{-x^2} \Rightarrow y = \frac{1}{e^{-x^2}} \int e^{-x^2}\left[\frac{e^{x^2}}{(x+1)^2}\right] dx$

$= e^{x^2} \int \frac{1}{(x+1)^2}\,dx = e^{x^2}\left[\frac{(x+1)^{-1}}{-1} + C\right] = -\frac{e^{x^2}}{x+1} + Ce^{x^2};\ y(0) = 5 \Rightarrow -\frac{1}{0+1} + C = 5 \Rightarrow -1 + C = 5$

$\Rightarrow C = 6 \Rightarrow y = 6e^{x^2} - \frac{e^{x^2}}{x+1}$

20. $\frac{dy}{dx} + xy = x \Rightarrow P(x) = x,\ Q(x) = x \Rightarrow \int P(x)\,dx = \int x\,dx = \frac{x^2}{2} \Rightarrow v(x) = e^{x^2/2} \Rightarrow y = \frac{1}{e^{x^2/2}} \int e^{x^2/2} \cdot x\,dx$

$= \frac{1}{e^{x^2/2}}\left(e^{x^2/2} + C\right) = 1 + \frac{C}{e^{x^2/2}};\ y(0) = -6 \Rightarrow 1 + C = -6 \Rightarrow C = -7 \Rightarrow y = 1 - \frac{7}{e^{x^2/2}}$

21. $\frac{dy}{dt} - ky = 0 \Rightarrow P(t) = -k,\ Q(t) = 0 \Rightarrow \int P(t)\,dt = \int -k\,dt = -kt \Rightarrow v(t) = e^{-kt}$

$\Rightarrow y = \frac{1}{e^{-kt}} \int \left(e^{-kt}\right)(0)\,dt = e^{kt}(0 + C) = Ce^{kt};\ y(0) = y_0 \Rightarrow C = y_0 \Rightarrow y = y_0 e^{kt}$

22. $\frac{dv}{dt} + \frac{k}{m}v = 0 \Rightarrow P(t) = \frac{k}{m},\ Q(t) = 0 \Rightarrow \int P(t)\,dt = \int \frac{k}{m}\,dt = \frac{k}{m}t = \frac{kt}{m} \Rightarrow v(t) = e^{kt/m}$

$\Rightarrow y = \frac{1}{e^{kt/m}} \int e^{kt/m} \cdot 0\,dt = \frac{C}{e^{kt/m}};\ v(0) = v_0 \Rightarrow \frac{C}{e^{k(0)/m}} = v_0 \Rightarrow C = v_0 \Rightarrow v = v_0 e^{-(k/m)t}$

23. $x \int \frac{1}{x}\,dx = x\left(\ln|x| + C\right) = x\ln|x| + Cx \Rightarrow$ (b) is correct

24. (a) $\frac{dx}{dt} = 1000 + 0.10x = 0.01(10{,}000 +) \Rightarrow dx = 0.1(10{,}000 + x)\,dt \Rightarrow \frac{dx}{x + 10{,}000} = 0.1\,dt$

$\Rightarrow \int \frac{dx}{x + 10{,}000} = \int 0.1\,dt \Rightarrow \ln|x + 10{,}000| = 0.1t + C_1 \Rightarrow e^{\ln|x+10{,}000|} = e^{0.1t + C_1}$

$\Rightarrow |x + 10{,}000| = e^{0.1t}e^{C_1} \Rightarrow x + 10{,}000 = \pm C_2 e^{0.1t} \Rightarrow x(t) = -10{,}000 + Ce^{0.1t}$, where $C_2 = e^{C_1}$ and $C = \pm C_2$. Apply the initial condition: $x(0) = 1000 = -10{,}000 + Ce^0 \Rightarrow C = 11{,}000$

$\Rightarrow x(t) = -10{,}000 + 11{,}000e^{0.1t}$

(b) $100{,}000 = -10{,}000 + 11{,}000e^{0.1t} \Rightarrow e^{0.1t} = 10 \Rightarrow t = 10\ln(10) \approx 23.03 \approx 23$ years and 11 days.

25. Let $y(t)$ = the amount of salt in the container and $V(t)$ = the total volume of liquid in the tank at time t. Then, the departure rate is $\frac{y(t)}{V(t)}$ (the outflow rate).

(a) Rate entering $= \frac{2\text{ lb}}{\text{gal}} \cdot \frac{5\text{ gal}}{\text{min}} = 10$ lb/min

(b) Volume $= V(t) = 100$ gal $+ (5t$ gal $- 4t$ gal$) = (100 + t)$ gal

(c) The volume at time t is $(100 + t)$ gal. The amount of salt in the tank at time t is y lbs. So the concentration at any time t is $\frac{y}{100+t}$ lbs/gal. Then, Rate leaving $= \frac{y}{100+t}$ (lbs/gal) $\cdot$ 4 (gal/min)

$= \frac{4y}{100+t}$ lbs/min

(d) $\frac{dy}{dt} = 10 - \frac{4y}{100+t} \Rightarrow \frac{dy}{dt} + \left(\frac{4}{100+t}\right)y = 10 \Rightarrow P(t) = \frac{4}{100+t},\ Q(t) = 10 \Rightarrow \int P(t)\,dt = \int \frac{4}{100+t}\,dt$

$= 4\ln(100+t) \Rightarrow v(t) = e^{4\ln(100+t)} = (100+t)^4 \Rightarrow y = \frac{1}{(100+t)^4}\int (100+t)^4(10\,dt)$

$= \frac{10}{(100+t)^4}\left(\frac{(100+t)^5}{5} + C\right) = 2(100+t) + \frac{C}{(100+t)^4};\ y(0) = 50 \Rightarrow 2(100+0) + \frac{C}{(100+0)^4} = 50$

$\Rightarrow C = -(150)(100)^4 \Rightarrow y = 2(100+t) - \frac{(150)(100)^4}{(100+t)^4} \Rightarrow y = 2(100+t) - \frac{150}{\left(1+\frac{t}{100}\right)^4}$

(e) $y(25) = 2(100+25) - \frac{(150)(100)^4}{(100+25)^4} \approx 188.56$ lbs $\Rightarrow$ concentration $= \frac{y(25)}{\text{volume}} \approx \frac{188.6}{125} \approx 1.5$ lb/gal

26. (a) $\frac{dV}{dt} = (5-3) = 2 \Rightarrow V = 100 + 2t$

The tank is full when $V = 200 = 100 + 2t \Rightarrow t = 50$ min.

(b) Let $y(t)$ be the amount of concentrate in the tank at time t.

$\frac{dy}{dt} = \left(\frac{1}{2}\,\frac{\text{lb}}{\text{gal}}\right)\left(5\,\frac{\text{gal}}{\text{min}}\right) - \left(\frac{y}{100+2t}\,\frac{\text{lb}}{\text{gal}}\right)\left(3\,\frac{\text{gal}}{\text{min}}\right) \Rightarrow \frac{dy}{dt} = \frac{5}{2} - \frac{3}{2}\left(\frac{y}{50+t}\right) \Rightarrow \frac{dy}{dt} + \frac{3}{2(t+50)}y = \frac{5}{2}$

$Q(t) = \frac{5}{2};\ P(t) = \frac{3}{2}\left(\frac{1}{t+50}\right) \Rightarrow \int P(t)\,dt = \frac{3}{2}\int \frac{1}{t+50}\,dt = \frac{3}{2}\ln(t+50)$ since $t + 50 > 0$

$v(t) = e^{\int P(t)\,dt} = e^{\frac{3}{2}\ln(t+50)} = (t+50)^{3/2}$

$y(t) = \frac{1}{(t+50)^{3/2}}\int \left(\frac{5}{2}\right)(t+50)^{3/2}\,dt = (t+50)^{-3/2}\left[(t+50)^{5/2} + C\right] \Rightarrow y(t) = t + 50 + \frac{C}{(t+50)^{3/2}}$

Apply the initial condition (i.e., distilled water in the tank at $t = 0$):

$y(0) = 0 = 50 + \frac{C}{50^{3/2}} \Rightarrow C = -50^{5/2} \Rightarrow y(t) = t + 50 - \frac{50^{5/2}}{(t+50)^{3/2}}$. When the tank is full at $t = 50$,

$y(50) = 100 - \frac{50^{3/2}}{100^{3/2}} \approx 83.22$ pounds of concentrate.

27. Let y be the amount of fertilizer in the tank at time t. Then rate entering $= 1\,\frac{\text{lb}}{\text{gal}} \cdot 1\,\frac{\text{gal}}{\text{min}} = 1\,\frac{\text{lb}}{\text{min}}$ and the volume in the tank at time t is $V(t) = 100\text{ (gal)} + [1\text{ (gal/min)} - 3\text{ (gal/min)}]t\text{ min} = (100 - 2t)$ gal. Hence rate out $= \left(\frac{y}{100-2t}\right)3 = \frac{3y}{100-2t}$ lbs/min $\Rightarrow \frac{dy}{dt} = \left(1 - \frac{3y}{100-2t}\right)$ lbs/min $\Rightarrow \frac{dy}{dt} + \left(\frac{3}{100-2t}\right)y = 1$

$\Rightarrow P(t) = \frac{3}{100-2t},\ Q(t) = 1 \Rightarrow \int P(t)\,dt = \int \frac{3}{100-2t}\,dt = \frac{3\ln(100-2t)}{-2} \Rightarrow v(t) = e^{(-3\ln(100-2t))/2}$

$= (100-2t)^{-3/2} \Rightarrow y = \frac{1}{(100-2t)^{-3/2}}\int (100-2t)^{-3/2}\,dt = (100-2t)^{3/2}\left[\frac{-2(100-2t)^{-1/2}}{-2} + C\right]$

$= (100-2t) + C(100-2t)^{3/2};\ y(0) = 0 \Rightarrow [100 - 2(0)] + C[100 - 2(0)]^{3/2} \Rightarrow C(100)^{3/2} = -100$

$\Rightarrow C = -(100)^{-1/2} = -\frac{1}{10} \Rightarrow y = (100-2t) - \frac{(100-2t)^{3/2}}{10}$. Let $\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -2 - \frac{\left(\frac{3}{2}\right)(100-2t)^{1/2}(-2)}{10}$

$= -2 + \frac{3\sqrt{100-2t}}{10} = 0 \Rightarrow 20 = 3\sqrt{100-2t} \Rightarrow 400 = 9(100-2t) \Rightarrow 400 = 900 - 18t \Rightarrow -500 = -18t$

$\Rightarrow t \approx 27.8$ min, the time to reach the maximum. The maximum amount is then

$y(27.8) = [100 - 2(27.8)] - \frac{[100-2(27.8)]^{3/2}}{10} \approx 14.8$ lb

28. Let $y = y(t)$ be the amount of carbon monoxide (CO) in the room at time t. The amount of CO entering the room is $\left(\frac{4}{100} \times \frac{3}{10}\right) = \frac{12}{1000}$ ft^3/min, and the amount of CO leaving the room is $\left(\frac{y}{4500}\right)\left(\frac{3}{10}\right) = \frac{y}{15{,}000}$ ft^3/min. Thus, $\frac{dy}{dt} = \frac{12}{1000} - \frac{y}{15{,}000} \Rightarrow \frac{dy}{dt} + \frac{1}{15{,}000}y = \frac{12}{1000} \Rightarrow P(t) = \frac{1}{15{,}000}$, $Q(t) = \frac{12}{1000} \Rightarrow v(t) = e^{t/15{,}000}$

$\Rightarrow y = \frac{1}{e^{t/15{,}000}} \int \frac{12}{1000} e^{t/15{,}000}\, dt \Rightarrow y = e^{-t/15{,}000}\left(\frac{12 \cdot 15{,}000}{1000} e^{t/1500} + C\right) = e^{-t/15{,}000}\left(180e^{t/15{,}000} + C\right)$;

$y(0) = 0 \Rightarrow 0 = 1(180 + C) \Rightarrow C = -180 \Rightarrow y = 180 - 180e^{-t/15{,}000}$. When the concentration of CO is 0.01% in the room, the amount of CO satisfies $\frac{y}{4500} = \frac{.01}{100} \Rightarrow y = 0.45$ ft^3. When the room contains this amount we have $0.45 = 180 - 180e^{-t/15{,}000} \Rightarrow \frac{179.55}{180} = e^{-t/15{,}000} \Rightarrow t = -15{,}000 \ln\left(\frac{179.55}{180}\right) \approx 37.55$ min.

29. Steady State $= \frac{V}{R}$ and we want $i = \frac{1}{2}\left(\frac{V}{R}\right) \Rightarrow \frac{1}{2}\left(\frac{V}{R}\right) = \frac{V}{R}\left(1 - e^{-Rt/L}\right) \Rightarrow \frac{1}{2} = 1 - e^{-Rt/L} \Rightarrow -\frac{1}{2} = -e^{-Rt/L}$

$\Rightarrow \ln \frac{1}{2} = -\frac{Rt}{L} \Rightarrow -\frac{L}{R} \ln \frac{1}{2} = t \Rightarrow t = \frac{L}{R} \ln 2$ sec

30. (a) $\frac{di}{dt} + \frac{R}{L}i = 0 \Rightarrow \frac{1}{i}\, di = -\frac{R}{L}\, dt \Rightarrow \ln i = -\frac{Rt}{L} + C_1 \Rightarrow i = e^{C_1}e^{-Rt/L} = Ce^{-Rt/L}$; $i(0) = I \Rightarrow I = C$

$\Rightarrow i = Ie^{-Rt/L}$ amp

(b) $\frac{1}{2}I = Ie^{-Rt/L} \Rightarrow e^{-Rt/L} = \frac{1}{2} \Rightarrow -\frac{Rt}{L} = \ln \frac{1}{2} = -\ln 2 \Rightarrow t = \frac{L}{R} \ln 2$ sec

(c) $t = \frac{L}{R} \Rightarrow i = Ie^{(-R/L)(L/R)} = Ie^{-1}$ amp

31. (a) $t = \frac{3L}{R} \Rightarrow i = \frac{V}{R}\left(1 - e^{(-R/L)(3L/R)}\right) = \frac{V}{R}\left(1 - e^{-3}\right) \approx 0.9502\, \frac{V}{R}$ amp, or about 95% of the steady state value

(b) $t = \frac{2L}{R} \Rightarrow i = \frac{V}{R}\left(1 - e^{(-R/L)(2L/R)}\right) = \frac{V}{R}\left(1 - e^{-2}\right) \approx 0.8647\, \frac{V}{R}$ amp, or about 86% of the steady state value

32. (a) $\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L} \Rightarrow P(t) = \frac{R}{L}$, $Q(t) = \frac{V}{L} \Rightarrow \int P(t)\, dt = \int \frac{R}{L}\, dt = \frac{Rt}{L} \Rightarrow v(t) = e^{Rt/L}$

$\Rightarrow i = \frac{1}{e^{Rt/L}} \int e^{Rt/L}\left(\frac{V}{L}\right) dt = \frac{1}{e^{Rt/L}}\left[\frac{L}{R}e^{Rt/L}\left(\frac{V}{L}\right) + C\right] = \frac{V}{R} + Ce^{-(R/L)t}$

(b) $i(0) = 0 \Rightarrow \frac{V}{R} + C = 0 \Rightarrow C = -\frac{V}{R} \Rightarrow i = \frac{V}{R} - \frac{V}{R}e^{-Rt/L}$

(c) $i = \frac{V}{R} \Rightarrow \frac{di}{dt} = 0 \Rightarrow \frac{di}{dt} + \frac{R}{L}i = 0 + \left(\frac{R}{L}\right)\left(\frac{V}{R}\right) = \frac{V}{L} \Rightarrow i = \frac{V}{R}$ is a solution of Eq. (11); $i = Ce^{-(R/L)t}$

$\Rightarrow \frac{di}{dt} = -\frac{RC}{L}e^{-(R/L)t} \Rightarrow \frac{di}{dt} + \frac{R}{L}i = -\frac{RC}{L}e^{-(R/L)t} + \frac{R}{L}\left(Ce^{-(R/L)t}\right) = 0 \Rightarrow i = Ce^{-(R/L)t}$ satisfies $\frac{di}{dt} + \frac{R}{L}i = 0$

6.6 EULER'S METHOD; POPULATION MODELS

1. $y_1 = y_0 + x_0(1 - y_0)\ dx = 0 + 1(1 - 0)(0.2) = 0.2,$

   $y_2 = y_1 + x_1(1 - y_1)\ dx = 0.2 + 1.2(1 - 0.2)(0.2) = 0.392,$

   $y_3 = y_2 + x_2(1 - y_2)\ dx = 0.392 + 1.4(1 - 0.392)(0.2) = 0.5622;$

   $\frac{dy}{1-y} = x\ dx \Rightarrow -\ln|1-y| = \frac{x^2}{2} + C;\ x = 1,\ y = 0 \Rightarrow -\ln 1 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2} \Rightarrow \ln|1-y| = -\frac{x^2}{2} + \frac{1}{2}$

   $\Rightarrow y = 1 - e^{\left(1-x^2\right)/2} \Rightarrow y(1.2) \approx 0.1975,\ y(1.4) \approx 0.3812,\ y(1.6) \approx 0.5416$

2. $y_1 = y_0 + \left(1 - \frac{y_0}{x_0}\right) dx = -1 + \left(1 - \frac{-1}{2}\right)(0.5) = -0.25,$

   $y_2 = y_1 + \left(1 - \frac{y_1}{x_1}\right) dx = -0.25 + \left(1 - \frac{-0.25}{2.5}\right)(0.5) = 0.3,$

   $y_3 = y_2 + \left(1 - \frac{y_2}{x_2}\right) dx = 0.3 + \left(1 - \frac{0.3}{3}\right)(0.5) = 0.75;$

   $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = 1 \Rightarrow P(x) = \frac{1}{x},\ Q(x) = 1 \Rightarrow \int P(x)\ dx = \int \frac{1}{x}\ dx = \ln|x| = \ln x,\ x > 0 \Rightarrow v(x) = e^{\ln x} = x$

   $\Rightarrow y = \frac{1}{x}\int x \cdot 1\ dx = \frac{1}{x}\left(\frac{x^2}{2} + C\right);\ x = 2,\ y = -1 \Rightarrow -1 = 1 + \frac{C}{2} \Rightarrow C = -4 \Rightarrow y = \frac{x}{2} - \frac{4}{x} \Rightarrow y(2.5) = \frac{2.5}{2} - \frac{4}{2.5}$

   $= -0.35;\ y(3.0) = \frac{3}{2} - \frac{4}{3} \approx 0.1667,\ y(3.5) = \frac{3.5}{2} - \frac{4}{3.5} = \frac{4.25}{7} \approx 0.6071$

3. $y_1 = y_0 + (2x_0y_0 + 2y_0)\ dx = 3 + [2(0)(3) + 2(3)](0.2) = 4.2,$

   $y_2 = y_1 + (2x_1y_1 + 2y_1)\ dx = 4.2 + [2(0.2)(4.2) + 2(4.2)](0.2) = 6.216,$

   $y_3 = y_2 + (2x_2y_2 + 2y_2)\ dx = 6.216 + [2(0.4)(6.216) + 2(6.216)](0.2) = 9.6970;$

   $\frac{dy}{dx} = 2y(x+1) \Rightarrow \frac{dy}{y} = 2(x+1)\ dx \Rightarrow \ln|y| = (x+1)^2 + C;\ x = 0,\ y = 3 \Rightarrow \ln 3 = 1 + C \Rightarrow C = \ln 3 - 1$

   $\Rightarrow \ln y = (x+1)^2 + \ln 3 - 1 \Rightarrow y = e^{(x+1)^2 + \ln 3 - 1} = e^{\ln 3}e^{x^2+2x} = 3e^{x(x+2)} \Rightarrow y(0.2) \approx 4.6581,$

   $y(0.4) \approx 7.8351,\ y(0.6) \approx 14.2765$

4. $y_1 = y_0 + y_0^2(1 + 2x_0)\ dx = 1 + 1^2[1 + 2(-1)](0.5) = 0.5,$

   $y_2 = y_1 + y_1^2(1 + 2x_1)\ dx = 0.5 + (0.5)^2[1 + 2(-0.5)](0.5) = 0.5,$

   $y_3 = y_2 + y_2^2(1 + 2x_2)\ dx = 0.5 + (0.5)^2[1 + 2(0)](0.5) = 0.625;$

   $\frac{dy}{y^2} = (1 + 2x)\ dx \Rightarrow -\frac{1}{y} = x + x^2 + C;\ x = -1,\ y = 1 \Rightarrow -1 = -1 + (-1)^2 + C \Rightarrow C = -1 \Rightarrow \frac{1}{y} = 1 - x - x^2$

   $\Rightarrow y = \frac{1}{1 - x - x^2} \Rightarrow y(-0.5) = 0.8,\ y(0) = 1,\ y(0.5) = 4$

5. $y_1 = 1 + 1(.2) = 1.2$,

$y_2 = 1.2 + (1.2)(.2) = 1.44$,

$y_3 = 1.44 + (1.44)(.2) = 1.728$,

$y_4 = 1.728 + (1.728)(.2) = 2.0736$,

$y_5 = 2.0736 + (2.0736)(.2) = 2.48832$;

$\frac{dy}{y} = dx \Rightarrow \ln y = x + C_1 \Rightarrow y = Ce^x$; $y(0) = 1 \Rightarrow 1 = Ce^0 \Rightarrow C = 1 \Rightarrow y = e^x \Rightarrow y(1) = e \approx 2.7183$

6. $y_1 = 2 + \left(\frac{2}{1}\right)(.2) = 2.4$,

$y_2 = 2.4 + \left(\frac{2.4}{1.2}\right)(.2) = 2.8$,

$y_3 = 2.8 + \left(\frac{2.8}{1.4}\right)(.2) = 3.2$,

$y_4 = 3.2 + \left(\frac{3.2}{1.6}\right)(.2) = 3.6$,

$y_5 = 3.6 + \left(\frac{3.6}{1.8}\right)(.2) = 4$;

$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln y = \ln x + C \Rightarrow y = kx$; $y(1) = 2 \Rightarrow 2 = k \Rightarrow y = 2x \Rightarrow y(2) = 4$

7. Let $z_n = y_{n-1} + 2y_{n-1}(x_{n-1} + 1)\,dx$ and $y_n = y_{n-1} + (y_{n-1}(x_{n-1} + 1) + z_n(x_n + 1))\,dx$ with $x_0 = 0$, $y_0 = 3$, and $dx = 0.2$. The exact solution is $y = 3e^{x(x+2)}$. Using a programmable calculator or a spreadsheet (I used a spreadsheet) gives the values in the following table.

x	z	y-approx	y-exact	Error
0	---	3	3	0
0.2	4.2	4.608	4.658122	0.050122
0.4	6.81984	7.623475	7.835089	0.211614
0.6	11.89262	13.56369	14.27646	0.712777

8. Let $z_n = y_{n-1} + x_{n-1}(1 - y_{n-1})\,dx$ and $y_n = y_{n-1} + \left(\frac{x_{n-1}(1 - y_{n-1}) + x_n(1 - z_n)}{2}\right)dx$ with $x_0 = 1$, $y_0 = 0$, and $dx = 0.2$. The exact solution is $y = 1 - e^{(1-x^2)/2}$. Using a programmable calculator or a spreadsheet (I used a spreadsheet) gives the values in the following table.

x	z	y-approx	y-exact	Error
1	---	0	0	0
1.2	0.2	0.196	0.197481	0.001481
1.4	0.38896	0.378026	0.381217	0.003191
1.6	0.552178	0.536753	0.541594	0.004841

9. (a) $\frac{dP}{dt} = 0.0015P(150 - P) = \frac{0.225}{150}P(150 - P) = \frac{k}{M}P(M - P)$

Thus, $k = 0.225$ and $M = 150$, and $P = \frac{M}{1 + Ae^{-k/t}} = \frac{150}{1 + Ae^{-0.225t}}$

Initial condition: $P(0) = 6 \Rightarrow 6 = \frac{150}{1 + Ae^0} \Rightarrow 1 + A = 25 \Rightarrow A = 24$

Formula: $P = \dfrac{150}{1+24^{-0.225t}}$

(b) $100 = \dfrac{150}{1+24e^{-0.225t}} \Rightarrow 1 + 24e^{-0.2225t} = \dfrac{3}{2} \Rightarrow 24e^{-0.225t} = \dfrac{1}{2} \Rightarrow e^{-0.225t} = \dfrac{1}{48} \Rightarrow -0.225t = -\ln 48$

$\Rightarrow t = \dfrac{\ln 48}{0.225} \approx 17.21$ weeks

$125 = \dfrac{150}{1+24e^{-0.225t}} \Rightarrow 1 + 24e^{-0.2225t} = \dfrac{6}{5} \Rightarrow 24e^{-0.225t} = \dfrac{1}{5} \Rightarrow e^{-0.225t} = \dfrac{1}{120} \Rightarrow -0.225t = -\ln 120$

$\Rightarrow t = \dfrac{\ln 120}{0.225} \approx 21.28$

It will take about 17.21 weeks to reach 100 guppies, and about 21.28 weeks to reach 125 guppies.

10. (a) $\dfrac{dP}{dt} = 0.0004P(250 - P) = \dfrac{0.1}{250}P(250 - P) = \dfrac{k}{M}P(M - P)$

Thus, $k = 0.1$ and $M = 250$, and $P = \dfrac{M}{1+Ae^{-kt}} = \dfrac{250}{1+Ae^{-0.1t}}$

Initial condition: $P(0) = 28$, where $t = 0$ represents the year 1970.

$28 = \dfrac{250}{1+Ae^{0}} \Rightarrow 28(1 + A) = 250 \Rightarrow A = \dfrac{250}{28} - 1 = \dfrac{111}{14} \approx 7.9286$

Formula: $P(t) = \dfrac{250}{1+111e^{-0.1t}/14}$, or approximately $P(t) = \dfrac{250}{1+7.9286e^{-0.1t}}$

(b) The population $P(t)$ will round to 250 when $P(t) \geq 249.5 \Rightarrow 249.5 = \dfrac{250}{1+111e^{-0.1t}/14}$

$\Rightarrow 249.5\left(1 + \dfrac{111e^{-0.1t}}{14}\right) = 250 \Rightarrow \dfrac{(249.5)(111e^{-0.1t})}{14} = 0.5 \Rightarrow e^{-0.1t} = \dfrac{14}{55{,}389} \Rightarrow -0.1t = \ln\dfrac{14}{55{,}389}$

$\Rightarrow t = 10(\ln 55{,}389 - \ln 14) \approx 82.8$

It will take about 83 years.

11. (a) Using the general solution from Example 6, part (c),

$\dfrac{dy}{dt} = (0.08875 \times 10^{-7})(8 \times 10^{7} - y)y \Rightarrow y(t) = \dfrac{M}{1+Ae^{-rMt}} = \dfrac{8 \times 10^{7}}{1+Ae^{-(0.08875)(8)t}} = \dfrac{8 \times 10^{7}}{1+Ae^{-0.71t}}$

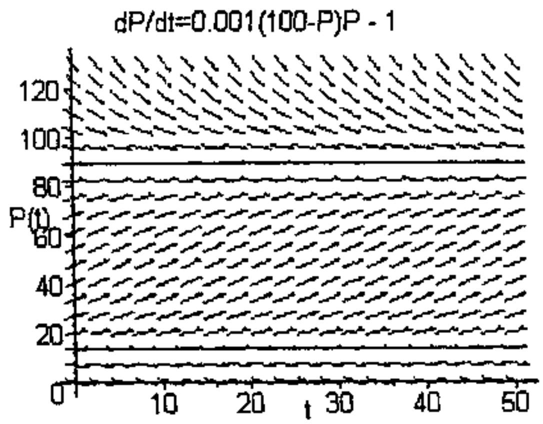
Apply the initial condition:

$y(0) = 1.6 \times 10^{7} = \dfrac{8 \times 10^{7}}{1+A} \Rightarrow A = \dfrac{8}{1.6} - 1 = 4 \Rightarrow y(1) = \dfrac{8 \times 10^{7}}{1+4e^{-0.71}} \approx 2.69671 \times 10^{7}$ kg.

(b) $y(t) = 4 \times 10^{7} = \dfrac{8 \times 10^{7}}{1+4e^{-0.71t}} \Rightarrow 4e^{-0.71t} = 1 \Rightarrow t = -\dfrac{\ln(1/4)}{0.71} \approx 1.95253$ years.

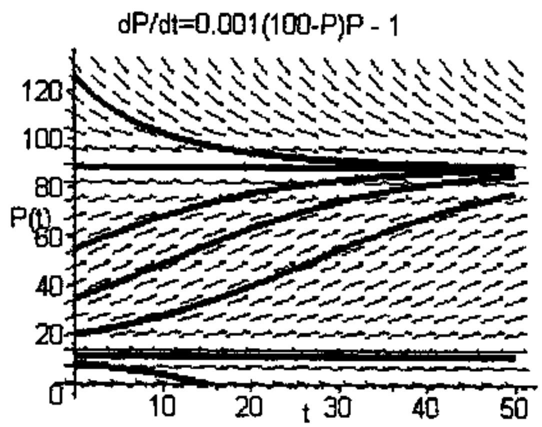
12. (a) If a part of the population leaves or is removed from the environment (e.g., a preserve or a region) each year, then c would represent the rate of reduction of the population due to this removal and/or migration. When grizzly bears become a nuisance (e.g., feeding on livestock) or threaten human safety, they are often relocated to other areas or even eliminated, but only after relocation efforts fail. In addition, bears are killed, sometimes accidentally and sometimes maliciously. For an environment that has a capacity of about 100 bears, a realistic value for c would probably be between 0 and 4.

(b)



Equilibrium solutions: $\frac{dP}{dt} = 0 = 0.001(100 - P)P - 1 \Rightarrow P^2 - 100P + 1000 = 0 \Rightarrow P_{eq} \approx 11.27$ (unstable) and $P_{eq} \approx 88.73$ (stable).

(c)



For $0 < P(0) \le 11$, the bear population will eventually disappear, for $12 \le P(0) \le 88$, the population will grow to about 89, for $P(0) = 89$, the population will remain at about 89, and for $P(0) > 89$, the population will decrease to about 89 bears.

13\. (a) $\frac{dy}{dt} = 1 + y \Rightarrow dy = (1+y)\,dt \Rightarrow \frac{dy}{1+y} = dt \Rightarrow \ln|1+y| = t + C_1 \Rightarrow e^{\ln|1+y|} = e^{t+C_1} \Rightarrow |1+y| = e^t e^{C_1}$

$1 + y = \pm C_2 e^t \Rightarrow y = Ce^t - 1$, where $C_2 = e^{C_1}$ and $C = \pm C_2$. Apply the initial condition: $y(0) = 1 = Ce^0 - 1 \Rightarrow C = 2 \Rightarrow y = 2e^t - 1$.

(b) $\frac{dy}{dt} = 0.5(400 - y)y \Rightarrow dy = 0.5(400 - y)y\,dt \Rightarrow \frac{dy}{y(400 - y)} = 0.5\,dt$. Using the partial fraction decomposition in Example 6, part (c), we obtain $\frac{1}{400}\left(\frac{1}{y} + \frac{1}{400 - y}\right)dy = 0.5\,dt \Rightarrow \left(\frac{1}{y} + \frac{1}{400 - y}\right)dy$

$= 200\,dt \Rightarrow \int \left(\frac{1}{y} - \frac{1}{y - 400}\right)dy = \int 200\,dt \Rightarrow \ln|y| - \ln|y - 400| = 200t + C_1 \Rightarrow \ln\left|\frac{y}{y - 400}\right| = 200t + C_1$

$\Rightarrow e^{\ln\left|\frac{y}{y-400}\right|} = e^{200t + C_1} = e^{200t}e^{C_1} \Rightarrow \left|\frac{y}{y - 400}\right| = C_2 e^{200t}$ (where $C_2 = e^{C_1}$) $\Rightarrow \frac{y}{y - 400} = \pm C_2 e^{200}$

$\Rightarrow \frac{y}{y-400} = Ce^{200t}$ (where $C = \pm C_2$) $\Rightarrow y = Ce^{200t}y - 400Ce^{200t} \Rightarrow (1 - Ce^{200t})y = -400Ce^{200t}$

$\Rightarrow y = \frac{400Ce^{200t}}{Ce^{200t} - 1} \Rightarrow y = \frac{400}{1 - \frac{1}{C}e^{-200t}} = \frac{400}{1 + Ae^{-200t}}$, where $A = -\frac{1}{C}$. Apply the initial condition:

$y(0) = 2 = \frac{400}{1 + Ae^0} \Rightarrow A = 199 \Rightarrow y(t) = \frac{400}{1 + 199e^{-200t}}$.

14. $\frac{dP}{dt} = r(M - P)P \Rightarrow dP = r(M - P)P\ dt \Rightarrow \frac{dP}{P(M - P)} = r\ dt$. Using the partial fraction decomposition in Example 6, part (c), we obtain $\frac{1}{M}\left(\frac{1}{P} + \frac{1}{M - P}\right) dP = r\ dt \Rightarrow \left(\frac{1}{P} + \frac{1}{M - P}\right) dP = rM\ dt \Rightarrow \int \left(\frac{1}{P} - \frac{1}{P - M}\right) dP$

$= \int rM\ dt \Rightarrow \ln|P| - \ln|P - M| = (rM)t + C_1 \Rightarrow \ln\left|\frac{P}{P - M}\right| = (rM)t + C_1 \Rightarrow e^{\ln\left|\frac{P}{P-M}\right|} = e^{(rM)t + C_1}$

$= e^{(rM)t}e^{C_1} \Rightarrow \left|\frac{P}{P - M}\right| = C_2e^{(rM)t}$ (where $C_2 = e^{C_1}$) $\Rightarrow \frac{P}{P - M} = \pm C_2e^{(rM)t} \Rightarrow \frac{P}{P - M} = Ce^{(rM)t}$ (where $C = \pm C_2$) $\Rightarrow P = Ce^{(rM)t}P - MCe^{(rM)t} \Rightarrow (1 - Ce^{(rM)t})P = -MCe^{(rM)t} \Rightarrow P = \frac{MCe^{(rM)t}}{Ce^{(rM)t} - 1}$

$\Rightarrow P = \frac{M}{1 - \frac{1}{C}e^{-(rM)t}} = \frac{M}{1 + Ae^{-(rM)t}}$, where $A = -\frac{1}{C}$.

15. (a) $\frac{dP}{dt} = kP^2 \Rightarrow \int P^{-2}\ dP = \int k\ dt \Rightarrow -P^{-1} = kt + C \Rightarrow P = \frac{-1}{kt + C}$

Initial condition: $P(0) = P_0 \Rightarrow P_0 = -\frac{1}{C} \Rightarrow C = -\frac{1}{P_0}$

Solution: $P = -\frac{1}{kt - (1/P_0)} = \frac{P_0}{1 - kP_0t}$

(b) There is a vertical asymptote at $t = \frac{1}{kP_0}$

16. (a) $\frac{dP}{dt} = r(M - P)(P - m) \Rightarrow \frac{dP}{dt} = r(1200 - P)(P - 100) \Rightarrow \frac{1}{(1200 - P)(P - 100)}\frac{dP}{dt} = r$

$\Rightarrow \frac{1100}{(1200 - P)(P - 100)}\frac{dP}{dt} = 1100r \Rightarrow \frac{(P - 100) + (1200 - P)}{(1200 - P)(P - 100)}\frac{dP}{dt} = 1100r$

$\Rightarrow \left(\frac{1}{1200 - P} + \frac{1}{P - 100}\right)\frac{dP}{dt} = 1100r$

$\left(\frac{1}{1200 - P} + \frac{1}{P - 100}\right) dP = 1100r\ dt \Rightarrow \int \left(\frac{1}{1200 - P} + \frac{1}{P - 100}\right) dP = \int 1100r\ dt$

$\Rightarrow \ln(1200 - P) + \ln(P - 100) = 1100rt + C_1 \Rightarrow \left|\frac{P - 100}{1200 - P}\right| = 1100rt + C_1 \Rightarrow \frac{P - 100}{1200 - P}$

$= \pm e^{C_1}e^{1100rt} \Rightarrow \frac{P - 100}{1200 - P} = Ce^{1100rt}$ where $C = \pm e^{C_1} \Rightarrow P - 100 = 1200Ce^{1100rt} - CPe^{1100rt}$

$\Rightarrow P(1 + Ce^{1100rt}) = 1200Ce^{1100rt} + 100 \Rightarrow P = \frac{1200Ce^{1100rt} + 100}{Ce^{1100rt} + 1} \Rightarrow \frac{1200 + \frac{100}{C}e^{-1100rt}}{1 + \frac{1}{C}e^{-1100rt}}$

$\Rightarrow P = \frac{1200 + 100Ae^{-1100rt}}{1 + Ae^{-1100rt}}$ where $A = \frac{1}{C}$.

(b) Apply the initial condition: $300 = \frac{1200 + 100A}{1 + A} = 300 + 300A = 1200 + 100A \Rightarrow A = \frac{9}{2}$

$\Rightarrow P = \frac{2400 + 900e^{-1100rt}}{2 + 9e^{-1100rt}}$. (Note that $P \to 1200$ as $t \to \infty$.)

(c) $\frac{dP}{dt} = r(M - P)(P - m) \Rightarrow \frac{dP}{dt} = r(M - P)(P - m) \Rightarrow \frac{1}{(M - P)(P - m)} \frac{dP}{dt} = r \Rightarrow \frac{M - m}{(M - P)(P - m)} \frac{dP}{dt}$

$= (M - m)r \Rightarrow \frac{(P - m) + (M - P)}{(M - P)(P - m)} \frac{dP}{dt} = (M - m)r \Rightarrow \left(\frac{1}{M - P} + \frac{1}{P - m}\right) \frac{dP}{dt} = (M - m)r$

$\Rightarrow \int \left(\frac{1}{M - P} + \frac{1}{P - m}\right) dP = \int (M - m)r\ dt \Rightarrow -\ln(M - P) + \ln(P - m) = (M - m)rt + C_1$

$\Rightarrow \ln\left|\frac{P - m}{M - P}\right| = (M - m)rt + C_1 \Rightarrow \frac{P - m}{M - P} = \pm e^{C_1} e^{(M-m)rt} \Rightarrow \frac{P - m}{M - P} = Ce^{(M-m)rt}$ where $C = \pm e^{C_1}$

$\Rightarrow P - m = MCe^{(M-m)rt} - CPe^{(M-m)rt} \Rightarrow P(1 + Ce^{(M-m)rt}) = MCe^{(M-m)rt} + m \Rightarrow P = \frac{MCe^{(M-m)rt} + m}{Ce^{(M-m)rt} + 1}$

$\Rightarrow P = \frac{M + \frac{m}{C}e^{-(M-m)rt}}{1 + \frac{1}{C}e^{-(M-m)rt}} \Rightarrow P = \frac{M + mAe^{-(M-m)rt}}{1 + Ae^{-(M-m)rt}}$ where $A = \frac{1}{C}$.

Apply the initial condition $P(0) = P_0$:

$P_0 = \frac{M + mA}{1 + A} \Rightarrow P_0 + P_0A = M + mA \Rightarrow A = \frac{M - P_0}{P_0 - m} \Rightarrow P = \frac{M(P_0 - m) + m(M - P_0)e^{-(M-m)rt}}{(P_0 - m) + (M - P_0)e^{-(M-m)rt}}$

(Note that $P \to M$ as $t \to \infty$ provided $P_0 > m$.)

17. $\frac{dy}{dx} = 2xe^{x^2}$, $y(0) = 2 \Rightarrow y_{n+1} = y_n + 2x_n e^{x_n^2}\ dx = y_n + 2x_n e^{x_n^2}(0.1) = y_n + 0.2x_n e^{x_n^2}$

On a TI-92 Plus calculator home screen, type the following commands:

2 STO> y: 0 STO> x:y (enter)
y+0.2*x*e^(x^2) STO> y: x+0.1 STO>x: y (enter, 10 times)

The last value displayed gives $y_{\text{Euler}}(1) \approx 3.45835$

The exact solution: $dy = 2xe^{x^2}\ dx \Rightarrow y = e^{x^2} + C$; $y(0) = 2 = e^0 + C \Rightarrow C = 1 \Rightarrow y = 1 + e^{x^2}$

$\Rightarrow y_{\text{exact}}(1) = 1 + e \approx 3.71828$

18. $\frac{dy}{dx} = y + e^x - 2$, $y(0) = 2 \Rightarrow y_{n+1} = y_n + (y_n + e^{x_n} - 2)\ dx = y_n + 0.5(y_n + e^{x_n} - 2)$

On a TI-92 Plus calculator home screen, type the following commands:

2 STO> y: 0 STO> x:y (enter)

y+0.5*$(y + e^x - 2)$ STO> y: x+0.5 STO>x: y (enter, 4 times)

The last value displayed gives $y_{\text{Euler}}(2) \approx 9.82187$

The exact solution: $\frac{dy}{dx} - y = e^x - 2 \Rightarrow P(x) = -1$, $Q(x) = e^x - 2 \Rightarrow \int P(x)\ dx = -x \Rightarrow v(x) = e^{-x}$

$\Rightarrow y = \frac{1}{e^{-x}} \int e^{-x}(e^x - 2)\ dx = e^x(x + 2e^{-x} + C)$; $y(0) = 2 \Rightarrow 2 = 2 + C \Rightarrow C = 0$

$\Rightarrow y = xe^x + 2 \Rightarrow y_{exact}(2) = 2e^2 + 2 \approx 16.7781$

19. $y_1 = -1 + \left[\frac{(-1)^2}{\sqrt{1}}\right](.5) = -.5,$

$y_2 = -.5 + \left[\frac{(-.5)^2}{\sqrt{1.5}}\right](.5) = -.39794,$

$y_3 = -.39794 + \left[\frac{(-.39794)^2}{\sqrt{2}}\right](.5) = -.34195,$

$y_4 = -.34195 + \left[\frac{(-.34195)^2}{\sqrt{2.5}}\right](.5) = -.30497,$

$y_5 = -.27812,\ y_6 = -.25745,\ y_7 = -.24088,\ y_8 = -.2272;$

$\frac{dy}{y^2} = \frac{dx}{\sqrt{x}} \Rightarrow -\frac{1}{y} = 2\sqrt{x} + C;\ y(1) = -1 \Rightarrow 1 = 2 + C \Rightarrow C = -1 \Rightarrow y = \frac{1}{1-2\sqrt{x}} \Rightarrow y(5) = \frac{1}{1-2\sqrt{5}} \approx -.2880$

20. $y_1 = 1 + (1 - e^0)\left(\frac{1}{3}\right) = 1,$

$y_2 = 1 + (1 - e^{2/3})\left(\frac{1}{3}\right) = 0.68409,$

$y_3 = 0.68409 + (0.68409 - e^{4/3})\left(\frac{1}{3}\right) = -0.35244,$

$y_4 = -0.35244 + (-0.35244 - e^{6/3})\left(\frac{1}{3}\right) = -2.93294,$

$y_5 = -2.93294 + (-2.93294 - e^{8/3})\left(\frac{1}{3}\right) = -8.70789,$

$y_6 = -8.70789 + (-8.70789 - e^{10/3})\left(\frac{1}{3}\right) = -20.95439;$

$y' - y = -e^{2x} \Rightarrow P(x) = -1,\ Q(x) = -e^{2x} \Rightarrow \int P(x)\,dx = -x \Rightarrow v(x) = e^{-x} \Rightarrow y = \frac{1}{e^{-x}} \int e^{-x}(-e^{2x})\,dx$

$= e^x(-e^x + C);\ y(0) = 1 \Rightarrow 1 = -1 + C \Rightarrow C = 2 \Rightarrow y = -e^{2x} + 2e^x \Rightarrow y(2) = -e^4 + 2e^2 \approx -39.8200$

21. (a) $\frac{dy}{dx} = 2y^2(x-1) \Rightarrow \frac{dy}{y^2} = 2(x-1)\,dx \Rightarrow \int y^{-2}\,dy = \int (2x-2)\,dx \Rightarrow -y^{-1} = x^2 - 2x + C$

Initial value: $y(2) = -\frac{1}{2} \Rightarrow 2 = 2^2 - 2(2) + C \Rightarrow C = 2$

Solution: $-y^{-1} = x^2 - 2x + 2$ or $y = -\frac{1}{x^2 - 2x + 2}$

$y(3) = -\frac{1}{3^2 - 2(3) + 2} = -\frac{1}{5} = -0.2$

(b) To find the approximation, set $y_1 = 2y^2(x-1)$ and use EULERT with initial values $x = 2$ and $y = -\frac{1}{2}$ and step size 0.2 for 5 points. This gives $y(3) \approx -0.1851$; error ≈ 0.0149.

(c) Use step size 0.1 for 10 points. This gives $y(3) \approx -0.1929$; error ≈ 0.0071.

(d) Use step size 0.05 for 20 points. This gives $y(3) \approx -0.1965$; error ≈ 0.0035.

22. (a) $\frac{dy}{dx} = y - 1 \Rightarrow \int \frac{dy}{y-1} = \int dx \Rightarrow \ln|y-1| = x + C \Rightarrow |y-1| = e^{x+C} \Rightarrow y - 1 = \pm e^C e^x$

$\Rightarrow y = Ae^x + 1$

Initial condition: $y(0) = 3 \Rightarrow 3 = Ae^0 + 1 \Rightarrow A = 2$

Solution: $y = 2e^x + 1$

$y(1) = 2e + 1 \approx 6.4366$

(b) To find the approximation, set $y_1 = y - 1$ and use a graphing calculator or CAS with initial values $x = 0$ and $y = 3$ and step size 0.2 for 5 points. This gives $y(1) \approx 5.9766$; error ≈ 0.4599.

(c) Use step size 0.1 for 10 points. This gives $y(1) \approx 6.1875$; error ≈ 0.2491.

(d) Use step size 0.05 for 20 points. This gives $y(1) \approx 6.3066$; error ≈ 0.1300.

23. The exact solution is $y = \frac{-1}{x^2 - 2x + 2}$, so $y(3) = -0.2$. To find the approximation, let $z_n = y_{n-1} + 2y_{n-1}^2(x_n - 1)\,dx$ and $y_n = y_{n-1} + (y_{n-1}^2(x_{n-1} - 1) + z_n^2(x_n^2 - 1))\,dx$ with initial values $x_0 = 2$ and $y_0 = -\frac{1}{2}$. Use a spreadsheet, graphing calculator, or CAS as indicated in parts (a) through (d).

(a) Use $dx = 0.2$ with 5 steps to obtain $y(3) \approx -0.2024 \Rightarrow$ error ≈ 0.0024.

(b) Use $dx = 0.1$ with 10 steps to obtain $y(3) \approx -0.2005 \Rightarrow$ error ≈ 0.0005.

(c) Use $dx = 0.05$ with 20 steps to obtain $y(3) \approx -0.2001 \Rightarrow$ error ≈ 0.0001.

(d) Each time the step size is cut in half, the error is reduced to approximately one-fourth of what it was for the larger step size.

24. The exact solution is $y = 2e^x + 1$, so $y(1) = 2e^1 + 1 \approx 6.4366$. To find the approximate solution let $z_n = y_{n-1} + (y_{n-1} - 1)\,dx$ and $y_n = y_{n-1} + \left(\frac{y_{n-1} + z_n - 2}{2}\right)dx$ with initial value $y_n = 3$. Use a spreadsheet, graphing calculator, or CAS as indicated in parts (a) through (d).

(a) Use $dx = 0.2$ with 5 steps to obtain $y(1) \approx 6.4054 \Rightarrow$ error ≈ 0.0311.

(b) Use $dx = 0.1$ with 10 steps to obtain $y(1) \approx 6.4282 \Rightarrow$ error ≈ 0.0084.

(c) Use $dx = 0.05$ with 20 steps to obtain $y(1) \approx 6.4344 \Rightarrow$ error ≈ 0.0022.

(d) Each time the step size is cut in half, the error is reduced to approximately one-fourth of what it was for the larger step size.

25-30. Example CAS commands:

Maple:

```
with(plots): with(DEtools):
a:=-4; b:=4;
eq:= D(y)(x) = x + y;
plot1:= dfieldplot(eq,[x,y], x=a..b, y=-4..4, scaling=CONSTRAINED);
display({plot1});
gen_sol:= dsolve({eq},y(x));
tograph:= {seq(subs(_C1=i, gen_sol), i = {-1,0,1,3,9})};
plot2:=implicitplot(tograph, x=a..b, y=-4..4, scaling=CONSTRAINED):
display({plot1,plot2}, title = `Direction Field and Solution Curves`);
eulerapprox:= proc(f,x0,y0,n) local i,j,h;
  x(0):= evalf(x0);
```

```
  y(0):= evalf(y0);
     h:= (b-a)/n;
for i from 1 to n do
  y(i):= evalf(y(i-1) + h*f(x(i-1),y(i-1))):  x(i):= x(i-1) + h  od;
[[x(j),y(j)] $j=0..n];
end;
rhs(eq);
f:= unapply(%,(x,y));
eulerapprox(f,0,-7/10,4);
plot3:= plot(%, style=LINE,scaling=CONSTRAINED, title=`Euler Approximation`):
display({plot3});
y(0):= 'y(0)';
partsol:= dsolve({eq, y(0)=-7/10}, y(x));
plot4: implicitplot(partsol,x=-1..8,y=-3..40,scaling=CONSTRAINED):
display({plot3, plot4}, title=`Actual Solution & Euler Approximation`);
```

Mathematica:

```
Need package for plotting vector fields:
<<  "Graphics`PlotField`"
Also load package to improve solving of ODE's:
<< Calculus`DSolve`
SetOptions[PlotVectorField, PlotPoints -> 6];
Clear[x,y,yp,h]
Note: here we define "eulerstep" to find the next Euler point, given the
current one, assuming that the variables "a" (initial point), "b" (final
point), and "n" (# of steps) have been defined, along with the function
"yp[x,y]" (which specifies the derivative).  Then the whole Euler solution
is given as a list of points {x,y} by:
  NestList[ eulerstep, N[{a,ya}], n ]
where "ya" is the initial value.
eulerstep[{x_,y_}]  :=  {x+h, y+h*N[yp[x,y]]  }
h  :=  N[(b-a)/n]

yp[x_,y_] := x + y
{a,b} = {0,1}; ya = 1;
{xmin,xmax} = {-4,4}; {ymin,ymax} = {-4,4};
p1 = PlotVectorField[{1,yp[x,y]},{x,xmin,xmax},{y,ymin,ymax},
 ScaleFunction -> (1&) ]
ode = y'[x] == yp[x,y[x]]
DSolve[ ode, y[x], x ]
gensol = y[x] /. First[%]
sols = Map[ (gensol /. C[1] -> #)&, {-2,-1,0,1,2} ]
p2 = Plot[ Evaluate[sols], {x,xmin,xmax} ]
Show[ {p1, p2}, PlotRange -> {Automatic,{ymin,ymax}} ]
DSolve[ {ode, y[a] == ya}, y[x], x ]
partsol = y[x] /. First[%]
p3 = Plot[ partsol, {x,a,b} ]
n = 10;
approx1 = NestList[ eulerstep, N[{a,ya}], n ];
p4 = ListPlot[ approx1, PlotJoined -> True ]
Show[{p4,p3)}]

Here's an alternate approach to plotting the two solutions (simpler but less
obvious):
```

```
Show[p3, Epilog -> {Line[approx]} ]

n = 25;
approx2 = NestList[ eulerstep, N[{a,ya}], n ];
p4 = ListPlot[ approx2, PlotJoined -> True ]
Show[{p4,p3}]
n = 50;
approx3 = NestList[ eulerstep, N[{a,ya}], n ];
p4 = ListPlot[ approx3, PlotJoined -> True ]
Show[{p4,p3}]
n = 100;
approx4 = NestList[ eulerstep, N[{a,ya}], n ];
p4 = ListPlot[ approx4, PlotJoined -> True ]
Show[{p4,p3}]
yb = partsol /. x -> b // N
err1 = Last[approx1][[2]] - yb
percent1 = err1/yb * 100
err2 = Last[approx2][[2]] - yb
percent2 = err2/yb * 100
err3 = Last[approx3][[2]] - yb
percent3 = err3/yb * 100
err4 = Last[approx4][[2]] - yb
percent4 = err4/yb * 100
```

31. Example CAS commands:

Maple:

```
with(plots): with(DEtools):
eq:= D(y)(x) = f - y;
eq1:= subs(f = 2*x, eq);
eq2:= subs(f=sin(2*x), eq);
eq3:= subs(f=3*exp(x/2), eq);
eq4:= subs(f=2*exp(-x/2)*cos(2*x), eq);
partsol1:= dsolve({eq1,y(0)=0}, y(x));
plot1:= implicitplot(partsol1, x=-2..6, y=-1..10, scaling=CONSTRAINED):
display(plot1);
partsol2:= dsolve({eq2, y(0)=0}, y(x));
plot2:= implicitplot(partsol2, x=-2..6, y=-1..4, scaling=CONSTRAINED):
display(plot2);
partsol3:= dsolve({eq3, y(0)=0}, y(x));
plot3:= implicitplot(partsol3, x=-2..6, y=-2..10, scaling=CONSTRAINED):
display(plot3);
partsol4:= dsolve({eq4, y(0)=0}, y(x));
plot4:= implicitplot(partsol4, x=-2..6, y=-3..2, scaling=CONSTRAINED):
display(plot4);
display({plot1,plot2,plot3,plot4});
```

Mathematica:

```
Clear[x,y,f]
ode = y'[x] + y[x] == f[x]
a = 0; ya = 0;
{xmin,xmax} = {-2,6};
f[x_] = 2x
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol1 = y[x] /. First[%]
```

```
Plot[ sol1, {x,xmin,xmax} ]
f[x_] = Sin[2x]
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol2 = y[x] /. First[%]
Plot[ sol2, {x,xmin,xmax} ]
f[x_] = 3 Exp[x/2]
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol3 = y[x] /. First[%]
Plot[ sol3, {x,xmin,xmax} ]
f[x_] = 2 Exp[-x/2] Cos[2x]
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol4 = y[x] /. First[%]
Plot[ sol4, {x,xmin,xmax} ]
Plot[ {sol1, sol2, sol3, sol4}, {x,xmin,xmax} ]
```

32. Example CAS commands:

Maple:

```
with(plots): with(DEtools):
a:=-3; b:=3;
eq:=D(y)(x)=(3*x^2+4*x+2)/(2*(y-1));
plot1:=dfieldplot(eq,[x,y],x=a..b,y=a..b,scaling=CONSTRAINED):
display({plot1});
right:=int(numer(rhs(eq)),x);
left:=int(denom(rhs(eq)),y);
sol:=left = right + C;
tograph:={seq(subs(C=i,sol),i={-6,-4,-2,0,2,4,6})};
plot2:=implicitplot(tograph,x=a..b,y=a..b, scaling=CONSTRAINED):
display(plot2);
DEplot(eq,y(x),x=a..b,{[0,-1]},y=a..b);
```

Mathematica:

```
yp[x_,y_] := (3x^2 + 4x + 2) / (2(y-1))
a = 0; ya = -1;
{xmin,xmax} = {-3,3}; {ymin,ymax} = {-3,3};
p1 = PlotVectorField[{1,yp[x,y]}, {x,xmin,xmax}, {y,ymin,ymax},
  ScaleFunction -> (1&) ]
impeqn =
  Integrate[Denominator[yp[x,y]],y] ==
  Integrate[Numerator[yp[x,y]],x] + C[1]
<< Graphics`ImplicitPlot`
eqns = Map[ (impeqn /. C[1] -> #)&,
  {-6,-4,-2,0,2,4,6} ];
p2 = ImplicitPlot[ Evaluate[eqns], {x,xmin,xmax} ]
Show[ {p1, p2} ]
impeqn /. {x -> 0, y -> -1}
Solve[%,C[1]]
parteqn = impeqn /. First[%]
ImplicitPlot[ Evaluate[parteqn], {x,xmin,xmax} ]
```

6.7 HYPERBOLIC FUNCTIONS

1. $\sinh x = -\frac{3}{4} \Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\left(-\frac{3}{4}\right)^2} = \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(-\frac{3}{4}\right)}{\left(\frac{5}{4}\right)} = -\frac{3}{5}$, $\coth x = \frac{1}{\tanh x} = -\frac{5}{3}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{4}{5}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = -\frac{4}{3}$

2. $\sinh x = \frac{4}{3} \Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{5}{3}\right)} = \frac{4}{5}$, $\coth x = \frac{1}{\tanh x} = \frac{5}{4}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{3}{5}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{3}{4}$

3. $\cosh x = \frac{17}{15}$, $x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \sqrt{\frac{289}{225} - 1} = \sqrt{\frac{64}{225}} = \frac{8}{15}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{8}{15}\right)}{\left(\frac{17}{15}\right)} = \frac{8}{17}$, $\coth x = \frac{1}{\tanh x} = \frac{17}{8}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{15}{17}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{15}{8}$

4. $\cosh x = \frac{13}{5}$, $x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\frac{169}{25} - 1} = \sqrt{\frac{144}{25}} = \frac{12}{5}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{12}{5}\right)}{\left(\frac{13}{5}\right)} = \frac{12}{13}$, $\coth x = \frac{1}{\tanh x} = \frac{13}{12}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{5}{13}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{5}{12}$

5. $2\cosh(\ln x) = 2\left(\frac{e^{\ln x} + e^{-\ln x}}{2}\right) = e^{\ln x} + \frac{1}{e^{\ln x}} = x + \frac{1}{x}$

6. $\sinh(2\ln x) = \frac{e^{2\ln x} - e^{-2\ln x}}{2} = \frac{e^{\ln x^2} - e^{\ln x^{-2}}}{2} = \frac{\left(x^2 - \frac{1}{x^2}\right)}{2} = \frac{x^4 - 1}{2x^2}$

7. $\cosh 5x + \sinh 5x = \frac{e^{5x} + e^{-5x}}{2} + \frac{e^{5x} - e^{-5x}}{2} = e^{5x}$

8. $\cosh 3x - \sinh 3x = \frac{e^{3x} + e^{-3x}}{2} - \frac{e^{3x} - e^{-3x}}{2} = e^{-3x}$

9. $(\sinh x + \cosh x)^4 = \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}\right)^4 = \left(e^x\right)^4 = e^{4x}$

10. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) = \ln\left(\cosh^2 x - \sinh^2 x\right) = \ln 1 = 0$

11. (a) $\sinh 2x = \sinh(x + x) = \sinh x \cosh x + \cosh x \sinh x = 2\sinh x \cosh x$

(b) $\cosh 2x = \cosh(x + x) = \cosh x \cosh x + \sinh x \sin x = \cosh^2 x + \sinh^2 x$

12. $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left[\left(e^x + e^{-x}\right) + \left(e^x - e^{-x}\right)\right]\left[\left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right)\right]$
$= \frac{1}{4}\left(2e^x\right)\left(2e^{-x}\right) = \frac{1}{4}\left(4e^0\right) = \frac{1}{4}(4) = 1$

13. $y = 6\sinh\frac{x}{3} \Rightarrow \frac{dy}{dx} = 6\left(\cosh\frac{x}{3}\right)\left(\frac{1}{3}\right) = 2\cosh\frac{x}{3}$

14. $y = \frac{1}{2}\sinh(2x+1) \Rightarrow \frac{dy}{dx} = \frac{1}{2}[\cosh(2x+1)](2) = \cosh(2x+1)$

15. $y = 2\sqrt{t}\tanh\sqrt{t} = 2t^{1/2}\tanh t^{1/2} \Rightarrow \frac{dy}{dt} = \left[\operatorname{sech}^2\left(t^{1/2}\right)\right]\left(\frac{1}{2}t^{-1/2}\right)\left(2t^{1/2}\right) + \left(\tanh t^{1/2}\right)\left(t^{-1/2}\right)$

$= \operatorname{sech}^2\sqrt{t} + \frac{\tanh\sqrt{t}}{\sqrt{t}}$

16. $y = t^2\tanh\frac{1}{t} = t^2\tanh t^{-1} \Rightarrow \frac{dy}{dt} = \left[\operatorname{sech}^2\left(t^{-1}\right)\right]\left(-t^{-2}\right)\left(t^2\right) + (2t)\left(\tanh t^{-1}\right) = -\operatorname{sech}^2\frac{1}{t} + 2t\tanh\frac{1}{t}$

17. $y = \ln(\sinh z) \Rightarrow \frac{dy}{dz} = \frac{\cosh z}{\sinh z} = \coth z$

18. $y = \ln(\cosh z) \Rightarrow \frac{dy}{dz} = \frac{\sinh z}{\cosh z} = \tanh z$

19. $y = (\operatorname{sech}\theta)(1 - \ln\operatorname{sech}\theta) \Rightarrow \frac{dy}{d\theta} = \left(-\frac{-\operatorname{sech}\theta\tanh\theta}{\operatorname{sech}\theta}\right)(\operatorname{sech}\theta) + (-\operatorname{sech}\theta\tanh\theta)(1 - \ln\operatorname{sech}\theta)$

$= \operatorname{sech}\theta\tanh\theta - (\operatorname{sech}\theta\tanh\theta)(1 - \ln\operatorname{sech}\theta) = (\operatorname{sech}\theta\tanh\theta)[1 - (1 - \ln\operatorname{sech}\theta)]$

$= (\operatorname{sech}\theta\tanh\theta)(\ln\operatorname{sech}\theta)$

20. $y = (\operatorname{csch}\theta)(1 - \ln\operatorname{csch}\theta) \Rightarrow \frac{dy}{d\theta} = (\operatorname{csch}\theta)\left(-\frac{-\operatorname{csch}\theta\coth\theta}{\operatorname{csch}\theta}\right) + (1 - \ln\operatorname{csch}\theta)(-\operatorname{csch}\theta\coth\theta)$

$= \operatorname{csch}\theta\coth\theta - (1 - \ln\operatorname{csch}\theta)(\operatorname{csch}\theta\coth\theta) = (\operatorname{csch}\theta\coth\theta)(1 - 1 + \ln\operatorname{csch}\theta) = (\operatorname{csch}\theta\coth\theta)(\ln\operatorname{csch}\theta)$

21. $y = \ln\cosh v - \frac{1}{2}\tanh^2 v \Rightarrow \frac{dy}{dv} = \frac{\sinh v}{\cosh v} - \left(\frac{1}{2}\right)(2\tanh v)\left(\operatorname{sech}^2 v\right) = \tanh v - (\tanh v)\left(\operatorname{sech}^2 v\right)$

$= (\tanh v)\left(1 - \operatorname{sech}^2 v\right) = (\tanh v)\left(\tanh^2 v\right) = \tanh^3 v$

22. $y = \ln\sinh v - \frac{1}{2}\coth^2 v \Rightarrow \frac{dy}{dv} = \frac{\cosh v}{\sinh v} - \left(\frac{1}{2}\right)(2\coth v)\left(-\operatorname{csch}^2 v\right) = \coth v + (\coth v)\left(\operatorname{csch}^2 v\right)$

$= (\coth v)\left(1 + \operatorname{csch}^2 v\right) = (\coth v)\left(\coth^2 v\right) = \coth^3 v$

23. $y = \left(x^2+1\right)\operatorname{sech}(\ln x) = \left(x^2+1\right)\left(\frac{2}{e^{\ln x} + e^{-\ln x}}\right) = \left(x^2+1\right)\left(\frac{2}{x + x^{-1}}\right) = \left(x^2+1\right)\left(\frac{2x}{x^2+1}\right) = 2x \Rightarrow \frac{dy}{dx} = 2$

24. $y = \left(4x^2-1\right)\operatorname{csch}(\ln 2x) = \left(4x^2-1\right)\left(\frac{2}{e^{\ln 2x} - e^{-\ln 2x}}\right) = \left(4x^2-1\right)\left(\frac{2}{2x - (2x)^{-1}}\right) = \left(4x^2-1\right)\left(\frac{4x}{4x^2-1}\right)$

$= 4x \Rightarrow \frac{dy}{dx} = 4$

25. $y = \sinh^{-1}\sqrt{x} = \sinh^{-1}\left(x^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)x^{-1/2}}{\sqrt{1 + \left(x^{1/2}\right)^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}} = \frac{1}{2\sqrt{x(1+x)}}$

26. $y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1}\left(2(x+1)^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{(2)\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{\left[2(x+1)^{1/2}\right]^2 - 1}} = \frac{1}{\sqrt{x+1}\sqrt{4x+3}} = \frac{1}{\sqrt{4x^2+7x+3}}$

27. $y = (1-\theta)\tanh^{-1}\theta \Rightarrow \frac{dy}{d\theta} = (1-\theta)\left(\frac{1}{1-\theta^2}\right) + (-1)\tanh^{-1}\theta = \frac{1}{1+\theta} - \tanh^{-1}\theta$

28. $y = \left(\theta^2+2\theta\right)\tanh^{-1}(\theta+1) \Rightarrow \frac{dy}{d\theta} = \left(\theta^2+2\theta\right)\left[\frac{1}{1-(\theta+1)^2}\right] + (2\theta+2)\tanh^{-1}(\theta+1)$

$= \frac{\theta^2+2\theta}{-\theta^2-2\theta} + (2\theta+2)\tanh^{-1}(\theta+1) = (2\theta+2)\tanh^{-1}(\theta+1) - 1$

29. $y = (1-t)\coth^{-1}\sqrt{t} = (1-t)\coth^{-1}\left(t^{1/2}\right) \Rightarrow \frac{dy}{dt} = (1-t)\left[\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1-\left(t^{1/2}\right)^2}\right] + (-1)\coth^{-1}\left(t^{1/2}\right) = \frac{1}{2\sqrt{t}} - \coth^{-1}\sqrt{t}$

30. $y = \left(1-t^2\right)\coth^{-1}t \Rightarrow \frac{dy}{dt} = \left(1-t^2\right)\left(\frac{1}{1-t^2}\right) + (-2t)\coth^{-1}t = 1 - 2t\coth^{-1}t$

31. $y = \cos^{-1}x - x\,\mathrm{sech}^{-1}x \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} - \left[x\left(\frac{-1}{x\sqrt{1-x^2}}\right) + (1)\,\mathrm{sech}^{-1}x\right] = \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} - \mathrm{sech}^{-1}x$

$= -\,\mathrm{sech}^{-1}x$

32. $y = \ln x + \sqrt{1-x^2}\,\mathrm{sech}^{-1}x = \ln x + \left(1-x^2\right)^{1/2}\mathrm{sech}^{-1}x \Rightarrow \frac{dy}{dx}$

$= \frac{1}{x} + \left(1-x^2\right)^{1/2}\left(\frac{-1}{x\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)\,\mathrm{sech}^{-1}x = \frac{1}{x} - \frac{1}{x} - \frac{x}{\sqrt{1-x^2}}\,\mathrm{sech}^{-1}x = \frac{-x}{\sqrt{1-x^2}}\,\mathrm{sech}^{-1}x$

33. $y = \mathrm{csch}^{-1}\left(\frac{1}{2}\right)^\theta \Rightarrow \frac{dy}{d\theta} = -\frac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^\theta}{\left(\frac{1}{2}\right)^\theta\sqrt{1+\left[\left(\frac{1}{2}\right)^\theta\right]^2}} = -\frac{\ln(1)-\ln(2)}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}} = \frac{\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$

34. $y = \mathrm{csch}^{-1}2^\theta \Rightarrow \frac{dy}{d\theta} = -\frac{(\ln 2)\,2^\theta}{2^\theta\sqrt{1+\left(2^\theta\right)^2}} = \frac{-\ln 2}{\sqrt{1+2^{2\theta}}}$

35. $y = \sinh^{-1}(\tan x) \Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1+(\tan x)^2}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{|\sec x|} = \frac{|\sec x|\,|\sec x|}{|\sec x|} = |\sec x|$

36. $y = \cosh^{-1}(\sec x) \Rightarrow \frac{dy}{dx} = \frac{(\sec x)(\tan x)}{\sqrt{\sec^2 x - 1}} = \frac{(\sec x)(\tan x)}{\sqrt{\tan^2 x}} = \frac{(\sec x)(\tan x)}{|\tan x|} = \sec x,\ 0 < x < \frac{\pi}{2}$

37. (a) If $y = \tan^{-1}(\sinh x) + C$, then $\frac{dy}{dx} = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \mathrm{sech}\,x$, which verifies the formula

(b) If $y = \sin^{-1}(\tanh x) + C$, then $\frac{dy}{dx} = \frac{\mathrm{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\mathrm{sech}^2 x}{\mathrm{sech}\,x} = \mathrm{sech}\,x$, which verifies the formula

38. If $y = \frac{x^2}{2}\,\mathrm{sech}^{-1}x - \frac{1}{2}\sqrt{1-x^2} + C$, then $\frac{dy}{dx} = x\,\mathrm{sech}^{-1}x + \frac{x^2}{2}\left(\frac{-1}{x\sqrt{1-x^2}}\right) + \frac{2x}{4\sqrt{1-x^2}} = x\,\mathrm{sech}^{-1}x$,

which verifies the formula

39. If $y = \frac{x^2-1}{2}\coth^{-1}x + \frac{x}{2} + C$, then $\frac{dy}{dx} = x\coth^{-1}x + \left(\frac{x^2-1}{2}\right)\left(\frac{1}{1-x^2}\right) + \frac{1}{2} = x\coth^{-1}x$, which verifies the formula

40. If $y = x\tanh^{-1}x + \frac{1}{2}\ln\left(1-x^2\right) + C$, then $\frac{dy}{dx} = \tanh^{-1}x + x\left(\frac{1}{1-x^2}\right) + \frac{1}{2}\left(\frac{-2x}{1-x^2}\right) = \tanh^{-1}x$, which verifies the formula

41. $\int \sinh 2x\, dx = \frac{1}{2}\int \sinh u\, du$, where $u = 2x$ and $du = 2\, dx$

$= \frac{\cosh u}{2} + C = \frac{\cosh 2x}{2} + C$

42. $\int \sinh \frac{x}{5}\, dx = 5\int \sinh u\, du$, where $u = \frac{x}{5}$ and $du = \frac{1}{5}\, dx$

$= 5\cosh u + C = 5\cosh\frac{x}{5} + C$

43. $\int 6\cosh\left(\frac{x}{2} - \ln 3\right) dx = 12\int \cosh u\, du$, where $u = \frac{x}{2} - \ln 3$ and $du = \frac{1}{2}\, dx$

$= 12\sinh u + C = 12\sinh\left(\frac{x}{2} - \ln 3\right) + C$

44. $\int 4\cosh(3x - \ln 2)\, dx = \frac{4}{3}\int \cosh u\, du$, where $u = 3x - \ln 2$ and $du = 3\, dx$

$= \frac{4}{3}\sinh u + C = \frac{4}{3}\sinh(3x - \ln 2) + C$

45. $\int \tanh\frac{x}{7}\, dx = 7\int \frac{\sinh u}{\cosh u}\, du$, where $u = \frac{x}{7}$ and $du = \frac{1}{7}\, dx$

$= 7\ln|\cosh u| + C_1 = 7\ln\left|\cosh\frac{x}{7}\right| + C_1 = 7\ln\left|\frac{e^{x/7}+e^{-x/7}}{2}\right| + C_1 = 7\ln\left|e^{x/7}+e^{-x/7}\right| - 7\ln 2 + C_1$

$= 7\ln\left(e^{x/7}+e^{-x/7}\right) + C$, since $e^{x/7}+e^{-x/7} > 0$ for all x.

46. $\int \coth\frac{\theta}{\sqrt{3}}\, d\theta = \sqrt{3}\int \frac{\cosh u}{\sinh u}\, du$, where $u = \frac{\theta}{\sqrt{3}}$ and $du = \frac{d\theta}{\sqrt{3}}$

$= \sqrt{3}\ln|\sinh u| + C_1 = \sqrt{3}\ln\left|\sinh\frac{\theta}{\sqrt{3}}\right| + C_1 = \sqrt{3}\ln\left|\frac{e^{\theta/\sqrt{3}}-e^{-\theta/\sqrt{3}}}{2}\right| + C_1$

$= \sqrt{3}\ln\left|e^{\theta/\sqrt{3}}-e^{-\theta/\sqrt{3}}\right| - \sqrt{3}\ln 2 + C_1 = \sqrt{3}\ln\left|e^{\theta/\sqrt{3}}-e^{-\theta/\sqrt{3}}\right| + C$

47. $\int \operatorname{sech}^2\left(x-\frac{1}{2}\right) dx = \int \operatorname{sech}^2 u\, du$, where $u = \left(x-\frac{1}{2}\right)$ and $du = dx$

$= \tanh u + C = \tan\left(x-\frac{1}{2}\right) + C$

48. $\int \operatorname{csch}^2(5-x)\, dx = -\int \operatorname{csch}^2 u\, du$, where $u = (5-x)$ and $du = -dx$

$= -(-\coth u) + C = \coth u + C = \coth(5-x) + C$

49. $\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}}{\sqrt{t}}\,dt = 2\int \operatorname{sech} u \tanh u\,du$, where $u = \sqrt{t} = t^{1/2}$ and $du = \frac{dt}{2\sqrt{t}}$

$= 2(-\operatorname{sech} u) + C = -2\operatorname{sech}\sqrt{t} + C$

50. $\int \frac{\operatorname{csch}(\ln t)\coth(\ln t)}{t}\,dt = \int \operatorname{csch} u \coth u\,du$, where $u = \ln t$ and $du = \frac{dt}{t}$

$= -\operatorname{csch} u + C = -\operatorname{csch}(\ln t) + C$

51. $\int_{\ln 2}^{\ln 4} \coth x\,dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\,dx = \int_{3/4}^{15/8} \frac{1}{u}\,du = [\ln|u|]_{3/4}^{15/8} = \ln\left|\frac{15}{8}\right| - \ln\left|\frac{3}{4}\right| = \ln\left|\frac{15}{8}\cdot\frac{4}{3}\right| = \ln\frac{5}{2}$,

where $u = \sinh x$, $du = \cosh x\,dx$, the lower limit is $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2-\left(\frac{1}{2}\right)}{2} = \frac{3}{4}$ and the upper

limit is $\sinh(\ln 4) = \frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4-\left(\frac{1}{4}\right)}{2} = \frac{15}{8}$

52. $\int_0^{\ln 2} \tanh 2x\,dx = \int_0^{\ln 2} \frac{\sinh 2x}{\cosh 2x}\,dx = \frac{1}{2}\int_1^{17/8} \frac{1}{u}\,du = \frac{1}{2}[\ln|u|]_1^{17/8} = \frac{1}{2}\left[\ln\left(\frac{17}{8}\right) - \ln 1\right] = \frac{1}{2}\ln\frac{17}{8}$, where

$u = \cosh 2x$, $du = 2\sinh(2x)\,dx$, the lower limit is $\cosh 0 = 1$ and the upper limit is $\cosh(2\ln 2) = \cosh(\ln 4)$

$= \frac{e^{\ln 4} + e^{-\ln 4}}{2} = \frac{4+\left(\frac{1}{4}\right)}{2} = \frac{17}{8}$

53. $\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\,d\theta = \int_{-\ln 4}^{-\ln 2} 2e^{\theta}\left(\frac{e^{\theta}+e^{-\theta}}{2}\right)d\theta = \int_{-\ln 4}^{-\ln 2} \left(e^{2\theta}+1\right)d\theta = \left[\frac{e^{2\theta}}{2} + \theta\right]_{-\ln 4}^{-\ln 2}$

$= \left(\frac{e^{-2\ln 2}}{2} - \ln 2\right) - \left(\frac{e^{-2\ln 4}}{2} - \ln 4\right) = \left(\frac{1}{8} - \ln 2\right) - \left(\frac{1}{32} - \ln 4\right) = \frac{3}{32} - \ln 2 + 2\ln 2 = \frac{3}{32} + \ln 2$

54. $\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\,d\theta = \int_0^{\ln 2} 4e^{-\theta}\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)d\theta = 2\int_0^{\ln 2} \left(1-e^{-2\theta}\right)d\theta = 2\left[\theta + \frac{e^{-2\theta}}{2}\right]_0^{\ln 2}$

$= 2\left[\left(\ln 2 + \frac{e^{-2\ln 2}}{2}\right) - \left(0 + \frac{e^{-0}}{2}\right)\right] = 2\left(\ln 2 + \frac{1}{8} - \frac{1}{2}\right) = 2\ln 2 + \frac{1}{4} - 1 = \ln 4 - \frac{3}{4}$

55. $\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta\,d\theta = \int_{-1}^{1} \cosh u\,du = [\sinh u]_{-1}^{1} = \sinh(1) - \sinh(-1) = \left(\frac{e^1 - e^{-1}}{2}\right) - \left(\frac{e^{-1} - e^1}{2}\right)$

$= \frac{e - e^{-1} - e^{-1} + e}{2} = e - e^{-1}$, where $u = \tan\theta$, $du = \sec^2\theta\,d\theta$, the lower limit is $\tan\left(-\frac{\pi}{4}\right) = -1$ and the upper limit is $\tan\left(\frac{\pi}{4}\right) = 1$

56. $\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\, d\theta = 2\int_0^1 \sinh u\, du = 2[\cosh u]_0^1 = 2(\cosh 1 - \cosh 0) = 2\left(\frac{e+e^{-1}}{2} - 1\right)$
$= e + e^{-1} - 2$, where $u = \sin\theta$, $du = \cos\theta\, d\theta$, the lower limit is $\sin 0 = 0$ and the upper limit is $\sin\left(\frac{\pi}{2}\right) = 1$

57. $\int_1^2 \frac{\cosh(\ln t)}{t}\, dt = \int_0^{\ln 2} \cosh u\, du = [\sinh u]_0^{\ln 2} = \sinh(\ln 2) - \sinh(0) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} - 0 = \frac{2-\frac{1}{2}}{2} = \frac{3}{4}$, where $u = \ln t$, $du = \frac{1}{t}\, dt$, the lower limit is $\ln 1 = 0$ and the upper limit is $\ln 2$

58. $\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}}\, dx = 16\int_1^2 \cosh u\, du = 16[\sinh u]_1^2 = 16(\sinh 2 - \sinh 1) = 16\left[\left(\frac{e^2 - e^{-2}}{2}\right) - \left(\frac{e - e^{-1}}{2}\right)\right]$
$= 8\left(e^2 - e^{-2} - e + e^{-1}\right)$, where $u = \sqrt{x} = x^{1/2}$, $du = \frac{1}{2}x^{-1/2} = \frac{dx}{2\sqrt{x}}$, the lower limit is $\sqrt{1} = 1$ and the upper limit is $\sqrt{4} = 2$

59. $\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right) dx = \int_{-\ln 2}^{0} \frac{\cosh x + 1}{2}\, dx = \frac{1}{2}\int_{-\ln 2}^{0} (\cosh x + 1)\, dx = \frac{1}{2}[\sinh x + x]_{-\ln 2}^{0}$
$= \frac{1}{2}[(\sinh 0 + 0) - (\sinh(-\ln 2) - \ln 2)] = \frac{1}{2}\left[(0+0) - \left(\frac{e^{-\ln 2} - e^{\ln 2}}{2} - \ln 2\right)\right] = \frac{1}{2}\left[-\frac{\left(\frac{1}{2}\right) - 2}{2} + \ln 2\right]$
$= \frac{1}{2}\left(1 - \frac{1}{4} + \ln 2\right) = \frac{3}{8} + \frac{1}{2}\ln 2 = \frac{3}{8} + \ln\sqrt{2}$

60. $\int_0^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right) dx = \int_0^{\ln 10} 4\left(\frac{\cosh x - 1}{2}\right) dx = 2\int_0^{\ln 10} (\cosh x - 1)\, dx = 2[\sinh x - x]_0^{\ln 10}$
$= 2[(\sinh(\ln 10) - \ln 10) - (\sinh 0 - 0)] = e^{\ln 10} - e^{-\ln 10} - 2\ln 10 = 10 - \frac{1}{10} - 2\ln 10 = 9.9 - 2\ln 10$

61. $\sinh^{-1}\left(\frac{-5}{12}\right) = \ln\left(-\frac{5}{12} + \sqrt{\frac{25}{144} + 1}\right) = \ln\left(\frac{2}{3}\right)$

62. $\cosh^{-1}\left(\frac{5}{3}\right) = \ln\left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1}\right) = \ln 3$

63. $\tanh^{-1}\left(-\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1-(1/2)}{1+(1/2)}\right) = -\frac{\ln 3}{2}$

64. $\coth^{-1}\left(\frac{5}{4}\right) = \frac{1}{2}\ln\left(\frac{(9/4)}{(1/4)}\right) = \frac{1}{2}\ln 9 = \ln 3$

65. $\operatorname{sech}^{-1}\left(\frac{3}{5}\right) = \ln\left(\frac{1+\sqrt{1-(9/25)}}{(3/5)}\right) = \ln 3$

66. $\operatorname{csch}^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \ln\left(-\sqrt{3} + \frac{\sqrt{4/3}}{(1/\sqrt{3})}\right) = \ln\left(-\sqrt{3} + 2\right)$

67. (a) $\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}} = \left[\sinh^{-1}\frac{x}{2}\right]_0^{2\sqrt{3}} = \sinh^{-1}\sqrt{3} - \sinh 0 = \sinh^{-1}\sqrt{3}$

(b) $\sinh^{-1}\sqrt{3} = \ln\left(\sqrt{3} + \sqrt{3+1}\right) = \ln\left(\sqrt{3}+2\right)$

68. (a) $\int_0^{1/3} \frac{6\,dx}{\sqrt{1+9x^2}} = 2\int_0^1 \frac{dx}{\sqrt{a^2+u^2}}$, where $u = 3x$, $du = 3\,dx$, $a = 1$

$= \left[2\sinh^{-1} u\right]_0^1 = 2\left(\sinh^{-1} 1 - \sinh^{-1} 0\right) = 2\sin^{-1} 1$

(b) $2\sinh^{-1} 1 = 2\ln\left(1+\sqrt{1^2+1}\right) = 2\ln\left(1+\sqrt{2}\right)$

69. (a) $\int_{5/4}^{2} \frac{1}{1-x^2}\,dx = \left[\coth^{-1} x\right]_{5/4}^{2} = \coth^{-1} 2 - \coth^{-1}\frac{5}{4}$

(b) $\coth^{-1} 2 - \coth^{-1}\frac{5}{4} = \frac{1}{2}\left[\ln 3 - \ln\left(\frac{9/4}{1/4}\right)\right] = \frac{1}{2}\ln\frac{1}{3}$

70. (a) $\int_0^{1/2} \frac{1}{1-x^2}\,dx = \left[\tanh^{-1} x\right]_0^{1/2} = \tanh^{-1}\frac{1}{2} - \tanh^{-1} 0 = \tanh^{-1}\frac{1}{2}$

(b) $\tanh^{-1}\frac{1}{2} = \frac{1}{2}\ln\left(\frac{1+(1/2)}{1-(1/2)}\right) = \frac{1}{2}\ln 3$

71. (a) $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = \int_{4/5}^{12/13} \frac{du}{u\sqrt{a^2-u^2}}$, where $u = 4x$, $du = 4\,dx$, $a = 1$

$= \left[-\operatorname{sech}^{-1} u\right]_{4/5}^{12/13} = -\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5}$

(b) $-\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5} = -\ln\left(\frac{1+\sqrt{1-(12/13)^2}}{(12/13)}\right) + \ln\left(\frac{1+\sqrt{1-(4/5)^2}}{(4/5)}\right)$

$= -\ln\left(\frac{13+\sqrt{169-144}}{12}\right) + \ln\left(\frac{5+\sqrt{25-16}}{4}\right) = \ln\left(\frac{5+3}{4}\right) - \ln\left(\frac{13+5}{12}\right) = \ln 2 - \ln\frac{3}{2}$

$= \ln\left(2\cdot\frac{2}{3}\right) = \ln\frac{4}{3}$

72. (a) $\int_1^2 \frac{dx}{x\sqrt{4+x^2}} = \left[-\frac{1}{2}\operatorname{csch}^{-1}\left|\frac{x}{2}\right|\right]_1^2 = -\frac{1}{2}\left(\operatorname{csch}^{-1} 1 - \operatorname{csch}^{-1}\frac{1}{2}\right) = \frac{1}{2}\left(\operatorname{csch}^{-1}\frac{1}{2} - \operatorname{csch}^{-1} 1\right)$

(b) $\frac{1}{2}\left(\operatorname{csch}^{-1}\frac{1}{2} - \operatorname{csch}^{-1} 1\right) = \frac{1}{2}\left[\ln\left(2+\frac{\sqrt{5/4}}{(1/2)}\right) - \ln\left(1+\sqrt{2}\right)\right] = \frac{1}{2}\ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)$

73. (a) $\int_0^{\pi} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^0 \frac{1}{\sqrt{1+u^2}}\,du = \left[\sinh^{-1} u\right]_0^0 = \sinh^{-1} 0 - \sinh^{-1} 0 = 0$, where $u = \sin x$, $du = \cos x\,dx$

(b) $\sinh^{-1} 0 - \sinh^{-1} 0 = \ln\left(0 + \sqrt{0+1}\right) - \ln\left(0 + \sqrt{0+1}\right) = 0$

74. (a) $\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}} = \int_0^1 \frac{du}{\sqrt{a^2+u^2}}$, where $u = \ln x$, $du = \frac{1}{x}\,dx$, $a = 1$

$= \left[\sinh^{-1} u\right]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0 = \sinh^{-1} 1$

(b) $\sinh^{-1} 1 - \sinh^{-1} 0 = \ln\left(1 + \sqrt{1^2+1}\right) - \ln\left(0 + \sqrt{0^2+1}\right) = \ln\left(1+\sqrt{2}\right)$

75. (a) Let $E(x) = \frac{f(x)+f(-x)}{2}$ and $O(x) = \frac{f(x)-f(-x)}{2}$. Then $E(x) + O(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$ $= \frac{2f(x)}{2} = f(x)$. Also, $E(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(x)+f(-x)}{2} = E(x) \Rightarrow E(x)$ is even, and $O(-x) = \frac{f(-x)-f(-(-x))}{2} = -\frac{f(x)-f(-x)}{2} = -O(x) \Rightarrow O(x)$ is odd. Consequently, $f(x)$ can be written as a sum of an even and an odd function.

(b) $f(x) = \frac{f(x)+f(-x)}{2}$ because $\frac{f(x)-f(-x)}{2} = 0$ and $f(x) = \frac{f(x)-f(-x)}{2}$ because $\frac{f(x)+f(-x)}{2} = 0$; thus $f(x) = \frac{2f(x)}{2} + 0$ and $f(x) = 0 + \frac{2f(x)}{2}$

76. $y = \sinh^{-1} x \Rightarrow x = \sinh y \Rightarrow x = \frac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - \frac{1}{e^y} \Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y - 1 = 0$ $\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} \Rightarrow e^y = x + \sqrt{x^2+1} \Rightarrow \sinh^{-1} x = y = \ln\left(x + \sqrt{x^2+1}\right)$. Since $e^y > 0$, we cannot choose $e^y = x - \sqrt{x^2+1}$ because $x - \sqrt{x^2+1} < 0$.

77. (a) $m\frac{dv}{dt} = mg - kv^2 \Rightarrow \frac{m\frac{dv}{dt}}{mg-kv^2} = 1 \Rightarrow \frac{\frac{1}{g}\frac{dv}{dt}}{1-\frac{kv^2}{mg}} = 1 \Rightarrow \frac{\sqrt{\frac{k}{mg}}\,dv}{1-\sqrt{v\left(\frac{k}{mg}\right)}^2} = \sqrt{\frac{kg}{m}}\,dt \Rightarrow \int \frac{\sqrt{\frac{k}{mg}}\,dv}{1-\sqrt{v\left(\frac{k}{mg}\right)}^2}\,dv$ $= \int \sqrt{\frac{kg}{m}}\,dt \Rightarrow \tanh^{-1}\left(\sqrt{\frac{k}{mg}}\,v\right) = \sqrt{\frac{kg}{m}}t + C \Rightarrow v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}t + C\right)$; $v(0) = 0 \Rightarrow C = 0$ $\Rightarrow v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}t\right)$

(b) $\lim_{t\to\infty} v = \lim_{t\to\infty} \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}t\right) = \sqrt{\frac{mg}{k}}\lim_{t\to\infty}\tanh\left(\sqrt{\frac{kg}{m}}t\right) = \sqrt{\frac{mg}{k}}(1) = \sqrt{\frac{mg}{k}}$

(c) $\sqrt{\frac{160}{0.005}} = \sqrt{\frac{1{,}600{,}000}{5}} = \frac{400}{\sqrt{5}} = 80\sqrt{5} \approx 178.89$ ft/sec

78. (a) $s(t) = a\cos kt + b\sin kt \Rightarrow \frac{ds}{dt} = -ak\sin kt + bk\cos kt \Rightarrow \frac{d^2s}{dt^2} = -ak^2\cos kt - bk^2\sin kt$ $= -k^2(a\cos kt + \sin kt) = -k^2 s(t) \Rightarrow$ acceleration is proportional to s. The negative constant $-k^2$ implies that the acceleration is directed toward the origin.

(b) $s(t) = a\cosh kt + b\sinh kt \Rightarrow \frac{ds}{dt} = ak\sinh kt + bk\cosh kt \Rightarrow \frac{d^2s}{dt^2} = ak^2\cosh kt + bk^2\sinh kt$
$= k^2(a\cosh kt + \sinh kt) = k^2 s(t) \Rightarrow$ acceleration is proportional to s. The positive constant k^2 implies that the acceleration is directed away from the origin.

79. $\frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \Rightarrow y = \int \frac{-1}{x\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx \Rightarrow y = \text{sech}^{-1}(x) - \sqrt{1-x^2} + C$; $x = 1$ and $y = 0 \Rightarrow C = 0 \Rightarrow y = \text{sech}^{-1}(x) - \sqrt{1-x^2}$

80. To find the length of the curve: $y = \frac{1}{a}\cosh ax \Rightarrow y' = \sinh ax \Rightarrow s = \int_0^b \sqrt{1+(\sinh ax)^2}\,dx$
$\Rightarrow s = \int_0^b \cosh ax\,dx = \left[\frac{1}{a}\sinh ax\right]_0^b = \frac{1}{a}\sinh ab$. Then the area under the curve is $A = \int_0^b \frac{1}{a}\cosh ax\,dx$
$= \left[\frac{1}{a^2}\sinh ax\right]_0^b = \frac{1}{a^2}\sinh ab = \left(\frac{1}{a}\right)\left(\frac{1}{a}\sinh ab\right)$ which is the area of the rectangle of height $\frac{1}{a}$ and length s as claimed.

81. $V = \pi \int_0^2 \left(\cosh^2 x - \sinh^2 x\right) dx = \pi \int_0^2 1\,dx = 2\pi$

82. $V = 2\pi \int_0^{\ln\sqrt{3}} \text{sech}^2 x\,dx = 2\pi[\tanh x]_0^{\ln\sqrt{3}} = 2\pi\left[\frac{\sqrt{3}-(1/\sqrt{3})}{\sqrt{3}+(1/\sqrt{3})}\right] = \pi$

83. $y = \frac{1}{2}\cosh 2x \Rightarrow y' = \sinh 2x \Rightarrow L = \int_0^{\ln\sqrt{5}} \sqrt{1+(\sinh 2x)^2}\,dx = \int_0^{\ln\sqrt{5}} \cosh 2x\,dx = \left[\frac{1}{2}\sinh 2x\right]_0^{\ln\sqrt{5}}$
$= \left[\frac{1}{2}\left(\frac{e^{2x}-e^{-2x}}{2}\right)\right]_0^{\ln\sqrt{5}} = \frac{1}{4}\left(5 - \frac{1}{5}\right) = \frac{6}{5}$

84. (a) Let the point located at $(\cosh x, 0)$ be called T. Then $A(u)$ = area of the triangle ΔOTP minus the area under the curve $y = \sqrt{x^2-1}$ from A to T $\Rightarrow A(u) = \frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2-1}\,dx$.

(b) $A(u) = \frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2-1}\,dx \Rightarrow A'(u) = \frac{1}{2}\left(\cosh^2 u + \sinh^2 u\right) - \left(\sqrt{\cosh^2 u - 1}\right)(\sinh u)$
$= \frac{1}{2}\cosh^2 u + \frac{1}{2}\sinh^2 u - \sinh^2 u = \frac{1}{2}\left(\cosh^2 u - \sinh^2 u\right) = \left(\frac{1}{2}\right)(1) = \frac{1}{2}$

(c) $A'(u) = \frac{1}{2} \Rightarrow A(u) = \frac{u}{2} + C$, and from part (a) we have $A(0) = 0 \Rightarrow C = 0 \Rightarrow A(u) = \frac{u}{2} \Rightarrow u = 2A$

85. (a) $y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right) \Rightarrow \tan\phi = \frac{dy}{dx} = \left(\frac{H}{w}\right)\left[\frac{w}{H}\sinh\left(\frac{w}{H}x\right)\right] = \sinh\left(\frac{w}{H}x\right)$

(b) The tension at P is given by $T\cos\phi = H \Rightarrow T = H\sec\phi = H\sqrt{1+\tan^2\phi} = H\sqrt{1+\left(\sinh\frac{w}{H}x\right)^2}$

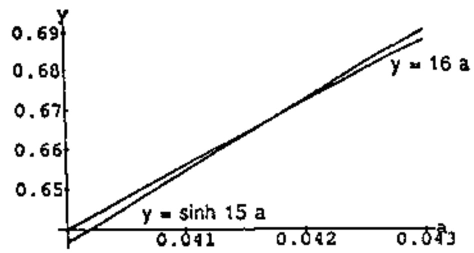
$= H\cosh\left(\frac{w}{H}x\right) = w\left(\frac{H}{w}\right)\cosh\left(\frac{w}{H}x\right) = wy$

86. $s = \frac{1}{a}\sinh ax \Rightarrow \sinh ax = as \Rightarrow ax = \sinh^{-1}as \Rightarrow x = \frac{1}{a}\sinh^{-1}as$; $y = \frac{1}{a}\cosh ax = \frac{1}{a}\sqrt{\cosh^2 ax}$

$= \frac{1}{a}\sqrt{\sinh^2 ax + 1} = \frac{1}{a}\sqrt{a^2s^2+1} = \sqrt{s^2+\frac{1}{a^2}}$

87. (a) Since the cable is 32 ft long, $s = 16$ and $x = 15$. From Exercise 88, $x = \frac{1}{a}\sinh^{-1}as \Rightarrow 15a = \sinh^{-1}16a$ $\Rightarrow \sinh 15a = 16a$.

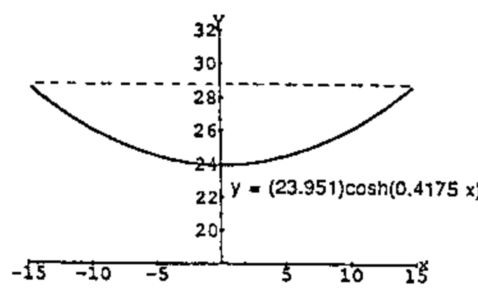
(b) The intersection is near $(0.042, 0.672)$.



(c) Newton's method indicates that at $a \approx 0.0417525$ the curves $y = 16a$ and $y = \sinh 15a$ intersect.

(d) $T = wy \approx (2\text{ lb})\left(\frac{1}{0.0417525}\right) \approx 47.90$ lb

(e) The sag is about 4.8 ft.



CHAPTER 6 PRACTICE EXERCISES

1. $y = 10e^{-x/5} \Rightarrow \frac{dy}{dx} = (10)\left(-\frac{1}{5}\right)e^{-x/5} = -2e^{-x/5}$

2. $y = \sqrt{2}\,e^{\sqrt{2}x} \Rightarrow \frac{dy}{dx} = (\sqrt{2})(\sqrt{2})e^{\sqrt{2}x} = 2e^{\sqrt{2}x}$

3. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} \Rightarrow \frac{dy}{dx} = \frac{1}{4}\left[x\left(4e^{4x}\right) + e^{4x}(1)\right] - \frac{1}{16}\left(4e^{4}\right) = xe^{4x} + \frac{1}{4}e^{4x} - \frac{1}{4}e^{4x} = xe^{4x}$

4. $y = x^2e^{-2/x} = x^2e^{-2x^{-1}} \Rightarrow \frac{dy}{dx} = x^2\left[\left(2x^{-2}\right)e^{-2x^{-1}}\right] + e^{-2x^{-1}}(2x) = (2+2x)e^{-2x^{-1}} = 2e^{-2/x}(1+x)$

5. $y = \ln\left(\sin^2\theta\right) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sin\theta)(\cos\theta)}{\sin^2\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta$

6. $y = \ln\left(\sec^2\theta\right) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sec\theta)(\sec\theta\tan\theta)}{\sec^2\theta} = 2\tan\theta$

7. $y = \log_2\left(\frac{x^2}{2}\right) = \frac{\ln\left(\frac{x^2}{2}\right)}{\ln 2} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 2}\left(\frac{x}{\left(\frac{x^2}{2}\right)}\right) = \frac{2}{(\ln 2)x}$

8. $y = \log_5(3x-7) = \frac{\ln(3x-7)}{\ln 5} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\ln 5}\right)\left(\frac{3}{3x-7}\right) = \frac{3}{(\ln 5)(3x-7)}$

9. $y = 8^{-t} \Rightarrow \frac{dy}{dt} = 8^{-t}(\ln 8)(-1) = -8^{-t}(\ln 8)$

10. $y = 9^{2t} \Rightarrow \frac{dy}{dt} = 9^{2t}(\ln 9)(2) = 9^{2t}(2\ln 9)$

11. $y = 5x^{3.6} \Rightarrow \frac{dy}{dx} = 5(3.6)x^{2.6} = 18x^{2.6}$

12. $y = \sqrt{2}\,x^{-\sqrt{2}} \Rightarrow \frac{dy}{dx} = \left(\sqrt{2}\right)\left(-\sqrt{2}\right)x^{\left(-\sqrt{2}-1\right)} = -2x^{\left(-\sqrt{2}-1\right)}$

13. $y = (x+2)^{x+2} \Rightarrow \ln y = \ln(x+2)^{x+2} = (x+2)\ln(x+2) \Rightarrow \frac{y'}{y} = (x+2)\left(\frac{1}{x+2}\right) + (1)\ln(x+2)$

$\Rightarrow \frac{dy}{dx} = (x+2)^{x+2}\left[\ln(x+2)+1\right]$

14. $y = 2(\ln x)^{x/2} \Rightarrow \ln y = \ln\left[2(\ln x)^{x/2}\right] = \ln(2) + \left(\frac{x}{2}\right)\ln(\ln x) \Rightarrow \frac{y'}{y} = 0 + \left(\frac{x}{2}\right)\left[\frac{\left(\frac{1}{x}\right)}{\ln x}\right] + (\ln(\ln x))\left(\frac{1}{2}\right)$

$\Rightarrow y' = \left[\frac{1}{2\ln x} + \left(\frac{1}{2}\right)\ln(\ln x)\right]2(\ln x)^{x/2} = (\ln x)^{x/2}\left[\ln(\ln x) + \frac{1}{\ln x}\right]$

15. $y = \sin^{-1}\sqrt{1-u^2} = \sin^{-1}\left(1-u^2\right)^{1/2} \Rightarrow \frac{dy}{du} = \frac{\frac{1}{2}\left(1-u^2\right)^{-1/2}(-2u)}{\sqrt{1-\left[\left(1-u^2\right)^{1/2}\right]^2}} = \frac{-u}{\sqrt{1-u^2}\sqrt{1-\left(1-u^2\right)}} = \frac{-u}{|u|\sqrt{1-u^2}}$

$= \frac{-u}{u\sqrt{1-u^2}} = \frac{-1}{\sqrt{1-u^2}},\ 0 < u < 1$

16. $y = \sin^{-1}\left(\frac{1}{\sqrt{v}}\right) = \sin^{-1}v^{-1/2} \Rightarrow \frac{dy}{dv} = \frac{-\frac{1}{2}v^{-3/2}}{\sqrt{1-\left(v^{-1/2}\right)^2}} = \frac{-1}{2v^{3/2}\sqrt{1-v^{-1}}} = \frac{-1}{2v^{3/2}\sqrt{\frac{v-1}{v}}} = \frac{-\sqrt{v}}{2v^{3/2}\sqrt{v-1}}$

$= \frac{-1}{2v\sqrt{v-1}}$

17. $y = \ln\left(\cos^{-1} x\right) \Rightarrow y' = \frac{\left(\frac{-1}{\sqrt{1-x^2}}\right)}{\cos^{-1} x} = \frac{-1}{\sqrt{1-x^2}\cos^{-1} x}$

18. $y = z\cos^{-1} z - \sqrt{1-z^2} = z\cos^{-1} z - \left(1-z^2\right)^{1/2} \Rightarrow \frac{dy}{dz} = \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} - \left(\frac{1}{2}\right)\left(1-z^2\right)^{-1/2}(-2z)$
$= \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} + \frac{z}{\sqrt{1-z^2}} = \cos^{-1} z$

19. $y = t\tan^{-1} t - \left(\frac{1}{2}\right)\ln t \Rightarrow \frac{dy}{dt} = \tan^{-1} t + t\left(\frac{1}{1+t^2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{t}\right) = \tan^{-1} t + \frac{t}{1+t^2} - \frac{1}{2t}$

20. $y = \left(1+t^2\right)\cot^{-1} 2t \Rightarrow \frac{dy}{dt} = 2t\cot^{-1} 2t + \left(1+t^2\right)\left(\frac{-2}{1+4t^2}\right)$

21. $y = z\sec^{-1} z - \sqrt{z^2-1} = z\sec^{-1} z - \left(z^2-1\right)^{1/2} \Rightarrow \frac{dy}{dz} = z\left(\frac{1}{|z|\sqrt{z^2-1}}\right) + \left(\sec^{-1} z\right)(1) - \frac{1}{2}\left(z^2-1\right)^{-1/2}(2z)$
$= \frac{z}{|z|\sqrt{z^2-1}} - \frac{z}{\sqrt{z^2-1}} + \sec^{-1} z = \frac{1-z}{\sqrt{z^2-1}} + \sec^{-1} z,\ z > 1$

22. $y = 2\sqrt{x-1}\sec^{-1}\sqrt{x} = 2(x-1)^{1/2}\sec^{-1}(x)^{1/2}$
$\Rightarrow \frac{dy}{dx} = 2\left[\left(\frac{1}{2}\right)(x-1)^{-1/2}\sec^{-1}(x)^{1/2} + (x-1)^{1/2}\left(\frac{\left(\frac{1}{2}\right)x^{-1/2}}{\sqrt{x}\sqrt{x-1}}\right)\right] = 2\left(\frac{\sec^{-1}\sqrt{x}}{2\sqrt{x-1}} + \frac{1}{2x}\right) = \frac{\sec^{-1}\sqrt{x}}{\sqrt{x-1}} + \frac{1}{x}$

23. $y = \csc^{-1}(\sec\theta) \Rightarrow \frac{dy}{d\theta} = \frac{-\sec\theta\tan\theta}{|\sec\theta|\sqrt{\sec^2\theta-1}} = -\frac{\tan\theta}{|\tan\theta|} = -1,\ 0 < \theta < \frac{\pi}{2}$

24. $y = \left(1+x^2\right)e^{\tan^{-1} x} \Rightarrow y' = 2xe^{\tan^{-1} x} + \left(1+x^2\right)\left(\frac{e^{\tan^{-1} x}}{1+x^2}\right) = 2xe^{\tan^{-1} x} + e^{\tan^{-1} x}$

25. $y = \frac{2\left(x^2+1\right)}{\sqrt{\cos 2x}} \Rightarrow \ln y = \ln\left(\frac{2\left(x^2+1\right)}{\sqrt{\cos 2x}}\right) = \ln(2) + \ln\left(x^2+1\right) - \frac{1}{2}\ln(\cos 2x) \Rightarrow \frac{y'}{y} = 0 + \frac{2x}{x^2+1} - \left(\frac{1}{2}\right)\frac{(-2\sin 2x)}{\cos 2x}$
$\Rightarrow y' = \left(\frac{2x}{x^2+1} + \tan 2x\right)y = \frac{2\left(x^2+1\right)}{\sqrt{\cos 2x}}\left(\frac{2x}{x^2+1} + \tan 2x\right)$

26. $y = \sqrt[10]{\frac{3x+4}{2x-4}} \Rightarrow \ln y = \ln\sqrt[10]{\frac{3x+4}{2x-4}} = \frac{1}{10}[\ln(3x+4) - \ln(2x-4)] \Rightarrow \frac{y'}{y} = \frac{1}{10}\left(\frac{3}{3x+4} - \frac{2}{2x-4}\right)$
$\Rightarrow y' = \frac{1}{10}\left(\frac{3}{3x+4} - \frac{1}{x-2}\right)y = \sqrt[10]{\frac{3x+4}{2x-4}}\left(\frac{1}{10}\right)\left(\frac{3}{3x+4} - \frac{1}{x-2}\right)$

27. $y = \left[\frac{(t+1)(t-1)}{(t-2)(t+3)}\right]^5 \Rightarrow \ln y = 5\left[\ln(t+1) + \ln(t-1) - \ln(t-2) - \ln(t+3)\right] \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{dt}\right)$

$$= 5\left(\frac{1}{t+1}+\frac{1}{t-1}-\frac{1}{t-2}-\frac{1}{t+3}\right) \Rightarrow \frac{dy}{dt} = 5\left[\frac{(t+1)(t-1)}{(t-2)(t+3)}\right]^5\left(\frac{1}{t+1}+\frac{1}{t-1}-\frac{1}{t-2}-\frac{1}{t+3}\right)$$

28. $y = \frac{2u2^u}{\sqrt{u^2+1}} \Rightarrow \ln y = \ln 2 + \ln u + u \ln 2 - \frac{1}{2}\ln(u^2+1) \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{du}\right) = \frac{1}{u} + \ln 2 - \frac{1}{2}\left(\frac{2u}{u^2+1}\right)$

$\Rightarrow \frac{dy}{du} = \frac{2u2^u}{\sqrt{u^2+1}}\left(\frac{1}{u} + \ln 2 - \frac{u}{u^2+1}\right)$

29. $y = (\sin\theta)^{\sqrt{\theta}} \Rightarrow \ln y = \sqrt{\theta}\ln(\sin\theta) \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{d\theta}\right) = \sqrt{\theta}\left(\frac{\cos\theta}{\sin\theta}\right) + \frac{1}{2}\theta^{-1/2}\ln(\sin\theta)$

$\Rightarrow \frac{dy}{d\theta} = (\sin\theta)^{\sqrt{\theta}}\left(\sqrt{\theta}\cot\theta + \frac{\ln(\sin\theta)}{2\sqrt{\theta}}\right) = \frac{1}{\sqrt{\theta}}(\sin\theta)^{\sqrt{\theta}}\left(\theta\cot\theta + \ln\sqrt{\sin\theta}\right)$

30. $y = (\ln x)^{1/\ln x} \Rightarrow \ln y = \left(\frac{1}{\ln x}\right)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{\ln x}\right)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) + \ln(\ln x)\left[\frac{-1}{(\ln x)^2}\right]\left(\frac{1}{x}\right)$

$\Rightarrow y' = (\ln x)^{1/\ln x}\left[\frac{1-\ln(\ln x)}{x(\ln x)^2}\right]$

31. $\int e^x\sin(e^x)\,dx = \int \sin u\,du$, where $u = e^x$ and $du = e^x\,dx$

$= -\cos u + C = -\cos(e^x) + C$

32. $\int e^t\cos(3e^t-2)\,dt = \frac{1}{3}\int \cos u\,du$, where $u = 3e^t - 2$ and $du = 3e^t\,dt$

$= \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3e^t-2) + C$

33. $\int e^x\sec^2(e^x-7)\,dx = \int \sec^2 u\,du$, where $u = e^x - 7$ and $du = e^x\,dx$

$= \tan u + C = \tan(e^x-7) + C$

34. $\int e^y\csc(e^y+1)\cot(e^y+1)\,dy = \int \csc u\cot u\,du$, where $u = e^y + 1$ and $du = e^y\,dy$

$= -\csc u + C = -\csc(e^y+1) + C$

35. $\int (\sec^2 x)e^{\tan x}\,dx = \int e^u\,du$, where $u = \tan x$ and $du = \sec^2 x\,dx$

$= e^u + C = e^{\tan x} + C$

36. $\int (\csc^2 x)e^{\cot x}\,dx = -\int e^u\,du$, where $u = \cot x$ and $du = -\csc^2 x\,dx$

$= -e^u + C = -e^{\cot x} + C$

37. $\int_{-1}^{1} \frac{1}{3x-4}\,dx = \frac{1}{3}\int_{-7}^{-1} \frac{1}{u}\,du$, where $u = 3x-4$, $du = 3\,dx$; $x=-1 \Rightarrow u=-7$, $x=1 \Rightarrow u=-1$

$= \frac{1}{3}\left[\ln|u|\right]_{-7}^{-1} = \frac{1}{3}\left[\ln|-1| - \ln|-7|\right] = \frac{1}{3}[0 - \ln 7] = -\frac{\ln 7}{3}$

38. $\int_{1}^{e} \frac{\sqrt{\ln x}}{x}\,dx = \int_{0}^{1} u^{1/2}\,du$, where $u = \ln x$, $du = \frac{1}{x}\,dx$; $x=1 \Rightarrow u=0$, $x=e \Rightarrow u=1$

$= \left[\frac{2}{3}u^{3/2}\right]_0^1 = \left[\frac{2}{3}1^{3/2} - \frac{2}{3}0^{3/2}\right] = \frac{2}{3}$

39. $\int_{0}^{\pi} \tan\left(\frac{x}{3}\right)dx = \int_{0}^{\pi} \frac{\sin\left(\frac{x}{3}\right)}{\cos\left(\frac{x}{3}\right)}\,dx = -3\int_{1}^{1/2} \frac{1}{u}\,du$, where $u = \cos\left(\frac{x}{3}\right)$, $du = -\frac{1}{3}\sin\left(\frac{x}{3}\right)dx$; $x=0 \Rightarrow u=1$, $x=\pi$ $\Rightarrow u = \frac{1}{2}$

$= -3\left[\ln|u|\right]_1^{1/2} = -3\left[\ln\left|\frac{1}{2}\right| - \ln|1|\right] = -3\ln\frac{1}{2} = \ln 2^3 = \ln 8$

40. $\int_{1/6}^{1/4} 2\cot \pi x\,dx = 2\int_{1/6}^{1/4} \frac{\cos \pi x}{\sin \pi x}\,dx = \frac{2}{\pi}\int_{1/2}^{1/\sqrt{2}} \frac{1}{u}\,du$, where $u = \sin \pi x$, $du = \pi\cos \pi x\,dx$; $x = \frac{1}{6} \Rightarrow u = \frac{1}{2}$, $x = \frac{1}{4}$ $\Rightarrow u = \frac{1}{\sqrt{2}}$

$= \frac{2}{\pi}\left[\ln|u|\right]_{1/2}^{1/\sqrt{2}} = \frac{2}{\pi}\left[\ln\left|\frac{1}{\sqrt{2}}\right| - \ln\left|\frac{1}{2}\right|\right] = \frac{2}{\pi}\left[\ln 1 - \frac{1}{2}\ln 2 - \ln 1 + \ln 2\right] = \frac{2}{\pi}\left[\frac{1}{2}\ln 2\right] = \frac{\ln 2}{\pi}$

41. $\int_{0}^{4} \frac{2t}{t^2-25}\,dt = \int_{-25}^{-9} \frac{1}{u}\,du$, where $u = t^2-25$, $du = 2t\,dt$; $t=0 \Rightarrow u=-25$, $t=4 \Rightarrow u=-9$

$= \left[\ln|u|\right]_{-25}^{-9} = \ln|-9| - \ln|-25| = \ln 9 - \ln 25 = \ln\frac{9}{25}$

42. $\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1-\sin t}\,dt = -\int_{2}^{1/2} \frac{1}{u}\,du$, where $u = 1-\sin t$, $du = -\cos t\,dt$; $t = -\frac{\pi}{2} \Rightarrow u=2$, $t = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}$

$= -\left[\ln|u|\right]_2^{1/2} = -\left[\ln\left|\frac{1}{2}\right| - \ln|2|\right] = -\ln 1 + \ln 2 + \ln 2 = 2\ln 2 = \ln 4$

43. $\int \frac{\tan(\ln v)}{v}\,dv = \int \tan u\,du = \int \frac{\sin u}{\cos u}\,du$, where $u = \ln v$ and $du = \frac{1}{v}\,dv$

$= -\ln|\cos u| + C = -\ln|\cos(\ln v)| + C$

44. $\int \frac{1}{v\ln v}\,dv = \int \frac{1}{u}\,du$, where $u = \ln v$ and $du = \frac{1}{v}\,dv$

$= \ln|u| + C = \ln|\ln v| + C$

45. $\int \frac{(\ln x)^{-3}}{x}\,dx = \int u^{-3}\,du$, where $u = \ln x$ and $du = \frac{1}{x}\,dx$

$= \frac{u^{-2}}{-2} + C = -\frac{1}{2}(\ln x)^{-2} + C$

46. $\int \frac{\ln(x-5)}{x-5}\,dx = \int u\,du$, where $u = \ln(x-5)$ and $du = \frac{1}{x-5}\,dx$

$= \frac{u^2}{2} + C = \frac{[\ln(x-5)]^2}{2} + C$

47. $\int \frac{1}{r}\csc^2(1+\ln r)\,dr = \int \csc^2 u\,du$, where $u = 1 + \ln r$ and $du = \frac{1}{r}\,dr$

$= -\cot u + C = -\cot(1+\ln r) + C$

48. $\int \frac{\cos(1-\ln v)}{v}\,dv = -\int \cos u\,du$, where $u = 1 - \ln v$ and $du = -\frac{1}{v}\,dv$

$= -\sin u + C = -\sin(1-\ln v) + C$

49. $\int x3^{x^2}\,dx = \frac{1}{2}\int 3^u\,du$, where $u = x^2$ and $du = 2x\,dx$

$= \frac{1}{2\ln 3}(3^u) + C = \frac{1}{2\ln 3}\left(3^{x^2}\right) + C$

50. $\int 2^{\tan x}\sec^2 x\,dx = \int 2^u\,du$, where $u = \tan x$ and $du = \sec^2 x\,dx$

$= \frac{1}{\ln 2}(2^u) + C = \frac{2^{\tan x}}{\ln 2} + C$

51. $\int_1^7 \frac{3}{x}\,dx = 3\int_1^7 \frac{1}{x}\,dx = 3[\ln|x|]_1^7 = 3(\ln 7 - \ln 1) = 3\ln 7$

52. $\int_1^{32} \frac{1}{5x}\,dx = \frac{1}{5}\int_1^{32} \frac{1}{x}\,dx = \frac{1}{5}[\ln|x|]_1^{32} = \frac{1}{5}(\ln 32 - \ln 1) = \frac{1}{5}\ln 32 = \ln\left(\sqrt[5]{32}\right) = \ln 2$

53. $\int_1^4 \left(\frac{x}{8} + \frac{1}{2x}\right)dx = \frac{1}{2}\int_1^4 \left(\frac{1}{4}x + \frac{1}{x}\right)dx = \frac{1}{2}\left[\frac{1}{8}x^2 + \ln|x|\right]_1^4 = \frac{1}{2}\left[\left(\frac{16}{8} + \ln 4\right) - \left(\frac{1}{8} + \ln 1\right)\right] = \frac{15}{16} + \frac{1}{2}\ln 4$

$= \frac{15}{16} + \ln\sqrt{4} = \frac{15}{16} + \ln 2$

54. $\int_1^8 \left(\frac{2}{3x} - \frac{8}{x^2}\right)dx = \frac{2}{3}\int_1^8 \left(\frac{1}{x} - 12x^{-2}\right)dx = \frac{2}{3}[\ln|x| + 12x^{-1}]_1^8 = \frac{2}{3}\left[\left(\ln 8 + \frac{12}{8}\right) - (\ln 1 + 12)\right]$

$= \frac{2}{3}\left(\ln 8 + \frac{3}{2} - 12\right) = \frac{2}{3}\left(\ln 8 - \frac{21}{2}\right) = \frac{2}{3}(\ln 8) - 7 = \ln\left(8^{2/3}\right) - 7 = \ln 4 - 7$

55. $\int_{-2}^{-1} e^{-(x+1)}\,dx = -\int_{1}^{0} e^{u}\,du$, where $u = -(x+1)$, $du = -dx$; $x = -2 \Rightarrow u = 1$, $x = -1 \Rightarrow u = 0$

$= -\left[e^{u}\right]_{1}^{0} = -\left(e^{0} - e^{1}\right) = e - 1$

56. $\int_{-\ln 2}^{0} e^{2w}\,dw = \frac{1}{2}\int_{\ln(1/4)}^{0} e^{u}\,du$, where $u = 2w$, $du = 2\,dw$; $w = -\ln 2 \Rightarrow u = \ln\frac{1}{4}$, $w = 0 \Rightarrow u = 0$

$= \frac{1}{2}\left[e^{u}\right]_{\ln(1/4)}^{0} = \frac{1}{2}\left[e^{0} - e^{\ln(1/4)}\right] = \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}$

57. $\int_{0}^{\ln 5} e^{r}\left(3e^{r}+1\right)^{-3/2}\,dr = \frac{1}{3}\int_{4}^{16} u^{-3/2}\,du$, where $u = 3e^{r}+1$, $du = 3e^{r}$; $r = 0 \Rightarrow u = 4$, $r = \ln 5 \Rightarrow u = 16$

$= -\frac{2}{3}\left[u^{-1/2}\right]_{4}^{16} = -\frac{2}{3}\left(16^{-1/2} - 4^{-1/2}\right) = \left(-\frac{2}{3}\right)\left(\frac{1}{4} - \frac{1}{2}\right) = \left(-\frac{2}{3}\right)\left(-\frac{1}{4}\right) = \frac{1}{6}$

58. $\int_{0}^{\ln 9} e^{\theta}\left(e^{\theta}-1\right)^{1/2}\,d\theta = \int_{0}^{8} u^{1/2}\,du$, where $u = e^{\theta} - 1$, $du = e^{\theta}\,d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \ln 9 \Rightarrow u = 8$

$= \frac{2}{3}\left[u^{3/2}\right]_{0}^{8} = \frac{2}{3}\left(8^{3/2} - 0^{3/2}\right) = \frac{2}{3}\left(2^{9/2} - 0\right) = \frac{2^{11/2}}{3} = \frac{32\sqrt{2}}{3}$

59. $\int_{1}^{e} \frac{1}{x}(1 + 7\ln x)^{-1/3}\,dx = \frac{1}{7}\int_{1}^{8} u^{-1/3}\,du$, where $u = 1 + 7\ln x$, $du = \frac{7}{x}\,dx$, $x = 1 \Rightarrow u = 1$, $x = e \Rightarrow u = 8$

$= \frac{3}{14}\left[u^{2/3}\right]_{1}^{8} = \frac{3}{14}\left(8^{2/3} - 1^{2/3}\right) = \left(\frac{3}{14}\right)(4-1) = \frac{9}{14}$

60. $\int_{e}^{e^2} \frac{1}{x\sqrt{\ln x}}\,dx = \int_{e}^{e^2} (\ln x)^{-1/2}\,\frac{1}{x}\,dx = \int_{1}^{2} u^{-1/2}\,du$, where $u = \ln x$, $du = \frac{1}{x}\,dx$; $x = e \Rightarrow u = 1$, $x = e^2 \Rightarrow u = 2$

$= 2\left[u^{1/2}\right]_{1}^{2} = 2\left(\sqrt{2} - 1\right) = 2\sqrt{2} - 2$

61. $\int_{1}^{3} \frac{[\ln(v+1)]^2}{v+1}\,dv = \int_{1}^{3} [\ln(v+1)]^2\,\frac{1}{v+1}\,dv = \int_{\ln 2}^{\ln 4} u^2\,du$, where $u = \ln(v+1)$, $du = \frac{1}{v+1}dv$;
$v = 1 \Rightarrow u = \ln 2$, $v = 3 \Rightarrow u = \ln 4$;

$= \frac{1}{3}\left[u^3\right]_{\ln 2}^{\ln 4} = \frac{1}{3}\left[(\ln 4)^3 - (\ln 2)^3\right] = \frac{1}{3}\left[(2\ln 2)^3 - (\ln 2)^3\right] = \frac{(\ln 2)^3}{3}(8-1) = \frac{7}{3}(\ln 2)^3$

62. $\int_{2}^{4} (1 + \ln t)(t\ln t)\,dt = \int_{2}^{4} (t\ln t)(1 + \ln t)\,dt = \int_{2\ln 2}^{4\ln 4} u\,du$, where $u = t\ln t$, $du = \left((t)\left(\frac{1}{t}\right) + (\ln t)(1)\right)dt$
$= (1 + \ln t)\,dt$; $t = 2 \Rightarrow u = 2\ln 2$, $t = 4$
$\Rightarrow u = 4\ln 4$

$= \frac{1}{2}\left[u^2\right]_{2\ln 2}^{4\ln 4} = \frac{1}{2}\left[(4\ln 4)^2 - (2\ln 2)^2\right] = \frac{1}{2}\left[(8\ln 2)^2 - (2\ln 2)^2\right] = \frac{(2\ln 2)^2}{2}(16-1) = 30(\ln 2)^2$

63. $\int_1^8 \frac{\log_4 \theta}{\theta}\, d\theta = \frac{1}{\ln 4}\int_1^8 (\ln \theta)\left(\frac{1}{\theta}\right) d\theta = \frac{1}{\ln 4}\int_0^{\ln 8} u\, du$, where $u = \ln \theta$, $du = \frac{1}{\theta}\, d\theta$, $\theta = 1 \Rightarrow u = 0$, $\theta = 8 \Rightarrow u = \ln 8$

$= \frac{1}{2 \ln 4}\left[u^2\right]_0^{\ln 8} = \frac{1}{\ln 16}\left[(\ln 8)^2 - 0^2\right] = \frac{(3 \ln 2)^2}{4 \ln 2} = \frac{9 \ln 2}{4}$

64. $\int_1^e \frac{8(\ln 3)(\log_3 \theta)}{\theta}\, d\theta = \int_1^e \frac{8(\ln 3)(\ln \theta)}{\theta(\ln 3)}\, d\theta = 8\int_1^e (\ln \theta)\left(\frac{1}{\theta}\right) d\theta = 8\int_0^1 u\, du$, where $u = \ln \theta$, $du = \frac{1}{\theta}\, d\theta$;

$\theta = 1 \Rightarrow u = 0$, $\theta = e \Rightarrow u = 1$

$= 4\left[u^2\right]_0^1 = 4\left(1^2 - 0^2\right) = 4$

65. $\int_{-3/4}^{3/4} \frac{6}{\sqrt{9 - 4x^2}}\, dx = 3\int_{-3/4}^{3/4} \frac{2}{\sqrt{3^2 - (2x)^2}}\, dx = 3\int_{-3/2}^{3/2} \frac{1}{\sqrt{3^2 - u^2}}\, du$, where $u = 2x$, $du = 2\, dx$;

$x = -\frac{3}{4} \Rightarrow u = -\frac{3}{2}$, $x = \frac{3}{4} \Rightarrow u = \frac{3}{2}$

$= 3\left[\sin^{-1}\left(\frac{u}{3}\right)\right]_{-3/2}^{3/2} = 3\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right] = 3\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = 3\left(\frac{\pi}{3}\right) = \pi$

66. $\int_{-1/5}^{1/5} \frac{6}{\sqrt{4 - 25x^2}}\, dx = \frac{6}{5}\int_{-1/5}^{1/5} \frac{5}{\sqrt{2^2 - (5x)^2}}\, dx = \frac{6}{5}\int_{-1}^{1} \frac{1}{\sqrt{2^2 - u^2}}\, du$, where $u = 5x$, $du = 5\, dx$;

$x = -\frac{1}{5} \Rightarrow u = -1$, $x = \frac{1}{5} \Rightarrow u = 1$

$= \frac{6}{5}\left[\sin^{-1}\left(\frac{u}{2}\right)\right]_{-1}^{1} = \frac{6}{5}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \frac{6}{5}\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = \frac{6}{5}\left(\frac{\pi}{3}\right) = \frac{2\pi}{5}$

67. $\int_{-2}^{2} \frac{3}{4 + 3t^2}\, dt = \sqrt{3}\int_{-2}^{2} \frac{\sqrt{3}}{2^2 + \left(\sqrt{3}t\right)^2}\, dt = \sqrt{3}\int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{1}{2^2 + u^2}\, du$, where $u = \sqrt{3}t$, $du = \sqrt{3}\, dt$;

$t = -2 \Rightarrow u = -2\sqrt{3}$, $t = 2 \Rightarrow u = 2\sqrt{3}$

$= \sqrt{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right]_{-2\sqrt{3}}^{2\sqrt{3}} = \frac{\sqrt{3}}{2}\left[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}\left(-\sqrt{3}\right)\right] = \frac{\sqrt{3}}{2}\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \frac{\pi}{\sqrt{3}}$

68. $\int_{\sqrt{3}}^{3} \frac{1}{3 + t^2}\, dt = \int_{\sqrt{3}}^{3} \frac{1}{\left(\sqrt{3}\right)^2 + t^2}\, dt = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_{\sqrt{3}}^{3} = \frac{1}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1} 1\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{3}\pi}{36}$

69. $\int \frac{1}{y\sqrt{4y^2 - 1}}\, dy = \int \frac{2}{(2y)\sqrt{(2y)^2 - 1}}\, dy = \int \frac{1}{u\sqrt{u^2 - 1}}\, du$, where $u = 2y$ and $du = 2\, dy$

$= \sec^{-1}|u| + C = \sec^{-1}|2y| + C$

70. $\int \frac{24}{y\sqrt{y^2-16}}\,dy = 24\int \frac{1}{y\sqrt{y^2-4^2}}\,dy = 24\left(\frac{1}{4}\sec^{-1}\left|\frac{y}{4}\right|\right)+C = 6\sec^{-1}\left|\frac{y}{4}\right|+C$

71. $\int_{\sqrt{2}/3}^{2/3} \frac{1}{|y|\sqrt{9y^2-1}}\,dy = \int_{\sqrt{2}/3}^{2/3} \frac{3}{|3y|\sqrt{(3y)^2-1}}\,dy = \int_{\sqrt{2}}^{2} \frac{1}{|u|\sqrt{u^2-1}}\,du$, where $u=3y$, $du=3\,dy$;
$y=\frac{\sqrt{2}}{3} \Rightarrow u=\sqrt{2}$, $y=\frac{2}{3}\Rightarrow u=2$

$= \left[\sec^{-1} u\right]_{\sqrt{2}}^{2} = \left[\sec^{-1} 2 - \sec^{-1}\sqrt{2}\right] = \frac{\pi}{3}-\frac{\pi}{4} = \frac{\pi}{12}$

72. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{1}{|y|\sqrt{5y^2-3}}\,dy = \int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{\sqrt{5}}{-\sqrt{5}y\sqrt{(\sqrt{5}y)^2-(\sqrt{3})^2}}\,dy = \int_{-2}^{-\sqrt{6}} \frac{1}{-u\sqrt{u^2-(\sqrt{3})^2}}\,du$,
where $u=\sqrt{5}y$, $du=\sqrt{5}\,dy$; $u=-\frac{2}{\sqrt{5}}\Rightarrow u=-2$, $y=-\frac{\sqrt{6}}{5}\Rightarrow u=-\sqrt{6}$

$=\left[-\frac{1}{\sqrt{3}}\sec^{-1}\left|\frac{u}{\sqrt{3}}\right|\right]_{-2}^{-\sqrt{6}} = \frac{-1}{\sqrt{3}}\left[\sec^{-1}\sqrt{2}-\sec^{-1}\frac{2}{\sqrt{3}}\right] = \frac{-1}{\sqrt{3}}\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{-1}{\sqrt{3}}\left[\frac{3\pi}{12}-\frac{2\pi}{12}\right] = \frac{-\pi}{12\sqrt{3}} = \frac{-\sqrt{3}\pi}{36}$

73. $\int \frac{1}{\sqrt{-2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x^2+2x+1)}}\,dx = \int \frac{1}{\sqrt{1-(x+1)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du$, where $u=x+1$ and $du=dx$

$= \sin^{-1} u + C = \sin^{-1}(x+1)+C$

74. $\int \frac{1}{\sqrt{-x^2+4x-1}}\,dx = \int \frac{1}{\sqrt{3-(x^2-4x+4)}}\,dx = \int \frac{1}{\sqrt{(\sqrt{3})^2-(x-2)^2}}\,dx = \int \frac{1}{\sqrt{(\sqrt{3})^2-u^2}}\,du$
where $u=x-2$ and $du=dx$

$= \sin^{-1}\left(\frac{u}{\sqrt{3}}\right)+C = \sin^{-1}\left(\frac{x-2}{\sqrt{3}}\right)+C$

75. $\int_{-2}^{-1} \frac{2}{v^2+4v+5}\,dv = 2\int_{-2}^{-1} \frac{1}{1+(v^2+4v+4)}\,dv = 2\int_{-2}^{-1} \frac{1}{1+(v+2)^2}\,dv = 2\int_{0}^{1} \frac{1}{1+u^2}\,du$,
where $u=v+2$, $du=dv$; $v=-2\Rightarrow u=0$, $v=-1\Rightarrow u=1$

$= 2\left[\tan^{-1} u\right]_0^1 = 2\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 2\left(\frac{\pi}{4}-0\right) = \frac{\pi}{2}$

76. $\int_{-1}^{1} \frac{3}{4v^2+4v+4}\,dv = \frac{3}{4}\int_{-1}^{1} \frac{1}{\frac{3}{4}+\left(v^2+v+\frac{1}{4}\right)}\,dv = \frac{3}{4}\int_{-1}^{1} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+\left(v+\frac{1}{2}\right)^2}\,dv = \frac{3}{4}\int_{-1/2}^{3/2} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+u^2}\,du$

where $u=v+\frac{1}{2}$, $du=dv$; $v=-1\Rightarrow u=-\frac{1}{2}$, $v=1\Rightarrow u=\frac{3}{2}$

$= \frac{3}{4}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\right]_{-1/2}^{3/2} = \frac{\sqrt{3}}{2}\left[\tan^{-1}\sqrt{3}-\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \frac{\sqrt{3}}{2}\left[\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right] = \frac{\sqrt{3}}{2}\left(\frac{2\pi}{6}+\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\cdot\frac{\pi}{2}$

$= \frac{\sqrt{3}\pi}{4}$

77. $\int \frac{1}{(t+1)\sqrt{t^2+2t-8}}\,dt = \int \frac{1}{(t+1)\sqrt{(t^2+2t+1)-9}}\,dt = \int \frac{1}{(t+1)\sqrt{(t+1)^2-3^2}}\,dt = \int \frac{1}{u\sqrt{u^2-3^2}}\,du$

where $u = t+1$ and $du = dt$

$= \frac{1}{3}\sec^{-1}\left|\frac{u}{3}\right| + C = \frac{1}{3}\sec^{-1}\left|\frac{t+1}{3}\right| + C$

78. $\int \frac{1}{(3t+1)\sqrt{9t^2+6t}}\,dt = \int \frac{1}{(3t+1)\sqrt{(9t^2+6t+1)-1}}\,dt = \int \frac{1}{(3t+1)\sqrt{(3t+1)^2-1^2}}\,dt = \frac{1}{3}\int \frac{1}{u\sqrt{u^2-1}}\,du$

where $u = 3t+1$ and $du = 3\,dt$

$= \frac{1}{3}\sec^{-1}|u| + C = \frac{1}{3}\sec^{-1}|3t+1| + C$

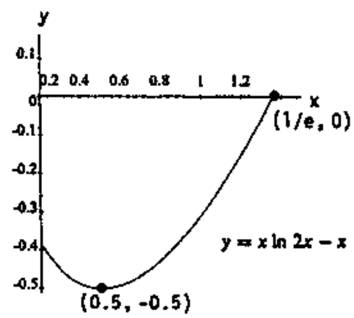
79. $\frac{df}{dx} = e^x + 1 \Rightarrow \left(\frac{df^{-1}}{dx}\right)_{x=f(\ln 2)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=\ln 2}} \Rightarrow \left(\frac{df^{-1}}{dx}\right)_{x=f(\ln 2)} = \frac{1}{(e^x+1)_{x=\ln 2}} = \frac{1}{2+1} = \frac{1}{3}$

80. $y = f(x) \Rightarrow y = 1 + \frac{1}{x} \Rightarrow \frac{1}{x} = y - 1 \Rightarrow x = \frac{1}{y-1} \Rightarrow f^{-1}(x) = \frac{1}{x-1}$; $f^{-1}(f(x)) = \frac{1}{\left(1+\frac{1}{x}\right)-1} = \frac{1}{\left(\frac{1}{x}\right)} = x$ and

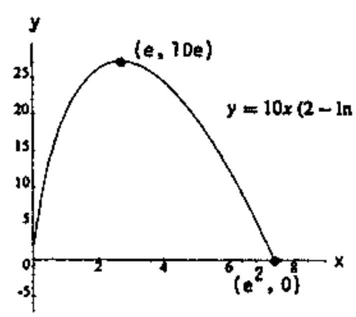
$f(f^{-1}(x)) = 1 + \frac{1}{\left(\frac{1}{x-1}\right)} = 1 + (x-1) = x$; $\left.\frac{df^{-1}}{dx}\right|_{f(x)} = \left.\frac{-1}{(x-1)^2}\right|_{f(x)} = \frac{-1}{\left[\left(1+\frac{1}{x}\right)-1\right]^2} = -x^2$;

$f'(x) = -\frac{1}{x^2} \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{f(x)} = \frac{1}{f'(x)}$

81. $y = x\ln 2x - x \Rightarrow y' = x\left(\frac{2}{2x}\right) + \ln(2x) - 1 = \ln 2x$; solving $y' = 0 \Rightarrow x = \frac{1}{2}$; $y' > 0$ for $x > \frac{1}{2}$ and $y' < 0$ for $x < \frac{1}{2} \Rightarrow$ relative minimum of $-\frac{1}{2}$ at $x = \frac{1}{2}$; $f\left(\frac{1}{2e}\right) = -\frac{1}{e}$ and $f\left(\frac{e}{2}\right) = 0 \Rightarrow$ absolute minimum is $-\frac{1}{2}$ at $x = \frac{1}{2}$ and the absolute maximum is 0 at $x = \frac{e}{2}$



82. $y = 10x(2 - \ln x) \Rightarrow y' = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right)$
$= 20 - 10\ln x - 10 = 10(1 - \ln x)$; solving $y' = 0$
$\Rightarrow x = e$; $y' < 0$ for $x > e$ and $y' > 0$ for $x < e$
$\Rightarrow$ relative maximum at $x = e$ of $10e$;
$y(e^2) = 10e^2(2 - 2\ln e) = 0 \Rightarrow$ absolute minimum is 0 at $x = e^2$ and the absolute maximum is $10e$ at $x = e$



83. $A = \int_1^e \frac{2 \ln x}{x}\, dx = \int_0^1 2u\, du = \left[u^2\right]_0^1 = 1$, where

$u = \ln x$ and $du = \frac{1}{x}\, dx$; $x = 1 \Rightarrow u = 0$, $x = e \Rightarrow u = 1$

84. (a) $A_1 = \int_{10}^{20} \frac{1}{x}\, dx = \left[\ln |x|\right]_{10}^{20} = \ln 20 - \ln 10 = \ln \frac{20}{10} = \ln 2$, and $A_2 = \int_1^2 \frac{1}{x}\, dx = \left[\ln |x|\right]_1^2 = \ln 2 - \ln 1 = \ln 2$

(b) $A_1 = \int_{ka}^{kb} \frac{1}{x}\, dx = \left[\ln |x|\right]_{ka}^{kb} = \ln kb - \ln ka = \ln \frac{kb}{ka} = \ln \frac{b}{a} = \ln b - \ln a$, and $A_2 = \int_a^b \frac{1}{x}\, dx = \left[\ln |x|\right]_a^b$

$= \ln b - \ln a$

85. $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$; $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \left(\frac{1}{x}\right)\sqrt{x} = \frac{1}{\sqrt{x}} \Rightarrow \left.\frac{dy}{dt}\right|_{e^2} = \frac{1}{e}$ m/sec

86. $y = 9e^{-x/3} \Rightarrow \frac{dy}{dx} = -3e^{-x/3}$; $\frac{dx}{dt} = \frac{(dy/dt)}{(dy/dx)} \Rightarrow \frac{dx}{dt} = \frac{\left(-\frac{1}{4}\right)\sqrt{9-y}}{-3e^{-x/3}}$; $x = 9 \Rightarrow y = 9e^{-3}$

$$\Rightarrow \left.\frac{dx}{dt}\right|_{x=9} = \frac{\left(-\frac{1}{4}\right)\sqrt{9-\frac{9}{e^3}}}{\left(-\frac{3}{e^3}\right)} = \frac{1}{4}\sqrt{e^3}\,\sqrt{e^3-1} \approx 4.895 \approx 5 \text{ ft/sec}$$

87. (a) $y = \frac{\ln x}{\sqrt{x}} \Rightarrow y' = \frac{1}{x\sqrt{x}} - \frac{\ln x}{2x^{3/2}} = \frac{2 - \ln x}{2x\sqrt{x}}$

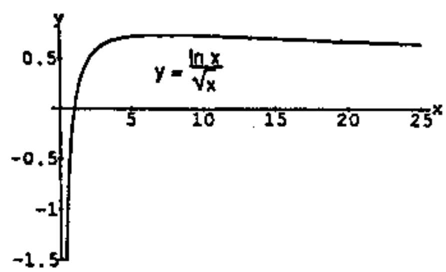
$\Rightarrow y'' = -\frac{3}{4}x^{-5/2}(2 - \ln x) - \frac{1}{2}x^{-5/2} = x^{-5/2}\left(\frac{3}{4}\ln x - 2\right)$;

solving $y' = 0 \Rightarrow \ln x = 2 \Rightarrow x = e^2$; $y' < 0$ for $x > e^2$ and

and $y' > 0$ for $x < e^2 \Rightarrow$ a maximum of $\frac{2}{e}$; $y'' = 0$

$\Rightarrow \ln x = \frac{8}{3} \Rightarrow x = e^{8/3}$; the curve is concave down on

$\left(0, e^{8/3}\right)$ and concave up on $\left(e^{8/3}, \infty\right)$



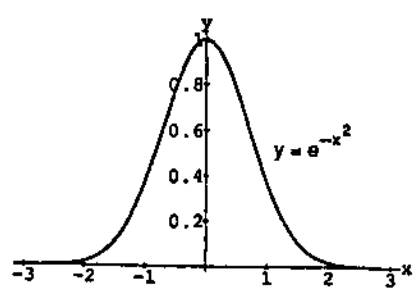
(b) $y = e^{-x^2} \Rightarrow y' = -2xe^{-x^2} \Rightarrow y'' = -2e^{-x^2} + 4x^2e^{-x^2}$;

solving $y' = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$ for

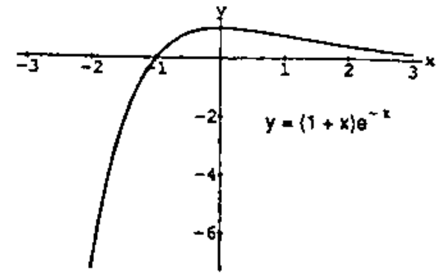
$x < 0 \Rightarrow$ a maximum at $x = 0$ of $e^0 = 1$; there are points

of inflection at $x = \pm\frac{1}{\sqrt{2}}$; the curve is concave down for

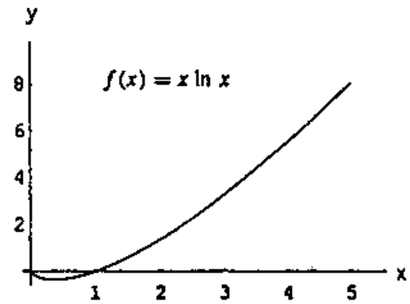
$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and concave up otherwise



(c) $y = (1+x)e^{-x} \Rightarrow y' = e^{-x} - (1+x)e^{-x} = -xe^{-x}$
$\Rightarrow y'' = -e^{-x} + xe^{-x} = (x-1)e^{-x}$; solving $y' = 0$
$\Rightarrow -xe^{-x} = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$
for $x < 0 \Rightarrow$ a maximum at $x = 0$ of $(1+0)e^0 = 1$;
there is a point of inflection at $x = 1$ and the curve is
concave up for $x > 1$ and concave down for $x < 1$



88. $y = x \ln x \Rightarrow y' = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$; solving $y' = 0$
$\Rightarrow \ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$; $y' > 0$ for
$x > e^{-1}$ and $y' < 0$ for $x < e^{-1} \Rightarrow$ an absolute minimum
of $e^{-1} \ln e^{-1} = -\frac{1}{e}$ at $x = e^{-1}$



89. Since the half life is 5700 years and $A(t) = A_0 e^{kt}$ we have $\frac{A_0}{2} = A_0 e^{5700k} \Rightarrow \frac{1}{2} = e^{5700k} \Rightarrow \ln(0.5) = 5700k$
$\Rightarrow k = \frac{\ln(0.5)}{5700}$. With 10% of the original carbon-14 remaining we have $0.1A_0 = A_0 e^{\frac{\ln(0.5)}{5700}t} \Rightarrow 0.1 = e^{\frac{\ln(0.5)}{5700}t}$
$\Rightarrow \ln(0.1) = \frac{\ln(0.5)}{5700}t \Rightarrow t = \frac{(5700)\ln(0.1)}{\ln(0.5)} \approx 18{,}935$ years (rounded to the nearest year).

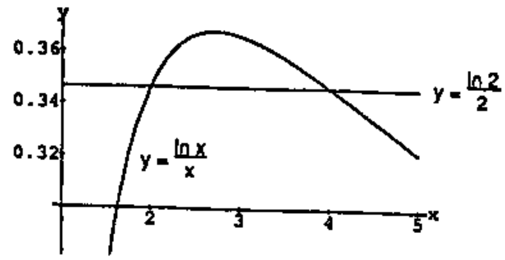
90. $T - T_s = (T_o - T_s)e^{-kt} \Rightarrow 180 - 40 = (220-40)e^{-k/4}$, time in hours, $\Rightarrow k = -4\ln\left(\frac{7}{9}\right) = 4\ln\left(\frac{9}{7}\right) \Rightarrow 70 - 40$
$= (220-40)e^{-4\ln(9/7)t} \Rightarrow t = \frac{\ln 6}{4\ln\left(\frac{9}{7}\right)} \approx 1.78$ hr ≈ 107 min, the total time $\Rightarrow$ the time it took to cool from
180° F to 70° F was $107 - 15 = 92$ min

91. $A = xy = xe^{-x^2} \Rightarrow \frac{dA}{dx} = e^{-x^2} + (x)(-2x)e^{-x^2} = e^{-x^2}(1-2x^2)$. Solving $\frac{dA}{dx} = 0 \Rightarrow 1 - 2x^2 = 0$
$\Rightarrow x = \frac{1}{\sqrt{2}}$; $\frac{dA}{dx} < 0$ for $x > \frac{1}{\sqrt{2}}$ and $\frac{dA}{dx} > 0$ for $0 < x < \frac{1}{\sqrt{2}} \Rightarrow$ absolute maximum of $\frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2e}}$ at
$x = \frac{1}{\sqrt{2}}$ units long by $y = e^{-1/2} = \frac{1}{\sqrt{e}}$ units high.

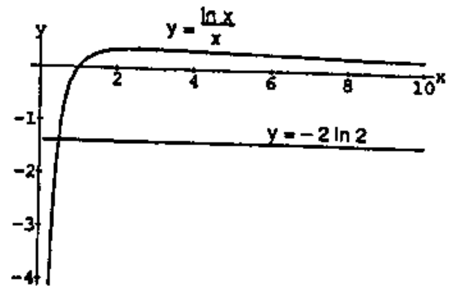
92. $A = xy = x\left(\frac{\ln x}{x^2}\right) = \frac{\ln x}{x} \Rightarrow \frac{dA}{dx} = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1-\ln x}{x^2}$. Solving $\frac{dA}{dx} = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow x = e$;
$\frac{dA}{dx} < 0$ for $x > e$ and $\frac{dA}{dx} > 0$ for $x < e \Rightarrow$ absolute maximum of $\frac{\ln e}{e} = \frac{1}{e}$ at $x = e$ units long and $y = \frac{1}{e^2}$ units
high.

93. $K = \ln(5x) - \ln(3x) = \ln 5 + \ln x - \ln 3 - \ln x = \ln 5 - \ln 3 = \ln\frac{5}{3}$

94. (a) No, there are two intersections: one at $x = 2$ and the other at $x = 4$



(b) Yes, because there is only one intersection point.



95. $\theta = \pi - \cot^{-1}\left(\frac{x}{60}\right) - \cot^{-1}\left(\frac{5}{3} - \frac{x}{30}\right)$, $0 < x < 50 \Rightarrow \frac{d\theta}{dx} = \frac{\left(\frac{1}{60}\right)}{1 + \left(\frac{x}{60}\right)^2} + \frac{\left(-\frac{1}{30}\right)}{1 + \left(\frac{50 - x}{30}\right)^2}$

$= 30\left[\frac{2}{60^2 + x^2} - \frac{1}{30^2 + (50 - x)^2}\right]$; solving $\frac{d\theta}{dx} = 0 \Rightarrow x^2 - 200x + 3200 = 0 \Rightarrow x = 100 \pm 20\sqrt{17}$, but

$100 + 20\sqrt{17}$ is not in the domain; $\frac{d\theta}{dx} > 0$ for $x < 20\left(5 - \sqrt{17}\right)$ and $\frac{d\theta}{dx} < 0$ for $20\left(5 - \sqrt{17}\right) < x < 50$ $\Rightarrow x = 20\left(5 - \sqrt{17}\right) \approx 17.54$ m maximizes θ

96. $v = x^2 \ln\left(\frac{1}{x}\right) = x^2 (\ln 1 - \ln x) = -x^2 \ln x \Rightarrow \frac{dv}{dx} = -2x \ln x - x^2\left(\frac{1}{x}\right) = -x(2 \ln x + 1)$; solving $\frac{dv}{dx} = 0$ $\Rightarrow 2 \ln x + 1 = 0 \Rightarrow \ln x = -\frac{1}{2} \Rightarrow x = e^{-1/2}$; $\frac{dv}{dx} < 0$ for $x > e^{-1/2}$ and $\frac{dv}{dx} > 0$ for $x < e^{-1/2} \Rightarrow$ a relative maximum at $x = e^{-1/2}$; $\frac{r}{h} = x$ and $r = 1 \Rightarrow h = e^{1/2} = \sqrt{e} \approx 1.65$ cm

97. (a) Force = Mass times Acceleration (Newton's Second Law) or $F = ma$. Let $a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\,\frac{dv}{ds}$. Then $ma = -mgR^2s^{-2} \Rightarrow a = -gR^2s^{-2} \Rightarrow v\,\frac{dv}{ds} = -gR^2s^{-2} \Rightarrow v\,dv = -gR^2s^{-2}\,ds \Rightarrow \int v\,dv = \int -gR^2s^{-2}\,ds$ $\Rightarrow \frac{v^2}{2} = \frac{gR^2}{s} + C \Rightarrow v^2 = \frac{2gR^2}{s} + 2C_1 = \frac{2gR^2}{s} + C$. When $t = 0$, $v = v_0$ and $s = R \Rightarrow v_0^2 = \frac{2gR^2}{R} + C$ $\Rightarrow C = v_0^2 - 2gR \Rightarrow v^2 = \frac{2gR^2}{s} + v_0^2 - 2gR$

(b) If $v_0 = \sqrt{2gR}$, then $v^2 = \frac{2gR^2}{s} \Rightarrow v = \sqrt{\frac{2gR^2}{s}}$, since $v \ge 0$ if $v_0 \ge \sqrt{2gR}$. Then $\frac{ds}{dt} = \frac{\sqrt{2gR^2}}{\sqrt{s}}$

$\Rightarrow \sqrt{s}\,ds = \sqrt{2gR^2}\,dt \Rightarrow \int s^{1/2}\,ds = \int \sqrt{2gR^2}\,dt \Rightarrow \frac{2}{3}s^{3/2} = \left(\sqrt{2gR^2}\right)t + C_1 \Rightarrow s^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)t + C$;

$t = 0$ and $s = R \Rightarrow R^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)(0) + C \Rightarrow C = R^{3/2} \Rightarrow s^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)t + R^{3/2}$

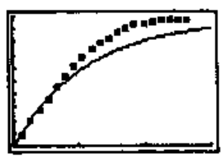
$= \left(\frac{3}{2}R\sqrt{2g}\right)t + R^{3/2} = R^{3/2}\left[\left(\frac{3}{2}R^{-1/2}\sqrt{2g}\right)t + 1\right] = R^{3/2}\left[\left(\frac{3\sqrt{2gR}}{2R}\right)t + 1\right]$

$= R^{3/2}\left[\left(\frac{3v_0}{2R}\right)t + 1\right] \Rightarrow s = R\left[1 + \left(\frac{3v_0}{2R}\right)t\right]^{2/3}$

98. $\frac{v_0 m}{k} = \text{coasting distance} \Rightarrow \frac{(0.86)(30.84)}{k} = 0.97 \Rightarrow k \approx 27.343$

$s(t) = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right) \Rightarrow s(t) = 0.97\left(1 - e^{-(27.343/30.84)t}\right) \Rightarrow s(t) = 0.97\left(1 - e^{-0.8866t}\right)$

A graph of the model is shown superimposed on a graph of the data.



[0, 3] by [0,1]

99. $\frac{dy}{dx} = e^{-x-y-2} \Rightarrow \frac{dy}{dx} = e^{-x-2} \cdot e^{-y} \Rightarrow e^y\,dy = e^{-x-2}\,dx = \int e^y\,dy = \int e^{-x-2}\,dx \Rightarrow e^y = -e^{-x-2} + C$; $x = 0$ and $y = -2 \Rightarrow e^{-2} = -e^{-2} + C \Rightarrow C = 2e^{-2} \Rightarrow e^y = -e^{-x-2} + 2e^{-2} \Rightarrow \ln\left(e^y\right) = \ln\left(-e^{-x-2} + 2e^{-2}\right)$

$\Rightarrow y = \ln\left(-e^{-x-2} + 2e^{-2}\right)$

100. $\frac{dy}{dx} = -\frac{y \ln y}{1 + x^2} \Rightarrow \frac{dy}{y \ln y} = -\frac{dx}{1 + x^2} \Rightarrow \int \frac{\left(\frac{1}{y}\right)}{\ln y}\,dy = -\int \frac{1}{1 + x^2}\,dx \Rightarrow \ln(\ln y) = -\tan^{-1}x + C$; $x = 0$ and $y = e^2 \Rightarrow \ln\left(\ln e^2\right) = -\tan^{-1}0 + C \Rightarrow \ln 2 = C \Rightarrow \ln(\ln y) = -\tan^{-1}x + \ln 2$

$\Rightarrow e^{\ln(\ln y)} = e^{(-\tan^{-1}x + \ln 2)} \Rightarrow \ln y = e^{(-\tan^{-1}x + \ln 2)} \Rightarrow e^{\ln y} = e^{e^{(-\tan^{-1}x + \ln 2)}} \Rightarrow y = e^{e^{(-\tan^{-1}x + \ln 2)}}$ or

$y = \exp\left(\exp\left(-\tan^{-1}x + \ln 2\right)\right)$

101. $\frac{dy}{dx} + \left(\frac{2}{x+1}\right)y = \frac{x}{x+1} \Rightarrow P(x) = \frac{2}{x+1}$, $Q(x) = \frac{x}{x+1} \Rightarrow \int P(x)\,dx = \int \left(\frac{2}{x+1}\right)dx = 2\ln|x+1| = \ln(x+1)^2$

$\Rightarrow v(x) = e^{\ln(x+1)^2} = (x+1)^2 \Rightarrow y = \frac{1}{(x+1)^2}\int (x+1)^2\left(\frac{x}{x+1}\right)dx = \frac{1}{(x+1)^2}\int \left(x^2 + x\right)dx$

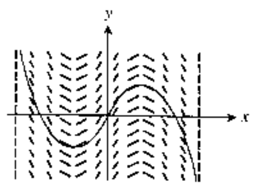
$= \left(\frac{1}{x+1}\right)^2\left(\frac{x^3}{3}+\frac{x^2}{2}+C\right)$; $x = 0$ and $y = 1 \Rightarrow 1 = 0+0+C \Rightarrow C = 1 \Rightarrow y = \frac{1}{(x+1)^2}\cdot\left(\frac{x^3}{3}+\frac{x^2}{2}+1\right)$

102. $\frac{dy}{dx}+\left(\frac{2}{x}\right)y = \frac{x^2+1}{x} \Rightarrow P(x) = \frac{2}{x}$, $Q(x) = \frac{x^2+1}{x} \Rightarrow \int P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2 \Rightarrow v(x) = e^{\ln x^2}$

$= x^2 \Rightarrow y = \frac{1}{x^2}\int x^2\left(\frac{x^2+1}{x}\right)dx = \frac{1}{x^2}\int (x^3+x)\,dx = \frac{1}{x^2}\left(\frac{x^4}{4}+\frac{x^2}{2}+C\right) = \frac{x^2}{4}+\frac{1}{2}+\frac{C}{x^2}$; $x = 1$ and $y = 1$

$\Rightarrow 1 = \frac{1}{4}+\frac{1}{2}+C \Rightarrow C = \frac{1}{4} \Rightarrow y = \frac{x^2}{4}+\frac{1}{2}+\frac{1}{4x^2}$

103.



104. To find the approximate values let $y_n = y_{n-1} + (y_{n-1} + \cos x_{n-1})(0.1)$ with $x_0 = 0$, $y_0 = 0$, and 20 steps. Use a spreadsheet, graphing calculator, or CAS to obtain the values in the following table.

x	y	x	y
0	0	1.1	1.6241
0.1	0.1000	1.2	1.8319
0.2	0.2095	1.3	2.0513
0.3	0.3285	1.4	2.2832
0.4	0.4568	1.5	2.5285
0.5	0.5946	1.6	2.7884
0.6	0.7418	1.7	3.0643
0.7	0.8986	1.8	3.3579
0.8	1.0649	1.9	3.6709
0.9	1.2411	2.0	4.0057
1.0	1.4273		

105. To find the approximate solution let $z_n = y_{n-1} + ((2-y_{n-1})(2x_{n-1}+3))(0.1)$ and

$y_n = y_{n-1} + \left(\frac{(2-y_{n-1})(2x_{n-1}+3)+(2-z_n)(2x_n+3)}{2}\right)(0.1)$ with initial values $x_0 = -3$, $y_0 = 1$,

and 20 steps. Use a spreadsheet, graphing calculator, or CAS to obtain the values in the following table.

x	y	x	y
−3	1	−1.9	−5.9686
−2.9	0.6680	−1.8	−6.5456
−2.8	0.2599	−1.7	−6.9831
−2.7	−0.2294	−1.6	−7.2562
−2.6	−0.8011	−1.5	−7.3488
−2.5	−1.4509	−1.4	−7.2553
−2.4	−2.1687	−1.3	−6.9813
−2.3	−2.9374	−1.2	−6.5430
−2.2	−3.7333	−1.1	−5.9655
−2.1	−4.5268	−1.0	−5.2805
−2.0	−5.2840		

106. To estimate y(3), let $z_n = y_{n-1} + \left(\frac{x_{n-1} - 2y_{n-1}}{x_{n-1} + 1}\right)(0.05)$ and $y_n = y_{n-1} + \frac{1}{2}\left(\frac{x_{n-1} - 2y_{n-1}}{x_{n-1} + 1} + \frac{x_n - 2z_n}{x_n + 1}\right)(0.05)$ with initial values $x_0 = 0$, $y_0 = 1$, and 60 steps. Use a spreadsheet, programmable calculator, or CAS to obtain $y(3) \approx 0.9063$.

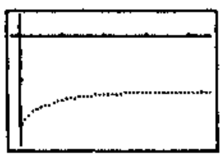
107. To estimate y(4), let $y_n = y_{n-1} + \left(\frac{x_{n-1}^2 - 2y_{n-1} + 1}{x_{n-1}}\right)(0.05)$ with initial values $x_0 = 1$, $y_0 = 1$, and 60 steps.

Use a spreadsheet, programmable calculator, or CAS to obtain $y(4) \approx 4.4974$.

108. Let $y_n = y_{n-1} + \left(\frac{1}{e^{x_{n-1} + y_{n-1} + 2}}\right)(dx)$ with starting values $x_0 = 0$ and $y_0 = -2$, and steps of 0.1 and −0.1.

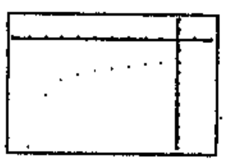
Use a spreadsheet, programmable calculator, or CAS to generate the following graphs.

(a)



[−0.2, 4.5] by [−2.5, 0.5]

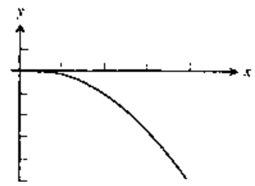
(b) Note that we choose a small interval of x-values because the y-values decrease very rapidly and our calculator cannot handle the calculations for $x \le -1$. (This occurs because the analytic solution is $y = -2 + \ln(2 - e^{-x})$, which has an asymptote at $x = -\ln 2 \approx -0.69$. Obviously, the Euler approximations are misleading for $x \le -0.7$.)



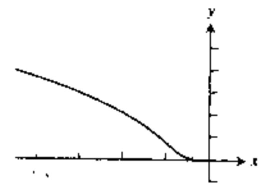
[−1, 0.2] by [−10, 2]

109. Let $z_n = y_{n-1} - \left(\frac{x_{n-1}^2 + y_{n-1}}{e^{y_{n-1}} + x_{n-1}}\right)(dx)$ and $y_n = y_{n-1} + \frac{1}{2}\left(\frac{x_{n-1}^2 + y_{n-1}}{e^{y_{n-1}} + x_{n-1}} + \frac{x_n^2 + z_n}{e^{z_n} + x_n}\right)(dx)$ with starting values $x_0 = 0$, $y_0 = 0$, and steps of 0.1 and −0.1. Use a spreadsheet, programmable calculator, or CAS to generate the following graphs.

(a)



(b)



110. (a) $\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{800}\right) \Rightarrow \frac{dP}{dt} = 0.002P\left(\frac{800 - P}{800}\right) \Rightarrow \frac{800}{P(800 - P)}\,dP = 0.002\,dt \Rightarrow \frac{(800 - P) + P}{P(800 - P)} = 0.002\,dt$

$\Rightarrow \int \left(\frac{1}{P} + \frac{1}{800 - P}\right) dP = 0.002\,dt \Rightarrow \ln|P| - \ln|800 - P| = 0.002t + C \Rightarrow \ln\left|\frac{P}{800 - P}\right| = 0.002t + C$

$\Rightarrow \ln\left|\frac{800 - P}{P}\right| = -0.002t - C \Rightarrow \left|\frac{800 - P}{P}\right| = e^{-0.002t - C} \Rightarrow \frac{800 - P}{P} = \pm e^{-C}e^{-0.002t}$

$\Rightarrow \frac{800}{P} - 1 = Ae^{-0.002t} \Rightarrow P = \frac{800}{1 + Ae^{-0.002t}}$

Initial condition: $P(0) = 50 \Rightarrow 50 = \frac{800}{1 + Ae^{0}} \Rightarrow 1 + A = 16 \Rightarrow A = 15$

Solution: $P = \frac{800}{1 + 15e^{-.002t}}$

(b) $\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{800}\right)$, $P(0) = 50 \Rightarrow P_{n+1} = P_n + 0.002P_n\left(1 - \frac{P_n}{800}\right)dt = P_n + 0.001P_n\left(1 - \frac{P_n}{800}\right)$

On a TI-02 Plus calculator home screen, type the following commands:

50 STO> p:0 STO> t:p (enter)
p+0.001*p*(1 – p/800) STO> p:t+0.5 STO> t:p (enter, 40 times)

The last value displayed gives $P_{Euler}(20) \approx 51.9073$

From part (a), $P_{exact}(20) = \frac{800}{1 + 15e^{-0.002(20)}} \approx 51.9081 \Rightarrow \left|\frac{P_{Euler}(20) - P_{exact}(20)}{P_{exact}(20)}\right| \times 100\%$

$= \left|\frac{51.9073 - 51.9081}{51.9081}\right| \times 100\% \approx 0.154\%$

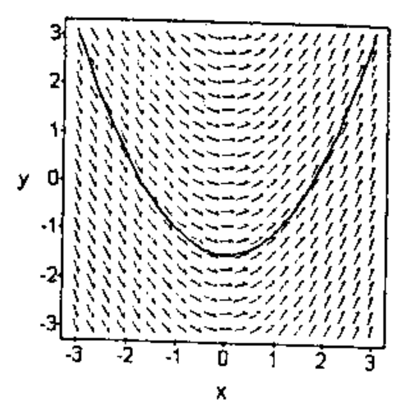
111.

x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−0.8	−0.56	−0.28	0.04	0.4

$\frac{dy}{dx} = x \Rightarrow dy = x\,dx \Rightarrow y = \frac{x^2}{2} + C$; $x = 1$ and $y = -1$

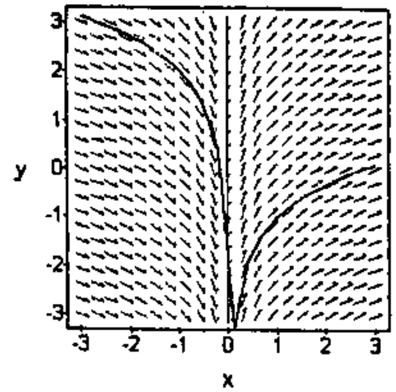
$\Rightarrow -1 = \frac{1}{2} + C \Rightarrow C = -\frac{3}{2} \Rightarrow y\text{ (exact)} = \frac{x^2}{2} - \frac{3}{2}$

$\Rightarrow y(2) = \frac{2^2}{2} - \frac{3}{2} = \frac{1}{2}$ is the exact value



112.

x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−0.8	−0.6333	−0.4904	−0.3654	−0.2544



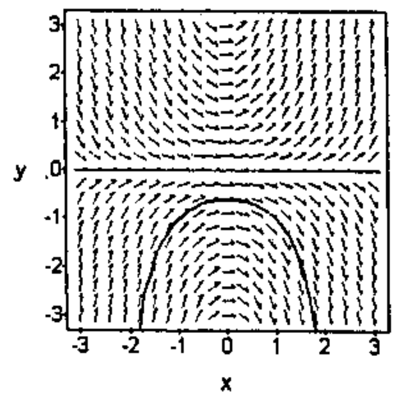
$\frac{dy}{dx} = \frac{1}{x} \Rightarrow dy = \frac{1}{x}\,dx \Rightarrow y = \ln|x| + C$; $x = 1$ and $y = -1$

$\Rightarrow -1 = \ln 1 + C \Rightarrow C = -1 \Rightarrow y \text{ (exact)} = \ln|x| - 1$

$\Rightarrow y(2) = \ln 2 - 1 \approx -0.3069$ is the exact value

113.

x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−1.2	−1.488	−1.9046	−2.5141	−3.4192



$\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x\,dx \Rightarrow \ln|y| = \frac{x^2}{2} + C \Rightarrow y = e^{\frac{x^2}{2} + C} = e^{x^2/2} \cdot e^C$

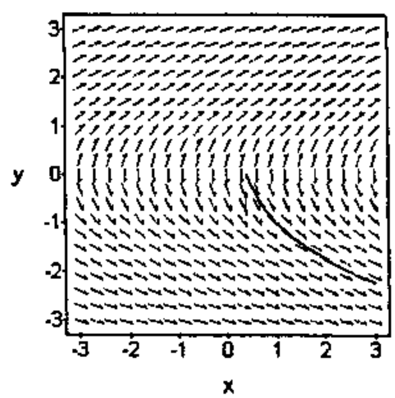
$= C_1 e^{x^2/2}$; $x = 1$ and $y = -1 \Rightarrow -1 = C_1 e^{1/2} \Rightarrow C_1 = -e^{-1/2}$

$\Rightarrow y \text{ (exact)} = -e^{-1/2} \cdot e^{x^2/2} = -e^{\left(x^2-1\right)/2} \Rightarrow y(2) = -e^{3/2}$

≈ -4.4817 is the exact value

114.

x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−1.2	−1.3667	−1.5130	−1.6452	−1.7668



$\frac{dy}{dx} = \frac{1}{y} \Rightarrow y\,dy = dx \Rightarrow \frac{y^2}{2} = x + C$; $x = 1$ and $y = -1$

$\Rightarrow \frac{1}{2} = 1 + C \Rightarrow C = -\frac{1}{2} \Rightarrow y^2 = 2x - 1 \Rightarrow y \text{ (exact)} = -\sqrt{2x-1}$

$\Rightarrow y(2) = -\sqrt{3} \approx -1.7321$ is the exact value

CHAPTER 6 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. $A_1 = \int_1^e \frac{2\log_2 x}{x}\,dx = \frac{2}{\ln 2}\int_1^e \frac{\ln x}{x}\,dx = \left[\frac{(\ln x)^2}{\ln 2}\right]_1^e = \frac{1}{\ln 2}$; $A_2 = \int_1^e \frac{2\log_4 x}{4}\,dx = \frac{2}{\ln 4}\int_1^e \frac{\ln x}{x}\,dx$

$= \left[\frac{(\ln x)^2}{2\ln 2}\right]_1^e = \frac{1}{2\ln 2} \Rightarrow A_1 : A_2 = 2 : 1$

2. $\ln x^{(x^x)} = x^x \ln x$ and $\ln(x^x)^x = x \ln x^x = x^2 \ln x$; then, $x^x \ln x = x^2 \ln x \Rightarrow x^x = x^2 \Rightarrow x \ln x = 2 \ln x$ $\Rightarrow x = 2$. Therefore, $x^{(x^x)} = (x^x)^x$ when $x = 2$.

3. $f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)}\, g'(x)$, where $g'(x) = \frac{x}{1+x^4} \Rightarrow f'(2) = e^0\left(\frac{2}{1+16}\right) = \frac{2}{17}$

4. (a) $\frac{df}{dx} = \frac{2 \ln e^x}{e^x} \cdot e^x = 2x$

(b) $f(0) = \int_1^1 \frac{2 \ln t}{t}\, dt = 0$

(c) $\frac{df}{dx} = 2x \Rightarrow f(x) = x^2 + C$; $f(0) = 0 \Rightarrow C = 0 \Rightarrow f(x) = x^2 \Rightarrow$ the graph of $f(x)$ is a parabola

5. (a) The figure shows that $\frac{\ln e}{e} > \frac{\ln \pi}{\pi} \Rightarrow \pi \ln e > e \ln \pi \Rightarrow \ln e^\pi > \ln \pi^e \Rightarrow e^\pi > \pi^e$

(b) $y = \frac{\ln x}{x} \Rightarrow y' = \left(\frac{1}{x}\right)\left(\frac{1}{x}\right) - \frac{\ln x}{x^2} \Rightarrow \frac{1 - \ln x}{x^2}$; solving $y' = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$; $y' < 0$ for $x > e$ and $y' > 0$ for $0 < x < e \Rightarrow$ an absolute maximum occurs at $x = e$

6. The area of the shaded region is $\int_0^1 \sin^{-1} x\, dx = \int_0^1 \sin^{-1} y\, dy$, which is the same as the area of the region to the left of the curve $y = \sin x$ (and part of the rectangle formed by the coordinate axes and dashed lines $y = 1$, $x = \frac{\pi}{2}$). The area of the rectangle is $\frac{\pi}{2} = \int_0^1 \sin^{-1} y\, dy + \int_0^{\pi/2} \sin x\, dx$, so we have $\frac{\pi}{2} = \int_0^1 \sin^{-1} x\, dx + \int_0^{\pi/2} \sin x\, dx \Rightarrow \int_0^{\pi/2} \sin x\, dx = \frac{\pi}{2} - \int_0^1 \sin^{-1} x\, dx$.

7. (a) slope of L_3 < slope of L_2 < slope of $L_1 \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$

(b) area of small (shaded) rectangle < area under curve < area of large rectangle

$\Rightarrow \frac{1}{b}(b - a) < \int_a^b \frac{1}{x}\, dx < \frac{1}{a}(b - a) \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$

8. Method 1: Use a CAS or a numerical integral function on a calculator or spreadsheet to define $y_1 = x^2 \ln x$ and $y_2 = \frac{d}{dx}\left(\frac{x^3 \ln x}{3} - \frac{x^3}{9} + C\right)$, then compare the graph of y_1 with that of y_2. The graphs should be the same.
Method 2: Use a CAS or a numerical integration function on a calculator or spreadsheet to define $y_1 = \int_a^x t^2 \ln t\, dt$ and $y_2 = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$ (you pick an $a > 0$ and any value for C), then compare the graph of y_1 with that of y_2. The graphs should be the same except for a vertical translation.

9. Use the Fundamental Theorem of Calculus.

$$y' = \frac{d}{dx}\left(\int_0^x \sin t^2\, dt\right) + \frac{d}{dx}\left(x^3 + x + 2\right) = \left(\sin x^2\right) + \left(3x^2 + 1\right)$$

$$y'' = \frac{d}{dx}\left(\sin x^2 + 3x^2 + 1\right) = \left(\cos x^2\right)(2x) + 6x = 2x \cos\left(x^2\right) + 6x$$

Thus, the differential equation is satisfied. Verify the initial conditions:

$$y'(0) = \left(\sin 0^2\right) + 3(0)^2 + 1 = 1 \text{ and } y(0) = \int_0^0 \sin\left(t^2\right) dt + 0^3 + 0 + 2 = 2$$

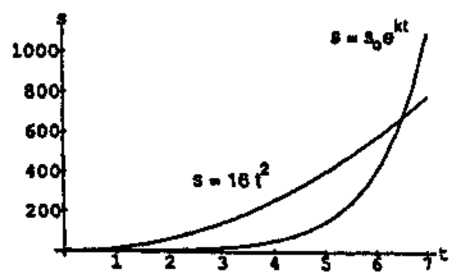
10. (a) $f'(x) = \frac{d}{dx}\int_0^x u(t)\, dt = u(x)$ and $g'(x) = \frac{d}{dx}\int_3^x u(t)\, dt = u(x)$

(b) $C = f(x) - g(x) = \int_0^x u(t)\, dt - \int_3^x u(t)\, dt = \int_0^x u(t)\, dt + \int_x^3 u(t)\, dt = \int_0^3 u(t)\, dt$

11. $V = \pi \int_{1/4}^{4} \left(\frac{1}{2\sqrt{x}}\right)^2 dx = \frac{\pi}{4}\int_{1/4}^{4} \frac{1}{x}\, dx = \frac{\pi}{4}\left[\ln|x|\right]_{1/4}^{4} = \frac{\pi}{4}\left(\ln 4 - \ln \frac{1}{4}\right) = \frac{\pi}{4}\ln 16 = \frac{\pi}{4}\ln\left(2^4\right) = \pi \ln 2$

12. $\frac{ds}{dt} = ks \Rightarrow \frac{ds}{s} = k\, dt \Rightarrow \ln s = kt + C \Rightarrow s = s_0 e^{kt}$

$\Rightarrow$ the 14th century model of free fall was exponential; note that the motion starts too slowly at first and then becomes too fast after about 7 seconds



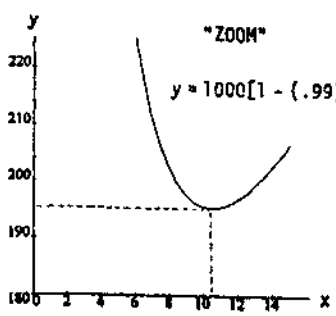
13. (a) $L = k\left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right) \Rightarrow \frac{dL}{d\theta} = k\left(\frac{b\csc^2\theta}{R^4} - \frac{b\csc\theta\cot\theta}{r^4}\right)$; solving $\frac{dL}{d\theta} = 0$

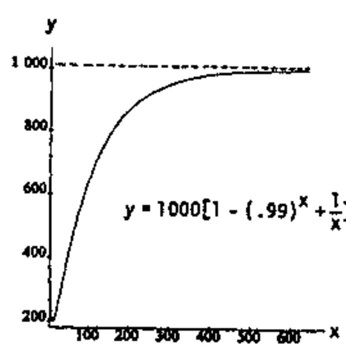
$\Rightarrow r^4 b\csc^2\theta - bR^4\csc\theta\cot\theta = 0 \Rightarrow (b\csc\theta)\left(r^4\csc\theta - R^4\cot\theta\right) = 0$; but $b\csc\theta \neq 0$ since

$\theta \neq \frac{\pi}{2} \Rightarrow r^4\csc\theta - R^4\cot\theta = 0 \Rightarrow \cos\theta = \frac{r^4}{R^4} \Rightarrow \theta = \cos^{-1}\left(\frac{r^4}{R^4}\right)$, the critical value of θ

(b) $\theta = \cos^{-1}\left(\frac{5}{6}\right)^4 \approx \cos^{-1}(0.48225) \approx 61^\circ$

14. Two views of the graph of $y = 1000\left[1 - (.99)^x + \frac{1}{x}\right]$ are shown below.





At about $x = 11$ there is a minimum. There is no maximum; however, the curve is asymptotic to $y = 1000$. The curve is near 1000 when $x \geq 643$.

15. (a) $\frac{dy}{dt} = k\,\frac{A}{V}(c - y) \Rightarrow dy = -k\,\frac{A}{V}(y - c)\,dt \Rightarrow \frac{dy}{y - c} = -k\,\frac{A}{V}\,dt \Rightarrow \int \frac{dy}{y-c} = -\int k\,\frac{A}{V}\,dt \Rightarrow \ln|y - c|$

$= -k\,\frac{A}{V}t + C_1 \Rightarrow y - c = \pm e^{C_1}e^{-k\frac{A}{V}t}$. Apply the initial condition, $y(0) = y_0 \Rightarrow y_0 = c + C \Rightarrow C = y_0 - c$

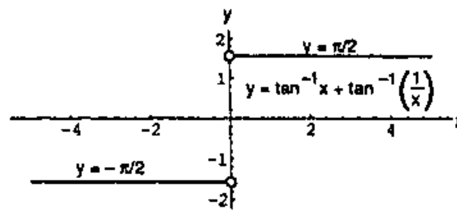
$\Rightarrow y = c + (y_0 - c)e^{-k\frac{A}{V}t}$.

(b) Steady state solution: $y_\infty = \lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[c + (y_0 - c)e^{-k\frac{A}{V}t}\right] = c + (y_0 - c)(0) = c$

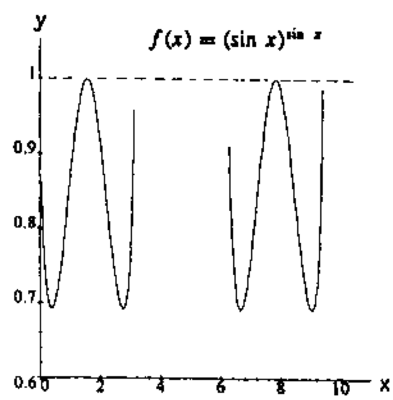
16. $y = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) \Rightarrow y' = \frac{1}{1+x^2} + \frac{\left(-\frac{1}{x^2}\right)}{\left(1+\frac{1}{x^2}\right)}$

$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \Rightarrow \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ is a constant and the constant is $\frac{\pi}{2}$ for $x > 0$; it is $-\frac{\pi}{2}$ for $x < 0$ since $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ is odd. Next the $\lim_{x\to 0^+}\left[\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right]$

$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$ and $\lim_{x\to 0^-}\left(\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right) = 0 + \left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}$



17. In the interval $\pi < x < 2\pi$ the function $\sin x < 0$ $\Rightarrow (\sin x)^{\sin x}$ is not defined for all values in that interval or its translation by 2π.



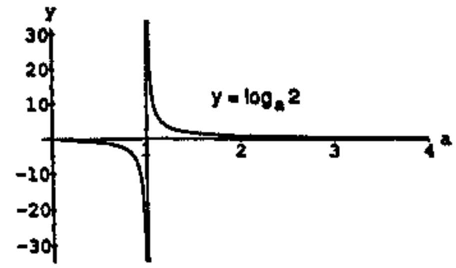
18. (a) $\lim\limits_{a \to 0^+} \log_a 2 = \lim\limits_{a \to 0^+} \frac{\ln 2}{\ln a} = 0;$

$\lim\limits_{a \to 1^-} \log_a 2 = \lim\limits_{a \to 1^-} \frac{\ln 2}{\ln a} = -\infty;$

$\lim\limits_{a \to 1^+} \log_a 2 = \lim\limits_{a \to 1^+} \frac{\ln 2}{\ln 1} = \infty;$

$\lim\limits_{a \to \infty} \log_a 2 = \lim\limits_{a \to \infty} \frac{\ln 2}{\ln a} = 0$

(b)



NOTES:

CHAPTER 7 TECHNIQUES OF INTEGRATION, L'HÔPITAL'S RULE, AND IMPROPER INTEGRALS

7.1 BASIC INTEGRATION FORMULAS

1. $\int \frac{16x\,dx}{\sqrt{8x^2+1}}$; $\begin{bmatrix} u = 8x^2+1 \\ du = 16x\,dx \end{bmatrix} \to \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{8x^2+1} + C$

2. $\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$; $\begin{bmatrix} u = 1+3\sin x \\ du = 3\cos x\,dx \end{bmatrix} \to \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1+3\sin x} + C$

3. $\int 3\sqrt{\sin v}\cos v\,dv$; $\begin{bmatrix} u = \sin v \\ du = \cos v\,dv \end{bmatrix} \to \int 3\sqrt{u}\,du = 3 \cdot \frac{2}{3} u^{3/2} + C = 2(\sin v)^{3/2} + C$

4. $\int \cot^3 y \csc^2 y\,dy$; $\begin{bmatrix} u = \cot y \\ du = -\csc^2 y\,dy \end{bmatrix} \to \int u^3(-du) = -\frac{u^4}{4} + C = \frac{-\cot^4 y}{4} + C$

5. $\int_0^1 \frac{16x\,dx}{8x^2+2}$; $\begin{bmatrix} u = 8x^2+2 \\ du = 16x\,dx \\ x = 0 \Rightarrow u = 2, \quad x = 1 \Rightarrow u = 10 \end{bmatrix} \to \int_2^{10} \frac{du}{u} = [\ln|u|]_2^{10} = \ln 10 - \ln 2 = \ln 5$

6. $\int_{\pi/4}^{\pi/3} \frac{\sec^2 z\,dz}{\tan z}$; $\begin{bmatrix} u = \tan z \\ du = \sec^2 z\,dz \\ z = \frac{\pi}{4} \Rightarrow u = 1,\, z = \frac{\pi}{3} \Rightarrow u = \sqrt{3} \end{bmatrix} \to \int_1^{\sqrt{3}} \frac{1}{u}\,du = [\ln|u|]_1^{\sqrt{3}} = \ln\sqrt{3} - \ln 1 = \ln\sqrt{3}$

7. $\int \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)}$; $\begin{bmatrix} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2\,du = \frac{dx}{\sqrt{x}} \end{bmatrix} \to \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln(\sqrt{x}+1) + C$

8. $\int \frac{dx}{x-\sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x}-1)}$; $\begin{bmatrix} u = \sqrt{x}-1 \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2\,du = \frac{dx}{\sqrt{x}} \end{bmatrix} \to \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln\left|\sqrt{x}-1\right| + C$

9. $\int \cot(3-7x)\,dx;\ \begin{bmatrix} u = 3-7x \\ du = -7\,dx \end{bmatrix} \Rightarrow -\frac{1}{7}\int \cot u\,du = -\frac{1}{7}\ln|\sin u| + C = -\frac{1}{7}\ln|\sin(3-7x)| + C$

10. $\int \csc(\pi x - 1)\,dx;\ \begin{bmatrix} u = \pi x - 1 \\ du = \pi\,dx \end{bmatrix} \Rightarrow \int \csc u \cdot \frac{du}{\pi} = \frac{-1}{\pi}\ln|\csc u + \cot u| + C$

$= -\frac{1}{\pi}\ln|\csc(\pi x - 1) + \cot(\pi x - 1)| + C$

11. $\int e^{\theta}\csc(e^{\theta}+1)\,d\theta;\ \begin{bmatrix} u = e^{\theta}+1 \\ du = e^{\theta}\,d\theta \end{bmatrix} \Rightarrow \int \csc u\,du = -\ln|\csc u + \cot u| + C = -\ln|\csc(e^{\theta}+1) + \cot(e^{\theta}+1)| + C$

12. $\int \frac{\cot(3+\ln x)}{x}\,dx;\ \begin{bmatrix} u = 3 + \ln x \\ du = \frac{dx}{x} \end{bmatrix} \Rightarrow \int \cot u\,du = \ln|\sin u| + C = \ln|\sin(3+\ln x)| + C$

13. $\int \sec\frac{t}{3}\,dt;\ \begin{bmatrix} u = \frac{t}{3} \\ du = \frac{dt}{3} \end{bmatrix} \Rightarrow \int 3\sec u\,du = 3\ln|\sec u + \tan u| + C = 3\ln\left|\sec\frac{t}{3} + \tan\frac{t}{3}\right| + C$

14. $\int x\sec(x^2-5)\,dx;\ \begin{bmatrix} u = x^2-5 \\ du = 2x\,dx \end{bmatrix} \Rightarrow \int \frac{1}{2}\sec u\,du = \frac{1}{2}\ln|\sec u + \tan u| + C$

$= \frac{1}{2}\ln|\sec(x^2-5) + \tan(x^2-5)| + C$

15. $\int \csc(s-\pi)\,ds;\ \begin{bmatrix} u = s-\pi \\ du = ds \end{bmatrix} \Rightarrow \int \csc u\,du = -\ln|\csc u + \cot u| + C = -\ln|\csc(s-\pi) + \cot(s-\pi)| + C$

16. $\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta;\ \begin{bmatrix} u = \frac{1}{\theta} \\ du = \frac{-d\theta}{\theta^2} \end{bmatrix} \Rightarrow \int -\csc u\,du = \ln|\csc u + \cot u| + C = \ln\left|\csc\frac{1}{\theta} + \cot\frac{1}{\theta}\right| + C$

17. $\int_0^{\sqrt{\ln 2}} 2xe^{x^2}\,dx;\ \begin{bmatrix} u = x^2 \\ du = 2x\,dx \\ x = 0 \Rightarrow u = 0,\ x = \sqrt{\ln 2} \Rightarrow u = \ln 2 \end{bmatrix} \Rightarrow \int_0^{\ln 2} e^u\,du = [e^u]_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1$

18. $\int_{\pi/2}^{\pi} \sin(y)\,e^{\cos y}\,dy;\ \begin{bmatrix} u = \cos y \\ du = -\sin y\,dy \\ x = \frac{\pi}{2} \Rightarrow u = 0,\ x = \pi \Rightarrow u = -1 \end{bmatrix} \Rightarrow \int_0^{-1} -e^u\,du = \int_{-1}^{0} e^u\,du = [e^u]_{-1}^{0} = 1 - e^{-1} = \frac{e-1}{e}$

19. $\int e^{\tan v}\sec^2 v\,dv;\ \begin{bmatrix} u = \tan v \\ du = \sec^2 v\,dv \end{bmatrix} \Rightarrow \int e^u\,du = e^u + C = e^{\tan v} + C$

20. $\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}};\begin{bmatrix} u=\sqrt{t} \\ du=\frac{dt}{2\sqrt{t}} \end{bmatrix} \Rightarrow \int 2e^u\,du = 2e^u + C = 2e^{\sqrt{t}} + C$

21. $\int 3^{x+1}\,dx;\begin{bmatrix} u=x+1 \\ du=dx \end{bmatrix} \Rightarrow \int 3^u\,du = \left(\frac{1}{\ln 3}\right)3^u + C = \frac{3^{(x+1)}}{\ln 3} + C$

22. $\int \frac{2^{\ln x}}{x}\,dx;\begin{bmatrix} u=\ln x \\ du=\frac{dx}{x} \end{bmatrix} \Rightarrow \int 2^u\,du = \frac{2^u}{\ln 2} + C = \frac{2^{\ln x}}{\ln 2} + C$

23. $\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}};\begin{bmatrix} u=\sqrt{w} \\ du=\frac{dw}{2\sqrt{w}} \end{bmatrix} \Rightarrow \int 2^u\,du = \frac{2^u}{\ln 2} + C = \frac{2^{\sqrt{w}}}{\ln 2} + C$

24. $\int 10^{2\theta}\,d\theta;\begin{bmatrix} u=2\theta \\ du=2\,d\theta \end{bmatrix} \Rightarrow \int \frac{1}{2}10^u\,du = \frac{10^u}{2\ln 10} + C = \frac{1}{2}\left(\frac{10^{2\theta}}{\ln 10}\right) + C$

25. $\int \frac{9\,du}{1+9u^2};\begin{bmatrix} x=3u \\ dx=3\,du \end{bmatrix} \Rightarrow \int \frac{3\,dx}{1+x^2} = 3\tan^{-1}x + C = 3\tan^{-1}3u + C$

26. $\int \frac{4\,dx}{1+(2x+1)^2};\begin{bmatrix} u=2x+1 \\ du=2\,dx \end{bmatrix} \Rightarrow \int \frac{2\,du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}(2x+1) + C$

27. $\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}};\begin{bmatrix} u=3x \\ du=3\,dx \\ x=0 \Rightarrow u=0,\ x=\frac{1}{6} \Rightarrow u=\frac{1}{2} \end{bmatrix} \Rightarrow \int_0^{1/2} \frac{1}{3}\frac{du}{\sqrt{1-u^2}} = \left[\frac{1}{3}\sin^{-1}u\right]_0^{1/2} = \frac{1}{3}\left(\frac{\pi}{6}-0\right) = \frac{\pi}{18}$

28. $\int_0^1 \frac{dt}{\sqrt{4-t^2}} = \left[\sin^{-1}\frac{t}{2}\right]_0^1 = \sin^{-1}\left(\frac{1}{2}\right) - 0 = \frac{\pi}{6}$

29. $\int \frac{2s\,ds}{\sqrt{1-s^4}};\begin{bmatrix} u=s^2 \\ du=2s\,ds \end{bmatrix} \Rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}s^2 + C$

30. $\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}};\begin{bmatrix} u=2\ln x \\ du=\frac{2\,dx}{x} \end{bmatrix} \Rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(2\ln x) + C$

31. $\int \frac{6\,dx}{x\sqrt{25x^2-1}} = \int \frac{6\,dx}{5x\sqrt{x^2-\frac{1}{25}}} = \frac{6}{5}\cdot 5\ \sec^{-1}|5x| + C = 6\ \sec^{-1}|5x| + C$

32. $\int \frac{dr}{r\sqrt{r^2-9}} = \frac{1}{3}\sec^{-1}\left|\frac{r}{3}\right| + C$

33. $\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x\,dx}{e^{2x}+1};\ \begin{bmatrix} u = e^x \\ du = e^x\,dx \end{bmatrix} \Rightarrow \int \frac{du}{u^2+1} = \tan^{-1}u + C = \tan^{-1}e^x + C$

34. $\int \frac{dy}{\sqrt{e^{2y}-1}} = \int \frac{e^y\,dy}{e^y\sqrt{(e^y)^2-1}};\ \begin{bmatrix} u = e^y \\ du = e^y\,dy \end{bmatrix} \Rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}e^y + C$

35. $\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)};\ \begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \\ x=1 \Rightarrow u=0,\ x=e^{\pi/3} \Rightarrow u=\frac{\pi}{3} \end{bmatrix} \Rightarrow \int_0^{\pi/3} \frac{du}{\cos u} = \int_0^{\pi/3} \sec u\,du = \left[\ln|\sec u + \tan u|\right]_0^{\pi/3}$

$= \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln|\sec 0 + \tan 0| = \ln(2+\sqrt{3}) - \ln(1) = \ln(2+\sqrt{3})$

36. $\int \frac{\ln x\,dx}{x+4x\ln^2 x} = \int \frac{\ln x\,dx}{x(1+4\ln^2 x)};\ \begin{bmatrix} u = \ln^2 x \\ du = \frac{2}{x}\ln x\,dx \end{bmatrix} \Rightarrow \int \frac{1}{2}\frac{du}{1+4u} = \frac{1}{8}\ln|1+4u| + C = \frac{1}{8}\ln(1+4\ln^2 x) + C$

37. $\int_1^2 \frac{8\,dx}{x^2-2x+2} = 8\int_1^2 \frac{dx}{1+(x-1)^2};\ \begin{bmatrix} u = x-1 \\ du = dx \\ x=1 \Rightarrow u=0,\ x=2 \Rightarrow u=1 \end{bmatrix} \Rightarrow 8\int_0^1 \frac{du}{1+u^2} = 8\left[\tan^{-1}u\right]_0^1$

$= 8(\tan^{-1}1 - \tan^{-1}0) = 8\left(\frac{\pi}{4}-0\right) = 2\pi$

38. $\int_2^4 \frac{2\,dx}{x^2-6x+10} = 2\int_2^4 \frac{dx}{(x-3)^2+1};\ \begin{bmatrix} u = x-3 \\ du = dx \\ x=2 \Rightarrow u=-1,\ x=4 \Rightarrow u=1 \end{bmatrix} \Rightarrow 2\int_{-1}^1 \frac{du}{u^2+1} = 2\left[\tan^{-1}u\right]_{-1}^1$

$= 2\left[\tan^{-1}1 - \tan^{-1}(-1)\right] = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

39. $\int \frac{dt}{\sqrt{-t^2+4t-3}} = \int \frac{dt}{\sqrt{1-(t-2)^2}};\ \begin{bmatrix} u = t-2 \\ du = dt \end{bmatrix} \Rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(t-2) + C$

40. $\int \frac{d\theta}{\sqrt{2\theta-\theta^2}} = \int \frac{d\theta}{\sqrt{1-(\theta-1)^2}};\ \begin{bmatrix} u = \theta-1 \\ du = d\theta \end{bmatrix} \Rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(\theta-1) + C$

41. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}}; \begin{bmatrix} u = x+1 \\ du = dx \end{bmatrix} \Rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}|x+1| + C,$
$|u| = |x+1| > 1$

42. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}; \begin{bmatrix} u = x-2 \\ du = dx \end{bmatrix} \Rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C$
$= \sec^{-1}|x+2| + C, |u| = |x+2| > 1$

43. $\int (\sec x + \cot x)^2\,dx = \int (\sec^2 x + 2\sec x \cot x + \cot^2 x)\,dx = \int \sec^2 x\,dx + \int 2\csc x\,dx + \int (\csc^2 x - 1)\,dx$
$= \tan x - 2\ln|\csc x + \cot x| - \cot x - x + C$

44. $\int (\csc x - \tan x)^2\,dx = \int (\csc^2 x - 2\csc x \tan x + \tan^2 x)\,dx = \int \csc^2 x\,dx - \int 2\sec x\,dx + \int (\sec^2 x - 1)\,dx$
$= -\cot x - 2\ln|\sec x + \tan x| + \tan x - x + C$

45. $\int \csc x \sin 3x\,dx = \int (\csc x)(\sin 2x \cos x + \sin x \cos 2x)\,dx = \int (\csc x)(2\sin x \cos^2 x + \sin x \cos 2x)\,dx$
$= \int (2\cos^2 x + \cos 2x)\,dx = \int [(1+\cos 2x) + \cos 2x]\,dx = \int (1 + 2\cos 2x)\,dx = x + \sin 2x + C$

46. $\int (\sin 3x \cos 2x - \cos 3x \sin 2x)\,dx = \int \sin(3x-2x)\,dx = \int \sin x\,dx = -\cos x + C$

47. $\int \frac{x}{x+1}\,dx = \int \left(1 - \frac{1}{x+1}\right)dx = x - \ln|x+1| + C$

48. $\int \frac{x^2}{x^2+1}\,dx = \int \left(1 - \frac{1}{x^2+1}\right)dx = x - \tan^{-1} x + C$

49. $\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\,dx = \int_{\sqrt{2}}^{3} \left(2x + \frac{2x}{x^2-1}\right)dx = \left[x^2 + \ln|x^2-1|\right]_{\sqrt{2}}^{3} = (9 + \ln 8) - (2 + \ln 1) = 7 + \ln 8$

50. $\int_{-1}^{3} \frac{4x^2-7}{2x+3}\,dx = \int_{-1}^{3} \left[(2x-3) + \frac{2}{2x+3}\right]dx = \left[x^2 - 3x + \ln|2x+3|\right]_{-1}^{3} = (9 - 9 + \ln 9) - (1 + 3 + \ln 1) = \ln 9 - 4$

51. $\int \frac{4t^3 - t^2 + 16t}{t^2+4}\,dt = \int \left[(4t-1) + \frac{4}{t^2+4}\right]dt = 2t^2 - t + 2\tan^{-1}\left(\frac{t}{2}\right) + C$

52. $\int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta-5}\,d\theta = \int \left[(\theta^2 - \theta + 1) + \frac{5}{2\theta-5}\right]d\theta = \frac{\theta^3}{3} - \frac{\theta^2}{2} + \theta + \frac{5}{2}\ln|2\theta-5| + C$

53. $\int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1} x + \sqrt{1-x^2} + C$

54. $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx = \int \frac{dx}{2\sqrt{x-1}} + \int \frac{dx}{x} = (x-1)^{1/2} + \ln|x| + C$

55. $\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx = \int_0^{\pi/4} \left(\sec^2 x + \sec x \tan x\right) dx = [\tan x + \sec x]_0^{\pi/4} = \left(1+\sqrt{2}\right) - (0+1) = \sqrt{2}$

56. $\int_0^{1/2} \frac{2-8x}{1+4x^2}\,dx = \int_0^{1/2} \left(\frac{2}{1+4x^2} - \frac{8x}{1+4x^2}\right) dx = \left[\tan^{-1}(2x) - \ln\left|1+4x^2\right|\right]_0^{1/2}$
$= \left(\tan^{-1} 1 - \ln 2\right) - \left(\tan^{-1} 0 - \ln 1\right) = \frac{\pi}{4} - \ln 2$

57. $\int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)}{\left(1-\sin^2 x\right)}\,dx = \int \frac{(1-\sin x)}{\cos^2 x}\,dx = \int \left(\sec^2 x - \sec x \tan x\right) dx = \tan x - \sec x + C$

58. $1+\cos x = 1+\cos\left(2\cdot\frac{x}{2}\right) = 2\cos^2\frac{x}{2} \Rightarrow \int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2\left(\frac{x}{2}\right)} = \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right) dx = \tan\frac{x}{2} + C$

59. $\int \frac{1}{\sec\theta + \tan\theta}\,d\theta = \int \frac{\cos\theta}{1+\sin\theta}\,d\theta; \begin{bmatrix} u = 1+\sin\theta \\ du = \cos\theta\,d\theta \end{bmatrix} \Rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|1+\sin\theta| + C$

60. $\int \frac{1}{\csc\theta + \cot\theta}\,d\theta = \int \frac{\sin\theta}{1+\cos\theta}\,d\theta; \begin{bmatrix} u = 1+\cos\theta \\ du = -\sin\theta\,d\theta \end{bmatrix} \Rightarrow \int \frac{-du}{u} = -\ln|u| + C = -\ln|1+\cos\theta| + C$

61. $\int \frac{1}{1-\sec x}\,dx = \int \frac{\cos x}{\cos x - 1}\,dx = \int \left(1 + \frac{1}{\cos x - 1}\right) dx = \int \left(1 - \frac{1+\cos x}{\sin^2 x}\right) dx = \int \left(1 - \csc^2 x - \frac{\cos x}{\sin^2 x}\right) dx$

$= \int \left(1 - \csc^2 x - \csc x \cot x\right) dx = x + \cot x + \csc x + C$

62. $\int \frac{1}{1-\csc x}\,dx = \int \frac{\sin x}{\sin x - 1}\,dx = \int \left(1 + \frac{1}{\sin x - 1}\right) dx = \int \left(1 + \frac{\sin x + 1}{(\sin x - 1)(\sin x + 1)}\right) dx$

$= \int \left(1 - \frac{1+\sin x}{\cos^2 x}\right) dx = \int \left(1 - \sec^2 x - \frac{\sin x}{\cos^2 x}\right) dx = \int \left(1 - \sec^2 x - \sec x \tan x\right) dx = x - \tan x - \sec x + C$

63. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx = \int_0^{2\pi} \left|\sin\frac{x}{2}\right| dx; \begin{bmatrix} \sin\frac{x}{2} \ge 0 \\ \text{for } 0 \le \frac{x}{2} \le \pi \end{bmatrix} \Rightarrow \int_0^{2\pi} \sin\left(\frac{x}{2}\right) dx = \left[-2\cos\frac{x}{2}\right]_0^{2\pi} = -2(\cos\pi - \cos 0)$
$= (-2)(-2) = 4$

64. $\int_0^{\pi} \sqrt{1-\cos 2x}\, dx = \int_0^{\pi} \sqrt{2}\,|\sin x|\, dx; \begin{bmatrix} \sin x \ge 0 \\ \text{for } 0 \le x \le \pi \end{bmatrix} \Rightarrow \sqrt{2} \int_0^{\pi} \sin x\, dx = \left[-\sqrt{2}\cos x\right]_0^{\pi}$

$= -\sqrt{2}(\cos \pi - \cos 0) = 2\sqrt{2}$

65. $\int_{\pi/2}^{\pi} \sqrt{1+\cos 2t}\, dt = \int_{\pi/2}^{\pi} \sqrt{2}\,|\cos t|\, dt; \begin{bmatrix} \cos t \le 0 \\ \text{for } \frac{\pi}{2} \le t \le \pi \end{bmatrix} \Rightarrow \int_{\pi/2}^{\pi} -\sqrt{2}\cos t\, dt = \left[-\sqrt{2}\sin t\right]_{\pi/2}^{\pi}$

$= -\sqrt{2}\left(\sin \pi - \sin \frac{\pi}{2}\right) = \sqrt{2}$

66. $\int_{-\pi}^{0} \sqrt{1+\cos t}\, dt = \int_{-\pi}^{0} \sqrt{2}\left|\cos \frac{t}{2}\right| dt; \begin{bmatrix} \cot \frac{t}{2} \ge 0 \\ \text{for } -\pi \le t \le 0 \end{bmatrix} \Rightarrow \int_{-\pi}^{0} \sqrt{2}\cos \frac{t}{2}\, dt = \left[2\sqrt{2}\sin \frac{t}{2}\right]_{-\pi}^{0}$

$= 2\sqrt{2}\left[\sin 0 - \sin\left(-\frac{\pi}{2}\right)\right] = 2\sqrt{2}$

67. $\int_{-\pi}^{0} \sqrt{1-\cos^2\theta}\, d\theta = \int_{-\pi}^{0} |\sin \theta|\, d\theta; \begin{bmatrix} \sin \theta \le 0 \\ \text{for } -\pi \le \theta \le 0 \end{bmatrix} \Rightarrow \int_{-\pi}^{0} -\sin \theta\, d\theta = [\cos \theta]_{-\pi}^{0} = \cos 0 - \cos(-\pi)$

$= 1 - (-1) = 2$

68. $\int_{\pi/2}^{\pi} \sqrt{1-\sin^2\theta}\, d\theta = \int_{\pi/2}^{\pi} |\cos \theta|\, d\theta; \begin{bmatrix} \cos \theta \le 0 \\ \text{for } \frac{\pi}{2} \le \theta \le \pi \end{bmatrix} \Rightarrow \int_{\pi/2}^{\pi} -\cos \theta\, d\theta = [-\sin \theta]_{\pi/2}^{\pi} = -\sin \pi + \sin \frac{\pi}{2} = 1$

69. $\int_{-\pi/4}^{\pi/4} \sqrt{\tan^2 y + 1}\, dy = \int_{-\pi/4}^{\pi/4} |\sec y|\, dy; \begin{bmatrix} \sec y \ge 0 \\ \text{for } -\frac{\pi}{4} \le y \le \frac{\pi}{4} \end{bmatrix} \Rightarrow \int_{-\pi/4}^{\pi/4} \sec y\, dy = \left[\ln|\sec y + \tan y|\right]_{-\pi/4}^{\pi/4}$

$= \ln\left|\sqrt{2}+1\right| - \ln\left|\sqrt{2}-1\right|$

70. $\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\, dy = \int_{-\pi/4}^{0} |\tan y|\, dy; \begin{bmatrix} \tan y \le 0 \\ \text{for } -\frac{\pi}{4} \le y \le 0 \end{bmatrix} \Rightarrow \int_{-\pi/4}^{0} -\tan y\, dy = \left[\ln|\cos y|\right]_{-\pi/4}^{0} = -\ln\left(\frac{1}{\sqrt{2}}\right)$

$= \ln \sqrt{2}$

71. $\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\, dx = \int_{\pi/4}^{3\pi/4} \left(\csc^2 x - 2\csc x \cot x + \cot^2 x\right) dx = \int_{\pi/4}^{3\pi/4} \left(2\csc^2 x - 1 - 2\csc x \cot x\right) dx$

$= \left[-2\cot x - x + 2\csc x\right]_{\pi/4}^{3\pi/4} = \left(-2\cot \frac{3\pi}{4} - \frac{3\pi}{4} + 2\csc \frac{3\pi}{4}\right) - \left(-2\cot \frac{\pi}{4} - \frac{\pi}{4} + 2\csc \frac{\pi}{4}\right)$

$= \left[-2(-1) - \frac{3\pi}{4} + 2\left(\sqrt{2}\right)\right] - \left[-2(1) - \frac{\pi}{4} + 2\left(\sqrt{2}\right)\right] = 4 - \frac{\pi}{2}$

72. $\int_0^{\pi/4}(\sec x+4\cos x)^2\,dx=\int_0^{\pi/4}\left[\sec^2 x+8+16\left(\frac{1+\cos 2x}{2}\right)\right]dx=\left[\tan x+16x+4\sin 2x\right]_0^{\pi/4}$
$=\left(\tan\frac{\pi}{4}+4\pi+4\sin\frac{\pi}{2}\right)-(\tan 0+0+4\sin 0)=5+4\pi$

73. $\int \cos\theta\,\csc(\sin\theta)\,d\theta;\begin{bmatrix}u=\sin\theta\\ du=\cos\theta\,d\theta\end{bmatrix}\Rightarrow\int \csc u\,du=-\ln|\csc u+\cot u|+C$
$=-\ln|\csc(\sin\theta)+\cot(\sin\theta)|+C$

74. $\int\left(1+\frac{1}{x}\right)\cot(x+\ln x)\,dx;\begin{bmatrix}u=x+\ln x\\ du=\left(1+\frac{1}{x}\right)dx\end{bmatrix}\Rightarrow\int\cot u\,du=\ln|\sin u|+C=\ln|\sin(x+\ln x)|+C$

75. $\int(\csc x-\sec x)(\sin x+\cos x)\,dx=\int(1+\cot x-\tan x-1)\,dx=\int\cot x\,dx-\int\tan x\,dx$
$=\ln|\sin x|+\ln|\cos x|+C$

76. $\int 3\sinh\left(\frac{x}{2}+\ln 5\right)dx=\begin{bmatrix}u=\frac{x}{2}+\ln 5\\ 2\,du=dx\end{bmatrix}=6\int\sinh u\,du=6\cosh u+C=6\cosh\left(\frac{x}{2}+\ln 5\right)+C$

77. $\int\frac{6\,dy}{\sqrt{y}(1+y)};\begin{bmatrix}u=\sqrt{y}\\ du=\frac{1}{2\sqrt{y}}\,dy\end{bmatrix}\Rightarrow\int\frac{12\,du}{1+u^2}=12\tan^{-1}u+C=12\tan^{-1}\sqrt{y}+C$

78. $\int\frac{dx}{x\sqrt{4x^2-1}}=\int\frac{2\,dx}{2x\sqrt{(2x)^2-1}};\begin{bmatrix}u=2x\\ du=2\,dx\end{bmatrix}\Rightarrow\int\frac{du}{u\sqrt{u^2-1}}=\sec^{-1}|u|+C=\sec^{-1}|2x|+C$

79. $\int\frac{7\,dx}{(x-1)\sqrt{x^2-2x-48}}=\int\frac{7\,dx}{(x-1)\sqrt{(x-1)^2-49}};\begin{bmatrix}u=x-1\\ du=dx\end{bmatrix}\Rightarrow\int\frac{7\,du}{u\sqrt{u^2-49}}=7\cdot\frac{1}{7}\sec^{-1}\left|\frac{u}{7}\right|+C$
$=\sec^{-1}\left|\frac{x-1}{7}\right|+C$

80. $\int\frac{dx}{(2x+1)\sqrt{4x^2+4x}}=\int\frac{dx}{(2x+1)\sqrt{(2x+1)^2-1}};\begin{bmatrix}u=2x+1\\ du=2\,dx\end{bmatrix}\Rightarrow\int\frac{du}{2u\sqrt{u^2-1}}=\frac{1}{2}\sec^{-1}|u|+C$
$=\frac{1}{2}\sec^{-1}|2x+1|+C$

81. $\int\sec^2 t\tan(\tan t)\,dt;\begin{bmatrix}u=\tan t\\ du=\sec^2 t\,dt\end{bmatrix}\Rightarrow\int\tan u\,du=-\ln|\cos u|+C=\ln|\sec u|+C=\ln|\sec(\tan t)|+C$

82. $\int\frac{dx}{x\sqrt{3+x^2}}=-\frac{1}{\sqrt{3}}\operatorname{csch}^{-1}\left|\frac{x}{\sqrt{3}}\right|+C$ (from Table 6.15)

83. (a) $\int \cos^3\theta \, d\theta = \int (\cos\theta)(1-\sin^2\theta)\, d\theta; \begin{bmatrix} u = \sin\theta \\ du = \cos\theta \, d\theta \end{bmatrix} \Rightarrow \int (1-u^2)\, du = u - \frac{u^3}{3} + C = \sin\theta - \frac{1}{3}\sin^3\theta + C$

(b) $\int \cos^5\theta \, d\theta = \int (\cos\theta)(1-\sin^2\theta)^2\, d\theta = \int (1-u^2)^2\, du = \int (1-2u^2+u^4)\, du = u - \frac{2}{3}u^3 + \frac{u^5}{5} + C$

$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C$

(c) $\int \cos^9\theta \, d\theta = \int (\cos^8\theta)(\cos\theta)\, d\theta = \int (1-\sin^2\theta)^4(\cos\theta)\, d\theta$

84. (a) $\int \sin^3\theta \, d\theta = \int (1-\cos^2\theta)(\sin\theta)\, d\theta; \begin{bmatrix} u = \cos\theta \\ du = -\sin\theta \, d\theta \end{bmatrix} \Rightarrow \int (1-u^2)(-du) = \frac{u^3}{3} - u + C$

$= -\cos\theta + \frac{1}{3}\cos^3\theta + C$

(b) $\int \sin^5\theta \, d\theta = \int (1-\cos^2\theta)^2(\sin\theta)\, d\theta = \int (1-u^2)^2(-du) = \int (-1+2u^2-u^4)\, du$

$= -\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + C$

(c) $\int \sin^7\theta \, d\theta = \int (1-u^2)^3(-du) = \int (-1+3u^2-3u^4+u^6)\, du = -\cos\theta + \cos^3\theta - \frac{3}{5}\cos^5\theta + \frac{\cos^7\theta}{7} + C$

(d) $\int \sin^{13}\theta \, d\theta = \int (\sin^{12}\theta)(\sin\theta)\, d\theta = \int (1-\cos^2\theta)^6(\sin\theta)\, d\theta$

85. (a) $\int \tan^3\theta \, d\theta = \int (\sec^2\theta - 1)(\tan\theta)\, d\theta = \int \sec^2\theta \tan\theta \, d\theta - \int \tan\theta \, d\theta = \frac{1}{2}\tan^2\theta - \int \tan\theta \, d\theta$

$= \frac{1}{2}\tan^2\theta + \ln|\cos\theta| + C$

(b) $\int \tan^5\theta \, d\theta = \int (\sec^2\theta - 1)(\tan^3\theta)\, d\theta = \int \tan^3\theta \sec^2\theta \, d\theta - \int \tan^3\theta \, d\theta = \frac{1}{4}\tan^4\theta - \int \tan^3\theta \, d\theta$

(c) $\int \tan^7\theta \, d\theta = \int (\sec^2\theta - 1)(\tan^5\theta)\, d\theta = \int \tan^5\theta \sec^2\theta \, d\theta - \int \tan^5\theta \, d\theta = \frac{1}{6}\tan^6\theta - \int \tan^5\theta \, d\theta$

(d) $\int \tan^{2k+1}\theta \, d\theta = \int (\sec^2\theta - 1)(\tan^{2k-1}\theta)\, d\theta = \int \tan^{2k-1}\theta \sec^2\theta \, d\theta - \int \tan^{2k-1}\theta \, d\theta;$

$\begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta \, d\theta \end{bmatrix} \Rightarrow \int u^{2k-1}\, du - \int \tan^{2k-1}\theta \, d\theta = \frac{1}{2k}u^{2k} - \int \tan^{2k-1}\theta \, d\theta = \frac{1}{2k}\tan^{2k}\theta - \int \tan^{2k-1}\theta \, d\theta$

86. (a) $\int \cot^3\theta \, d\theta = \int (\csc^2\theta - 1)(\cot\theta)\, d\theta = \int \cot\theta \csc^2\theta \, d\theta - \int \cot\theta \, d\theta = -\frac{1}{2}\cot^2\theta - \int \cot\theta \, d\theta$

$= -\frac{1}{2}\cot^2\theta - \ln|\sin\theta| + C$

(b) $\int \cot^5\theta \, d\theta = \int (\csc^2\theta - 1)(\cot^3\theta)\, d\theta = \int \cot^3\theta \csc^2\theta \, d\theta - \int \cot^3\theta \, d\theta = -\frac{1}{4}\cot^4\theta - \int \cot^3\theta \, d\theta$

(c) $\int \cot^7\theta \, d\theta = \int (\csc^2\theta - 1)(\cot^5\theta)\, d\theta = \int \cot^5\theta \csc^2\theta \, d\theta - \int \cot^5\theta \, d\theta = -\frac{1}{6}\cot^6\theta - \int \cot^5\theta \, d\theta$

(d) $\int \cot^{2k+1}\theta\, d\theta = \int \left(\csc^2\theta - 1\right)\left(\cot^{2k-1}\theta\right) d\theta = \int \cot^{2k-1}\theta \csc^2\theta\, d\theta - \int \cot^{2k-1}\theta\, d\theta;$

$\begin{bmatrix} u = \cot\theta \\ du = -\csc^2\theta\, d\theta \end{bmatrix} \Rightarrow -\int u^{2k-1}\, du - \int \cot^{2k-1}\theta\, d\theta = -\frac{1}{2k}u^{2k} - \int \cot^{2k-1}\theta\, d\theta$

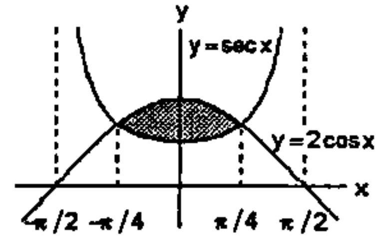
$= -\frac{1}{2k}\cot^{2k}\theta - \int \cot^{2k-1}\theta\, d\theta$

87. $A = \int_{-\pi/4}^{\pi/4} (2\cos x - \sec x)\, dx = \left[2\sin x - \ln|\sec x + \tan x|\right]_{-\pi/4}^{\pi/4}$

$= \left[\sqrt{2} - \ln\left(\sqrt{2}+1\right)\right] - \left[-\sqrt{2} - \ln\left(\sqrt{2}-1\right)\right]$

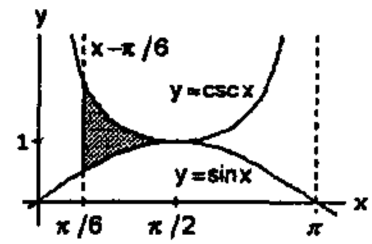
$= 2\sqrt{2} - \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\sqrt{2} - \ln\left(\frac{\left(\sqrt{2}+1\right)^2}{2-1}\right)$

$= 2\sqrt{2} - \ln\left(3+2\sqrt{2}\right)$



88. $A = \int_{\pi/6}^{\pi/2} (\csc x - \sin x)\, dx = \left[-\ln|\csc x + \cot x| + \cos x\right]_{\pi/6}^{\pi/2}$

$= -\ln|1+0| + \ln\left|2+\sqrt{3}\right| - \frac{\sqrt{3}}{2} = \ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}$



89. $V = \int_{-\pi/4}^{\pi/4} \pi(2\cos x)^2\, dx - \int_{-\pi/4}^{\pi/4} \pi\sec^2 x\, dx = 4\pi \int_{-\pi/4}^{\pi/4} \cos^2 x\, dx - \pi \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$

$= 2\pi \int_{-\pi/4}^{\pi/4} (1+\cos 2x)\, dx - \pi[\tan x]_{-\pi/4}^{\pi/4} = 2\pi\left[x + \frac{1}{2}\sin 2x\right]_{-\pi/4}^{\pi/4} - \pi[1-(-1)]$

$= 2\pi\left[\left(\frac{\pi}{4}+\frac{1}{2}\right) - \left(-\frac{\pi}{4}-\frac{1}{2}\right)\right] - 2\pi = 2\pi\left(\frac{\pi}{2}+1\right) - 2\pi = \pi^2$

90. $V = \int_{\pi/6}^{\pi/2} \pi\csc^2 x\, dx - \int_{\pi/6}^{\pi/2} \pi\sin^2 x\, dx = \pi \int_{\pi/6}^{\pi/2} \csc^2 x\, dx - \frac{\pi}{2} \int_{\pi/6}^{\pi/2} (1-\cos 2x)\, dx$

$= \pi[-\cot x]_{\pi/6}^{\pi/2} - \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/6}^{\pi/2} = \pi\left[0-\left(-\sqrt{3}\right)\right] - \frac{\pi}{2}\left[\left(\frac{\pi}{2}-0\right) - \left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]$

$= \pi\sqrt{3} - \frac{\pi}{2}\left(\frac{2\pi}{6} + \frac{\sqrt{3}}{4}\right) = \pi\left(\frac{7\sqrt{3}}{8} - \frac{\pi}{6}\right)$

91. $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = -\frac{\sin x}{\cos x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1$; $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_0^{\pi/3} \sqrt{1+(\sec^2 x - 1)}\, dx = \int_0^{\pi/3} \sec x\, dx = [\ln|\sec x + \tan x|]_0^{\pi/3} = \ln\left|2+\sqrt{3}\right| - \ln|1+0| = \ln(2+\sqrt{3})$

92. $y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1$; $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_0^{\pi/4} \sec x\, dx = [\ln|\sec x + \tan x|]_0^{\pi/4} = \ln\left|\sqrt{2}+1\right| - \ln|1+0| = \ln(\sqrt{2}+1)$

93. $\int \csc x\, dx = \int (\csc x)(1)\, dx = \int (\csc x)\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right) dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx$;

$\begin{bmatrix} u = \csc x + \cot x \\ du = (-\csc x \cot x - \csc^2 x)\, dx \end{bmatrix} \Rightarrow \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C$

94. $\left[(x^2-1)(x+1)\right]^{-2/3} = \left[(x-1)(x+1)^2\right]^{-2/3} = (x-1)^{-2/3}(x+1)^{-4/3} = (x+1)^{-2}\left[(x-1)^{-2/3}(x+1)^{2/3}\right]$

$= (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} = (x+1)^{-2}\left(1-\frac{2}{x+1}\right)^{-2/3}$

(a) $\int \left[(x^2-1)(x+1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(1-\frac{2}{x+1}\right)^{-2/3} dx$; $\begin{bmatrix} u = \frac{1}{x+1} \\ du = -\frac{1}{(x+1)^2}\, dx \end{bmatrix}$

$\Rightarrow \int -(1-2u)^{-2/3}\, du = \frac{3}{2}(1-2u)^{1/3} + C = \frac{3}{2}\left(1-\frac{2}{x+1}\right)^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(b) $\int \left[(x^2-1)(x+1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} dx$; $u = \left(\frac{x-1}{x+1}\right)^k$

$\Rightarrow du = k\left(\frac{x-1}{x+1}\right)^{k-1}\frac{[(x+1)-(x-1)]}{(x+1)^2}\, dx = 2k\frac{(x-1)^{k-1}}{(x+1)^{k+1}}\, dx$; $dx = \frac{(x+1)^2}{2k}\left(\frac{x+1}{x-1}\right)^{k-1} du$

$= \frac{(x+1)^2}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du$; then, $\int \left(\frac{x-1}{x+1}\right)^{-2/3}\frac{1}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du = \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{(1/3-k)} du$

$= \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{k(1/3k-1)} du = \frac{1}{2k}\int u^{(1/3k-1)}\, du = \frac{1}{2k}(3k)\, u^{1/3k} + C = \frac{3}{2}u^{1/3k} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(c) $\int \left[(x^2-1)(x+1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} dx$;

$\begin{bmatrix} u = \tan^{-1} x \\ x = \tan u \\ dx = \frac{du}{\cos^2 u} \end{bmatrix} \Rightarrow \int \frac{1}{(\tan u + 1)^2}\left(\frac{\tan u - 1}{\tan u + 1}\right)^{-2/3}\left(\frac{du}{\cos^2 u}\right) = \int \frac{1}{(\sin u + \cos u)^2}\left(\frac{\sin u - \cos u}{\sin u + \cos u}\right)^{-2/3} du$;

$$\left[\begin{array}{l} \sin u+\cos u=\sin u+\sin\left(\frac{\pi}{2}-u\right)=2\sin\frac{\pi}{4}\cos\left(u-\frac{\pi}{4}\right) \\ \sin u-\cos u=\sin u-\sin\left(\frac{\pi}{2}-u\right)=2\cos\frac{\pi}{4}\sin\left(u-\frac{\pi}{4}\right) \end{array}\right] \Rightarrow \int \frac{1}{2\cos^2\left(u-\frac{\pi}{4}\right)}\left[\frac{\sin\left(u-\frac{\pi}{4}\right)}{\cos\left(u-\frac{\pi}{4}\right)}\right]^{-2/3} du$$

$$= \frac{1}{2}\int \tan^{-2/3}\left(u-\frac{\pi}{4}\right)\sec^2\left(u-\frac{\pi}{4}\right)du = \frac{3}{2}\tan^{1/3}\left(u-\frac{\pi}{4}\right)+C = \frac{3}{2}\left[\frac{\tan u-\tan\frac{\pi}{4}}{1+\tan u\tan\frac{\pi}{4}}\right]^{1/3}+C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(d) $u=\tan^{-1}\sqrt{x} \Rightarrow \tan u=\sqrt{x} \Rightarrow \tan^2 u = x \Rightarrow dx = 2\tan u\left(\frac{1}{\cos^2 u}\right)du = \frac{2\sin u}{\cos^3 u}du = -\frac{2d(\cos u)}{\cos^3 u}$;

$x-1=\tan^2 u-1=\frac{\sin^2 u-\cos^2 u}{\cos^2 u}=\frac{1-2\cos^2 u}{\cos^2 u}$; $x+1=\tan^2 u+1=\frac{\cos^2 u+\sin^2 u}{\cos^2 u}=\frac{1}{\cos^2 u}$;

$$\int (x-1)^{-2/3}(x+1)^{-4/3}dx = \int \frac{\left(1-2\cos^2 u\right)^{-2/3}}{\left(\cos^2 u\right)^{-2/3}}\cdot\frac{1}{\left(\cos^2 u\right)^{-4/3}}\cdot\frac{-2d(\cos u)}{\cos^3 u}$$

$$= \int\left(1-2\cos^2 u\right)^{-2/3}\cdot(-2)\cdot\cos u\cdot d(\cos u) = \frac{1}{2}\int\left(1-2\cos^2 u\right)^{-2/3}\cdot d\left(1-2\cos^2 u\right)$$

$$= \frac{3}{2}\left(1-2\cos^2 u\right)^{1/3}+C = \frac{3}{2}\left[\frac{\left(\frac{1-2\cos^2 u}{\cos^2 u}\right)}{\left(\frac{1}{\cos^2 u}\right)}\right]^{1/3}+C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(e) $u=\tan^{-1}\left(\frac{x-1}{2}\right) \Rightarrow \frac{x-1}{2}=\tan u \Rightarrow x+1=2(\tan u+1) \Rightarrow dx=\frac{2\,du}{\cos^2 u}=2d(\tan u)$;

$$\int (x-1)^{-2/3}(x+1)^{-4/3}dx = \int (\tan u)^{-2/3}(\tan u+1)^{-4/3}\cdot 2^{-2}\cdot 2\cdot d(\tan u)$$

$$= \frac{1}{2}\int\left(1-\frac{1}{\tan u+1}\right)^{-2/3}d\left(1-\frac{1}{\tan u+1}\right) = \frac{3}{2}\left(1-\frac{1}{\tan u+1}\right)^{1/3}+C = \frac{3}{2}\left(1-\frac{2}{x+1}\right)^{1/3}+C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(f) $\left[\begin{array}{c} u=\cos^{-1}x \\ x=\cos u \\ dx=-\sin u\,du \end{array}\right] \Rightarrow -\int\frac{\sin u\,du}{\sqrt[3]{\left(\cos^2 u-1\right)^2(\cos u+1)^2}} = -\int\frac{\sin u\,du}{\left(\sin^{4/3}u\right)\left(4\cos\frac{u}{2}\right)^{4/3}}$

$$= -\int\frac{du}{(\sin u)^{1/3}\left(4\cos\frac{u}{2}\right)^{4/3}} = -\int\frac{du}{2\left(\sin\frac{u}{2}\right)^{4/3}\left(\cos\frac{u}{2}\right)^{5/3}} = -\frac{1}{2}\int\left(\frac{\cos\frac{u}{2}}{\sin\frac{u}{2}}\right)^{1/3}\frac{du}{\left(\cos^2\frac{u}{2}\right)}$$

$$= -\int\tan^{-1/3}\left(\frac{u}{2}\right)d\left(\tan\frac{u}{2}\right) = -\frac{3}{2}\tan^{2/3}\frac{u}{2}+C = \frac{3}{2}\left(-\tan^2\frac{u}{2}\right)^{1/3}+C = \frac{3}{2}\left(\frac{\cos u-1}{\cos u+1}\right)^{1/3}+C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(g) $\int \left[(x^2-1)(x+1)\right]^{-2/3} dx$; $\begin{bmatrix} u = \cosh^{-1} x \\ x = \cosh u \\ dx = \sinh u \end{bmatrix} \Rightarrow \int \frac{\sinh u \, du}{\sqrt[3]{\left(\cosh^2 u - 1\right)^2 (\cosh u + 1)^2}}$

$$= \int \frac{\sinh u \, du}{\sqrt[3]{\left(\sinh^4 u\right)(\cosh u + 1)^2}} = \int \frac{du}{\sqrt[3]{(\sinh u)\left(4 \cosh^4 \frac{u}{2}\right)}} = \frac{1}{2} \int \frac{du}{\sqrt[3]{\sinh\left(\frac{u}{2}\right) \cosh^5\left(\frac{u}{2}\right)}}$$

$$= \int \left(\tanh \frac{u}{2}\right)^{-1/3} d\left(\tanh \frac{u}{2}\right) = \frac{3}{2}\left(\tanh \frac{u}{2}\right)^{2/3} + C = \frac{3}{2}\left(\frac{\cosh x - 1}{\cosh x + 1}\right)^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$$

7.2 INTEGRATION BY PARTS

1. $u = x$, $du = dx$; $dv = \sin \frac{x}{2} dx$, $v = -2 \cos \frac{x}{2}$;

$$\int x \sin \frac{x}{2} dx = -2x \cos \frac{x}{2} - \int \left(-2 \cos \frac{x}{2}\right) dx = -2x \cos\left(\frac{x}{2}\right) + 4 \sin\left(\frac{x}{2}\right) + C$$

2. $u = \theta$, $du = d\theta$; $dv = \cos \pi\theta \, d\theta$, $v = \frac{1}{\pi} \sin \pi\theta$;

$$\int \theta \cos \pi\theta \, d\theta = \frac{\theta}{\pi} \sin \pi\theta - \int \frac{1}{\pi} \sin \pi\theta \, d\theta = \frac{\theta}{\pi} \sin \pi\theta + \frac{1}{\pi^2} \cos \pi\theta + C$$

3.

$$\begin{array}{lcl} & & \cos t \\ t^2 & \xrightarrow{(+)} & \sin t \\ 2t & \xrightarrow{(-)} & -\cos t \\ 2 & \xrightarrow{(+)} & -\sin t \\ 0 & & \end{array}$$

$$\int t^2 \cos t \, dt = t^2 \sin t + 2t \cos t - 2 \sin t + C$$

4.

$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

5. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x \, dx$, $v = \frac{x^2}{2}$;

$$\int_1^2 x \ln x \, dx = \left[\frac{x^2}{2} \ln x\right]_1^2 - \int_1^2 \frac{x^2}{2} \frac{dx}{x} = 2 \ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2 \ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

6. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x^3\,dx$, $v = \frac{x^4}{4}$;

$$\int_1^e x^3 \ln x\,dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$$

7. $u = \tan^{-1} y$, $du = \frac{dy}{1+y^2}$; $dv = dy$, $v = y$;

$$\int \tan^{-1} y\,dy = y\tan^{-1} y - \int \frac{y\,dy}{(1+y^2)} = y\tan^{-1} y - \frac{1}{2}\ln\left(1+y^2\right) + C = y\tan^{-1} y - \ln\sqrt{1+y^2} + C$$

8. $u = \sin^{-1} y$, $du = \frac{dy}{\sqrt{1-y^2}}$; $dv = dy$, $v = y$;

$$\int \sin^{-1} y\,dy = y\sin^{-1} y - \int \frac{y\,dy}{\sqrt{1-y^2}} = y\sin^{-1} y + \sqrt{1-y^2} + C$$

9. $u = x$, $du = dx$; $dv = \sec^2 x\,dx$, $v = \tan x$;

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C$$

10. $\int 4x\sec^2 2x\,dx$; $[y = 2x] \to \int y\sec^2 y\,dy = y\tan y - \int \tan y\,dy = y\tan y - \ln|\sec y| + C$

$= 2x\tan 2x - \ln|\sec 2x| + C$

11.
$$\begin{array}{lll} & & e^x \\ x^3 & \xrightarrow{(+)} & e^x \\ 3x^2 & \xrightarrow{(-)} & e^x \\ 6x & \xrightarrow{(+)} & e^x \\ 6 & \xrightarrow{(-)} & e^x \\ 0 & & \end{array}$$

$$\int x^3e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = \left(x^3 - 3x^2 + 6x - 6\right)e^x + C$$

12.
$$\begin{array}{lll} & & e^{-p} \\ p^4 & \xrightarrow{(+)} & -e^{-p} \\ 4p^3 & \xrightarrow{(-)} & e^{-p} \\ 12p^2 & \xrightarrow{(+)} & -e^{-p} \\ 24p & \xrightarrow{(-)} & e^{-p} \\ 24 & \xrightarrow{(+)} & e^{-p} \\ 0 & & \end{array}$$

$$\int p^4e^{-p}\,dp = -p^4e^{-p} - 4p^3e^{-p} - 12p^2e^{-p} - 24pe^{-p} - 24e^{-p} + C$$
$$= \left(-p^4 - 4p^3 - 12p^2 - 24p - 24\right)e^{-p} + C$$

13.

$$\begin{array}{lll} & & e^x \\ x^2-5x & \xrightarrow{(+)} & e^x \\ 2x-5 & \xrightarrow{(-)} & e^x \\ 2 & \xrightarrow{(+)} & e^x \\ 0 & & \end{array}$$

$$\int (x^2-5x)e^x\,dx = (x^2-5x)e^x - (2x-5)e^x + 2e^x + C = x^2e^x - 7xe^x + 7e^x + C$$
$$= (x^2-7x+7)e^x + C$$

14.

$$\begin{array}{lll} & & e^r \\ r^2+r+1 & \xrightarrow{(+)} & e^r \\ 2r+1 & \xrightarrow{(-)} & e^r \\ 2 & \xrightarrow{(+)} & e^r \\ 0 & & \end{array}$$

$$\int (r^2+r+1)e^r\,dr = (r^2+r+1)e^r - (2r+1)\,e^r + 2e^r + C$$
$$= \left[(r^2+r+1) - (2r+1) + 2\right]e^r + C = (r^2-r+2)e^r + C$$

15.

$$\begin{array}{lll} & & e^x \\ x^5 & \xrightarrow{(+)} & e^x \\ 5x^4 & \xrightarrow{(-)} & e^x \\ 20x^3 & \xrightarrow{(+)} & e^x \\ 60x^2 & \xrightarrow{(-)} & e^x \\ 120x & \xrightarrow{(+)} & e^x \\ 120 & \xrightarrow{(-)} & e^x \\ 0 & & \end{array}$$

$$\int x^5e^x\,dx = x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + C$$
$$= (x^5-5x^4+20x^3-60x^2+120x-120)e^x + C$$

16.

$$\begin{array}{lll} & & e^{4t} \\ t^2 & \xrightarrow{(+)} & \frac{1}{4}e^{4t} \\ 2t & \xrightarrow{(-)} & \frac{1}{16}e^{4t} \\ 2 & \xrightarrow{(+)} & \frac{1}{64}e^{4t} \\ 0 & & \end{array}$$

$$\int t^2e^{4t}\,dt = \frac{t^2}{4}e^{4t} - \frac{2t}{16}e^{4t} + \frac{2}{64}e^{4t} + C = \frac{t^2}{4}e^{4t} - \frac{t}{8}e^{4t} + \frac{1}{32}e^{4t} + C$$
$$\left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32}\right)e^{4t} + C$$

17.

$$\begin{array}{lll} & & \sin 2\theta \\ \theta^2 & \xrightarrow{(+)} & -\frac{1}{2}\cos 2\theta \\ 2\theta & \xrightarrow{(-)} & -\frac{1}{4}\sin 2\theta \end{array}$$

$2 \xrightarrow{(+)} \frac{1}{8}\cos 2\theta$

0

$$\int_0^{\pi/2} \theta^2 \sin 2\theta \, d\theta = \left[-\frac{\theta^2}{2}\cos 2\theta + \frac{\theta}{2}\sin 2\theta + \frac{1}{4}\cos 2\theta\right]_0^{\pi/2}$$

$$= \left[-\frac{\pi^2}{8}\cdot(-1) + \frac{\pi}{4}\cdot 0 + \frac{1}{4}\cdot(-1)\right] - \left[0 + 0 + \frac{1}{4}\cdot 1\right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2 - 4}{8}$$

18. $\cos 2x$

$x^3 \xrightarrow{(+)} \frac{1}{2}\sin 2x$

$3x^2 \xrightarrow{(-)} -\frac{1}{4}\cos 2x$

$6x \xrightarrow{(+)} -\frac{1}{8}\sin 2x$

$6 \xrightarrow{(-)} \frac{1}{16}\cos 2x$

0

$$\int_0^{\pi/2} x^3 \cos 2x \, dx = \left[\frac{x^3}{2}\sin 2x + \frac{3x^2}{4}\cos 2x - \frac{3x}{4}\sin 2x - \frac{3}{8}\cos 2x\right]_0^{\pi/2}$$

$$= \left[\frac{\pi^3}{16}\cdot 0 + \frac{3\pi^2}{16}\cdot(-1) - \frac{3\pi}{8}\cdot 0 - \frac{3}{8}\cdot(-1)\right] - \left[0 + 0 - 0 - \frac{3}{8}\cdot 1\right] = -\frac{3\pi^2}{16} + \frac{3}{4} = \frac{3(4-\pi^2)}{16}$$

19. $u = \sec^{-1} t$, $du = \frac{dt}{t\sqrt{t^2-1}}$; $dv = t\, dt$, $v = \frac{t^2}{2}$;

$$\int_{2/\sqrt{3}}^{2} t \sec^{-1} t \, dt = \left[\frac{t^2}{2}\sec^{-1} t\right]_{2/\sqrt{3}}^{2} - \int_{2/\sqrt{3}}^{2} \left(\frac{t^2}{2}\right)\frac{dt}{t\sqrt{t^2-1}} = \left(2\cdot\frac{\pi}{3} - \frac{2}{3}\cdot\frac{\pi}{6}\right) - \int_{2/\sqrt{3}}^{2} \frac{t\, dt}{2\sqrt{t^2-1}}$$

$$= \frac{5\pi}{9} - \left[\frac{1}{2}\sqrt{t^2-1}\right]_{2/\sqrt{3}}^{2} = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \sqrt{\frac{4}{3}-1}\right) = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{5\pi}{9} - \frac{\sqrt{3}}{3} = \frac{5\pi - 3\sqrt{3}}{9}$$

20. $u = \sin^{-1}(x^2)$, $du = \frac{2x\, dx}{\sqrt{1-x^4}}$; $dv = 2x\, dx$, $v = x^2$;

$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx = \left[x^2 \sin^{-1}(x^2)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \cdot \frac{2x\, dx}{\sqrt{1-x^4}} = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) + \int_0^{1/\sqrt{2}} \frac{d(1-x^4)}{2\sqrt{1-x^4}}$$

$$= \frac{\pi}{12} + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

21. $I = \int e^\theta \sin\theta\, d\theta$; $[u = \sin\theta,\ du = \cos\theta\, d\theta;\ dv = e^\theta\, d\theta,\ v = e^\theta] \Rightarrow I = e^\theta \sin\theta - \int e^\theta \cos\theta\, d\theta$;

$[u = \cos\theta,\ du = -\sin\theta\, d\theta;\ dv = e^\theta\, d\theta,\ v = e^\theta] \Rightarrow I = e^\theta \sin\theta - \left(e^\theta \cos\theta + \int e^\theta \sin\theta\, d\theta\right)$

$= e^\theta \sin\theta - e^\theta \cos\theta - I + C' \Rightarrow 2I = (e^\theta \sin\theta - e^\theta \cos\theta) + C' \Rightarrow I = \frac{1}{2}(e^\theta \sin\theta - e^\theta \cos\theta) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

22. $I = \int e^{-y} \cos y\ dy$; $\left[u = \cos y,\ du = -\sin y\ dy;\ dv = e^{-y}\ dy,\ v = -e^{-y}\right]$

$\Rightarrow I = -e^{-y} \cos y - \int \left(-e^{-y}\right)(-\sin y)\ dy = -e^{-y} \cos y - \int e^{-y} \sin y\ dy$; $\left[u = \sin y,\ du = \cos y\ dy;\right.$

$\left.dv = e^{-y}\ dy,\ v = -e^{-y}\right] \Rightarrow I = -e^{-y} \cos y - \left(-e^{-y} \sin y - \int \left(-e^{y}\right) \cos y\ dy\right) = -e^{-y} \cos y + e^{-y} \sin y - I + C'$

$\Rightarrow 2I = e^{-y}(\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2}\left(e^{-y} \sin y - e^{-y} \cos y\right) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

23. $I = \int e^{2x} \cos 3x\ dx$; $\left[u = \cos 3x;\ du = -3 \sin 3x\ dx,\ dv = e^{2x}\ dx;\ v = \frac{1}{2}e^{2x}\right]$

$\Rightarrow I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x\ dx$; $\left[u = \sin 3x,\ du = 3 \cos 3x,\ dv = e^{2x}\ dx;\ v = \frac{1}{2}e^{2x}\right]$

$\Rightarrow I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{2}\left(\frac{1}{2}e^{2x} \sin 3x - \frac{3}{2} \int e^{2x} \cos 3x\ dx\right) = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4}I + C'$

$\Rightarrow \frac{13}{4}I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{4}e^{2x} \sin 3x + C' \Rightarrow \frac{e^{2x}}{13}(3 \sin 3x + 2 \cos 3x) + C$, where $C = \frac{4}{13}C'$

24. $\int e^{-2x} \sin 2x\ dx$; $[y = 2x] \rightarrow \frac{1}{2} \int e^{-y} \sin y\ dy = I$; $\left[u = \sin y,\ du = \cos y\ dy;\ dv = e^{-y}\ dy,\ v = -e^{-y}\right]$

$\Rightarrow I = \frac{1}{2}\left(-e^{-y} \sin y + \int e^{-y} \cos y\ dy\right)\left[u = \cos u,\ du = -\sin y;\ dv = e^{-y}\ dy,\ v = -e^{-y}\right]$

$\Rightarrow I = -\frac{1}{2}e^{-y} \sin y + \frac{1}{2}\left(-e^{-y} \cos y - \int \left(-e^{-y}\right)(-\sin y)\ dy\right) = -\frac{1}{2}e^{-y}(\sin y + \cos y) - I + C'$

$\Rightarrow 2I = -\frac{1}{2}e^{-y}(\sin y + \cos y) + C' \Rightarrow I = -\frac{1}{4}e^{-y}(\sin y + \cos y) + C = -\frac{e^{-2x}}{4}(\sin 2x + \cos 2x) + C$, where $C = \frac{C'}{2}$

25. $\int e^{\sqrt{3s+9}}\ ds$; $\begin{bmatrix} 3s + 9 = x^2 \\ ds = \frac{2}{3}x\ dx \end{bmatrix} \rightarrow \int e^{x} \cdot \frac{2}{3}x\ dx = \frac{2}{3} \int xe^{x}\ dx$; $\left[u = x,\ du = dx;\ dv = e^{x}\ dx,\ v = e^{x}\right]$;

$\frac{2}{3} \int xe^{x}\ dx = \frac{2}{3}\left(xe^{x} - \int e^{x}\ dx\right) = \frac{2}{3}\left(xe^{x} - e^{x}\right) + C = \frac{2}{3}\left(\sqrt{3s+9}\, e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$

26. $u = x,\ du = dx;\ dv = \sqrt{1-x}\ dx,\ v = -\frac{2}{3}\sqrt{(1-x)^3}$;

$$\int_0^1 x\sqrt{1-x}\ dx = \left[-\frac{2}{3}\sqrt{(1-x)^3}\, x\right]_0^1 + \frac{2}{3} \int_0^1 \sqrt{(1-x)^3}\ dx = \frac{2}{3}\left[-\frac{2}{5}(1-x)^{5/2}\right]_0^1 = \frac{4}{15}$$

27. $u = x,\ du = dx;\ dv = \tan^2 x\ dx,\ v = \int \tan^2 x\ dx = \int \frac{\sin^2 x}{\cos^2 x}\ dx = \int \frac{1 - \cos^2 x}{\cos^2 x}\ dx = \int \frac{dx}{\cos^2 x} - \int dx$

$= \tan x - x$; $\int_0^{\pi/3} x \tan^2 x\ dx = \left[x(\tan x - x)\right]_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x)\ dx = \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \left[\ln|\cos x| + \frac{x^2}{2}\right]_0^{\pi/3}$

$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln \frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}$

28. $u = \ln(x + x^2)$, $du = \frac{(2x+1)\,dx}{x + x^2}$; $dv = dx$, $v = x$; $\int \ln(x + x^2)\,dx = x \ln(x + x^2) - \int \frac{2x+1}{x(x+1)} \cdot x\,dx$

$= x \ln(x + x^2) - \int \frac{(2x+1)\,dx}{x+1} = x \ln(x + x^2) - \int \frac{2(x+1) - 1}{x+1}\,dx = x \ln(x + x^2) - 2x + \ln|x+1| + C$

29. $\int \sin(\ln x)\,dx; \begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = e^u\,du \end{bmatrix} \to \int (\sin u)\,e^u\,du$. From Exercise 21, $\int (\sin u)\,e^u\,du = e^u\left(\frac{\sin u - \cos u}{2}\right) + C$

$= \frac{1}{2}[-x \cos(\ln x) + x \sin(\ln x)] + C$

30. $\int z(\ln z)^2\,dz; \begin{bmatrix} u = \ln z \\ du = \frac{1}{z}\,dz \\ dz = e^u\,du \end{bmatrix} \to \int e^u \cdot u^2 \cdot e^u\,du = \int e^{2u} \cdot u^2\,du;$

$$\begin{array}{ll} & e^{2u} \\ u^2 \xrightarrow{(+)} & \frac{1}{2}e^{2u} \\ 2u \xrightarrow{(-)} & \frac{1}{4}e^{2u} \\ 2 \xrightarrow{(+)} & \frac{1}{8}e^{2u} \\ 0 & \end{array}$$

$\int u^2 e^{2u}\,du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}[2u^2 - 2u + 1] + C$

$= \frac{z^2}{4}[2(\ln z)^2 - 2\ln z + 1] + C$

31. $y = \int x^2 e^{4x}\,dx$

Let $u = x^2$ $\qquad dv = e^{4x}\,dx$

$du = 2x\,dx$ $\qquad v = \frac{1}{4}e^{4x}$

$y = (x^2)\left(\frac{1}{4}e^{4x}\right) - \int \left(\frac{1}{4}e^{4x}\right)(2x\,dx)$

$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\int xe^{4x}\,dx$

Let $u = x$ $\qquad dv = e^{4x}\,dx$

$du = dx$ $\qquad v = \frac{1}{4}e^{4x}$

$y = \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\left[(x)\left(\frac{1}{4}e^{4x}\right) - \int \left(\frac{1}{4}e^{4x}\right)dx\right]$

$$y = \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C$$

$$y = \left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right)e^{4x} + C$$

32. $y = \int x^2 \ln x \, dx$

Let $u = \ln x$ $\qquad dv = x^2 \, dx$

$\quad du = \frac{1}{x} \, dx$ $\qquad v = \frac{1}{3}x^3$

$$y = (\ln x)\left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{3}x^3\right)\left(\frac{1}{x} \, dx\right)$$

$$y = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx$$

$$y = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

33. Let $w = \sqrt{\theta}$. Then $dw = \frac{d\theta}{2\sqrt{\theta}}$, so $d\theta = 2\sqrt{\theta} \, dw = 2w \, dw$.

$$\int \sin \sqrt{\theta} \, d\theta = \int (\sin w)(2w \, dw) = 2\int w \sin w \, dw$$

Let $u = w$ $\qquad dv = \sin w \, dw$

$\quad du = dw$ $\qquad v = -\cos w$

$$\begin{aligned}\int w \sin w \, dw &= -w \cos w + \int \cos w \, dw \\ &= -w \cos w + \sin w + C\end{aligned}$$

$$\begin{aligned}\int \sin \sqrt{\theta} \, d\theta &= 2\int w \sin w \, dw \\ &= -2w \cos w + 2 \sin w + C \\ &= -2\sqrt{\theta} \cos \sqrt{\theta} + 2 \sin \sqrt{\theta} + C\end{aligned}$$

34. $y = \int \theta \sec \theta \tan \theta \, d\theta$

Let $u = \theta$ $\qquad dv = \sec \theta \tan \theta \, d\theta$

$\quad du = d\theta$ $\qquad v = \sec \theta$

$$y = \theta \sec \theta - \int \sec \theta \, d\theta$$

$$y = \theta \sec \theta - \ln|\sec \theta + \tan\theta| + C$$

35. (a) $u = x$, $du = dx$; $dv = \sin x \, dx$, $v = -\cos x$;

$$S_1 = \int_0^{\pi} x \sin x \, dx = [-x \cos x]_0^{\pi} + \int_0^{\pi} \cos x \, dx = \pi + [\sin x]_0^{\pi} = \pi$$

(b) $S_2 = -\int_{\pi}^{2\pi} x \sin x \, dx = \left[[-x \cos x]_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos x \, dx \right] = -\left[-3\pi + [\sin x]_{\pi}^{2\pi}\right] = 3\pi$

(c) $S_3 = \int_{2\pi}^{3\pi} x \sin x \, dx = [-x \cos x]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} \cos x \, dx = 5\pi + [\sin x]_{2\pi}^{3\pi} = 5\pi$

(d) $S_{n+1} = (-1)^{n+1} \int_{n\pi}^{(n+1)\pi} x \sin x \, dx = (-1)^{n+1}\left[[-x \cos x]_{n\pi}^{(n+1)\pi} + [\sin x]_{n\pi}^{(n+1)\pi} \right]$

$= (-1)^{n+1}\left[-(n+1)\pi(-1)^n + n\pi(-1)^{n+1}\right] + 0 = (2n+1)\pi$

36. (a) $u = x$, $du = dx$; $dv = \cos x \, dx$, $v = \sin x$;

$S_1 = -\int_{\pi/2}^{3\pi/2} x \cos x \, dx = -\left[[x \sin x]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin x \, dx \right] = -\left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) - [\cos x]_{\pi/2}^{3\pi/2} = 2\pi$

(b) $S_2 = \int_{3\pi/2}^{5\pi/2} x \cos x \, dx = [x \sin x]_{3\pi/2}^{5\pi/2} - \int_{3\pi/2}^{5\pi/2} \sin x \, dx = \left[\frac{5\pi}{2} - \left(-\frac{3\pi}{2}\right)\right] - [\cos x]_{3\pi/2}^{5\pi/2} = 4\pi$

(c) $S_3 = -\int_{5\pi/2}^{7\pi/2} x \cos x \, dx = -\left[[x \sin x]_{5\pi/2}^{7\pi/2} - \int_{5\pi/2}^{7\pi/2} \sin x \, dx \right] = -\left(-\frac{7\pi}{2} - \frac{5\pi}{2}\right) - [\cos x]_{5\pi/2}^{7\pi/2} = 6\pi$

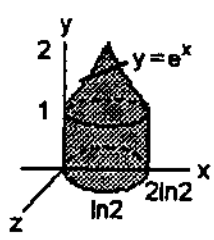
(d) $S_n = (-1)^n \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} x \cos x \, dx = (-1)^n \left[[x \sin x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} - \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \sin x \, dx \right]$

$= (-1)^n \left[\frac{(2n+1)\pi}{2}(-1)^n - \frac{(2n-1)\pi}{2}(-1)^{n-1} \right] - [\cos x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} = \frac{1}{2}(2n\pi + \pi + 2n\pi - \pi) = 2n\pi$

37. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x) e^x \, dx = 2\pi \ln 2 \int_0^{\ln 2} e^x \, dx - 2\pi \int_0^{\ln 2} x e^x \, dx$

$= (2\pi \ln 2)\left[e^x\right]_0^{\ln 2} - 2\pi\left(\left[x e^x\right]_0^{\ln 2} - \int_0^{\ln 2} e^x \, dx \right)$

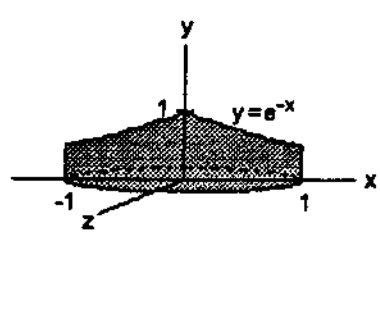
$= 2\pi \ln 2 - 2\pi\left(2 \ln 2 + \left[e^x\right]_0^{\ln 2}\right) = -2\pi \ln 2 + 2 = 2\pi(1 - \ln 2)$



38. (a) $V = \int_0^1 2\pi x e^{-x} \, dx = 2\pi\left(\left[-x e^{-x}\right]_0^1 + \int_0^1 e^{-x} \, dx \right)$

$= 2\pi\left(-\frac{1}{e} + \left[-e^{-x}\right]_0^1\right) = 2\pi\left(-\frac{1}{e} - \frac{1}{e} + 1\right)$

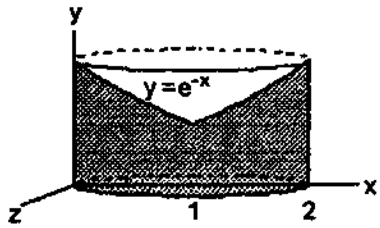
$= 2\pi - \frac{4\pi}{e}$



(b) $V=\int_0^1 2\pi(1-x)e^{-x}\,dx$; $u=1-x$, $du=-dx$; $dv=e^{-x}\,dx$,

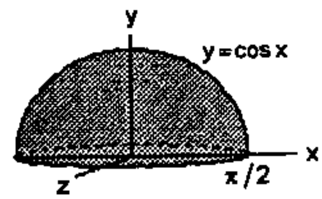
$v=-e^{-x}$; $V=2\pi\left[\left[(1-x)\left(-e^{-x}\right)\right]_0^1-\int_0^1 e^{-x}\,dx\right]$

$=2\pi\left[[0-1(-1)]+\left[e^{-x}\right]_0^1\right]=2\pi\left(1+\frac{1}{e}-1\right)=\frac{2\pi}{e}$



39. (a) $V=\int_0^{\pi/2} 2\pi x\cos x\,dx=2\pi\left([x\sin x]_0^{\pi/2}-\int_0^{\pi/2}\sin x\,dx\right)$

$=2\pi\left(\frac{\pi}{2}+[\cos x]_0^{\pi/2}\right)=2\pi\left(\frac{\pi}{2}+0-1\right)=\pi(\pi-2)$



(b) $V=\int_0^{\pi/2} 2\pi\left(\frac{\pi}{2}-x\right)\cos x\,dx$; $u=\frac{\pi}{2}-x$, $du=-dx$; $dv=\cos x\,dx$, $v=\sin x$;

$V=2\pi\left[\left(\frac{\pi}{2}-x\right)\sin x\right]_0^{\pi/2}+2\pi\int_0^{\pi/2}\sin x\,dx=0+2\pi[-\cos x]_0^{\pi/2}=2\pi(0+1)=2\pi$

40. (a) $V=\int_0^{\pi} 2\pi x(x\sin x)\,dx$;

$$\begin{array}{lll} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$\Rightarrow V=2\pi\int_0^{\pi} x^2\sin x\,dx=2\pi\left[-x^2\cos x+2x\sin x+2\cos x\right]_0^{\pi}=2\pi\left(\pi^2-4\right)$

(b) $V=\int_0^{\pi} 2\pi(\pi-x)x\sin x\,dx=2\pi^2\int_0^{\pi} x\sin x\,dx-2\pi\int_0^{\pi} x^2\sin x\,dx=2\pi^2\left[-x\cos x+\sin x\right]_0^{\pi}-\left(2\pi^3-8\pi\right)$

$=8\pi$

41. $av(y)=\frac{1}{2\pi}\int_0^{2\pi} 2e^{-t}\cos t\,dt=\frac{1}{\pi}\left[e^{-t}\left(\frac{\sin t-\cos t}{2}\right)\right]_0^{2\pi}$

(see Exercise 22) $\Rightarrow av(y)=\frac{1}{2\pi}\left(1-e^{-2\pi}\right)$

42. $av(y)=\frac{1}{2\pi}\int_0^{2\pi} 4e^{-t}(\sin t-\cos t)\,dt=\frac{2}{\pi}\int_0^{2\pi} e^{-t}\sin t\,dt-\frac{2}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt$

$=\frac{2}{\pi}\left[e^{-t}\left(\frac{-\sin t-\cos t}{2}\right)-e^{-t}\left(\frac{\sin t-\cos t}{2}\right)\right]_0^{2\pi}=\frac{2}{\pi}\left[-e^{-t}\sin t\right]_0^{2\pi}=0$

43. Let $u = x^n$ $\qquad dv = \cos x\, dx$

$du = nx^{n-1}\, dx \qquad v = \sin x$

$$\int x^n \cos x\, dx = x^n \sin x - \int (\sin x)(nx^{n-1}\, dx) = x^n \sin x - n\int x^{n-1} \sin x\, dx$$

44. Let $u = x^n$ $\qquad dv = \sin x\, dx$

$du = nx^{n-1}\, dx \qquad v = -\cos x$

$$\int x^n \sin x\, dx = (x^n)(-\cos x) - \int (-\cos x)(nx^{n-1}\, dx) = -x^n \cos x + n\int x^{n-1} \cos x\, dx$$

45. Let $u = x^n$ $\qquad dv = e^{ax}\, dx$

$du = nx^{n-1}\, dx \qquad v = \frac{1}{a}e^{ax}$

$$\int x^n e^{ax}\, dx = (x^n)\left(\tfrac{1}{a}e^{ax}\right) - \int \left(\tfrac{1}{a}e^{ax}\right)(nx^{n-1}\, dx) = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx,\ a \neq 0$$

46. Let $u = (\ln x)^n$ $\qquad dv = dx$

$du = \frac{n(\ln x)^{n-1}}{x}\, dx \qquad v = x$

$$\int (\ln x)^n\, dx = x(\ln x)^n - \int x\left[\frac{n(\ln x)^{n-1}}{x}\right] dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$$

47. (a) Let $y = f^{-1}(x)$. Then $x = f(y)$, so $dx = f'(y)\, dy$.

Hence, $\int f^{-1}(x)\, dx = \int (y)[f'(y)\, dy] = \int yf'(y)\, dy$

(b) Let $u = y$ $\qquad dv = f'(y)\, dy$

$du = dy \qquad v = f(y)$

$$\int yf'(y)\, dy = yf(y) - \int f(y)\, dy = f^{-1}(x)(x) - \int f(y)\, dy$$

Hence, $\int f^{-1}(x)\, dx = \int yf'(y)\, dy = xf^{-1}(x) - \int f(y)\, dy.$

48. Let $u = f^{-1}(x)$ $\qquad dv = dx$

$du = \left(\frac{d}{dx}f^{-1}(x)\right) dx \qquad v = x$

$$\int f^{-1}(x)\, dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right) dx$$

49. (a) Using $y = f^{-1}(x) = \sin^{-1} x$ and $f(y) = \sin y$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have:

$$\int \sin^{-1} x\, dx = x \sin^{-1} x - \int \sin y\, dy = x \sin^{-1} x + \cos y + C = x \sin^{-1} x + \cos(\sin^{-1} x) + C$$

(b) $\int \sin^{-1} x\, dx = x \sin^{-1} x - \int x\left(\frac{d}{dx} \sin^{-1} x\right) dx = x \sin^{-1} x - \int x \frac{1}{\sqrt{1-x^2}}\, dx$

$u = 1 - x^2$, $du = -2x\, dx$ $= x \sin^{-1} x + \frac{1}{2}\int u^{-1/2}\, du = x \sin^{-1} x + u^{1/2} + C = x \sin^{-1} x + \sqrt{1-x^2} + C$

(c) $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

50. (a) Using $y = f^{-1}(x) = \tan^{-1} x$ and $f(y) = \tan y$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$, we have:

$$\int \tan^{-1} x\, dx = x \tan^{-1} x - \int \tan y\, dy = x \tan^{-1} x - \ln|\sec y| + C = x \tan^{-1} x + \ln|\cos y| + C$$

$$= x \tan^{-1} x + \ln\left|\cos(\tan^{-1}(x)\right| dx + C$$

(b) $\int \tan^{-1} x\, dx = x \tan^{-1} x - \int x\left(\frac{d}{dx} \tan^{-1} x\right) dx = x \tan^{-1} x - \int x\left(\frac{1}{1+x^2}\right) dx$

$u = 1 + x^2$, $du = 2x\, dx$ $= x \tan^{-1} x - \frac{1}{2}\int u^{-1}\, du = x \tan^{-1} x - \frac{1}{2}\ln|u| + C = x \tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + C$

(c) $\ln\left|\cos(\tan^{-1} x)\right| = \ln\left|\frac{1}{\sqrt{1+x^2}}\right| = -\frac{1}{2}\ln\left(1+x^2\right)$

51. (a) Using $y = f^{-1}(x) = \cos^{-1} x$ and $f(y) = \cos x$, $0 \le x \le \pi$, we have:

$$\int \cos^{-1} x\, dx = x \cos^{-1} x - \int \cos y\, dy = x \cos^{-1} x - \sin y + C = x \cos^{-1} x - \sin(\cos^{-1} x) + C$$

(b) $\int \cos^{-1} x\, dx = x \cos^{-1} x - \int x\left(\frac{d}{dx} \cos^{-1} x\right) dx = x \cos^{-1} x - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right) dx$

$u = 1 - x^2$, $du = -2x\, dx$ $= x \cos^{-1} x - \frac{1}{2}\int u^{-1/2}\, du = x \cos^{-1} x - u^{1/2} + C = x \cos^{-1} x - \sqrt{1-x^2} + C$

(c) $\sin(\cos^{-1} x) = \sqrt{1-x^2}$

52. (a) Using $y = f^{-1}(x) = \log_2 x$ and $f(y) = 2^y$, we have

$$\int \log_2 x\, dx = x \log_2 x - \int 2^y\, dy = x \log_2 x - \frac{2^y}{\ln 2} + C = x \log_2 x - \frac{1}{\ln 2}\, 2^{\log_2 x}$$

(b) $\int \log_2 x\, dx = x \log_2 x - \int x\left(\frac{d}{dx} \log_2 x\right) dx = x \log_2 x - \int x\left(\frac{1}{x \ln 2}\right) dx = x \log_2 x - \int \frac{dx}{\ln 2}$

$$= x \log_2 x - \left(\frac{1}{\ln 2}\right)x + C$$

(c) $2^{\log_2 x} = x$

7.3 PARTIAL FRACTIONS

1. $\frac{5x-13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x - 13 = A(x-2) + B(x-3) = (A+B)x - (2A+3B)$

$\Rightarrow \left.\begin{array}{c} A + B = 5 \\ 2A + 3B = 13 \end{array}\right\} \Rightarrow -B = (10-13) \Rightarrow B = 3 \Rightarrow A = 2$; thus, $\frac{5x-13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}$

2. $\frac{5x-7}{x^2-3x+2}=\frac{5x-7}{(x-2)(x-1)}=\frac{A}{x-2}+\frac{B}{x-1}\Rightarrow 5x-7=A(x-1)+B(x-2)=(A+B)x-(A+2B)$

$\Rightarrow \left.\begin{array}{c}A+B=5\\A+2B=7\end{array}\right\}\Rightarrow B=2\Rightarrow A=3$; thus, $\frac{5x-7}{x^2-3x+2}=\frac{3}{x-2}+\frac{2}{x-1}$

3. $\frac{x+4}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}\Rightarrow x+4=A(x+1)+B=Ax+(A+B)\ \Rightarrow \left.\begin{array}{c}A=1\\A+B=4\end{array}\right\}\Rightarrow A=1$ and $B=3$;

thus, $\frac{x+4}{(x+1)^2}=\frac{1}{x+1}+\frac{3}{(x+1)^2}$

4. $\frac{2x+2}{x^2-2x+1}=\frac{2x+2}{(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}\Rightarrow 2x+2=A(x-1)+B=Ax+(-A+B)\ \Rightarrow \left.\begin{array}{c}A=2\\-A+B=2\end{array}\right\}$

$\Rightarrow A=2$ and $B=4$; thus, $\frac{2x+2}{x^2-2x+1}=\frac{2}{x-1}+\frac{4}{(x-1)^2}$

5. $\frac{z+1}{z^2(z-1)}=\frac{A}{z}+\frac{B}{z^2}+\frac{C}{z-1}\Rightarrow z+1=Az(z-1)+B(z-1)+Cz^2\Rightarrow z+1=(A+C)z^2+(-A+B)z-B$

$\Rightarrow \left.\begin{array}{c}A+C=0\\-A+B=1\\-B=1\end{array}\right\}\Rightarrow B=-1\Rightarrow A=-2\Rightarrow C=2$; thus, $\frac{z+1}{z^2(z-1)}=\frac{-2}{z}+\frac{-1}{z^2}+\frac{2}{z-1}$

6. $\frac{z}{z^3-z^2-6z}=\frac{1}{z^2-z-6}=\frac{1}{(z-3)(z+2)}=\frac{A}{z-3}+\frac{B}{z+2}\Rightarrow 1=A(z+2)+B(z-3)=(A+B)z+(2A-3B)$

$\Rightarrow \left.\begin{array}{c}A+B=0\\2A-3B=1\end{array}\right\}\Rightarrow -5B=1\Rightarrow B=-\frac{1}{5}\Rightarrow A=\frac{1}{5}$; thus, $\frac{z}{z^3-z^2-6z}=\frac{\frac{1}{5}}{z-3}+\frac{-\frac{1}{5}}{z+2}$

7. $\frac{t^2+8}{t^2-5t+6}=1+\frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6}=\frac{5t+2}{(t-3)(t-2)}=\frac{A}{t-3}+\frac{B}{t-2}$

$\Rightarrow 5t+2=A(t-2)+B(t-3)=(A+B)t+(-2A-3B)\Rightarrow \left.\begin{array}{c}A+B=5\\-2A-3B=2\end{array}\right\}\Rightarrow -B=(10+2)=12$

$\Rightarrow B=-12\Rightarrow A=17$; thus, $\frac{t^2+8}{t^2-5t+6}=1+\frac{17}{t-3}+\frac{-12}{t-2}$

8. $\frac{t^4+9}{t^4+9t^2}=1+\frac{-9t^2+9}{t^4+9t^2}=1+\frac{-9t^2+9}{t^2(t^2+9)}$ (after long division); $\frac{-9t^2+9}{t^2(t^2+9)}=\frac{A}{t}+\frac{B}{t^2}+\frac{Ct+D}{t^2+9}$

$\Rightarrow -9t^2+9=At(t^2+9)+B(t^2+9)+(Ct+D)t^2=(A+C)t^3+(B+D)t^2+9At+9B$

$\Rightarrow \left.\begin{array}{c}A+C=0\\B+D=-9\\9A=0\\9B=9\end{array}\right\}\Rightarrow A=0\Rightarrow C=0;\ B=1\Rightarrow D=-10$; thus, $\frac{t^4+9}{t^4+9t^2}=1+\frac{1}{t^2}+\frac{-10}{t^2+9}$

9. $\frac{1}{1-x^2}=\frac{A}{1-x}+\frac{B}{1+x}\Rightarrow 1=A(1+x)+B(1-x)$; $x=1\Rightarrow A=\frac{1}{2}$; $x=-1\Rightarrow B=\frac{1}{2}$;

$\int\frac{dx}{1-x^2}=\frac{1}{2}\int\frac{dx}{1-x}+\frac{1}{2}\int\frac{dx}{1+x}=\frac{1}{2}\left[\ln|1+x|-\ln|1-x|\right]+C$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + Bx$; $x = 0 \Rightarrow A = \frac{1}{2}$; $x = -2 \Rightarrow B = -\frac{1}{2}$;

$$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}[\ln|x| - \ln|x+2|] + C$$

11. $\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1) + B(x+6)$; $x = 1 \Rightarrow B = \frac{5}{7}$; $x = -6 \Rightarrow A = \frac{-2}{-7} = \frac{2}{7}$;

$$\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln\left|(x+6)^2(x-1)^5\right| + C$$

12. $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3) + B(x-4)$; $x = 3 \Rightarrow B = \frac{7}{-1} = -7$; $x = 4 \Rightarrow A = \frac{9}{1} = 9$;

$$\int \frac{2x+1}{x^2-7x+12}\,dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3} = 9\ln|x-4| - 7\ln|x-3| + C = \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$$

13. $\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3)$; $y = -1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4}$; $y = 3 \Rightarrow A = \frac{3}{4}$;

$$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$$

$$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \frac{\ln 15}{2}$$

14. $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By$; $y = 0 \Rightarrow A = 4$; $y = -1 \Rightarrow B = \frac{3}{-1} = -3$;

$$\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^1 \frac{dy}{y} - 3\int_{1/2}^1 \frac{dy}{y+1} = [4\ln|y| - 3\ln|y+1|]_{1/2}^1 = (4\ln 1 - 3\ln 2) - \left(4\ln\frac{1}{2} - 3\ln\frac{3}{2}\right)$$

$$= \ln\frac{1}{8} - \ln\frac{1}{16} + \ln\frac{27}{8} = \ln\left(\frac{27}{8}\cdot\frac{1}{8}\cdot 16\right) = \ln\frac{27}{4}$$

15. $\frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2)$; $t = 0 \Rightarrow A = -\frac{1}{2}$; $t = -2$

$\Rightarrow B = \frac{1}{6}$; $t = 1 \Rightarrow C = \frac{1}{3}$; $\int \frac{dt}{t^3+t^2-2t} = -\frac{1}{2}\int \frac{dt}{t} + \frac{1}{6}\int \frac{dt}{t+2} + \frac{1}{3}\int \frac{dt}{t-1}$

$$= -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C$$

16. $\frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) = Cx(x+2)$; $x = 0 \Rightarrow A = \frac{3}{-8}$; $x = -2$

$\Rightarrow B = \frac{1}{16}$; $x = 2 \Rightarrow C = \frac{5}{16}$; $\int \frac{x+3}{2x^3-8x}\,dx = -\frac{3}{8}\int \frac{dx}{x} + \frac{1}{16}\int \frac{dx}{x+2} + \frac{5}{16}\int \frac{dx}{x-2}$

$$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + C = \frac{1}{16}\ln\left|\frac{(x-2)^5(x+2)}{x^6}\right| + C$$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+3 = A(x+1) + B$

$= Ax + (A+B) \Rightarrow A = 3,\ A+B = 2 \Rightarrow A = 3,\ B = -1;\ \int_0^1 \frac{x^3\,dx}{x^2+2x+1}$

$= \int_0^1 (x-2)\,dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$

$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x+2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1) + B$

$= Ax + (-A+B) \Rightarrow A = 3,\ -A+B = -2 \Rightarrow A = 3,\ B = 1;\ \int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$

$= \int_{-1}^0 (x+2)\,dx + 3\int_{-1}^0 \frac{dx}{x-1} + \int_{-1}^0 \frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}\right]_{-1}^0$

$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$

19. $\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$

$x = -1 \Rightarrow C = \frac{1}{4};\ x = 1 \Rightarrow D = \frac{1}{4};$ coefficient of $x^3 = A + B \Rightarrow A + B = 0;$ constant $= A - B + C + D$

$\Rightarrow A - B + C + D = 1 \Rightarrow A - B = \frac{1}{2};$ thus, $A = \frac{1}{4} \Rightarrow B = -\frac{1}{4};\ \int \frac{dx}{(x^2-1)^2}$

$= \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{(x+1)^2} + \frac{1}{4}\int \frac{dx}{(x-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C$

20. $\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1);\ x = -1$

$\Rightarrow C = -\frac{1}{2};\ x = 1 \Rightarrow A = \frac{1}{4};$ coefficient of $x^2 = A + B \Rightarrow A + B = 1 \Rightarrow B = \frac{3}{4};\ \int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

$= \frac{1}{4}\int \frac{dx}{x-1} + \frac{3}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2} = \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C$

$= \frac{\ln\left|(x-1)(x+1)^3\right|}{4} + \frac{1}{2(x+1)} + C$

21. $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1);\ x = -1 \Rightarrow A = \frac{1}{2};$ coefficient of x^2

$= A + B \Rightarrow A + B = 0 \Rightarrow B = -\frac{1}{2};$ constant $= A + C \Rightarrow A + C = 1 \Rightarrow C = \frac{1}{2};\ \int_0^1 \frac{dx}{(x+1)(x^2+1)}$

$$= \frac{1}{2}\int_0^1 \frac{dx}{x+1} + \frac{1}{2}\int_0^1 \frac{(-x+1)}{x^2+1}\,dx = \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x\right]_0^1$$

$$= \left(\frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\tan^{-1}1\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 + \frac{1}{2}\tan^{-1}0\right) = \frac{1}{4}\ln 2 + \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{(\pi + 2\ln 2)}{8}$$

22. $\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Rightarrow 3t^2+t+4 = A(t^2+1) + (Bt+C)t$; $t=0 \Rightarrow A=4$; coefficient of t^2

$= A + B \Rightarrow A + B = 3 \Rightarrow B = -1$; coefficient of $t = C \Rightarrow C = 1$; $\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$

$$= 4\int_1^{\sqrt{3}} \frac{dt}{t} + \int_1^{\sqrt{3}} \frac{(-t+1)}{t^2+1}\,dt = \left[4\ln|t| - \frac{1}{2}\ln(t^2+1) + \tan^{-1}t\right]_1^{\sqrt{3}}$$

$$= \left(4\ln\sqrt{3} - \frac{1}{2}\ln 4 + \tan^{-1}\sqrt{3}\right) - \left(4\ln 1 - \frac{1}{2}\ln 2 + \tan^{-1}1\right) = 2\ln 3 - \ln 2 + \frac{\pi}{3} + \frac{1}{2}\ln 2 + \frac{\pi}{4}$$

$$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12} = \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}$$

23. $\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2} \Rightarrow y^2+2y+1 = (Ay+B)(y^2+1) + Cy + D$

$= Ay^3 + By^2 + (A+C)y + (B+D) \Rightarrow A = 0,\ B = 1;\ A + C = 2 \Rightarrow C = 2;\ B + D = 1 \Rightarrow D = 0;$

$$\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy = \int \frac{1}{y^2+1}\,dy + 2\int \frac{y}{(y^2+1)^2}\,dy = \tan^{-1}y - \frac{1}{y^2+1} + C$$

24. $\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2} \Rightarrow 8x^2+8x+2 = (Ax+B)(4x^2+1) + Cx + D$

$= 4Ax^3 + 4Bx^2 + (A+C)x + (B+D);\ A = 0,\ B = 2;\ A + C = 8 \Rightarrow C = 8;\ B + D = 2 \Rightarrow D = 0;$

$$\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx = 2\int \frac{dx}{4x^2+1} + 8\int \frac{x\,dx}{(4x^2+1)^2} = \tan^{-1}2x - \frac{1}{4x^2+1} + C$$

25. $\frac{2s+2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3} \Rightarrow 2s+2$

$= (As+B)(s-1)^3 + C(s^2+1)(s-1)^2 + D(s^2+1)(s-1) + E(s^2+1)$

$= [As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B] + C(s^4 - 2s^3 + 2s^2 - 2s + 1) + D(s^3 - s^2 + s - 1)$
$\quad + E(s^2+1)$

$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$

$$\Rightarrow \left.\begin{array}{l} A + C = 0 \\ -3A + B - 2C + D = 0 \\ 3A - 3B + 2C - D + E = 0 \\ -A + 3B - 2C + D = 2 \\ -B + C - D + E = 2 \end{array}\right\} \text{summing all equations} \Rightarrow 2E = 4 \Rightarrow E = 2;$$

summing eqs (2) and (3) $\Rightarrow -2B + 2 = 0 \Rightarrow B = 1$; summing eqs (3) and (4) $\Rightarrow 2A + 2 = 2 \Rightarrow A = 0$; $C = 0$ from eq (1); then $-1 + 0 - D + 2 = 2$ from eq (5) $\Rightarrow D = -1$;

$$\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1} s + C$$

26. $\frac{s^4+81}{s(s^2+9)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \Rightarrow s^4 + 81 = A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s$

$= A(s^4 + 18s^2 + 81) + (Bs^4 + Cs^3 + 9Bs^2 + 9Cs) + Ds^2 + Es$

$= (A+B)s^4 + Cs^3 + (18A + 9B + D)s^2 + (9C+E)s + 81A \Rightarrow 81A = 81$ or $A = 1$; $A + B = 1 \Rightarrow B = 0$;

$C = 0$; $9C + E = 0 \Rightarrow E = 0$; $18A + 9B + D = 0 \Rightarrow D = -18$; $\int \frac{s^4+81}{s(s^2+9)^2}\,ds = \int \frac{ds}{s} - 18\int \frac{s\,ds}{(s^2+9)^2}$

$= \ln|s| + \frac{9}{(s^2+9)} + C$

27. $\frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3 + 5\theta^2 + 8\theta + 4 = (A\theta+B)(\theta^2+2\theta+2) + C\theta + D$

$= A\theta^3 + (2A+B)\theta^2 + (2A+2B+C)\theta + (2B+D) \Rightarrow A = 2$; $2A + B = 5 \Rightarrow B = 1$; $2A + 2B + C = 8 \Rightarrow C = 2$;

$2B + D = 4 \Rightarrow D = 2$; $\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta = \int \frac{2\theta+1}{(\theta^2+2\theta+2)}\,d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2}\,d\theta$

$= \int \frac{2\theta+2}{\theta^2+2\theta+2}\,d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{d(\theta^2+2\theta+2)}{(\theta^2+2\theta+2)^2} = \int \frac{d(\theta^2+2\theta+2)}{\theta^2+2\theta+2} - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2}$

$= \frac{-1}{\theta^2+2\theta+2} + \ln(\theta^2+2\theta+2) - \tan^{-1}(\theta+1) + C$

28. $\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{(\theta^2+1)^2} + \frac{E\theta+F}{(\theta^2+1)^3} \Rightarrow \theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1$

$= (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + E\theta + F = (A\theta+B)(\theta^4+2\theta^2+1) + (C\theta^3 + D\theta^2 + C\theta + D) + E\theta + F$

$= (A\theta^5 + B\theta^4 + 2A\theta^3 + 2B\theta^2 + A\theta + B) + (C\theta^3 + D\theta^2 + C\theta + D) + E\theta + F$

$= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + (B+D+F) \Rightarrow A = 0$; $B = 1$; $2A + C = -4$

$\Rightarrow C = -4$; $2B + D = 2 \Rightarrow D = 0$; $A + C + E = -3 \Rightarrow E = 1$; $B + D + F = 1 \Rightarrow F = 0$;

$$\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta = \int \frac{d\theta}{\theta^2+1} - 4\int \frac{\theta\,d\theta}{(\theta^2+1)^2} + \int \frac{\theta\,d\theta}{(\theta^2+1)^3} = \tan^{-1}\theta + 2(\theta^2+1)^{-1} - \tfrac{1}{4}(\theta^2+1)^{-2} + C$$

29. $\frac{2x^3-2x^2+1}{x^2-x} = 2x + \frac{1}{x^2-x} = 2x + \frac{1}{x(x-1)}$; $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + Bx$; $x = 0 \Rightarrow A = -1$;

$x = 1 \Rightarrow B = 1$; $\int \frac{2x^3-2x^2+1}{x^2-x} = \int 2x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C$

30. $\frac{x^4}{x^2-1}=(x^2+1)+\frac{1}{x^2-1}=(x^2+1)+\frac{1}{(x+1)(x-1)}$; $\frac{1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}\Rightarrow 1=A(x-1)+B(x+1)$;

$x=-1\Rightarrow A=-\frac{1}{2}$; $x=1\Rightarrow B=\frac{1}{2}$; $\int\frac{x^4}{x^2-1}\,dx=\int(x^2+1)\,dx-\frac{1}{2}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{x-1}$

$=\frac{1}{3}x^3+x-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C=\frac{x^3}{3}+x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$

31. $\frac{9x^3-3x+1}{x^3-x^2}=9+\frac{9x^2-3x+1}{x^2(x-1)}$ (after long division); $\frac{9x^2-3x+1}{x^2(x-1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$

$\Rightarrow 9x^2-3x+1=Ax(x-1)+B(x-1)+Cx^2$; $x=1\Rightarrow C=7$; $x=0\Rightarrow B=-1$; $A+C=9\Rightarrow A=2$;

$\int\frac{9x^3-3x+1}{x^3-x^2}\,dx=\int 9\,dx+2\int\frac{dx}{x}-\int\frac{dx}{x^2}+7\int\frac{dx}{x-1}=9x+2\ln|x|+\frac{1}{x}+7\ln|x-1|+C$

32. $\frac{16x^3}{4x^2-4x+1}=(4x+4)+\frac{12x-4}{4x^2-4x+1}$; $\frac{12x-4}{(2x-1)^2}=\frac{A}{2x-1}+\frac{B}{(2x-1)^2}\Rightarrow 12x-4=A(2x-1)+B$

$\Rightarrow A=6$; $-A+B=-4\Rightarrow B=2$; $\int\frac{16x^3}{4x^2-4x+1}\,dx=4\int(x+1)\,dx+6\int\frac{dx}{2x-1}+2\int\frac{dx}{(2x-1)^2}$

$=2(x+1)^2+3\ln|2x-1|-\frac{1}{2x-1}+C_1=2x^2+4x+3\ln|2x-1|-(2x-1)^{-1}+C$, where $C=2+C_1$

33. $\frac{y^4+y^2-1}{y^3+y}=y-\frac{1}{y(y^2+1)}$; $\frac{1}{y(y^2+1)}=\frac{A}{y}+\frac{By+C}{y^2+1}\Rightarrow 1=A(y^2+1)+(By+C)y=(A+B)y^2+Cy+A$

$\Rightarrow A=1$; $A+B=0\Rightarrow B=-1$; $C=0$; $\int\frac{y^4+y^2-1}{y^3+y}\,dy=\int y\,dy-\int\frac{dy}{y}+\int\frac{y\,dy}{y^2+1}$

$=\frac{y^2}{2}-\ln|y|+\frac{1}{2}\ln(1+y^2)+C$

34. $\frac{2y^4}{y^3-y^2+y-1}=2y+2+\frac{2}{y^3-y^2+y-1}$; $\frac{2}{y^3-y^2+y-1}=\frac{2}{(y^2+1)(y-1)}=\frac{A}{y-1}+\frac{By+C}{y^2+1}$

$\Rightarrow 2=A(y^2+1)+(By+C)(y-1)=(Ay^2+A)+(By^2+Cy-By-C)=(A+B)y^2+(-B+C)y+(A-C)$

$\Rightarrow A+B=0$, $-B+C=0$ or $C=B$, $A-C=A-B=2\Rightarrow A=1$, $B=-1$, $C=-1$;

$\int\frac{2y^4}{y^3-y^2+y-1}\,dy=2\int(y+1)\,dy+\int\frac{dy}{y-1}-\int\frac{y}{y^2+1}\,dy-\int\frac{dy}{y^2+1}$

$=(y+1)^2+\ln|y-1|-\frac{1}{2}\ln(y^2+1)-\tan^{-1}y+C_1=y^2+2y+\ln|y-1|-\frac{1}{2}\ln(y^2+1)-\tan^{-1}y+C$,

where $C=C_1+1$

35. $\int\frac{e^t\,dt}{e^{2t}+3e^t+2}=[e^t=y]\int\frac{dy}{y^2+3y+2}=\int\frac{dy}{y+1}-\int\frac{dy}{y+2}=\ln\left|\frac{y+1}{y+2}\right|+C=\ln\left|\frac{e^t+1}{e^t+2}\right|+C$

36. $\int\frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt$; $[e^t=y]\to\int\frac{y^3+2y-1}{y^2+1}\,dy=\int\left(y+\frac{y-1}{y^2+1}\right)dy=\frac{y^2}{2}+\frac{1}{2}\ln(y^2+1)-\tan^{-1}y+C$

$= \frac{1}{2}e^{2t} - \tan^{-1}(e^t) + \frac{1}{2}\ln(e^{2t}+1) + C$

37. $\int \frac{\cos y\, dy}{\sin^2 y + \sin y - 6}$; $[\sin y = t, \cos y\, dy = dt] \to \int \frac{dy}{t^2+t-6} = \frac{1}{5}\int \left(\frac{1}{t-2} - \frac{1}{t+3}\right) dt = \frac{1}{5}\ln\left|\frac{t-2}{t+3}\right| + C$

$= \frac{1}{5}\ln\left|\frac{\sin y - 2}{\sin y + 3}\right| + C$

38. $\int \frac{\sin\theta\, d\theta}{\cos^2\theta + \cos\theta - 2}$; $[\cos\theta = y] \to -\int \frac{dy}{y^2+y-2} = \frac{1}{3}\int \frac{dy}{y+2} - \frac{1}{3}\int \frac{dy}{y-1} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C = \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C$

$= \frac{1}{3}\ln\left|\frac{2+\cos\theta}{1-\cos\theta}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

39. $\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\, dx = \int \frac{\tan^{-1}(2x)}{4x^2+1}\, dx - 3\int \frac{x}{(x-2)^2}\, dx$

$= \frac{1}{2}\int \tan^{-1}(2x)\, d\left(\tan^{-1}(2x)\right) - 3\int \frac{dx}{x-2} - 6\int \frac{dx}{(x-2)^2} = \frac{\left(\tan^{-1}2x\right)^2}{4} - 3\ln|x-2| + \frac{6}{x-2} + C$

40. $\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\, dx = \int \frac{\tan^{-1}(3x)}{9x^2+1}\, dx + \int \frac{x}{(x+1)^2}\, dx$

$= \frac{1}{3}\int \tan^{-1}(3x)\, d\left(\tan^{-1}(3x)\right) + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{\left(\tan^{-1}3x\right)^2}{6} + \ln|x+1| + \frac{1}{x+1} + C$

41. $(t^2-3t+2)\frac{dx}{dt} = 1$; $x = \int \frac{dt}{t^2-3t+2} = \int \frac{dt}{t-2} - \int \frac{dt}{t-1} = \ln\left|\frac{t-2}{t-1}\right| + C$; $\frac{t-2}{t-1} = Ce^x$; $t = 3$ and $x = 0$

$\Rightarrow \frac{1}{2} = C \Rightarrow \frac{t-2}{t-1} = \frac{1}{2}e^x \Rightarrow x = \ln\left|2\left(\frac{t-2}{t-1}\right)\right| = \ln|t-2| - \ln|t-1| + \ln 2$

42. $(3t^4+4t^2+1)\frac{dx}{dt} = 2\sqrt{3}$; $x = 2\sqrt{3}\int \frac{dt}{3t^4+4t^2+1} = \sqrt{3}\int \frac{dt}{t^2+\frac{1}{3}} - \sqrt{3}\int \frac{dt}{t^2+1}$

$= 3\tan^{-1}(\sqrt{3}t) - \sqrt{3}\tan^{-1}t + C$; $t = 1$ and $x = \frac{-\pi\sqrt{3}}{4} \Rightarrow -\frac{\sqrt{3}\pi}{4} = \pi - \frac{\sqrt{3}}{4}\pi + C \Rightarrow C = -\pi$

$\Rightarrow x = 3\tan^{-1}(\sqrt{3}t) - \sqrt{3}\tan^{-1}t - \pi$

43. $(t^2+2t)\frac{dx}{dt} = 2x+2$; $\frac{1}{2}\int \frac{dx}{x+1} = \int \frac{dt}{t^2+2t} \Rightarrow \frac{1}{2}\ln|x+1| = \frac{1}{2}\int \frac{dt}{t} - \frac{1}{2}\int \frac{dt}{t+2} \Rightarrow \ln|x+1| = \ln\left|\frac{t}{t+2}\right| + C$;

$t = 1$ and $x = 1 \Rightarrow \ln 2 = \ln\frac{1}{3} + C \Rightarrow C = \ln 2 + \ln 3 = \ln 6 \Rightarrow \ln|x+1| = \ln 6\left|\frac{t}{t+2}\right| \Rightarrow x+1 = \frac{6t}{t+2}$

$\Rightarrow x = \frac{6t}{t+2} - 1,\ t > 0$

44. $(t+1)\frac{dx}{dt} = x^2 + 1 \Rightarrow \int \frac{dx}{x^2+1} = \int \frac{dt}{t+1} \Rightarrow \tan^{-1} x = \ln|t+1| + C$; $t = 0$ and $x = \frac{\pi}{4} \Rightarrow \tan^{-1}\frac{\pi}{4} = \ln|1| + C$

$\Rightarrow C = \tan^{-1}\frac{\pi}{4} = 1 \Rightarrow \tan^{-1} x = \ln|t+1| + 1 \Rightarrow x = \tan(\ln(t+1)+1),\ t > -1$

45. $\frac{1}{y^2 - y}\,dy = e^x\,dx \Rightarrow \int \frac{1}{y(y-1)}\,dy = \int e^x\,dx = e^x + C$

$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1} \Rightarrow 1 = A(y-1) + B(y) = (A+B)y - A$

Equating coefficients of like terms gives

$A + B = 0$ and $-A = 1$

Solving the system simultaneously yields $A = -1$, $B = 1$.

$\int \frac{1}{y(y-1)}\,dy = \int -\frac{1}{y}\,dy + \int \frac{1}{y-1}\,dy = -\ln|y| + \ln|y-1| + C_2 \Rightarrow -\ln|y| + \ln|y-1| = e^x + C$

Substitute $x = 0$, $y = 2$.

$-\ln 2 + 0 = 1 + C$ or $C = -1 - \ln 2$

The solution to the initial value problem is

$-\ln|y| + \ln|y-1| = e^x - 1 - \ln 2.$

46. $\frac{1}{(y+1)^2}\,dy = \sin\theta\,d\theta \Rightarrow \int \frac{1}{(y+1)^2}\,dy = \int \sin\theta\,d\theta \Rightarrow -\frac{1}{y+1} = -\cos\theta + C$

Substitute $\theta = \frac{\pi}{2}$, $y = 0 \Rightarrow -1 = 0 + C$ or $C = -1$

The solution to the initial value problem is

$-\frac{1}{y+1} = -\cos\theta - 1 \Rightarrow y + 1 = \frac{1}{\cos\theta + 1} \Rightarrow y = \frac{1}{\cos\theta + 1} - 1$

47. $dy = \frac{dx}{x^2 - 3x + 2}$; $x^2 - 3x + 2 = (x-2)(x-1) \Rightarrow \frac{1}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x-2)$

$\Rightarrow 1 = (A+B)x - A - 2B$

Equating coefficients of like terms gives

$A + B = 0$, $-A - 2B = 1$

Solving the system simultaneously yields $A = 1$, $B = -1$.

$\int dy = \int \frac{dx}{x^2 - 3x + 2} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1}$

$y = \ln|x-2| - \ln|x-1| + C$

Substitute $x = 3$, $y = 0 \Rightarrow 0 = 0 - \ln 2 + C$ or $C = \ln 2$

The solution to the initial value problems is

$y = \ln|x-2| - \ln|x-1| + \ln 2$

48. $\frac{ds}{2s+2} = \frac{dt}{t^2+2t} \Rightarrow \int \frac{ds}{2s+2} = \frac{1}{2}\int \frac{ds}{s+1} = \frac{1}{2}\ln|s+1| + C_1$

$t^2+2t=t(t+2) \Rightarrow \frac{1}{t^2+2t}=\frac{A}{t}+\frac{B}{t+2} \Rightarrow 1=A(t+2)+Bt \Rightarrow 1=(A+B)t+2A$

Equating coefficients of like terms gives $A+B=0$ and $2A=1$

Solving the system simultaneously yields $A=\frac{1}{2}$, $B=-\frac{1}{2}$.

$$\int \frac{dt}{t^2+2t}=\int \frac{1/2}{t}\,dt-\int \frac{1/2}{t+2}\,dt=\frac{1}{2}\ln|t|-\frac{1}{2}\ln|t+2|+C_2 \Rightarrow \frac{1}{2}\ln|s+1|=\frac{1}{2}\ln|t|-\frac{1}{2}\ln|t+2|+C_3$$

$\Rightarrow \ln|s+1|=\ln|t|-\ln|t+2|+C$

Substitute $t=1$, $s=1 \Rightarrow \ln 2=0-\ln 3+C$ or $C=\ln 2+\ln 3=\ln 6$

The solution to the initial value problem is

$\ln|s+1|=\ln|t|-\ln|t+2|+\ln 6 \Rightarrow \ln|s+1|=\ln\left|\frac{6t}{t+2}\right| \Rightarrow |s+1|=\left|\frac{6t}{t+2}\right|$.

49. $V=\pi\int_{0.5}^{2.5} y^2\,dx=\pi\int_{0.5}^{2.5}\frac{9}{3x-x^2}\,dx=3\pi\left(\int_{0.5}^{2.5}\left(-\frac{1}{x-3}+\frac{1}{x}\right)\right)dx=\left[3\pi\ln\left|\frac{x}{x-3}\right|\right]_{0.5}^{2.5}=3\pi\ln 25$

50. $V=2\pi\int_0^1 xy\,dx=2\pi\int_0^1\frac{2x}{(x+1)(2-x)}\,dx=4\pi\int_0^1\left(-\frac{1}{3}\left(\frac{1}{x+1}\right)+\frac{2}{3}\left(\frac{1}{2-x}\right)\right)dx$

$=\left[-\frac{4\pi}{3}(\ln|x+1|+2\ln|2-x|)\right]_0^1=\frac{4\pi}{3}(\ln 2)$

51. (a) $\frac{dx}{dt}=kx(N-x) \Rightarrow \int\frac{dx}{x(N-x)}=\int k\,dt \Rightarrow \frac{1}{N}\int\frac{dx}{x}+\frac{1}{N}\int\frac{dx}{N-x}=\int k\,dt \Rightarrow \frac{1}{N}\ln\left|\frac{x}{N-x}\right|=kt+C;$

$k=\frac{1}{250}$, $N=1000$, $t=0$ and $x=2 \Rightarrow \frac{1}{1000}\ln\left|\frac{2}{998}\right|=C \Rightarrow \frac{1}{1000}\ln\left|\frac{x}{1000-x}\right|=\frac{t}{250}+\frac{1}{1000}\ln\left(\frac{1}{499}\right)$

$\Rightarrow \ln\left|\frac{499x}{1000-x}\right|=4t \Rightarrow \frac{499x}{1000-x}=e^{4t} \Rightarrow 499x=e^{4t}(1000-x) \Rightarrow (499+e^{4t})x=1000e^{4t} \Rightarrow x=\frac{1000e^{4t}}{499+e^{4t}}$

(b) $x=\frac{1}{2}N=500 \Rightarrow 500=\frac{1000e^{4t}}{499+e^{4t}} \Rightarrow 500\cdot 499+500e^{4t}=1000e^{4t} \Rightarrow e^{4t}=499 \Rightarrow t=\frac{1}{4}\ln 499 \approx 1.55$ days

52. $\frac{dx}{dt}=k(a-x)(b-x) \Rightarrow \frac{dx}{(a-x)(b-x)}=k\,dt$

(a) $a=b$: $\int\frac{dx}{(a-x)^2}=\int k\,dt \Rightarrow \frac{1}{a-x}=kt+C$; $t=0$ and $x=0 \Rightarrow \frac{1}{a}=C \Rightarrow \frac{1}{a-x}=kt+\frac{1}{a}$

$\Rightarrow \frac{1}{a-x}=\frac{akt+1}{a} \Rightarrow a-x=\frac{a}{akt+1} \Rightarrow x=a-\frac{a}{akt+1}=\frac{a^2kt}{akt+1}$

(b) $a\neq b$: $\int\frac{dx}{(a-x)(b-x)}=\int k\,dt \Rightarrow \frac{1}{b-a}\int\frac{dx}{a-x}-\frac{1}{b-a}\int\frac{dx}{b-x}=\int k\,dt \Rightarrow \frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right|=kt+C;$

$t=0$ and $x=0 \Rightarrow \frac{1}{b-a}\ln\frac{b}{a}=C \Rightarrow \ln\left|\frac{b-x}{a-x}\right|=(b-a)kt+\ln\left(\frac{b}{a}\right) \Rightarrow \frac{b-x}{a-x}=\frac{b}{a}e^{(b-a)kt}$

$$\Rightarrow x = \frac{ab\left[1 - e^{(a-b)kt}\right]}{a - be^{(a-b)kt}}$$

7.4 TRIGONOMETRIC SUBSTITUTIONS

1. $y = 3\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dy = \frac{3\,d\theta}{\cos^2\theta},\ 9 + y^2 = 9\left(1 + \tan^2\theta\right) = \frac{9}{\cos^2\theta} \Rightarrow \frac{1}{\sqrt{9 + y^2}} = \frac{|\cos\theta|}{3} = \frac{\cos\theta}{3}$

$\left(\text{because } \cos\theta > 0 \text{ when } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$;

$$\int \frac{dy}{\sqrt{9 + y^2}} = 3\int \frac{\cos\theta\,d\theta}{3\cos^2\theta} = \int \frac{d\theta}{\cos\theta} = \ln|\sec\theta + \tan\theta| + C' = \ln\left|\frac{\sqrt{9 + y^2}}{3} + \frac{y}{3}\right| + C' = \ln\left|\sqrt{9 + y^2} + y\right| + C$$

2. $\int \frac{3\,dy}{\sqrt{1 + 9y^2}};\ [3y = x] \Rightarrow \int \frac{dx}{\sqrt{1 + x^2}};\ x = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2},\ dx = \frac{dt}{\cos^2 t},\ \sqrt{1 + x^2} = \frac{1}{\cos t}$;

$$\int \frac{dx}{\sqrt{1 + x^2}} = \int \frac{dt}{\cos^2 t\left(\frac{1}{\cos t}\right)} = \ln|\sec t + \tan t| + C = \ln\left|\sqrt{x^2 + 1} + x\right| + C = \ln\left|\sqrt{1 + 9y^2} + 3y\right| + C$$

3. $t = 5\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = 5\cos\theta\,d\theta,\ \sqrt{25 - t^2} = 5\cos\theta$;

$$\int \sqrt{25 - t^2}\,dt = \int (5\cos\theta)(5\cos\theta)\,d\theta = 25\int \cos^2\theta\,d\theta = 25\int \frac{1 + \cos 2\theta}{2}\,d\theta = 25\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C$$
$$= \frac{25}{2}\left(\theta + \sin\theta\cos\theta\right) + C = \frac{25}{2}\left[\sin^{-1}\left(\frac{t}{5}\right) + \left(\frac{t}{5}\right)\left(\frac{\sqrt{25 - t^2}}{5}\right)\right] + C = \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25 - t^2}}{2} + C$$

4. $t = \frac{1}{3}\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\cos\theta\,d\theta,\ \sqrt{1 - 9t^2} = \cos\theta$;

$$\int \sqrt{1 - 9t^2}\,dt = \frac{1}{3}\int (\cos\theta)(\cos\theta)\,d\theta = \frac{1}{3}\int \cos^2\theta\,d\theta = \frac{1}{6}\left(\theta + \sin\theta\cos\theta\right) + C = \frac{1}{6}\left[\sin^{-1}(3t) + 3t\sqrt{1 - 9t^2}\right] + C$$

5. $x = \frac{7}{2}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{7}{2}\sec\theta\tan\theta\,d\theta,\ \sqrt{4x^2 - 49} = \sqrt{49\sec^2\theta - 49} = 7\tan\theta$;

$$\int \frac{dx}{\sqrt{4x^2 - 49}} = \int \frac{\left(\frac{7}{2}\sec\theta\tan\theta\right)d\theta}{7\tan\theta} = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2 - 49}}{7}\right| + C$$

6. $x = \frac{3}{5}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{3}{5}\sec\theta\tan\theta\,d\theta,\ \sqrt{25x^2 - 9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta$;

$$\int \frac{5\,dx}{\sqrt{25x^2 - 9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta\right)d\theta}{3\tan\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2 - 9}}{3}\right| + C$$

7. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ \sqrt{x^2-1} = \tan\theta;$

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\tan\theta} = \int \frac{d\theta}{\sec\theta} = \sin\theta + C = \tan\theta\cos\theta + C = \frac{\sqrt{x^2-1}}{x} + C$$

8. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ \sqrt{x^2-1} = \tan\theta;$

$$\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = \int \frac{2\tan\theta\sec\theta\, d\theta}{\sec^3\theta\tan\theta} = 2\int \cos^2\theta\, d\theta = 2\int\left(\frac{1+\cos 2\theta}{2}\right) d\theta = \theta + \sin\theta\cos\theta + C$$

$$= \theta + \tan\theta\cos^2\theta + C = \sec^{-1}x + \sqrt{x^2-1}\left(\frac{1}{x}\right)^2 + C = \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C$$

9. $x = 2\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{2\,d\theta}{\cos^2\theta},\ \sqrt{x^2+4} = \frac{2}{\cos\theta};$

$$\int \frac{x^3\,dx}{\sqrt{x^2+4}} = \int \frac{(8\tan^3\theta)(\cos\theta)\,d\theta}{\cos^2\theta} = 8\int \frac{\sin^3\theta\,d\theta}{\cos^4\theta} = 8\int \frac{(\cos^2\theta-1)(-\sin\theta)\,d\theta}{\cos^4\theta};$$

$$[t = \cos\theta] \Rightarrow 8\int \frac{t^2-1}{t^4}\,dt = 8\int\left(\frac{1}{t^2} - \frac{1}{t^4}\right)dt = 8\left(-\frac{1}{t} + \frac{1}{3t^3}\right) + C = 8\left(-\sec\theta + \frac{\sec^3\theta}{3}\right) + C$$

$$= 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{(x^2+4)^{3/2}}{8\cdot 3}\right) + C = \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$$

10. $x = \tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \sec^2\theta\, d\theta,\ \sqrt{x^2+1} = \sec\theta;$

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sec\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$$

11. $w = 2\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 2\cos\theta\, d\theta,\ \sqrt{4-w^2} = 2\cos\theta;$

$$\int \frac{8\,dw}{w^2\sqrt{4-w^2}} = \int \frac{8\cdot 2\cos\theta\,d\theta}{4\sin^2\theta\cdot 2\cos\theta} = 2\int \frac{d\theta}{\sin^2\theta} = -2\cot\theta + C = \frac{-2\sqrt{4-w^2}}{w} + C$$

12. $w = 3\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 3\cos\theta\, d\theta,\ \sqrt{9-w^2} = 3\cos\theta;$

$$\int \frac{\sqrt{9-w^2}}{w^2}\,dw = \int \frac{3\cos\theta\cdot 3\cos\theta\,d\theta}{9\sin^2\theta} = \int \cot^2\theta\,d\theta = \int\left(\frac{1-\sin^2\theta}{\sin^2\theta}\right)d\theta = \int(\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$$

13. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ (x^2-1)^{3/2} = \tan^3\theta;$

$$\int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\tan^3\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = -\left(\frac{1}{\tan\theta}\right)\left(\frac{1}{\cos\theta}\right) + C$$

$$= -\left(\frac{1}{\sqrt{x^2-1}}\right)(x) + C = -\frac{x}{\sqrt{x^2-1}} + C$$

14. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ \left(x^2 - 1\right)^{5/2} = \tan^5\theta;$

$$\int \frac{x^2\, dx}{\left(x^2-1\right)^{5/2}} = \int \frac{\sec^2\theta \cdot \sec\theta\tan\theta\, d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\, d\theta = -\frac{1}{3\sin^3\theta} + C = -\frac{1}{3}\left(\frac{1}{\tan^3\theta}\right)\left(\frac{1}{\cos^3\theta}\right) + C$$

$$= -\frac{1}{3}\left[\frac{1}{\left(x^2-1\right)^{3/2}}\right]\left(x^3\right) + C = -\frac{x^3}{3\left(x^2-1\right)^{3/2}} + C$$

15. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\, d\theta,\ \left(1 - x^2\right)^{3/2} = \cos^3\theta;$

$$\int \frac{\left(1-x^2\right)^{3/2}\, dx}{x^6} = \int \frac{\cos^3\theta \cdot \cos\theta\, d\theta}{\sin^6\theta} = \int \cot^4\theta \csc^2\theta\, d\theta = -\frac{\cot^5\theta}{5} + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$$

16. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\, d\theta,\ \left(1 - x^2\right)^{1/2} = \cos\theta;$

$$\int \frac{\left(1-x^2\right)^{1/2}\, dx}{x^4} = \int \frac{\cos\theta \cdot \cos\theta\, d\theta}{\sin^4\theta} = \int \cot^2\theta \csc^2\theta\, d\theta = -\frac{\cot^3\theta}{3} + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C$$

17. $x = \frac{1}{2}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{1}{2}\sec^2\theta\, d\theta,\ \left(4x^2 + 1\right)^2 = \sec^4\theta;$

$$\int \frac{8\, dx}{\left(4x^2+1\right)^2} = \int \frac{8\left(\frac{1}{2}\sec^2\theta\right) d\theta}{\sec^4\theta} = 4\int \cos^2\theta\, d\theta = 2(\theta + \sin\theta\cos\theta) + C = 2\left(\theta + \tan\theta + \cos^2\theta\right) + C$$

$$= 2\tan^{-1} 2x + \frac{4x}{\left(4x^2+1\right)} + C$$

18. $t = \frac{1}{3}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\sec^2\theta\, d\theta,\ 9t^2 + 1 = \sec^2\theta;$

$$\int \frac{6\, dt}{\left(9t^2+1\right)^2} = \int \frac{6\left(\frac{1}{3}\sec^2\theta\right) d\theta}{\sec^4\theta} = 2\int \cos^2\theta\, d\theta = \theta + \sin\theta\cos\theta + C = \theta + \tan\theta\cos^2\theta + C$$

$$= \tan^{-1} 3t + \frac{3t}{\left(9t^2+1\right)} + C$$

19. Let $e^t = 3\tan\theta,\ t = \ln(3\tan\theta),\ dt = \frac{\sec^2\theta}{\tan\theta}\, d\theta,\ \sqrt{e^{2t}+9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta;$

$$\int_0^{\ln 4} \frac{e^t\, dt}{\sqrt{e^{2t}+9}} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\tan\theta \cdot \sec^2\theta\, d\theta}{\tan\theta \cdot 3\sec\theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec\theta\, d\theta = \left[\ln|\sec\theta + \tan\theta|\right]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}$$

$$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3} + \frac{1}{3}\right) = \ln 9 - \ln\left(1 + \sqrt{10}\right)$$

20. Let $e^t = \tan\theta$, $t = \ln(\tan\theta)$, $\frac{3}{4} \le \theta \le \frac{4}{3}$, $dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta$, $1 + e^{2t} = 1 + \tan^2\theta = \sec^2\theta$;

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{\left(1+e^{2t}\right)^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan\theta)\left(\frac{\sec^2\theta}{\tan\theta}\right)d\theta}{\sec^3\theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta\,d\theta = [\sin\theta]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$$

21. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t} + 4t\sqrt{t}}$; $\left[u = 2\sqrt{t},\ du = \frac{1}{\sqrt{t}}\,dt\right] \Rightarrow \int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2}$; $u = \tan\theta$, $\frac{\pi}{6} < \theta < \frac{\pi}{4}$, $du = \sec^2\theta\,d\theta$, $1 + u^2 = \sec^2\theta$;

$$\int_{1/\sqrt{3}}^{1} \frac{4\,du}{u\left(1+u^2\right)} = \int_{\pi/6}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\sec^2\theta} = [2\theta]_{\pi/6}^{\pi/4} = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{6}$$

22. $y = e^{\tan\theta}$, $dy = e^{\tan\theta}\sec^2\theta\,d\theta$, $\sqrt{1 + (\ln y)^2} = \sqrt{1 + \tan^2\theta} = \sec\theta$;

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\,d\theta = \int_0^{\pi/4} \sec\theta\,d\theta = \left[\ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} = \ln\left(1+\sqrt{2}\right)$$

23. $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$;

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\tan\theta} = \theta + C = \sec^{-1}|x| + C$$

24. $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $1 + x^2 = \sec^2\theta$;

$$\int \frac{dx}{x^2+1} = \int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \theta + C = \tan^{-1}x + C$$

25. $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$;

$$\int \frac{x\,dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\cdot\sec\theta\tan\theta\,d\theta}{\tan\theta} = \int \sec^2\theta\,d\theta = \tan\theta + C = \sqrt{x^2-1} + C$$

26. $x = \sin\theta$, $dx = \cos\theta\,d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$;

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{\cos\theta} = \theta + C = \sin^{-1}x + C$$

27. $x\frac{dy}{dx} = \sqrt{x^2-4}$; $dy = \sqrt{x^2-4}\,\frac{dx}{x}$; $y = \int \frac{\sqrt{x^2-4}}{x}\,dx$; $\left[\begin{array}{c} x = 2\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 2\sec\theta\tan\theta\,d\theta \\ \sqrt{x^2-4} = 2\tan\theta \end{array}\right]$

$$\Rightarrow y = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)\,d\theta}{2\sec\theta} = 2\int \tan^2\theta\,d\theta = 2\int\left(\sec^2\theta - 1\right)d\theta = 2(\tan\theta - \theta) + C$$

$= 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right] + C$; $x = 2$ and $y = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow y = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\frac{x}{2}\right]$

28. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1$, $dy = \frac{dx}{\sqrt{x^2-9}}$; $y = \int \frac{dx}{\sqrt{x^2-9}}$; $\left[\begin{array}{c} x = 3\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 3\sec\theta\tan\theta\,d\theta \\ \sqrt{x^2-9} = 3\tan\theta \end{array}\right] \Rightarrow y = \int \frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta}$

$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$; $x = 5$ and $y = \ln 3 \Rightarrow \ln 3 = \ln 3 + C \Rightarrow C = 0$

$\Rightarrow y = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|$

29. $(x^2+4)\frac{dy}{dx} = 3$, $dy = \frac{3\,dx}{x^2+4}$; $y = 3\int \frac{dx}{x^2+4} = \frac{3}{2}\tan^{-1}\frac{x}{2} + C$; $x = 2$ and $y = 0 \Rightarrow 0 = \frac{3}{2}\tan^{-1}1 + C$

$\Rightarrow C = -\frac{3\pi}{8} \Rightarrow y = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{3\pi}{8}$

30. $(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}$, $dy = \frac{dx}{(x^2+1)^{3/2}}$; $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $(x^2+1)^{3/2} = \sec^3\theta$;

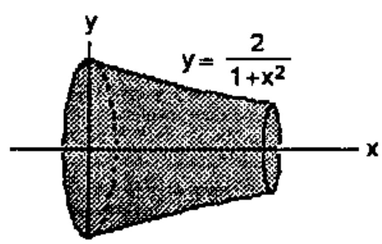
$y = \int \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int \cos\theta\,d\theta = \sin\theta + C = \tan\theta\cos\theta + C = \frac{\tan\theta}{\sec\theta} + C = \frac{x}{\sqrt{x^2+1}} + C$; $x = 0$ and $y = 1$

$\Rightarrow 1 = 0 + C \Rightarrow y = \frac{x}{\sqrt{x^2+1}} + 1$

31. $A = \int_0^3 \frac{\sqrt{9-x^2}}{3}\,dx$; $x = 3\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$, $dx = 3\cos\theta\,d\theta$, $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\cos\theta$;

$A = \int_0^{\pi/2} \frac{3\cos\theta\cdot 3\cos\theta\,d\theta}{3} = 3\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{3}{2}\left[\theta + \sin\theta\cos\theta\right]_0^{\pi/2} = \frac{3\pi}{4}$

32. $V = \int_0^1 \pi\left(\frac{2}{1+x^2}\right)^2 dx = 4\pi\int_0^1 \frac{dx}{(x^2+1)^2}$;



$x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $x^2+1 = \sec^2\theta$;

$V = 4\pi\int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{\sec^4\theta} = 4\pi\int_0^{\pi/4}\cos^2\theta\,d\theta = 2\pi\int_0^{\pi/4}(1+\cos 2\theta)\,d\theta = 2\pi\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \pi\left(\frac{\pi}{2}+1\right)$

33. (a) From the figure, $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$

(b) From part (a), $z = \frac{\sin x}{1+\cos x} \Rightarrow z(1+\cos x) = \sin x \Rightarrow z^2(1+\cos x)^2 = \sin^2 x$

$\Rightarrow z^2(1+\cos x)^2 - (1-\cos x)(1+\cos x) = 0 \Rightarrow (1+\cos x)(z^2 + z^2 \cos x - 1 + \cos x) = 0$

$1 + \cos x = 0$ or $(z^2+1)\cos x = 1 - z^2$

$\cos x = -1$ $\cos x = \frac{1-z^2}{1+z^2}$

$\cos x = -1$ does not make sense in this case.

(c) From part (b), $\cos x = \frac{1-z^2}{1+z^2} \Rightarrow \sin^2 x = 1 - \frac{(1-z^2)^2}{(1+z^2)^2} = \frac{(1+z^2)^2 - (1-z^2)^2}{(1+z^2)^2}$

$= \frac{1+2z^2+z^4-1+2z^2-z^4}{(1+z^2)^2} = \frac{4z^2}{(1+z^2)^2} \Rightarrow \sin x = \pm\frac{2z}{1+z^2}$

Only $\sin x = \frac{2z}{1+z^2}$ makes sense in this case.

(d) $z = \tan\frac{x}{2}$, $dz = \left(\frac{1}{2}\sec^2\frac{x}{2}\right)dx \Rightarrow dz = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow dz = \frac{1}{2}(1+z^2)\,dx \Rightarrow dx = \frac{2\,dz}{1+z^2}$

34. $$\int \frac{dx}{1+\sin x} = \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}} = \int \frac{2\,dz}{z^2+2z+1} = \int \frac{2\,dz}{(z+1)^2} = -\frac{2}{z+1} + C = -\frac{2}{\tan\frac{x}{2}+1} + C$$

35. $$\int \frac{dx}{1-\cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{1-\frac{1-z^2}{1+z^2}} = \int \frac{dz}{z^2} = -\frac{1}{z} + C = -\frac{1}{\tan\frac{x}{2}} + C$$

36. $$\int \frac{d\theta}{1-\sin\theta} = \int \frac{\frac{2\,dz}{1+z^2}}{1-\frac{2z}{1+z^2}} = \int \frac{2\,dz}{z^2-2z+1} = \int \frac{2\,dz}{(z-1)^2} = -\frac{2}{z-1} + C = -\frac{2}{\tan\frac{\theta}{2}-1} + C$$

$$= \frac{2}{1-\tan\frac{\theta}{2}} + C$$

37. $$\int \frac{dt}{1+\sin t+\cos t} = \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}} = \int \frac{dz}{z+1} = \ln|z+1| + C = \ln\left|\tan\frac{t}{2}+1\right| + C$$

38. $$\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{2+\left(\frac{1-z^2}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{2+2z^2+1-z^2} = \int_0^1 \frac{2\,dz}{z^2+3} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{z}{\sqrt{3}}\right]_0^1 = \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$$

$$= \frac{\pi}{3\sqrt{3}} = \frac{\sqrt{3}\pi}{9}$$

39. $$\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta+\sin\theta} = \int_1^{\sqrt{3}} \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{\left[\frac{2z(1-z^2)}{(1+z^2)^2}+\left(\frac{2z}{1+z^2}\right)\right]} = \int_1^{\sqrt{3}} \frac{2(1-z^2)\,dz}{2z-2z^3+2z+2z^3} = \int_1^{\sqrt{3}} \frac{1-z^2}{2z}\,dz$$

$$= \left[\frac{1}{2}\ln z - \frac{z^2}{4}\right]_1^{\sqrt{3}} = \left(\frac{1}{2}\ \ln\ \sqrt{3} - \frac{3}{4}\right) - \left(0 - \frac{1}{4}\right) = \frac{\ln 3}{4} - \frac{1}{2} = \frac{1}{4}(\ln 3 - 2) = \frac{1}{2}\left(\ln\ \sqrt{3} - 1\right)$$

40. $\int \frac{dt}{\sin t - \cos t} = \int \frac{\left(\frac{2\ dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2} - \frac{1-z^2}{1+z^2}\right)} = \int \frac{2\ dz}{2z - 1 + z^2} = \int \frac{2\ dz}{(z+1)^2 - 2} = \frac{1}{\sqrt{2}} \ln \left|\frac{z+1-\sqrt{2}}{z+1+\sqrt{2}}\right| + C$

$$= \frac{1}{\sqrt{2}} \ln \left|\frac{\tan\left(\frac{t}{2}\right) + 1 - \sqrt{2}}{\tan\left(\frac{t}{2}\right) + 1 + \sqrt{2}}\right| + C$$

41. $\int \frac{\cos t\ dt}{1 - \cos t} = \int \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\ dz}{1+z^2}\right)}{1 - \left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\left(1-z^2\right) dz}{\left(1+z^2\right)^2 - \left(1+z^2\right)\left(1-z^2\right)} = \int \frac{2\left(1-z^2\right) dz}{\left(1+z^2\right)\left(1+z^2-1+z^2\right)}$

$$= \int \frac{\left(1-z^2\right) dz}{\left(1+z^2\right)z^2} = \int \frac{dz}{z^2\left(1+z^2\right)} - \int \frac{dz}{1+z^2} = \int \frac{dz}{z^2} - 2\int \frac{dz}{z^2+1} = -\frac{1}{z} - 2\tan^{-1} z + C = -\cot\left(\frac{t}{2}\right) - t + C$$

7.5 INTEGRAL TABLES, COMPUTER ALGEBRA SYSTEMS, AND MONTE CARLO INTEGRATION

1. $\int \frac{dx}{x\sqrt{x-3}} = \frac{2}{\sqrt{3}} \tan^{-1} \sqrt{\frac{x-3}{3}} + C$

   (We used FORMULA 13(a) with $a = 1$, $b = -3$)

2. $\int \frac{x\ dx}{\sqrt{x-2}} = \int \frac{(x-2)\ dx}{\sqrt{x-2}} + 2\int \frac{dx}{\sqrt{x-2}} = \int \left(\sqrt{x-2}\right)^1 dx + 2\int \left(\sqrt{x-2}\right)^{-1} dx$

   $$= \left(\frac{2}{1}\right)\frac{\left(\sqrt{x-2}\right)^3}{3} + 2\left(\frac{2}{1}\right)\frac{\left(\sqrt{x-2}\right)^1}{1} = \sqrt{x-2}\left[\frac{2(x-2)}{3} + 4\right] + C$$

   (We used FORMULA 11 with $a = 1$, $b = -2$, $n = 1$ and $a = 1$, $b = -2$, $n = -1$)

3. $\int x\sqrt{2x-3}\ dx = \frac{1}{2}\int (2x-3)\sqrt{2x-3}\ dx + \frac{3}{2}\int \sqrt{2x-3}\ dx = \frac{1}{2}\int \left(\sqrt{2x-3}\right)^3 dx + \frac{3}{2}\int \left(\sqrt{2x-3}\right)^1 dx$

   $$= \left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\frac{\left(\sqrt{2x-3}\right)^5}{5} + \left(\frac{3}{2}\right)\left(\frac{2}{2}\right)\frac{\left(\sqrt{2x-3}\right)^3}{3} + C = \frac{(2x-3)^{3/2}}{2}\left[\frac{2x-3}{5} + 1\right] + C = \frac{(2x-3)^{3/2}(x+1)}{5} + C$$

   (We used FORMULA 11 with $a = 2$, $b = -3$, $n = 3$ and $a = 2$, $b = -3$, $n = 1$)

4. $\int \frac{\sqrt{9-4x}}{x^2}\ dx = -\frac{\sqrt{9-4x}}{x} + \frac{(-4)}{2}\int \frac{dx}{x\sqrt{9-4x}} + C$

   (We used FORMULA 14 with $a = -4$, $b = 9$)

$= -\frac{\sqrt{9-4x}}{x} - 2\left(\frac{1}{\sqrt{9}}\right) \ln \left|\frac{\sqrt{9-4x}-\sqrt{9}}{\sqrt{9-4x}+\sqrt{9}}\right| + C$

(We used FORMULA 13(b) with $a = -4$, $b = 9$)

$= \frac{-\sqrt{9-4x}}{x} - \frac{2}{3} \ln \left|\frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3}\right| + C$

5. $\int x\sqrt{4x-x^2}\,dx = \int x\sqrt{2\cdot 2x - x^2}\,dx = \frac{(x+2)(2x-3\cdot 2)\sqrt{2\cdot 2\cdot x - x^2}}{6} + \frac{2^3}{2}\sin^{-1}\left(\frac{x-2}{2}\right) + C$

$= \frac{(x+2)(2x-6)\sqrt{4x-x^2}}{6} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$

(We used FORMULA 51 with $a = 2$)

6. $\int \frac{dx}{x\sqrt{7+x^2}} = \int \frac{dx}{x\sqrt{\left(\sqrt{7}\right)^2 + x^2}} = -\frac{1}{\sqrt{7}} \ln \left|\frac{\sqrt{7}+\sqrt{\left(\sqrt{7}\right)^2+x^2}}{x}\right| + C = -\frac{1}{\sqrt{7}} \ln \left|\frac{\sqrt{7}+\sqrt{7+x^2}}{x}\right| + C$

(We used FORMULA 26 with $a = \sqrt{7}$)

7. $\int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{\sqrt{2^2-x^2}}{x}\,dx = \sqrt{2^2-x^2} - 2\ln\left|\frac{2+\sqrt{2^2-x^2}}{x}\right| + C = \sqrt{4-x^2} - 2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + C$

(We used FORMULA 31 with $a = 2$)

8. $\int \sqrt{25-p^2}\,dp = \int \sqrt{5^2-p^2}\,dp = \frac{p}{2}\sqrt{5^2-p^2} + \frac{5^2}{2}\sin^{-1}\frac{p}{5} + C = \frac{p}{2}\sqrt{25-p^2} + \frac{25}{2}\sin^{-1}\frac{p}{5} + C$

(We used FORMULA 29 with $a = 5$)

9. $\int \frac{r^2}{\sqrt{4-r^2}}\,dr = \int \frac{r^2}{\sqrt{2^2-r^2}}\,dr = \frac{2^2}{2}\sin^{-1}\left(\frac{r}{2}\right) - \frac{1}{2}r\sqrt{2^2-r^2} + C = 2\sin^{-1}\left(\frac{r}{2}\right) - \frac{1}{2}r\sqrt{4-r^2} + C$

(We used FORMULA 33 with $a = 2$)

10. $\int \frac{d\theta}{5+4\sin 2\theta} = \frac{-2}{2\sqrt{25-16}}\tan^{-1}\left[\sqrt{\frac{5-4}{5+4}}\tan\left(\frac{\pi}{4}-\frac{2\theta}{2}\right)\right] + C = -\frac{1}{3}\tan^{-1}\left[\frac{1}{3}\tan\left(\frac{\pi}{4}-\theta\right)\right] + C$

(We used FORMULA 70 with $b = 5$, $c = 4$, $a = 2$)

11. $\int e^{2t}\cos 3t\,dt = \frac{e^{2t}}{2^2+3^2}(2\cos 3t + 3\sin 3t) + C = \frac{e^{2t}}{13}(2\cos 3t + 3\sin 3t) + C$

(We used FORMULA 108 with $a = 2$, $b = 3$)

12. $\int x\cos^{-1}x\,dx = \int x^1\cos^{-1}x\,dx = \frac{x^{1+1}}{1+1}\cos^{-1}x + \frac{1}{1+1}\int \frac{x^{1+1}\,dx}{\sqrt{1-x^2}} = \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

(We used FORMULA 100 with $a = 1$, $n = 1$)

$= \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\left(\frac{1}{2}\sin^{-1}x\right) - \frac{1}{2}\left(\frac{1}{2}x\sqrt{1-x^2}\right) + C = \frac{x^2}{2}\cos^{-1}x + \frac{1}{4}\sin^{-1}x - \frac{1}{4}x\sqrt{1-x^2} + C$

(We used FORMULA 33 with $a = 1$)

13. $\int \frac{ds}{(9-s^2)^2} = \int \frac{ds}{(3^3-s^2)^2} = \frac{s}{2\cdot 3^2\cdot(3^2-s^2)} + \frac{1}{2\cdot 3^2}\int \frac{ds}{3^2-s^2}$

(We used FORMULA 19 with $a = 3$)

$= \frac{s}{18(9-s^2)} + \frac{1}{18}\left(\frac{1}{2\cdot 3}\ln\left|\frac{s+3}{s-3}\right|\right) + C = \frac{s}{18(9-s^2)} + \frac{1}{108}\ln\left|\frac{s+3}{s-3}\right| + C$

(We used FORMULA 18 with $a = 3$)

14. $\int \frac{\sqrt{4x+9}}{x^2}\,dx = -\frac{\sqrt{4x+9}}{x} + \frac{4}{2}\int \frac{dx}{x\sqrt{4x+9}}$

(We used FORMULA 14 with $a = 4$, $b = 9$)

$= -\frac{\sqrt{4x+9}}{x} + 2\left(\frac{1}{\sqrt{9}}\ln\left|\frac{\sqrt{4x+9}-\sqrt{9}}{\sqrt{4x+9}+\sqrt{9}}\right|\right) + C = -\frac{\sqrt{4x+9}}{x} + \frac{2}{3}\ln\left|\frac{\sqrt{4x+9}-3}{\sqrt{4x+9}+3}\right| + C$

(We used FORMULA 13(b) with $a = 4$, $b = 9$)

15. $\int \frac{\sqrt{3t-4}}{t}\,dt = 2\sqrt{3t-4} + (-4)\int \frac{dt}{t\sqrt{3t-4}}$

(We used FORMULA 12 with $a = 3$, $b = -4$)

$= 2\sqrt{3t-4} - 4\left(\frac{2}{\sqrt{4}}\tan^{-1}\sqrt{\frac{3t-4}{4}}\right) + C = 2\sqrt{3t-4} - 4\tan^{-1}\sqrt{\frac{3t-4}{4}} + C$

(We used FORMULA 13(a) with $a = 3$, $b = -4$)

16. $\int x^2\tan^{-1}x\,dx = \frac{x^{2+1}}{2+1}\tan^{-1}x - \frac{1}{2+1}\int \frac{x^{2+1}}{1+x^2}\,dx = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$

(We used FORMULA 101 with $a = 1$, $n = 2$);

$\int \frac{x^3}{1+x^2}\,dx = \int x\,dx - \int \frac{x\,dx}{1+x^2} = \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + C \Rightarrow \int x^2\tan^{-1}x\,dx$

$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C$

17. $\int \sin 3x\cos 2x\,dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$

(We used FORMULA 62(a) with $a = 3$, $b = 2$)

18. $\int 8\sin 4t\sin\frac{t}{2}\,dx = \frac{8}{7}\sin\left(\frac{7t}{2}\right) - \frac{8}{9}\sin\left(\frac{9t}{2}\right) + C = 8\left[\frac{\sin\left(\frac{7t}{2}\right)}{7} - \frac{\sin\left(\frac{9t}{2}\right)}{9}\right] + C$

(We used FORMULA 62(b) with $a = 4$, $b = \frac{1}{2}$)

19. $\int \cos\frac{\theta}{3}\cos\frac{\theta}{4}\,d\theta = 6\sin\left(\frac{\theta}{12}\right) + \frac{6}{7}\sin\left(\frac{7\theta}{12}\right) + C$

(We used FORMULA 62(c) with $a = \frac{1}{3}$, $b = \frac{1}{4}$)

20. $\int \cos\frac{\theta}{2}\cos 7\theta\,d\theta = \frac{1}{13}\sin\left(\frac{13\theta}{2}\right) + \frac{1}{15}\sin\left(\frac{15\theta}{2}\right) + C = \frac{\sin\left(\frac{13\theta}{2}\right)}{13} + \frac{\sin\left(\frac{15\theta}{2}\right)}{15} + C$

(We used FORMULA 62(c) with $a = \frac{1}{2}$, $b = 7$)

21. $\int \frac{x^3+x+1}{\left(x^2+1\right)^2}\,dx = \int \frac{x\,dx}{x^2+1} + \int \frac{dx}{\left(x^2+1\right)^2} = \frac{1}{2}\int \frac{d\left(x^2+1\right)}{x^2+1} + \int \frac{dx}{\left(x^2+1\right)^2}$

$= \frac{1}{2}\ln\left|x^2+1\right| + \frac{x}{2\left(1+x^2\right)} + \frac{1}{2}\tan^{-1}x + C$

(For the second integral we used FORMULA 17 with $a = 1$)

22. $\int \frac{x^2+6x}{\left(x^2+3\right)^2}\,dx = \int \frac{dx}{x^2+3} + \int \frac{6x\,dx}{\left(x^2+3\right)^2} - \int \frac{3\,dx}{\left(x^2+3\right)^2} = \int \frac{dx}{x^2+\left(\sqrt{3}\right)^2} + 3\int \frac{d\left(x^2+3\right)}{\left(x^2+3\right)^2} - 3\int \frac{dx}{\left[x^2+\left(\sqrt{3}\right)^2\right]^2}$

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{\left(x^2+3\right)} - 3\left(\frac{x}{2\left(\sqrt{3}\right)^2\left(\left(\sqrt{3}\right)^2+x^2\right)} + \frac{1}{2\left(\sqrt{3}\right)^3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right) + C$

(For the first integral we used FORMULA 16 with $a = \sqrt{3}$; for the third integral we used FORMULA 17 with $a = \sqrt{3}$)

$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{x^2+3} - \frac{x}{2\left(x^2+3\right)} + C$

23. $\int \sin^{-1}\sqrt{x}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \Rightarrow 2\int u^1\sin^{-1}u\,du = 2\left(\frac{u^{1+1}}{1+1}\sin^{-1}u - \frac{1}{1+1}\int \frac{u^{1+1}}{\sqrt{1-u^2}}\,du\right)$

$= u^2\sin^{-1}u - \int \frac{u^2\,du}{\sqrt{1-u^2}}$

(We used FORMULA 99 with $a = 1$, $n = 1$)

$= u^2\sin^{-1}u - \left(\frac{1}{2}\sin^{-1}u - \frac{1}{2}u\sqrt{1-u^2}\right) + C = \left(u^2 - \frac{1}{2}\right)\sin^{-1}u + \frac{1}{2}u\sqrt{1-u^2} + C$

(We used FORMULA 33 with $a = 1$)

$= \left(x - \frac{1}{2}\right)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2} + C$

24. $\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \Rightarrow \int \frac{\cos^{-1}u}{u}\cdot 2u\,du = 2\int \cos^{-1}u\,du = 2\left(u\cos^{-1}u - \frac{1}{1}\sqrt{1-u^2}\right) + C$

(We used FORMULA 97 with $a = 1$)

$= 2\left(\sqrt{x}\cos^{-1}\sqrt{x} - \sqrt{1-x}\right) + C$

25. $\int (\cot t)\sqrt{1-\sin^2 t}\, dt = \int \frac{\sqrt{1-\sin^2 t}\,(\cos t)\, dt}{\sin t}; \begin{bmatrix} u = \sin t \\ du = \cos t\, dt \end{bmatrix} \Rightarrow \int \frac{\sqrt{1-u^2}\, du}{u}$

$= \sqrt{1-u^2} - \ln\left|\frac{1+\sqrt{1-u^2}}{u}\right| + C$

(We used FORMULA 31 with $a = 1$)

$= \sqrt{1-\sin^2 t} - \ln\left|\frac{1+\sqrt{1-\sin^2 t}}{\sin t}\right| + C$

26. $\int \frac{dt}{(\tan t)\sqrt{4-\sin^2 t}} = \int \frac{\cos t\, dt}{(\sin t)\sqrt{4-\sin^2 t}}; \begin{bmatrix} u = \sin t \\ du = \cos t\, dt \end{bmatrix} \Rightarrow \int \frac{du}{u\sqrt{4-u^2}} = -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-u^2}}{u}\right| + C$

(We used FORMULA 34 with $a = 2$)

$= -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-\sin^2 t}}{\sin t}\right| + C$

27. $\int \frac{dy}{y\sqrt{3+(\ln y)^2}}; \begin{bmatrix} u = \ln y \\ y = e^u \\ dy = e^u\, du \end{bmatrix} \Rightarrow \int \frac{e^u\, du}{e^u\sqrt{3+u^2}} = \int \frac{du}{\sqrt{3+u^2}} = \ln\left|u + \sqrt{3+u^2}\right| + C$

$= \ln\left|\ln y + \sqrt{3+(\ln y)^2}\right| + C$

(We used FORMULA 20 with $a = \sqrt{3}$)

28. $\int \frac{\cos\theta\, d\theta}{\sqrt{5+\sin^2\theta}}; \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \Rightarrow \int \frac{du}{\sqrt{5+u^2}} = \ln\left|u+\sqrt{5+u^2}\right| + C = \ln\left|\sin\theta + \sqrt{5+\sin^2\theta}\right| + C$

(We used FORMULA 20 with $a = \sqrt{5}$)

29. $\int \frac{3\, dr}{\sqrt{9r^2-1}}; \begin{bmatrix} u = 3r \\ du = 3\, dr \end{bmatrix} \Rightarrow \int \frac{du}{\sqrt{u^2-1}} = \ln\left|u+\sqrt{u^2-1}\right| + C = \ln\left|3r+\sqrt{9r^2-1}\right| + C$

(We used FORMULA 36 with $a = 1$)

30. $\int \frac{3\, dy}{\sqrt{1+9y^2}}; \begin{bmatrix} u = 3y \\ du = 3\, dy \end{bmatrix} \Rightarrow \int \frac{du}{\sqrt{1+u^2}} = \ln\left|u+\sqrt{1+u^2}\right| + C = \ln\left|3y+\sqrt{1+9y^2}\right| + C$

(We used FORMULA 20 with $a = 1$)

31. $\int \cos^{-1}\sqrt{x}\,dx;\begin{bmatrix} t=\sqrt{x} \\ x=t^2 \\ dx=2t\,dt \end{bmatrix} \Rightarrow 2\int t\cos^{-1}t\,dt = 2\left(\frac{t^2}{2}\cos^{-1}t+\frac{1}{2}\int \frac{t^2}{\sqrt{1-t^2}}\,dt\right) = t^2\cos^{-1}t+\int \frac{t^2}{\sqrt{1-t^2}}\,dt$

(We used FORMULA 100 with $a=1$, $n=1$)

$= t^2\cos^{-1}t+\frac{1}{2}\sin^{-1}t-\frac{1}{2}t\sqrt{1-t^2}+C$

(We used FORMULA 33 with $a=1$)

$= x\cos^{-1}\sqrt{x}+\frac{1}{2}\sin^{-1}\sqrt{x}-\frac{1}{2}\sqrt{x}\sqrt{1-x}+C = x\cos^{-1}\sqrt{x}+\frac{1}{2}\sin^{-1}\sqrt{x}-\frac{1}{2}\sqrt{x-x^2}+C$

32. $\int \tan^{-1}\sqrt{y}\,dy;\begin{bmatrix} t=\sqrt{y} \\ y=t^2 \\ dy=2t\,dt \end{bmatrix} \Rightarrow 2\int t\tan^{-1}t\,dt = 2\left[\frac{t^2}{2}\tan^{-1}t-\frac{1}{2}\int \frac{t^2}{1+t^2}\,dt\right] = t^2\tan^{-1}t-\int \frac{t^2}{1+t^2}\,dt$

(We used FORMULA 101 with $n=1$, $a=1$)

$= t^2\tan^{-1}t-\int \frac{t^2+1}{t^2+1}\,dt+\int \frac{dt}{1+t^2} = t^2\tan^{-1}t-t+\tan^{-1}t+C = y\tan^{-1}\sqrt{y}+\tan^{-1}\sqrt{y}-\sqrt{y}+C$

33. $\int xe^{3x}\,dx = \frac{e^{3x}}{3^2}(3x-1)+C = \frac{e^{3x}}{9}(3x-1)+C$

(We used FORMULA 104 with $a=3$)

34. $\int x^3e^{x/2}\,dx = 2x^3e^{x/2}-3\cdot 2\int x^2e^{x/2}\,dx = 2x^3e^{x/2}-6\left(2x^2e^{x/2}-2\cdot 2\int xe^{x/2}\,dx\right)$

$= 2x^3e^{x/2}-12x^2e^{x/2}+24\cdot 4e^{x/2}\left(\frac{x}{2}-1\right)+C = 2x^3e^{x/2}-12x^2e^{x/2}+96e^{x/2}\left(\frac{x}{2}-1\right)+C$

(We used FORMULA 105 with $a=\frac{1}{2}$ twice and FORMULA 104 with $a=\frac{1}{2}$)

35. $\int x^2 2^x\,dx = \frac{x^2 2^x}{\ln 2}-\frac{2}{\ln 2}\int x2^x\,dx = \frac{x^2 2^x}{\ln 2}-\frac{2}{\ln 2}\left(\frac{x2^x}{\ln 2}-\frac{1}{\ln 2}\int 2^x\,dx\right) = \frac{x^2 2^x}{\ln 2}-\frac{2}{\ln 2}\left[\frac{x2^x}{\ln 2}-\frac{2^x}{(\ln 2)^2}\right]+C$

(We used FORMULA 106 with $a=1$, $b=2$)

36. $\int x\pi^x\,dx = \frac{x\pi^x}{\ln \pi}-\frac{1}{\ln \pi}\int \pi^x\,dx = \frac{x\pi^x}{\ln \pi}-\frac{1}{\ln \pi}\left(\frac{\pi^x}{\ln \pi}\right)+C = \frac{x\pi^x}{\ln \pi}-\frac{\pi^x}{\ln \pi}-\frac{\pi^x}{(\ln \pi)^2}+C$

(We used FORMULA 106 with $n=1$, $b=\pi$, $a=1$)

37. $\int \frac{1}{8}\sinh^5 3x\,dx = \frac{1}{8}\left(\frac{\sinh^4 3x\cosh 3x}{5\cdot 3}-\frac{5-1}{5}\int \sinh^3 3x\,dx\right)$

$= \frac{\sinh^4 3x\cosh 3x}{120}-\frac{1}{10}\left(\frac{\sinh^2 3x\cosh 3x}{3\cdot 3}-\frac{3-1}{3}\int \sinh 3x\,dx\right)$

(We used FORMULA 117 with $a=3$, $n=5$ and $a=1$, $n=3$)

$= \frac{\sinh^4 3x \cosh 3x}{120} - \frac{\sinh^2 3x \cosh 3x}{90} + \frac{2}{30}\left(\frac{1}{3}\cosh 3x\right) + C$

$= \frac{1}{120}\sinh^4 3x \cosh 3x - \frac{1}{90}\sinh^2 3x \cosh 3x + \frac{2}{90}\cosh 3x + C$

38. $\int \frac{\cosh^4 \sqrt{x}}{\sqrt{x}}\,dx; \left[\begin{array}{c} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{array}\right] \Rightarrow 2\int \cosh^4 u\,du = 2\left(\frac{\cosh^3 u \sinh u}{4} + \frac{4-1}{4}\int \cosh^2 u\,du\right)$

$= \frac{\cosh^3 u \sinh u}{2} + \frac{3}{2}\left(\frac{\sinh 2u}{4} + \frac{u}{2}\right) + C$

(We used FORMULA 118 with $a = 1$, $n = 2$ and FORMULA 116 with $a = 1$)

$= \frac{1}{2}\cosh^3 \sqrt{x} \sinh \sqrt{x} + \frac{3}{8}\sinh 2\sqrt{x} + \frac{3}{4}\sqrt{x} + C$

39. $\int x^2 \cosh 3x\,dx = \frac{x^2}{3}\sinh 3x - \frac{2}{3}\int x \sinh 3x\,dx = \frac{x^2}{3}\sinh 3x - \frac{2}{3}\left(\frac{x}{3}\cosh 3x - \frac{1}{3}\int \cosh 3x\,dx\right)$

(We used FORMULA 122 with $a = 3$, $n = 2$ and FORMULA 121 with $a = 3$, $n = 1$)

$= \frac{x^2}{3}\sinh 3x - \frac{2x}{9}\cosh 3x + \frac{2}{27}\sinh 3x + C$

40. $\int x \sinh 5x\,dx = \frac{x}{5}\cosh 5x - \frac{1}{25}\sinh 5x + C$

(We used FORMULA 119 with $a = 5$, $n = 1$)

41. $u = ax + b \Rightarrow x = \frac{u-b}{a} \Rightarrow dx = \frac{du}{a}$;

$\int \frac{x\,dx}{(ax+b)^2} = \int \frac{(u-b)}{au^2}\frac{du}{a} = \frac{1}{a^2}\int\left(\frac{1}{u} - \frac{b}{u^2}\right)du = \frac{1}{a^2}\left[\ln|u| + \frac{b}{u}\right] + C = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$

42. $x = a\sin\theta \Rightarrow a^2 - x^2 = a^2\cos^2\theta \Rightarrow -2x\,dx = -2a^2\cos\theta\sin\theta\,d\theta \Rightarrow dx = a\cos\theta\,d\theta$;

$\int \sqrt{a^2 - x^2}\,dx = \int a\cos\theta(a\cos\theta)\,d\theta = a^2\int \cos^2\theta\,d\theta = \frac{a^2}{2}\int(1 + \cos 2\theta)\,d\theta = \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$

$= \frac{a^2}{2}(\theta + \cos\theta\sin\theta) + C = \frac{a^2}{2}\left(\theta + \sqrt{1 - \sin^2\theta}\cdot\sin\theta\right) + C = \frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} + \frac{\sqrt{a^2 - x^2}}{a}\cdot\frac{x}{a}\right) + C$

$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$

43. $\int x^n(\ln ax)^m\,dx = \int(\ln ax)^m\,d\left(\frac{x^{n+1}}{n+1}\right) = \frac{x^{n+1}(\ln ax)^m}{n+1} - \int\left(\frac{x^{n+1}}{n+1}\right)m(\ln ax)^{m-1}\left(\frac{1}{x}\right)dx$

$= \frac{x^{n+1}(\ln ax)^m}{n+1} - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx$

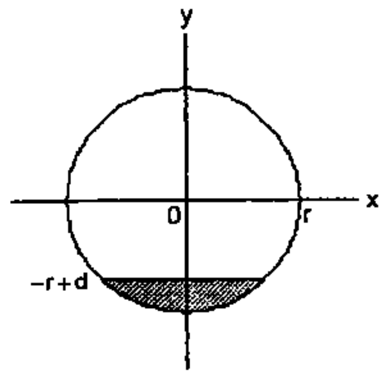
$\left(\text{We used integration by parts } \int u\,dv = uv - \int v\,du \text{ with } u = (\ln ax)^m,\ v = \frac{x^{n+1}}{n+1}\right)$

44. $\int x^n \sin^{-1} ax\, dx = \int \sin^{-1} ax\, d\left(\frac{x^{n+1}}{n+1}\right) = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \int \left(\frac{x^{n+1}}{n+1}\right) \frac{a}{\sqrt{1-(ax)^2}}\, dx$

$= \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1}\, dx}{\sqrt{1-a^2x^2}},\ n \neq -1$

$\left(\text{We used integration by parts } \int u\, dv = uv - \int v\, du \text{ with } u = \sin^{-1} ax,\ v = \frac{x^{n+1}}{n+1}\right)$

45. (a) The volume of the filled part equals the length of the tank times the area of the shaded region shown in the accompanying figure. Consider a layer of gasoline of thickness dy located at height y where $-r < y < -r + d$. The width of this layer is $2\sqrt{r^2 - y^2}$. Therefore, $A = 2 \int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy$ and $V = L \cdot A = 2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy$



(b) $2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy = 2L\left[\frac{y\sqrt{r^2-y^2}}{2} + \frac{r^2}{2} \sin^{-1} \frac{y}{r}\right]_{-r}^{-r+d}$

(We used FORMULA 29 with $a = r$)

$= 2L\left[\frac{(d-r)}{2}\sqrt{2rd - d^2} + \frac{r^2}{2} \sin^{-1}\left(\frac{d-r}{r}\right) + \frac{r^2}{2}\left(\frac{\pi}{2}\right)\right] = 2L\left[\left(\frac{d-r}{2}\right)\sqrt{2rd-d^2} + \left(\frac{r^2}{2}\right)\left(\sin^{-1}\left(\frac{d-r}{r}\right) + \frac{\pi}{2}\right)\right]$

46. The integrand $f(x) = \sqrt{x - x^2}$ is nonnegative, so the integral is maximized by integrating over the function's entire domain, which runs from $x = 0$ to $x = 1$

$\Rightarrow \int_0^1 \sqrt{x - x^2}\, dx = \int_0^1 \sqrt{2 \cdot \frac{1}{2}x - x^2}\, dx = \left[\frac{\left(x - \frac{1}{2}\right)}{2}\sqrt{2 \cdot \frac{1}{2}x - x^2} + \frac{\left(\frac{1}{2}\right)^2}{2} \sin^{-1}\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$

$\left(\text{We used FORMULA 48 with } a = \frac{1}{2}\right)$

$= \left[\frac{\left(x - \frac{1}{2}\right)}{2}\sqrt{x - x^2} + \frac{1}{8} \sin^{-1}(2x-1)\right]_0^1 = \frac{1}{8} \cdot \frac{\pi}{2} - \frac{1}{8}\left(-\frac{\pi}{2}\right) = \frac{\pi}{8}$

CAS EXPLORATIONS

For MAPLE use the int(f(x),x) command, and for MATHEMATICA use the command Integrate[f(x),x], as discussed in the text.

47. (e) $\int x^n \ln x\, dx = \frac{x^{n+1} \ln x}{n+1} - \frac{1}{n+1} \int x^n\, dx,\ n \neq -1$

(We used FORMULA 110 with $a = 1$, $m = 1$)

$$= \frac{x^{n+1} \ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C$$

48. (e) $\int x^{-n} \ln x \, dx = \frac{x^{-n+1} \ln x}{-n+1} - \frac{1}{(-n)+1} \int x^{-n} \, dx, \ n \neq 1$

(We used FORMULA 110 with $a = 1$, $m = 1$, $n = -n$)

$$= \frac{x^{1-n} \ln x}{1-n} - \frac{1}{1-n}\left(\frac{x^{1-n}}{1-n}\right) + C = \frac{x^{1-n}}{1-n}\left(\ln x - \frac{1}{1-n}\right) + C$$

49. (a) Neither MAPLE nor MATHEMATICA can find this integral for arbitrary n.

(b) MAPLE and MATHEMATICA get stuck at about $n = 5$.

(c) Let $x = \frac{\pi}{2} - u \Rightarrow dx = -du$; $x = 0 \Rightarrow u = \frac{\pi}{2}$, $x = \frac{\pi}{2} \Rightarrow u = 0$;

$$I = \int_0^{\pi/2} \frac{\sin^n x \, dx}{\sin^n x + \cos^n x} = \int_{\pi/2}^{0} \frac{-\sin^n\left(\frac{\pi}{2} - u\right) du}{\sin^n\left(\frac{\pi}{2} - u\right) + \cos^n\left(\frac{\pi}{2} - u\right)} = \int_0^{\pi/2} \frac{\cos^n u \, du}{\cos^n u + \sin^n u} = \int_0^{\pi/2} \frac{\cos^n x \, dx}{\cos^n x + \sin^n x}$$

$$\Rightarrow I + I = \int_0^{\pi/2} \left(\frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\right) dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

The following *Mathematica* module is used to obtain the Monte Carlo estimates of area in Problems 50 through 55.

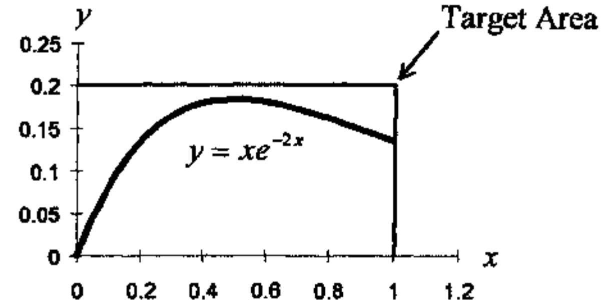
```
monte[f_, indvar_, m_, a_, b_, n_List] :=
 Module[{g, x, xr, yr, area, lim, areaavg, y1, y2},
  g = f/. indvar -> x;
  lim = Length[n];
  area - Table[0, {k, 1, lim}];
  For[k = 1, k <= lim, k++,
    For[counter = 0; i = 1, i <= n[[k]], i++,
     xr = a+ (b - a)*Random[];
     yr = m*Random[];
    If[yr <= g/. x -> xr, counter = counter + 1];];
  area[[k]] = m*(b - a)*counter/n[[k]]];
  areaavg = (Sum[n[[i]]*area[[i]], {i, 1, lim}]) /
    Sum[n[i]], {i, 1, lim}];
  y1 = Integrate[g, {x, a, b}] // N;
  y2 = Integrate[g, {x, a, b}];
  Print[area];
  Print[areaavg];
  Print[y2];
  Print y1  ;
```

The following command executes the preceding module. The arguments are the integrand function, the independent variable, an upper bound on the integrand function, the lower limit of integration, the upper limit of integration, and a list of the numbers of random points to generate in each estimation.

```
monte[z*Sqrt[1 - z], z, 0.5, 0, 1, {100, 200, 300, 400,
  500, 600, 700, 800, 900, 1000, 2000, 3000, 4000,
  5000, 6000, 8000, 10000, 15000, 20000, 30000
```

The preceding command is for Problem 51.

50.



Select M = 0.2

The area approximations will vary depending on the random number generator and seed value that is used

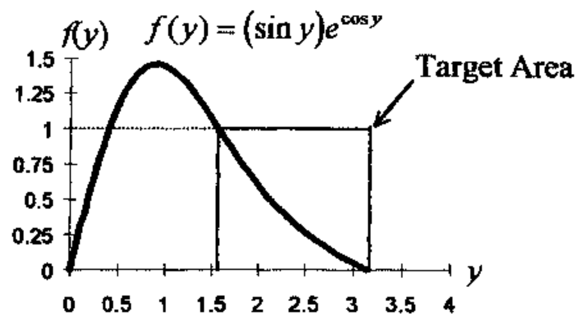
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.154	2000	0.1492
200	0.151	3000	0.147867
300	0.148	4000	0.1497
400	0.149	5000	0.14712
500	0.1528	6000	0.148433
600	0.151667	8000	0.147925
700	0.149429	10,000	0.14796
800	0.1435	15,000	0.148147
900	0.146444	20,000	0.14824
1000	0.1408	30,000	0.147687

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_0^1 xe^{-2x}\,dx \approx \left(\sum_{i=1}^{20} n_i \cdot \mathrm{area(i)}\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.147987 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{(1-3e^2)}{4} \approx 0.148499$.

51.



Select M = 1

The area approximations will vary depending on the random number generator and seed value that is used

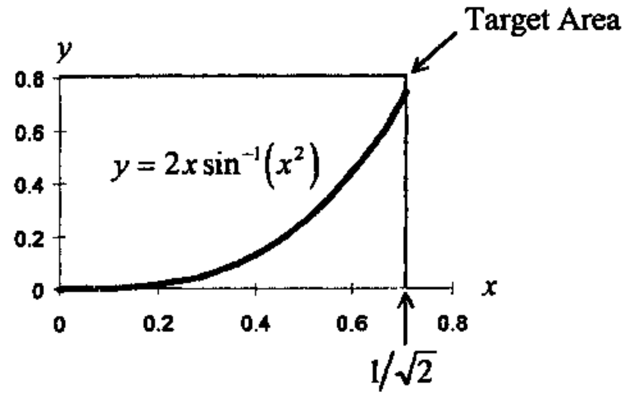
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.722566	2000	0.628319
200	0.628319	3000	0.646121
300	0.586431	4000	0.642456
400	0.581195	5000	0.636487
500	0.637743	6000	0.627533
600	0.560251	8000	0.643437
700	0.583439	10,000	0.62235
800	0.577268	15,000	0.625386
900	0.5621337	20,000	0.635073
1000	0.655022	30,000	0.638895

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_{\pi/2}^{\pi} (\sin y)e^{\cos y}\,dy \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area(i)}\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.63298 \text{ by Monte Carlo.}$$

The actual value of the integral is $1 - \frac{1}{e} \approx 0.632121$.

52.



Select M = 0.8
The area approximations will vary depending on the random number generator and seed value that is used

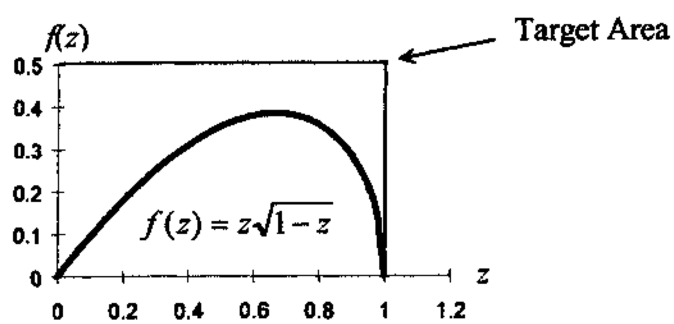
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.152735	2000	0.129542
200	0.10748	3000	0.133879
300	0.118794	4000	0.125724
400	0.130108	5000	0.123206
500	0.139159	6000	0.130956
600	0.129165	8000	0.128693
700	0.118794	10,000	0.127279
800	0.123744	15,000	0.129844
900	0.121308	20,000	0.129712
1000	0.122188	30,000	0.128335

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_{0}^{1/\sqrt{2}} 2x\sin^{-1}(x^2)\,dx \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area(i)}\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.128523 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{\pi - 12 + 6\sqrt{3}}{12} \approx 0.127825$.

53.



Select M = 0.5

The area approximations will vary depending on the random number generator and seed value that is used

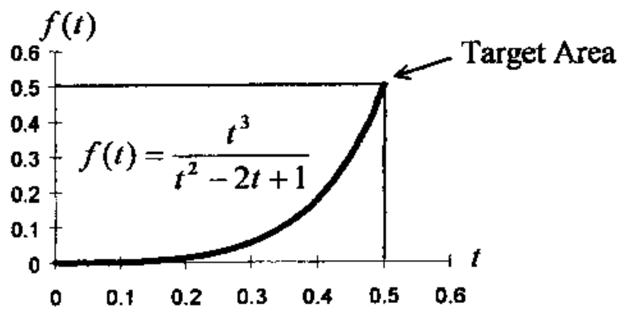
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.28	2000	0.259
200	0.265	3000	0.262167
300	0.278333	4000	0.259625
400	0.2625	5000	0.2724
500	0.261	6000	0.270583
600	0.27	8000	0.265875
700	0.254286	10,000	0.26495
800	0.270625	15,000	0.2668
900	0.277778	20,000	0.268275
1000	0.2685	30,000	0.265875

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_0^1 z\sqrt{1-z}\,dz \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area(i)}\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.266465 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{4}{15} \approx 0.266667$.

54.



Select M = 0.5

The area approximations will vary depending on the random number generator and seed value that is used

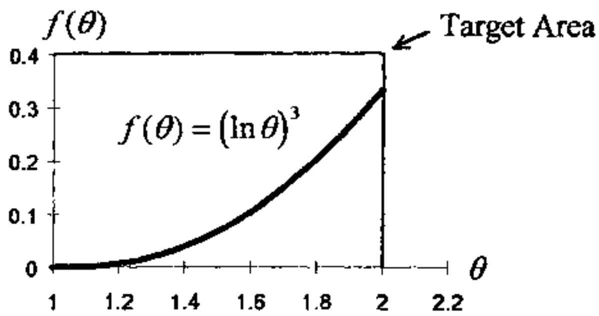
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.0375	2000	0.04725
200	0.06	3000	0.0435
300	0.06	4000	0.0480625
400	0.0425	5000	0.046
500	0.0435	6000	0.04525
600	0.05125	8000	0.0445937
700	0.0439286	10,000	0.047375
800	0.053125	15,000	0.0449
900	0.0472222	20,000	0.0446375
1000	0.0425	30,000	0.0458

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_0^{1/2} \frac{t^3\,dt}{t^2-2t+1} \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area(i)}\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.0456313 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{17}{8} - 3\ln 2 \approx 0.0455585$.

55.



Select M = 0.4

The area approximations will vary depending on the random number generator and seed value that is used

Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.096	2000	0.095
200	0.104	3000	0.103467
300	0.0986667	4000	0.0999
400	0.095	5000	0.10096
500	0.0992	6000	0.1048
600	0.096	8000	0.10105
700	0.0908571	10,000	0.104
800	0.0985	15,000	0.0995733
900	0.1	20,000	0.1013
1000	0.104	30,000	0.100707

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_1^2 (\ln \theta)^3 \, d\theta \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i) \right) \Big/ \left(\sum_{i=1}^{20} n(i) \right) = 0.101054 \text{ by Monte Carlo.}$$

The actual value of the integral is $6 + 2\left[(\ln 2)^3 - 3(\ln 2)^2 + 6 \ln 2 - 6\right] \approx 0.101097$.

7.6 L'HÔPITAL'S RULE

1. l'Hôpital: $\lim\limits_{x \to 2} \frac{x-2}{x^2-4} = \frac{1}{2x}\Big|_{x=2} = \frac{1}{4}$ or $\lim\limits_{x \to 2} \frac{x-2}{x^2-4} = \lim\limits_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim\limits_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$

2. l'Hôpital: $\lim\limits_{x \to 0} \frac{\sin 5x}{x} = \frac{5 \cos 5x}{1}\Big|_{x=0} = 5$ or $\lim\limits_{x \to 0} \frac{\sin 5x}{x} = 5 \lim\limits_{\substack{x \to 0 \\ 5x \to 0}} \frac{\sin 5x}{5x} = 5 \cdot 1 = 5$

3. l'Hôpital: $\lim\limits_{x \to \infty} \frac{5x^3 - 3x}{7x^2 + 1} = \lim\limits_{x \to \infty} \frac{10x - 3}{14x} = \lim\limits_{x \to \infty} \frac{10}{14} = \frac{5}{7}$ or $\lim\limits_{x \to \infty} \frac{5x^2 - 3x}{7x^2 + 1} = \lim\limits_{x \to \infty} \frac{5 - \frac{3}{x}}{7 + \frac{1}{x}} = \frac{5}{7}$

4. l'Hôpital: $\lim\limits_{x \to 1} \frac{x^3 - 1}{4x^3 - x - 3} = \lim\limits_{x \to 1} \frac{3x^2}{12x^2 - 1} = \frac{3}{11}$ or $\lim\limits_{x \to 1} \frac{x^3 - 1}{4x^3 - x - 3} = \lim\limits_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(4x^2+4x+3)}$

$= \lim\limits_{x \to 1} \frac{(x^2+x+1)}{(4x^2+4x+3)} = \frac{3}{11}$

5. l'Hôpital: $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \frac{\sin x}{2x} = \lim\limits_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$ or $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \left[\frac{1-\cos x}{x^2}\left(\frac{1+\cos x}{1+\cos x}\right)\right]$

$= \lim\limits_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim\limits_{x\to 0} \left[\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x}\right)\left(\frac{1}{1+\cos x}\right)\right] = \frac{1}{2}$

6. l'Hôpital: $\lim\limits_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim\limits_{x\to\infty} \frac{4x+3}{3x^2+1} = \lim\limits_{x\to\infty} \frac{4}{6x^2} = 0$ or $\lim\limits_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim\limits_{x\to\infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{0}{1} = 0$

7. $\lim\limits_{\theta\to 0} \frac{\sin \theta^2}{\theta} = \lim\limits_{\theta\to 0} \frac{2\theta \cos \theta^2}{1} = (2)(0) \cos (0)^2 = 0$

8. $\lim\limits_{\theta\to\pi/2} \frac{1-\sin\theta}{1+\cos 2\theta} = \lim\limits_{\theta\to\pi/2} \frac{-\cos\theta}{-2\sin 2\theta} = \lim\limits_{\theta\to\pi/2} \frac{\sin\theta}{-4\cos 2\theta} = \frac{\sin \pi/2}{-4\cos\pi} = \frac{1}{4}$

9. $\lim\limits_{t\to 0} \frac{\cos t - 1}{e^t - t - 1} = \lim\limits_{t\to 0} \frac{-\sin t}{e^t - 1} = \lim\limits_{t\to 0} \frac{-\cos t}{e^t} = -1$

10. $\lim\limits_{t\to 1} \frac{t-1}{\ln t - \sin \pi t} = \lim\limits_{t\to 1} \frac{1}{\frac{1}{t} - \pi \cos \pi t} = \frac{1}{1-\pi(-1)} = \frac{1}{\pi+1}$

11. $\lim\limits_{x\to\infty} \frac{\ln(x+1)}{\log_2 t} = \lim\limits_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x \ln 2}} = \lim\limits_{x\to\infty} \frac{x \ln 2}{x+1} = \lim\limits_{x\to\infty} \ln 2 = \ln 2$

12. $\lim\limits_{x\to\infty} \frac{\log_2 x}{\log_3 (x+3)} = \lim\limits_{x\to\infty} \frac{\frac{1}{x \ln 2}}{\frac{1}{(x+3)\ln 3}} = \lim\limits_{x\to\infty} \frac{(x+3)\ln 3}{x \ln 2} = \lim\limits_{x\to\infty} \frac{x \ln 3 + 3 \ln 3}{x \ln 2} = \lim\limits_{x\to\infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2}$

13. $\lim\limits_{y\to 0^+} \frac{\ln(y^2+2y)}{\ln y} = \lim\limits_{y\to 0^+} \frac{\frac{2y+2}{y^2+2y}}{\frac{1}{y}} = \lim\limits_{y\to 0^+} \frac{y(2y+2)}{y^2+2y} = \lim\limits_{y\to 0^+} \frac{2y^2+2y}{y^2+2y} = \lim\limits_{y\to 0^+} \frac{4y+2}{2y+2} = \frac{4(0)+2}{2(0)+2} = \frac{2}{2} = 1$

14. $\lim\limits_{y\to\pi/2} \left(\frac{\pi}{2} - y\right) \tan y = \lim\limits_{y\to\pi/2} \frac{\left(\frac{\pi}{2}-y\right)\sin y}{\cos y} = \lim\limits_{y\to\pi/2} \frac{\left(\frac{\pi}{2}-y\right)\cos y + (-1)\sin y}{-\sin y} = \frac{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)\cos\frac{\pi}{2} + (-1)\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}}$

$= \frac{(-1)(1)}{-(1)} = 1$

15. $\lim\limits_{x\to 0^+} x \ln x = \lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+} \frac{-x^2}{x} = \lim\limits_{x\to 0^+} -x = 0$

16. $\lim\limits_{x\to\infty} x \tan \frac{1}{x} = \lim\limits_{x\to\infty} \frac{\tan \frac{1}{x}}{\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{-\frac{1}{x^2}\sec^2 \frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \sec^2 \frac{1}{x} = \sec^2 0 = 1$

17. $\lim\limits_{x\to 0^+} (\csc x - \cot x + \cos x) = \lim\limits_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} + \cos x\right) = \lim\limits_{x\to 0^+} \frac{1 - \cos x + \cos x \sin x}{\sin x}$

$= \lim\limits_{x\to 0^+} \frac{\sin x + \cos x \cos x - \sin x \sin x}{\cos x} = 1$

18. $\lim\limits_{x\to\infty} (\ln 2x - \ln(x+1)) = \lim\limits_{x\to\infty} \ln\left(\frac{2x}{x+1}\right)$; Let $f(x) = \frac{2x}{x+1} \Rightarrow \lim\limits_{x\to\infty} \frac{2x}{x+1} = \lim\limits_{x\to\infty} \frac{2}{1} = 2$. Therefore,

$\lim\limits_{x\to\infty} (\ln 2x - \ln \sin x) = \lim\limits_{x\to\infty} \ln f(x) = \ln 2$

19. $\lim\limits_{x\to 0^+} (\ln x - \ln \sin x) = \lim\limits_{x\to 0^+} \ln \frac{x}{\sin x}$; let $f(x) = \frac{x}{\sin x} \Rightarrow \lim\limits_{x\to 0^+} \frac{x}{\sin x} = \lim\limits_{x\to 0^+} \frac{1}{\cos x} = 1$. Therefore,

$\lim\limits_{x\to 0^+} (\ln x - \ln \sin x) = \lim\limits_{x\to 0^+} \ln f(x) = \ln 1 = 0$

20. $\lim\limits_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) = \lim\limits_{x\to 0^+} \frac{1-\sqrt{x}}{x} = \infty$

21. The limit leads to the indeterminate form 1^∞. Let $f(x) = (e^x + x)^{1/x} \Rightarrow \ln (e^x + x)^{1/x} = \frac{\ln (e^x + x)}{x}$

$\Rightarrow \lim\limits_{x\to 0} \frac{\ln (e^x + x)}{x} = \lim\limits_{x\to 0} \frac{\frac{e^x+1}{e^x+x}}{1} = 2 \Rightarrow \lim\limits_{x\to 0} (e^x + x)^{1/x} \lim\limits_{x\to 0} e^{\ln f(x)} = e^2$

22. The limit leads to the indeterminate form ∞^0. Let $f(x) = \left(\frac{1}{x^2}\right)^x \Rightarrow \ln\left(\frac{1}{x^2}\right)^x = x \ln\left(\frac{1}{x^2}\right) = \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}}$

$\Rightarrow \lim\limits_{x\to 0} \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}} = \lim\limits_{x\to 0} \frac{\frac{-2/x^3}{1/x^2}}{-1/x^2} = \lim\limits_{x\to 0} 2x = 0 \Rightarrow \lim\limits_{x\to 0} \left(\frac{1}{x^2}\right)^x = \lim\limits_{x\to 0} e^{\ln f(x)} = e^0 = 1$

23. $\lim\limits_{x\to\pm\infty} \frac{3x-5}{2x^2-x+2} = \lim\limits_{x\to\pm\infty} \frac{3}{4x-1} = 0$

24. $\lim\limits_{x\to 0} \frac{\sin 7x}{\tan 11x} = \lim\limits_{x\to 0} \frac{7\cos 7x}{11\sec^2 11x} = \frac{7}{11}$

25. The limit leads to the indeterminate form ∞^0. Let $f(x) = (\ln x)^{1/x} \Rightarrow \ln(\ln x)^{1/x} = \frac{\ln(\ln x)}{x}$

$\Rightarrow \lim\limits_{x\to\infty} \frac{\ln(\ln x)}{x} = \lim\limits_{x\to\infty} \frac{\frac{1/x}{\ln x}}{1} = \lim\limits_{x\to\infty} \frac{1}{x \ln x} = 0 \Rightarrow \lim\limits_{x\to\infty} (\ln x)^{1/x} = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^0 = 1$

26. The limit leads to the indeterminate form ∞^0. Let $f(x) = (1+2x)^{1/(2\ln x)} \Rightarrow \ln(1+2x)^{1/(2\ln x)} = \frac{\ln(1+2x)}{2\ln x}$

$$\Rightarrow \lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x} = \lim_{x\to\infty} \frac{\left(\frac{2}{1+2x}\right)}{\left(\frac{2}{x}\right)} = \lim_{x\to\infty} \frac{x}{1+2x} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2} \Rightarrow \lim_{x\to\infty} (1+2x)^{1/(2\ln x)}$$

$$= \lim_{x\to\infty} e^{\ln f(x)} = e^{1/2} = \sqrt{e}$$

27. The limit leads to the indeterminate form 0^0. Let $f(x) = (x^2-2x+1)^{x-1}$

$$\Rightarrow \ln(x^2-2x+1)^{x-1} = (x-1)\ln(x^2-2x+1) = \frac{\ln(x^2-2x+1)}{\left(\frac{1}{x-1}\right)} \Rightarrow \lim_{x\to 1} \frac{\ln(x^2-2x+1)}{\left(\frac{1}{x-1}\right)} = \lim_{x\to 1} \frac{\left(\frac{2x-2}{x^2-2x+1}\right)}{\left(-\frac{1}{(x-1)^2}\right)}$$

$$= \lim_{x\to 1} \frac{\left(\frac{2(x-1)}{(x-1)^2}\right)}{\left(-\frac{1}{(x-1)^2}\right)} = \lim_{x\to 1} -2(x-1) = 0 \Rightarrow \lim_{x\to 1} (x^2-2x+1)^{x-1} = \lim_{x\to 1} e^{\ln f(x)} = e^0 = 1$$

28. The limit leads to the indeterminate form 0^0. Let $f(x) = (\cos x)^{\cos x} \Rightarrow \ln(\cos x)^{\cos x}$

$$= (\cos x)\ln(\cos x) = \frac{\ln(\cos x)}{\sec x} \Rightarrow \lim_{x\to\pi/2^-} \frac{\ln(\cos x)}{\sec x} = \lim_{x\to\pi/2^-} \frac{\left(\frac{-\sin x}{\cos x}\right)}{\sec x \tan x} = \lim_{x\to\pi/2^-} \frac{-\tan x}{\sec x \tan x}$$

$$= \lim_{x\to\pi/2^-} -\cos x = 0 \Rightarrow \lim_{x\to\pi/2^-} (\cos x)^{\cos x} = \lim_{x\to\pi/2^-} e^{\ln f(x)} = e^0 = 1$$

29. The limit leads to the indeterminate form 1^∞. Let $f(x) = (1+x)^{1/x} \Rightarrow \ln(1+x)^{1/x} = \frac{\ln(1+x)}{x}$

$$\Rightarrow \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \lim_{x\to 0^+} \frac{\left(\frac{1}{1+x}\right)}{1} = 1 \Rightarrow \lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e$$

30. The limit leads to the indeterminate form 1^∞. Let $f(x) = x^{1/(x-1)} \Rightarrow \ln x^{1/(x-1)} = \frac{\ln x}{x-1}$

$$\Rightarrow \lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1 \Rightarrow \lim_{x\to 1} x^{1/(x-1)} = \lim_{x\to 1} e^{\ln f(x)} = e^1 = e$$

31. The limit leads to the indeterminate form 0^0. Let $f(x) = (\sin x)^x \Rightarrow \ln(\sin x)^x = x\ln(\sin x) = \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)}$

$$\Rightarrow \lim_{x\to 0^+} \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} = \lim_{x\to 0^+} \frac{\left(\frac{\cos x}{\sin x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} \frac{-x^2\cos x}{\sin x} = \lim_{x\to 0^+} \frac{x^2\sin x - 2x\cos x}{\cos x} = 0$$

$$\Rightarrow \lim_{x\to 0^+} (\sin x)^x = \lim_{x\to 0^+} e^{\ln f(x)} = e^0 = 1$$

32. The limit leads to the indeterminate form 0^0. Let $f(x) = (\sin x)^{\tan x} \Rightarrow \ln(\sin x)^{\tan x}$

$$= \tan x \ln(\sin x) = \frac{\ln(\sin x)}{\cot x} \Rightarrow \lim_{x\to 0^+} \frac{\ln(\sin x)}{\cot x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to 0^+} (-\sin x \cos x) = 0$$

$$\Rightarrow \lim_{x\to 0^+} (\sin x)^{\tan x} = \lim_{x\to 0^+} e^{\ln f(x)} = e^0 = 1$$

33. The limit leads to the indeterminate form $1^{-\infty}$. Let $f(x) = x^{1/(1-x)} \Rightarrow \ln x^{1/(1-x)} = \frac{\ln x}{1-x}$

$$\Rightarrow \lim_{x\to 1^+} \frac{\ln x}{1-x} = \lim_{x\to 1^+} \frac{\frac{1}{x}}{-1} = -1 \Rightarrow \lim_{x\to 1^+} x^{1/(1-x)} = \lim_{x\to 1^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$$

34. $\lim\limits_{x\to\infty} x^2e^{-x} = \lim\limits_{x\to\infty} \frac{x^2}{e^x} = \lim\limits_{x\to\infty} \frac{2x}{e^x} = \lim\limits_{x\to\infty} \frac{2}{e^x} = 0$

35. $\lim\limits_{x\to\infty} \int_x^{2x} \frac{1}{t}\,dt = \lim\limits_{x\to\infty} \left[\ln|t|\right]_x^{2x} = \lim\limits_{x\to\infty} \ln\left(\frac{2x}{x}\right) = \ln 2$

36. $\lim\limits_{x\to\infty} \frac{\int_1^x \ln t\,dt}{x\ln x} = \lim\limits_{x\to\infty} \frac{\ln x}{\ln x + 1} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = 1$

37. $\lim\limits_{\theta\to 0} \frac{\cos\theta - 1}{e^\theta - \theta - 1} = \lim\limits_{\theta\to 0} \frac{-\sin\theta}{e^\theta - 1} = \lim\limits_{\theta\to 0} \frac{-\cos\theta}{e^\theta} = -1$

38. $\lim\limits_{t\to\infty} \frac{e^t + t^2}{e^t - 1} = \lim\limits_{t\to\infty} \frac{e^t + 2t}{e^t} = \lim\limits_{t\to\infty} \frac{e^t + 2}{e^t} = \lim\limits_{t\to\infty} \frac{e^t}{e^t} = 1$

39. $\lim\limits_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} = \sqrt{\lim\limits_{x\to\infty} \frac{9x+1}{x+1}} = \sqrt{\lim\limits_{x\to\infty} \frac{9}{1}} = \sqrt{9} = 3$

40. $\lim\limits_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = \sqrt{\frac{1}{\lim\limits_{x\to 0^+} \frac{\sin x}{x}}} = \sqrt{\frac{1}{1}} = 1$

41. $\lim\limits_{x\to \pi/2^-} \frac{\sec x}{\tan x} = \lim\limits_{x\to \pi/2^-} \left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right) = \lim\limits_{x\to \pi/2^-} \frac{1}{\sin x} = 1$

42. $\lim\limits_{x\to 0^+} \frac{\cot x}{\csc x} = \lim\limits_{x\to 0^+} \frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)} = \lim\limits_{x\to 0^+} \cos x = 1$

43. Part (b) is correct because part (a) is neither in the $\frac{0}{0}$ nor $\frac{\infty}{\infty}$ form and so l'Hôpital's rule may not be used.

44. Answers may vary.

(a) $f(x) = 3x + 1$; $g(x) = x$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{3x+1}{x} = \lim_{x\to\infty} \frac{3}{1} = 3$$

(b) $f(x) = x + 1$; $g(x) = x^2$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x+1}{x^2} = \lim_{x\to\infty} \frac{1}{2x} = 0$$

(c) $f(x) = x^2$; $g(x) = x + 1$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x+1} = \lim_{x\to\infty} \frac{2x}{1} = \infty$$

45. If $f(x)$ is to be continuous at $x = 0$, then $\lim\limits_{x\to 0} f(x) = f(0) \Rightarrow c = f(0) = \lim\limits_{x\to 0} \frac{9x - 3\sin 3x}{5x^3} = \lim\limits_{x\to 0} \frac{9 - 9\cos 3x}{15x^2}$ $= \lim\limits_{x\to 0} \frac{27\sin 3x}{30x} = \lim\limits_{x\to 0} \frac{81\cos 3x}{30} = \frac{27}{10}$.

46. (a) For $x \neq 0$, $f'(x) = \frac{d}{dx}(x+2) = 1$ and $g'(x) = \frac{d}{dx}(x+1) = 1$. Therefore, $\lim\limits_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{1}{1} = 1$, while

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{x+2}{x+1} = \frac{0+2}{0+1} = 2.$$

(b) This does not contradict l'Hôpital's rule because neither f nor g is differentiable at $x = 0$ (as evidenced by the fact that neither is continuous at $x = 0$), so l'Hôpital's rule does not apply.

47. (a) The limit leads to the indeterminate form 1^∞. Let $f(k) = \left(1 + \frac{r}{k}\right)^{kt} \Rightarrow \ln f(k) = kt \ln\left(1 + \frac{r}{k}\right) = \frac{t \ln\left(1 + \frac{r}{k}\right)}{\frac{1}{k}}$

$$\Rightarrow \lim_{k\to\infty} \frac{t \ln\left(1 + \frac{r}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty} \frac{t\left(-\frac{r}{k^2}\right)\left(1 + \frac{r}{k}\right)^{-1}}{-\frac{1}{k^2}} = \lim_{k\to\infty} \frac{rt}{1 + \frac{r}{k}} = \frac{rt}{1} = rt$$

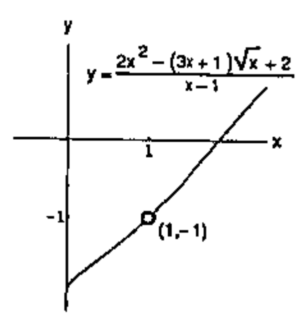
$$\Rightarrow \lim_{k\to\infty} A_0\left(1 + \frac{r}{k}\right)^{kt} = A_0 \lim_{k\to\infty} \left(1 + \frac{r}{k}\right)^{kt} = A_0 \lim_{k\to\infty} e^{\ln f(k)} = A_0 e^{rt}$$

(b) Part (a) shows that as the number of compoundings per year increases toward infinity, the limit of interest compounded k times per year is interest compounded continuously.

48. The graph indicates a limit near -1. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x\to 1} \frac{2x^2 - (3x+1)\sqrt{x} + 2}{x - 1}$

$$= \lim_{x\to 1} \frac{2x^2 - 3x^{3/2} - x^{1/2} + 2}{x - 1} = \lim_{x\to 1} \frac{4x - \frac{9}{2}x^{1/2} - \frac{1}{2}x^{-1/2}}{1}$$

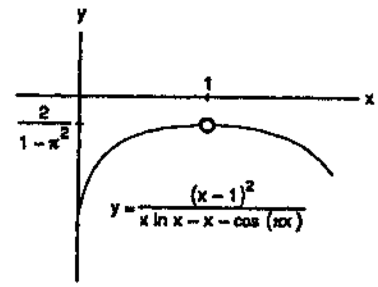
$$= \frac{4 - \frac{9}{2} - \frac{1}{2}}{1} = \frac{4 - 5}{1} = -1$$



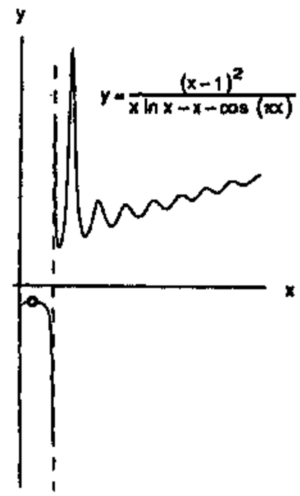
49. (a) The graph indicates a limit near -0.225. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x\to 1} \frac{(x-1)^2}{x \ln x - x - \cos(\pi x)}$

$= \lim\limits_{x\to 1} \frac{2(x-1)}{\ln x + 1 - 1 + \pi \sin(\pi x)} = \lim\limits_{x\to 1} \frac{2}{\frac{1}{x} + \pi^2 \cos(\pi x)}$

$= \frac{2}{1+\pi^2(-1)} = \frac{2}{1-\pi^2}$



(b) The graph of $y = \frac{(x-1)^2}{x \ln x - x - \cos(\pi x)}$ has a vertical asymptote near $x = 2.552$.



50. (a) $\ln f(x)^{g(x)} = g(x) \ln f(x)$

$\lim\limits_{x\to c} (g(x) \ln f(x)) = \left(\lim\limits_{x\to c} g(x)\right)\left(\lim\limits_{x\to c} \ln f(x)\right) = \infty(-\infty) = -\infty$

$\lim\limits_{x\to c} f(x)^{g(x)} = \lim\limits_{x\to c} e^{\ln f(x)^{g(x)}} = e^{-\infty} = 0$

(b) $\lim\limits_{x\to c} (g(x) \ln f(x)) = \left(\lim\limits_{x\to c} g(x)\right)\left(\lim\limits_{x\to c} \ln f(x)\right) = (-\infty)(-\infty) = \infty$

$\lim\limits_{x\to c} f(x)^{g(x)} = \lim\limits_{x\to c} e^{\ln f(x)^{g(x)}} = e^{\infty} = \infty$

51. (a) Because the difference in the numerator is so small compared to the values being subtracted, any calculator or computer with limited precision will give the incorrect result that $1 - \cos x^6$ is 0 for even moderately small values of x. For example, at $x = 0.1$, $\cos x^6 \approx 0.9999999999995$ (13 places), so on a 10-place calculator, $\cos x^6 = 1$ and $1 - \cos x^6 = 0$.

(b) Same reason as in part (a) applies.

(c) $\lim\limits_{x\to 0} \frac{1-\cos x^6}{x^{12}} = \lim\limits_{x\to 0} \frac{6x^5 \sin x^6}{12x^{11}} = \lim\limits_{x\to 0} \frac{\sin x^6}{2x^6} = \lim\limits_{x\to 0} \frac{6x^5 \cos x^6}{12x^5} = \lim\limits_{x\to 0} \frac{\cos x^6}{2} = \frac{1}{2}$

(d) The graph and/or table on a grapher shows the value of the function to be 0 for x-values moderately close to 0, but the limit is 1/2. The calculator is giving unreliable information because there is significant round-off error in computing values of this function on a limited precision device.

52. (b) The limit leads to the indeterminate form $\infty - \infty$:

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)=\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)\left(\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}\right)=\lim_{x\to\infty}\left(\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}\right)=\lim_{x\to\infty}\left(\frac{-x}{x+\sqrt{x^2+x}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{-1}{1+\sqrt{1+\frac{1}{x}}}\right)=\frac{-1}{1+\sqrt{1+0}}=-\frac{1}{2}$$

53. (a) $f(x)=e^{x\ln(1+1/x)}$

$1+\frac{1}{x}>0$ when $x<-1$ or $x>0$

Domain: $(-\infty,-1)\cup(0,\infty)$

(b) The form is 0^{-1}, so $\lim\limits_{x\to-1} f(x)=\infty$

(c) $$\lim_{x\to-\infty} x\ln\left(1+\frac{1}{x}\right)=\lim_{x\to-\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}=\lim_{x\to-\infty}\frac{\left(-\frac{1}{x^2}\right)\left(1+\frac{1}{x}\right)^{-1}}{-\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{1}{1+\frac{1}{x}}=1$$

$$\Rightarrow \lim_{x\to-\infty} f(x)=\lim_{x\to-\infty} e^{x\ln(1+1/x)}=e$$

54. (a) $y=x^{1/x}\Rightarrow \ln y=\frac{\ln x}{x}\Rightarrow \frac{y'}{y}=\frac{\left(\frac{1}{x}\right)(x)-\ln x}{x^2}\Rightarrow y'=\left(\frac{1-\ln x}{x^2}\right)\left(x^{1/x}\right)$. The sign pattern is

$y'=\underset{0}{|}+++++\underset{e}{|}----$ which indicates a maximum value of $y=e^{1/e}$ when $x=e$

(b) $y=x^{1/x^2}\Rightarrow \ln y=\frac{\ln x}{x^2}\Rightarrow \frac{y'}{y}=\frac{\left(\frac{1}{x}\right)(x^2)-2x\ln x}{x^4}\Rightarrow y'=\left(\frac{1-2\ln x}{x^3}\right)\left(x^{1/x^2}\right)$. The sign pattern is

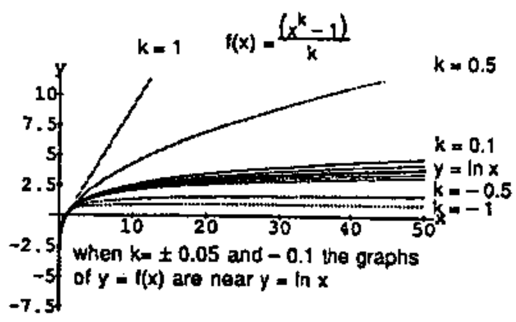
$y'=\underset{0}{|}+++\underset{\sqrt{e}}{|}----$ which indicates a maximum of $y=e^{1/2e}$ when $x=\sqrt{e}$

(c) $y=x^{1/x^n}\Rightarrow \ln y=\frac{\ln x}{x^n}=\frac{\left(\frac{1}{x}\right)(x^n)-(\ln x)(nx^{n-1})}{x^{2n}}\Rightarrow y'=\frac{x^{n-1}(1-n\ln x)}{x^{2n}}\cdot x^{1/x^n}$. The sign pattern is

$y'=\underset{0}{|}+++\underset{\sqrt[n]{e}}{|}----$ which indicates a maximum of $y=e^{1/ne}$ when $x=\sqrt[n]{e}$

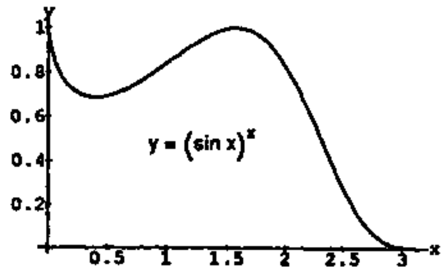
(d) $\lim\limits_{x\to\infty} x^{1/x^n}=\lim\limits_{x\to\infty}\left(e^{\ln x}\right)^{1/x^n}=\lim\limits_{x\to\infty} e^{(\ln x)/x^n}=\exp\left(\lim\limits_{x\to\infty}\frac{\ln x}{x^n}\right)=\exp\left(\lim\limits_{x\to\infty}\left(\frac{1}{nx^n}\right)\right)=e^0=1$

55. (a)



(b) $\lim\limits_{k \to 0} \frac{x^k - 1}{k} = \lim\limits_{k \to 0} \frac{x^k \ln x}{1} = \ln x$

56. (a) We should assign the value 1 to $f(x) = (\sin x)^x$ to make it continuous at $x = 0$.

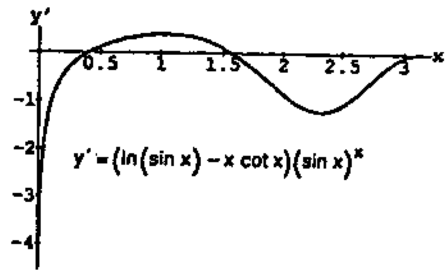


(b) $\ln f(x) = x \ln(\sin x) = \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} \Rightarrow \lim\limits_{x \to 0^+} \ln f(x) = \lim\limits_{x \to 0^+} \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to 0^+} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{\left(-\frac{1}{x^2}\right)}$

$= \lim\limits_{x \to 0} \frac{-x^2}{\tan x} = \lim\limits_{x \to 0} \frac{-2x}{\sec^2 x} = 0 \Rightarrow \lim\limits_{x \to 0} f(x) = e^0 = 1$

(c) The maximum value of f(x) is close to 2 near the point $x \approx 1.55$ (see the graph in part (a)).

(d) The root in question is near 1.57.



(e) $y' = 0 \Rightarrow (\ln(\sin x) - x \cot x)(\sin x)^x = 0 \Rightarrow \ln(\sin x) - x \cot x = 0$. Let $g(x) = \ln(\sin x) - x \cot x$ $\Rightarrow g'(x) = \cot x - \cot x + x \csc^2 x = x \csc^2 x$. Using Newton's method, $g(x) = 0 \Rightarrow x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$ $= x_n - \frac{\ln(\sin x_n) - x_n \cot x_n}{x_n \csc^2 x_n}$. Then $x_1 = 1.55 \Rightarrow x_2 = 1.57093 \Rightarrow x_3 = 1.57080 \Rightarrow x_4 = 1.57080$ $\Rightarrow x_k = 1.57080$, $k \geq 3$.

(f)

x	1.55	1.57	1.57080
$(\sin x)^x$	0.999664854	0.999999502	1

7.7 IMPROPER INTEGRALS

1. (a) The integral is improper because of an infinite limit of integration.

(b) $\int_0^\infty \frac{dx}{x^2+1} = \lim\limits_{b \to \infty} \int_0^b \frac{dx}{x^2+1} = \lim\limits_{b \to \infty} \left[\tan^{-1} x\right]_0^b = \lim\limits_{b \to \infty} (\tan^{-1} b - 0) = \frac{\pi}{2}$

The integral converges.

(c) $\frac{\pi}{2}$

2. (a) The integral is improper because the integrand has an infinite discontinuity at $x = 0$.

(b) $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \int_b^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \left[2\sqrt{x}\right]_b^1 = \lim_{b\to 0^+} \left(2 - 2\sqrt{b}\right) = 2$

The integral converges.

(c) 2

3. (a) The integral involves improper integrals because the integrand has an infinite discontinuity at $x = 0$.

(b) $\int_{-8}^1 \frac{dx}{x^{1/3}} = \int_{-8}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}}$

$\int_{-8}^0 \frac{dx}{x^{1/3}} = \lim_{b\to 0^-} \int_{-8}^b \frac{dx}{x^{1/3}} = \lim_{b\to 0^-} \left[\frac{3}{2}x^{2/3}\right]_{-8}^b = \lim_{b\to 0^-} \left(\frac{3}{2}b^{2/3} - 6\right) = -6$

$\int_0^1 \frac{dx}{x^{1/3}} = \lim_{b\to 0^+} \int_b^1 \frac{dx}{x^{1/3}} = \lim_{b\to 0^+} \left[\frac{3}{2}x^{2/3}\right]_b^1 = \lim_{b\to 0^+} \left(\frac{3}{2} - \frac{3}{2}b^{2/3}\right) = \frac{3}{2}$

$\int_{-8}^1 \frac{dx}{x^{1/3}} = -6 + \frac{3}{2} = -\frac{9}{2}$

The integral converges.

(c) $-\frac{9}{2}$

4. (a) The integral is improper because of two infinite limits of integration.

(b) $\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \int_{-\infty}^{0} \frac{2x\,dx}{(x^2+1)^2} + \int_{0}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$

$\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \lim_{b\to -\infty} \int_b^0 \frac{2x\,dx}{(x^2+1)^2} = \lim_{b\to -\infty} \left[-(x^2+1)^{-1}\right]_b^0 = \lim_{b\to -\infty} \left[-1 + (b^2+1)^{-1}\right] = -1$

$\int_0^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \lim_{b\to \infty} \int_0^b \frac{2x\,dx}{(x^2+1)^2} = \lim_{b\to \infty} \left[-(x^2+1)^{-1}\right]_0^b$

$\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = -1 + 1 = 0$

The integral converges.

(c) 0

5. (a) The integral is improper because the integrand has an infinite discontinuity at 0.

(b) $\int_0^{\ln 2} x^{-2}e^{1/x}\,dx = \lim_{b\to 0^+} \int_b^{\ln 2} x^{-2}e^{1/x}\,dx = \lim_{b\to 0^+} \left[-e^{1/x}\right]_b^{\ln 2} = \lim_{b\to 0^+} \left[-e^{1/\ln 2} + e^{1/b}\right] = \infty$

The integral diverges.

(c) No value

6. (a) The integral is improper because the integrand has an infinite discontinuity at $x = 0$.

(b) $\int_0^{\pi/2} \cot\theta\,d\theta = \lim_{b\to 0^+} \int_b^{\pi/2} \cot\theta\,d\theta = \lim_{b\to 0^+} \int_b^{\pi/2} \frac{\cos\theta\,d\theta}{\sin\theta} = \lim_{b\to 0^+} \left[\ln|\sin\theta|\right]_b^{\pi/2} = \lim_{b\to 0^+} (0 - \ln|\sin b|) = \infty$

The integral diverges.

(c) No value

7. $\int_1^{\infty} \frac{dx}{x^{1.001}} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x^{1.001}} = \lim_{b\to\infty} \left[-1000x^{-0.001}\right]_1^b = \lim_{b\to\infty} \left(\frac{-1000}{b^{0.001}} + 1000\right) = 1000$

8. $\int_{-1}^{1} \frac{dx}{x^{2/3}} = \int_{-1}^{0} \frac{dx}{x^{2/3}} + \int_0^1 \frac{dx}{x^{2/3}} = \lim_{b\to 0^-} \left[3x^{1/3}\right]_{-1}^b + \lim_{c\to 0^+} \left[3x^{1/3}\right]_c^1$

$= \lim_{b\to 0^-} \left[3b^{1/3} - 3(-1)^{1/3}\right] + \lim_{c\to 0^+} \left[3(1)^{1/3} - 3c^{1/3}\right] = (0+3) + (3-0) = 6$

9. $\int_0^4 \frac{dr}{\sqrt{4-r}} = \lim_{b\to 4^-} \left[-2\sqrt{4-r}\right]_0^b = \lim_{b\to 4^-} \left[-2\sqrt{4-b} - (-2\sqrt{4})\right] = 0 + 4 = 4$

10. $\int_0^1 \frac{dr}{r^{0.999}} = \lim_{b\to 0^+} \left[1000r^{0.001}\right]_b^1 = \lim_{b\to 0^+} \left(1000 - 1000b^{0.001}\right) = 1000 - 0 = 1000$

11. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-} \left[\sin^{-1} x\right]_0^b = \lim_{b\to 1^-} \left(\sin^{-1} b - \sin^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

12. $\int_{-\infty}^{2} \frac{2\,dx}{x^2+4} = \lim_{b\to -\infty} \left[\tan^{-1}\frac{x}{2}\right]_b^2 = \lim_{b\to -\infty} \left(\tan^{-1} 1 - \tan^{-1} b\right) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$

13. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1} = \int_{-\infty}^{-2} \frac{dx}{x-1} - \int_{-\infty}^{-2} \frac{dx}{x+1} = \lim_{b\to -\infty} \left[\ln|x-1|\right]_b^{-2} - \lim_{b\to -\infty} \left[\ln|x+1|\right]_b^{-2} = \lim_{b\to -\infty} \left[\ln\left|\frac{x-1}{x+1}\right|\right]_b^{-2}$

$= \lim_{b\to -\infty} \left(\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right) = \ln 3 - \ln\left(\lim_{b\to -\infty} \frac{b-1}{b+1}\right) = \ln 3 - \ln 1 = \ln 3$

14. $\int_2^\infty \frac{3\,dt}{t^2-1} = \int_2^\infty \frac{3\,dt}{t-1} - \int_2^\infty \frac{3\,dt}{t} = \lim_{b\to\infty} [3\ln(t-1) - 3\ln t]_2^b = \lim_{b\to\infty} \left[3\ln\left(\frac{t-1}{t}\right)\right]_2^b$

$= 3\lim_{b\to\infty} \left[\ln\left(\frac{b-1}{b}\right) - \ln\left(\frac{1}{2}\right)\right] = 3\lim_{b\to\infty} \left[\ln\left(\frac{1-\frac{1}{b}}{1}\right) + \ln 2\right] = 3(\ln 1 + \ln 2) = 3\ln 2$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$; $\begin{bmatrix} u = \theta^2 + 2\theta \\ du = 2(\theta+1)\,d\theta \end{bmatrix} \Rightarrow \int_0^3 \frac{du}{2\sqrt{u}} = \lim_{b\to0^+} \int_b^3 \frac{du}{2\sqrt{u}} = \lim_{b\to0^+} [\sqrt{u}]_b^3 = \lim_{b\to0^+} \left(\sqrt{3}-\sqrt{b}\right)$

$= \sqrt{3} - 0 = \sqrt{3}$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds = \frac{1}{2}\int_0^2 \frac{2s\,ds}{\sqrt{4-s^2}} + \int_0^2 \frac{ds}{\sqrt{4-s^2}}$; $\begin{bmatrix} u = 4-s^2 \\ du = -2s\,ds \end{bmatrix} \Rightarrow -\frac{1}{2}\int_4^0 \frac{du}{\sqrt{u}} + \lim_{c\to2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}}$

$= \lim_{b\to0^+} \int_b^4 \frac{du}{2\sqrt{u}} + \lim_{c\to2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to0^+} [\sqrt{u}]_b^4 + \lim_{c\to2^-} \left[\sin^{-1}\frac{s}{2}\right]_0^c$

$= \lim_{b\to0^+} \left(2-\sqrt{b}\right) + \lim_{c\to2^-} \left(\sin^{-1}\frac{c}{2} - \sin^{-1} 0\right) = (2-0) + \left(\frac{\pi}{2} - 0\right) = \frac{4+\pi}{2}$

17. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{a\to0^+} \int_a^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{a\to0^+}\left[\lim_{b\to\infty} \int_a^b \frac{dx}{(1+x)\sqrt{x}}\right]$; $\begin{bmatrix} u = \sqrt{x} \\ 2du = \frac{dx}{\sqrt{x}} \end{bmatrix} \Rightarrow \lim_{a\to0^+}\left[\lim_{b\to\infty} \int_{\sqrt{a}}^{\sqrt{b}} \frac{2du}{1+u^2}\right]$

$= \lim_{a\to0^+}\left[\lim_{b\to\infty}\left(\left(2\tan^{-1} u\right)_{\sqrt{a}}^{\sqrt{b}}\right)\right] = 2\lim_{a\to0^+}\left[\lim_{b\to\infty}\left(\tan^{-1}\sqrt{b} - \tan^{-1}\sqrt{a}\right)\right] = 2\lim_{a\to0^+}\left[\frac{\pi}{2} - \tan^{-1}\sqrt{a}\right] = \pi + 0 = \pi$

18. $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$

$= \lim_{b\to1^+} \left[\sec^{-1}|x|\right]_b^2 + \lim_{c\to\infty} \left[\sec^{-1}|x|\right]_2^c = \lim_{b\to1^+} \left(\sec^{-1} 2 - \sec^{-1} b\right) + \lim_{c\to\infty} \left(\sec^{-1} c - \sec^{-1} 2\right)$

$= \left(\frac{\pi}{3} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{2}$

19. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}} = \lim_{b\to1^+} \left[\sec^{-1} s\right]_b^2 = \sec^{-1} 2 - \lim_{b\to1^+} \sec^{-1} b = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

20. $\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6} = \lim_{b\to\infty} \left[\ln\left|\frac{\theta+2}{\theta+3}\right|\right]_{-1}^b = \lim_{b\to\infty} \left[\ln\left|\frac{b+2}{b+3}\right|\right] - \ln\left|\frac{-1+2}{-1+3}\right| = 0 - \ln\left(\frac{1}{2}\right) = \ln 2$

21. $\int_2^\infty \frac{2\,dv}{v^2-v} = \lim_{b\to\infty} \left[2\ln\left|\frac{v-1}{v}\right|\right]_2^b = \lim_{b\to\infty} \left(2\ln\left|\frac{b-1}{b}\right| - 2\ln\left|\frac{2-1}{2}\right|\right) = 2\ln(1) - 2\ln\left(\frac{1}{2}\right) = 0 + 2\ln 2 = \ln 4$

22. $\int_2^{\infty} \frac{2\,dt}{t^2-1} = \lim_{b\to\infty} \left[\ln\left|\frac{t-1}{t+1}\right|\right]_2^b = \lim_{b\to\infty} \left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right) = \ln(1) - \ln\left(\frac{1}{3}\right) = 0 + \ln 3 = \ln 3$

23. $\int_0^2 \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to 2^-} \left[\sin^{-1} \frac{s}{2}\right]_0^b = \lim_{b\to 2^-} \left(\sin^{-1} \frac{b}{2}\right) - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

24. $\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}} = \lim_{b\to 1^-} \left[2 \sin^{-1}(r^2)\right]_0^b = \lim_{b\to 1^-} \left[2 \sin^{-1}(b^2)\right] - 2 \sin^{-1} 0 = 2\cdot\frac{\pi}{2} - 0 = \pi$

25. $\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1} v)} = \lim_{b\to\infty} \left[\ln\left|1+\tan^{-1} v\right|\right]_0^b = \lim_{b\to\infty} \left[\ln\left|1+\tan^{-1} b\right|\right] - \ln\left|1+\tan^{-1} 0\right|$

$= \ln\left(1+\frac{\pi}{2}\right) - \ln(1+0) = \ln\left(1+\frac{\pi}{2}\right)$

26. $\int_0^{\infty} \frac{16 \tan^{-1} x}{1+x^2}\,dx = \lim_{b\to\infty} \left[8\left(\tan^{-1} x\right)^2\right]_0^b = \lim_{b\to\infty} \left[8\left(\tan^{-1} b\right)^2\right] - 8\left(\tan^{-1} 0\right)^2 = 8\left(\frac{\pi}{2}\right)^2 - 8(0) = 2\pi^2$

27. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}} = \lim_{b\to 0^-} \int_{-1}^b \frac{dx}{\sqrt{-x}} + \lim_{c\to 0^+} \int_c^4 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^-} \left[-2\sqrt{-x}\right]_{-1}^b + \lim_{c\to 0^+} \left[2\sqrt{x}\right]_c^4$

$= \lim_{b\to 0^-} \left(-2\sqrt{-b}\right) - \left(-2\sqrt{-(-1)}\right) + 2\sqrt{4} - 6 \lim_{c\to 0^+} 2\sqrt{c} = 0 + 2 + 2\cdot 2 - 0 = 6$

28. $\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{b\to 1^-} \left[-2\sqrt{1-x}\right]_0^b + \lim_{c\to 1^+} \left[2\sqrt{x-1}\right]_c^2$

$= \lim_{b\to 1^-} \left(-2\sqrt{1-b}\right) - \left(-2\sqrt{1-0}\right) + 2\sqrt{2-1} - \lim_{c\to 1^+} \left(2\sqrt{c-1}\right) = 0 + 2 + 2 - 0 = 4$

29. $\int_{-\infty}^0 \theta e^{\theta}\,d\theta = \lim_{b\to -\infty} \left[\theta e^{\theta} - e^{\theta}\right]_b^0 = \left(0\cdot e^0 - e^0\right) - \lim_{b\to -\infty} \left[be^b - e^b\right] = -1 - \lim_{b\to -\infty} \left(\frac{b-1}{e^{-b}}\right)$

$= -1 - \lim_{b\to -\infty} \left(\frac{1}{-e^{-b}}\right)$ (l'Hôpital's rule for $\frac{\infty}{\infty}$ form)

$= -1 - 0 = -1$

30. $\int_0^{\infty} 2e^{-\theta} \sin\theta\,d\theta = \lim_{b\to\infty} \int_0^b 2e^{-\theta} \sin\theta\,d\theta$

$= \lim_{b\to\infty} 2\left[\frac{e^{-\theta}}{1+1}\left(-\sin\theta - \cos\theta\right)\right]_0^b$ (FORMULA 107 with $a = -1$, $b = 1$)

$$= \lim_{b\to\infty} \frac{-2(\sin b + \cos b)}{2e^b} + \frac{2(\sin 0 + \cos 0)}{2e^0} = 0 + \frac{2(0+1)}{2} = 1$$

31. $\int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^x\,dx + \int_0^{\infty} e^{-x}\,dx = \lim_{b\to-\infty}\left[1 - e^b\right] + \lim_{b\to-\infty}\left[-e^b + 1\right] = (1-0) + (0+1) = 2$

32. $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \int_{-\infty}^{0} 2xe^{-x^2}\,dx + \int_0^{\infty} 2xe^{-x^2}\,dx = \lim_{b\to\infty}\left[-e^{-x^2}\right]_b^0 + \lim_{c\to\infty}\left[-e^{-x^2}\right]_0^c$

$$= \lim_{b\to-\infty}\left[-1-\left(-e^{-b^2}\right)\right] + \lim_{c\to\infty}\left[-e^{-c^2} - (-1)\right] = (-1-0) + (0+1) = 0$$

33. $\int_0^1 x\ln x\,dx = \lim_{b\to0^+}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_b^1 = \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right) - \lim_{b\to0^+}\left(\frac{b^2}{2}\ln b - \frac{b^2}{4}\right) = -\frac{1}{4} - \lim_{b\to0^+}\frac{\ln b}{\left(\frac{2}{b^2}\right)} + 0$

$$= -\frac{1}{4} - \lim_{b\to0^+}\frac{\left(\frac{1}{b}\right)}{\left(-\frac{4}{b^3}\right)} = -\frac{1}{4} + \lim_{b\to0^+}\left(\frac{b^2}{4}\right) = -\frac{1}{4} + 0 = -\frac{1}{4}$$

34. $\int_0^1 (-\ln x)\,dx = \lim_{b\to0^+}\left[x - x\ln x\right]_b^1 = [1 - 1\ln 1] - \lim_{b\to0^+}[b - b\ln b] = 1 - 0 + \lim_{b\to0^+}\frac{\ln b}{\left(\frac{1}{b}\right)} = 1 - \lim_{b\to0^+}\frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$

$$= 1 + \lim_{b\to0^+} b = 1 + 0 = 1$$

35. $\int_0^{\pi/2} \tan\theta\,d\theta = \lim_{b\to\frac{\pi}{2}^-}\left[-\ln|\cos\theta|\right]_0^b = \lim_{b\to\frac{\pi}{2}^-}\left[-\ln|\cos b|\right] + \ln 1 = \lim_{b\to\frac{\pi}{2}^-}\left[-\ln|\cos b|\right] = -\infty,$

the integral diverges

36. $\int_0^{\pi/2} \cot\theta\,d\theta = \lim_{b\to0^+}\left[\ln|\sin\theta|\right]_b^{\pi/2} = \ln 1 - \lim_{b\to0^+}\left[\ln|\sin b|\right] = -\lim_{b\to0^+}\left[\ln|\sin b|\right] = -\infty,$

the integral diverges

37. $\int_0^{\pi} \frac{\sin\theta\,d\theta}{\sqrt{\pi-\theta}}$; $[\pi - \theta = x] \Rightarrow -\int_{\pi}^{0} \frac{\sin x\,dx}{\sqrt{x}} = \int_0^{\pi}\frac{\sin x\,dx}{\sqrt{x}}$. Since $0 \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ for all $0 \le x \le \pi$ and $\int_0^{\pi}\frac{dx}{\sqrt{x}}$ converges, then $\int_0^{\pi}\frac{\sin x}{\sqrt{x}}\,dx$ converges by the Direct Comparison Test.

38. $\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{(\pi - 2\theta)^{1/3}}$; $\begin{bmatrix} x = \pi - 2\theta \\ \theta = \frac{\pi}{2} - \frac{x}{2} \\ d\theta = -\frac{dx}{2} \end{bmatrix} \Rightarrow \int_{2\pi}^{0} \frac{-\cos\left(\frac{\pi}{2} - \frac{x}{2}\right)dx}{2x^{1/3}} = \int_0^{2\pi}\frac{\sin\left(\frac{x}{2}\right)dx}{2x^{1/3}}$. Since $0 \le \frac{\sin\frac{x}{2}}{x^{1/3}} \le \frac{1}{x^{1/3}}$ for all

$0 \le x \le 2\pi$ and $\int_0^{2\pi} \frac{dx}{2x^{1/3}}$ converges, then $\int_0^{2\pi} \frac{\sin \frac{x}{2}\, dx}{2x^{1/3}}$ converges by the Direct Comparison Test.

39. $\int_0^{\ln 2} x^{-2} e^{-1/x}\, dx; \left[\frac{1}{x} = y\right] \Rightarrow \int_{\infty}^{1/\ln 2} \frac{y^2 e^{-y}\, dy}{-y^2} = \int_{1/\ln 2}^{\infty} e^{-y}\, dy = \lim\limits_{b\to\infty} \left[-e^{-y}\right]_{1/\ln 2}^{b} = \lim\limits_{b\to\infty} \left[-e^{-b}\right] - \left[-e^{-1/\ln 2}\right]$

$= 0 + e^{-1/\ln 2} = e^{-1/\ln 2}$, so the integral converges.

40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx; \left[y = \sqrt{x}\right] \Rightarrow 2 \int_0^1 e^{-y}\, dy = 2 - 2e$, so the integral converges.

41. $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$. Since for $0 \le t \le \pi$, $0 \le \frac{1}{\sqrt{t} + \sin t} \le \frac{1}{\sqrt{t}}$ and $\int_0^{\pi} \frac{dt}{\sqrt{t}}$ converges, then the original integral converges as well by the Direct Comparison Test.

42. $\int_0^1 \frac{dt}{t - \sin t}$; let $f(t) = \frac{1}{t - \sin t}$ and $g(t) = \frac{1}{t^3}$, then $\lim\limits_{t\to 0} \frac{f(t)}{g(t)} = \lim\limits_{t\to 0} \frac{t^3}{t - \sin t} = \lim\limits_{t\to 0} \frac{3t^2}{1 - \cos t} = \lim\limits_{t\to 0} \frac{6t}{\sin t}$

$= \lim\limits_{t\to 0} \frac{6}{\cos t} = 6$. Now, $\int_0^1 \frac{dt}{t^3} = \lim\limits_{b\to 0^+} \left[-\frac{1}{2t^2}\right]_b^1 = -\frac{1}{2} - \lim\limits_{b\to 0^+} \left[-\frac{1}{2b^2}\right] = +\infty$, which diverges $\Rightarrow \int_0^1 \frac{dt}{t - \sin t}$ diverges by the Limit Comparison Test.

43. $\int_0^2 \frac{dx}{1 - x^2} = \int_0^1 \frac{dx}{1 - x^2} + \int_1^2 \frac{dx}{1 - x^2}$ and $\int_0^1 \frac{dx}{1 - x^2} = \lim\limits_{b\to 1^-} \left[\frac{1}{2} \ln \left|\frac{1+x}{1-x}\right|\right]_0^b = \lim\limits_{b\to 1^-} \left[\frac{1}{2} \ln \left|\frac{1+b}{1-b}\right|\right] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1 - x^2}$ diverges as well.

44. $\int_0^2 \frac{dx}{1 - x} = \int_0^1 \frac{dx}{1 - x} + \int_1^2 \frac{dx}{1 - x}$ and $\int_0^1 \frac{dx}{1 - x} = \lim\limits_{b\to 1^-} \left[-\ln(1 - x)\right]_0^b = \lim\limits_{b\to 1^-} \left[-\ln(1 - b)\right] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1 - x}$ diverges as well.

45. $\int_{-1}^1 \ln|x|\, dx = \int_{-1}^0 \ln(-x)\, dx + \int_0^1 \ln x\, dx$; $\int_0^1 \ln x\, dx = \lim\limits_{b\to 0^+} \left[x \ln x - x\right]_b^1 = [1 \cdot 0 - 1] - \lim\limits_{b\to 0^+} [b \ln b - b]$

$= -1 - 0 = -1$; $\int_{-1}^0 \ln(-x)\, dx = -1 \Rightarrow \int_{-1}^1 \ln|x|\, dx = -2$ converges.

46. $\int_{-1}^{1} (-x \ln|x|)\,dx = \int_{-1}^{0} [-x \ln(-x)]\,dx + \int_{0}^{1} (-x \ln x)\,dx = \lim_{b\to 0^+} \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_b^1 - \lim_{c\to 0^+} \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_c^1$

$= \left[\frac{1}{2}\ln 1 - \frac{1}{4}\right] - \lim_{b\to 0^+} \left[\frac{b^2}{2}\ln b - \frac{b^2}{4}\right] - \left[\frac{1}{2}\ln 1 - \frac{1}{4}\right] + \lim_{c\to 0^+} \left[\frac{c^2}{2}\ln c - \frac{c^2}{4}\right] = -\frac{1}{4} - 0 + \frac{1}{4} + 0 = 0 \Rightarrow$ the integral converges (see Exercise 33 for the limit calculations).

47. $\int_{1}^{\infty} \frac{dx}{1+x^3}$; $0 \le \frac{1}{x^3+1} \le \frac{1}{x^3}$ for $1 \le x < \infty$ and $\int_{1}^{\infty} \frac{dx}{x^3}$ converges $\Rightarrow \int_{1}^{\infty} \frac{dx}{1+x^3}$ converges by the Direct Comparison Test.

48. $\int_{4}^{\infty} \frac{dx}{\sqrt{x}-1}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x}-1}\right)}{\left(\frac{1}{\sqrt{x}}\right)} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}-1} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{\sqrt{x}}} = \frac{1}{1-0} = 1$ and $\int_{4}^{\infty} \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \left[2\sqrt{x}\right]_4^b = \infty$, which diverges $\Rightarrow \int_{4}^{\infty} \frac{dx}{\sqrt{x}-1}$ diverges by the Limit Comparison Test.

49. $\int_{2}^{\infty} \frac{dv}{\sqrt{v-1}}$; $\lim_{v\to\infty} \frac{\left(\frac{1}{\sqrt{v-1}}\right)}{\left(\frac{1}{\sqrt{v}}\right)} = \lim_{v\to\infty} \frac{\sqrt{v}}{\sqrt{v-1}} = \lim_{v\to\infty} \frac{1}{\sqrt{1-\frac{1}{v}}} = \frac{1}{\sqrt{1-0}} = 1$ and $\int_{2}^{\infty} \frac{dv}{\sqrt{v}} = \lim_{b\to\infty} \left[2\sqrt{v}\right]_2^b = \infty$, which diverges $\Rightarrow \int_{2}^{\infty} \frac{dv}{\sqrt{v-1}}$ diverges by the Limit Comparison Test.

50. $\int_{4}^{\infty} \frac{2\,dt}{t^{3/2}+1}$; $\lim_{t\to\infty} \frac{t^{3/2}}{t^{3/2}+1} = 1$ and $\int_{4}^{\infty} \frac{2\,dt}{t^{3/2}} = \lim_{b\to\infty} \left[-4t^{-1/2}\right]_4^b = \lim_{b\to\infty} \left(\frac{-4}{\sqrt{b}} + 2\right) = 2 \Rightarrow \int_{4}^{\infty} \frac{2\,dt}{t^{3/2}}$ converges $\Rightarrow \int_{4}^{\infty} \frac{2\,dt}{t^{3/2}+1}$ converges by the Direct Comparison Test.

51. $\int_{0}^{\infty} \frac{dx}{\sqrt{x^6+1}} = \int_{0}^{1} \frac{dx}{\sqrt{x^6+1}} + \int_{1}^{\infty} \frac{dx}{\sqrt{x^6+1}} < \int_{0}^{1} \frac{dx}{\sqrt{x^6+1}} + \int_{1}^{\infty} \frac{dx}{x^3}$ and $\int_{1}^{\infty} \frac{dx}{x^3} = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_1^b$ $= \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2} \Rightarrow \int_{0}^{\infty} \frac{dx}{\sqrt{x^6+1}}$ converges by the Direct Comparison Test.

52. $\int_{2}^{\infty} \frac{dx}{\sqrt{x^2-1}}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$; $\int_{2}^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} [\ln b]_2^b = \infty$,

which diverges $\Rightarrow \int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges by the Limit Comparison Test.

53. $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$; $\lim\limits_{x\to\infty} \frac{\left(\frac{\sqrt{x}}{x^2}\right)}{\left(\frac{\sqrt{x+1}}{x^2}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+1}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}} = 1$; $\int_1^\infty \frac{\sqrt{x}}{x^2}\,dx = \int_1^\infty \frac{dx}{x^{3/2}}$

$= \lim\limits_{b\to\infty} \left[-2x^{-1/2}\right]_1^b = \lim\limits_{b\to\infty} \left(\frac{-2}{\sqrt{x}}+2\right) = 2 \Rightarrow \int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ converges by the Limit Comparison Test.

54. $\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{x}{\sqrt{x^4-1}}\right)}{\left(\frac{x}{\sqrt{x^4}}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{x^4}}{\sqrt{x^4-1}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^4}}} = 1$; $\int_2^\infty \frac{x\,dx}{\sqrt{x^4}} = \int_2^\infty \frac{dx}{x} = \lim\limits_{b\to\infty} [\ln x]_2^b = \infty$,

which diverges $\Rightarrow \int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$ diverges by the Limit Comparison Test.

55. $\int_\pi^\infty \frac{2+\cos x}{x}\,dx$; $0 < \frac{1}{x} \le \frac{2+\cos x}{x}$ for $x \ge \pi$ and $\int_\pi^\infty \frac{dx}{x} = \lim\limits_{b\to\infty} [\ln x]_\pi^b = \infty$, which diverges

$\Rightarrow \int_\pi^\infty \frac{2+\cos x}{x}\,dx$ diverges by the Direct Comparison Test.

56. $\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$; $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$ for $x \ge \pi$ and $\int_\pi^\infty \frac{2}{x^2}\,dx = \lim\limits_{b\to\infty} \left[-\frac{2}{x}\right]_\pi^b = \lim\limits_{b\to\infty} \left(-\frac{2}{b}+\frac{2}{\pi}\right) = \frac{2}{\pi}$

$\Rightarrow \int_\pi^\infty \frac{2\,dx}{x^2}$ converges $\Rightarrow \int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$ converges by the Direct Comparison Test.

57. $\int_0^\infty \frac{d\theta}{1+e^\theta}$; $0 \le \frac{1}{1+e^\theta} \le \frac{1}{e^\theta}$ for $0 \le \theta < \infty$ and $\int_0^\infty \frac{d\theta}{e^\theta} = \lim\limits_{b\to 0} \left[-e^{-\theta}\right]_0^b = \lim\limits_{b\to\infty} \left(-e^{-b}+1\right) = 1$

$\Rightarrow \int_0^\infty \frac{d\theta}{e^\theta}$ converges $\Rightarrow \int_0^\infty \frac{d\theta}{1+e^\theta}$ converges by the Direct Comparison Test.

58. $\int_2^\infty \frac{dx}{\ln x}$; $0 < \frac{1}{x} < \frac{1}{\ln x}$ for $x > 2$ and $\int_2^\infty \frac{dx}{x}$ diverges $\Rightarrow \int_2^\infty \frac{dx}{\ln x}$ diverges by the Direct Comparison Test.

59. $\int_1^\infty \frac{e^x}{x}\,dx$; $0 < \frac{1}{x} < \frac{e^x}{x}$ for $x > 1$ and $\int_1^\infty \frac{dx}{x}$ diverges $\Rightarrow \int_1^\infty \frac{e^x\,dx}{x}$ diverges by the Direct Comparison Test.

60. $\int_{e^e}^{\infty} \ln(\ln x)\, dx$; $[x = e^y] \to \int_e^{\infty} (\ln y)\, e^y\, dy$; $0 < \ln y < (\ln y)\, e^y$ for $y \ge e$ and $\int_e^{\infty} \ln y\, dy = \lim\limits_{b\to\infty} [y \ln y - y]_e^b$ $= \infty$, which diverges $\Rightarrow \int_e^{\infty} \ln e^y\, dy$ diverges $\Rightarrow \int_{e^e}^{\infty} \ln(\ln x)\, dx$ diverges by the Direct Comparison Test.

61. $\int_1^{\infty} \frac{dx}{\sqrt{e^x - x}}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{\sqrt{e^x - x}}\right)}{\left(\frac{1}{\sqrt{e^x}}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{e^x}}{\sqrt{e^x - x}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1 - \frac{x}{e^x}}} = \frac{1}{\sqrt{1-0}} = 1$; $\int_1^{\infty} \frac{dx}{\sqrt{e^x}} = \int_1^{\infty} e^{-x/2}\, dx$ $= \lim\limits_{b\to\infty} \left[-2e^{-x/2}\right]_1^b = \lim\limits_{b\to\infty} \left(-2e^{-b/2} + 2e^{-1/2}\right) = \frac{2}{\sqrt{e}} \Rightarrow \int_1^{\infty} e^{-x/2}\, dx$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{\sqrt{e^x - x}}$ converges by the Limit Comparison Test.

62. $\int_1^{\infty} \frac{dx}{e^x - 2^x}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{e^x - 2^x}\right)}{\left(\frac{1}{e^x}\right)} = \lim\limits_{x\to\infty} \frac{e^x}{e^x - 2^x} = \lim\limits_{x\to\infty} \frac{1}{1 - \left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = 1$ and $\int_1^{\infty} \frac{dx}{e^x} = \lim\limits_{b\to\infty} \left[-e^{-x}\right]_1^b$ $= \lim\limits_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = \frac{1}{e} \Rightarrow \int_1^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{e^x - 2^x}$ converges by the Limit Comparison Test.

63. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}} = 2\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}}$; $\lim\limits_{x\to\infty} \frac{x^2}{\sqrt{x^4+1}} = 1$; $\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}} = \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{\infty} \frac{dx}{\sqrt{x^4+1}}$ $< \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{\infty} \frac{dx}{x^2}$ and $\int_1^{\infty} \frac{dx}{x^2} = \lim\limits_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim\limits_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$ converges by the Direct Comparison Test.

64. $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = 2\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$; $0 < \frac{1}{e^x + e^{-x}} < \frac{1}{e^x}$ for $x > 0$; $\int_0^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow 2\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$ converges by the Direct Comparison Test.

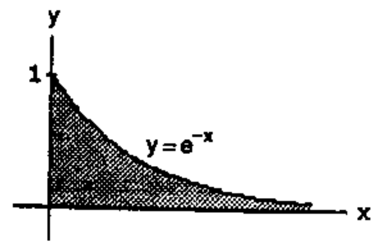
65. (a) $\int_1^2 \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \to \int_0^{\ln 2} \frac{dt}{t^p} = \lim\limits_{b\to 0^+} \left[\frac{1}{-p+1} t^{1-p}\right]_b^{\ln 2} = \lim\limits_{b\to 0^+} \frac{b^{1-p}}{p-1} + \frac{1}{1-p}(\ln 2)^{1-p}$

$\Rightarrow$ the integral converges for $p < 1$ and diverges for $p \ge 1$

(b) $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \to \int_{\ln 2}^{\infty} \frac{dt}{t^p}$ and this integral is essentially the same as in Exercise 67(a): it converges for $p > 1$ and diverges for $p \le 1$

66. $\int_0^{\infty} \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \left[\ln(x^2+1)\right]_0^b = \lim_{b\to\infty} \left[\ln(b^2+1)\right] - 0 = \lim_{b\to\infty} \ln(b^2+1) = \infty \Rightarrow$ the integral $\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\,dx$ diverges. But $\lim_{b\to\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \left[\ln(x^2+1)\right]_{-b}^{b} = \lim_{b\to\infty} \left[\ln(b^2+1) - \ln(b^2+1)\right] = \lim_{b\to\infty} \ln\left(\frac{b^2+1}{b^2+1}\right)$ $= \lim_{b\to\infty} (\ln 1) = 0$

67. $A = \int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_0^b = \lim_{b\to\infty} \left(-e^{-b}\right) - \left(-e^{-0}\right)$ $= 0 + 1 = 1$



68. $V = \int_0^{\infty} 2\pi x e^{-x}\,dx = 2\pi \int_0^{\infty} x e^{-x}\,dx = 2\pi \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^b = 2\pi\left[\lim_{b\to\infty} \left(-be^{-b} - e^{-b}\right) - 1\right] = 2\pi$

69. $V = \int_0^{\infty} \pi\left(e^{-x}\right)^2 dx = \pi \int_0^{\infty} e^{-2x}\,dx = \pi \lim_{b\to\infty} \left[-\frac{1}{2}e^{-2x}\right]_0^b = \pi \lim_{b\to\infty} \left(-\frac{1}{2}e^{-2b} + \frac{1}{2}\right) = \frac{\pi}{2}$

70. $A = \int_0^{\pi/2} (\sec x - \tan x)\,dx = \lim_{b\to\frac{\pi}{2}^-} \left[\ln|\sec x + \tan x| - \ln|\sec x|\right]_0^b = \lim_{b\to\frac{\pi}{2}^-} \left(\ln\left|1 + \frac{\tan b}{\sec b}\right| - \ln|1+0|\right)$ $= \lim_{b\to\frac{\pi}{2}^-} \ln|1 + \sin b| = \ln 2$

71. $\int_3^{\infty} \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \neq \int_3^{\infty} \frac{dx}{x-2} - \int_3^{\infty} \frac{dx}{x}$, since the left hand integral converges but both of the right hand integrals diverge.

72. (a) The statement is true since $\int_{-\infty}^{b} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^b f(x)\,dx$, $\int_b^{\infty} f(x)\,dx = \int_a^{\infty} f(x)\,dx - \int_a^b f(x)\,dx$

and $\int_a^b f(x)\,dx$ exists since $f(x)$ is integrable on every interval $[a, b]$.

(b) $\int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^b f(x)\,dx - \int_a^b f(x)\,dx + \int_a^{\infty} f(x)\,dx$

$= \int_{-\infty}^{b} f(x)\,dx + \int_b^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_b^{\infty} f(x)\,dx$

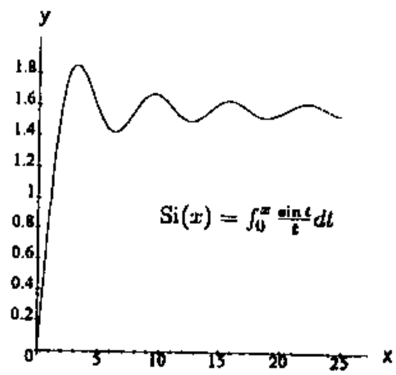
73. (a) $\int_1^{\infty} e^{-3x}\,dx = \lim_{b\to\infty} \left[-\frac{1}{3}e^{-3x}\right]_3^b = \lim_{b\to\infty} \left(-\frac{1}{3}e^{-3b}\right) - \left(-\frac{1}{3}e^{-3\cdot 3}\right) = 0 + \frac{1}{3}\cdot e^{-9} = \frac{1}{3}e^{-9}$

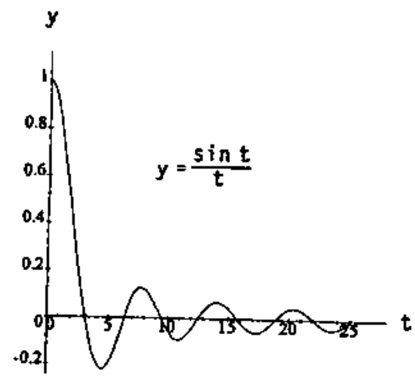
$\approx 0.0000411 < 0.000042$. Since $e^{-x^2} \le e^{-3x}$ for $x > 3$, then $\int_3^{\infty} e^{-x^2}\,dx < 0.000042$ and therefore

$\int_0^{\infty} e^{-x^2}\,dx$ can be replaced by $\int_0^{3} e^{-x^2}\,dx$ without introducing an error greater than 0.000042.

(b) $\int_0^{3} e^{-x^2}\,dx \cong 0.88621$

74. (a)

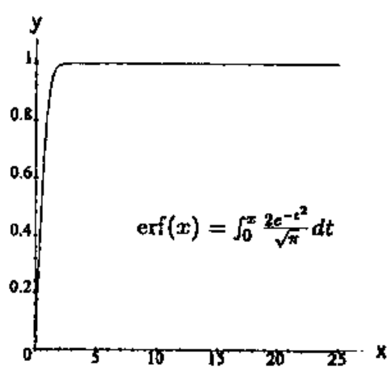




(b) Maple command:

> int((sin(t))/t, t=0..infinity); $\left(\text{answer is } \frac{\pi}{2}\right)$

75. (a)



(b) Maple commands:

> f:= 2*exp(-t^2)/sqrt(Pi);
> int(f, t=0..infinity); (answer is 1)

76. (a) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

[-3, 3] by [0, 0.5]

f is increasing on $(-\infty, 0]$. f is decreasing on $[0, \infty)$. f has a local maximum at $(0, f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)$

(b) Maple commands:

>f:=exp(-x^2/2)(sqrt(2*pi);
>int(f,x=-1..1); ≈ 0.683
>int(f,x=-2..2); ≈ 0.954
>int(f,x=-3..3); ≈ 0.997

(c) Part (b) suggests that as b increases, the integral approaches 1. We can make $\int_{-b}^{b} f(x)\,dx$ as close to 1 as we want by choosing $b > 1$ large enough. Also, we can make $\int_{b}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{-b} f(x)\,dx$ as small as we want by choosing b large enough enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$.) Thus, $\int_{b}^{\infty} f(x)\,dx < \int_{b}^{\infty} e^{-x/2}\,dx$.

$$\int_{b}^{\infty} e^{-x/2}\,dx = \lim_{c\to\infty} \int_{b}^{c} e^{-x/2}\,dx = \lim_{c\to\infty} \left[-2e^{-x/2}\right]_{b}^{c} = \lim_{c\to\infty} \left[-2e^{-c/2} + 2e^{-b/2}\right] = 2e^{-b/2}$$

As $b \to \infty$, $2e^{-b/2} \to 0$, for for large enough b, $\int_{b}^{\infty} f(x)\,dx$ is as small as we want. Likewise, for large enough b, $\int_{-\infty}^{-b} f(x)\,dx$ is as small as we want.

77-80. Use the MAPLE or MATHEMATICA integration commands, as discussed in the text.

CHAPTER 7 PRACTICE EXERCISES

1. $\int x\sqrt{4x^2-9}\,dx;\ \begin{bmatrix} u = 4x^2-9 \\ du = 8x\,dx \end{bmatrix} \Rightarrow \frac{1}{8}\int \sqrt{u}\,du = \frac{1}{8}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{12}\left(4x^2-9\right)^{3/2} + C$

2. $\int x(2x+1)^{1/2}\,dx;\ \begin{bmatrix} u = 2x+1 \\ du = 2\,dx \end{bmatrix} \Rightarrow \frac{1}{2}\int \left(\frac{u-1}{2}\right)\sqrt{u}\,du = \frac{1}{4}\left(\int u^{3/2}\,du - \int u^{1/2}\,du\right) = \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$

$= \frac{(2x+1)^{5/2}}{10} - \frac{(2x+1)^{3/2}}{6} + C$

3. $\int \frac{x\,dx}{\sqrt{8x^2+1}};\ \begin{bmatrix} u = 8x^2+1 \\ du = 16x\,dx \end{bmatrix} \Rightarrow \frac{1}{16}\int \frac{du}{\sqrt{u}} = \frac{1}{16}\cdot 2u^{1/2} + C = \frac{\sqrt{8x^2+1}}{8} + C$

4. $\int \frac{y\,dy}{25+y^2};\ \begin{bmatrix} u = 25+y^2 \\ du = 2y\,dy \end{bmatrix} \Rightarrow \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(25+y^2\right) + C$

5. $\int \frac{t^3\,dt}{\sqrt{9-4t^4}};\ \begin{bmatrix} u = 9-4t^4 \\ du = -16t^3\,dt \end{bmatrix} \Rightarrow -\frac{1}{16}\int \frac{du}{\sqrt{u}} = -\frac{1}{16}\cdot 2u^{1/2} + C = -\frac{\sqrt{9-4t^4}}{8} + C$

6. $\int z^{2/3}\left(z^{5/3}+1\right)^{2/3}\,dz;\ \begin{bmatrix} u = z^{5/3}+1 \\ du = \frac{5}{3}z^{2/3}\,dz \end{bmatrix} \Rightarrow \frac{3}{5}\int u^{2/3}\,du = \frac{3}{5}\cdot\frac{3}{5}u^{5/3} + C = \frac{9}{25}\left(z^{5/3}+1\right)^{5/3} + C$

7. $\int \frac{\sin 2\theta\,d\theta}{(1-\cos 2\theta)^2};\ \begin{bmatrix} u = 1-\cos 2\theta \\ du = 2\sin 2\theta\,d\theta \end{bmatrix} \Rightarrow \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u} + C = -\frac{1}{2(1-\cos 2\theta)} + C$

8. $\int \frac{\cos 2t\,dt}{1+\sin 2t};\ \begin{bmatrix} u = 1+\sin 2t \\ du = 2\cos 2t\,dt \end{bmatrix} \Rightarrow \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+\sin 2t| + C$

9. $\int (\sin 2x)\,e^{\cos 2x}\,dx;\ \begin{bmatrix} u = \cos 2x \\ du = -2\sin 2x\,dx \end{bmatrix} \Rightarrow -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{\cos 2x} + C$

10. $\int e^\theta \sec^2\left(e^\theta\right)d\theta;\ \begin{bmatrix} u = e^\theta \\ du = e^\theta\,d\theta \end{bmatrix} \Rightarrow \int \sec^2 u\,du = \tan u + C = \tan\left(e^\theta\right) + C$

11. $\int 2^{x-1}\,dx = \frac{2^{x-1}}{\ln 2} + C$

12. $\int \frac{dv}{v\ln v};\ \begin{bmatrix} u = \ln v \\ du = \frac{1}{v}\,dv \end{bmatrix} \Rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|\ln v| + C$

13. $\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)}$; $\begin{bmatrix} u = 2+\tan^{-1}x \\ du = \frac{dx}{x^2+1} \end{bmatrix} \Rightarrow \int \frac{du}{u} = \ln|u| + C = \ln\left|2+\tan^{-1}x\right| + C$

14. $\int \frac{2\,dx}{\sqrt{1-4x^2}}$; $\begin{bmatrix} u = 2x \\ du = 2\,dx \end{bmatrix} \Rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(2x) + C$

15. $\int \frac{dt}{\sqrt{16-9t^2}} = \frac{1}{4}\int \frac{dt}{\sqrt{1-\left(\frac{3t}{4}\right)^2}}$; $\begin{bmatrix} u = \frac{3}{4}t \\ du = \frac{3}{4}\,dt \end{bmatrix} \Rightarrow \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1}u + C = \frac{1}{3}\sin^{-1}\left(\frac{3t}{4}\right) + C$

16. $\int \frac{dt}{9+t^2} = \frac{1}{9}\int \frac{dt}{1+\left(\frac{t}{3}\right)^2}$; $\begin{bmatrix} u = \frac{1}{3}t \\ du = \frac{1}{3}dt \end{bmatrix} \Rightarrow \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}u + C = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$

17. $\int \frac{4\,dx}{5x\sqrt{25x^2-16}} = \frac{4}{25}\int \frac{dx}{x\sqrt{x^2-\frac{16}{25}}} = \frac{1}{5}\sec^{-1}\left|\frac{5x}{4}\right| + C$

18. $\int \frac{dx}{\sqrt{4x-x^2-3}} = \int \frac{d(x-2)}{\sqrt{1-(x-2)^2}} = \sin^{-1}(x-2) + C$

19. $\int \frac{dy}{y^2-4y+8} = \int \frac{d(y-2)}{(y-2)^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{y-2}{2}\right) + C$

20. $\int \frac{dv}{(v+1)\sqrt{v^2+2v}} = \int \frac{d(v+1)}{(v+1)\sqrt{(v+1)^2-1}} = \sec^{-1}|v+1| + C$

21. $\int \cos^2 3x\,dx = \int \frac{1+\cos 6x}{2}\,dx = \frac{x}{2} + \frac{\sin 6x}{12} + C$

22. $\int \sin^3\frac{\theta}{2}\,d\theta = \int \left(1-\cos^2\frac{\theta}{2}\right)\left(\sin\frac{\theta}{2}\right)d\theta$; $\begin{bmatrix} u = \cos\frac{\theta}{2} \\ du = -\frac{1}{2}\sin\frac{\theta}{2}\,d\theta \end{bmatrix} \Rightarrow -2\int \left(1-u^2\right)du = \frac{2u^3}{3} - 2u + C$

$= \frac{2}{3}\cos^3\frac{\theta}{2} - 2\cos\frac{\theta}{2} + C$

23. $\int \tan^3 2t\, dt = \int (\tan 2t)(\sec^2 2t - 1)\, dt = \int \tan 2t \sec^2 2t\, dt - \int \tan 2t\, dt; \begin{bmatrix} u = 2t \\ du = 2\, dt \end{bmatrix}$

$\Rightarrow \frac{1}{2}\int \tan u \sec^2 u\, du - \frac{1}{2}\int \tan u\, du = \frac{1}{4}\tan^2 u + \frac{1}{2}\ln|\cos u| + C = \frac{1}{4}\tan^2 2t + \frac{1}{2}\ln|\cos 2t| + C$

$= \frac{1}{4}\tan^2 2t - \frac{1}{2}\ln|\sec 2t| + C$

24. $\int \frac{dx}{2\sin x\cos x} = \int \frac{dx}{\sin 2x} = \int \csc 2x\, dx = -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$

25. $\int \frac{2\, dx}{\cos^2 x - \sin^2 x} = \int \frac{2\, dx}{\cos 2x}; \begin{bmatrix} u = 2x \\ du = 2\, dx \end{bmatrix} \Rightarrow \int \frac{du}{\cos u} = \int \sec u\, du = \ln|\sec u + \tan u| + C$

$= \ln|\sec 2x + \tan 2x| + C$

26. $\int_{\pi/4}^{\pi/2} \sqrt{\csc^2 y - 1}\, dy = \int_{\pi/4}^{\pi/2} \cot y\, dy = [\ln|\sin y|]_{\pi/4}^{\pi/2} = \ln 1 - \ln \frac{1}{\sqrt{2}} = \ln\sqrt{2}$

27. $\int_{\pi/4}^{3\pi/4} \sqrt{\cot^2 t + 1}\, dt = \int_{\pi/4}^{3\pi/4} \csc t\, dt = [-\ln|\csc t + \cot t|]_{\pi/4}^{3\pi/4} = -\ln\left|\csc\frac{3\pi}{4} + \cot\frac{3\pi}{4}\right| + \ln\left|\csc\frac{\pi}{4} + \cot\frac{\pi}{4}\right|$

$= -\ln\left|\sqrt{2} - 1\right| + \ln\left|\sqrt{2} + 1\right| = \ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| = \ln\left|\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{2-1}\right| = \ln(3 + 2\sqrt{2})$

28. $\int_0^{2\pi} \sqrt{1 - \sin^2 \frac{x}{2}}\, dx = \int_0^{2\pi} \left|\cos\frac{x}{2}\right| dx = \int_0^{\pi} \cos\frac{x}{2}\, dx - \int_{\pi}^{2\pi} \cos\frac{x}{2}\, dx = \left[2\sin\frac{x}{2}\right]_0^{\pi} - \left[2\sin\frac{x}{2}\right]_{\pi}^{2\pi}$

$= (2 - 0) - (0 - 2) = 4$

29. $\int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos 2t}\, dt = \sqrt{2}\int_{-\pi/2}^{\pi/2} |\sin t|\, dt = 2\sqrt{2}\int_0^{\pi/2} \sin t\, dt = [-2\sqrt{2}\cos t]_0^{\pi/2} = 2\sqrt{2}\,[0 - (-1)] = 2\sqrt{2}$

30. $\int_{\pi}^{2\pi} \sqrt{1 + \cos 2t}\, dt = \sqrt{2}\int_{\pi}^{2\pi} |\cos t|\, dt = -\sqrt{2}\int_{\pi}^{3\pi/2} \cos t\, dt + \sqrt{2}\int_{3\pi/2}^{2\pi} \cos t\, dt$

$= -\sqrt{2}\,[\sin t]_{\pi}^{3\pi/2} + \sqrt{2}\,[\sin t]_{3\pi/2}^{2\pi} = -\sqrt{2}\,(-1 - 0) + \sqrt{2}\,[0 - (-1)] = 2\sqrt{2}$

31. $\int \frac{x^2\, dx}{x^2 + 4} = x - \int \frac{4\, dx}{x^2 + 4} = x - 2\tan^{-1}\left(\frac{x}{2}\right) + C$

32. $\int \frac{x^3\, dx}{9 + x^2} = \int \left[\frac{x(x^2 + 9) - 9x}{x^2 + 9}\right] dx = \int \left(x - \frac{9x}{x^2 + 9}\right) dx = \frac{x^2}{2} - \frac{9}{2}\ln(9 + x^2) + C$

33. $\int \frac{2y-1}{y^2+4}\,dy = \int \frac{2y\,dy}{y^2+4} - \int \frac{dy}{y^2+4} = \ln(y^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) + C$

34. $\int \frac{y+4}{y^2+1}\,dy = \int \frac{y\,dy}{y^2+1} + 4\int \frac{dy}{y^2+1} = \frac{1}{2}\ln(y^2+1) + 4\tan^{-1}y + C$

35. $\int \frac{t+2}{\sqrt{4-t^2}}\,dt = \int \frac{t\,dt}{\sqrt{4-t^2}} + 2\int \frac{dt}{\sqrt{4-t^2}} = -\sqrt{4-t^2} + 2\sin^{-1}\left(\frac{t}{2}\right) + C$

36. $\int \frac{2t^2+\sqrt{1-t^2}}{t\sqrt{1-t^2}}\,dt = \int \frac{2t\,dt}{\sqrt{1-t^2}} + \int \frac{dt}{t} = -2\sqrt{1-t^2} + \ln|t| + C$

37. $\int \frac{\tan x\,dx}{\tan x + \sec x} = \int \frac{\sin x\,dx}{\sin x + 1} = \int \frac{(\sin x)(1-\sin x)}{1-\sin^2 x}\,dx = \int \frac{\sin x - 1 + \cos^2 x}{\cos^2 x}\,dx$

$= -\int \frac{d(\cos x)}{\cos^2 x} - \int \frac{dx}{\cos^2 x} + \int dx = \frac{1}{\cos x} - \tan x + x + C = x - \tan x + \sec x + C$

38. $\int x\csc(x^2+3)\,dx = \frac{1}{2}\int \csc(x^2+3)\,d(x^2+3) = -\frac{1}{2}\ln\left|\csc(x^2+3)+\cot(x^2+3)\right| + C$

39. $\int \cot\left(\frac{x}{4}\right)dx = 4\int \cot\left(\frac{x}{4}\right)d\left(\frac{x}{4}\right) = 4\ln\left|\sin\left(\frac{x}{4}\right)\right| + C$

40. $\int x\sqrt{1-x}\,dx; \begin{bmatrix} u = 1-x \\ du = -dx \end{bmatrix} \Rightarrow -\int (1-u)\sqrt{u}\,du = \int \left(u^{3/2}-u^{1/2}\right)du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C = -2\left[\frac{\left(\sqrt{1-x}\right)^3}{3} - \frac{\left(\sqrt{1-x}\right)^5}{5}\right] + C$

41. $\int \left(16+z^2\right)^{-3/2}dz; \begin{bmatrix} z = 4\tan\theta \\ dz = 4\sec^2\theta\,d\theta \end{bmatrix} \Rightarrow \int \frac{4\sec^2\theta\,d\theta}{64\sec^3\theta\,d\theta} = \frac{1}{16}\int \cos\theta\,d\theta = \frac{1}{16}\sin\theta + C = \frac{z}{16\sqrt{16+z^2}} + C$

$= \frac{z}{16\left(16+z^2\right)^{1/2}} + C$

42. $\int \frac{dy}{\sqrt{25+y^2}} = \frac{1}{5}\int \frac{dy}{\sqrt{1+\left(\frac{y}{5}\right)^2}} = \int \frac{du}{\sqrt{1+u^2}}, \left[u = \frac{y}{5}\right]; \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\,d\theta \end{bmatrix} \Rightarrow \int \frac{\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}} = \int \sec\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\sqrt{1+u^2}+u\right| + C_1 = \ln\left|\sqrt{1+\left(\frac{y}{5}\right)^2} + \frac{y}{5}\right| + C_1 = \ln\left|\frac{\sqrt{25+y^2}+y}{5}\right| + C_1$

$= \ln\left|y + \sqrt{25+y^2}\right| + C$

43. $\int \frac{dx}{x^2\sqrt{1-x^2}}; \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} \Rightarrow \int \frac{\cos\theta\, d\theta}{\sin^2\theta\cos\theta} = \int \csc^2\theta\, d\theta = -\cot\theta + C = -\frac{\cos\theta}{\sin\theta} + C = \frac{-\sqrt{1-x^2}}{x} + C$

44. $\int \frac{x^2\, dx}{\sqrt{1-x^2}}; \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} \Rightarrow \int \frac{\sin^2\theta\cos\theta\, d\theta}{\cos\theta} = \int \sin^2\theta\, d\theta = \int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$

$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta = \frac{\sin^{-1} x}{2} - \frac{x\sqrt{1-x^2}}{2} + C$

45. $\int \frac{dx}{\sqrt{x^2-9}}; \begin{bmatrix} x = 3\sec\theta \\ dx = 3\sec\theta\tan\theta\, d\theta \end{bmatrix} \Rightarrow \int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta - 9}} = \int \frac{3\sec\theta\tan\theta\, d\theta}{3\tan\theta} = \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\frac{x}{3} + \sqrt{\left(\frac{x}{3}\right)^2 - 1}\right| + C_1 = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C_1 = \ln\left|x + \sqrt{x^2-9}\right| + C$

46. $\int \frac{12\, dx}{(x^2-1)^{3/2}}; \begin{bmatrix} x = \sec\theta \\ dx = \sec\theta\tan\theta\, d\theta \end{bmatrix} \Rightarrow \int \frac{12\sec\theta\tan\theta\, d\theta}{\tan^3\theta} = \int \frac{12\cos\theta\, d\theta}{\sin^2\theta}; \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \to \int \frac{12\, du}{u^2}$

$= -\frac{12}{u} + C = -\frac{12}{\sin\theta} + C = -\frac{12\sec\theta}{\tan\theta} + C = -\frac{12x}{\sqrt{x^2-1}} + C$

47. $u = \ln(x+1)$, $du = \frac{dx}{x+1}$; $dv = dx$, $v = x$;

$\int \ln(x+1)\, dx = x\ln(x+1) - \int \frac{x}{x+1}\, dx = x\ln(x+1) - \int dx + \int \frac{dx}{x+1} = x\ln(x+1) - x + \ln(x+1) + C_1$

$= (x+1)\ln(x+1) - x + C_1 = (x+1)\ln(x+1) - (x+1) + C$, where $C = C_1 + 1$

48. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x^2\, dx$, $v = \frac{1}{3}x^3$;

$\int x^2 \ln x\, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3\left(\frac{1}{x}\right) dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

49. $u = \tan^{-1} 3x$, $du = \frac{3\, dx}{1+9x^2}$; $dv = dx$, $v = x$;

$\int \tan^{-1} 3x\, dx = x\tan^{-1} 3x - \int \frac{3x\, dx}{1+9x^2}; \begin{bmatrix} y = 1+9x^2 \\ dy = 18x\, dx \end{bmatrix} \Rightarrow x\tan^{-1} 3x - \frac{1}{6}\int \frac{dy}{y}$

$= x\tan^{-1}(3x) - \frac{1}{6}\ln(1+9x^2) + C$

50. $u = \cos^{-1}\left(\frac{x}{2}\right)$, $du = \frac{-dx}{\sqrt{4-x^2}}$; $dv = dx$, $v = x$;

$\int \cos^{-1}\left(\frac{x}{2}\right) dx = x\cos^{-1}\left(\frac{x}{2}\right) + \int \frac{x\, dx}{\sqrt{4-x^2}}; \begin{bmatrix} y = 4-x^2 \\ dy = -2x\, dx \end{bmatrix} \Rightarrow x\cos^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\int \frac{dy}{\sqrt{y}}$

$= x\cos^{-1}\left(\frac{x}{2}\right) - \sqrt{4-x^2} + C = x\cos^{-1}\left(\frac{x}{2}\right) - 2\sqrt{1-\left(\frac{x}{2}\right)^2} + C$

51.
$$\begin{array}{lll} & & e^x \\ (x+1)^2 & \xrightarrow{(+)} & e^x \\ 2(x+1) & \xrightarrow{(-)} & e^x \\ 2 & \xrightarrow{(+)} & e^x \\ 0 & & \end{array} \Rightarrow \int (x+1)^2 e^x\,dx = \left[(x+1)^2 - 2(x+1) + 2\right]e^x + C$$

52.
$$\begin{array}{lll} & & \sin(1-x) \\ x^2 & \xrightarrow{(+)} & \cos(1-x) \\ 2x & \xrightarrow{(-)} & -\sin(1-x) \\ 2 & \xrightarrow{(+)} & -\cos(1-x) \\ 0 & & \end{array} \Rightarrow \int x^2 \sin(1-x)\,dx = x^2\cos(1-x) + 2x\sin(1-x) - 2\cos(1-x) + C$$

53. $u = \cos 2x$, $du = -2\sin 2x\,dx$; $dv = e^x\,dx$, $v = e^x$;

$I = \int e^x \cos 2x\,dx = e^x \cos 2x + 2\int e^x \sin 2x\,dx$;

$u = \sin 2x$, $du = 2\cos 2x\,dx$; $dv = e^x\,dx$, $v = e^x$;

$I = e^x\cos 2x + 2\left[e^x \sin 2x - 2\int e^x\cos 2x\,dx\right] = e^x\cos 2x + 2e^x \sin 2x - 4I \Rightarrow I = \frac{e^x\cos 2x}{5} + \frac{2e^x \sin 2x}{5} + C$

54. $u = \sin 3x$, $du = 3\cos 3x\,dx$; $dv = e^{-2x}\,dx$, $v = -\frac{1}{2}e^{-2x}$;

$I = \int e^{-2x}\sin 3x\,dx = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\int e^{-2x}\cos 3x\,dx$;

$u = \cos 3x$, $du = -3\sin 3x\,dx$; $dv = e^{-2x}\,dx$, $v = -\frac{1}{2}e^{-2x}$;

$I = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\left[-\frac{1}{2}e^{-2x}\cos 3x - \frac{3}{2}\int e^{-2x}\sin 3x\,dx\right] = -\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x - \frac{9}{4}I$

$\Rightarrow I = \frac{4}{13}\left(-\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x\right) + C = -\frac{2}{13}e^{-2x}\sin 3x - \frac{3}{13}e^{-2x}\cos 3x + C$

55. $\int \frac{x\,dx}{x^2-3x+2} = \int \frac{2\,dx}{x-2} - \int \frac{dx}{x-1} = 2\ln|x-2| - \ln|x-1| + C$

56. $\int \frac{dx}{x(x+1)^2} = \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{x}{(x+1)^2}\right)dx = \ln|x| - 2\ln|x+1| + \left(\ln|x+1| + \frac{1}{x+1}\right) + C$

$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + C$

57. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$; $[\cos\theta = y] \Rightarrow -\int \frac{dy}{y^2+y-2} = -\frac{1}{3}\int \frac{dy}{y-1} + \frac{1}{3}\int \frac{dy}{y+2} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C$

$= \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

58. $\int \frac{3x^2+4x+4}{x^3+x}\,dx = \int \frac{4}{x}\,dx - \int \frac{x-4}{x^2+1}\,dx = 4\ln|x| - \frac{1}{2}\ln(x^2+1) + 4\tan^{-1}x + C$

59. $\int \frac{(v+3)\,dv}{2v^3-8v} = \frac{1}{2}\int\left(-\frac{3}{4v} + \frac{5}{8(v-2)} + \frac{1}{8(v+2)}\right)dv = -\frac{3}{8}\ln|v| + \frac{5}{16}\ln|v-2| + \frac{1}{16}\ln|v+2| + C$

$= \frac{1}{16}\ln\left|\frac{(v-2)^5(v+2)}{v^6}\right| + C$

60. $\int \frac{dt}{t^4+4t^2+3} = \frac{1}{2}\int \frac{dt}{t^2+1} - \frac{1}{2}\int \frac{dt}{t^2+3} = \frac{1}{2}\tan^{-1}t - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{1}{2}\tan^{-1}t - \frac{\sqrt{3}}{6}\tan^{-1}\frac{t}{\sqrt{3}} + C$

61. $\int \frac{x^3+x^2}{x^2+x-2}\,dx = \int\left(x + \frac{2x}{x^2+x-2}\right)dx = \int x\,dx + \frac{2}{3}\int\frac{dx}{x-1} + \frac{4}{3}\int\frac{dx}{x+2}$

$= \frac{x^2}{2} + \frac{4}{3}\ln|x+2| + \frac{2}{3}\ln|x-1| + C$

62. $\int \frac{x^3+4x^2}{x^2+4x+3}\,dx = \int\left(x - \frac{3x}{x^2+4x+3}\right)dx = \int x\,dx + \frac{3}{2}\int\frac{dx}{x+1} - \frac{9}{2}\int\frac{dx}{x+3}$

$= \frac{x^2}{2} - \frac{9}{2}\ln|x+3| + \frac{3}{2}\ln|x+1| + C$

63. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx = \int\left[(2x-3) + \frac{x}{x^2+2x-8}\right]dx = \int (2x-3)\,dx + \frac{1}{3}\int\frac{dx}{x-2} + \frac{2}{3}\int\frac{dx}{x+4}$

$= x^2 - 3x + \frac{2}{3}\ln|x+4| + \frac{1}{3}\ln|x-2| + C$

64. $\int \frac{dx}{x(3\sqrt{x+1})}; \begin{bmatrix} u = \sqrt{x+1} \\ du = \frac{dx}{2\sqrt{x+1}} \\ dx = 2u\,du \end{bmatrix} \to \frac{2}{3}\int\frac{u\,du}{(u^2-1)u} = \frac{1}{3}\int\frac{du}{u-1} - \frac{1}{3}\int\frac{du}{u+1} = \frac{1}{3}\ln|u-1| - \frac{1}{3}\ln|u+1| + C$

$= \frac{1}{3}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

65. $\int \frac{ds}{e^s-1}; \begin{bmatrix} u = e^s - 1 \\ du = e^s\,ds \\ ds = \frac{du}{u+1} \end{bmatrix} \to \int\frac{du}{u(u-1)} = \int\frac{du}{u-1} - \int\frac{du}{u} = \ln\left|\frac{u-1}{u}\right| + C = \ln\left|\frac{e^s-1}{e^s}\right| + C = \ln|1-e^{-s}| + C$

66. $\int \frac{ds}{\sqrt{e^s+1}}$; $\begin{bmatrix} u = \sqrt{e^s+1} \\ du = \frac{e^s\,ds}{2\sqrt{e^s+1}} \\ ds = \frac{2u\,du}{u^2-1} \end{bmatrix} \Rightarrow \int \frac{2u\,du}{u(u^2-1)} = 2\int \frac{du}{(u+1)(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u+1} = \ln\left|\frac{u-1}{u+1}\right| + C$

$= \ln\left|\frac{\sqrt{e^s+1}-1}{\sqrt{e^s+1}+1}\right| + C$

67. (a) $\int \frac{y\,dy}{\sqrt{16-y^2}} = -\frac{1}{2}\int \frac{d(16-y^2)}{\sqrt{16-y^2}} = -\sqrt{16-y^2} + C$

(b) $\int \frac{y\,dy}{\sqrt{16-y^2}}$; $[y = 4\sin x] \Rightarrow 4\int \frac{\sin x\cos x\,dx}{\cos x} = -4\cos x + C = -\frac{4\sqrt{16-y^2}}{4} + C = -\sqrt{16-y^2} + C$

68. (a) $\int \frac{x\,dx}{\sqrt{4+x^2}} = \frac{1}{2}\int \frac{d(4+x^2)}{\sqrt{4+x^2}} = \sqrt{4+x^2} + C$

(b) $\int \frac{x\,dx}{\sqrt{4+x^2}}$; $[x = 2\tan y] \Rightarrow \int \frac{2\tan y \cdot 2\sec^2 y\,dy}{2\sec y} = 2\int \sec y\tan y\,dy = 2\sec y + C = \sqrt{4+x^2} + C$

69. (a) $\int \frac{x\,dx}{4-x^2} = -\frac{1}{2}\int \frac{d(4-x^2)}{4-x^2} = -\frac{1}{2}\ln|4-x^2| + C$

(b) $\int \frac{x\,dx}{4-x^2}$; $[x = 2\sin\theta] \Rightarrow \int \frac{2\sin\theta \cdot 2\cos\theta\,d\theta}{4\cos^2\theta} = \int \tan\theta\,d\theta = -\ln|\cos\theta| + C = -\ln\sqrt{4-x^2} + C$

$= -\frac{1}{2}\ln|4-x^2| + C$

70. (a) $\int \frac{t\,dt}{\sqrt{4t^2-1}} = \frac{1}{8}\int \frac{d(4t^2-1)}{\sqrt{4t^2-1}} = \frac{1}{4}\sqrt{4t^2-1} + C$

(b) $\int \frac{t\,dt}{\sqrt{4t^2-1}}$; $\left[t = \frac{1}{2}\sec\theta\right] \Rightarrow \int \frac{\frac{1}{2}\sec\theta\tan\theta \cdot \frac{1}{2}\sec\theta\,d\theta}{\tan\theta} = \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{\tan\theta}{4} + C = \frac{\sqrt{4t^2-1}}{4} + C$

71. $\int \frac{x\,dx}{9-x^2}$; $\begin{bmatrix} u = 9-x^2 \\ du = -2x\,dx \end{bmatrix} \Rightarrow -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C = \ln\frac{1}{\sqrt{u}} + C = \ln\frac{1}{\sqrt{9-x^2}} + C$

72. $\int \frac{dx}{x(9-x^2)} = \frac{1}{9}\int \frac{dx}{x} + \frac{1}{18}\int \frac{dx}{3-x} - \frac{1}{18}\int \frac{dx}{3+x} = \frac{1}{9}\ln|x| - \frac{1}{18}\ln|3-x| - \frac{1}{18}\ln|3+x| + C$

$= \frac{1}{9}\ln|x| - \frac{1}{18}\ln|9-x^2| + C$

73. $\int \frac{dx}{9-x^2} = \frac{1}{6}\int \frac{dx}{3-x} + \frac{1}{6}\int \frac{dx}{3+x} = -\frac{1}{6}\ln|3-x| + \frac{1}{6}\ln|3+x| + C = \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$

74. $\int \frac{dx}{\sqrt{9-x^2}}; \left[\begin{array}{c} x = 3\sin\theta \\ dx = 3\cos\theta\, d\theta \end{array}\right] \Rightarrow \int \frac{3\cos\theta}{3\cos\theta}\, d\theta = \int d\theta = \theta + C = \sin^{-1}\frac{x}{3} + C$

75. $\int \frac{x\, dx}{1+\sqrt{x}}; \left[\begin{array}{c} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{array}\right] \Rightarrow \int \frac{u^2 \cdot 2u\, du}{1+u} = \int \left(2u^2 - 2u + 2 - \frac{2}{1+u}\right) du = \frac{2}{3}u^3 - u^2 + 2u - 2\ln|1+u| + C$

$= \frac{2x^{3/2}}{3} - x + 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

76. $\int \frac{dx}{x\left(x^2+1\right)^2}; \left[\begin{array}{c} x = \tan\theta \\ dx = \sec^2\theta\, d\theta \end{array}\right] \Rightarrow \int \frac{\sec^2\theta\, d\theta}{\tan\theta\sec^4\theta} = \int \frac{\cos^3\theta\, d\theta}{\sin\theta} = \int \left(\frac{1-\sin^2\theta}{\sin\theta}\right) d(\sin\theta)$

$= \ln|\sin\theta| - \frac{1}{2}\sin^2\theta + C = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| - \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)^2 + C$

77. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx; \left[\begin{array}{c} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{array}\right] \Rightarrow \int \frac{\cos u \cdot 2u\, du}{u} = 2\int \cos u\, du = 2\sin u + C = 2\sin\sqrt{x} + C$

78. $\int \frac{dx}{\sqrt{-2x-x^2}} = \int \frac{d(x+1)}{\sqrt{1-(x+1)^2}} = \sin^{-1}(x+1) + C$

79. $\int \frac{du}{\sqrt{1+u^2}}; [u = \tan\theta] \Rightarrow \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \ln|\sec\theta + \tan\theta| + C = \ln\left|\sqrt{1+u^2} + u\right| + C$

80. $\int \frac{2-\cos x+\sin x}{\sin^2 x}\, dx = \int 2\csc^2 x\, dx - \int \frac{\cos x\, dx}{\sin^2 x} + \int \csc x\, dx = -2\cot x + \frac{1}{\sin x} - \ln|\csc x + \cot x| + C$

$= -2\cot x + \csc x - \ln|\csc x + \cot x| + C$

81. $\int \frac{9\, dv}{81-v^4} = \frac{1}{2}\int \frac{dv}{v^2+9} + \frac{1}{12}\int \frac{dv}{3-v} + \frac{1}{12}\int \frac{dv}{3+v} = \frac{1}{12}\ln\left|\frac{3+v}{3-v}\right| + \frac{1}{6}\tan^{-1}\frac{v}{3} + C$

82.

$$\begin{array}{lll} & \cos(2\theta+1) & \\ \theta \xrightarrow{(+)} & \frac{1}{2}\sin(2\theta+1) & \\ 1 \xrightarrow{(-)} & -\frac{1}{4}\cos(2\theta+1) & \\ 0 & & \int \theta\cos(2\theta+1)\, d\theta = \frac{\theta}{2}\sin(2\theta+1) + \frac{1}{4}\cos(2\theta+1) + C \end{array}$$

83. $\int \frac{x^3\,dx}{x^2-2x+1} = \int \left(x+2+\frac{3x+2}{x^2-2x+1}\right) dx = \int (x+2)\,dx + 3\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$

$= \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C$

84. $\int \frac{d\theta}{\sqrt{1+\sqrt{\theta}}};\left[\begin{array}{c} x = 1+\sqrt{\theta} \\ dx = \frac{d\theta}{2\sqrt{\theta}} \\ d\theta = 2(x-1)\,dx \end{array}\right] \Rightarrow \int \frac{2(x-1)\,dx}{\sqrt{x}} = 2\int \sqrt{x}\,dx - 2\int \frac{dx}{\sqrt{x}} = \frac{4}{3}x^{3/2} - 4x^{1/2} + C$

$= \frac{4}{3}\left(1+\sqrt{\theta}\right)^{3/2} - 4\left(1+\sqrt{\theta}\right)^{1/2} + C = 4\left[\frac{\left(\sqrt{1+\sqrt{\theta}}\right)^3}{3} - \sqrt{1+\sqrt{\theta}}\right] + C$

85. $\int \frac{2\sin\sqrt{x}\,dx}{\sqrt{x}\sec\sqrt{x}};\left[\begin{array}{c} y = \sqrt{x} \\ dy = \frac{dx}{2\sqrt{x}} \end{array}\right] \Rightarrow \int \frac{2\sin y \cdot 2y\,dy}{y\sec y} = \int 2\sin 2y\,dy = -\cos(2y) + C = -\cos\left(2\sqrt{x}\right) + C$

86. $\int \frac{x^5\,dx}{x^4-16} = \int \left(x+\frac{16x}{x^4-16}\right) dx = \frac{x^2}{2} + \int \left(\frac{2x}{x^2-4} - \frac{2x}{x^2+4}\right) dx = \frac{x^2}{2} + \ln\left|\frac{x^2-4}{x^2+4}\right| + C$

87. $\int \frac{d\theta}{\theta^2-2\theta+4} = \int \frac{d\theta}{(\theta-1)^2+3} = \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{\theta-1}{\sqrt{3}}\right) + C$

88. $\int \frac{dr}{(r+1)\sqrt{r^2+2r}} = \int \frac{d(r+1)}{(r+1)\sqrt{(r+1)^2-1}} = \sec^{-1}|r+1| + C$

89. $\int \frac{\sin 2\theta\,d\theta}{(1+\cos 2\theta)^2} = -\frac{1}{2}\int \frac{d(1+\cos 2\theta)}{(1+\cos 2\theta)^2} = \frac{1}{2(1+\cos 2\theta)} + C = \frac{1}{4}\sec^2\theta + C$

90. $\int \frac{dx}{\left(x^2-1\right)^2} = \int \frac{dx}{\left(1-x^2\right)^2} = \frac{x}{2\left(1-x^2\right)} + \frac{1}{2}\int \frac{dx}{1-x^2}$ (FORMULA 19)

$= \frac{x}{2\left(1-x^2\right)} + \frac{1}{4}\int \frac{dx}{1-x} + \frac{1}{4}\int \frac{dx}{1+x} = \frac{x}{2\left(1-x^2\right)} - \frac{1}{4}\ln|1-x| + \frac{1}{4}\ln|1+x| + C = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2\left(x^2-1\right)} + C$

91. $\int \frac{x\,dx}{\sqrt{2-x}};\left[\begin{array}{c} y = 2-x \\ dy = -dx \end{array}\right] \Rightarrow -\int \frac{(2-y)\,dy}{\sqrt{y}} = \frac{2}{3}y^{3/2} - 4y^{1/2} + C = \frac{2}{3}(2-x)^{3/2} - 4(2-x)^{1/2} + C$

$= 2\left[\frac{\left(\sqrt{2-x}\right)^3}{3} - 2\sqrt{2-x}\right] + C$

92. $\int \frac{dy}{y^2-2y+2} = \int \frac{d(y-1)}{(y-1)^2+1} = \tan^{-1}(y-1) + C$

93. $\int \ln \sqrt{x-1}\, dx; \begin{bmatrix} y = \sqrt{x-1} \\ dy = \frac{dx}{2\sqrt{x-1}} \end{bmatrix} \Rightarrow \int \ln y \cdot 2y\, dy;\ u = \ln y,\ du = \frac{dy}{y};\ dv = 2y\, dy,\ v = y^2$

$\Rightarrow \int 2y \ln y\, dy = y^2 \ln y - \int y\, dy = y^2 \ln y - \frac{1}{2}y^2 + C = (x-1)\ln\sqrt{x-1} - \frac{1}{2}(x-1) + C_1$

$= \frac{1}{2}[(x-1)\ln|x-1| - x] + \left(C_1 + \frac{1}{2}\right) = \frac{1}{2}[x \ln|x-1| - x - \ln|x-1|] + C$

94. $\int \frac{x\, dx}{\sqrt{8-2x^2-x^4}} = \frac{1}{2}\int \frac{d(x^2+1)}{\sqrt{9-(x^2+1)^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x^2+1}{3}\right) + C$

95. $\int \frac{z+1}{z^2(z^2+4)}\, dz = \frac{1}{4}\int \left(\frac{1}{z} + \frac{1}{z^2} - \frac{z+1}{z^2+4}\right) dz = \frac{1}{4}\ln|z| - \frac{1}{4z} - \frac{1}{8}\ln(z^2+4) - \frac{1}{8}\tan^{-1}\frac{z}{2} + C$

96. $\int x^3 e^{x^2}\, dx = \frac{1}{2}\int x^2 e^{x^2}\, d(x^2) = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C = \frac{(x^2-1)e^{x^2}}{2} + C$

97. $u = \tan^{-1} x,\ du = \frac{dx}{1+x^2};\ dv = \frac{dx}{x^2},\ v = -\frac{1}{x};$

$\int \frac{\tan^{-1} x\, dx}{x^2} = -\frac{1}{x}\tan^{-1} x + \int \frac{dx}{x(1+x^2)} = -\frac{1}{x}\tan^{-1} x + \int \frac{dx}{x} - \int \frac{x\, dx}{1+x^2}$

$= -\frac{1}{x}\tan^{-1} x + \ln|x| - \frac{1}{2}\ln(1+x^2) + C = -\frac{\tan^{-1} x}{x} + \ln|x| - \ln\sqrt{1+x^2} + C$

98. $\int \frac{e^t\, dt}{e^{2t}+3e^t+2};\ [e^t = x] \Rightarrow \int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln|x+1| - \ln|x+2| + C$

$= \ln\left|\frac{x+1}{x+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

99. $\int \frac{1-\cos 2x}{1+\cos 2x}\, dx = \int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C$

100. $\int \frac{\cos(\sin^{-1} x)\, dx}{\sqrt{1-x^2}}; \begin{bmatrix} u = \sin^{-1} x \\ du = \frac{dx}{\sqrt{1-x^2}} \end{bmatrix} \Rightarrow \int \cos u\, du = \sin u + C = \sin(\sin^{-1} x) + C = x + C$

101. $\int \frac{\cos x\, dx}{\sin^3 x - \sin x} = -\int \frac{\cos x\, dx}{(\sin x)(1-\sin^2 x)} = -\int \frac{\cos x\, dx}{(\sin x)(\cos^2 x)} = -\int \frac{2\, dx}{\sin 2x} = -2\int \csc 2x\, dx$

$= \ln|\csc(2x) + \cot(2x)| + C$

102. $\int \frac{e^t\, dt}{1+e^t} = \ln(1+e^t) + C$

103. $\int_1^{\infty} \frac{\ln y\ dy}{y^3}; \begin{bmatrix} x = \ln y \\ dx = \frac{dy}{y} \\ dy = e^x\ dx \end{bmatrix} \Rightarrow \int_0^{\infty} \frac{x \cdot e^x}{e^{3x}}\ dx = \int_0^{\infty} xe^{-2x}\ dx = \lim_{b\to\infty} \left[-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^b$

$= \lim_{b\to\infty} \left(\frac{-b}{2e^{2b}} - \frac{1}{4e^{2b}}\right) - \left(0 - \frac{1}{4}\right) = \frac{1}{4}$

104. $\int \frac{\cot v\ dv}{\ln(\sin v)} = \int \frac{\cos v\ dv}{(\sin v)\ln(\sin v)}; \begin{bmatrix} u = \ln(\sin v) \\ du = \frac{\cos v\ dv}{\sin v} \end{bmatrix} \Rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|\ln(\sin v)| + C$

105. $\int \frac{dx}{(2x-1)\sqrt{x^2-x}} = \int \frac{2\ dx}{(2x-1)\sqrt{4x^2-4x}} = \int \frac{2\ dx}{(2x-1)\sqrt{(2x-1)^2-1}}; \begin{bmatrix} u = 2x-1 \\ du = 2\ dx \end{bmatrix} \Rightarrow \int \frac{du}{u\sqrt{u^2-1}}$

$= \sec^{-1}|u| + C = \sec^{-1}|2x-1| + C$

106. $\int e^{\ln\sqrt{x}}\ dx = \int \sqrt{x}\ dx = \frac{2}{3}x^{3/2} + C$

107. $\int e^{\theta}\sqrt{3+4e^{\theta}}\ d\theta; \begin{bmatrix} u = 4e^{\theta} \\ du = 4e^{\theta}\ d\theta \end{bmatrix} \Rightarrow \frac{1}{4}\int \sqrt{3+u}\ du = \frac{1}{4}\cdot\frac{2}{3}(3+u)^{3/2} + C = \frac{1}{6}\left(3+4e^{\theta}\right)^{3/2} + C$

108. $\int \frac{dv}{\sqrt{e^{2v}-1}}; \begin{bmatrix} x = e^v \\ dx = e^v\ dv \end{bmatrix} \Rightarrow \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + C = \sec^{-1}(e^v) + C$

109. $\int (27)^{3\theta+1}\ d\theta = \frac{1}{3}\int (27)^{3\theta+1}\ d(3\theta+1) = \frac{1}{3\ln 27}(27)^{3\theta+1} + C = \frac{1}{3}\left(\frac{27^{3\theta+1}}{\ln 27}\right) + C$

110.

$$\begin{array}{lcl} & & \sin x \\ x^5 & \xrightarrow{(+)} & -\cos x \\ 5x^4 & \xrightarrow{(-)} & -\sin x \\ 20x^3 & \xrightarrow{(+)} & \cos x \\ 60x^2 & \xrightarrow{(-)} & \sin x \\ 120x & \xrightarrow{(+)} & -\cos x \\ 120 & \xrightarrow{(-)} & -\sin x \\ 0 & & \end{array}$$

$\int x^5 \sin x\ dx = -x^5\cos x + 5x^4\sin x + 20x^3\cos x - 60x^2\sin x - 120x\cos x + 120\sin x + C$

111. $\int \frac{dr}{1+\sqrt{r}}; \begin{bmatrix} u = \sqrt{r} \\ du = \frac{dr}{2\sqrt{r}} \end{bmatrix} \Rightarrow \int \frac{2u\ du}{1+u} = \int \left(2 - \frac{2}{1+u}\right) du = 2u - 2\ln|1+u| + C = 2\sqrt{r} - 2\ln\left(1+\sqrt{r}\right) + C$

112. $\int \frac{8\ dy}{y^3(y+2)} = \int \frac{dy}{y} - \int \frac{2\ dy}{y^2} + \int \frac{4\ dy}{y^3} - \int \frac{dy}{(y+2)} = \ln\left|\frac{y}{y+2}\right| + \frac{2}{y} - \frac{2}{y^2} + C$

113. $\int \frac{8\ dm}{m\sqrt{49m^2-4}} = \frac{8}{7}\int \frac{dm}{m\sqrt{m^2-\left(\frac{2}{7}\right)^2}} = 4\sec^{-1}\left(\frac{7m}{2}\right) + C$

114. $\int \frac{dt}{t(1+\ln t)\sqrt{(\ln t)(2+\ln t)}}; \begin{bmatrix} u = \ln t \\ du = \frac{dt}{t} \end{bmatrix} \Rightarrow \int \frac{du}{(1+u)\sqrt{u(2+u)}} = \int \frac{du}{(u+1)\sqrt{(u+1)^2-1}}$

$= \sec^{-1}|u+1| + C = \sec^{-1}|\ln t + 1| + C$

115. $\lim\limits_{t\to 0} \frac{t - \ln(1+2t)}{t^2} = \lim\limits_{t\to 0} \frac{1 - \frac{2}{1+2t}}{2t} = \infty$ for $t \to 0^-$ and $-\infty$ for $t \to 0^+$

The limit does not exist.

116. $\lim\limits_{t\to 0} \frac{\tan 3t}{\tan 5t} = \lim\limits_{t\to 0} \frac{3\sec^2 3t}{5\sec^2 5t} = \frac{3}{5}$

117. $\lim\limits_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim\limits_{x\to 0} \frac{x\cos x + \sin x}{\sin x} = \lim\limits_{x\to 0} \frac{-x\sin x + \cos x + \cos x}{\cos x} = 2$

118. The limit leads to the indeterminate form 1^∞. $f(x) = x^{1/(1-x)} \Rightarrow \ln f(x) = \frac{\ln x}{1-x}$

$\Rightarrow \lim\limits_{x\to 1} \frac{\ln x}{1-x} = \lim\limits_{x\to 1} \frac{1/x}{-1} = -1 \Rightarrow \lim\limits_{x\to 1} x^{1/(1-x)} = \lim\limits_{x\to 1} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$

119. The limit leads to the indeterminate form ∞^0. $f(x) = x^{1/x} \Rightarrow \ln f(x) = \frac{\ln x}{x} \Rightarrow \lim\limits_{x\to\infty} \frac{\ln x}{x} = \lim\limits_{x\to\infty} \frac{1/x}{1} = 0$

$\Rightarrow \lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^0 = 1$

120. The limit leads to the indeterminate form 1^∞. $f(x) = \left(1+\frac{3}{x}\right)^x \Rightarrow \ln f(x) = x\ln\left(1+\frac{3}{x}\right) = \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}$

$\Rightarrow \lim\limits_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{\frac{-3/x^2}{1+3/x}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \frac{3x}{x+3} = 3 \Rightarrow \lim\limits_{x\to\infty} \left(1+\frac{3}{x}\right)^x = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^3$

121. $\lim\limits_{r\to\infty} \frac{\cos r}{\ln r} = 0$ since $|\cos r| \le 1$ and $\ln r \to \infty$ as $r \to \infty$.

122. $\lim\limits_{\theta\to\pi/2} \left(\theta - \frac{\pi}{2}\right)\sec\theta = \lim\limits_{\theta\to\pi/2} \frac{\theta - \frac{\pi}{2}}{\cos\theta} = \lim\limits_{\theta\to\pi/2} \frac{1}{-\sin\theta} = -1$

123. $\lim\limits_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)=\lim\limits_{x\to 1}\left[\frac{\ln x-x+1}{(x-1)\ln x}\right]=\lim\limits_{x\to 1}\frac{\frac{1}{x}-1}{\frac{x-1}{x}+\ln x}=\lim\limits_{x\to 1}\frac{1-x}{x-1+x\ln x}=\lim\limits_{x\to 1}\frac{-1}{1+x/x+\ln x}=-\frac{1}{2}$

124. The limit leads to the indeterminate form ∞^0. $f(x)=\left(1+\frac{1}{x}\right)^x\Rightarrow \ln f(x)=x\ln\left(1+\frac{1}{x}\right)=\frac{\ln(1+1/x)}{1/x}$

$\Rightarrow \lim\limits_{x\to 0^+}\frac{\ln(1+1/x)}{1/x}=\lim\limits_{x\to 0^+}\frac{\frac{-1/x^2}{1+1/x}}{-1/x^2}=\lim\limits_{x\to 0^+}\frac{x}{x+1}=0\Rightarrow\lim\limits_{x\to 0^+}\left(1+\frac{1}{x}\right)^x=\lim\limits_{x\to 0^+}e^{\ln f(x)}=e^0=1$

125. The limit leads to the indeterminate form 0^0. $f(\theta)=(\tan\theta)^\theta\Rightarrow\ln f(\theta)=\theta\ln(\tan\theta)=\frac{\ln(\tan\theta)}{1/\theta}$

$\Rightarrow\lim\limits_{x\to 0^+}\frac{\ln(\tan\theta)}{1/\theta}=\lim\limits_{x\to 0^+}\frac{\frac{\sec^2\theta}{\tan\theta}}{-\frac{1}{\theta^2}}=\lim\limits_{x\to 0^+}-\frac{\theta^2}{\sin\theta\cos\theta}=\lim\limits_{x\to 0^+}\frac{-2\theta}{-\sin^2\theta+\cos^2\theta}=0$

$\Rightarrow\lim\limits_{x\to 0^+}(\tan\theta)^\theta=\lim\limits_{x\to 0^+}e^{\ln f(\theta)}=e^0=1$

126. $\lim\limits_{\theta\to\infty}\theta^2\sin\left(\frac{1}{\theta}\right)=\lim\limits_{t\to 0^+}\frac{\sin t}{t^2}=\lim\limits_{t\to 0^+}\frac{\cos t}{2t}=\infty$

127. $\lim\limits_{x\to\infty}\frac{x^3-3x^2+1}{2x^2+x-3}=\lim\limits_{x\to\infty}\frac{3x^2-6x}{4x+1}=\lim\limits_{x\to\infty}\frac{6x-6}{4}=\infty$

128. $\lim\limits_{x\to\infty}\frac{3x^2-x+1}{x^4-x^3+2}=\lim\limits_{x\to\infty}\frac{6x-1}{4^3-3x^2}=\lim\limits_{x\to\infty}\frac{6}{12x^2-6x}=0$

129. $\int_0^3\frac{dx}{\sqrt{9-x^2}}=\lim\limits_{b\to 3^-}\int_0^b\frac{dx}{\sqrt{9-x^2}}=\lim\limits_{b\to 3^-}\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^b=\lim\limits_{b\to 3^-}\sin^{-1}\left(\frac{b}{3}\right)-\sin^{-1}\left(\frac{0}{3}\right)=\frac{\pi}{2}-0=\frac{\pi}{2}$

130. $\int_0^1\ln x\,dx=\lim\limits_{b\to 0^+}[x\ln x-x]_b^1=(1\cdot\ln 1-1)-\lim\limits_{b\to 0^+}[b\ln b-b]=-1-\lim\limits_{b\to 0^+}\frac{\ln b}{\left(\frac{1}{b}\right)}=-1-\lim\limits_{b\to 0^+}\frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$

$=-1+0=-1$

131. $\int_{-1}^1\frac{dy}{y^{2/3}}=\int_{-1}^0\frac{dy}{y^{2/3}}+\int_0^1\frac{dy}{y^{2/3}}=2\int_0^1\frac{dy}{y^{2/3}}=2\cdot 3\lim\limits_{b\to 0^+}\left[y^{1/3}\right]_b^1=6\left(1-\lim\limits_{b\to 0^+}b^{1/3}\right)=6$

132. $\int_{-2}^0\frac{d\theta}{(\theta+1)^{3/5}}=\int_{-2}^{-1}\frac{d\theta}{(\theta+1)^{3/5}}+\int_{-1}^0\frac{d\theta}{(\theta+1)^{3/5}}=\lim\limits_{b\to -1^-}\left[\frac{5}{2}(\theta+1)^{2/5}\right]_{-2}^b+\lim\limits_{b\to -1^+}\left[\frac{5}{2}(\theta+1)^{2/5}\right]_b^0$

$=\lim\limits_{b\to -1^-}\left(\frac{5}{2}(b+1)^{2/5}-\frac{5}{2}\right)+\lim\limits_{b\to -1^+}\left(\frac{5}{2}-\frac{5}{2}(b+1)^{2/5}\right)=-\frac{5}{2}+\frac{5}{2}=0$

133. $\int_3^\infty \frac{2\,du}{u^2-2u} = \int_3^\infty \frac{du}{u-2} - \int_3^\infty \frac{du}{u} = \lim_{b\to\infty} \left[\ln\left|\frac{u-2}{u}\right|\right]_3^b = \lim_{b\to\infty} \left[\ln\left|\frac{b-2}{b}\right|\right] - \ln\left|\frac{3-2}{3}\right| = 0 - \ln\left(\frac{1}{3}\right) = \ln 3$

134. $\int_1^\infty \frac{3v-1}{4v^3-v^2}\,dv = \int_1^\infty \left(\frac{1}{v} + \frac{1}{v^2} - \frac{4}{4v-1}\right) dv = \lim_{b\to\infty} \left[\ln v - \frac{1}{v} - \ln(4v-1)\right]_1^b$

$= \lim_{b\to\infty} \left[\ln\left(\frac{b}{4b-1}\right) - \frac{1}{b}\right] - (\ln 1 - 1 - \ln 3) = \ln\frac{1}{4} + 1 + \ln 3 = 1 + \ln\frac{3}{4}$

135. $\int_0^\infty x^2 e^{-x}\,dx = \lim_{b\to\infty} \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^b = \lim_{b\to\infty} \left(-b^2e^{-b} - 2be^{-b} - 2e^{-b}\right) - (-2) = 0 + 2 = 2$

136. $\int_{-\infty}^0 xe^{3x}\,dx = \lim_{b\to-\infty} \left[\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]_b^0 = -\frac{1}{9} - \lim_{b\to-\infty} \left(\frac{b}{3}e^{3b} - \frac{1}{9}e^{3b}\right) = -\frac{1}{9} - 0 = -\frac{1}{9}$

137. $\int_{-\infty}^\infty \frac{dx}{4x^2+9} = 2\int_0^\infty \frac{dx}{4x^2+9} = \frac{1}{2}\int_0^\infty \frac{dx}{x^2+\frac{9}{4}} = \frac{1}{2}\lim_{b\to\infty} \left[\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)\right]_0^b = \frac{1}{2}\lim_{b\to\infty} \left[\frac{2}{3}\tan^{-1}\left(\frac{2b}{3}\right)\right] - \frac{1}{3}\tan^{-1}(0)$

$= \frac{1}{2}\left(\frac{2}{3}\cdot\frac{\pi}{2}\right) - 0 = \frac{\pi}{6}$

138. $\int_{-\infty}^\infty \frac{4\,dx}{x^2+16} = 2\int_0^\infty \frac{4\,dx}{x^2+16} = 2\lim_{b\to\infty} \left[\tan^{-1}\left(\frac{x}{4}\right)\right]_0^b = 2\lim_{b\to\infty} \left[\tan^{-1}\left(\frac{b}{4}\right)\right] - \tan^{-1}(0) = 2\left(\frac{\pi}{2}\right) - 0 = \pi$

139. $\lim_{\theta\to\infty} \frac{\theta}{\sqrt{\theta^2+1}} = 1$ and $\int_6^\infty \frac{d\theta}{\theta}$ diverges $\Rightarrow \int_6^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$ diverges

140. $I = \int_0^\infty e^{-u}\cos u\,du = \lim_{b\to\infty} \left[-e^{-u}\cos u\right]_0^b - \int_0^\infty e^{-u}\sin u\,du = 1 - \lim_{b\to\infty} \left[e^{-u}\sin u\right]_0^b + \int_0^\infty \left(-e^{-u}\right)\cos u\,du$

$\Rightarrow I = 1 + 0 + I \Rightarrow 2I = 1 \Rightarrow I = \frac{1}{2}$ converges

141. $\int_1^\infty \frac{\ln z}{z}\,dz = \int_1^e \frac{\ln z}{z}\,dz + \int_e^\infty \frac{\ln z}{z}\,dz = \left[(\ln z)^2\right]_1^e + \lim_{b\to\infty} \left[(\ln z)^2\right]_e^b = \left(1^2 - 0\right) + \lim_{b\to\infty} \left[(\ln b)^2 - 1\right]$

$= \infty \Rightarrow$ diverges

142. $0 < \frac{e^{-t}}{\sqrt{t}} \le e^{-t}$ for $t \ge 1$ and $\int_1^\infty e^{-t}\,dt$ converges $\Rightarrow \int_1^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$ converges

143. $0 < \frac{e^{-x}}{3+e^{-2x}} = \frac{1}{3e^{x}+e^{-x}} < \frac{1}{e^{x}+e^{-x}}$ and $\int_{-\infty}^{\infty} \frac{dx}{e^{x}+e^{-x}} = 2\int_{0}^{\infty} \frac{dx}{e^{x}+e^{-x}} < \int_{0}^{\infty} \frac{2\,dx}{e^{x}}$ converges

$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-x}}{3+e^{-2x}}\,dx$ converges

144. $\int_{-\infty}^{\infty} \frac{dx}{x^2(1+e^x)} = \int_{-\infty}^{-1} \frac{dx}{x^2(1+e^x)} + \int_{-1}^{0} \frac{dx}{x^2(1+e^x)} + \int_{0}^{1} \frac{dx}{x^2(1+e^x)} + \int_{1}^{\infty} \frac{dx}{x^2(1+e^x)}$;

$\lim\limits_{x\to 0} \frac{\left(\frac{1}{x^2}\right)}{\left[\frac{1}{x^2(1+e^x)}\right]} = \lim\limits_{x\to 0} \frac{x^2(1+e^x)}{x^2} = \lim\limits_{x\to 0} (1+e^x) = 2$ and $\int_0^1 \frac{dx}{x^2}$ diverges $\Rightarrow \int_0^1 \frac{dx}{x^2(1+e^x)}$ diverges

$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2(1+e^x)}$ diverges

145. $\frac{1}{y^2-y}\,dy = e^x\,dx \Rightarrow \int \frac{1}{y(y-1)}\,dy = \int e^x\,dx = e^x + C;\ \frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1} \Rightarrow 1 = A(y-1) + B(y)$

$= (A+B)y - A$

Equating coefficients of like terms gives $A + B = 0$ and $-A = 1$. Solving the system simultaneously yields $A = -1$, $B = 1$.

$\int \frac{1}{y(y-1)}\,dy = \int -\frac{1}{y}\,dy + \int \frac{1}{y-1}\,dy = -\ln|y| + \ln|y-1| + C_2 \Rightarrow -\ln|y| + \ln|y-1| = e^x + C$

Substitute $x = 0$, $y = 2 \Rightarrow -\ln 2 + 0 = 1 + C$ or $C = -1 - \ln 2$.

The solution to the initial value problem is $-\ln|y| + \ln|y-1| = e^x - 1 - \ln 2$.

146. $\frac{1}{(y+1)^2}\,dy = \sin\theta\,d\theta;\ \int \frac{1}{(y+1)^2}\,dy = \int \sin\theta\,d\theta \Rightarrow -\frac{1}{y+1} = -\cos\theta + C$

Substitute $x = \frac{\pi}{2}$, $y = 0 \Rightarrow -1 = 0 + C$ or $C = -1$.

The solution to the initial value problem is $-\frac{1}{y+1} = -\cos\theta - 1 \Rightarrow y + 1 = \frac{1}{\cos\theta + 1} \Rightarrow y = \frac{1}{\cos\theta + 1} - 1$

147. $dy = \frac{dx}{x^2 - 3x + 2};\ x^2 - 3x + 2 = (x-2)(x-1) \Rightarrow \frac{1}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x-2)$

$\Rightarrow 1 = (A+B)x - A - 2B$

Equating coefficients of like terms gives $A + B = 0$, $-A - 2B = 1$. Solving the system simultaneously yields $A = 1$, $B = -1$.

$\int dy = \int \frac{dx}{x^2-3x+2} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1} \Rightarrow y = \ln|x-2| - \ln|x-1| + C$

Substitute $x = 3$, $y = 0 \Rightarrow 0 = 0 - \ln 2 + C$ or $C = \ln 2$.

The solution to the initial value problem is $y = \ln|x-2| - \ln|x-1| + \ln 2$.

148. $\frac{ds}{2x+2} = \frac{dt}{t^2+2t}$; $\int \frac{ds}{2x+2} = \frac{1}{2}\int \frac{ds}{s+1} = \frac{1}{2}\ln|s+1| + C_1$; $t^2 + 2t = t(t+2) \Rightarrow \frac{1}{t^2+2t} = \frac{A}{t} + \frac{B}{t+2}$

$\Rightarrow 1 = A(t+2) + Bt \Rightarrow 1 = (A+B)t + 2A.$

Equating coefficients of like terms gives $A + B = 0$ and $2A = 1$. Solving the system simultaneously yields $A = \frac{1}{2}$, $B = -\frac{1}{2}$.

$\int \frac{dt}{t^2+2t} = \int \frac{1/2}{t}\,dt - \int \frac{1/2}{t+2}\,dt \;\; = \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| + C_2 \Rightarrow \frac{1}{2}\ln|s+1| = \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| + C_3$

$\Rightarrow \ln|s+1| = \ln|t| - \ln|t+2| + C$

Substitute $t = 1$, $x = 1 \Rightarrow \ln 2 = 0 - \ln 3 + C$ or $C = \ln 2 + \ln 3 = \ln 6$.

The solution to the initial value problem is $\ln|s+1| = \ln|t| - \ln|t+2| + \ln 6 \Rightarrow \ln|s+1| = \ln\left|\frac{6t}{t+2}\right|$

$\Rightarrow |s+1| = \left|\frac{6t}{t+2}\right|$

CHAPTER 7 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. $u = \left(\sin^{-1} x\right)^2$, $du = \frac{2\sin^{-1} x\,dx}{\sqrt{1-x^2}}$; $dv = dx$, $v = x$;

$\int \left(\sin^{-1} x\right)^2 dx = x\left(\sin^{-1} x\right)^2 - \int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}}$;

$u = \sin^{-1} x$, $du = \frac{dx}{\sqrt{1-x^2}}$; $dv = -\frac{2x\,dx}{\sqrt{1-x^2}}$, $v = 2\sqrt{1-x^2}$;

$\int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}} = 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - \int 2\,dx = 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - 2x + C$; therefore

$\int \left(\sin^{-1} x\right)^2 dx = x\left(\sin^{-1} x\right)^2 + 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - 2x + C$

2. $\frac{1}{x} = \frac{1}{x}$,

$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$,

$\frac{1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)}$,

$\frac{1}{x(x+1)(x+2)(x+3)} = \frac{1}{6x} - \frac{1}{2(x+1)} + \frac{1}{2(x+2)} - \frac{1}{6(x+3)}$,

$\frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{24x} - \frac{1}{6(x+1)} + \frac{1}{4(x+2)} - \frac{1}{6(x+3)} + \frac{1}{24(x+4)} \Rightarrow$ the following pattern:

$\frac{1}{x(x+1)(x+2)\cdots(x+m)} = \sum_{k=0}^{m} \frac{(-1)^k}{(k!)(m-k)!(x+k)}$; therefore $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

$$= \sum_{k=0}^{m} \left[\frac{(-1)^k}{(k!)(m-k)!} \ln|x+k| \right] + C$$

3. $u = \sin^{-1} x$, $du = \frac{dx}{\sqrt{1-x^2}}$; $dv = x\,dx$, $v = \frac{x^2}{2}$;

$$\int x \sin^{-1} x\, dx = \frac{x^2}{2} \sin^{-1} x - \int \frac{x^2\, dx}{2\sqrt{1-x^2}}; \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} \Rightarrow \int x \sin^{-1} x\, dx = \frac{x^2}{2} \sin^{-1} x - \int \frac{\sin^2\theta \cos\theta\, d\theta}{2\cos\theta}$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2}\int \sin^2\theta\, d\theta = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C = \frac{x^2}{2} \sin^{-1} x + \frac{\sin\theta\cos\theta - \theta}{4} + C$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{x\sqrt{1-x^2} - \sin^{-1} x}{4} + C$$

4. $\int \sin^{-1} \sqrt{y}\, dy; \begin{bmatrix} z = \sqrt{y} \\ dz = \frac{dy}{2\sqrt{y}} \end{bmatrix} \Rightarrow \int 2z \sin^{-1} z\, dz$; from Exercise 3, $\int z \sin^{-1} z\, dz$

$$= \frac{z^2 \sin^{-1} z}{2} + \frac{z\sqrt{1-z^2} - \sin^{-1} z}{4} + C \Rightarrow \int \sin^{-1}\sqrt{y}\, dy = y \sin^{-1}\sqrt{y} + \frac{\sqrt{y}\sqrt{1-y} - \sin^{-1}\sqrt{y}}{2} + C$$

$$= y \sin^{-1}\sqrt{y} + \frac{\sqrt{y-y^2}}{2} - \frac{\sin^{-1}\sqrt{y}}{2} + C$$

5. $\int \frac{d\theta}{1-\tan^2\theta} = \int \frac{\cos^2\theta}{\cos^2\theta - \sin^2\theta}\, d\theta = \int \frac{1+\cos 2\theta}{2\cos 2\theta}\, d\theta = \frac{1}{2}\int (\sec 2\theta + 1)\, d\theta = \frac{\ln|\sec 2\theta + \tan 2\theta| + 2\theta}{4} + C$

6. $u = \ln\left(\sqrt{x} + \sqrt{1+x}\right)$, $du = \left(\frac{dx}{\sqrt{x}+\sqrt{1+x}}\right)\left(\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{1+x}}\right) = \frac{dx}{2\sqrt{x}\sqrt{1+x}}$; $dv = dx$, $v = x$;

$$\int \ln\left(\sqrt{x}+\sqrt{1+x}\right) dx = x \ln\left(\sqrt{x}+\sqrt{1+x}\right) - \frac{1}{2}\int \frac{x\, dx}{\sqrt{x}\sqrt{1+x}} = \frac{1}{2}\int \frac{x\, dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}};$$

$$\begin{bmatrix} x + \frac{1}{2} = \frac{1}{2}\sec\theta \\ dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta \end{bmatrix} \Rightarrow \frac{1}{4}\int \frac{(\sec\theta - 1)\cdot\sec\theta\tan\theta\, d\theta}{\left(\frac{1}{2}\tan\theta\right)} = \frac{1}{2}\int \left(\sec^2\theta - \sec\theta\right) d\theta$$

$$= \frac{\tan\theta - \ln|\sec\theta + \tan\theta|}{2} + C = \frac{2\sqrt{x^2+x} - \ln\left|2x+1+2\sqrt{x^2+x}\right|}{2} + C$$

$$\Rightarrow \int \ln\left(\sqrt{x}+\sqrt{1+x}\right) dx = x \ln\left(\sqrt{x}+\sqrt{1+x}\right) - \frac{2\sqrt{x^2+x} - \ln\left|2x+1+2\sqrt{x^2+x}\right|}{4} + C$$

7. $\int \frac{dt}{t-\sqrt{1-t^2}}$; $\begin{bmatrix} t = \sin\theta \\ dt = \cos\theta\, d\theta \end{bmatrix} \Rightarrow \int \frac{\cos\theta\, d\theta}{\sin\theta - \cos\theta} = \int \frac{d\theta}{\tan\theta - 1}$; $\begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \\ d\theta = \frac{du}{u^2+1} \end{bmatrix} \Rightarrow \int \frac{du}{(u-1)(u^2+1)}$

$= \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u^2+1} - \frac{1}{2}\int \frac{u\, du}{u^2+1} = \frac{1}{2}\ln\left|\frac{u-1}{\sqrt{u^2+1}}\right| - \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\ln\left|\frac{\tan\theta - 1}{\sec\theta}\right| - \frac{1}{2}\theta + C$

$= \frac{1}{2}\ln\left(t - \sqrt{1-t^2}\right) - \frac{1}{2}\sin^{-1}t + C$

8. $\int \frac{\left(2e^{2x} - e^x\right)dx}{\sqrt{3e^{2x} - 6e^x - 1}}$; $\begin{bmatrix} u = e^x \\ du = e^x\, dx \end{bmatrix} \Rightarrow \int \frac{(2u-1)\, du}{\sqrt{3u^2 - 6u - 1}} = \frac{1}{\sqrt{3}}\int \frac{(2u-1)\, du}{\sqrt{(u-1)^2 - \frac{4}{3}}}$;

$\begin{bmatrix} u - 1 = \frac{2}{\sqrt{3}}\sec\theta \\ du = \frac{2}{\sqrt{3}}\sec\theta\tan\theta\, d\theta \end{bmatrix} \Rightarrow \frac{1}{\sqrt{3}}\int \left(\frac{4}{\sqrt{3}}\sec\theta + 1\right)(\sec\theta)\, d\theta = \frac{4}{3}\int \sec^2\theta\, d\theta + \frac{1}{\sqrt{3}}\int \sec\theta\, d\theta$

$= \frac{4}{3}\tan\theta + \frac{1}{\sqrt{3}}\ln|\sec\theta + \tan\theta| + C_1 = \frac{4}{3}\cdot\sqrt{\frac{3}{4}(u-1)^2 - 1} + \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}}{2}(u-1) + \sqrt{\frac{3}{4}(u-1)^2 - 1}\right| + C_1$

$= \frac{2}{3}\sqrt{3u^2 - 6u - 1} + \frac{1}{\sqrt{3}}\ln\left|u - 1 + \sqrt{(u-1)^2 - \frac{4}{3}}\right| + \left(C_1 + \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{\sqrt{3}}\left[2\sqrt{e^{2x} - 2e^x - \frac{1}{3}} + \ln\left|e^x - 1 + \sqrt{e^{2x} - 2e^x - \frac{1}{3}}\right|\right] + C$

9. $\int \frac{1}{x^4+4}\, dx = \int \frac{1}{\left(x^2+2\right)^2 - 4x^2}\, dx = \int \frac{1}{(x^2+2x+2)(x^2-2x+2)}\, dx$

$= \frac{1}{16}\int \left[\frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1} - \frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1}\right]dx$

$= \frac{1}{16}\ln\left|\frac{x^2+2x+2}{x^2-2x+2}\right| + \frac{1}{8}\left[\tan^{-1}(x+1) + \tan^{-1}(x-1)\right] + C$

10. $\int \frac{1}{x^6-1}\, dx = \frac{1}{6}\int \left(\frac{1}{x-1} - \frac{1}{x+1} + \frac{x-2}{x^2-x+1} - \frac{x+2}{x^2+x+1}\right)dx$

$= \frac{1}{6}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{12}\int \left[\frac{2x-1}{x^2-x+1} - \frac{3}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{2x+1}{x^2+x+1} - \frac{3}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\right]dx$

$= \frac{1}{6}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{12}\left[\ln\left|\frac{x^2-x+1}{x^2+x+1}\right| - 2\sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) - 2\sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right] + C$

11. $\lim\limits_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim\limits_{b\to 1^-} \left[\sin^{-1} x\right]_0^b = \lim\limits_{b\to 1^-} \left(\sin^{-1} b - \sin^{-1} 0\right) = \lim\limits_{b\to 1^-} \left(\sin^{-1} b - 0\right) = \lim\limits_{b\to 1^-} \sin^{-1} b = \frac{\pi}{2}$

12. $\lim\limits_{x\to\infty} \frac{1}{x} \int_0^x \tan^{-1} t\,dt = \lim\limits_{x\to\infty} \frac{\int_0^x \tan^{-1} t\,dt}{x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$= \lim\limits_{x\to\infty} \frac{\tan^{-1} x}{1} = \frac{\pi}{2}$

13. $y = \left(\cos\sqrt{x}\right)^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln\left(\cos\sqrt{x}\right)$ and $\lim\limits_{x\to 0^+} \frac{\ln\left(\cos\sqrt{x}\right)}{x} = \lim\limits_{x\to 0^+} \frac{-\sin\sqrt{x}}{2\sqrt{x}\cos\sqrt{x}} = \frac{-1}{2}\lim\limits_{x\to 0^+} \frac{\tan\sqrt{x}}{\sqrt{x}}$

$= -\frac{1}{2}\lim\limits_{x\to 0^+} \frac{\frac{1}{2}x^{-1/2}\sec^2\sqrt{x}}{\frac{1}{2}x^{-1/2}} = -\frac{1}{2} \Rightarrow \lim\limits_{x\to 0^+} \left(\cos\sqrt{x}\right)^{1/x} = e^{-1/2} = \frac{1}{\sqrt{e}}$

14. $y = \left(x+e^x\right)^{2/x} \Rightarrow \ln y = \frac{2\ln\left(x+e^x\right)}{x} \Rightarrow \lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \frac{2\left(1+e^x\right)}{x+e^x} = \lim\limits_{x\to\infty} \frac{2e^x}{1+e^x} = \lim\limits_{x\to\infty} \frac{2e^x}{e^x} = 2$

$\Rightarrow \lim\limits_{x\to\infty} \left(x+e^x\right)^{2/x} = \lim\limits_{x\to\infty} e^y = e^2$

15. $\lim\limits_{x\to\infty} \int_{-x}^{x} \sin t\,dt = \lim\limits_{x\to\infty} \left[-\cos t\right]_{-x}^{x} = \lim\limits_{x\to\infty} \left[-\cos x + \cos(-x)\right] = \lim\limits_{x\to\infty} \left(-\cos x + \cos x\right) = \lim\limits_{x\to\infty} 0 = 0$

16. $\lim\limits_{x\to 0^+} \int_x^1 \frac{\cos t}{t^2}\,dt$; $\lim\limits_{t\to 0^+} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{\cos t}{t^2}\right)} = \lim\limits_{t\to 0^+} \frac{1}{\cos t} = 1 \Rightarrow \lim\limits_{x\to 0^+} \int_x^1 \frac{\cos t}{t^2}\,dt$ diverges since $\int_0^1 \frac{dt}{t^2}$ diverges; thus

$\lim\limits_{x\to 0^+} x\int_x^1 \frac{\cos t}{t^2}\,dt$ is an indeterminate $0\cdot\infty$ form and we apply l'Hôpital's rule:

$$\lim_{x\to 0^+} x\int_x^1 \frac{\cos t}{t^2}\,dt = \lim_{x\to 0^+} \frac{-\int_1^x \frac{\cos t}{t^2}\,dt}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{-\left(\frac{\cos x}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} \cos x = 1$$

17. $\frac{dy}{dx} = \sqrt{\cos 2x} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cos 2x = 2\cos^2 x$; $L = \int_0^{\pi/4} \sqrt{1+\left(\sqrt{\cos 2t}\right)^2}\,dt = \sqrt{2}\int_0^{\pi/4} \sqrt{\cos^2 t}\,dt$

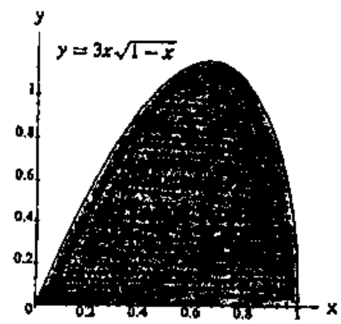
$= \sqrt{2}\left[\sin t\right]_0^{\pi/4} = 1$

18. $\frac{dy}{dx} = \frac{-2x}{1-x^2} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{\left(1-x^2\right)^2 + 4x^2}{\left(1-x^2\right)^2} = \frac{1+2x^2+x^4}{\left(1-x^2\right)^2} = \left(\frac{1+x^2}{1-x^2}\right)^2$; $L = \int_0^{1/2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= \int_0^{1/2} \left(\frac{1+x^2}{1-x^2}\right) dx = \int_0^{1/2} \left(-1+\frac{2}{1-x^2}\right) dx = \int_0^{1/2} \left(-1+\frac{1}{1+x}+\frac{1}{1-x}\right) dx = \left[-x+\ln\left|\frac{1+x}{1-x}\right|\right]_0^{1/2}$$

$$= \left(-\tfrac{1}{2}+\ln 3\right)-(0+\ln 1) = \ln 3 - \tfrac{1}{2}$$

19. $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi xy\ dx$

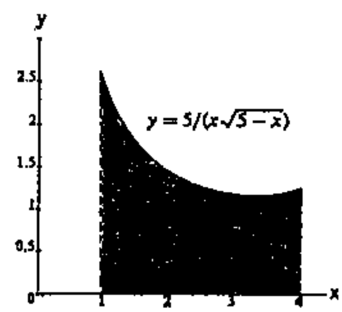


$$= 6\pi \int_0^1 x^2\sqrt{1-x}\ dx; \begin{bmatrix} u = 1-x \\ du = -dx \\ x^2 = (1-u)^2 \end{bmatrix}$$

$$\Rightarrow -6\pi \int_1^0 (1-u)^2\sqrt{u}\ du = -6\pi \int_1^0 \left(u^{1/2}-2u^{3/2}+u^{5/2}\right) du$$

$$= -6\pi\left[\tfrac{2}{3}u^{3/2}-\tfrac{4}{5}u^{5/2}+\tfrac{2}{7}u^{7/2}\right]_1^0 = 6\pi\left(\tfrac{2}{3}-\tfrac{4}{5}+\tfrac{2}{7}\right) = 6\pi\left(\frac{70-84+30}{105}\right) = 6\pi\left(\frac{16}{105}\right) = \frac{32\pi}{35}$$

20. $V = \int_a^b \pi y^2\ dx = \pi \int_1^4 \frac{25\ dx}{x^2(5-x)} = \pi \int_1^4 \left(\frac{dx}{x}+\frac{5\ dx}{x^2}+\frac{dx}{5-x}\right)$

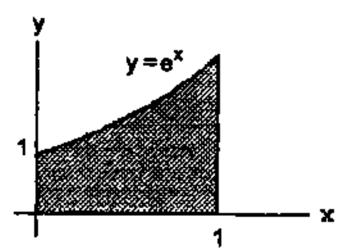


$$= \pi\left[\ln\left|\frac{x}{5-x}\right|-\frac{5}{x}\right]_1^4 = \pi\left(\ln 4-\tfrac{5}{4}\right)-\pi\left(\ln\tfrac{1}{4}-5\right)$$

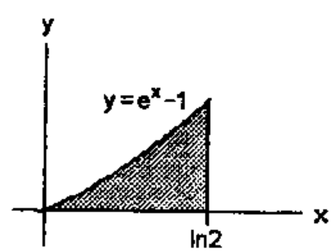
$$= \frac{15\pi}{4}+2\pi\ \ln 4$$

21. $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi xe^x\ dx$

$$= 2\pi\left[xe^x - e^x\right]_0^1 = 2\pi$$



22. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x)\left(e^x-1\right) dx$



$$= 2\pi \int_0^{\ln 2} \left[(\ln 2)\, e^x - \ln 2 - xe^x + x\right] dx$$

$$= 2\pi\left[(\ln 2)\, e^x - (\ln 2)x - xe^x + e^x + \frac{x^2}{2}\right]_0^{\ln 2}$$

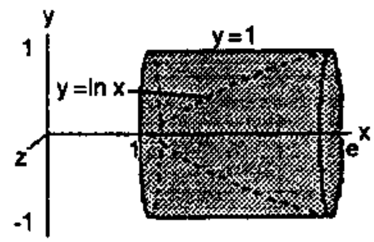
$$= 2\pi\left[2\ \ln 2 - (\ln 2)^2 - 2\ \ln 2 + 2 + \frac{(\ln 2)^2}{2}\right] - 2\pi(\ln 2 + 1) = 2\pi\left[-\frac{(\ln 2)^2}{2} - \ln 2 + 1\right]$$

23. (a) $V = \int_1^e \pi\left[1-(\ln x)^2\right] dx$

$= \pi\left[x - x(\ln x)^2\right]_1^e - 2\pi \int_1^e \ln x \, dx$ (FORMULA 110)

$= \pi\left[x - x(\ln x)^2 + 2(x \ln x - x)\right]_1^e$

$= \pi\left[-x - x(\ln x)^2 + 2x \ln x\right]_1^e = \pi\left[-e - e + 2e - (-1)\right] = \pi$



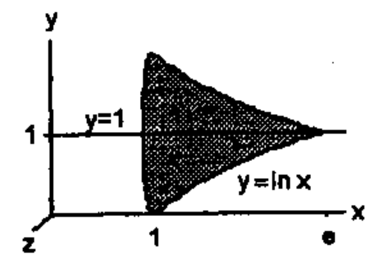
(b) $V = \int_1^e \pi(1-\ln x)^2 \, dx = \pi \int_1^e \left[1 - 2\ln x + (\ln x)^2\right] dx$

$= \pi\left[x - 2(x \ln x - x) + x(\ln x)^2\right]_1^e - 2\pi \int_1^e \ln x \, dx$

$= \pi\left[x - 2(x \ln x - x) + x(\ln x)^2 - 2(x \ln x - x)\right]_1^e$

$= \pi\left[5x - 4x \ln x + x(\ln x)^2\right]_1^e = \pi\left[(5e - 4e + e) - (5)\right]$

$= \pi(2e - 5)$



24. (a) $V = \pi \int_0^1 \left[(e^y)^2 - 1\right] dy = \pi \int_0^1 \left(e^{2y} - 1\right) dy = \pi\left[\frac{e^{2y}}{2} - y\right]_0^1 = \pi\left[\frac{e^2}{2} - 1 - \left(\frac{1}{2}\right)\right] = \frac{\pi\left(e^2 - 3\right)}{2}$

(b) $V = \pi \int_0^1 \left(e^y - 1\right)^2 dy = \pi \int_0^1 \left(e^{2y} - 2e^y + 1\right) dy = \pi\left[\frac{e^{2y}}{2} - 2e^y + y\right]_0^1 = \pi\left[\left(\frac{e^2}{2} - 2e + 1\right) - \left(\frac{1}{2} - 2\right)\right]$

$= \pi\left(\frac{e^2}{2} - 2e + \frac{5}{2}\right) = \frac{\pi\left(e^2 - 4e + 5\right)}{2}$

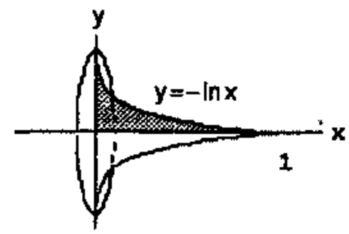
25. (a) $\lim\limits_{x\to 0^+} x \ln x = 0 \Rightarrow \lim\limits_{x\to 0^+} f(x) = 0 = f(0) \Rightarrow$ f is continuous

(b) $V = \int_0^2 \pi x^2 (\ln x)^2 \, dx; \begin{bmatrix} u = (\ln x)^2 \\ du = (2 \ln x)\frac{dx}{x} \\ dv = x^2 \\ v = \frac{x^3}{3} \end{bmatrix} \Rightarrow \pi\left(\lim\limits_{b\to 0^+} \left[\frac{x^3}{3}(\ln x)^2\right]_b^2 - \int_0^2 \left(\frac{x^3}{3}\right)(2 \ln x)\frac{dx}{x}\right)$

$= \pi\left[\left(\frac{8}{3}\right)(\ln 2)^2 - \left(\frac{2}{3}\right) \lim\limits_{b\to 0^+} \left[\frac{x^3}{3} \ln x - \frac{x^3}{9}\right]_b^2\right] = \pi\left[\frac{8(\ln 2)^2}{3} - \frac{16(\ln 2)}{9} + \frac{16}{27}\right]$

26. $V = \int_0^1 \pi(-\ln x)^2\,dx = \pi\left(\lim\limits_{b\to 0}\,[x(\ln x)^2]_b^1 - 2\int_0^1 \ln x\,dx\right)$

$= -2\pi \lim\limits_{b\to 0}\,[x\ln x - x]_b^1 = 2\pi$



27. $u = \frac{1}{1+y},\ du = -\frac{dy}{(1+y)^2};\ dv = ny^{n-1}\,dy,\ v = y^n;$

$\lim\limits_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}\,dy = \lim\limits_{n\to\infty}\left(\left[\frac{y^n}{1+y}\right]_0^1 + \int_0^1 \frac{y^n}{1+y^2}\,dy\right) = \frac{1}{2} + \lim\limits_{n\to\infty}\int_0^1 \frac{y^n}{1+y^2}\,dy$. Now, $0 \le \frac{y^n}{1+y^2} \le y^n$

$\Rightarrow 0 \le \lim\limits_{n\to\infty}\int_0^1 \frac{y^n}{1+y^2}\,dy \le \lim\limits_{n\to\infty}\int_0^1 y^n\,dy = \lim\limits_{n\to\infty}\left[\frac{y^{n+1}}{n+1}\right]_0^1 = \lim\limits_{n\to\infty}\frac{1}{n+1} = 0 \Rightarrow \lim\limits_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}\,dy$

$= \frac{1}{2} + 0 = \frac{1}{2}$

28. $u = x^2 - a^2 \Rightarrow du = 2x\,dx;$

$\int x\left(\sqrt{x^2-a^2}\right)^n dx = \frac{1}{2}\int\left(\sqrt{u}\right)^n du = \frac{1}{2}\int u^{n/2}\,du = \frac{1}{2}\left(\frac{u^{n/2+1}}{\frac{n}{2}+1}\right) + C,\ n \ne -2$

$= \frac{u^{(n+2)/2}}{n+2} + C = \frac{\left(\sqrt{u}\right)^{n+2}}{n+2} + C = \frac{\left(\sqrt{x^2-a^2}\right)^{n+2}}{n+2} + C$

29. $\frac{\pi}{6} = \sin^{-1}\frac{1}{2} = \left[\sin^{-1}\frac{x}{2}\right]_0^1 = \int_0^1 \frac{dx}{\sqrt{4-x^2}} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \int_0^1 \frac{dx}{\sqrt{4-2x^2}} = \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} \frac{du}{\sqrt{4-u^2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{u}{2}\right]_0^{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{8}$

30. $\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right)dx = \lim\limits_{b\to\infty}\int_1^b \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right)dx = \lim\limits_{b\to\infty}\left[\frac{a}{2}\ln(x^2+1) - \frac{1}{2}\ln x\right]_1^b = \lim\limits_{b\to\infty}\left[\frac{1}{2}\ln\frac{(x^2+1)^a}{x}\right]_1^b$

$= \lim\limits_{b\to\infty}\frac{1}{2}\left[\ln\frac{(b^2+1)^a}{b} - \ln 2^a\right];\ \lim\limits_{b\to\infty}\frac{(b^2+1)^a}{b} > \lim\limits_{b\to\infty}\frac{b^{2a}}{b} = \lim\limits_{b\to\infty} b^{2\left(a-\frac{1}{2}\right)} = \infty$ if $a > \frac{1}{2} \Rightarrow$ the improper integral diverges if $a > \frac{1}{2}$; for $a = \frac{1}{2}$: $\lim\limits_{b\to\infty}\frac{\sqrt{b^2+1}}{b} = \lim\limits_{b\to\infty}\sqrt{1+\frac{1}{b^2}} = 1 \Rightarrow \lim\limits_{b\to\infty}\frac{1}{2}\left[\ln\frac{(b^2+1)^{1/2}}{b} - \ln 2^{1/2}\right]$

$= \frac{1}{2}\left(\ln 1 - \frac{1}{2}\ln 2\right) = -\frac{\ln 2}{4}$; if $a < \frac{1}{2}$: $0 \le \lim\limits_{b\to\infty}\frac{(b^2+1)^a}{b} < \lim\limits_{b\to\infty}\frac{(b+1)^{2a}}{b+1} = \lim\limits_{b\to\infty}(b+1)^{2a-1} = 0$

$\Rightarrow \lim\limits_{b\to\infty}\ln\frac{(b^2+1)^a}{b} = -\infty \Rightarrow$ the improper integral diverges if $a < \frac{1}{2}$; in summary, the improper integral

$\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right)dx$ converges only when $a = \frac{1}{2}$ and has the value $-\frac{\ln 2}{4}$

31. Let $u = f(x) \Rightarrow du = f'(x)\,dx$ and $dv = dx \Rightarrow v = x$;

$$\int_{\pi/2}^{3\pi/2} f(x)\,dx = \left[x\,f(x)\right]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} xf'(x)\,dx = \left[\frac{3\pi}{2} f\left(\frac{3\pi}{2}\right) - \frac{\pi}{2} f\left(\frac{\pi}{2}\right)\right] - \int_{\pi/2}^{3\pi/2} \cos x\,dx$$

$$= \frac{3\pi}{2} b - \frac{\pi}{2} a - [\sin x]_{\pi/2}^{3\pi/2} = \frac{\pi}{2}(3b-a) - [-1-1] = \frac{\pi}{2}(3b-a) + 2$$

32. $\int_0^a \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^a = \tan^{-1} a$; $\int_a^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \left[\tan^{-1} x\right]_a^b = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} a\right) = \frac{\pi}{2} - \tan^{-1} a$;

therefore, $\tan^{-1} a = \frac{\pi}{2} - \tan^{-1} a \Rightarrow \tan^{-1} a = \frac{\pi}{4} \Rightarrow a = 1$ for $a > 0$.

33. $L = 4 \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dy$; $x^{2/3} + y^{2/3} = 1 \Rightarrow y = \left(1 - x^{2/3}\right)^{3/2} \Rightarrow \frac{dy}{dx} = -\frac{3}{2}\left(1-x^{2/3}\right)^{1/2}\left(x^{-1/3}\right)\left(\frac{2}{3}\right)$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1-x^{2/3}}{x^{2/3}} \Rightarrow L = 4\int_0^1 \sqrt{1+\left(\frac{1-x^{2/3}}{x^{2/3}}\right)}\,dx = 4\int_0^1 \frac{dx}{x^{1/3}} = 6\left[x^{2/3}\right]_0^1 = 6$$

34. $\left(\frac{dy}{dx}\right)^2 = \frac{1}{4x} \Rightarrow \frac{dy}{dx} = \frac{\pm 1}{2\sqrt{x}} \Rightarrow y = \pm\sqrt{x},\ 0 \le x \le 4$

35. $P(x) = ax^2 + bx + c$, $P(0) = c = 1$ and $P'(0) = 0 \Rightarrow b = 0 \Rightarrow P(x) = ax^2 + 1$. Next,
$\frac{ax^2+1}{x^3(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$; for the integral to be a rational function, we must have $A = 0$ and $D = 0$. Thus, $ax^2 + 1 = Bx(x-1)^2 + C(x-1)^2 + Ex^3 = (B+E)x^3 + (C-2B)x^2 + (B-2C)x + C$
$\Rightarrow C = 1$; B-2C $= 0 \Rightarrow B = 2$; $C - 2B = a \Rightarrow a = -3$; therefore, $P(x) = -3x^2 + 1$

36. The integral $\int_{-1}^1 \sqrt{1-x^2}\,dx$ is the area enclosed by the x-axis and the semicircle $y = \sqrt{1-x^2}$. This area is half the circle's area, or $\frac{\pi}{2}$ and multiplying by 2 gives π. The length of the circular arc $y = \sqrt{1-x^2}$ from $x = -1$ to $x = 1$ is $L = \int_{-1}^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{-1}^1 \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \frac{1}{2}(2\pi) = \pi$ since L is half the circle's circumference. In conclusion, $2\int_{-1}^1 \sqrt{1-x^2}\,dx = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$.

37. $A = \int_1^{\infty} \frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \le 1$ (Exercise 67 in Section 7.6). Thus, $p \le 1$ for infinite area.

The volume of the solid of revolution about the x-axis is $V = \int_1^{\infty} \pi\left(\frac{1}{x^p}\right)^2 dx = \pi \int_1^{\infty} \frac{dx}{x^{2p}}$ which converges if $2p > 1$ and diverges if $2p \le 1$. Thus we want $p > \frac{1}{2}$ for finite volume. In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $\frac{1}{2} < p \le 1$.

38. The area is given by the integral $A = \int_0^1 \frac{dx}{x^p}$;

$p = 1$: $A = \lim_{b \to 0^+} [\ln x]_b^1 = -\lim_{b \to 0^+} \ln b = \infty$, diverges;

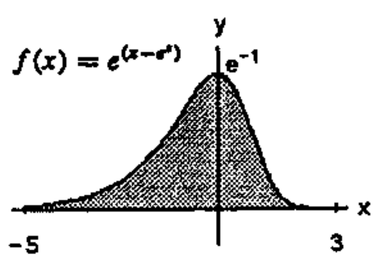
$p > 1$: $A = \lim_{b \to 0^+} \left[x^{1-p}\right]_b^1 = 1 - \lim_{b \to 0^+} b^{1-p} = -\infty$, diverges;

$p < 1$: $A = \lim_{b \to 0^+} \left[x^{1-p}\right]_b^1 = 1 - \lim_{b \to 0^+} b^{1-p} = 1 - 0$, converges; thus, $p \ge 1$ for infinite area.

The volume of the solid of revolution about the x-axis is $V_x = \pi \int_0^1 \frac{dx}{x^{2p}}$ which converges if $2p < 1$ or $p < \frac{1}{2}$, and diverges if $p \ge \frac{1}{2}$. Thus, V_x is infinite whenever the area is infinite ($p \ge 1$).

The volume of the solid of revolution about the y-axis is $V_y = \pi \int_1^{\infty} [R(y)]^2\, dy = \pi \int_1^{\infty} \frac{dy}{y^{2/p}}$ which converges if $\frac{2}{p} > 1 \Leftrightarrow p < 2$ (see Exercise 39). In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $1 \le p < 2$, as described above.

39. (a)



(b) $\int_{-\infty}^{\infty} e^{(x-e^x)}\, dx = \int_{-\infty}^{\infty} e^{(-e^x)} e^x\, dx$

$$= \lim_{a\to-\infty} \int_a^0 e^{(-e^x)} e^x \, dx + \lim_{b\to+\infty} \int_0^b e^{(-e^x)} e^x \, dx;$$

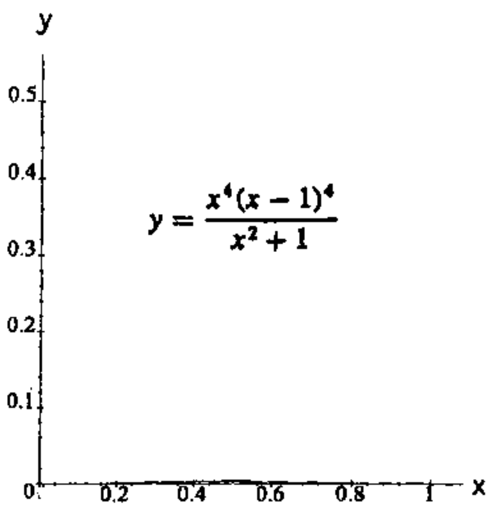
$$\begin{bmatrix} u = e^x \\ du = e^x \, dx \end{bmatrix} \Rightarrow \lim_{a\to-\infty} \int_{e^a}^{1} e^{-u} \, du + \lim_{b\to+\infty} \int_1^{e^b} e^{-u} \, du$$

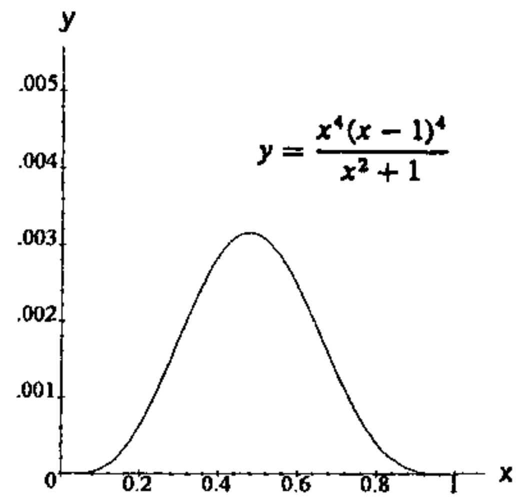
$$= \lim_{a\to-\infty} \left[-e^{-u}\right]_{e^a}^{1} + \lim_{b\to-\infty} \left[-e^{-u}\right]_1^{e^b} = \lim_{a\to-\infty} \left[-\tfrac{1}{e} + e^{-(e^a)}\right] + \lim_{b\to+\infty} \left[-e^{-\left(e^b\right)} + \tfrac{1}{e}\right] = \left(-\tfrac{1}{e} + e^0\right) + \left(0 + \tfrac{1}{e}\right) = 1$$

40. (a) $\int_0^1 \frac{x^4(x-1)^4}{x^2+1} \, dx = \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}\right) dx = \frac{22}{7} - \pi$

(b) $\frac{\frac{22}{7} - \pi}{\pi} \cdot 100\% \cong 0.04\%$

(c) The area is less than 0.003





41. e^{2x} $(+)$ $\cos 3x$

$2e^{2x}$ $(-)$ $\frac{1}{3}\sin 3x$

$4e^{2x}$ $\xrightarrow{(+)}$ $-\frac{1}{9}\cos 3x$

$I = \frac{e^{2x}}{3} \sin 3x + \frac{2e^{2x}}{9} \cos 3x - \frac{4}{9} I \Rightarrow \frac{13}{9} I = \frac{e^{2x}}{9}(3 \sin 3x + 2 \cos 3x) \Rightarrow I = \frac{e^{2x}}{13}(3 \sin 3x + 2 \cos 3x) + C$

42. e^{3x} $(+)$ $\sin 4x$

$3e^{3x}$ $(-)$ $-\frac{1}{4}\cos 4x$

$9e^{3x}$ $\xrightarrow{(+)}$ $-\frac{1}{16}\sin 4x$

$I = -\frac{e^{3x}}{4} \cos 4x + \frac{3e^{3x}}{16} \sin 4x - \frac{9}{16} I \Rightarrow \frac{25}{16} I = \frac{e^{3x}}{16}(3 \sin 4x - 4 \cos 4x) \Rightarrow I = \frac{e^{3x}}{25}(3 \sin 4x - 4 \cos 4x) + C$

43. $\sin 3x \qquad (+) \qquad \sin x$

$3 \cos 3x \qquad (-) \quad -\cos x$

$-9 \sin 3x \xrightarrow{(+)} -\sin x$

$I = -\sin 3x \cos x + 3 \cos 3x \sin x + 9I \Rightarrow -8I = -\sin 3x \cos x + 3 \cos 3x \sin x$

$\Rightarrow I = \frac{\sin 3x \cos x - 3 \cos 3x \sin x}{8} + C$

44. $\cos 5x \qquad (+) \qquad \sin 4x$

$-5 \sin 5x \qquad (-) \quad -\frac{1}{4} \cos 4x$

$-25 \cos 5x \xrightarrow{(+)} -\frac{1}{16} \sin 4x$

$I = -\frac{1}{4} \cos 5x \cos 4x - \frac{5}{16} \sin 5x \sin 4x + \frac{25}{16} I \Rightarrow -\frac{9}{16} I = -\frac{1}{4} \cos 5x \cos 4x - \frac{5}{16} \sin 5x \sin 4x$

$\Rightarrow I = \frac{1}{9}(4 \cos 5x \cos 4x) \Rightarrow I = \frac{1}{9}(4 \cos 5x \cos 4x + 5 \sin 5x \sin 4x) + C$

45. $e^{ax} \qquad (+) \qquad \sin bx$

$ae^{ax} \qquad (-) \quad -\frac{1}{b} \cos bx$

$a^2 e^{ax} \xrightarrow{(+)} -\frac{1}{b^2} \sin bx$

$I = -\frac{e^{ax}}{b} \cos bx + \frac{ae^{ax}}{b^2} \sin bx - \frac{a^2}{b^2} I \Rightarrow \left(\frac{a^2 + b^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(a \sin bx - b \cos bx)$

$\Rightarrow I = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C$

46. $e^{ax} \qquad (+) \qquad \cos bx$

$ae^{ax} \qquad (-) \quad \frac{1}{b} \sin bx$

$a^2 e^{ax} \xrightarrow{(+)} -\frac{1}{b^2} \cos bx$

$I = -\frac{e^{ax}}{b} \sin bx + \frac{ae^{ax}}{b^2} \cos bx - \frac{a^2}{b^2} I \Rightarrow \left(\frac{a^2 + b^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(a \cos bx + b \sin bx)$

$\Rightarrow I = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C$

47. $\ln(ax) \qquad (+) \qquad 1$

$\frac{1}{x} \xrightarrow{(-)} x$

$I = x \ln(ax) - \int \left(\frac{1}{x}\right) x \, dx = x \ln(ax) - x + C$

48. $\ln(ax) \quad (+) \quad x^2$

$\frac{1}{x} \xrightarrow{(-)} \quad \frac{1}{3}x^3$

$I = \frac{1}{3}x^3 \ln(ax) - \int \left(\frac{1}{x}\right)\left(\frac{x^3}{3}\right) dx = \frac{1}{3}x^3 \ln(ax) - \frac{1}{9}x^3 + C$

49. (a) $\Gamma(1) = \int_0^{\infty} e^{-t}\, dt = \lim_{b\to\infty} \int_0^{b} e^{-t}\, dt = \lim_{b\to\infty} \left[-e^{-t}\right]_0^b = \lim_{b\to\infty} \left[-\frac{1}{e^b} - (-1)\right] = 0 + 1 = 1$

(b) $u = t^x$, $du = xt^{x-1}\, dt$; $dv = e^{-t}\, dt$, $v = -e^{-t}$; $x =$ fixed positive real

$\Rightarrow \Gamma(x+1) = \int_0^{\infty} t^x e^{-t}\, dt = \lim_{b\to\infty} \left[-t^x e^{-t}\right]_0^b + x \int_0^{\infty} t^{x-1} e^{-t}\, dt = \lim_{b\to\infty} \left(-\frac{b^x}{e^b} + 0^x e^0\right) + x\Gamma(x) = x\Gamma(x)$

(c) $\Gamma(n+1) = n\Gamma(n) = n!$:

$n = 0$: $\Gamma(0+1) = \Gamma(1) = 0!$;

$n = k$: Assume $\Gamma(k+1) = k!$ for some $k > 0$;

$n = k+1$: $\Gamma(k+1+1) = (k+1)\,\Gamma(k+1)$ from part (b)

$= (k+1)k!$ induction hypothesis

$= (k+1)!$ definition of factorial

Thus, $\Gamma(n+1) = n\Gamma(n) = n!$ for every positive integer n.

50. (a) $\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}$ and $n\Gamma(n) = n! \Rightarrow n! \approx n\left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{n}} = \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$

(b)

n	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$	calculator
10	3598695.619	3628800
20	2.4227868×10^{18}	2.432902×10^{18}
30	2.6451710×10^{32}	2.652528×10^{32}
40	8.1421726×10^{47}	8.1591528×10^{47}
50	3.0363446×10^{64}	3.0414093×10^{64}
60	8.3094383×10^{81}	8.3209871×10^{81}

(c)

n	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}\, e^{1/12n}$	calculator
10	3598695.619	3628810.051	3628800

NOTES:

CHAPTER 8 INFINITE SERIES

8.1 LIMITS OF SEQUENCES OF NUMBERS

1. $a_1 = \frac{1-1}{1^2} = 0$, $a_2 = \frac{1-2}{2^2} = -\frac{1}{4}$, $a_3 = \frac{1-3}{3^2} = -\frac{2}{9}$, $a_4 = \frac{1-4}{4^2} = -\frac{3}{16}$

2. $a_1 = \frac{1}{1!} = 1$, $a_2 = \frac{1}{2!} = \frac{1}{2}$, $a_3 = \frac{1}{3!} = \frac{1}{6}$, $a_4 = \frac{1}{4!} = \frac{1}{24}$

3. $a_1 = \frac{(-1)^2}{2-1} = 1$, $a_2 = \frac{(-1)^3}{4-1} = -\frac{1}{3}$, $a_3 = \frac{(-1)^4}{6-1} = \frac{1}{5}$, $a_4 = \frac{(-1)^5}{8-1} = -\frac{1}{7}$

4. $a_1 = \frac{2}{2^2} = \frac{1}{2}$, $a_2 = \frac{2^2}{2^3} = \frac{1}{2}$, $a_3 = \frac{2^3}{2^4} = \frac{1}{2}$, $a_4 = \frac{2^4}{2^5} = \frac{1}{2}$

5. $a_n = (-1)^{n+1}$, $n = 1, 2, \ldots$

6. $a_n = (-1)^{n+1}n^2$, $n = 1, 2, \ldots$

7. $a_n = n^2 - 1$, $n = 1, 2, \ldots$

8. $a_n = n - 4$, $n = 1, 2, \ldots$

9. $a_n = 4n - 3$, $n = 1, 2, \ldots$

10. $a_n = 4n - 2$, $n = 1, 2, \ldots$

11. $a_n = \frac{1 + (-1)^{n+1}}{2}$, $n = 1, 2, \ldots$

12. $a_n = \frac{n - \frac{1}{2} + (-1)^n\left(\frac{1}{2}\right)}{2} = \operatorname{int}\left(\frac{n}{2}\right)$, $n = 1, 2, \ldots$

13. $\lim\limits_{n\to\infty} 2 + (0.1)^n = 2 \Rightarrow$ converges (Table 8.1, #4)

14. $\lim\limits_{n\to\infty} \frac{n + (-1)^n}{n} = \lim\limits_{n\to\infty} 1 + \frac{(-1)^n}{n} = 1 \Rightarrow$ converges

15. $\lim\limits_{n\to\infty} \frac{1-2n}{1+2n} = \lim\limits_{n\to\infty} \frac{\left(\frac{1}{n}\right) - 2}{\left(\frac{1}{n}\right) + 2} = \lim\limits_{n\to\infty} \frac{-2}{2} = -1 \Rightarrow$ converges

16. $\lim\limits_{n\to\infty} \frac{1-5n^4}{n^4+8n^3} = \lim\limits_{n\to\infty} \frac{\left(\frac{1}{n^4}\right) - 5}{1 + \left(\frac{8}{n}\right)} = -5 \Rightarrow$ converges

17. $\lim\limits_{n\to\infty} \frac{n^2-2n+1}{n-1} = \lim\limits_{n\to\infty} \frac{(n-1)(n-1)}{n-1} = \lim\limits_{n\to\infty} (n-1) = \infty \Rightarrow$ diverges

18. $\lim\limits_{n\to\infty} \frac{n+3}{n^2+5n+6} = \lim\limits_{n\to\infty} \frac{n+3}{(n+3)(n+2)} = \lim\limits_{n\to\infty} \frac{1}{n+2} = 0 \Rightarrow$ converges

19. $\lim_{n\to\infty} \left(1+(-1)^n\right)$ does not exist $\Rightarrow$ diverges

20. $\lim_{n\to\infty} (-1)^n\left(1-\frac{1}{n}\right)$ does not exist $\Rightarrow$ diverges

21. $\lim_{n\to\infty} \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{1}{2}+\frac{1}{2n}\right)\left(1-\frac{1}{n}\right) = \frac{1}{2} \Rightarrow$ converges

22. $\lim_{n\to\infty} \frac{(-1)^{n+1}}{2n-1} = 0 \Rightarrow$ converges

23. $\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty} \left(\frac{2}{1+\frac{1}{n}}\right)} = \sqrt{2} \Rightarrow$ converges

24. $\lim_{n\to\infty} \sin\left(\frac{\pi}{2}+\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty} \left(\frac{\pi}{2}+\frac{1}{n}\right)\right) = \sin\frac{\pi}{2} = 1 \Rightarrow$ converges

25. $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ because $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \Rightarrow$ converges by the Sandwich Theorem for sequences

26. $\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$ because $0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n} \Rightarrow$ converges by the Sandwich Theorem for sequences

27. $\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 0 \Rightarrow$ converges (using l'Hôpital's rule)

28. $\lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} = \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{2\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{\left(\frac{2}{\sqrt{n}}\right)}{1+\left(\frac{1}{n}\right)} = 0 \Rightarrow$ converges

29. $\lim_{n\to\infty} \frac{\ln n}{n^{1/n}} = \frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} n^{1/n}} = \frac{\infty}{1} = \infty \Rightarrow$ diverges (Table 8.1, #2)

30. $\lim_{n\to\infty} [\ln n - \ln(n+1)] = \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{n+1}\right) = \ln 1 = 0 \Rightarrow$ converges

31. $\lim_{n\to\infty} \left(1+\frac{7}{n}\right)^n = e^7 \Rightarrow$ converges (Table 8.1, #5)

32. $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} \left[1+\frac{(-1)}{n}\right]^n = e^{-1} \Rightarrow$ converges (Table 8.1, #5)

33. $\lim_{n\to\infty} \sqrt[n]{10n} = \lim_{n\to\infty} 10^{1/n} \cdot n^{1/n} = 1 \cdot 1 = 1 \Rightarrow$ converges (Table 8.1, #3 and #2)

34. $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^2 = 1^2 = 1 \Rightarrow$ converges (Table 8.1, #2)

35. $\lim_{n\to\infty} \left(\frac{3}{n}\right)^{1/n} = \frac{\lim_{n\to\infty} 3^{1/n}}{\lim_{n\to\infty} n^{1/n}} = \frac{1}{1} = 1 \Rightarrow$ converges (Table 8.1, #3 and #2)

36. $\lim\limits_{n\to\infty} (n+4)^{1/(n+4)} = \lim\limits_{x\to\infty} x^{1/x} = 1 \Rightarrow$ converges; (let $x = n+4$, then use Table 8.1, #2)

37. $\lim\limits_{n\to\infty} \sqrt[n]{4^n n} = \lim\limits_{n\to\infty} 4\sqrt[n]{n} = 4\cdot 1 = 4 \Rightarrow$ converges (Table 8.1, #2)

38. $\lim\limits_{n\to\infty} \sqrt[n]{3^{2n+1}} = \lim\limits_{n\to\infty} 3^{2+(1/n)} = \lim\limits_{n\to\infty} 3^2\cdot 3^{1/n} = 9\cdot 1 = 9 \Rightarrow$ converges (Table 8.1, #3)

39. $\lim\limits_{n\to\infty} \frac{n!}{n^n} = \lim\limits_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots(n-1)(n)}{n\cdot n\cdot n\cdots n\cdot n} \le \lim\limits_{n\to\infty} \left(\frac{1}{n}\right) = 0$ and $\frac{n!}{n^n} \ge 0 \Rightarrow \lim\limits_{n\to\infty} \frac{n!}{n^n} = 0 \Rightarrow$ converges

40. $\lim\limits_{n\to\infty} \frac{(-4)^n}{n!} = 0 \Rightarrow$ converges (Table 8.1, #6)

41. $\lim\limits_{n\to\infty} \frac{n!}{10^{6n}} = \lim\limits_{n\to\infty} \frac{1}{\left(\frac{(10^6)^n}{n!}\right)} = \infty \Rightarrow$ diverges (Table 8.1, #6)

42. $\lim\limits_{n\to\infty} \frac{n!}{2^n 3^n} = \lim\limits_{n\to\infty} \frac{1}{\left(\frac{6^n}{n!}\right)} = \infty \Rightarrow$ diverges (Table 8.1, #6)

43. $\lim\limits_{n\to\infty} \left(\frac{1}{n}\right)^{1/(\ln n)} = \lim\limits_{n\to\infty} \exp\left(\frac{1}{\ln n}\ln\left(\frac{1}{n}\right)\right) = \lim\limits_{n\to\infty} \exp\left(\frac{\ln 1 - \ln n}{\ln n}\right) = e^{-1} \Rightarrow$ converges

44. $\lim\limits_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \ln\left(\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right) = \ln e = 1 \Rightarrow$ converges (Table 8.1, #5)

45. $\lim\limits_{n\to\infty} \left(\frac{3n+1}{3n-1}\right)^n = \lim\limits_{n\to\infty} \exp\left(n\ln\left(\frac{3n+1}{3n-1}\right)\right) = \lim\limits_{n\to\infty} \exp\left(\frac{\ln(3n+1)-\ln(3n-1)}{\frac{1}{n}}\right)$

$= \lim\limits_{n\to\infty} \exp\left(\frac{\frac{3}{3n+1}-\frac{3}{3n-1}}{\left(-\frac{1}{n^2}\right)}\right) = \lim\limits_{n\to\infty} \exp\left(\frac{6n^2}{(3n+1)(3n-1)}\right) = \exp\left(\frac{6}{9}\right) = e^{2/3} \Rightarrow$ converges

46. $\lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim\limits_{n\to\infty} \exp\left(n\ln\left(\frac{n}{n+1}\right)\right) = \lim\limits_{n\to\infty} \exp\left(\frac{\ln n - \ln(n+1)}{\left(\frac{1}{n}\right)}\right) = \lim\limits_{n\to\infty} \exp\left(\frac{\frac{1}{n}-\frac{1}{n+1}}{\left(-\frac{1}{n^2}\right)}\right)$

$= \lim\limits_{n\to\infty} \exp\left(-\frac{n^2}{n(n+1)}\right) = e^{-1} \Rightarrow$ converges

47. $\lim\limits_{n\to\infty} \left(\frac{x^n}{2n+1}\right)^{1/n} = \lim\limits_{n\to\infty} x\left(\frac{1}{2n+1}\right)^{1/n} = x\lim\limits_{n\to\infty} \exp\left(\frac{1}{n}\ln\left(\frac{1}{2n+1}\right)\right) = x\lim\limits_{n\to\infty} \exp\left(\frac{-\ln(2n+1)}{n}\right)$

$= x\lim\limits_{n\to\infty} \exp\left(\frac{-2}{2n+1}\right) = xe^0 = x,\ x > 0 \Rightarrow$ converges

48. $\lim\limits_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n = \lim\limits_{n\to\infty} \exp\left(n \ln\left(1-\frac{1}{n^2}\right)\right) = \lim\limits_{n\to\infty} \exp\left(\frac{\ln\left(1-\frac{1}{n^2}\right)}{\left(\frac{1}{n}\right)}\right) = \lim\limits_{n\to\infty} \exp\left[\frac{\left(\frac{2}{n^3}\right)/\left(1-\frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}\right]$

$= \lim\limits_{n\to\infty} \exp\left(\frac{-2n}{n^2-1}\right) = e^0 = 1 \Rightarrow$ converges

49. $\lim\limits_{n\to\infty} \frac{3^n \cdot 6^n}{2^{-n} \cdot n!} = \lim\limits_{n\to\infty} \frac{36^n}{n!} = 0 \Rightarrow$ converges (Table 8.1, #6)

50. $\lim\limits_{n\to\infty} \frac{n^2 \sin\left(\frac{1}{n}\right)}{2n-1} = \lim\limits_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{2}{n}-\frac{1}{n^2}\right)} = \lim\limits_{n\to\infty} \frac{-\left(\cos\left(\frac{1}{n}\right)\right)\left(\frac{1}{n^2}\right)}{\left(-\frac{2}{n^2}+\frac{2}{n^3}\right)} = \lim\limits_{n\to\infty} \frac{-\cos\left(\frac{1}{n}\right)}{-2+\left(\frac{2}{n}\right)} = \frac{1}{2} \Rightarrow$ converges

51. $\lim\limits_{n\to\infty} \tan^{-1} n = \frac{\pi}{2} \Rightarrow$ converges

52. $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} \tan^{-1} n = 0 \cdot \frac{\pi}{2} = 0 \Rightarrow$ converges

53. $\lim\limits_{n\to\infty} \left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}} = \lim\limits_{n\to\infty} \left(\left(\frac{1}{3}\right)^n + \left(\frac{1}{\sqrt{2}}\right)^n\right) = 0 \Rightarrow$ converges (Table 8.1, #4)

54. $\lim\limits_{n\to\infty} \sqrt[n]{n^2+n} = \lim\limits_{n\to\infty} \exp\left[\frac{\ln\left(n^2+n\right)}{n}\right] = \lim\limits_{n\to\infty} \exp\left(\frac{2n+1}{n^2+n}\right) = e^0 = 1 \Rightarrow$ converges

55. $\lim\limits_{n\to\infty} \frac{(\ln n)^5}{\sqrt{n}} = \lim\limits_{n\to\infty} \left[\frac{\left(\frac{5(\ln n)^4}{n}\right)}{\left(\frac{1}{2\sqrt{n}}\right)}\right] = \lim\limits_{n\to\infty} \frac{10(\ln n)^4}{\sqrt{n}} = \lim\limits_{n\to\infty} \frac{80(\ln n)^3}{\sqrt{n}} = \ldots = \lim\limits_{n\to\infty} \frac{3840}{\sqrt{n}} = 0 \Rightarrow$ converges

56. $\lim\limits_{n\to\infty} \left(n-\sqrt{n^2-n}\right) = \lim\limits_{n\to\infty} \left(n-\sqrt{n^2-n}\right)\left(\frac{n+\sqrt{n^2-n}}{n+\sqrt{n^2-n}}\right) = \lim\limits_{n\to\infty} \frac{n}{n+\sqrt{n^2-n}} = \lim\limits_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}}$

$= \frac{1}{2} \Rightarrow$ converges

57. $\left|\sqrt[n]{0.5}-1\right| < 10^{-3} \Rightarrow -\frac{1}{1000} < \left(\frac{1}{2}\right)^{1/n} - 1 < \frac{1}{1000} \Rightarrow \left(\frac{999}{1000}\right)^n < \frac{1}{2} < \left(\frac{1001}{1000}\right)^n \Rightarrow n > \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{999}{1000}\right)} \Rightarrow n > 692.8$

$\Rightarrow N = 692$; $a_n = \left(\frac{1}{2}\right)^{1/n}$ and $\lim\limits_{n\to\infty} a_n = 1$

58. $\left|\sqrt[n]{n}-1\right| < 10^{-3} \Rightarrow -\frac{1}{1000} < n^{1/n} - 1 < \frac{1}{1000} \Rightarrow \left(\frac{999}{1000}\right)^n < n < \left(\frac{1001}{1000}\right)^n \Rightarrow n > 9123 \Rightarrow N = 9123$;

$a_n = \sqrt[n]{n} = n^{1/n}$ and $\lim\limits_{n\to\infty} a_n = 1$

59. $(0.9)^n < 10^{-3} \Rightarrow n \ln(0.9) < -3 \ln 10 \Rightarrow n > \frac{-3 \ln 10}{\ln(0.9)} \approx 65.54 \Rightarrow N = 65$; $a_n = \left(\frac{9}{10}\right)^n$ and $\lim\limits_{n\to\infty} a_n = 0$

60. $\frac{2^n}{n!} < 10^{-7} \Rightarrow n! > 2^n 10^7$ and by calculator experimentation, $n > 14 \Rightarrow N = 14$; $a_n = \frac{2^n}{n!}$ and $\lim\limits_{n\to\infty} a_n = 0$

61. (a) $1^2 - 2(1)^2 = -1$, $3^2 - 2(2)^2 = 1$; let $f(a,b) = (a+2b)^2 - 2(a+b)^2 = a^2 + 4ab + 4b^2 - 2a^2 - 4ab - 2b^2$
$= 2b^2 - a^2$; $a^2 - 2b^2 = -1 \Rightarrow f(a,b) = 2b^2 - a^2 = 1$; $a^2 - 2b^2 = 1 \Rightarrow f(a,b) = 2b^2 - a^2 = -1$

(b) $r_n^2 - 2 = \left(\frac{a+2b}{a+b}\right)^2 - 2 = \frac{a^2+4ab+4b^2-2a^2-4ab-2b^2}{(a+b)^2} = \frac{-(a^2-2b^2)}{(a+b)^2} = \frac{\pm 1}{y_n^2} \Rightarrow r_n = \sqrt{2 \pm \left(\frac{1}{y_n}\right)^2}$

In the first and second fractions, $y_n \ge n$. Let $\frac{a}{b}$ represent the $(n-1)$th fraction where $\frac{a}{b} \ge 1$ and $b \ge n-1$ for n a positive integer ≥ 3. Now the nth fraction is $\frac{a+2b}{a+b}$ and $a + b \ge 2b \ge 2n - 2 \ge n \Rightarrow y_n \ge n$. Thus, $\lim\limits_{n\to\infty} r_n = \sqrt{2}$.

62. (a) $\lim\limits_{n\to\infty} nf\left(\frac{1}{n}\right) = \lim\limits_{\Delta x\to 0^+} \frac{f(\Delta x)}{\Delta x} = \lim\limits_{\Delta x\to 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = f'(0)$, where $\Delta x = \frac{1}{n}$

(b) $\lim\limits_{n\to\infty} n \tan^{-1}\left(\frac{1}{n}\right) = f'(0) = \frac{1}{1+0^2} = 1$, $f(x) = \tan^{-1} x$

(c) $\lim\limits_{n\to\infty} n\left(e^{1/n} - 1\right) = f'(0) = e^0 = 1$, $f(x) = e^x$

(d) $\lim\limits_{n\to\infty} n \ln\left(1 + \frac{2}{n}\right) = f'(0) = \frac{2}{1+2(0)} = 2$, $f(x) = \ln(1+2x)$

63. (a) If $a = 2n+1$, then $b = \lfloor\frac{a^2}{2}\rfloor = \lfloor\frac{4n^2+4n+1}{2}\rfloor = \lfloor 2n^2 + 2n + \frac{1}{2}\rfloor = 2n^2 + 2n$, $c = \lceil\frac{a^2}{2}\rceil = \lceil 2n^2 + 2n + \frac{1}{2}\rceil$
$= 2n^2 + 2n + 1$ and $a^2 + b^2 = (2n+1)^2 + \left(2n^2+2n\right)^2 = 4n^2 + 4n + 1 + 4n^4 + 8n^3 + 4n^2$
$= 4n^4 + 8n^3 + 8n^2 + 4n + 1 = \left(2n^2 + 2n + 1\right)^2 = c^2$.

(b) $\lim\limits_{a\to\infty} \frac{\lfloor\frac{a^2}{2}\rfloor}{\lceil\frac{a^2}{2}\rceil} = \lim\limits_{a\to\infty} \frac{2n^2+2n}{2n^2+2n+1} = 1$ or $\lim\limits_{a\to\infty} \frac{\lfloor\frac{a^2}{2}\rfloor}{\lceil\frac{a^2}{2}\rceil} = \lim\limits_{a\to\infty} \sin\theta = \lim\limits_{\theta\to\pi/2} \sin\theta = 1$

64. (a) $\lim\limits_{n\to\infty} (2n\pi)^{1/(2n)} = \lim\limits_{n\to\infty} \exp\left(\frac{\ln 2n\pi}{2n}\right) = \lim\limits_{n\to\infty} \exp\left(\frac{\left(\frac{2\pi}{2n\pi}\right)}{2}\right) = \lim\limits_{n\to\infty} \exp\left(\frac{1}{2n}\right) = e^0 = 1$;

$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$, Stirlings approximation $\Rightarrow \sqrt[n]{n!} \approx \left(\frac{n}{e}\right)(2n\pi)^{1/(2n)} \approx \frac{n}{e}$ for large values of n

(b)

n	$\sqrt[n]{n!}$	$\frac{n}{e}$
40	15.76852702	14.71517765
50	19.48325423	18.39397206
60	23.19189561	22.07276647

65. (a) $\lim\limits_{n\to\infty} \frac{\ln n}{n^c} = \lim\limits_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{cn^{c-1}} = \lim\limits_{n\to\infty} \frac{1}{cn^c} = 0$

(b) For all $\epsilon > 0$, there exists an $N = e^{-(\ln \epsilon)/c}$ such that $n > e^{-(\ln \epsilon)/c} \Rightarrow \ln n > -\frac{\ln \epsilon}{c} \Rightarrow \ln n^c > \ln\left(\frac{1}{\epsilon}\right)$ $\Rightarrow n^c > \frac{1}{\epsilon} \Rightarrow \frac{1}{n^c} < \epsilon \Rightarrow \left|\frac{1}{n^c} - 0\right| < \epsilon \Rightarrow \lim\limits_{n\to\infty} \frac{1}{n^c} = 0$

66. Let $\{a_n\}$ and $\{b_n\}$ be sequences both converging to L. Define $\{c_n\}$ by $c_{2n} = b_n$ and $c_{2n-1} = a_n$, where $n = 1, 2, 3, \ldots$. For all $\epsilon > 0$ there exists N_1 such that when $n > N_1$ then $|a_n - L| < \epsilon$ and there exists N_2 such that when $n > N_2$ then $|b_n - L| < \epsilon$. If $n > \max\{N_1, N_2\}$, then both inequalities hold and hence $|c_n - L| < \epsilon$, so $\{c_n\}$ converges to L.

67. $\lim\limits_{n\to\infty} n^{1/n} = \lim\limits_{n\to\infty} \exp\left(\frac{1}{n} \ln n\right) = \lim\limits_{n\to\infty} \exp\left(\frac{1}{n}\right) = e^0 = 1$

68. $\lim\limits_{n\to\infty} x^{1/n} = \lim\limits_{n\to\infty} \exp\left(\frac{1}{n} \ln x\right) = e^0 = 1$, because x remains fixed while n gets large

69. Assume the hypotheses of the theorem and let ϵ be a positive number. For all ϵ there exists a N_1 such that when $n > N_1$ then $|a_n - L| < \epsilon \Rightarrow -\epsilon < a_n - L < \epsilon \Rightarrow L - \epsilon < a_n$, and there exists a N_2 such that when $n > N_2$ then $|c_n - L| < \epsilon \Rightarrow -\epsilon < c_n - L < \epsilon \Rightarrow c_n < L + \epsilon$. If $n > \max\{N_1, N_2\}$, then $L - \epsilon < a_n \le b_n \le c_n < L + \epsilon \Rightarrow |b_n - L| < \epsilon \Rightarrow \lim\limits_{n\to\infty} b_n = L$.

70. $|a_n - L| < \delta \Rightarrow |f(a_n) - f(L)| < \epsilon \Rightarrow f(a_n) \to f(L)$

71. Let L be the limit of the convergent sequence $\{a_n\}$. Then by definition of convergence, for $\frac{\epsilon}{2}$ there corresponds an N such that for all m and n, $m > N \Rightarrow |a_m - L| < \frac{\epsilon}{2}$ and $n > N \Rightarrow |a_n - L| < \frac{\epsilon}{2}$. Now $|a_m - a_n| = |a_m - L + L - a_n| \le |a_m - L| + |L - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ whenever $m > N$ and $n > N$.

72. Given an $\epsilon > 0$, by definition of convergence there corresponds an N such that for all $n > N$, $|L_1 - a_n| < \epsilon$ and $|L_2 - a_n| < \epsilon$. Now $|L_2 - L_1| = |L_2 - a_n + a_n - L_1| \le |L_2 - a_n| + |a_n - L_1| < \epsilon + \epsilon = 2\epsilon$. $|L_2 - L_1| < 2\epsilon$ says that the difference between two fixed values is smaller than any positive number 2ϵ. The only nonnegative number smaller than every positive number is 0, so $|L_1 - L_2| = 0$ or $L_1 = L_2$.

73. Assume $a_n \to 0$. This implies that given an $\epsilon > 0$ there corresponds an N such that $n > N \Rightarrow |a_n - 0| < \epsilon$ $\Rightarrow |a_n| < \epsilon \Rightarrow ||a_n|| < \epsilon \Rightarrow ||a_n| - 0| < \epsilon \Rightarrow |a_n| \to 0$. On the other hand, assume $|a_n| \to 0$. This implies that given an $\epsilon > 0$ there corresponds an N such that for $n > N$, $||a_n| - 0| < \epsilon \Rightarrow ||a_n|| < \epsilon \Rightarrow |a_n| < \epsilon$ $\Rightarrow |a_n - 0| < \epsilon \Rightarrow a_n \to 0$.

74. (a) $S_1 = 6.815$, $S_2 = 6.4061$, $S_3 = 6.021734$, $S_4 = 5.66042996$, $S_5 = 5.320804162$, $S_6 = 5.001555913$, $S_7 = 4.701462558$, $S_8 = 4.419374804$, $S_9 = 4.154212316$, $S_{10} = 3.904959577$, $S_{11} = 3.670662003$, $S_{12} = 3.450422282$ so it will take Ford about 12 years to catch up

(b) $3.5 = 7.25(0.94)^n \Rightarrow (0.94)^n = \frac{3.5}{7.25}$

$\Rightarrow n \ln(0.94) = \ln \frac{3.5}{7.25} \Rightarrow n = \frac{\ln\left(\frac{3.5}{7.25}\right)}{\ln(0.94)}$

$\Rightarrow n \approx 11.764 \approx 12$

75-84. Example CAS Commands:

Maple:

```
a:= n -> (n)^(1/n);
j:= 9400: k:= 9800: A:= plot(a(n), n=j..k, style=POINT, symbol=CIRCLE):
f:= x -> 0.999: g:= x -> 1.001:
B:= plot({f(x), g(x)}, x=j..k):
with(plots): display({A,B});
```

Mathematica:

```
Clear[a,i,n]
a[n_] = n^(1/n)
atab = Table[ a[i], {i,25} ] // N;
ListPlot[ atab ]
L = Limit[ a[n], n -> Infinity ]

Note: for this a[n], the first n for which |a[n]-L|< 0.001 is n = 1!  Let's
find the next...

a[1] - L

First check several orders of magnitude, then zoom in by trial & error:

Table[ {i, N[a[10^i] - L]}, {i,10} ]
N[a[9000] - L]
N[a[9200] - L]
N[a[9123] - L]
N[a[9124] - L]

This is the first n for which |a[n]-L|< 0.001; for 0.0001, we get the rough
estimate:

N[a[120000] - L]
```

8.2 SUBSEQUENCES, BOUNDED SEQUENCES, AND PICARD'S METHOD

1. $a_1 = 1$, $a_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $a_3 = \frac{3}{2} + \frac{1}{2^2} = \frac{7}{4}$, $a_4 = \frac{7}{4} + \frac{1}{2^3} = \frac{15}{8}$, $a_5 = \frac{15}{8} + \frac{1}{2^4} = \frac{31}{16}$, $a_6 = \frac{63}{32}$, $a_7 = \frac{127}{64}$, $a_8 = \frac{255}{128}$, $a_9 = \frac{511}{256}$, $a_{10} = \frac{1023}{512}$

2. $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{\left(\frac{1}{2}\right)}{3} = \frac{1}{6}$, $a_4 = \frac{\left(\frac{1}{6}\right)}{4} = \frac{1}{24}$, $a_5 = \frac{\left(\frac{1}{24}\right)}{5} = \frac{1}{120}$, $a_6 = \frac{1}{720}$, $a_7 = \frac{1}{5040}$, $a_8 = \frac{1}{40,320}$, $a_9 = \frac{1}{362,880}$, $a_{10} = \frac{1}{3,628,800}$

3. $a_1 = 2,\ a_2 = \frac{(-1)^2(2)}{2} = 1,\ a_3 = \frac{(-1)^3(1)}{2} = -\frac{1}{2},\ a_4 = \frac{(-1)^4\left(-\frac{1}{2}\right)}{2} = -\frac{1}{4},\ a_5 = \frac{(-1)^5\left(-\frac{1}{4}\right)}{2} = \frac{1}{8},$
$a_6 = \frac{1}{16},\ a_7 = -\frac{1}{32},\ a_8 = -\frac{1}{64},\ a_9 = \frac{1}{128},\ a_{10} = \frac{1}{256}$

4. $a_1 = -2,\ a_2 = \frac{1\cdot(-2)}{2} = -1,\ a_3 = \frac{2\cdot(-1)}{3} = -\frac{2}{3},\ a_4 = \frac{3\cdot\left(-\frac{2}{3}\right)}{4} = -\frac{1}{2},\ a_5 = \frac{4\cdot\left(-\frac{1}{2}\right)}{5} = -\frac{2}{5},\ a_6 = -\frac{1}{3},$
$a_7 = -\frac{2}{7},\ a_8 = -\frac{1}{4},\ a_9 = -\frac{2}{9},\ a_{10} = -\frac{1}{5}$

5. $a_1 = 1,\ a_2 = 1,\ a_3 = 1 + 1 = 2,\ a_4 = 2 + 1 = 3,\ a_5 = 3 + 2 = 5,\ a_6 = 8,\ a_7 = 13,\ a_8 = 21,\ a_9 = 34,\ a_{10} = 55$

6. $a_1 = 2,\ a_2 = -1,\ a_3 = -\frac{1}{2},\ a_4 = \frac{\left(-\frac{1}{2}\right)}{-1} = \frac{1}{2},\ a_5 = \frac{\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)} = -1,\ a_6 = -2,\ a_7 = 2,\ a_8 = -1,\ a_9 = -\frac{1}{2},\ a_{10} = \frac{1}{2}$

7. (a) $f(x) = x^2 - a \Rightarrow f'(x) = 2x \Rightarrow x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} \Rightarrow x_{n+1} = \frac{2x_n^2 - \left(x_n^2 - a\right)}{2x_n} = \frac{x_n^2 + a}{2x_n} = \frac{\left(x_n + \frac{a}{x_n}\right)}{2}$

(b) $x_1 = 2,\ x_2 = 1.75,\ x_3 = 1.732142857,\ x_4 = 1.73205081,\ x_5 = 1.732050808$; we are finding the positive number where $x^2 - 3 = 0$; that is, where $x^2 = 3,\ x > 0$, or where $x = \sqrt{3}$.

8. $x_1 = 1.5,,\ x_2 = 1.416666667,\ x_3 = 1.414215686,\ x_4 = 1.414213562,\ x_5 = 1.414213562$; we are finding the positive number $x^2 - 2 = 0$; that is, where $x^2 = 2,\ x > 0$, or where $x = \sqrt{2}$.

9. (a) $f(x) = x^2 - 2$; the sequence converges to $1.414213562 \approx \sqrt{2}$

(b) $f(x) = \tan(x) - 1$; the sequence converges to $0.7853981635 \approx \frac{\pi}{4}$

(c) $f(x) = e^x$; the sequence $1, 0, -1, -2, -3, -4, -5, \ldots$ diverges

10. (a) $x_1 = 1,\ x_2 = 1 + \cos(1) = 1.540302306,\ x_3 = 1.540302306 + \cos(1 + \cos(1)) = 1.570791601,$
$x_4 = 1.570791601 + \cos(1.570791601) = 1.570796327 = \frac{\pi}{2}$ to 9 decimal places.

(b) After a few steps, the arc$\left(x_{n-1}\right)$ and line segment $\cos\left(x_{n-1}\right)$ are nearly the same as the quarter circle.

11. $a_{n+1} \ge a_n \Rightarrow \frac{3(n+1)+1}{(n+1)+1} > \frac{3n+1}{n+1} \Rightarrow \frac{3n+4}{n+2} > \frac{3n+1}{n+1} \Rightarrow 3n^2 + 3n + 4n + 4 > 3n^2 + 6n + n + 2$
$\Rightarrow 4 > 2$; the steps are reversible so the sequence is nondecreasing; $\frac{3n+1}{n+1} < 3 \Rightarrow 3n + 1 < 3n + 3$
$\Rightarrow 1 < 3$; the steps are reversible so the sequence is bounded above by 3

12. $a_{n+1} \ge a_n \Rightarrow \frac{(2(n+1)+3)!}{((n+1)+1)!} > \frac{(2n+3)!}{(n+1)!} \Rightarrow \frac{(2n+5)!}{(n+2)!} > \frac{(2n+3)!}{(n+1)!} \Rightarrow \frac{(2n+5)!}{(2n+3)!} > \frac{(n+2)!}{(n+1)!}$
$\Rightarrow (2n+5)(2n+4) > n + 2$; the steps are reversible so the sequence is nondecreasing; the sequence is not bounded since $\frac{(2n+3)!}{(n+1)!} = (2n+3)(2n+2)\cdots(n+2)$ can become as large as we please

13. $a_{n+1} \le a_n \Rightarrow \frac{2^{n+1}3^{n+1}}{(n+1)!} \le \frac{2^n 3^n}{n!} \Rightarrow \frac{2^{n+1}3^{n+1}}{2^n 3^n} \le \frac{(n+1)!}{n!} \Rightarrow 2\cdot 3 \le n + 1$ which is true for $n \ge 5$; the steps are reversible so the sequence is decreasing after a_5, but it is not nondecreasing for all its terms; $a_1 = 6,\ a_2 = 18,$

$a_3 = 36$, $a_4 = 54$, $a_5 = \frac{324}{5} = 64.8 \Rightarrow$ the sequence is bounded from above by 64.8

14. $a_{n+1} \ge a_n \Rightarrow 2 - \frac{2}{n+1} - \frac{1}{2^{n+1}} \ge 2 - \frac{2}{n} - \frac{1}{2^n} \Rightarrow \frac{2}{n} - \frac{2}{n+1} \ge \frac{1}{2^{n+1}} - \frac{1}{2^n} \Rightarrow \frac{2}{n(n+1)} \ge -\frac{1}{2^{n+1}}$; the steps are reversible so the sequence is nondecreasing; $2 - \frac{2}{n} - \frac{1}{2^n} \le 2 \Rightarrow$ the sequence is bounded from above

15. $a_n = 1 - \frac{1}{n}$ converges because $\frac{1}{n} \to 0$ by Example 6 in Section 8.1; also it is a nondecreasing sequence bounded above by 1

16. $a_n = n - \frac{1}{n}$ diverges because $n \to \infty$ and $\frac{1}{n} \to 0$ by Example 6 in Section 8.1, so the sequence is unbounded

17. $a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$ and $0 < \frac{1}{2^n} < \frac{1}{n}$; since $\frac{1}{n} \to 0$ (by Example 6 in Section 8.1) $\Rightarrow \frac{1}{2^n} \to 0$, the sequence converges; also it is a nondecreasing sequence bounded above by 1

18. $a_n = \frac{2^n - 1}{3^n} = \left(\frac{2}{3}\right)^n - \frac{1}{3^n}$; $0 < \left(\frac{2}{3}\right)^{n+1} < \left(\frac{2}{3}\right)^n$ and $0 < \frac{1}{3^n} < \frac{1}{n} \Rightarrow$ the sequence converges by definition of convergence

19. $a_n = \left((-1)^n + 1\right)\left(\frac{n+1}{n}\right)$ diverges because $a_n = 0$ for n odd, while for n even $a_n = 2\left(1 + \frac{1}{n}\right)$ converges to 2; it diverges by definition of divergence

20. $x_n = \max\{\cos 1, \cos 2, \cos 3, \ldots, \cos n\}$ and $x_{n+1} = \max\{\cos 1, \cos 2, \cos 3, \ldots, \cos(n+1)\} \ge x_n$ with $x_n \le 1$ so the sequence is nondecreasing and bounded above by $1 \Rightarrow$ the sequence converges.
upper bound and therefore diverges. Hence, $\{a_n\}$ also diverges.

21. $a_n \ge a_{n+1} \Leftrightarrow \frac{n+1}{n} \ge \frac{(n+1)+1}{n+1} \Leftrightarrow n^2 + 2n + 1 \ge n^2 + 2n \Leftrightarrow 1 \ge 0$ and $\frac{n+1}{n} \ge 1$; thus the sequence is nonincreasing and bounded below by $1 \Rightarrow$ it converges

22. $a_n \ge a_{n+1} \Leftrightarrow \frac{1+\sqrt{2n}}{\sqrt{n}} \ge \frac{1+\sqrt{2(n+1)}}{\sqrt{n+1}} \Leftrightarrow \sqrt{n+1} + \sqrt{2n^2+2n} \ge \sqrt{n} + \sqrt{2n^2+2n} \Leftrightarrow \sqrt{n+1} \ge \sqrt{n}$ and $\frac{1+\sqrt{2n}}{\sqrt{n}} \ge \sqrt{2}$; thus the sequence is nonincreasing and bounded below by $\sqrt{2} \Rightarrow$ it converges

23. $a_n \ge a_{n+1} \Leftrightarrow \frac{1-4^n}{2^n} \ge \frac{1-4^{n+1}}{2^{n+1}} \Leftrightarrow 2^{n+1} - 2^{n+1}4^n \ge 2^n - 2^n 4^{n+1} \Leftrightarrow 2^{n+1} - 2^n \ge 2^{n+1}4^n - 2^n 4^{n+1}$
$\Leftrightarrow 2 - 1 \ge 2 \cdot 4^n - 4^{n+1} \Leftrightarrow 1 \ge 4^n(2-4) \Leftrightarrow 1 \ge (-2)\cdot 4^n$; thus the sequence is nonincreasing. However, $a_n = \frac{1}{2^n} - \frac{4^n}{2^n} = \frac{1}{2^n} - 2^n$ which is not bounded below so the sequence diverges

24. $\frac{4^{n+1} + 3^n}{4^n} = 4 + \left(\frac{3}{4}\right)^n$ so $a_n \ge a_{n+1} \Leftrightarrow 4 + \left(\frac{3}{4}\right)^n \ge 4 + \left(\frac{3}{4}\right)^{n+1} \Leftrightarrow \left(\frac{3}{4}\right)^n \ge \left(\frac{3}{4}\right)^{n+1} \Leftrightarrow 1 \ge \frac{3}{4}$ and $4 + \left(\frac{3}{4}\right)^n \ge 4$; thus the sequence is nonincreasing and bounded below by $4 \Rightarrow$ it converges

25. Let k(n) and i(n) be two order-preserving functions whose domains are the set of positive integers and whose ranges are a subset of the positive integers. Consider the two subsequences $a_{k(n)}$ and $a_{i(n)}$, where $a_{k(n)} \to L_1$, $a_{i(n)} \to L_2$ and $L_1 \neq L_2$. Given an $\epsilon > 0$ there corresponds an N_1 such that for $k(n) > N_1$, $\left|a_{k(n)} - L_1\right| < \epsilon$, and an N_2 such that for $i(n) > N_2$, $\left|a_{i(n)} - L_2\right| < \epsilon$. Let $N = \max\{N_1, N_2\}$. Then for $n > N$, we have that $\left|a_n - L_1\right| < \epsilon$ and $\left|a_n - L_2\right| < \epsilon$. This implies $a_n \to L_1$ and $a_n \to L_2$ where $L_1 \neq L_2$. Since the limit of a sequence is unique (by Exercise 72, Section 8.1), a_n does not converge and hence diverges.

26. $a_{2k} \to L \Leftrightarrow$ given an $\epsilon > 0$ there corresponds an N_1 such that $\left[2k > N_1 \Rightarrow \left|a_{2k} - L\right| < \epsilon\right]$. Similarly, $a_{2k+1} \to L \Leftrightarrow \left[2k + 1 > N_2 \Rightarrow \left|a_{2k+1} - L\right| < \epsilon\right]$. Let $N = \max\{N_1, N_2\}$. Then $n > N \Rightarrow \left|a_n - L\right| < \epsilon$ whether n is even or odd, and hence $a_n \to L$.

27. $g(x) = \sqrt{x}$; $2 \to 1.00000132$ in 20 iterations; $.1 \to 0.9999956$ in 20 iterations; a root is 1

28. $g(x) = x^2$; $x_0 = .5 \to 0.0000152$ in 5 iterations; $-.5 \to 0.0000152$ in 5 iterations; a root is 0

29. $g(x) = -\cos x$; $x_0 = .1 \to x \approx -0.739085$

30. $g(x) = \cos x - 1$; $x_0 = .1 \to x = 0$

31. $g(x) = 0.1 + \sin x$; $x_0 = -2 \to x \approx 0.853750$

32. $g(x) = \left(4 - \sqrt{1+x}\right)^2$; $x_0 = 3.5 \to x = 3.515625$

33. $x_0 =$ initial guess $> 0 \Rightarrow x_1 = \sqrt{x_0} = (x_0)^{1/2} \Rightarrow x_2 = \sqrt{{x_0}^{1/2}} = {x_0}^{1/4}, \ldots \Rightarrow x_n = {x_0}^{1/(2n)} \Rightarrow x_n \to 1$ as $n \to \infty$

34. $x_0 =$ initial guess $\Rightarrow x_1 = x_0^2 \Rightarrow x_2 = \left(x_0^2\right)^2 = x_0^4, \ldots \Rightarrow x_n = {x_0}^{2n}$; $|x_0| < 1 \Rightarrow x_n \to 0$ as $n \to \infty$; $|x_0| > 1 \Rightarrow x_n \to \infty$ as $n \to \infty$

35-36. Example CAS Commands

Mathematica (with comments in text cells)

```
Clear[a];
a[1] = SetPrecision[1,20]
a[n_] := a[n] = SetPrecision[a[n-1] + (1/5)^(n-1),20];
```

The **SetPrecision**[ ] command allows you to see the specified number of digits rather than the default value of six.

The recursive definition, **a[n_]:=a[n]=...**, causes *Mathematica* to remember values of the sequence that were previously calculated. The alternative form, **a[n_]:=...** forces *Mathematica* to recalculate all the values of the sequence up to **a[n]**, for each value of **n**, as a result, the first form is computationally more efficient.

```
Clear[seq];
seq = Table[a[n], {n,1,25}]
ListPlot[seq,PlotRange→ {Min[seq],Max[seq]},
  PlotStyle→ {PointSize[0.020], RGBColor[1,0,0]},
  AxesLabel→ {"n","a[n]"}];
```

The sequence in Exercise 35 appears to converge to the limiting value of 1.25.

```
L = 1.25;
eps = 0.0001;
n = 1;
While[Abs[a[n] - L] ≥ eps, n++];
Print[n];
```

Maple:

```
> restart;
> Digits:=20;
```

Specifying a value for Digits allows you to see the specified number of digits of precision in the displayed results of numerical calculations.

```
> n:='n';
> recur:=proc(f,a1,n) local i,j;
> a(1)=evalf(a1);
> for i from 2 to n do
> a(i):=evalf(f(a(i-1),i-1))
> od;
> [[j,a(j)] $j=1..n];
> end;
> a:='a':i='i':f:=(a,i)->a+(1/5)^i;
> avals:=recur(f,1,25);
> plot(avals,style=POINT,symbol=CIRCLE);
```

The sequence in Exercise 35 appears to be converging to a limit value of 1.25.

```
> L:=1.25;
> n:=1;
> eps:=0.0001;
> for i from 1 to 25 while abs(avals[i,2]-L)>=eps do n:=n+1 od:
> print(n);
>
```

37. Example CAS Commands:

Maple:

```
n:='n':
recur:= proc(f,a0,n) local i,j;
a(0):= evalf(a0);
for i from 1 to n do
a(i):= evalf(f(a(i - 1)))
od;
[[j,a(j)] $j=1..n];
end;
a:= 'a': f:= a -> (1 + r/m)*a + b;
r:= 0.02015; m:= 12; b:= 50;
recur(f,1000,100):
plot(%,style=POINT,symbol=CIRCLE):
a(60);
```

Mathematica:

```
Clear[a,r,m,b]
a[n_] := (1 + r/m) a[n - 1] + b
(a)
a[0] = 1000; r = 0.02015; m = 12; b = 50;
atab = Table[ a[i], {i,0,50} ] // N;
ListPlot[ atab ]
a[60]
```

```
a[0] = 1000; r = 0.02015; m = 12; b = 50;
ak[n_] := (1+r/m)^n (a[0] + m b/r) - m b/r
atab = Table[ {a[i],ak[i]}, {i,0,50} ] // N
ak[n+1] == (1+r/m) ak[n] + b // Simplify
```

38. Example CAS Commands:

Maple:

```
n:= 'n':
iterate:= proc(f,a0,n) local i,j;
   a(0):= evalf(a0);
   for i from 1 to n do
   a(i):= evalf(f(a(i-1)))
   od;
   [[j, a(j)] $j= 1..n];
   end;
a:= 'a': f:= a -> r*a*(1-a);
r:= 3.75;
iterate(f, 0.301, 300):
plot(%, style=POINT, symbol=CIRCLE, title='LOGISTIC PLOT, r = 3.75, a = .301');
```

Mathematica:

```
Note: We could define a[n] recursively, but here we need only the first
several values so it's easier to use an iterated function:

Clear[a,r,n,i]
iter[ an_ ] = r an (1-an)
r = 3/4;
atab = NestList[ iter, 0.3, 100 ];
ListPlot[ atab ]

To plot several lists together:

<< Graphics'MultipleListPlot'
r = 3.65;
MultipleListPlot[
 NestList[ iter, 0.3, 300 ],
 NestList[ iter, 0.301, 300] ]
r = 3.75;
MultipleListPlot[
 NestList[ iter, 0.3, 300 ],
 NestList[ iter, 0.301, 300 ] ]
```

8.3 INFINITE SERIES

1. $s_n = \frac{a\left(1-r^n\right)}{(1-r)} = \frac{2\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\left(\frac{1}{3}\right)} \Rightarrow \lim\limits_{n\to\infty} s_n = \frac{2}{1-\left(\frac{1}{3}\right)} = 3$

2. $s_n = \frac{a(1-r^n)}{(1-r)} = \frac{\left(\frac{9}{100}\right)\left(1-\left(\frac{1}{100}\right)^n\right)}{1-\left(\frac{1}{100}\right)} \Rightarrow \lim\limits_{n\to\infty} s_n = \frac{\left(\frac{9}{100}\right)}{1-\left(\frac{1}{100}\right)} = \frac{1}{11}$

3. $s_n = \frac{a(1-r^n)}{(1-r)} = \frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)} \Rightarrow \lim\limits_{n\to\infty} s_n = \frac{1}{\left(\frac{3}{2}\right)} = \frac{2}{3}$

4. $s_n = \frac{1-(-2)^n}{1-(-2)}$, a geometric series where $|r| > 1 \Rightarrow$ divergence

5. $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2} \Rightarrow s_n = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right) = \frac{1}{2}-\frac{1}{n+2} \Rightarrow \lim\limits_{n\to\infty} s_n = \frac{1}{2}$

6. $\frac{5}{n(n+1)} = \frac{5}{n} - \frac{5}{n+1} \Rightarrow s_n = \left(5-\frac{5}{2}\right)+\left(\frac{5}{2}-\frac{5}{3}\right)+\left(\frac{5}{3}-\frac{5}{4}\right)+\ldots+\left(\frac{5}{n-1}-\frac{5}{n}\right)+\left(\frac{5}{n}-\frac{5}{n+1}\right) = 5-\frac{5}{n+1}$
$\Rightarrow \lim\limits_{n\to\infty} s_n = 5$

7. $1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+\ldots$, the sum of this geometric series is $\frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{1}{1+\left(\frac{1}{4}\right)} = \frac{4}{5}$

8. $\frac{7}{4}+\frac{7}{16}+\frac{7}{64}+\ldots$, the sum of this geometric series is $\frac{\left(\frac{7}{4}\right)}{1-\left(\frac{1}{4}\right)} = \frac{7}{3}$

9. $(5+1)+\left(\frac{5}{2}+\frac{1}{3}\right)+\left(\frac{5}{4}+\frac{1}{9}\right)+\left(\frac{5}{8}+\frac{1}{27}\right)+\ldots$, is the sum of two geometric series; the sum is
$\frac{5}{1-\left(\frac{1}{2}\right)}+\frac{1}{1-\left(\frac{1}{3}\right)} = 10+\frac{3}{2} = \frac{23}{2}$

10. $(5-1)+\left(\frac{5}{2}-\frac{1}{3}\right)+\left(\frac{5}{4}-\frac{1}{9}\right)+\left(\frac{5}{8}-\frac{1}{27}\right)+\ldots$, is the difference of two geometric series; the sum is
$\frac{5}{1-\left(\frac{1}{2}\right)}-\frac{1}{1-\left(\frac{1}{3}\right)} = 10-\frac{3}{2} = \frac{17}{2}$

11. $(1+1)+\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{4}+\frac{1}{25}\right)+\left(\frac{1}{8}-\frac{1}{125}\right)+\ldots$, is the sum of two geometric series; the sum is
$\frac{1}{1-\left(\frac{1}{2}\right)}+\frac{1}{1+\left(\frac{1}{5}\right)} = 2+\frac{5}{6} = \frac{17}{6}$

12. $2+\frac{4}{5}+\frac{8}{25}+\frac{16}{125}+\ldots = 2\left(1+\frac{2}{5}+\frac{4}{25}+\frac{8}{125}+\ldots\right)$; the sum of this geometric series is $2\left(\frac{1}{1-\left(\frac{2}{5}\right)}\right) = \frac{10}{3}$

13. $\frac{4}{(4n-3)(4n+1)} = \frac{1}{4n-3} - \frac{1}{4n+1} \Rightarrow s_n = \left(1-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)+\ldots+\left(\frac{1}{4n-7}-\frac{1}{4n-3}\right)$
$+\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right) = 1-\frac{1}{4n+1} \Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left(1-\frac{1}{4n+1}\right) = 1$

14. $\frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} = \frac{A(2n+1)+B(2n-1)}{(2n-1)(2n+1)} \Rightarrow A(2n+1)+B(2n-1) = 6$

$\Rightarrow (2A+2B)n + (A-B) = 6 \Rightarrow \begin{cases} 2A+2B = 0 \\ A - \ B = 6 \end{cases} \Rightarrow \begin{cases} A+B = 0 \\ A-B = 6 \end{cases} \Rightarrow 2A = 6 \Rightarrow A = 3$ and $B = -3$. Hence,

$\sum\limits_{n=1}^{k} \frac{6}{(2n-1)(2n+1)} = 3\sum\limits_{n=1}^{k}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = 3\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \ldots - \frac{1}{2(k-1)+1} + \frac{1}{2k-1} - \frac{1}{2k+1}\right)$

$= 3\left(1 - \frac{1}{2k+1}\right) \Rightarrow$ the sum is $\lim\limits_{k\to\infty} 3\left(1 - \frac{1}{2k+1}\right) = 3$

15. $\frac{40n}{(2n-1)^2(2n+1)^2} = \frac{A}{(2n-1)} + \frac{B}{(2n-1)^2} + \frac{C}{(2n+1)} + \frac{D}{(2n+1)^2}$

$= \frac{A(2n-1)(2n+1)^2 + B(2n+1)^2 + C(2n+1)(2n-1)^2 + D(2n-1)^2}{(2n-1)^2(2n+1)^2}$

$\Rightarrow A(2n-1)(2n+1)^2 + B(2n+1)^2 + C(2n+1)(2n-1)^2 + D(2n-1)^2 = 40n$

$\Rightarrow A\left(8n^3 + 4n^2 - 2n - 1\right) + B\left(4n^2 + 4n + 1\right) + C\left(8n^3 - 4n^2 - 2n + 1\right) = D\left(4n^2 - 4n + 1\right) = 40n$

$\Rightarrow (8A+8C)n^3 + (4A+4B-4C+4D)n^2 + (-2A+4B-2C-4D)n + (-A+B+C+D) = 40n$

$\Rightarrow \begin{cases} 8A+8C = 0 \\ 4A+4B-4C+4D = 0 \\ -2A+4B-2C-4D = 40 \\ -A+B+C+D = 0 \end{cases} \Rightarrow \begin{cases} 8A+8C = 0 \\ A+B-C+D = 0 \\ -A+2B-C-2D = 20 \\ -A+B+C+D = 0 \end{cases} \Rightarrow \begin{cases} B+D = 0 \\ 2B-2D = 20 \end{cases} \Rightarrow 4B = 20 \Rightarrow B = 5$ and

$D = -5 \Rightarrow \begin{cases} A+C = 0 \\ -A+5+C-5 = 0 \end{cases} \Rightarrow C = 0$ and $A = 0$. Hence, $\sum\limits_{n=1}^{k}\left[\frac{40n}{(2n-1)^2(2n+1)^2}\right]$

$= 5\sum\limits_{n=1}^{k}\left[\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2}\right] = 5\left(\frac{1}{1} - \frac{1}{9} + \frac{1}{9} - \frac{1}{25} + \frac{1}{25} - \ldots - \frac{1}{(2(k-1)+1)^2} + \frac{1}{(2k-1)^2} - \frac{1}{(2k+1)^2}\right)$

$= 5\left(1 - \frac{1}{(2k+1)^2}\right) \Rightarrow$ the sum is $\lim\limits_{n\to\infty} 5\left(1 - \frac{1}{(2k+1)^2}\right) = 5$

16. $\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2} \Rightarrow s_n = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{16}\right) + \ldots + \left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right] + \left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]$

$\Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left[1 - \frac{1}{(n+1)^2}\right] = 1$

17. $s_n = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \ldots + \left(\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n}}\right) + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}$

$\Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1$

18. $s_n = \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3}\right) + \left(\frac{1}{\ln 5} - \frac{1}{\ln 4}\right) + \ldots + \left(\frac{1}{\ln(n+1)} - \frac{1}{\ln n}\right) + \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$

$= -\frac{1}{\ln 2} + \frac{1}{\ln(n+2)} \Rightarrow \lim\limits_{n\to\infty} s_n = -\frac{1}{\ln 2}$

19. convergent geometric series with sum $\frac{1}{1-\left(\frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{2}}{\sqrt{2}-1} = 2+\sqrt{2}$

20. divergent geometric series with $|r| = \sqrt{2} > 1$

21. convergent geometric series with sum $\frac{\left(\frac{3}{2}\right)}{1-\left(-\frac{1}{2}\right)} = 1$

22. $\cos(n\pi) = (-1)^n \Rightarrow$ convergent geometric series with sum $\frac{1}{1-\left(-\frac{1}{5}\right)} = \frac{5}{6}$

23. convergent geometric series with sum $\frac{1}{1-\left(\frac{1}{e^2}\right)} = \frac{e^2}{e^2-1}$

24. $\lim_{n\to\infty} \ln \frac{1}{n} = -\infty \neq 0 \Rightarrow$ diverges

25. convergent geometric series with sum $\frac{1}{1-\left(\frac{1}{x}\right)} = \frac{x}{x-1}$

26. difference of two geometric series with sum $\frac{1}{1-\left(\frac{2}{3}\right)} - \frac{1}{1-\left(\frac{1}{3}\right)} = 3 - \frac{3}{2} = \frac{3}{2}$

27. $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{-1}{n}\right)^n = e^{-1} \neq 0 \Rightarrow$ diverges

28. convergent geometric series with sum $\frac{1}{1-\left(\frac{e}{\pi}\right)} = \frac{\pi}{\pi - e}$

29. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} [\ln(n) - \ln(n+1)] \Rightarrow s_n = [\ln(1)-\ln(2)]+[\ln(2)-\ln(3)]+[\ln(3)-\ln(4)]+\dots$
$+[\ln(n-1)-\ln(n)]+[\ln(n)-\ln(n+1)] = \ln(1)-\ln(n+1) = -\ln(n+1) \Rightarrow \lim_{n\to\infty} s_n = -\infty, \Rightarrow$ diverges

30. divergent geometric series with $|r| = \frac{e^\pi}{\pi^e} \approx \frac{23.141}{22.459} > 1$

31. $\lim_{n\to\infty} \frac{n!}{1000^n} = \infty \neq 0 \Rightarrow$ diverges

32. $\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n \cdot n \cdots n}{1 \cdot 2 \cdots n} > \lim_{n\to\infty} n = \infty \Rightarrow$ diverges

33. $\sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$; $a = 1$, $r = -x$; converges to $\frac{1}{1-(-x)} = \frac{1}{1+x}$ for $|x| < 1$

34. $\sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \left(-x^2\right)^n$; $a = 1$, $r = -x^2$; converges to $\frac{1}{1+x^2}$ for $|x| < 1$

35. $a = 3$, $r = \frac{x-1}{2}$; converges to $\frac{3}{1-\left(\frac{x-1}{2}\right)} = \frac{6}{3-x}$ for $-1 < \frac{x-1}{2} < 1$ or $-1 < x < 3$

36. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{-1}{3+\sin x}\right)^n$; $a = \frac{1}{2}$, $r = \frac{-1}{3+\sin x}$; converges to $\frac{\left(\frac{1}{2}\right)}{1-\left(\frac{-1}{3+\sin x}\right)}$

$= \frac{3+\sin x}{2(4+\sin x)} = \frac{3+\sin x}{8+2\sin x}$ for all x $\left(\text{since } \frac{1}{4} \le \frac{1}{3+\sin x} \le \frac{1}{2} \text{ for all } x\right)$

37. $a = 1$, $r = 2x$; converges to $\frac{1}{1-2x}$ for $|2x| < 1$ or $|x| < \frac{1}{2}$

38. $a = 1$, $r = -\frac{1}{x^2}$; converges to $\frac{1}{1-\left(\frac{-1}{x^2}\right)} = \frac{x^2}{x^2+1}$ for $|x^2| < 1$ or $|x| < 1$

39. $a = 1$, $r = \frac{3-x}{2}$; converges to $\frac{1}{1-\left(\frac{3-x}{2}\right)} = \frac{2}{x-1}$ for $\left|\frac{3-x}{2}\right| < 1$ or $1 < x < 5$

40. $a = 1$, $r = \ln x$; converges to $\frac{1}{1-\ln x}$ for $|\ln x| < 1$ or $e^{-1} < x < e$

41. $0.\overline{23} = \sum_{n=0}^{\infty} \frac{23}{100}\left(\frac{1}{10^2}\right)^n = \frac{\left(\frac{23}{100}\right)}{1-\left(\frac{1}{100}\right)} = \frac{23}{99}$

42. $0.\overline{234} = \sum_{n=0}^{\infty} \frac{234}{1000}\left(\frac{1}{10^3}\right)^n = \frac{\left(\frac{234}{1000}\right)}{1-\left(\frac{1}{1000}\right)} = \frac{234}{999}$

43. $0.\overline{7} = \sum_{n=0}^{\infty} \frac{7}{10}\left(\frac{1}{10}\right)^n = \frac{\left(\frac{7}{10}\right)}{1-\left(\frac{1}{10}\right)} = \frac{7}{9}$

44. $1.\overline{414} = 1 + \sum_{n=0}^{\infty} \frac{414}{1000}\left(\frac{1}{10^3}\right)^n = 1 + \frac{\left(\frac{414}{1000}\right)}{1-\left(\frac{1}{1000}\right)} = 1 + \frac{414}{999} = \frac{1413}{999}$

45. $1.24\overline{123} = \frac{124}{100} + \sum_{n=0}^{\infty} \frac{123}{10^5}\left(\frac{1}{10^3}\right)^n = \frac{124}{100} + \frac{\left(\frac{123}{10^5}\right)}{1-\left(\frac{1}{10^3}\right)} = \frac{124}{100} + \frac{123}{10^5 - 10^2} = \frac{124}{100} + \frac{123}{99{,}900} = \frac{123{,}999}{99{,}900} = \frac{41{,}333}{33{,}300}$

46. $3.\overline{142857} = 3 + \sum_{n=0}^{\infty} \frac{142{,}857}{10^6}\left(\frac{1}{10^6}\right)^n = 3 + \frac{\left(\frac{142{,}857}{10^6}\right)}{1-\left(\frac{1}{10^6}\right)} = 3 + \frac{142{,}857}{10^6 - 1} = \frac{2{,}857{,}140}{999{,}999} = \frac{317{,}460}{111{,}111}$

47. distance $= 4 + 2\left[(4)\left(\frac{3}{4}\right) + (4)\left(\frac{3}{4}\right)^2 + \ldots\right] = 4 + 2\left(\frac{3}{1-\left(\frac{3}{4}\right)}\right) = 28$ m

48. time $= \sqrt{\frac{4}{4.9}} + 2\sqrt{\left(\frac{4}{4.9}\right)\left(\frac{3}{4}\right)} + 2\sqrt{\left(\frac{4}{4.9}\right)\left(\frac{3}{4}\right)^2} + 2\sqrt{\left(\frac{4}{4.9}\right)\left(\frac{3}{4}\right)^3} + \ldots = \sqrt{\frac{4}{4.9}} + 2\sqrt{\frac{4}{4.9}}\left[\sqrt{\frac{3}{4}} + \sqrt{\left(\frac{3}{4}\right)^2} + \ldots\right]$

$$= \frac{2}{\sqrt{4.9}} + \left(\frac{4}{\sqrt{4.9}}\right)\left[\frac{\sqrt{\frac{3}{4}}}{1-\sqrt{\frac{3}{4}}}\right] = \frac{2}{\sqrt{4.9}} + \left(\frac{4}{\sqrt{4.9}}\right)\left(\frac{\sqrt{3}}{2-\sqrt{3}}\right) = \frac{(4-2\sqrt{3})+4\sqrt{3}}{\sqrt{4.9}(2-\sqrt{3})} = \frac{4+2\sqrt{3}}{\sqrt{4.9}(2-\sqrt{3})} \approx 12.58 \text{ sec}$$

49. $\text{area} = 2^2 + (\sqrt{2})^2 + (1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \ldots = 4 + 2 + 1 + \frac{1}{2} + \ldots = \frac{4}{1-\frac{1}{2}} = 8 \text{ m}^2$

50. $\text{area} = 2\left[\frac{\pi\left(\frac{1}{2}\right)^2}{2}\right] + 4\left[\frac{\pi\left(\frac{1}{4}\right)^2}{2}\right] + 8\left[\frac{\pi\left(\frac{1}{8}\right)^2}{2}\right] + \ldots = \pi\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\right) = \pi\left(\frac{\left(\frac{1}{4}\right)}{1-\left(\frac{1}{2}\right)}\right) = \frac{\pi}{2}$

51. (a) $L_1 = 3$, $L_2 = 3\left(\frac{4}{3}\right)$, $L_3 = 3\left(\frac{4}{3}\right)^2$, $\ldots$, $L_n = 3\left(\frac{4}{3}\right)^{n-1} \Rightarrow \lim_{n\to\infty} L_n = \lim_{n\to\infty} 3\left(\frac{4}{3}\right)^{n-1} = \infty$

(b) $A_1 = \frac{1}{2}(1)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$, $A_2 = A_1 + 3\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{6}\right) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}$, $A_3 = A_2 + 12\left(\frac{1}{2}\right)\left(\frac{1}{9}\right)\left(\frac{\sqrt{3}}{18}\right)$

$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{27}$, $A_4 = A_3 + 48\left(\frac{1}{2}\right)\left(\frac{1}{27}\right)\left(\frac{\sqrt{3}}{54}\right)$, $\ldots$, $A_n = \frac{\sqrt{3}}{4} + \frac{27\sqrt{3}}{64}\left(\frac{4}{9}\right)^2 + \frac{27\sqrt{3}}{64}\left(\frac{4}{9}\right)^3 + \ldots$

$$= \frac{\sqrt{3}}{4} + \sum_{n=2}^{\infty} \frac{27\sqrt{3}}{64}\left(\frac{4}{9}\right)^n = \frac{\sqrt{3}}{4} + \frac{\left(\frac{27\sqrt{3}}{64}\right)\left(\frac{4}{9}\right)^2}{1-\left(\frac{4}{9}\right)} = \frac{\sqrt{3}}{4} + \frac{\left(\frac{27\sqrt{3}}{64}\right)\left(\frac{16}{9}\right)}{9-4} = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4\cdot 5} = \frac{5\sqrt{3}+3\sqrt{3}}{20} = \frac{2\sqrt{3}}{5}$$

52. Each term of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ represents the area of one of the squares shown in the figure, and all of the squares lie inside the rectangle of width 1 and length $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$. Since the squares do not fill the rectangle completely, and the area of the rectangle is 2, we have $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.

53. (a) $\sum_{n=-2}^{\infty} \frac{1}{(n+4)(n+5)}$ (b) $\sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)}$ (c) $\sum_{n=5}^{\infty} \frac{1}{(n-3)(n-2)}$

54. (a) one example is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = 1$

(b) one example is $-\frac{3}{2} - \frac{3}{4} - \frac{3}{8} - \frac{3}{16} - \ldots = \frac{\left(-\frac{3}{2}\right)}{1-\left(\frac{1}{2}\right)} = -3$

(c) one example is $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \ldots$; the series $\frac{k}{2} + \frac{k}{4} + \frac{k}{8} + \ldots = \frac{\left(\frac{k}{2}\right)}{1-\left(\frac{1}{2}\right)} = k$ where k is any positive or negative number.

55. $1+e^b+e^{2b}+\ldots=\frac{1}{1-e^b}=9 \Rightarrow \frac{1}{9}=1-e^b \Rightarrow e^b=\frac{8}{9} \Rightarrow b=\ln\left(\frac{8}{9}\right)$

56. $s_n=1+2r+r^2+2r^3+r^4+2r^5+\ldots+r^{2n}+2r^{2n+1}$, $n=0, 1, \ldots$
$\Rightarrow s_n=\left(1+r^2+r^4+\ldots+r^{2n}\right)+\left(2r+2r^3+2r^5+\ldots+2r^{2n+1}\right) \Rightarrow \lim_{n\to\infty} s_n=\frac{1}{1-r^2}+\frac{2r}{1-r^2}$
$=\frac{1+2r}{1-r^2}$, if $|r^2|<1$ or $|r|<1$

57. $L-s_n=\frac{a}{1-r}-\frac{a\left(1-r^n\right)}{1-r}=\frac{ar^n}{1-r}$

58. Let $a_n=b_n=\left(\frac{1}{2}\right)^n$. Then $\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=1$, while $\sum_{n=1}^{\infty}\left(a_nb_n\right)=\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n=\frac{1}{3}$.

59. Let $a_n=\left(\frac{1}{4}\right)^n$ and $b_n=\left(\frac{1}{2}\right)^n$. Then $A=\sum_{n=1}^{\infty} a_n=\frac{1}{3}$, $B=\sum_{n=1}^{\infty} b_n=1$ and $\sum_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=1\neq\frac{A}{B}$.

60. Let $a_n=b_n=\left(\frac{1}{2}\right)^n$. Then $\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=1$, while $\sum_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)=\sum_{n=1}^{\infty}(1)$ diverges.

61. Yes: $\sum\left(\frac{1}{a_n}\right)$ diverges. The reasoning: $\sum a_n$ converges $\Rightarrow a_n\to 0 \Rightarrow \frac{1}{a_n}\to\infty \Rightarrow \sum\left(\frac{1}{a_n}\right)$ diverges by the nth-Term Test.

62. Since the sum of a finite number of terms is finite, adding or subtracting a finite number of terms from a series that diverges does not change the divergence of the series.

63. Let $A_n=a_1+a_2+\ldots+a_n$ and $\lim_{n\to\infty} A_n=A$. Assume $\sum\left(a_n+b_n\right)$ converges to S. Let
$S_n=(a_1+b_1)+(a_2+b_2)+\ldots+(a_n+b_n) \Rightarrow S_n=(a_1+a_2+\ldots+a_n)+(b_1+b_2+\ldots+b_n)$
$\Rightarrow b_1+b_2+\ldots+b_n=S_n-A_n \Rightarrow \lim_{n\to\infty}\left(b_1+b_2+\ldots+b_n\right)=S-A \Rightarrow \sum b_n$ converges. This contradicts the assumption that $\sum b_n$ diverges; therefore, $\sum\left(a_n+b_n\right)$ diverges.

8.4 SERIES OF NONNEGATIVE TERMS

1. diverges by the Integral Test; $\int_1^n \frac{5}{x+1}\,dx=\ln(n+1)-\ln 2 \Rightarrow \int_1^{\infty}\frac{5}{x+1}\,dx\to\infty$

2. diverges by the Integral Test: $\int_1^n \frac{dx}{2x-1}=\frac{1}{2}\ln(2n-1)\to\infty$ as $n\to\infty$

3. diverges by the Integral Test: $\int_2^n \frac{\ln x}{x}\,dx = \frac{1}{2}\left(\ln^2 n - \ln 2\right) \Rightarrow \int_2^\infty \frac{\ln x}{x}\,dx \to \infty$

4. diverges by the Integral Test: $\int_2^\infty \frac{\ln x}{\sqrt{x}}\,dx; \begin{bmatrix} t = \ln x \\ dt = \frac{dx}{x} \\ dx = e^t\,dt \end{bmatrix} \Rightarrow \int_{\ln 2}^\infty te^{t/2}\,dt = \lim_{b\to\infty} \left[2te^{t/2} - 4e^{t/2}\right]_{\ln 2}^b$

$= \lim_{b\to\infty} \left[2e^{b/2}(b-2) - 2e^{(\ln 2)/2}(\ln 2 - 2)\right] = \infty$

5. converges by the Integral Test: $\int_1^\infty \frac{e^x}{1+e^{2x}}\,dx; \begin{bmatrix} u = e^x \\ du = e^x\,dx \end{bmatrix} \Rightarrow \int_e^\infty \frac{1}{1+u^2}\,du = \lim_{b\to\infty} \left[\tan^{-1} u\right]_e^b$

$= \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} e\right) = \frac{\pi}{2} - \tan^{-1} e \approx 0.35$

6. diverges by the Integral Test: $\int_1^n \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}; \begin{bmatrix} u = \sqrt{x}+1 \\ du = \frac{dx}{\sqrt{x}} \end{bmatrix} \Rightarrow \int_2^{\sqrt{n}+1} \frac{du}{u} = \ln\left(\sqrt{n}+1\right) - \ln 2$

$\to \infty$ as $n \to \infty$

7. converges by the Integral Test: $\int_3^\infty \frac{\left(\frac{1}{x}\right)}{(\ln x)\sqrt{(\ln x)^2 - 1}}\,dx; \begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \end{bmatrix} \Rightarrow \int_{\ln 3}^\infty \frac{1}{u\sqrt{u^2-1}}\,du$

$= \lim_{b\to\infty} \left[\sec^{-1} |u|\right]_{\ln 3}^b = \lim_{b\to\infty} \left[\sec^{-1} b - \sec^{-1}(\ln 3)\right] = \lim_{b\to\infty} \left[\cos^{-1}\left(\frac{1}{b}\right) - \sec^{-1}(\ln 3)\right]$

$= \cos^{-1}(0) - \sec^{-1}(\ln 3) = \frac{\pi}{2} - \sec^{-1}(\ln 3) \approx 1.1439$

8. converges by the Integral Test: $\int_1^\infty \frac{1}{x\left(1+\ln^2 x\right)}\,dx = \int_1^\infty \frac{\left(\frac{1}{x}\right)}{1+(\ln x)^2}\,dx; \begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \end{bmatrix} \Rightarrow \int_0^\infty \frac{1}{1+u^2}\,du$

$= \lim_{b\to\infty} \left[\tan^{-1} u\right]_0^b = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

9. diverges by the Direct Comparison Test since $n \ge 1 \Rightarrow \sqrt{n} \ge \sqrt[3]{n} \Rightarrow 3\sqrt{n} \ge 2\sqrt{n} + \sqrt[3]{n}$

$\Rightarrow \frac{1}{2\sqrt{n}+\sqrt[3]{n}} \ge \frac{1}{3} \cdot \frac{1}{\sqrt{n}}$, and the p-series $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ diverges

10. diverges by the Direct Comparison Test since $n + n + n > n + \sqrt{n} + 0 \Rightarrow \frac{3}{n+\sqrt{n}} > \frac{1}{n}$, which is the nth term of the divergent series $\sum_{n=1}^\infty \frac{1}{n}$

11. converges by the Direct Comparison Test; $\frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$, which is the nth term of a convergent geometric series

12. converges by the Direct Comparison Test; $\frac{1+\cos n}{n^2} \le \frac{2}{n^2}$ and the p-series $\sum \frac{1}{n^2}$ converges

13. converges by the Direct Comparison Test; $\left(\frac{n}{3n+1}\right)^n < \left(\frac{n}{3n}\right)^n < \left(\frac{1}{3}\right)^n$, the nth term of a convergent geometric series

14. diverges by the Direct Comparison Test; $n > \ln n \Rightarrow \ln n > \ln \ln n \Rightarrow \frac{1}{\ln n} < \frac{1}{\ln(\ln n)}$ and the series $\sum\limits_{n=3}^{\infty} \frac{1}{n}$ diverges

15. diverges by the Limit Comparison Test (part 3) when compared with $\sum\limits_{n=2}^{\infty} \frac{1}{n}$, a divergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{(\ln n)^2}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{(\ln n)^2} = \lim_{n\to\infty} \frac{1}{2(\ln n)\left(\frac{1}{n}\right)} = \frac{1}{2}\lim_{n\to\infty} \frac{n}{\ln n} = \frac{1}{2}\lim_{n\to\infty} \frac{1}{\left(\frac{1}{n}\right)} = \frac{1}{2}\lim_{n\to\infty} n = \infty$$

16. converges by the Limit Comparison Test (part 2) when compared with $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p-series:

$$\lim_{n\to\infty} \frac{\left[\frac{(\ln n)^2}{n^3}\right]}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{n\to\infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{1} = 2\lim_{n\to\infty} \frac{\ln n}{n} = 0 \qquad \text{(Table 8.1)}$$

17. converges by the Limit Comparison Test (part 2) when compared with $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p-series:

$$\lim_{n\to\infty} \frac{\left[\frac{(\ln n)^3}{n^3}\right]}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{(\ln n)^3}{n} = \lim_{n\to\infty} \frac{3(\ln n)^2\left(\frac{1}{n}\right)}{1} = 3\lim_{n\to\infty} \frac{(\ln n)^2}{n} = 3\lim_{n\to\infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{1} = 6\lim_{n\to\infty} \frac{\ln n}{n}$$

$$= 6 \cdot 0 = 0 \qquad \text{(Table 8.1)}$$

18. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$$\lim_{n\to\infty} \frac{\left[\frac{1}{\sqrt{n}\ln n}\right]}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2\sqrt{n}}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty$$

19. converges by the Limit Comparison Test (part 2) with $\frac{1}{n^{5/4}}$, the nth term of a convergent p-series:

$$\lim_{n\to\infty} \frac{\left[\frac{(\ln n)^2}{n^{3/2}}\right]}{\left(\frac{1}{n^{5/4}}\right)} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^{1/4}} = \lim_{n\to\infty} \frac{\left(\frac{2\ln n}{n}\right)}{\left(\frac{1}{4n^{3/4}}\right)} = 8\lim_{n\to\infty} \frac{\ln n}{n^{1/4}} = 8\lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{4n^{3/4}}\right)} = 32\lim_{n\to\infty} \frac{1}{n^{1/4}} = 32 \cdot 0 = 0$$

20. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{1+\ln n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{1+\ln n} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} n = \infty$$

21. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left[\frac{(n+1)^{\sqrt{2}}}{2^{n+1}}\right]}{\left[\frac{n^{\sqrt{2}}}{2^n}\right]} = \lim_{n\to\infty} \frac{(n+1)^{\sqrt{2}}}{2^{n+1}} \cdot \frac{2^n}{n^{\sqrt{2}}}$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{\sqrt{2}} \left(\frac{1}{2}\right) = \frac{1}{2} < 1$

22. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)^2}{e^{n+1}}\right)}{\left(\frac{n^2}{e^n}\right)} = \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \left(\frac{1}{e}\right) = \frac{1}{e} < 1$

23. diverges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)!}{e^{n+1}}\right)}{\left(\frac{n!}{e^n}\right)} = \lim_{n\to\infty} \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \lim_{n\to\infty} \frac{n+1}{e} = \infty$

24. diverges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)!}{10^{n+1}}\right)}{\left(\frac{n!}{10^n}\right)} = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \lim_{n\to\infty} \frac{n}{10} = \infty$

25. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)^{10}}{10^{n+1}}\right)}{\left(\frac{n^{10}}{10^n}\right)} = \lim_{n\to\infty} \frac{(n+1)^{10}}{10^{n+1}} \cdot \frac{10^n}{n^{10}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{10} \left(\frac{1}{10}\right)$

$= \frac{1}{10} < 1$

26. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\ln(n+1)}{2^{n+1}} \cdot \frac{2^n}{n \ln(n)} = \frac{1}{2} < 1$

27. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)(n+3)}{(n+1)!} \cdot \frac{n!}{(n+1)(n+2)} = 0 < 1$

28. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3}{e^{n+1}} \cdot \frac{e^n}{n^3} = \frac{1}{e} < 1$

29. converges by the nth-Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{\left((\ln n)^n\right)^{1/n}}{\left(n^n\right)^{1/n}} = \lim_{n\to\infty} \frac{\ln n}{n}$

$= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0 < 1$

30. converges by the nth-Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n} = \lim_{n\to\infty} \left(\left(\frac{1}{n} - \frac{1}{n^2}\right)^n\right)^{1/n}$

$= \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 < 1$

31. converges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\ln n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1$

32. converges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^{n/2}}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt{\ln n}} = \frac{\lim_{n\to\infty} \sqrt[n]{n}}{\lim_{n\to\infty} \sqrt{\ln n}} = 0 < 1$

$\left(\lim_{n\to\infty} \sqrt[n]{n} = 1\right)$

33. diverges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^2}} = \lim_{n\to\infty} \frac{n!}{n^2} = \infty > 1$

34. diverges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(2^n)^2}} = \lim_{n\to\infty} \frac{n}{4} = \infty > 1$

35. converges; a geometric series with $r = \frac{1}{e} < 1$

36. diverges; by the nth-Term Test for Divergence, $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

37. diverges; $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series

38. converges; $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = -2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series

39. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$\lim_{n\to\infty} \frac{\left(\frac{1}{(1+\ln n)^2}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{(1+\ln n)^2} = \lim_{n\to\infty} \frac{1}{\left[\frac{2(1+\ln n)}{n}\right]}$ (by L'Hôpital's Rule) $= \lim_{n\to\infty} \frac{n}{2(1+\ln n)}$

$= \lim_{n\to\infty} \frac{1}{\left(\frac{2}{n}\right)}$ (by L'Hôpital's Rule) $= \lim_{n\to\infty} \frac{n}{2} = \infty$

40. diverges by the Integral Test: $\int_2^{\infty} \frac{\ln(x+1)}{x+1}\,dx = \int_{\ln 3}^{\infty} u\,du = \lim_{b\to\infty} \left[\frac{1}{2}u^2\right]_{\ln 3}^{b} = \lim_{b\to\infty} \frac{1}{2}\left(b^2 - \ln^2 3\right) = \infty$

41. converges by the Direct Comparison Test with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series: $n^2 - 1 > n$ for

$n \geq 2 \Rightarrow n^2(n^2-1) > n^3 \Rightarrow n\sqrt{n^2-1} > n^{3/2} \Rightarrow \frac{1}{n^{3/2}} > \frac{1}{n\sqrt{n^2-1}}$

42. diverges; $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim\limits_{n\to\infty} \left(1+\frac{-2}{n}\right)^n = e^{-2} \neq 0$

43. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}} \cdot \frac{3!\,n!\,3^n}{(n+3)!} = \lim\limits_{n\to\infty} \frac{n+4}{3(n+1)} = \frac{1}{3} < 1$

44. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)2^{n+1}(n+2)!}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n2^n(n+1)!}$

$= \lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{2}{3}\right)\left(\frac{n+2}{n+1}\right) = \frac{2}{3} < 1$

45. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!} = \lim\limits_{n\to\infty} \frac{n+1}{(2n+3)(2n+2)} = 0 < 1$

46. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim\limits_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$

$= \lim\limits_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$

47. converges by the Integral Test: $\int_1^\infty \frac{8\tan^{-1}x}{1+x^2}\,dx$; $\begin{bmatrix} u = \tan^{-1}x \\ du = \frac{dx}{1+x^2} \end{bmatrix} \Rightarrow \int_{\pi/4}^{\pi/2} 8u\,du = \left[4u^2\right]_{\pi/4}^{\pi/2} = 4\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) = \frac{3\pi^2}{4}$

48. diverges by the Integral Test: $\int_1^\infty \frac{x}{x^2+1}\,dx$; $\begin{bmatrix} u = x^2+1 \\ du = 2x\,dx \end{bmatrix} \Rightarrow \frac{1}{2}\int_2^\infty \frac{du}{u} = \lim\limits_{b\to\infty} \left[\frac{1}{2}\ln u\right]_2^b$

$= \lim\limits_{b\to\infty} \frac{1}{2}(\ln b - \ln 2) = \infty$

49. converges by the Integral Test: $\int_1^\infty \operatorname{sech} x\,dx = 2\lim\limits_{b\to\infty} \int_1^b \frac{e^x}{1+(e^x)^2}\,dx = 2\lim\limits_{b\to\infty} \left[\tan^{-1} e^x\right]_1^b$

$= 2\lim\limits_{b\to\infty} \left(\tan^{-1} e^b - \tan^{-1} e\right) = \pi - 2\tan^{-1} e$

50. converges by the Integral Test: $\int_1^\infty \operatorname{sech}^2 x\,dx = \lim\limits_{b\to\infty} \int_1^b \operatorname{sech}^2 x\,dx = \lim\limits_{b\to\infty} [\tanh x]_1^b = \lim\limits_{b\to\infty} (\tanh b - \tanh 1)$

$= 1 - \tanh 1$

51. converges by the Direct Comparison Test: $\frac{2+(-1)^n}{(1.25)^n} = \left(\frac{4}{5}\right)^n [2+(-1)^n] \leq \left(\frac{4}{5}\right)^n (3)$

52. diverges; $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(1-\frac{1}{3n}\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{\left(-\frac{1}{3}\right)}{n}\right)^n = e^{-1/3} \approx 0.72 \neq 0$

53. converges by the Direct Comparison Test: $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$ for $n \geq 2$

54. diverges by the Direct Comparison Test: $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 3$

55. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$, the nth term of a convergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{10n+1}{n(n+1)(n+2)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{10n^2+n}{n^2+3n+2} = \lim_{n\to\infty} \frac{20n+1}{2n+3} = \lim_{n\to\infty} \frac{20}{2} = 10$$

56. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$, the nth term of a convergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{5n^3-3n}{n^2(n-2)(n^2+5)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{5n^3-3n}{n^3-2n^2+5n-10} = \lim_{n\to\infty} \frac{15n^2-3}{3n^2-4n+5} = \lim_{n\to\infty} \frac{30n}{6n-4} = 5$$

57. converges by the Direct Comparison Test: $\frac{\tan^{-1} n}{n^{1.1}} < \frac{\frac{\pi}{2}}{n^{1.1}}$ and $\sum\limits_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{1.1}} = \frac{\pi}{2}\sum\limits_{n=1}^{\infty} \frac{1}{n^{1.1}}$ is the product of a convergent p-series and a nonzero constant

58. converges by the Direct Comparison Test: $\sec^{-1} n < \frac{\pi}{2} \Rightarrow \frac{\sec^{-1} n}{n^{1.3}} < \frac{\left(\frac{\pi}{2}\right)}{n^{1.3}}$ and $\sum\limits_{n=1}^{\infty} \frac{\left(\frac{\pi}{2}\right)}{n^{1.3}} = \frac{\pi}{2}\sum\limits_{n=1}^{\infty} \frac{1}{n^{1.3}}$ is the product of a convergent p-series and a nonzero constant

59. diverges by the nth-Term Test for divergence; $\lim\limits_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim\limits_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim\limits_{x\to 0} \frac{\sin x}{x} = 1 \neq 0$

60. converges by the Integral Test: $\int_1^{\infty} \frac{2}{1+e^x}\,dx;\ \begin{bmatrix} u = e^x \\ du = e^x\,dx \\ dx = \frac{1}{u}\,du \end{bmatrix} \Rightarrow \int_e^{\infty} \frac{2}{u(1+u)}\,du = \int_e^{\infty} \left(\frac{2}{u} - \frac{2}{u+1}\right) du$

$$= \lim_{b\to\infty}\left[2\ln\frac{u}{u+1}\right]_e^b = \lim_{b\to\infty} 2\ln\left(\frac{b}{b+1}\right) - 2\ln\left(\frac{e}{e+1}\right) = 2\ln 1 - 2\ln\left(\frac{e}{e+1}\right) = -2\ln\left(\frac{e}{e+1}\right)$$

61. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\left(\frac{1+\sin n}{n}\right)a_n}{a_n} = 0 < 1$

62. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1+\tan^{-1} n}{n}\right)a_n}{a_n} = \lim_{n\to\infty} \frac{1+\tan^{-1} n}{n} = 0$ since the numerator approaches $1+\frac{\pi}{2}$ while the denominator tends to ∞

63. diverges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{3n-1}{2n+1}\right)a_n}{a_n} = \lim_{n\to\infty} \frac{3n-1}{2n+1} = \frac{3}{2} > 1$

64. diverges; $a_{n+1} = \frac{n}{n+1} a_n \Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n} a_{n-1}\right) \Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1} a_{n-2}\right)$
$\Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\cdots\left(\frac{1}{2}\right)a_1 \Rightarrow a_{n+1} = \frac{a_1}{n+1} \Rightarrow a_{n+1} = \frac{3}{n+1}$, which is a constant times the general term of the diverging harmonic series

65. diverges by the nth-Term Test: $a_1 = \frac{1}{3}$, $a_2 = \sqrt[2]{\frac{1}{3}}$, $a_3 = \sqrt[3]{\sqrt[2]{\frac{1}{3}}} = \sqrt[6]{\frac{1}{3}}$, $a_4 = \sqrt[4]{\sqrt[3]{\sqrt[2]{\frac{1}{3}}}} = \sqrt[4!]{\frac{1}{3}}, \ldots,$
$a_n = \sqrt[n!]{\frac{1}{3}} \Rightarrow \lim_{n\to\infty} a_n = 1$ because $\left\{\sqrt[n!]{\frac{1}{3}}\right\}$ is a subsequence of $\left\{\sqrt[n]{\frac{1}{3}}\right\}$ whose limit is 1 by Table 8.1

66. converges by the Direct Comparison Test: $a_1 = \frac{1}{2}$, $a_2 = \left(\frac{1}{2}\right)^2$, $a_3 = \left(\left(\frac{1}{2}\right)^2\right)^3 = \left(\frac{1}{2}\right)^6$, $a_4 = \left(\left(\frac{1}{2}\right)^6\right)^4 = \left(\frac{1}{2}\right)^{24}, \ldots$
$\Rightarrow a_n = \left(\frac{1}{2}\right)^{n!} < \left(\frac{1}{2}\right)^n$ which is the nth-term of a convergent geometric series

67. (a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then there exists an integer N such that for all $n > N$, $\left|\frac{a_n}{b_n} - 0\right| < 1 \Rightarrow -1 < \frac{a_n}{b_n} < 1$
$\Rightarrow a_n < b_n$. Thus, if $\sum b_n$ converges, then $\sum a_n$ converges by the Direct Comparison Test.

(b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then there exists an integer N such that for all $n > N$, $\frac{a_n}{b_n} > 1 \Rightarrow a_n > b_n$. Thus, if $\sum b_n$ diverges, then $\sum a_n$ diverges by the Direct Comparison Test.

68. Yes, $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges by the Direct Comparison Test because $\frac{a_n}{n} < a_n$

69. $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \Rightarrow$ there exists an integer N such that for all $n > N$, $\frac{a_n}{b_n} > 1 \Rightarrow a_n > b_n$. If $\sum a_n$ converges, then $\sum b_n$ converges by the Direct Comparison Test

70. $\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0 \Rightarrow$ there exists an integer N such that for all $n > N$, $0 \le a_n < 1 \Rightarrow a_n^2 < a_n$
$\Rightarrow \sum a_n^2$ converges by the Direct Comparison Test

71. $\int_1^{\infty} \left(\frac{a}{x+2} - \frac{1}{x+4}\right) dx = \lim_{b\to\infty} \left[a \ln|x+2| - \ln|x+4|\right]_1^b = \lim_{b\to\infty} \ln \frac{(b+2)^a}{b+4} - \ln\left(\frac{3^a}{5}\right);$

$\lim\limits_{b\to\infty} \frac{(b+2)^a}{b+4} = a \lim\limits_{b\to\infty} (b+2)^{a-1} = \begin{cases} \infty, & a > 1 \\ 1, & a = 1 \end{cases} \Rightarrow$ the series converges to $\ln\left(\frac{5}{3}\right)$ if $a = 1$ and diverges to ∞ if $a > 1$. If $a < 1$, the terms of the series eventually become negative and the Integral Test does not apply. From that point on, however, the series behaves like a negative multiple of the harmonic series, and so it diverges.

72. $\int\limits_3^\infty \left(\frac{1}{x-1} - \frac{2a}{x+1}\right) dx = \lim\limits_{b\to\infty} \left[\ln\left|\frac{x-1}{(x+1)^{2a}}\right|\right]_3^b = \lim\limits_{b\to\infty} \ln \frac{b-1}{(b+1)^{2a}} - \ln\left(\frac{2}{4^{2a}}\right)$; $\lim\limits_{b\to\infty} \frac{b-1}{(b+1)^{2a}}$

$= \lim\limits_{b\to\infty} \frac{1}{2a(b+1)^{2a-1}} = \begin{cases} 1, & a = \frac{1}{2} \\ \infty, & a < \frac{1}{2} \end{cases} \Rightarrow$ the series converges to $\ln\left(\frac{4}{2}\right) = \ln 2$ if $a = \frac{1}{2}$ and diverges to ∞ if $a < \frac{1}{2}$. If $a > \frac{1}{2}$, the terms of the series eventually become negative and the Integral Test does not apply. From that point on, however, the series behaves like a negative multiple of the harmonic series, and so it diverges.

73. Let $A_n = \sum\limits_{k=1}^{n} a_k$ and $B_n = \sum\limits_{k=1}^{n} 2^k a_{(2^k)}$, where $\{a_k\}$ is a nonincreasing sequence of positive terms converging to 0. Note that $\{A_n\}$ and $\{B_n\}$ are nondecreasing sequences of positive terms. Now,

$B_n = 2a_2 + 4a_4 + 8a_8 + \ldots + 2^n a_{(2^n)} = 2a_2 + (2a_4 + 2a_4) + (2a_8 + 2a_8 + 2a_8 + 2a_8) + \ldots$

$+ \underbrace{\left(2a_{(2^n)} + 2a_{(2^n)} + \ldots + 2a_{(2^n)}\right)}_{2^{n-1}\text{ terms}} \le 2a_1 + 2a_2 + (2a_3 + 2a_4) + (2a_5 + 2a_6 + 2a_7 + 2a_8) + \ldots$

$+ \left(2a_{(2^{n-1})} + 2a_{(2^{n-1}+1)} + \ldots + 2a_{(2^n)}\right) = 2A_{(2^n)} \le 2\sum\limits_{k=1}^{\infty} a_k$. Therefore if $\sum a_k$ converges, then $\{B_n\}$ is bounded above $\Rightarrow \sum 2^k a_{(2^k)}$ converges. Conversely,

$A_n = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \ldots + a_n < a_1 + 2a_2 + 4a_4 + \ldots + 2^n a_{(2^n)} = a_1 + B_n < a_1 + \sum\limits_{k=1}^{\infty} 2^k a_{(2^k)}$.

Therefore, if $\sum\limits_{k=1}^{\infty} 2^k a_{(2^k)}$ converges, then $\{A_n\}$ is bounded above and hence converges.

74. (a) $a_{(2^n)} = \frac{1}{2^n \ln(2^n)} = \frac{1}{2^n \cdot n(\ln 2)} \Rightarrow \sum\limits_{n=2}^{\infty} 2^n a_{(2^n)} = \sum\limits_{n=2}^{\infty} 2^n \frac{1}{2^n \cdot n(\ln 2)} = \frac{1}{\ln 2} \sum\limits_{n=2}^{\infty} \frac{1}{n}$, which diverges $\Rightarrow \sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

(b) $a_{(2^n)} = \frac{1}{2^{np}} \Rightarrow \sum\limits_{n=1}^{\infty} 2^n a_{(2^n)} = \sum\limits_{n=1}^{\infty} 2^n \cdot \frac{1}{2^{np}} = \sum\limits_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}} = \sum\limits_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$, a geometric series that converges if $\frac{1}{2^{p-1}} < 1$ or $p > 1$, but diverges if $p \ge 1$.

75. (a) $\int\limits_2^\infty \frac{dx}{x(\ln x)^p}$; $\begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \end{bmatrix} \Rightarrow \int\limits_{\ln 2}^{\infty} u^{-p}\, du = \lim\limits_{b\to\infty} \left[\frac{u^{-p+1}}{-p+1}\right]_{\ln 2}^{b} = \lim\limits_{b\to\infty} \left(\frac{1}{1-p}\right)\left[b^{-p+1} - (\ln 2)^{-p+1}\right]$

$= \begin{cases} \frac{1}{p-1}(\ln 2)^{-p+1}, & p > 1 \\ \infty, & p < 1 \end{cases} \Rightarrow$ the improper integral converges if $p > 1$ and diverges if $p < 1$. For $p = 1$: $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty} \left[\ln(\ln x)\right]_2^b = \lim_{b\to\infty} \left[\ln(\ln b) - \ln(\ln 2)\right] = \infty$, so the improper integral diverges if $p = 1$.

(b) Since the series and the integral converge or diverge together, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$.

76. (a) $p = 1 \Rightarrow$ the series diverges

(b) $p = 1.01 \Rightarrow$ the series converges

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n^3)} = \frac{1}{3} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$; $p = 1 \Rightarrow$ the series diverges

(d) $p = 3 \Rightarrow$ the series converges

77. Ratio: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(\ln(n+1))^p} \cdot \frac{(\ln n)^p}{1} = \left[\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}\right]^p = \left[\lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)}\right]^p = \left(\lim_{n\to\infty} \frac{n+1}{n}\right)^p$

$= (1)^p = 1 \Rightarrow$ no conclusion

Root: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^p}} = \frac{1}{\left(\lim_{n\to\infty} (\ln n)^{1/n}\right)^p}$; let $f(n) = (\ln n)^{1/n}$, then $\ln f(n) = \frac{\ln(\ln n)}{n}$

$\Rightarrow \lim_{n\to\infty} \ln f(n) = \lim_{n\to\infty} \frac{\ln(\ln n)}{n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n \ln n}\right)}{1} = \lim_{n\to\infty} \frac{1}{n \ln n} = 0 \Rightarrow \lim_{n\to\infty} (\ln n)^{1/n}$

$= \lim_{n\to\infty} e^{\ln f(n)} = e^0 = 1$; therefore $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\left(\lim_{n\to\infty} (\ln n)^{1/n}\right)^p} = \frac{1}{(1)^p} = 1 \Rightarrow$ no conclusion

78. $a_n \le \frac{n}{2^n}$ for every n and the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges by the Ratio Test since $\lim_{n\to\infty} \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} < 1$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test

79. Ratio: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1^p = 1 \Rightarrow$ no conclusion

Root: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{1}{\left(\sqrt[n]{n}\right)^p} = \frac{1}{(1)^p} = 1 \Rightarrow$ no conclusion

80. Example CAS commands:

<u>Maple</u>:

```
s:= k -> sum(1/(n^3*(sin^2)(n)), n=1..k);
limit(s(k), k=infinity);
plot(s(k), k=1..100, style=POINT, symbol=CIRCLE);
plot(s(k), k=1..200, style=POINT, symbol=CIRCLE);
plot(s(k), k=1..400, style=POINT, symbol=CIRCLE);
evalf(355/113);
```

Mathematica:

```
Clear[a,k,n,s]
a[n_] = 1/ (n^3 Sin[n]^2)
s[k_] = Sum[ a[n], {n,1,k} ]

Note: To make Mathematica smart about  limits, load the package:

<< Calculus`Limit`
Limit[ s[k], k -> Infinity ]

But Mathematica still cannot find the limit...

Note:  For plotting many partial sums, it is far more efficient to do the
calculations numerically rather than exactly.  So we redefine s[k] (where
the "s[k_] := s[k] = ..." causes Mathematica to remember previous results)

Clear[s]
s[k_] := s[k] = s[k-1] + N[a[k] ]
s[1] = N[a[1]]
ListPlot[ Table[ s[k], {k,100} ] ]
ListPlot[ Table[ s[k], {k,200} ] ]
ListPlot[ Table[ s[k], {k,400} ] ]

Note: Change PlotRange so Mathematica does not cut off the jump.

Show[ %, PlotRange -> All ]
N[ 355/113 ]
N[ Pi - 355/113 ]
Sin[ 355 ] // N
a[ 355 ] // N
```

8.5 ALTERNATING SERIES, ABSOLUTE AND CONDITIONAL CONVERGENCE

1. converges absolutely $\Rightarrow$ converges by the Absolute Convergence Test since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series

2. converges absolutely $\Rightarrow$ converges by the Absolute Convergence Test since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent p-series

3. diverges by the nth-Term Test since for $n > 10 \Rightarrow \frac{n}{10} > 1 \Rightarrow \lim_{n\to\infty} \left(\frac{n}{10}\right)^n \neq 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{n}{10}\right)^n$ diverges

4. diverges by the nth-Term Test since $\lim_{n\to\infty} \frac{10^n}{n^{10}} = \lim_{n\to\infty} \frac{10^n(\ln 10)^{10}}{10!} = \infty$ (after 10 applications of L'Hôpital's rule)

5. converges by the Alternating Series Test because $f(x) = \ln x$ is an increasing function of $x \Rightarrow \frac{1}{\ln x}$ is decreasing $\Rightarrow u_n \geq u_{n+1}$ for $n \geq 1$; also $u_n \geq 0$ for $n \geq 1$ and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$

6. converges by the Alternating Series Test since $f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{1 - \ln x}{x^2} < 0$ when $x > e \Rightarrow f(x)$ is decreasing $\Rightarrow u_n \geq u_{n+1}$; also $u_n \geq 0$ for $n \geq 1$ and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{\ln n}{n} = \lim\limits_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0$

7. diverges by the nth-Term Test since $\lim\limits_{n\to\infty} \frac{\ln n}{\ln n^2} = \lim\limits_{n\to\infty} \frac{\ln n}{2 \ln n} = \lim\limits_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0$

8. converges by the Alternating Series Test since $f(x) = \ln\left(1 + x^{-1}\right) \Rightarrow f'(x) = \frac{-1}{x(x+1)} < 0$ for $x > 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n \geq u_{n+1}$; also $u_n \geq 0$ for $n \geq 1$ and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) = \ln\left(\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)\right) = \ln 1 = 0$

9. converges by the Alternating Series Test since $f(x) = \frac{\sqrt{x} + 1}{x + 1} \Rightarrow f'(x) = \frac{1 - x - 2\sqrt{x}}{2\sqrt{x}\,(x+1)^2} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n \geq u_{n+1}$; also $u_n \geq 0$ for $n \geq 1$ and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{\sqrt{n} + 1}{n + 1} = 0$

10. diverges by the nth-Term Test since $\lim\limits_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n} + 1} = \lim\limits_{n\to\infty} \frac{3\sqrt{1 + \frac{1}{n}}}{1 + \left(\frac{1}{\sqrt{n}}\right)} = 3 \neq 0$

11. converges absolutely since $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ a convergent geometric series

12. converges absolutely by the Direct Comparison Test since $\left|\frac{(-1)^{n+1}(0.1)^n}{n}\right| = \frac{1}{(10)^n n} < \left(\frac{1}{10}\right)^n$ which is the nth term of a convergent geometric series

13. The series $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}}$ converges by the Alternating Series Test since $\left(\frac{1}{\sqrt{n+1}}\right) > \left(\frac{1}{\sqrt{n+2}}\right)$ and $\left(\frac{1}{\sqrt{n+1}}\right) \to 0$. The series diverges absolutely by the Integral Test: $\int\limits_1^{\infty} \frac{1}{\sqrt{x+1}}\, dx = \lim\limits_{b\to\infty} 2\sqrt{x+1}\,\Big|_1^b$ $= \lim\limits_{b\to\infty} \left[2\sqrt{b+1} - 2\sqrt{2}\right] = \infty.$

14. converges conditionally since $\frac{1}{1 + \sqrt{n}} > \frac{1}{1 + \sqrt{n+1}} > 0$ and $\lim\limits_{n\to\infty} \frac{1}{1 + \sqrt{n}} = 0 \Rightarrow$ convergence; but $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{1}{1 + \sqrt{n}}$ is a divergent series since $\frac{1}{1 + \sqrt{n}} \geq \frac{1}{2\sqrt{n}}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series

15. converges absolutely since $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{n}{n^3 + 1}$ and $\frac{n}{n^3 + 1} < \frac{1}{n^2}$ which is the nth-term of a converging p-series

16. diverges by the nth-Term Test since $\lim_{n\to\infty} \frac{n!}{2^n} = \infty$ (Table 8.1)

17. converges conditionally since $\frac{1}{n+3} > \frac{1}{(n+1)+3} > 0$ and $\lim_{n\to\infty} \frac{1}{n+3} = 0 \Rightarrow$ convergence; but $\sum_{n=1}^{\infty} |a_n|$ $= \sum_{n=1}^{\infty} \frac{1}{n+3}$ diverges because $\frac{1}{n+3} \geq \frac{1}{4n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series

18. converges absolutely because the series $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$ converges by the Direct Comparison Test since $\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$

19. diverges by the nth-Term Test since $\lim_{n\to\infty} \frac{3+n}{5+n} = 1 \neq 0$

20. converges conditionally since $f(x) = \ln x$ is an increasing function of $x \Rightarrow \frac{1}{3 \ln x} = \frac{1}{\ln(x^3)}$ is decreasing $\Rightarrow \frac{1}{3 \ln n} > \frac{1}{3 \ln (n+1)} > 0$ for $n \geq 2$ and $\lim_{n\to\infty} \frac{1}{3 \ln n} = 0 \Rightarrow$ convergence; but $\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{\ln(n^3)}$ $= \sum_{n=2}^{\infty} \frac{1}{3 \ln n}$ diverges because $\frac{1}{3 \ln n} > \frac{1}{3n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges

21. converges conditionally since $f(x) = \frac{1}{x^2} + \frac{1}{x} \Rightarrow f'(x) = -\left(\frac{2}{x^3} + \frac{1}{x^2}\right) < 0 \Rightarrow f(x)$ is decreasing and hence $u_n > u_{n+1} > 0$ for $n \geq 1$ and $\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{n}\right) = 0 \Rightarrow$ convergence; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1+n}{n^2}$ $= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n}$ is the sum of a convergent and divergent series, and hence diverges

22. converges absolutely by the Direct Comparison Test since $\left|\frac{(-2)^{n+1}}{n+5^n}\right| = \frac{2^{n+1}}{n+5^n} < 2\left(\frac{2}{5}\right)^n$ which is the nth term of a convergent geometric series

23. converges absolutely by the Ratio Test: $\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \left[\frac{(n+1)^2\left(\frac{2}{3}\right)^{n+1}}{n^2\left(\frac{2}{3}\right)^n}\right] = \frac{2}{3} < 1$

24. diverges by the nth-Term Test since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 10^{1/n} = 1 \neq 0$ (Table 8.1)

25. converges absolutely by the Integral Test since $\int_1^{\infty} \left(\tan^{-1} x\right)\left(\frac{1}{1+x^2}\right) dx = \lim_{b\to\infty} \left[\frac{\left(\tan^{-1} x\right)^2}{2}\right]_1^b$ $= \lim_{b\to\infty} \left[\left(\tan^{-1} b\right)^2 - \left(\tan^{-1} 1\right)^2\right] = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] = \frac{3\pi^2}{32}$

26. converges conditionally since $f(x) = \frac{1}{x \ln x} \Rightarrow f'(x) = -\frac{[\ln(x) + 1]}{(x \ln x)^2} < 0 \Rightarrow f(x)$ is decreasing

$\Rightarrow u_n > u_{n+1} > 0$ for $n \geq 2$ and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0 \Rightarrow$ convergence; but by the Integral Test,

$$\int_2^\infty \frac{dx}{x \ln x} = \lim_{b\to\infty} \int_2^b \left(\frac{\left(\frac{1}{x}\right)}{\ln x}\right) dx = \lim_{b\to\infty} [\ln(\ln x)]_2^b = \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$$

$\Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges

27. diverges by the nth-Term Test since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

28. converges conditionally since $f(x) = \frac{\ln x}{x - \ln x} \Rightarrow f'(x) = \frac{\left(\frac{1}{x}\right)(x - \ln x) - (\ln x)\left(1 - \frac{1}{x}\right)}{(x - \ln x)^2}$

$= \frac{1 - \left(\frac{\ln x}{x}\right) - \ln x + \left(\frac{\ln x}{x}\right)}{(x - \ln x)^2} = \frac{1 - \ln x}{(x - \ln x)^2} < 0 \Rightarrow u_n \geq u_{n+1} > 0$ when $n > e$ and $\lim_{n\to\infty} \frac{\ln n}{n - \ln n}$

$= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1 - \left(\frac{1}{n}\right)} = 0 \Rightarrow$ convergence; but $n - \ln n < n \Rightarrow \frac{1}{n - \ln n} > \frac{1}{n} \Rightarrow \frac{\ln n}{n - \ln n} > \frac{1}{n}$ so that

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\ln n}{n - \ln n}$ diverges by the Direct Comparison Test

29. converges absolutely by the Ratio Test: $\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \frac{(100)^{n+1}}{(n+1)!} \cdot \frac{n!}{(100)^n} = \lim_{n\to\infty} \frac{100}{n+1} = 0 < 1$

30. converges absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$ is a convergent geometric series

31. converges absolutely by the Direct Comparison Test since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 1}$ and

$\frac{1}{n^2 + 2n + 1} < \frac{1}{n^2}$ which is the nth-term of a convergent p-series

32. converges absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{\ln n}{\ln n^2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{\ln n}{2 \ln n}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series

33. converges absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series

34. converges conditionally since $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is the convergent alternating harmonic series, but

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

35. converges absolutely by the Root Test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{(n+1)^n}{(2n)^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$

36. converges absolutely by the Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{((n+1)!)^2}{((2n+2)!)} \cdot \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$

37. diverges by the nth-Term Test since $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{(2n)!}{2^n n!\, n} = \lim_{n\to\infty} \frac{(n+1)(n+2)\cdots(2n)}{2^n n}$
$= \lim_{n\to\infty} \frac{(n+1)(n+2)\cdots(n+(n-1))}{2^{n-1}} > \lim_{n\to\infty} \left(\frac{n+1}{2}\right)^{n-1} = \infty \neq 0$

38. converges absolutely by the Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!\,3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{n!\,n!\,3^n}$
$= \lim_{n\to\infty} \frac{(n+1)^2\, 3}{(2n+2)(2n+3)} = \frac{3}{4} < 1$

39. converges conditionally since $\frac{\sqrt{n+1}-\sqrt{n}}{1} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ and $\left\{\frac{1}{\sqrt{n+1}+\sqrt{n}}\right\}$ is a decreasing sequence of positive terms which converges to $0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ converges; but $n > \frac{1}{3} \Rightarrow 3n > 1$ $\Rightarrow 4n > n+1 \Rightarrow 2\sqrt{n} > \sqrt{n+1} \Rightarrow 3\sqrt{n} > \sqrt{n+1}+\sqrt{n} \Rightarrow \frac{1}{3\sqrt{n}} < \frac{1}{\sqrt{n+1}+\sqrt{n}} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges by the Direct Comparison Test

40. diverges by the nth-Term Test since $\lim_{n\to\infty} \left(\sqrt{n^2+n}-n\right) = \lim_{n\to\infty} \left(\sqrt{n^2+n}-n\right) \cdot \left(\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\right)$
$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2} \neq 0$

41. diverges by the nth-Term Test since $\lim_{n\to\infty} \left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right) = \lim_{n\to\infty} \left[\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)\left(\frac{\sqrt{n+\sqrt{n}}+\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}}\right)\right]$
$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}+1} = \frac{1}{2} \neq 0$

42. converges conditionally since $\left\{\frac{1}{\sqrt{n}+\sqrt{n+1}}\right\}$ is a decreasing sequence of positive terms converging to 0

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges; but $\lim_{n\to\infty} \frac{\left(\frac{1}{\sqrt{n}+\sqrt{n+1}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = 1$

so that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent p-series

43. converges absolutely by the Direct Comparison Test since $\operatorname{sech}(n) = \frac{2}{e^n + e^{-n}} = \frac{2e^n}{e^{2n}+1} < \frac{2e^n}{e^{2n}} = \frac{2}{e^n}$ which is the nth term of a convergent geometric series

44. converges absolutely by the Integral Test since $\int_1^{\infty} \operatorname{csch} x \, dx = \int_1^{\infty} \left(\frac{2}{e^x - e^{-x}} \cdot \frac{e^x}{e^x}\right) dx = -2 \int_1^{\infty} \frac{e^x}{1-(e^x)^2} \, dx$

$= -2 \lim_{b\to\infty} \int_1^b \frac{e^x}{1-(e^x)^2} \, dx = -2 \lim_{b\to\infty} \left[\coth^{-1} e^x\right]_1^b = -2 \lim_{b\to\infty} \left[\coth^{-1}(e^b) - \coth^{-1} e\right]$

$= -2 \lim_{b\to\infty} \left[\frac{1}{2} \ln\left(\frac{e^b+1}{e^n-1}\right) - \frac{1}{2} \ln\left(\frac{e+1}{e-1}\right)\right] = \ln\left(\frac{e+1}{e-1}\right) - \ln\left(\lim_{n\to\infty} \left(\frac{e^b+1}{e^b-1}\right)\right) = \ln\left(\frac{e+1}{e-1}\right) - \ln 1 \approx 0.77$

$\Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \operatorname{csch} n$ converges

45. $|\text{error}| < \left|(-1)^6\left(\frac{1}{5}\right)\right| = 0.2$

46. $|\text{error}| < \left|(-1)^6\left(\frac{1}{10^5}\right)\right| = 0.00001$

47. $|\text{error}| < \left|(-1)^6 \frac{(0.01)^5}{5}\right| = 2 \times 10^{-11}$

48. $|\text{error}| < \left|(-1)^4 t^4\right| = t^4 < 1$

49. $\frac{1}{(2n)!} < \frac{5}{10^6} \Rightarrow (2n)! > \frac{10^6}{5} = 200{,}000 \Rightarrow n \ge 5 \Rightarrow 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.54030$

50. $\frac{1}{n!} < \frac{5}{10^6} \Rightarrow \frac{10^6}{5} < n! \Rightarrow n \ge 9 \Rightarrow 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} \approx 0.367881944$

51. (a) $a_n \ge a_{n+1}$ fails since $\frac{1}{3} < \frac{1}{2}$

(b) Since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n\right] = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is the sum of two absolutely convergent series, we can rearrange the terms of the original series to find its sum:

$$\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right) = \frac{\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)} - \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = \frac{1}{2} - 1 = -\frac{1}{2}$$

52. $s_{20} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{19} - \frac{1}{20} \approx 0.6687714032 \Rightarrow s_{20} + \frac{1}{2} \cdot \frac{1}{21} \approx 0.692580927$

53. The unused terms are $\sum\limits_{j=n+1}^{\infty} (-1)^{j+1} a_j = (-1)^{n+1}\left(a_{n+1} - a_{n+2}\right) + (-1)^{n+3}\left(a_{n+3} - a_{n+4}\right) + \ldots$
$= (-1)^{n+1}\left[\left(a_{n+1} - a_{n+2}\right) + \left(a_{n+3} - a_{n+4}\right) + \ldots\right]$. Each grouped term is positive, so the remainder has the same sign as $(-1)^{n+1}$, which is the sign of the first unused term.

54. $s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} = \sum\limits_{k=1}^{n} \frac{1}{k(k+1)} = \sum\limits_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$
$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$ which are the first 2n terms of the first series, hence the two series are the same. Yes, for
$s_n = \sum\limits_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$
$\Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 \Rightarrow$ both series converge to 1. The sum of the first $2n + 1$ terms of the first series is $\left(1 - \frac{1}{n+1}\right) + \frac{1}{n+1} = 1$. Their sum is $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$.

55. Using the Direct Comparison Test, since $|a_n| \ge a_n$ and $\sum\limits_{n=1}^{\infty} a_n$ diverges we must have that $\sum\limits_{n=1}^{\infty} |a_n|$ diverges.

56. $|a_1 + a_2 + \ldots + a_n| \le |a_1| + |a_2| + \ldots + |a_n|$ for *all* n; then $\sum\limits_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ converges and these imply that $\left|\sum\limits_{n=1}^{\infty} a_n\right| \le \sum\limits_{n=1}^{\infty} |a_n|$

57. (a) $\sum\limits_{n=1}^{\infty} |a_n + b_n|$ converges by the Direct Comparison Test since $|a_n + b_n| \le |a_n| + |b_n|$ and hence $\sum\limits_{n=1}^{\infty} (a_n + b_n)$ converges absolutely

(b) $\sum\limits_{n=1}^{\infty} |b_n|$ converges $\Rightarrow \sum\limits_{n=1}^{\infty} -b_n$ converges absolutely; since $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely and $\sum\limits_{n=1}^{\infty} -b_n$ converges absolutely, we have $\sum\limits_{n=1}^{\infty} \left[a_n + (-b_n)\right] = \sum\limits_{n=1}^{\infty} (a_n - b_n)$ converges absolutely by part (a)

(c) $\sum\limits_{n=1}^{\infty} |a_n|$ converges $\Rightarrow |k| \sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} |ka_n|$ converges $\Rightarrow \sum\limits_{n=1}^{\infty} ka_n$ converges absolutely

58. If $a_n = b_n = (-1)^n \frac{1}{\sqrt{n}}$, then $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges, but $\sum\limits_{n=1}^{\infty} a_n b_n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges

59. $s_1 = -\frac{1}{2}$, $s_2 = -\frac{1}{2} + 1 = \frac{1}{2}$,

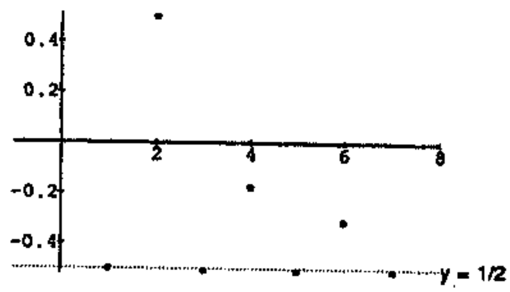
$s_3 = -\frac{1}{2} + 1 - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} - \frac{1}{18} - \frac{1}{20} - \frac{1}{22} \approx -0.5099,$

$s_4 = s_3 + \frac{1}{3} \approx -0.1766,$

$s_5 = s_4 - \frac{1}{24} - \frac{1}{26} - \frac{1}{28} - \frac{1}{30} - \frac{1}{32} - \frac{1}{34} - \frac{1}{36} - \frac{1}{38} - \frac{1}{40} - \frac{1}{42} - \frac{1}{44} \approx -0.512,$

$s_6 = s_5 + \frac{1}{5} \approx -0.312,$

$s_7 = s_6 - \frac{1}{46} - \frac{1}{48} - \frac{1}{50} - \frac{1}{52} - \frac{1}{54} - \frac{1}{56} - \frac{1}{58} - \frac{1}{60} - \frac{1}{62} - \frac{1}{64} - \frac{1}{66} \approx -0.51106$



60. (a) Since $\sum |a_n|$ converges, say to M, for $\epsilon > 0$ there is an integer N_1 such that $\left|\sum_{n=1}^{N_1-1} |a_n| - M\right| < \frac{\epsilon}{2}$

$\Leftrightarrow \left|\sum_{n=1}^{N_1-1} |a_n| - \left(\sum_{n=1}^{N_1-1} |a_n| + \sum_{n=N_1}^{\infty} |a_n|\right)\right| < \frac{\epsilon}{2} \Leftrightarrow \left|-\sum_{n=N_1}^{\infty} |a_n|\right| < \frac{\epsilon}{2} \Leftrightarrow \sum_{n=N_1}^{\infty} |a_n| < \frac{\epsilon}{2}$. Also, $\sum a_n$ converges to L $\Leftrightarrow$ for $\epsilon > 0$ there is an integer N_2 (which we can choose greater than or equal to N_1) such that $\left|s_{N_2} - L\right| < \frac{\epsilon}{2}$. Therefore, $\sum_{n=N_1}^{\infty} |a_n| < \frac{\epsilon}{2}$ and $\left|s_{N_2} - L\right| < \frac{\epsilon}{2}$.

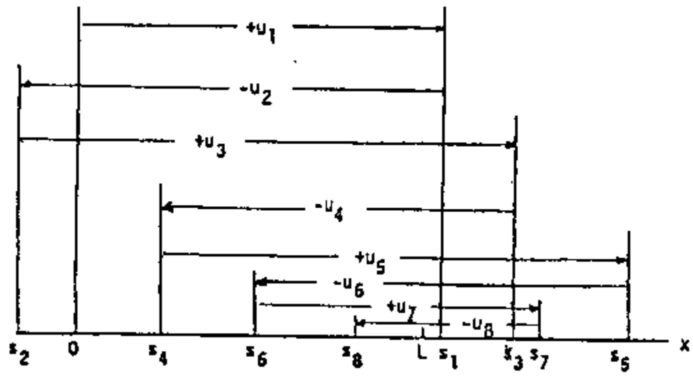
(b) The series $\sum_{n=1}^{\infty} |a_n|$ converges absolutely, say to M. Thus, there exists N_1 such that $\left|\sum_{n=1}^{k} |a_n| - M\right| < \epsilon$ whenever $k > N_1$. Now all of the terms in the sequence $\{|b_n|\}$ appear in $\{|a_n|\}$. Sum together all of the terms in $\{|b_n|\}$, in order, until you include all of the terms $\{|a_n|\}_{n=1}^{N_1}$, and let N_2 be the largest index in the sum $\sum_{n=1}^{N_2} |b_n|$ so obtained. Then $\left|\sum_{n=1}^{N_2} |b_n| - M\right| < \epsilon$ as well $\Rightarrow \sum_{n=1}^{\infty} |b_n|$ converges to M.

61. (a) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges and $\frac{1}{2}\sum_{n=1}^{\infty} a_n + \frac{1}{2}\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{a_n + |a_n|}{2}$ converges where $b_n = \frac{a_n + |a_n|}{2} = \begin{cases} a_n, & \text{if } a_n \ge 0 \\ 0, & \text{if } a_n < 0 \end{cases}$.

(b) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges and $\frac{1}{2}\sum_{n=1}^{\infty} a_n - \frac{1}{2}\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{a_n - |a_n|}{2}$ converges where $c_n = \frac{a_n - |a_n|}{2} = \begin{cases} 0, & \text{if } a_n \ge 0 \\ a_n, & \text{if } a_n < 0 \end{cases}$.

62. The terms in this conditionally convergent series were not added in the order given.

63. Here is an example figure when $N = 5$. Notice that $u_3 > u_2 > u_1$ and $u_3 > u_5 > u_4$, but $u_n \ge u_{n+1}$ for $n \ge 5$.



8.6 POWER SERIES

1. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} 1$, a divergent series
   (a) the radius is 1; the interval of convergence is $-1 < x < 1$
   (b) the interval of absolute convergence is $-1 < x < 1$
   (c) there are no values for which the series converges conditionally

2. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x+5)^{n+1}}{(x+5)^n}\right| < 1 \Rightarrow |x+5| < 1 \Rightarrow -6 < x < -4$; when $x = -6$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = -4$ we have $\sum_{n=1}^{\infty} 1$, a divergent series
   (a) the radius is 1; the interval of convergence is $-6 < x < -4$
   (b) the interval of absolute convergence is $-6 < x < -4$
   (c) there are no values for which the series converges conditionally

3. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(4x+1)^{n+1}}{(4x+1)^n}\right| < 1 \Rightarrow |4x+1| < 1 \Rightarrow -1 < 4x+1 < 1 \Rightarrow -\frac{1}{2} < x < 0$; when $x = -\frac{1}{2}$ we have $\sum_{n=1}^{\infty} (-1)^n(-1)^n = \sum_{n=1}^{\infty} (-1)^{2n} = \sum_{n=1}^{\infty} 1^n$, a divergent series; when $x = 0$ we have $\sum_{n=1}^{\infty} (-1)^n(1)^n$ $= \sum_{n=1}^{\infty} (-1)^n$, a divergent series
   (a) the radius is $\frac{1}{4}$; the interval of convergence is $-\frac{1}{2} < x < 0$
   (b) the interval of absolute convergence is $-\frac{1}{2} < x < 0$
   (c) there are no values for which the series converges conditionally

4. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n}\right| < 1 \Rightarrow |3x-2| \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow |3x-2| < 1$

$\Rightarrow -1 < 3x-2 < 1 \Rightarrow \frac{1}{3} < x < 1$; when $x = \frac{1}{3}$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is the alternating harmonic series and is conditionally convergent; when $x = 1$ we have $\sum\limits_{n=1}^{\infty} \frac{1}{n}$, the divergent harmonic series

(a) the radius is $\frac{1}{3}$; the interval of convergence is $\frac{1}{3} \le x < 1$

(b) the interval of absolute convergence is $\frac{1}{3} < x < 1$

(c) the series converges conditionally at $x = \frac{1}{3}$

5. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n}\right| < 1 \Rightarrow \frac{|x-2|}{10} < 1 \Rightarrow |x-2| < 10 \Rightarrow -10 < x-2 < 10$

$\Rightarrow -8 < x < 12$; when $x = -8$ we have $\sum\limits_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = 12$ we have $\sum\limits_{n=1}^{\infty} 1$, a divergent series

(a) the radius is 10; the interval of convergence is $-8 < x < 12$

(b) the interval of absolute convergence is $-8 < x < 12$

(c) there are no values for which the series converges conditionally

6. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{(2x)^{n+1}}{(2x)^n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} |2x| < 1 \Rightarrow |2x| < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$; when $x = -\frac{1}{2}$ we have $\sum\limits_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = \frac{1}{2}$ we have $\sum\limits_{n=1}^{\infty} 1$, a divergent series

(a) the radius is $\frac{1}{2}$; the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$

(b) the interval of absolute convergence is $-\frac{1}{2} < x < \frac{1}{2}$

(c) there are no values for which the series converges conditionally

7. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{(n+3)} \cdot \frac{(n+2)}{nx^n}\right| < 1 \Rightarrow |x| \lim\limits_{n\to\infty} \frac{(n+1)(n+2)}{(n+3)(n)} < 1 \Rightarrow |x| < 1$

$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$, a divergent series by the nth-term Test; when $x = 1$ we have $\sum\limits_{n=1}^{\infty} \frac{n}{n+2}$, a divergent series

(a) the radius is 1; the interval of convergence is $-1 < x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) there are no values for which the series converges conditionally

8. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x+2)^{n+1}}{n+1} \cdot \frac{n}{(x+2)^n}\right| < 1 \Rightarrow |x+2| \lim_{n\to\infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow |x+2| < 1$

$\Rightarrow -1 < x+2 < 1 \Rightarrow -3 < x < -1$; when $x = -3$ we have $\sum_{n=1}^{\infty} \frac{1}{n}$, a divergent series; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, a convergent series

(a) the radius is 1; the interval of convergence is $-3 < x \le -1$

(b) the interval of absolute convergence is $-3 < x < -1$

(c) the series converges conditionally at $x = -1$

9. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)\sqrt{n+1}\, 3^{n+1}} \cdot \frac{n\sqrt{n}\, 3^n}{x^n}\right| < 1 \Rightarrow \frac{|x|}{3}\left(\lim_{n\to\infty} \frac{n}{n+1}\right)\left(\sqrt{\lim_{n\to\infty} \frac{n}{n+1}}\right) < 1$

$\Rightarrow \frac{|x|}{3}(1)(1) < 1 \Rightarrow |x| < 3 \Rightarrow -3 < x < 3$; when $x = -3$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$, an absolutely convergent series; when $x = 3$ we have $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, a convergent p-series

(a) the radius is 3; the interval of convergence is $-3 \le x \le 3$

(b) the interval of absolute convergence is $-3 \le x \le 3$

(c) there are no values for which the series converges conditionally

10. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-1)^n}\right| < 1 \Rightarrow |x-1| \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} < 1 \Rightarrow |x-1| < 1$

$\Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$; when $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$, a conditionally convergent series; when $x = 2$ we have $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, a divergent series

(a) the radius is 1; the interval of convergence is $0 \le x < 2$

(b) the interval of absolute convergence is $0 < x < 2$

(c) the series converges conditionally at $x = 0$

11. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left(\frac{1}{n+1}\right) < 1$ for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

12. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n}\right| < 1 \Rightarrow 3|x| \lim_{n\to\infty} \left(\frac{1}{n+1}\right) < 1$ for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

13. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{2n+3}}{(n+1)!}\cdot\frac{n!}{x^{2n+1}}\right| < 1 \Rightarrow x^2 \lim_{n\to\infty} \left(\frac{1}{n+1}\right) < 1$ for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

14. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(2x+3)^{2n+3}}{(n+1)!}\cdot\frac{n!}{(2x+3)^{2n+1}}\right| < 1 \Rightarrow (2x+3)^2 \lim_{n\to\infty} \left(\frac{1}{n+1}\right) < 1$ for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

15. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}}\cdot\frac{\sqrt{n^2+3}}{x^n}\right| < 1 \Rightarrow |x|\sqrt{\lim_{n\to\infty} \frac{n^2+3}{n^2+2n+4}} < 1 \Rightarrow |x| < 1$

$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$, a conditionally convergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$, a divergent series

(a) the radius is 1; the interval of convergence is $-1 \le x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) the series converges conditionally at $x = -1$

16. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}}\cdot\frac{\sqrt{n^2+3}}{x^n}\right| < 1 \Rightarrow |x|\sqrt{\lim_{n\to\infty} \frac{n^2+3}{n^2+2n+4}} < 1 \Rightarrow |x| < 1$

$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$, a divergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+3}$, a conditionally convergent series

(a) the radius is 1; the interval of convergence is $-1 < x \le 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) the series converges conditionally at $x = 1$

17. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+1)(x+3)^{n+1}}{5^{n+1}}\cdot\frac{5^n}{n(x+3)^n}\right| < 1 \Rightarrow \frac{|x+3|}{5} \lim_{n\to\infty} \left(\frac{n+1}{n}\right) < 1 \Rightarrow \frac{|x+3|}{5} < 1$

$\Rightarrow |x+3| < 5 \Rightarrow -5 < x+3 < 5 \Rightarrow -8 < x < 2$; when $x = -8$ we have $\sum_{n=1}^{\infty} \frac{n(-5)^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n n$, a divergent series; when $x = 2$ we have $\sum_{n=1}^{\infty} \frac{n5^n}{5^n} = \sum_{n=1}^{\infty} n$, a divergent series

(a) the radius is 5; the interval of convergence is $-8 < x < 2$

(b) the interval of absolute convergence is $-8 < x < 2$

(c) there are no values for which the series converges conditionally

18. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{4^{n+1}(n^2+2n+2)} \cdot \frac{4^n(n^2+1)}{nx^n}\right| < 1 \Rightarrow \frac{|x|}{4} \lim_{n\to\infty} \left|\frac{(n+1)(n^2+1)}{n(n^2+2n+2)}\right| < 1 \Rightarrow |x| < 4$

$\Rightarrow -4 < x < 4$; when $x = -4$ we have $\sum_{n=1}^{\infty} \frac{n(-1)^n}{n^2+1}$, a conditionally convergent series; when $x = 4$ we have $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$, a divergent series

(a) the radius is 4; the interval of convergence is $-4 \le x < 4$

(b) the interval of absolute convergence is $-4 < x < 4$

(c) the series converges conditionally at $x = -4$

19. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\sqrt{n+1}\,x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\sqrt{n}\,x^n}\right| < 1 \Rightarrow \frac{|x|}{3} \sqrt{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)} < 1 \Rightarrow \frac{|x|}{3} < 1 \Rightarrow |x| < 3$

$\Rightarrow -3 < x < 3$; when $x = -3$ we have $\sum_{n=1}^{\infty} (-1)^n\sqrt{n}$, a divergent series; when $x = 3$ we have $\sum_{n=1}^{\infty} \sqrt{n}$, a divergent series

(a) the radius is 3; the interval of convergence is $-3 < x < 3$

(b) the interval of absolute convergence is $-3 < x < 3$

(c) there are no values for which the series converges conditionally

20. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\sqrt[n+1]{n+1}\,(2x+5)^{n+1}}{\sqrt[n]{n}\,(2x+5)^n}\right| < 1 \Rightarrow |2x+5| \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}\right) < 1$

$\Rightarrow |2x+5| \left(\frac{\lim_{t\to\infty} \sqrt[t]{t}}{\lim_{n\to\infty} \sqrt[n]{n}}\right) < 1 \Rightarrow |2x+5| < 1 \Rightarrow -1 < 2x+5 < 1 \Rightarrow -3 < x < -2$; when $x = -3$ we have $\sum_{n=1}^{\infty} (-1)^n\sqrt[n]{n}$, a divergent series since $\lim_{n\to\infty} \sqrt[n]{n} = 1$; when $x = -2$ we have $\sum_{n=1}^{\infty} \sqrt[n]{n}$, a divergent series

(a) the radius is $\frac{1}{2}$; the interval of convergence is $-3 < x < -2$

(b) the interval of absolute convergence is $-3 < x < -2$

(c) there are no values for which the series converges conditionally

21. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\left(1+\frac{1}{n+1}\right)^{n+1} x^{n+1}}{\left(1+\frac{1}{n}\right)^n x^n}\right| < 1 \Rightarrow |x| \left(\frac{\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}\right) < 1 \Rightarrow |x|\left(\frac{e}{e}\right) < 1 \Rightarrow |x| < 1$

$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} (-1)^n\left(1+\frac{1}{n}\right)^n$, a divergent series by the nth-Term Test since

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$; when $x = 1$ we have $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$, a divergent series

(a) the radius is 1; the interval of convergence is $-1 < x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) there are no values for which the series converges conditionally

22. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\ln(n+1)x^{n+1}}{x^n \ln n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow |x| < 1$ $\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} (-1)^n \ln n$, a divergent series by the nth-Term Test since $\lim_{n\to\infty} \ln n \neq 0$; when $x = 1$ we have $\sum_{n=1}^{\infty} \ln n$, a divergent series

(a) the radius is 1; the interval of convergence is $-1 < x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) there are no values for which the series converges conditionally

23. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}x^{n+1}}{n^n x^n}\right| < 1 \Rightarrow |x| \left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n\right)\left(\lim_{n\to\infty} (n+1)\right) < 1$ $\Rightarrow e|x| \lim_{n\to\infty} (n+1) < 1 \Rightarrow$ only $x = 0$ satisfies this inequality

(a) the radius is 0; the series converges only for $x = 0$

(b) the series converges absolutely only for $x = 0$

(c) there are no values for which the series converges conditionally

24. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+1)!\,(x-4)^{n+1}}{n!\,(x-4)^n}\right| < 1 \Rightarrow |x-4| \lim_{n\to\infty} (n+1) < 1 \Rightarrow$ only $x = 4$ satisfies this inequality

(a) the radius is 0; the series converges only for $x = 4$

(b) the series converges absolutely only for $x = 4$

(c) there are no values for which the series converges conditionally

25. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x+2)^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n2^n}{(x+2)^n}\right| < 1 \Rightarrow \frac{|x+2|}{2} \lim_{n\to\infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow \frac{|x+2|}{2} < 1 \Rightarrow |x+2| < 2$ $\Rightarrow -2 < x+2 < 2 \Rightarrow -4 < x < 0$; when $x = -4$ we have $\sum_{n=1}^{\infty} \frac{-1}{n}$, a divergent series; when $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, the alternating harmonic series which converges conditionally

(a) the radius is 2; the interval of convergence is $-4 < x \leq 0$

(b) the interval of absolute convergence is $-4 < x < 0$

(c) the series converges conditionally at $x = 0$

26. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{(-2)^{n+1}(n+2)(x-1)^{n+1}}{(-2)^n(n+1)(x-1)^n}\right| < 1 \Rightarrow 2\,|x-1| \lim\limits_{n\to\infty} \left(\frac{n+2}{n+1}\right) < 1 \Rightarrow 2\,|x-1| < 1$

$\Rightarrow |x-1| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x-1 < \frac{1}{2} \Rightarrow \frac{1}{2} < x < \frac{3}{2}$; when $x = \frac{1}{2}$ we have $\sum\limits_{n=1}^{\infty} (n+1)$, a divergent series; when $x = \frac{3}{2}$ we have $\sum\limits_{n=1}^{\infty} (-1)^n(n+1)$, a divergent series

(a) the radius is $\frac{1}{2}$; the interval of convergence is $\frac{1}{2} < x < \frac{3}{2}$

(b) the interval of absolute convergence is $\frac{1}{2} < x < \frac{3}{2}$

(c) there are no values for which the series converges conditionally

27. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{x^n}\right| < 1 \Rightarrow |x| \left(\lim\limits_{n\to\infty} \frac{n}{n+1}\right)\left(\lim\limits_{n\to\infty} \frac{\ln n}{\ln(n+1)}\right)^2 < 1$

$\Rightarrow |x|(1)\left(\lim\limits_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)}\right)^2 < 1 \Rightarrow |x|\left(\lim\limits_{n\to\infty} \frac{n+1}{n}\right)^2 < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ which converges absolutely; when $x = 1$ we have $\sum\limits_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$ which converges

(a) the radius is 1; the interval of convergence is $-1 \le 1 \le 1$

(b) the interval of absolute convergence is $-1 \le x \le 1$

(c) there are no values for which the series converges conditionally

28. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n\ln(n)}{x^n}\right| < 1 \Rightarrow |x| \left(\lim\limits_{n\to\infty} \frac{n}{n+1}\right)\left(\lim\limits_{n\to\infty} \frac{\ln(n)}{\ln(n+1)}\right) < 1$

$\Rightarrow |x|(1)(1) < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$, a convergent alternating series; when $x = 1$ we have $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ which diverges by Exercise 75, Section 8.4

(a) the radius is 1; the interval of convergence is $-1 \le x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) the series converges conditionally at $x = -1$

29. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{(4x-5)^{2n+3}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{(4x-5)^{2n+1}}\right| < 1 \Rightarrow (4x-5)^2\left(\lim\limits_{n\to\infty} \frac{n}{n+1}\right)^{3/2} < 1 \Rightarrow (4x-5)^2 < 1$

$\Rightarrow |4x-5| < 1 \Rightarrow -1 < 4x-5 < 1 \Rightarrow 1 < x < \frac{3}{2}$; when $x = 1$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n^{3/2}} = \sum\limits_{n=1}^{\infty} \frac{-1}{n^{3/2}}$ which is absolutely convergent; when $x = \frac{3}{2}$ we have $\sum\limits_{n=1}^{\infty} \frac{(1)^{2n+1}}{n^{3/2}}$, a convergent p-series

(a) the radius is $\frac{1}{4}$; the interval of convergence is $1 \le x \le \frac{3}{2}$

(b) the interval of absolute convergence is $1 \le x \le \frac{3}{2}$

(c) there are no values for which the series converges conditionally

30. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(3x+1)^{n+2}}{2n+4} \cdot \frac{2n+2}{(3x+1)^{n+1}}\right| < 1 \Rightarrow |3x+1| \lim_{n\to\infty} \left(\frac{2n+2}{2n+4}\right) < 1 \Rightarrow |3x+1| < 1$

$\Rightarrow -1 < 3x+1 < 1 \Rightarrow -\frac{2}{3} < x < 0$; when $x = -\frac{2}{3}$ we have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$, a conditionally convergent series;

when $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{1}{2n+1}$, a divergent series

(a) the radius is $\frac{1}{3}$; the interval of convergence is $-\frac{2}{3} \le x < 0$

(b) the interval of absolute convergence is $-\frac{2}{3} < x < 0$

(c) the series converges conditionally at $x = -\frac{2}{3}$

31. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x+\pi)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x+\pi)^n}\right| < 1 \Rightarrow |x+\pi| \lim_{n\to\infty} \left|\sqrt{\frac{n}{n+1}}\right| < 1$

$\Rightarrow |x+\pi| \sqrt{\lim_{n\to\infty} \left(\frac{n}{n+1}\right)} < 1 \Rightarrow |x+\pi| < 1 \Rightarrow -1 < x+\pi < 1 \Rightarrow -1-\pi < x < 1-\pi$;

when $x = -1-\pi$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$, a conditionally convergent series; when $x = 1-\pi$ we have

$\sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, a divergent p-series

(a) the radius is 1; the interval of convergence is $(-1-\pi) \le x < (1-\pi)$

(b) the interval of absolute convergence is $-1-\pi < x < 1-\pi$

(c) the series converges conditionally at $x = -1-\pi$

32. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\left(x-\sqrt{2}\right)^{2n+3}}{2^{n+1}} \cdot \frac{2^n}{\left(x-\sqrt{2}\right)^{2n+1}}\right| < 1 \Rightarrow \frac{\left(x-\sqrt{2}\right)^2}{2} \lim_{n\to\infty} |1| < 1$

$\Rightarrow \frac{\left(x-\sqrt{2}\right)^2}{2} < 1 \Rightarrow \left(x-\sqrt{2}\right)^2 < 2 \Rightarrow \left|x-\sqrt{2}\right| < \sqrt{2} \Rightarrow -\sqrt{2} < x-\sqrt{2} < \sqrt{2} \Rightarrow 0 < x < 2\sqrt{2}$; when $x = 0$

we have $\sum_{n=1}^{\infty} \frac{\left(-\sqrt{2}\right)^{2n+1}}{2^n} = -\sum_{n=1}^{\infty} \frac{2^{n+1/2}}{2^n} = -\sum_{n=1}^{\infty} \sqrt{2}$ which diverges since $\lim_{n\to\infty} a_n \ne 0$; when $x = 2\sqrt{2}$ we

have $\sum_{n=1}^{\infty} \frac{\left(\sqrt{2}\right)^{2n+1}}{2^n} = \sum_{n=1}^{\infty} \frac{2^{n+1/2}}{2^n} = \sum_{n=1}^{\infty} \sqrt{2}$, a divergent series

(a) the radius is $\sqrt{2}$; the interval of convergence is $0 < x < 2\sqrt{2}$

(b) the interval of absolute convergence is $0 < x < 2\sqrt{2}$

(c) there are no values for which the series converges conditionally

33. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-1)^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{(x-1)^{2n}}\right| < 1 \Rightarrow \frac{(x-1)^2}{4} \lim_{n\to\infty} |1| < 1 \Rightarrow (x-1)^2 < 4 \Rightarrow |x-1| < 2$

$\Rightarrow -2 < x-1 < 2 \Rightarrow -1 < x < 3$; at $x = -1$ we have $\sum_{n=0}^{\infty} \frac{(-2)^{2n}}{4^n} = \sum_{n=0}^{\infty} \frac{4^n}{4^n} = \sum_{n=0}^{\infty} 1$, which diverges; at $x = 3$

we have $\sum_{n=0}^{\infty} \frac{2^{2n}}{4^n} = \sum_{n=0}^{\infty} \frac{4^n}{4^n} = \sum_{n=0}^{\infty} 1$, a divergent series; the interval of convergence is $-1 < x < 3$; the series $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n} = \sum_{n=0}^{\infty} \left(\left(\frac{x-1}{2}\right)^2\right)^n$ is a convergent geometric series when $-1 < x < 3$ and the sum is $\frac{1}{1-\left(\frac{x-1}{2}\right)^2} = \frac{1}{\left[\frac{4-(x-1)^2}{4}\right]} = \frac{4}{4-x^2+2x-1} = \frac{4}{3+2x-x^2}$

34. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x+1)^{2n+2}}{9^{n+1}} \cdot \frac{9^n}{(x+1)^{2n}}\right| < 1 \Rightarrow \frac{(x+1)^2}{9} \lim_{n\to\infty} |1| < 1 \Rightarrow (x+1)^2 < 9 \Rightarrow |x+1| < 3$ $\Rightarrow -3 < x+1 < 3 \Rightarrow -4 < x < 2$; when $x = -4$ we have $\sum_{n=0}^{\infty} \frac{(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty} 1$ which diverges; at $x = 2$ we have $\sum_{n=0}^{\infty} \frac{3^{2n}}{9^n} = \sum_{n=0}^{\infty} 1$ which also diverges; the interval of convergence is $-4 < x < 2$; the series $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty} \left(\left(\frac{x+1}{3}\right)^2\right)^n$ is a convergent geometric series when $-4 < x < 2$ and the sum is $\frac{1}{1-\left(\frac{x+1}{3}\right)^2} = \frac{1}{\left[\frac{9-(x+1)^2}{9}\right]} = \frac{9}{9-x^2-2x-1} = \frac{9}{8-2x-x^2}$

35. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\left(\sqrt{x}-2\right)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{\left(\sqrt{x}-2\right)^n}\right| < 1 \Rightarrow \left|\sqrt{x}-2\right| < 2 \Rightarrow -2 < \sqrt{x}-2 < 2 \Rightarrow 0 < \sqrt{x} < 4$ $\Rightarrow 0 < x < 16$; when $x = 0$ we have $\sum_{n=0}^{\infty} (-1)^n$, a divergent series; when $x = 16$ we have $\sum_{n=0}^{\infty} (1)^n$, a divergent series; the interval of convergence is $0 < x < 16$; the series $\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}-2}{2}\right)^n$ is a convergent geometric series when $0 < x < 16$ and its sum is $\frac{1}{1-\left(\frac{\sqrt{x}-2}{2}\right)} = \frac{1}{\left(\frac{2-\sqrt{x}+2}{2}\right)} = \frac{2}{4-\sqrt{x}}$

36. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(\ln x)^{n+1}}{(\ln x)^n}\right| < 1 \Rightarrow |\ln x| < 1 \Rightarrow -1 < \ln x < 1 \Rightarrow e^{-1} < x < e$; when $x = e^{-1}$ or e we obtain the series $\sum_{n=0}^{\infty} 1^n$ and $\sum_{n=0}^{\infty} (-1)^n$ which both diverge; the interval of convergence is $e^{-1} < x < e$; $\sum_{n=0}^{\infty} (\ln x)^n = \frac{1}{1-\ln x}$ when $e^{-1} < x < e$

37. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\left(\frac{x^2+1}{3}\right)^{n+1} \cdot \left(\frac{3}{x^2+1}\right)^n\right| < 1 \Rightarrow \frac{(x^2+1)}{3} \lim_{n\to\infty} |1| < 1 \Rightarrow \frac{x^2+1}{3} < 1 \Rightarrow x^2 < 2$ $\Rightarrow |x| < \sqrt{2} \Rightarrow -\sqrt{2} < x < \sqrt{2}$; at $x = \pm\sqrt{2}$ we have $\sum_{n=0}^{\infty} (1)^n$ which diverges; the interval of convergence is

$-\sqrt{2} < x < \sqrt{2}$; the series $\sum\limits_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ is a convergent geometric series when $-\sqrt{2} < x < \sqrt{2}$ and its sum is

$$\frac{1}{1-\left(\frac{x^2+1}{3}\right)} = \frac{1}{\left(\frac{3-x^2-1}{3}\right)} = \frac{3}{2-x^2}$$

38. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{\left(x^2-1\right)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{\left(x^2+1\right)^n}\right| < 1 \Rightarrow |x^2-1| < 2 \Rightarrow -\sqrt{3} < x < \sqrt{3}$; when $x = \pm\sqrt{3}$ we have $\sum\limits_{n=0}^{\infty} 1^n$, a divergent series; the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$; the series $\sum\limits_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$ is a convergent geometric series when $-\sqrt{3} < x < \sqrt{3}$ and its sum is $\frac{1}{1-\left(\frac{x^2-1}{2}\right)} = \frac{1}{\left(\frac{2-\left(x^2-1\right)}{2}\right)} = \frac{2}{3-x^2}$

39. $\lim\limits_{n\to\infty} \left|\frac{(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x-3)^n}\right| < 1 \Rightarrow |x-3| < 2 \Rightarrow 1 < x < 5$; when $x = 1$ we have $\sum\limits_{n=1}^{\infty} (1)^n$ which diverges; when $x = 5$ we have $\sum\limits_{n=1}^{\infty} (-1)^n$ which also diverges; the interval of convergence is $1 < x < 5$; the sum of this convergent geometric series is $\frac{1}{1+\left(\frac{x-3}{2}\right)} = \frac{2}{x-1}$. If $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \ldots + \left(-\frac{1}{2}\right)^n (x-3)^n + \ldots$ $= \frac{2}{x-1}$ then $f'(x) = -\frac{1}{2} + \frac{1}{2}(x-3) + \ldots + \left(-\frac{1}{2}\right)^n n(x-3)^{n-1} + \ldots$ is convergent when $1 < x < 5$, and diverges when $x = 1$ or 5. The sum for $f'(x)$ is $\frac{-2}{(x-1)^2}$, the derivative of $\frac{2}{x-1}$.

40. If $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \ldots + \left(-\frac{1}{2}\right)^n (x-3)^n + \ldots = \frac{2}{x-1}$ then $\int f(x)\,dx$ $= x - \frac{(x-3)^2}{4} + \frac{(x-3)^3}{12} + \ldots + \left(-\frac{1}{2}\right)^n \frac{(x-3)^{n+1}}{n+1} + \ldots$. At $x = 1$ the series $\sum\limits_{n=1}^{\infty} \frac{-2}{n+1}$ diverges; at $x = 5$ the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n 2}{n+1}$ converges. Therefore the interval of convergence is $1 < x \le 5$ and the sum is $2\ln|x-1| + (3 - \ln 4)$, since $\int \frac{2}{x-1}\,dx = 2\ln|x-1| + C$, where $C = 3 - \ln 4$ when $x = 3$.

41. (a) Differentiate the series for sin x to get $\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!} - \frac{11x^{10}}{11!} + \ldots$ $= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$. The series converges for all values of x since

$$\lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = |x| \lim_{n\to\infty} \left(\frac{1}{n+1}\right) = 0 < 1 \text{ for all } x$$

(b) $\sin 2x = 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \ldots = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \frac{2048x^{11}}{11!} + \ldots$

(c) $2\sin x\cos x = 2\Big[(0\cdot 1) + (0\cdot 0 + 1\cdot 1)x + \left(0\cdot\frac{-1}{2} + 1\cdot 0 + 0\cdot 1\right)x^2 + \left(0\cdot 0 - 1\cdot\frac{1}{2} + 0\cdot 0 - 1\cdot\frac{1}{3!}\right)x^3$

$+\left(0\cdot\frac{1}{4!} + 1\cdot 0 - 0\cdot\frac{1}{2} - 0\cdot\frac{1}{3!} + 0\cdot 1\right)x^4 + \left(0\cdot 0 + 1\cdot\frac{1}{4!} + 0\cdot 0 + \frac{1}{2}\cdot\frac{1}{3!} + 0\cdot 0 + 1\cdot\frac{1}{5!}\right)x^5$

$+\left(0\cdot\frac{1}{6!} + 1\cdot 0 + 0\cdot\frac{1}{4!} + 0\cdot\frac{1}{3!} + 0\cdot\frac{1}{2} + 0\cdot\frac{1}{5!} + 0\cdot 1\right)x^6 + \ldots\Big] = 2\left[x - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \ldots\right]$

$= 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \ldots$

42. (a) $\frac{d}{x}(e^x) = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \ldots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = e^x$; thus the derivative of e^x is e^x itself

(b) $\int e^x\,dx = e^x + C = x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + C$, which is the general antiderivative of e^x

(c) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots$; $e^{-x}\cdot e^x = 1\cdot 1 + (1\cdot 1 - 1\cdot 1)x + \left(1\cdot\frac{1}{2!} - 1\cdot 1 + \frac{1}{2!}\cdot 1\right)x^2$

$+\left(1\cdot\frac{1}{3!} - 1\cdot\frac{1}{2!} + \frac{1}{2!}\cdot 1 - \frac{1}{3!}\cdot 1\right)x^3 + \left(1\cdot\frac{1}{4!} - 1\cdot\frac{1}{3!} + \frac{1}{2!}\cdot\frac{1}{2!} - \frac{1}{3!}\cdot 1 + \frac{1}{4!}\cdot 1\right)x^4$

$+\left(1\cdot\frac{1}{5!} - 1\cdot\frac{1}{4!} + \frac{1}{2!}\cdot\frac{1}{3!} - \frac{1}{3!}\cdot\frac{1}{2!} + \frac{1}{4!}\cdot 1 - \frac{1}{5!}\cdot 1\right)x^5 + \ldots = 1 + 0 + 0 + 0 + 0 + 0 + \ldots$

43. (a) $\ln|\sec x| + C = \int \tan x\,dx = \int\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots\right)dx$

$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14{,}175} + \ldots + C$; $x = 0 \Rightarrow C = 0 \Rightarrow \ln|\sec x| = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14{,}175} + \ldots$,

converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(b) $\sec^2 x = \frac{d(\tan x)}{dx} = \frac{d}{dx}\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots\right) = 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \ldots$, converges

when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) $\sec^2 x = (\sec x)(\sec x) = \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right)$

$= 1 + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{5}{24} + \frac{1}{4} + \frac{5}{24}\right)x^4 + \left(\frac{61}{720} + \frac{5}{48} + \frac{5}{48} + \frac{61}{720}\right)x^6 + \ldots$

$= 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \ldots,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

44. (a) $\ln|\sec x + \tan x| + C = \int \sec x\,dx = \int\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right)dx$

$= x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{72{,}576} + \ldots + C$; $x = 0 \Rightarrow C = 0 \Rightarrow \ln|\sec x + \tan x|$

$= x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{72{,}576} + \ldots$, converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(b) $\sec x\tan x = \frac{d(\sec x)}{dx} = \frac{d}{dx}\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right) = x + \frac{5x^3}{6} + \frac{61x^5}{120} + \frac{277x^7}{1008} + \ldots$, converges

when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) $(\sec x)(\tan x) = \left(1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+\ldots\right)\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\ldots\right)$

$= x+\left(\frac{1}{3}+\frac{1}{2}\right)x^3+\left(\frac{2}{15}+\frac{1}{6}+\frac{5}{24}\right)x^5+\left(\frac{17}{315}+\frac{1}{15}+\frac{5}{72}+\frac{61}{720}\right)x^7+\ldots = x+\frac{5x^3}{6}+\frac{61x^5}{120}+\frac{277x^7}{1008}+\ldots,$

$-\frac{\pi}{2} < x < \frac{\pi}{2}$

45. (a) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-(k-1))\,a_n x^{n-k}$ and $f^{(k)}(0) = k!a_k$

$\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$; likewise if $f(x) = \sum_{n=0}^{\infty} b_n x^n$, then $b_k = \frac{f^{(k)}(0)}{k!} \Rightarrow a_k = b_k$ for every nonnegative integer k

(b) If $f(x) = \sum_{n=0}^{\infty} a_n x^n = 0$ for all x, then $f^{(k)}(x) = 0$ for all x $\Rightarrow$ from part (a) that $a_k = 0$ for every nonnegative integer k

46. $\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\ldots \Rightarrow x\left[\frac{1}{(1-x)^2}\right] = x\left(1+2x+3x^2+4x^3+\ldots\right) \Rightarrow \frac{x}{(1-x)^2}$

$= x+2x^2+3x^3+4x^4+\ldots \Rightarrow x\left[\frac{1+x}{(1-x)^3}\right] = x\left(1+4x+9x^2+16x^3+\ldots\right) \Rightarrow \frac{x+x^2}{(1-x)^3}$

$= x+4x^2+9x^3+16x^4+\ldots \Rightarrow \frac{\left(\frac{1}{2}+\frac{1}{4}\right)}{\left(\frac{1}{8}\right)} = \frac{1}{2}+\frac{4}{4}+\frac{9}{8}+\frac{16}{16}+\ldots \Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$

47. The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges conditionally at the left-hand endpoint of its interval of convergence $[-1,1]$; the series $\sum_{n=1}^{\infty} \frac{x^n}{(n^2)}$ converges absolutely at the left-hand endpoint of its interval of convergence $[-1,1]$

48. Answers will vary. For instance:

(a) $\sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$ (b) $\sum_{n=1}^{\infty} (x+1)^n$ (c) $\sum_{n=1}^{\infty} \left(\frac{x-3}{2}\right)^n$

8.7 TAYLOR AND MACLAURIN SERIES

1. $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$; $f(1) = \ln 1 = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2 \Rightarrow P_0(x) = 0$, $P_1(x) = (x-1)$, $P_2(x) = (x-1)-\frac{1}{2}(x-1)^2$, $P_3(x) = (x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3$

2. $f(x) = \ln(1+x)$, $f'(x) = \frac{1}{1+x} = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$, $f'''(x) = 2(1+x)^{-3}$; $f(0) = \ln 1 = 0$, $f'(0) = \frac{1}{1} = 1$, $f''(0) = -(1)^{-2} = -1$, $f'''(0) = 2(1)^{-3} = 2 \Rightarrow P_0(x) = 0$, $P_1(x) = x$, $P_2(x) = x-\frac{x^2}{2}$, $P_3(x)$ $= x-\frac{x^2}{2}+\frac{x^3}{3}$

3. $f(x) = (x+2)^{-1}$, $f'(x) = -(x+2)^{-2}$, $f''(x) = 2(x+2)^{-3}$, $f'''(x) = -6(x+2)^{-4}$; $f(0) = (2)^{-1} = \frac{1}{2}$, $f'(0) = -(2)^{-2}$ $= -\frac{1}{4}$, $f''(0) = 2(2)^{-3} = \frac{1}{4}$, $f'''(0) = -6(2)^{-4} = -\frac{3}{8} \Rightarrow P_0(x) = \frac{1}{2}$, $P_1(x) = \frac{1}{2} - \frac{x}{4}$, $P_2(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}$, $P_3(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$

4. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$; $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$, $f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \Rightarrow P_0 = \frac{\sqrt{2}}{2}$, $P_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$, $P_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$, $P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$

5. $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$; $f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$, $f''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$, $f'''\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow P_0(x) = \frac{1}{\sqrt{2}}$, $P_1(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$, $P_2(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2$, $P_3(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3$

6. $f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = \left(\frac{1}{2}\right)x^{-1/2}$, $f''(x) = \left(-\frac{1}{4}\right)x^{-3/2}$, $f'''(x) = \left(\frac{3}{8}\right)x^{-5/2}$; $f(4) = \sqrt{4} = 2$, $f'(4) = \left(\frac{1}{2}\right)4^{-1/2} = \frac{1}{4}$, $f''(4) = \left(-\frac{1}{4}\right)4^{-3/2} = -\frac{1}{32}$, $f'''(4) = \left(\frac{3}{8}\right)4^{-5/2} = \frac{3}{256} \Rightarrow P_0(x) = 2$, $P_1(x) = 2 + \frac{1}{4}(x-4)$, $P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$, $P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$

7. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$

8. $f(x) = (1+x)^{-1} \Rightarrow f'(x) = -(1+x)^{-2}$, $f''(x) = 2(1+x)^{-3}$, $f'''(x) = -3!(1+x)^{-4} \Rightarrow \ldots f^{(k)}(x)$ $= (-1)^k k!(1+x)^{-k-1}$; $f(0) = 1$, $f'(0) = -1$, $f''(0) = 2$, $f'''(0) = -3!, \ldots, f^{(k)}(0) = (-1)^k k!$ $\Rightarrow \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

9. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!} = 3x - \frac{3^3x^3}{3!} + \frac{3^5x^5}{5!} - \ldots$

10. $7\cos(-x) = 7\cos x = 7\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 7 - \frac{7x^2}{2!} + \frac{7x^4}{4!} - \frac{7x^6}{6!} + \ldots$, since the cosine is an even function

11. $\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x^2 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$ $= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

12. $\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right] = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots$
$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

13. $f(x) = x^4 - 2x^3 - 5x + 4 \Rightarrow f'(x) = 4x^3 - 6x^2 - 5,\ f''(x) = 12x^2 - 12x,\ f'''(x) = 24x - 12,\ f^{(4)}(x) = 24$
$\Rightarrow f^{(n)}(x) = 0$ if $n \ge 5$; $f(0) = 4,\ f'(0) = -5,\ f''(0) = 0,\ f'''(0) = -12,\ f^{(4)}(0) = 24,\ f^{(n)}(0) = 0$ if $n \ge 5$
$\Rightarrow x^4 - 2x^3 - 5x + 4 = 4 - 5x - \frac{12}{3!}x^3 + \frac{24}{4!}x^4 = x^4 - 2x^3 - 5x + 4$ itself

14. $f(x) = (x+1)^2 \Rightarrow f'(x) = 2(x+1);\ f''(x) = 2 \Rightarrow f^{(n)}(x) = 0$ if $n \ge 3$; $f(0) = 1,\ f'(0) = 2,\ f''(0) = 2,\ f^{(n)}(0) = 0$ if $n \ge 3 \Rightarrow (x+1)^2 = 1 + 2x + \frac{2}{2!}x^2 = 1 + 2x + x^2$

15. $f(x) = x^3 - 2x + 4 \Rightarrow f'(x) = 3x^2 - 2,\ f''(x) = 6x,\ f'''(x) = 6 \Rightarrow f^{(n)}(x) = 0$ if $n \ge 4$; $f(2) = 8,\ f'(2) = 10,$
$f''(2) = 12,\ f'''(2) = 6,\ f^{(n)}(2) = 0$ if $n \ge 4 \Rightarrow x^3 - 2x + 4 = 8 + 10(x-2) + \frac{12}{2!}(x-2)^2 + \frac{6}{3!}(x-2)^3$
$= 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$

16. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \Rightarrow f'(x) = 15x^4 - 4x^3 + 6x^2 + 2x,\ f''(x) = 60x^3 - 12x^2 + 12x + 2,$
$f'''(x) = 180x^2 - 24x + 12,\ f^{(4)}(x) = 360x - 24,\ f^{(5)}(x) = 360,\ f^{(n)}(x) = 0$ if $n \ge 6$; $f(-1) = -7,$
$f'(-1) = 23,\ f''(-1) = -82,\ f'''(-1) = 216,\ f^{(4)}(-1) = -384,\ f^{(5)}(-1) = 360,\ f^{(n)}(-1) = 0$ if $n \ge 6$
$\Rightarrow 3x^5 - x^4 + 2x^3 + x^2 - 2 = -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^5$
$= -7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^5$

17. $f(x) = x^{-2} \Rightarrow f'(x) = -2x^{-3},\ f''(x) = 3!\,x^{-4},\ f'''(x) = -4!\,x^{-5} \Rightarrow f^{(n)}(x) = (-1)^n(n+1)!\,x^{-n-2};$
$f(1) = 1,\ f'(1) = -2,\ f''(1) = 3!,\ f'''(1) = -4!,\ f^{(n)}(1) = (-1)^n(n+1)! \Rightarrow \frac{1}{x^2}$
$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n(n+1)(x-1)^n$

18. $f(x) = \frac{x}{1-x} \Rightarrow f'(x) = (1-x)^{-2},\ f''(x) = 2(1-x)^{-3},\ f'''(x) = 3!\,(1-x)^{-4} \Rightarrow f^{(n)}(x) = n!\,(1-x)^{-n-1};$
$f(0) = 0,\ f'(0) = 1,\ f''(0) = 2,\ f'''(0) = 3! \Rightarrow \frac{x}{1-x} = x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^{n+1}$

19. $f(x) = e^x \Rightarrow f'(x) = e^x,\ f''(x) = e^x \Rightarrow f^{(n)}(x) = e^x;\ f(2) = e^2,\ f'(2) = e^2, \ldots f^{(n)}(2) = e^2$
$\Rightarrow e^x = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^3}{3!}(x-2)^3 + \ldots = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$

20. $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2,\ f''(x) = 2^x(\ln 2)^2,\ f'''(x) = 2^x(\ln 2)^3 \Rightarrow f^{(n)}(x) = 2^x(\ln 2)^n;\ f(1) = 2,\ f'(1) = 2\ln 2,$
$f''(1) = 2(\ln 2)^2,\ f'''(1) = 2(\ln 2)^3, \ldots, f^{(n)}(1) = 2(\ln 2)^n$
$\Rightarrow 2^x = 2 + (2\ln 2)(x-1) + \frac{2(\ln 2)^2}{2}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 + \ldots = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n(x-1)^n}{n!}$

21. $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-5x} = 1 + (-5x) + \frac{(-5x)^2}{2!} + \ldots = 1 - 5x + \frac{5^2x^2}{2!} - \frac{5^3x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^n}{n!}$

22. $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x/2} = 1 + \left(\frac{-x}{2}\right) + \frac{\left(-\frac{x}{2}\right)^2}{2!} + \ldots = 1 - \frac{x}{2} + \frac{x^2}{2^2 2!} - \frac{x^3}{2^3 3!} + \ldots$
$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!}$

23. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \frac{\pi x}{2} = \frac{\pi x}{2} - \frac{\left(\frac{\pi x}{2}\right)^3}{3!} + \frac{\left(\frac{\pi x}{2}\right)^5}{5!} - \frac{\left(\frac{\pi x}{2}\right)^7}{7!} + \ldots$
$= \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{2^{2n+1}(2n+1)!}$

24. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos \sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{1/2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots$

25. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow xe^x = x\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \ldots$

26. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow x^2 \sin x = x^2\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!} = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \ldots$

27. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \frac{x^2}{2} - 1 + \cos x = \frac{x^2}{2} - 1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{x^2}{2} - 1 + 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$
$= \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

28. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin x - x + \frac{x^3}{3!} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) - x + \frac{x^3}{3!}$
$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots\right) - x + \frac{x^3}{3!} = \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

29. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow x \cos \pi x = x \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n+1}}{(2n)!} = x - \frac{\pi^2 x^3}{2!} + \frac{\pi^4 x^5}{4!} - \frac{\pi^6 x^7}{6!} + \ldots$

30. $\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2} = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \frac{1}{2} + \frac{1}{2}\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \ldots\right]$
$= 1 - \frac{(2x)^2}{2 \cdot 2!} + \frac{(2x)^4}{2 \cdot 4!} - \frac{(2x)^6}{2 \cdot 6!} + \frac{(2x)^8}{2 \cdot 8!} - \ldots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{2 \cdot (2n)!}$

31. $\sin^2 x = \left(\frac{1-\cos 2x}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots\right) = \frac{(2x)^2}{2\cdot 2!} - \frac{(2x)^4}{2\cdot 4!} + \frac{(2x)^6}{2\cdot 6!} - \ldots$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2x)^{2n}}{2\cdot(2n)!}$

32. $\frac{x^2}{1-2x} = x^2\left(\frac{1}{1-2x}\right) = x^2 \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^{n+2} = x^2 + 2x^3 + 2^2x^4 + 2^3x^5 + \ldots$

33. $x \ln(1+2x) = x \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n x^{n+1}}{n} = 2x^2 - \frac{2^2x^3}{2} + \frac{2^3x^4}{3} - \frac{2^4x^5}{4} + \ldots$

34. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots = \sum_{n=1}^{\infty} nx^{n-1}$

$= \sum_{n=0}^{\infty} (n+1)x^n$

35. By the Alternating Series Estimation Theorem, the error is less than $\frac{|x|^5}{5!} \Rightarrow |x|^5 < (5!)(5\times 10^{-4})$

$\Rightarrow |x|^5 < 600\times 10^{-4} \Rightarrow |x| < \sqrt[5]{6\times 10^{-2}} \approx 0.56968$

36. If $\cos x = 1 - \frac{x^2}{2}$ and $|x| < 0.5$, then the $|\text{error}| = |R_3(x)| = \left|\frac{\cos c}{4!}x^4\right| < \left|\frac{(.5)^4}{24}\right| = 0.0026$, where c is between 0 and x; since the next term in the series is positive, the approximation $1 - \frac{x^2}{2}$ is too small, by the Alternating Series Estimation Theorem

37. If $\sin x = x$ and $|x| < 10^{-3}$, then the $|\text{error}| = |R_2(x)| = \left|\frac{-\cos c}{3!}x^3\right| < \frac{(10^{-3})^3}{3!} \approx 1.67\times 10^{-10}$, where c is between 0 and x. The Alternating Series Estimation Theorem says $R_2(x)$ has the same sign as $-\frac{x^3}{3!}$. Moreover, $x < \sin x \Rightarrow 0 < \sin x - x = R_2(x) \Rightarrow x < 0 \Rightarrow -10^{-3} < x < 0$.

38. $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$. By the Alternating Series Estimation Theorem the $|\text{error}| < \left|\frac{-x^2}{8}\right| < \frac{(0.01)^2}{8}$

$= 1.25\times 10^{-5}$

39. (a) $|R_2(x)| = \left|\frac{e^c x^3}{3!}\right| < \frac{3^{(0.1)}(0.1)^3}{3!} < 1.87\times 10^{-4}$, where c is between 0 and x

(b) $|R_2(x)| = \left|\frac{e^c x^3}{3!}\right| < \frac{(0.1)^3}{3!} = 1.67\times 10^{-4}$, where c is between 0 and x

40. $|R_4(x)| < \left|\frac{\cosh c}{5!}x^5\right| = \left|\frac{e^c + e^{-c}}{2}\frac{x^5}{5!}\right| < \frac{1.65 + \frac{1}{1.65}}{2}\cdot\frac{(0.5)^5}{5!} = (1.3)\frac{(0.5)^5}{5!} \approx 0.000293653$

41. If we approximate e^h with $1 + h$ and $0 \le h \le 0.01$, then $|\text{error}| < \left|\frac{e^c h^2}{2}\right| \le \frac{e^{0.01}h\cdot h}{2} = \left(\frac{e^{0.01}(0.01)}{2}\right)h$

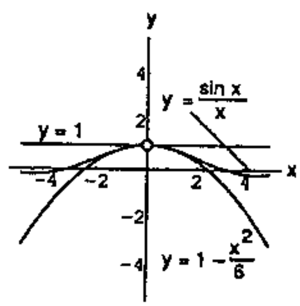
$= 0.005005h < 0.006h = (0.6\%)h$, where c is between 0 and h.

42. $|R_1| = \left|\frac{1}{(1+c)^2}\frac{x^2}{2!}\right| < \frac{x^2}{2} = \left|\frac{x}{2}\right| |x| < .01\,|x| = (1\%)\,|x| \Rightarrow \left|\frac{x}{2}\right| < .01 \Rightarrow 0 < |x| < .02$

43. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \Rightarrow \frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots;\ |\text{error}| < \frac{1}{2n+1} < .01$
$\Rightarrow 2n + 1 > 100 \Rightarrow n > 49$

44. (a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$, $s_1 = 1$ and $s_2 = 1 - \frac{x^2}{6}$; if L is the sum of the series representing $\frac{\sin x}{x}$, then by the Alternating Series Estimation Theorem, $L - s_1 = \frac{\sin x}{x} - 1 < 0$ and $L - s_2 = \frac{\sin x}{x} - \left(1 - \frac{x^2}{6}\right) > 0$. Therefore $1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1$

(b) The graph of $y = \frac{\sin x}{x}$, $x \neq 0$, is bounded below by the graph of $y = 1 - \frac{x^2}{6}$ and above by the graph of $y = 1$ as derived in part (a).



45. $f(x) = \ln(\cos x) \Rightarrow f'(x) = -\tan x$ and $f''(x) = -\sec^2 x$; $f(0) = 0$, $f'(0) = 0$, $f''(0) = -1$
$\Rightarrow L(x) = 0$ and $Q(x) = -\frac{x^2}{2}$

46. $f(x) = e^{\sin x} \Rightarrow f'(x) = (\cos x)e^{\sin x}$ and $f''(x) = (-\sin x)e^{\sin x} + (\cos x)^2 e^{\sin x}$; $f(0) = 1$, $f'(0) = 1$,
$f''(0) = 1 \Rightarrow L(x) = 1 + x$ and $Q(x) = 1 + x + \frac{x^2}{2}$

47. $f(x) = (1 - x^2)^{-1/2} \Rightarrow f'(x) = x(1 - x^2)^{-3/2}$ and $f''(x) = (1 - x^2)^{-3/2} + 3x^2(1 - x^2)^{-5/2}$; $f(0) = 1$,
$f'(0) = 0$, $f''(0) = 1 \Rightarrow L(x) = 1$ and $Q(x) = 1 = \frac{x^2}{2}$

48. $f(x) = \cosh x \Rightarrow f'(x) = \sinh x$ and $f''(x) = \cosh x$; $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1 \Rightarrow L(x) = 1$ and $Q(x) = 1 + \frac{x^2}{2}$

49. A special case of Taylor's Formula is $f(x) = f(a) + f'(c)(x - a)$. Let $x = b$ and this becomes
$f(b) - f(a) = f'(c)(b - a)$, the Mean Value Theorem

50. If f(x) is twice differentiable and at $x = a$ there is a point of inflection, then $f''(a) = 0$. Therefore,

$L(x) = Q(x) = f(a) + f'(a)(x-a)$.

51. (a) $f'' \le 0$, $f'(a) = 0$ and $x = a$ interior to the interval $I \Rightarrow f(x) - f(a) = \frac{f''(c_2)}{2}(x-a)^2 \le 0$ throughout I $\Rightarrow f(x) \le f(a)$ throughout I $\Rightarrow$ f has a local maximum at $x = a$

(b) similar reasoning gives $f(x) - f(a) = \frac{f''(c_2)}{2}(x-a)^2 \ge 0$ throughout I $\Rightarrow f(x) \ge f(a)$ throughout I $\Rightarrow$ f has a local minimum at $x = a$

52. (a) $f(x) = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f^{(3)}(x) = 6(1-x)^{-4}$ $\Rightarrow f^{(4)}(x) = 24(1-x)^{-5}$; therefore $\frac{1}{1-x} \approx 1 + x + x^2 + x^3$

(b) $|x| < 0.1 \Rightarrow \frac{10}{11} < \frac{1}{1-x} < \frac{10}{9} \Rightarrow \left|\frac{1}{(1-x)^5}\right| < \left(\frac{10}{9}\right)^5 \Rightarrow \left|\frac{x^4}{(1-x)^5}\right| < x^4\left(\frac{10}{9}\right)^5 \Rightarrow$ the error

$e_3 \le \left|\frac{\max f^{(4)}(x)\, x^4}{4!}\right| < (0.1)^4\left(\frac{10}{9}\right)^5 = 0.00016935 < 0.00017$, since $\left|\frac{f^{(4)}(x)}{4!}\right| = \left|\frac{1}{(1-x)^5}\right|$.

53. Let $P = x + \pi \Rightarrow |x| = |P - \pi| < .5 \times 10^{-n}$ since P approximates π accurate to n decimals. Then, $P + \sin P = (\pi + x) + \sin(\pi + x) = (\pi + x) - \sin x = \pi + (x - \sin x) \Rightarrow |(P + \sin P) - \pi|$ $= |\sin x - x| \le \frac{|x|^3}{3!} < \frac{0.125}{3!} \times 10^{-3n} < .5 \times 10^{-3n} \Rightarrow P + \sin P$ gives an approximation to π correct to 3n decimals.

54. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)a_n x^{n-k}$ and $f^{(k)}(0) = k!\,a_k$ $\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$ for k a nonnegative integer. Therefore, the coefficients of f(x) are identical with the corresponding coefficients in the Maclaurin series of f(x) and the statement follows.

55. <u>Note</u>: f even $\Rightarrow f(-x) = f(x) \Rightarrow -f'(-x) = f'(x) \Rightarrow f'(-x) = -f'(x) \Rightarrow f'$ odd;
f odd $\Rightarrow f(-x) = -f(x) \Rightarrow -f'(-x) = -f'(x) \Rightarrow f'(-x) = f'(x) \Rightarrow f'$ even;
also, f odd $\Rightarrow f(-0) = f(0) \Rightarrow 2f(0) = 0 \Rightarrow f(0) = 0$

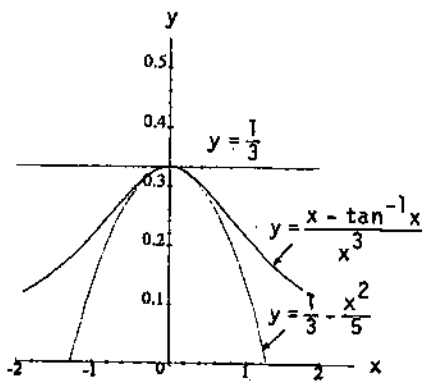
(a) If f(x) is even, then any odd-order derivative is odd and equal to 0 at $x = 0$. Therefore, $a_1 = a_3 = a_5 = \ldots = 0$; that is, the Maclaurin series for f contains only even powers.

(b) If f(x) is odd, then any even-order derivative is odd and equal to 0 at $x = 0$. Therefore, $a_0 = a_2 = a_4 = \ldots = 0$; that is, the Maclaurin series for f contains only odd powers.

56. (a) Suppose f(x) is a continuous periodic function with period p. Let x_0 be an arbitrary real number. Then f assumes a minimum m_1 and a maximum m_2 in the interval $[x_0, x_0 + p]$; i.e., $m_1 \le f(x) \le m_2$ for all x in $[x_0, x_0 + p]$. Since f is periodic it has exactly the same values on all other intervals $[x_0 + p, x_0 + 2p]$, $[x_0 + 2p, x_0 + 3p]$, …, and $[x_0 - p, x_0]$, $[x_0 - 2p, x_0 - p]$, …, and so forth. That is, for all real numbers $-\infty < x < \infty$ we have $m_1 \le f(x) \le m_2$. Now choose $M = \max\{|m_1|, |m_2|\}$. Then $-M \le -|m_1| \le m_1 \le f(x) \le m_2 \le |m_2| \le M \Rightarrow |f(x)| \le M$ for all x.

(b) The dominate term in the nth order Taylor polynomial generated by cos x about x = a is $\frac{\sin(a)}{n!}(x-a)^n$ or $\frac{\cos(a)}{n!}(x-a)^n$. In both cases, as $|x|$ increases the absolute value of these dominate terms tends to ∞, causing the graph of $P_n(x)$ to move away from cos x.

57. (a)



(b) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \Rightarrow \frac{x - \tan^{-1} x}{x^3} = \frac{1}{3} - \frac{x^2}{5} + \ldots$; from the Alternating Series Estimation Theorem, $\frac{x - \tan^{-1} x}{x^3} - \frac{1}{3} < 0$ $\Rightarrow \frac{x - \tan^{-1} x}{x^3} - \left(\frac{1}{3} - \frac{x^2}{5}\right) > 0 \Rightarrow \frac{1}{3} < \frac{x - \tan^{-1} x}{x^3} < \frac{1}{3} - \frac{x^2}{5}$; therefore, the $\lim\limits_{x \to 0} \frac{x - \tan^{-1} x}{x^3} = \frac{1}{3}$

58. $E(x) = f(x) - b_0 - b_1(x-a) - b_2(x-a)^2 - b_3(x-a)^3 - \ldots - b_n(x-a)^n$

$\Rightarrow 0 = E(a) = f(a) - b_0 \Rightarrow b_0 = f(a)$; from condition (b),

$$\lim_{x \to a} \frac{f(x) - f(a) - b_1(x-a) - b_2(x-a)^2 - b_3(x-a)^3 - \ldots - b_n(x-a)^n}{(x-a)^n} = 0$$

$$\Rightarrow \lim_{x \to a} \frac{f'(x) - b_1 - 2b_2(x-a) - 3b_3(x-a)^2 - \ldots - nb_n(x-a)^{n-1}}{n(x-a)^{n-1}} = 0$$

$$\Rightarrow b_1 = f'(a) \Rightarrow \lim_{x \to a} \frac{f''(x) - 2b_2 - 3!\,b_3(x-a) - \ldots - n(n-1)b_n(x-a)^{n-2}}{n(n-1)(x-a)^{n-2}} = 0$$

$$\Rightarrow b_2 = \frac{1}{2}f''(a) \Rightarrow \lim_{x \to a} \frac{f'''(x) - 3!\,b_3 - \ldots - n(n-1)(n-2)b_n(x-a)^{n-3}}{n(n-1)(n-2)(x-a)^{n-3}} = 0$$

$$= b_3 = \frac{1}{3!}f'''(a) \Rightarrow \lim_{x \to a} \frac{f^{(n)}(x) - n!\,b_n}{n!} = 0 \Rightarrow b_n = \frac{1}{n!}f^{(n)}(a); \text{ therefore,}$$

$$g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n = P_n(x)$$

59-64. Example CAS commands:

Maple:

```
f:= x -> (1 + x)^(3/2);
plot(f(x), x = -1..2);
mp:=proc(n):
convert(series(f(x),x=0,n),polynom) end:
p1:= mp(2); p2:= mp(3); p3:=mp(4);
der:=proc(n):
simplify(subs(x=z,diff(f(x),x$(n+1)))) end:
der(2); der(3); der(4);
plot(der(3),z=0..2, title = `3rd Derivative`);
Max:= 0.56: r:= (x,n) -> Max*x^(n+1)/(n+1)!;
r(x,2);
plot(r(x,2),x=0..2, title = `Maximum Remainder Term Using P2`);
plot({f(x),mp(3)}, x = -1..2, title = `Function and Taylor Polynomial P2`);
```

```
plot(f(x) - mp(3), x=-1..2, title = `Maximum Error Function `);
R:= (x,z,n) -> der(n)*x^(n+1)/(n+1)!;
R(x,z,3);
with(plots):
plot3d(R(x,z,3), x=-1..2, z=0..2);
```

Mathematica:

```
Clear[f,x,c]
f[x_] = (1+x)  (3/2)
{a,b} = {-1/2,2};
Plot[ f[x], {x,a,b} ]
p1[x_] = Series[ f[x], {x,0,1} ] // Normal
p2[x_] = Series[ f[x], {x,0,2} ] // Normal
p3[x_] = Series[ f[x], {x,0,3} ] // Normal
f''[c]
Plot[ f''[c], {c,a,b} ]
m1 = f''[a]
f'''[c]
Plot[ f'''[c], {c,a,b} ]
m2 = -f'''[a]
f''''[c]
Plot[ f''''[c], {c,a,b} ]
m3 = f''''[a]
r1 [x_] = m1 x^2/2!
Plot[ r1[x], {x,a,b} ]
r2 [x_] = m2 x^3/3!
Plot[ r2[x], {x,a,b} ]
r3 [x_] = m3 x^4/4!
Plot[ r3[x], {x,a,b} ]

Note: In estimating Rn from these graphs, consider only the portions where c
is between 0 and x.  (Mathematica has no simple way to plot only that
portion.)

Plot3D[ f''[c] x^2/2!, {x,a,b}, {c,a,b}, PlotRange -> All ]
Plot3D[ f'''[c] x^3/3!, {x,a,b}, {c,a,b}, PlotRange -> All ]
Plot3D[ f''''[c] x^4/4!, {x,a,b}, {c,a,b}, PlotRange -> All ]
Plot[ {f[x],p1[x],p2[x],p3[x]}, {x,a,b} ]
```

8.8 APPLICATIONS OF POWER SERIES

1. $(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^3}{3!} + \ldots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots$

2. $(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)x^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)x^3}{3!} + \ldots = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \ldots$

3. $(1-x)^{-1/2} = 1 - \frac{1}{2}(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^3}{3!} + \ldots = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots$

4. $(1-2x)^{1/2} = 1 + \frac{1}{2}(-2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-2x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^3}{3!} + \ldots = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \ldots$

5. $\left(1+\frac{x}{2}\right)^{-2} = 1 - 2\left(\frac{x}{2}\right) + \frac{(-2)(-3)\left(\frac{x}{2}\right)^2}{2!} + \frac{(-2)(-3)(-4)\left(\frac{x}{2}\right)^3}{3!} + \ldots = 1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3$

6. $\left(1-\frac{x}{2}\right)^{-2} = 1 - 2\left(-\frac{x}{2}\right) + \frac{(-2)(-3)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(-2)(-3)(-4)\left(-\frac{x}{2}\right)^3}{3!} + \ldots = 1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \ldots$

7. $\left(1+x^3\right)^{-1/2} = 1 - \frac{1}{2}x^3 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(x^3\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(x^3\right)^3}{3!} + \ldots = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \ldots$

8. $\left(1+x^2\right)^{-1/3} = 1 - \frac{1}{3}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(x^2\right)^2}{2!} + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\left(x^2\right)^3}{3!} + \ldots = 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6 + \ldots$

9. $\left(1+\frac{1}{x}\right)^{1/2} = 1 + \frac{1}{2}\left(\frac{1}{x}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{x}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{x}\right)^3}{3!} + \ldots = 1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3}$

10. $\left(1-\frac{2}{x}\right)^{1/3} = 1 + \frac{1}{3}\left(-\frac{2}{x}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{x}\right)^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{2}{x}\right)^3}{3!} + \ldots = 1 - \frac{2}{3x} - \frac{4}{9x^2} - \frac{40}{81x^3} - \ldots$

11. $(1+x)^4 = 1 + 4x + \frac{(4)(3)x^2}{2!} + \frac{(4)(3)(2)x^3}{3!} + \frac{(4)(3)(2)x^4}{4!} = 1 + 4x + 6x^2 + 4x^3 + x^4$

12. $\left(1+x^2\right)^3 = 1 + 3x^2 + \frac{(3)(2)\left(x^2\right)^2}{2!} + \frac{(3)(2)(1)\left(x^2\right)^3}{3!} = 1 + 3x^2 + 3x^4 + x^6$

13. $(1-2x)^3 = 1 + 3(-2x) + \frac{(3)(2)(-2x)^2}{2!} + \frac{(3)(2)(1)(-2x)^3}{3!} = 1 - 6x + 12x^2 - 8x^3$

14. $\left(1-\frac{x}{2}\right)^4 = 1 + 4\left(-\frac{x}{2}\right) + \frac{(4)(3)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(4)(3)(2)\left(-\frac{x}{2}\right)^3}{3!} + \frac{(4)(3)(2)(1)\left(-\frac{x}{2}\right)^4}{4!} = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

15. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} + y = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = 0$

$\Rightarrow a_1 + a_0 = 0$, $2a_2 + a_1 = 0$, $3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$. Since $y = 1$ when $x = 0$ we have $a_0 = 1$. Therefore $a_1 = -1$, $a_2 = \frac{-a_1}{2 \cdot 1} = \frac{1}{2}$, $a_3 = \frac{-a_2}{3} = -\frac{1}{3 \cdot 2}$, $\ldots$, $a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!}$

$\Rightarrow y = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = e^{-x}$

16. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} - 2y = (a_1 - 2a_0) + (2a_2 - 2a_1)x + (3a_3 - 2a_2)x^2 + \ldots + (na_n - 2a_{n-1})x^{n-1} + \ldots = 0$

$\Rightarrow a_1 - 2a_0 = 0$, $2a_2 - 2a_1 = 0$, $3a_3 - 2a_2 = 0$ and in general $na_n - 2a_{n-1} = 0$. Since $y = 1$ when $x = 0$ we have $a_0 = 1$. Therefore $a_1 = 2a_0 = 2(1) = 2$, $a_2 = \frac{2}{2}a_1 = \frac{2}{2}(2) = \frac{2^2}{2}$, $a_3 = \frac{2}{3}a_2 = \frac{2}{3}\left(\frac{2^2}{2}\right) = \frac{2^3}{3 \cdot 2}, \ldots,$

$a_n = \left(\frac{2}{n}\right)a_{n-1} = \left(\frac{2}{n}\right)\left(\frac{2^{n-1}}{n-1}\right)a_{n-2} = \frac{2^n}{n!} \Rightarrow y = 1 + 2x + \frac{2^2}{2}x^2 + \frac{2^3}{3!}x^3 + \ldots + \frac{2^n}{n!}x^n + \ldots$

$= 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots + \frac{(2x)^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = e^{2x}$

17. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} - y = (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = 1$

$\Rightarrow a_1 - a_0 = 1$, $2a_2 - a_1 = 0$, $3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$. Since $y = 0$ when $x = 0$ we have $a_0 = 0$. Therefore $a_1 = 1$, $a_2 = \frac{a_1}{2} = \frac{1}{2}$, $a_3 = \frac{a_2}{3} = \frac{1}{3 \cdot 2}$, $a_4 = \frac{a_3}{4} = \frac{1}{4 \cdot 3 \cdot 2}, \ldots, a_n = \frac{a_{n-1}}{n} = \frac{1}{n!}$

$\Rightarrow y = 0 + 1x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots$

$= \left(1 + 1x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots\right) - 1 = \sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 = e^x - 1$

18. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} + y = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = 1$

$\Rightarrow a_1 + a_0 = 1$, $2a_2 + a_1 = 0$, $3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$. Since $y = 2$ when $x = 0$ we have $a_0 = 2$. Therefore $a_1 = 1 - a_0 = -1$, $a_2 = \frac{-a_1}{2 \cdot 1} = \frac{1}{2}$, $a_3 = \frac{-a_2}{3} = -\frac{1}{3 \cdot 2}$, $\ldots, a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!}$

$\Rightarrow y = 2 - x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots = 1 + \left(1 - x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots\right)$

$= 1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 + e^{-x}$

19. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} - y = (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = x$

$\Rightarrow a_1 - a_0 = 0$, $2a_2 - a_1 = 1$, $3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$. Since $y = 0$ when $x = 0$ we have $a_0 = 0$. Therefore $a_1 = 0$, $a_2 = \frac{1 + a_1}{2} = \frac{1}{2}$, $a_3 = \frac{a_2}{3} = \frac{1}{3 \cdot 2}$, $a_4 = \frac{a_3}{4} = \frac{1}{4 \cdot 3 \cdot 2}, \ldots, a_n = \frac{a_{n-1}}{n} = \frac{1}{n!}$

$\Rightarrow y = 0 + 0x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + + \ldots + \frac{1}{n!}x^n + \ldots$

$= \left(1 + 1x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots\right) - 1 - x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x = e^x - x - 1$

20. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} + y = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = 2x$

$\Rightarrow a_1 + a_0 = 0$, $2a_2 + a_1 = 2$, $3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$. Since $y = -1$ when $x = 0$ we have $a_0 = -1$. Therefore $a_1 = 1$, $a_2 = \frac{2 - a_1}{2} = \frac{1}{2}$, $a_3 = \frac{-a_2}{3} = -\frac{1}{3 \cdot 2}$, $\ldots$, $a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!}$

$\Rightarrow y = -1 + 1x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots$

$= \left(1 - 1x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots\right) - 2 + 2x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} - 2 + 2x = e^{-x} + 2x - 2$

21. $y' - xy = a_1 + (2a_2 - a_0)x + (3a_3 - a_1)x + \ldots + (na_n - a_{n-2})x^{n-1} + \ldots = 0 \Rightarrow a_1 = 0$, $2a_2 - a_0 = 0$, $3a_3 - a_1 = 0$, $4a_4 - a_2 = 0$ and in general $na_n - a_{n-2} = 0$. Since $y = 1$ when $x = 0$, we have $a_0 = 1$. Therefore $a_2 = \frac{a_0}{2} = \frac{1}{2}$, $a_3 = \frac{a_1}{3} = 0$, $a_4 = \frac{a_2}{4} = \frac{1}{2 \cdot 4}$, $a_5 = \frac{a_3}{5} = 0$, $\ldots$, $a_{2n} = \frac{1}{2 \cdot 4 \cdot 6 \cdots 2n}$ and $a_{2n+1} = 0$

$\Rightarrow y = 1 + \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 + \frac{1}{2 \cdot 4 \cdot 6}x^6 + \ldots + \frac{1}{2 \cdot 4 \cdot 6 \cdots 2n}x^{2n} + \ldots = \sum\limits_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum\limits_{n=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^n}{n!} = e^{x^2/2}$

22. $y' - x^2y = a_1 + 2a_2x + (3a_3 - a_0)x^2 + (4a_4 - a_1)x^3 + \ldots + (na_n - a_{n-3})x^{n-1} + \ldots = 0 \Rightarrow a_1 = 0$, $a_2 = 0$, $3a_3 - a_0 = 0$, $4a_4 - a_1 = 0$ and in general $na_n - a_{n-3} = 0$. Since $y = 1$ when $x = 0$, we have $a_0 = 1$. Therefore $a_3 = \frac{a_0}{3} = \frac{1}{3}$, $a_4 = \frac{a_1}{4} = 0$, $a_5 = \frac{a_2}{5} = 0$, $a_6 = \frac{a_3}{6} = \frac{1}{3 \cdot 6}$, $\ldots$, $a_{3n} = \frac{1}{3 \cdot 6 \cdot 9 \cdots 3n}$, $a_{3n+1} = 0$ and $a_{3n+2} = 0$

$\Rightarrow y = 1 + \frac{1}{3}x^3 + \frac{1}{3 \cdot 6}x^6 + \frac{1}{3 \cdot 6 \cdot 9}x^9 + \ldots + \frac{1}{3 \cdot 6 \cdot 9 \cdots 3n}x^{3n} + \ldots = \sum\limits_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = \sum\limits_{n=0}^{\infty} \frac{\left(\frac{x^3}{3}\right)^n}{n!} = e^{x^3/3}$

23. $(1 - x)y' - y = (a_1 - a_0) + (2a_2 - a_1 - a_1)x + (3a_3 - 2a_2 - a_2)x^2 + (4a_4 - 3a_3 - a_3)x^3 + \ldots + (na_n - (n-1)a_{n-1} - a_{n-1})x^{n-1} + \ldots = 0 \Rightarrow a_1 - a_0 = 0$, $2a_2 - 2a_1 = 0$, $3a_3 - 3a_2 = 0$ and in general $(na_n - na_{n-1}) = 0$. Since $y = 2$ when $x = 0$, we have $a_0 = 2$. Therefore $a_1 = 2$, $a_2 = 2$, $\ldots$, $a_n = 2 \Rightarrow y = 2 + 2x + 2x^2 + \ldots = \sum\limits_{n=0}^{\infty} 2x^n = \frac{2}{1 - x}$

24. $(1 + x^2)y' + 2xy = a_1 + (2a_2 + 2a_0)x + (3a_3 + 2a_1 + a_1)x^2 + (4a_4 + 2a_2 + 2a_2)x^3 + \ldots + (na_n + na_{n-2})x^{n-1} + \ldots = 0 \Rightarrow a_1 = 0$, $2a_2 + 2a_0 = 0$, $3a_3 + 3a_1 = 0$, $4a_4 + 4a_2 = 0$ and in general $na_n + na_{n-2} = 0$. Since $y = 3$ when $x = 0$, we have $a_0 = 3$. Therefore $a_2 = -3$, $a_3 = 0$, $a_4 = 3$, $\ldots$, $a_{2n+1} = 0$, $a_{2n} = (-1)^n 3$

$\Rightarrow y = 3 - 3x^2 + 3x^4 - \ldots = \sum\limits_{n=0}^{\infty} 3(-1)^n x^{2n} = \sum\limits_{n=0}^{\infty} 3\left(-x^2\right)^n = \frac{3}{1 + x^2}$

25. $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \Rightarrow y'' = 2a_2 + 3\cdot 2a_3x + \ldots + n(n-1)a_nx^{n-2} + \ldots \Rightarrow y'' - y$
$= (2a_2 - a_0) + (3\cdot 2a_3 - a_1)x + (4\cdot 3a_4 - a_2)x^2 + \ldots + (n(n-1)a_n - a_{n-2})x^{n-2} + \ldots = 0 \Rightarrow 2a_2 - a_0 = 0,$
$3\cdot 2a_3 - a_1 = 0$, $4\cdot 3a_4 - a_2 = 0$ and in general $n(n-1)a_n - a_{n-2} = 0$. Since $y' = 1$ and $y = 0$ when $x = 0$, we have $a_0 = 0$ and $a_1 = 1$. Therefore $a_2 = 0$, $a_3 = \frac{1}{3\cdot 2}$, $a_4 = 0$, $a_5 = \frac{1}{5\cdot 4\cdot 3\cdot 2}$, $\ldots$, $a_{2n+1} = \frac{1}{(2n+1)!}$ and $a_{2n} = 0 \Rightarrow y = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$

26. $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \Rightarrow y'' = 2a_2 + 3\cdot 2a_3x + \ldots + n(n-1)a_nx^{n-2} + \ldots \Rightarrow y'' + y$
$= (2a_2 + a_0) + (3\cdot 2a_3 + a_1)x + (4\cdot 3a_4 + a_2)x^2 + \ldots + (n(n-1)a_n + a_{n-2})x^{n-2} + \ldots = 0 \Rightarrow 2a_2 + a_0 = 0,$
$3\cdot 2a_3 + a_1 = 0$, $4\cdot 3a_4 + a_2 = 0$ and in general $n(n-1)a_n + a_{n-2} = 0$. Since $y' = 0$ and $y = 1$ when $x = 0$, we have $a_0 = 1$ and $a_1 = 0$. Therefore $a_2 = -\frac{1}{2}$, $a_3 = 0$, $a_4 = \frac{1}{4\cdot 3\cdot 2}$, $a_5 = 0$, $\ldots$, $a_{2n+1} = 0$ and $a_{2n} = \frac{(-1)^n}{(2n)!}$
$\Rightarrow y = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x$

27. $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \Rightarrow y'' = 2a_2 + 3\cdot 2a_3x + \ldots + n(n-1)a_nx^{n-2} + \ldots \Rightarrow y'' + y$
$= (2a_2 + a_0) + (3\cdot 2a_3 + a_1)x + (4\cdot 3a_4 + a_2)x^2 + \ldots + (n(n-1)a_n + a_{n-2})x^{n-2} + \ldots = x \Rightarrow 2a_2 + a_0 = 0,$
$3\cdot 2a_3 + a_1 = 1$, $4\cdot 3a_4 + a_2 = 0$ and in general $n(n-1)a_n + a_{n-2} = 0$. Since $y' = 1$ and $y = 2$ when $x = 0$, we have $a_0 = 2$ and $a_1 = 1$. Therefore $a_2 = -1$, $a_3 = 0$, $a_4 = \frac{1}{4\cdot 3}$, $a_5 = 0$, $\ldots$, $a_{2n} = -2\cdot\frac{(-1)^{n+1}}{(2n)!}$ and $a_{2n+1} = 0 \Rightarrow y = 2 + x - x^2 + 2\cdot\frac{x^4}{4!} + \ldots = 2 + x - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{(2n)!}$

28. $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \Rightarrow y'' = 2a_2 + 3\cdot 2a_3x + \ldots + n(n-1)a_nx^{n-2} + \ldots \Rightarrow y'' - y$
$= (2a_2 - a_0) + (3\cdot 2a_3 - a_1)x + (4\cdot 3a_4 - a_2)x^2 + \ldots + (n(n-1)a_n - a_{n-2})x^{n-2} + \ldots = x \Rightarrow 2a_2 - a_0 = 0,$
$3\cdot 2a_3 - a_1 = 1$, $4\cdot 3a_4 - a_2 = 0$ and in general $n(n-1)a_n - a_{n-2} = 0$. Since $y' = 2$ and $y = -1$ when $x = 0$, we have $a_0 = -1$ and $a_1 = 2$. Therefore $a_2 = \frac{-1}{2}$, $a_3 = \frac{1}{2}$, $a_4 = \frac{-1}{2\cdot 3\cdot 4}$, $a_5 = \frac{1}{5\cdot 4\cdot 2} = \frac{3}{5!}$, $\ldots$, $a_{2n} = \frac{-1}{(2n)!}$ and $a_{2n+1} = \frac{3}{(2n+1)!} \Rightarrow y = -1 + 2x - \frac{1}{2}x^2 + \frac{3}{3!}x^3 - \ldots = -1 + 2x - \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \frac{3x^{2n+1}}{(2n+1)!}$

29. $y = a_0 + a_1(x-2) + a_2(x-2)^2 + \ldots + a_n(x-2)^n + \ldots$
$\Rightarrow y'' = 2a_2 + 3\cdot 2a_3(x-2) + \ldots + n(n-1)a_n(x-2)^{n-2} + \ldots \Rightarrow y'' - y$
$= (2a_2 - a_0) + (3\cdot 2a_3 - a_1)(x-2) + (4\cdot 3a_4 - a_2)(x-2)^2 + \ldots + (n(n-1)a_n - a_{n-2})(x-2)^{n-2} + \ldots$
$= -2 - (x-2) \Rightarrow 2a_2 - a_0 = -2$, $3\cdot 2a_3 - a_1 = -1$, $4\cdot 3a_4 - a_2$ and in general $n(n-1)a_n - a_{n-2} = 0$.
Since $y' = -2$ and $y = 0$ when $x = 2$, we have $a_0 = 0$ and $a_1 = -2$. Therefore $a_2 = \frac{-2}{2} = -1$,
$a_3 = \frac{-2-1}{3\cdot 2} = -\frac{3}{3\cdot 2}$, $a_4 = -\frac{2}{4\cdot 3\cdot 2}$, $a_5 = -\frac{3}{5\cdot 4\cdot 3\cdot 2}$, $\ldots$, $a_{2n} = -\frac{2}{(2n)!}$, $a_{2n+1} = -\frac{3}{(2n+1)!}$
$\Rightarrow y = -2(x-2) - \frac{2}{2!}(x-2)^2 - \frac{3}{3!}(x-2)^3 - \frac{2}{4!}(x-2)^4 - \frac{3}{5!}(x-2)^5 - \ldots$

$$= -2(x-2) - \sum_{n=1}^{\infty}\left[\frac{2(x-2)^{2n}}{(2n)!} + \frac{3(x-2)^{2n+1}}{(2n+1)!}\right]$$

30. $y'' - x^2y = 2a_2 + 6a_3x + (4\cdot 3a_4 - a_0)x^2 + \ldots + (n(n-1)a_n - a_{n-4})x^{n-2} + \ldots = 0 \Rightarrow 2a_2 = 0,\ 6a_3 = 0,$ $4\cdot 3a_4 - a_0 = 0,\ 5\cdot 4a_5 - a_1 = 0$, and in general $n(n-1)a_n - a_{n-4} = 0$. Since $y' = b$ and $y = a$ when $x = 0$, we have $a_0 = a,\ a_1 = b,\ a_2 = 0,\ a_3 = 0,\ a_4 = \frac{a}{3\cdot 4},\ a_5 = \frac{b}{4\cdot 5},\ a_6 = 0,\ a_7 = 0,\ a_8 = \frac{a}{3\cdot 4\cdot 7\cdot 8},\ a_9 = \frac{b}{4\cdot 5\cdot 8\cdot 9}$ $\Rightarrow y = a + bx + \frac{a}{3\cdot 4}x^4 + \frac{b}{4\cdot 5}x^5 + \frac{a}{3\cdot 4\cdot 7\cdot 8}x^8 + \frac{b}{4\cdot 5\cdot 8\cdot 9}x^9 + \ldots$

31. $y'' + x^2y = 2a_2 + 6a_3x + (4\cdot 3a_4 + a_0)x^2 + \ldots + (n(n-1)a_n + a_{n-4})x^{n-2} + \ldots = x \Rightarrow 2a_2 = 0,\ 6a_3 = 1,$ $4\cdot 3a_4 + a_0 = 0,\ 5\cdot 4a_5 + a_1 = 0$, and in general $n(n-1)a_n + a_{n-4} = 0$. Since $y' = b$ and $y = a$ when $x = 0$, we have $a_0 = a$ and $a_1 = b$. Therefore $a_2 = 0,\ a_3 = \frac{1}{2\cdot 3},\ a_4 = -\frac{a}{3\cdot 4},\ a_5 = -\frac{b}{4\cdot 5},\ a_6 = 0,\ a_7 = \frac{1}{2\cdot 3\cdot 6\cdot 7}$ $\Rightarrow y = a + bx + \frac{1}{2\cdot 3}x^3 - \frac{a}{3\cdot 4}x^4 - \frac{b}{4\cdot 5}x^5 - \frac{1}{2\cdot 3\cdot 6\cdot 7}x^7 + \frac{ax^8}{3\cdot 4\cdot 7\cdot 8} + \frac{bx^9}{4\cdot 5\cdot 8\cdot 9} + \cdots$

32. $y'' - 2y' + y = (2a_2 - 2a_1 + a_0) + (2\cdot 3a_3 - 4a_2 + a_1)x + (3\cdot 4a_4 - 2\cdot 3a_3 + a_2)x^2 + \ldots$ $+ ((n-1)na_n - 2(n-1)a_{n-1} + a_{n-2})x^{n-2} + \ldots = 0 \Rightarrow 2a_2 - 2a_1 + a_0 = 0,\ 2\cdot 3a_3 - 4a_2 + a_1 = 0,$ $3\cdot 4a_4 - 2\cdot 3a_3 + a_2 = 0$ and in general $(n-1)na_n - 2(n-1)a_{n-1} + a_{n-2} = 0$. Since $y' = 1$ and $y = 0$ when when $x = 0$, we have $a_0 = 0$ and $a_1 = 1$. Therefore $a_2 = 1,\ a_3 = \frac{1}{2},\ a_4 = \frac{1}{6},\ a_5 = \frac{1}{24}$ and $a_n = \frac{1}{(n-1)!}$ $\Rightarrow y = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \ldots = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} = x\sum_{n=0}^{\infty}\frac{x^n}{n!} = xe^x$

33. $F(x) = \int_0^x \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \ldots\right)dt = \left[\frac{t^3}{3} - \frac{t^7}{7\cdot 3!} + \frac{t^{11}}{11\cdot 5!} - \frac{t^{15}}{15\cdot 7!} + \ldots\right]_0^x \approx \frac{x^3}{3} - \frac{x^7}{7\cdot 3!}$

$\Rightarrow |\text{error}| < \frac{1}{11\cdot 5!} \approx 0.0008$

34. $F(x) = \int_0^x \left(t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \ldots\right)dt = \left[\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7\cdot 2!} - \frac{t^9}{9\cdot 3!} + \frac{t^{11}}{11\cdot 4!} - \frac{t^{13}}{13\cdot 5!} + \ldots\right]_0^x$

$\approx \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7\cdot 2!} - \frac{x^9}{9\cdot 3!} + \frac{x^{11}}{11\cdot 4!} \Rightarrow |\text{error}| < \frac{1}{13\cdot 5!} \approx 0.00064$

35. (a) $F(x) = \int_0^x \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots\right)dt = \left[\frac{t^2}{2} - \frac{t^4}{12} + \frac{t^6}{30} - \ldots\right]_0^x \approx \frac{x^2}{2} - \frac{x^4}{12} \Rightarrow |\text{error}| < \frac{(0.5)^6}{30} \approx .00052$

(b) $|\text{error}| < \frac{1}{33\cdot 34} \approx .00089$ so $F(x) \approx \frac{x^2}{2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \ldots + (-1)^{15}\frac{x^{32}}{31\cdot 32}$

36. (a) $F(x) = \int_0^x \left(1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \ldots\right)dt = \left[t - \frac{t^2}{2\cdot 2} + \frac{t^3}{3\cdot 3} - \frac{t^4}{4\cdot 4} + \frac{t^5}{5\cdot 5} - \ldots\right]_0^x \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2}$

$\Rightarrow |\text{error}| < \frac{(0.5)^6}{6^2} \approx .00043$

(b) $|\text{error}| < \frac{1}{32^2} \approx .00097$ so $F(x) \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots + (-1)^{31}\,\frac{x^{31}}{31^2}$

37. $\frac{1}{x^2}\left(e^x - (1+x)\right) = \frac{1}{x^2}\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots\right) - 1 - x\right) = \frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots \Rightarrow \lim\limits_{x\to 0} \frac{e^x - (1+x)}{x^2}$

$= \lim\limits_{x\to 0}\left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots\right) = \frac{1}{2}$

38. $\frac{1}{t^4}\left(1 - \cos t - \frac{t^2}{2}\right) = \frac{1}{t^4}\left[1 - \frac{t^2}{2} - \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots\right)\right] = -\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \ldots \Rightarrow \lim\limits_{t\to 0} \frac{1 - \cos t - \left(\frac{t^2}{2}\right)}{t^4}$

$= \lim\limits_{t\to 0}\left(-\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \ldots\right) = -\frac{1}{24}$

39. $x^2\left(-1 + e^{-1/x^2}\right) = x^2\left(-1 + 1 - \frac{1}{x^2} + \frac{1}{2x^4} - \frac{1}{6x^6} + \ldots\right) = -1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \ldots \Rightarrow \lim\limits_{x\to\infty} x^2\left(e^{-1/x^2} - 1\right)$

$= \lim\limits_{x\to\infty}\left(-1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \ldots\right) = -1$

40. $\frac{\tan^{-1} y - \sin y}{y^3 \cos y} = \frac{\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \ldots\right) - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right)}{y^3 \cos y} = \frac{\left(-\frac{y^3}{6} + \frac{23y^5}{5!} - \ldots\right)}{y^3 \cos y} = \frac{\left(-\frac{1}{6} + \frac{23y^2}{5!} - \ldots\right)}{\cos y}$

$\Rightarrow \lim\limits_{y\to 0} \frac{\tan^{-1} y - \sin y}{y^3 \cos y} = \lim\limits_{y\to 0} \frac{\left(-\frac{1}{6} + \frac{23y^2}{5!} - \ldots\right)}{\cos y} = -\frac{1}{6}$

41. $\frac{\ln(1+x^2)}{1 - \cos x} = \frac{\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \ldots\right)}{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)} = \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \ldots\right)}{\left(\frac{1}{2!} - \frac{x^2}{4!} + \ldots\right)} \Rightarrow \lim\limits_{x\to 0} \frac{\ln(1+x^2)}{1 - \cos x} = \lim\limits_{x\to 0} \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \ldots\right)}{\left(\frac{1}{2!} - \frac{x^2}{4!} + \ldots\right)} = 2!$

$= 2$

42. $(x+1)\sin\left(\frac{1}{x+1}\right) = (x+1)\left(\frac{1}{x+1} - \frac{1}{3!(x+1)^3} + \frac{1}{5!(x+1)^5} - \ldots\right) = 1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \ldots$

$\Rightarrow \lim\limits_{x\to\infty} (x+1)\sin\left(\frac{1}{x+1}\right) = \lim\limits_{x\to\infty}\left(1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \ldots\right) = 1$

43. $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$

44. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1}x^n}{n} + \ldots \Rightarrow |\text{error}| = \left|\frac{(-1)^{n-1}x^n}{n}\right| = \frac{1}{n10^n}$ when $x = 0.1$;

$\frac{1}{n10^n} < \frac{1}{10^8} \Rightarrow n10^n > 10^8$ when $n \geq 8 \Rightarrow 7$ terms

45. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \ldots \Rightarrow |\text{error}| = \left|\frac{(-1)^{n-1}x^{2n-1}}{2n-1}\right| = \frac{1}{2n-1}$ when $x = 1$;

$\frac{1}{2n-1} < \frac{1}{10^3} \Rightarrow n > \frac{1001}{2} = 500.5 \Rightarrow$ the first term not used is the 501^{st} $\Rightarrow$ we must use 500 terms

46. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \ldots$ and $\lim\limits_{n\to\infty} \left|\frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}}\right| = x^2 \lim\limits_{n\to\infty} \left|\frac{2n-1}{2n+1}\right| = x^2$

$\Rightarrow \tan^{-1} x$ converges for $|x| < 1$; when $x = -1$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ which is a convergent series; when $x = 1$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ which is a convergent series $\Rightarrow$ the series representing $\tan^{-1} x$ diverges for $|x| > 1$

47. (a) $\left(1 - x^2\right)^{-1/2} \approx 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} \Rightarrow \sin^{-1} x \approx x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}$;

$$\lim_{n\to\infty} \left|\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)x^{2n+3}}{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)(2n+3)} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n)(2n+1)}{1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n+1}}\right| < 1 \Rightarrow x^2 \lim_{n\to\infty} \left|\frac{(2n+1)(2n+1)}{(2n+2)(2n+3)}\right| < 1$$

$\Rightarrow |x| < 1 \Rightarrow$ the radius of convergence is 1

(b) $\frac{d}{dx}\left(\cos^{-1} x\right) = -\left(1 - x^2\right)^{-1/2} \Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \approx \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}\right) \approx \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{5x^7}{112}$

48. $$\left(1 - x^2\right)^{-1/2} = \left(1 + \left(-x^2\right)\right)^{-1/2} = (1)^{-1/2} + \left(-\frac{1}{2}\right)(1)^{-3/2}\left(-x^2\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1)^{-5/2}\left(-x^2\right)^2}{2!}$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1)^{-7/2}\left(-x^2\right)^3}{3!} + \ldots = 1 + \frac{x^2}{2} + \frac{1 \cdot 3x^4}{2^2 \cdot 2!} + \frac{1 \cdot 3 \cdot 5x^6}{2^3 \cdot 3!} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n \cdot n!}$$

$$\Rightarrow \sin^{-1} x = \int_0^x \left(1 - t^2\right)^{-1/2} dt = \int_0^x \left(1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n \cdot n!}\right) dt = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n+1}}{2 \cdot 4 \cdots (2n)(2n+1)},$$

where $|x| < 1$

49. $$\left[\tan^{-1} t\right]_x^{\infty} = \frac{\pi}{2} - \tan^{-1} x = \int_x^{\infty} \frac{dt}{1+t^2} = \int_x^{\infty} \left[\frac{\left(\frac{1}{t^2}\right)}{1 + \left(\frac{1}{t^2}\right)}\right] dt = \int_x^{\infty} \frac{1}{t^2}\left(1 - \frac{1}{t^2} + \frac{1}{t^4} - \frac{1}{t^6} + \ldots\right) dt$$

$$= \int_x^{\infty} \left(\frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \ldots\right) dt = \lim_{b\to\infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \ldots\right]_x^b = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \ldots$$

$$\Rightarrow \tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots,\ x > 1;\ \left[\tan^{-1} t\right]_{-\infty}^{x} = \tan^{-1} x + \frac{\pi}{2} = \int_{-\infty}^{x} \frac{dt}{1+t^2}$$

$= \lim_{b \to -\infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \dots \right]_b^x = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots \Rightarrow \tan^{-1} x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots,$
$x < -1$

50. (a) $\tan\left(\tan^{-1}(n+1) - \tan^{-1}(n-1)\right) = \frac{\tan\left(\tan^{-1}(n+1)\right) - \tan\left(\tan^{-1}(n-1)\right)}{1 + \tan\left(\tan^{-1}(n+1)\right)\tan\left(\tan^{-1}(n-1)\right)} = \frac{(n+1)-(n-1)}{1+(n+1)(n-1)} = \frac{2}{n^2}$

(b) $\sum_{n=1}^{N} \tan^{-1}\left(\frac{2}{n^2}\right) = \sum_{n=1}^{N} \left[\tan^{-1}(n+1) - \tan^{-1}(n-1)\right] = \left(\tan^{-1} 2 - \tan^{-1} 0\right) + \left(\tan^{-1} 3 - \tan^{-1} 1\right)$
$+ \left(\tan^{-1} 4 - \tan^{-1} 2\right) + \dots + \left(\tan^{-1}(N+1) - \tan^{-1}(N-1)\right) = \tan^{-1}(N+1) + \tan^{-1} N - \frac{\pi}{4}$

(c) $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{n^2}\right) = \lim_{N \to \infty} \left[\tan^{-1}(N+1) + \tan^{-1} N - \frac{\pi}{4}\right] = \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} = \frac{3\pi}{4}$

8.9 FOURIER SERIES

1. $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \, dx = \left(\frac{1}{\pi}\right) x \Big|_{-\pi}^{\pi} = 2$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \, dx = \frac{1}{\pi n} \sin nx \Big|_{-\pi}^{\pi} = 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx \, dx = -\frac{1}{n\pi} \cos nx \Big|_{-\pi}^{\pi} = \frac{1}{nx}[\cos(-n\pi) - \cos(n\pi)] = 0$

Therefore,

$f(x) = \frac{a_0}{2} = 1.$

2. $a_0 = \frac{1}{\pi} \int_{-\pi}^{0} -dx + \frac{1}{\pi} \int_{0}^{\pi} dx = \left(\frac{1}{\pi}\right)(-x) \Big|_{\pi}^{0} + \left(\frac{1}{\pi}\right) x \Big|_{0}^{\pi} = 0$

$a_n = \frac{1}{\pi} \int_{-\pi}^{0} -\cos nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} \cos nx \, dx = \left(-\frac{1}{n\pi}\right) \sin nx \Big|_{\pi}^{0} + \left(\frac{1}{n\pi}\right) \sin nx \Big|_{0}^{\pi} = 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{0} -\sin nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin nx \, dx = \left(\frac{1}{n\pi}\right) \cos nx \Big|_{-\pi}^{0} + \left(-\frac{1}{n\pi}\right) \cos nx \Big|_{0}^{\pi}$

$= \frac{1}{nx}[\cos 0 - \cos(-n\pi)] + \left(-\frac{1}{n\pi}\right)(\cos n\pi - \cos 0)$

$= \frac{1}{n\pi}\left[1 - (-1)^n\right] - \frac{1}{n\pi}\left[(-1)^n - 1\right] = \frac{2}{n\pi}\left[1 - (-1)^n\right]$

Therefore,

$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - (-1)^n\right] \sin nx.$

3. $a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} x\ dx = \frac{1}{2\pi}x^2\Big|_{-\pi}^{\pi} = 0.$ (Note: x is an odd function)

$a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} x \cos nx\ dx = 0.$ (because x cos nx is an odd function)

$b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} x \sin nx\ dx = \frac{2}{\pi}\int\limits_{0}^{\pi} x \sin nx\ dx$ (because x sin nx is even)

$= \frac{2}{\pi}\left(-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\Big|_0^{\pi}\right) = -\frac{2}{n}\cos nx = \frac{2}{n}(-1)^{n+1}$

Therefore,

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx.$$

4. $a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} (1-x)\ dx = \frac{1}{\pi}\left(x - \frac{1}{2}x^2\right)\Big|_{-\pi}^{\pi} = 2$

$a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} (1-x)\cos nx\ dx = \frac{1}{\pi}\left[\frac{1}{n}(1-x)\sin nx - \frac{1}{n^2}\cos nx\right]_{-\pi}^{\pi} = 0$

$b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} (1-x)\sin nx\ dx = -\frac{1}{\pi}\left[\frac{1}{n}(1-x)\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} = \frac{2\pi}{n\pi}\cos n\pi = \frac{2}{n}(-1)^n.$

Therefore,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin nx = 1 + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\sin nx = 1 - \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx.$$

(Compare this result with the Fourier series found in problems 1 and 3.)

5. $a_0 = \frac{1}{4\pi}\int\limits_{-\pi}^{\pi} x^2\ dx = \frac{1}{2\pi}\int\limits_{0}^{\pi} x^2\ dx = \frac{\pi^2}{6}$

$a_n = \frac{1}{4\pi}\int\limits_{-\pi}^{\pi} x^2\cos nx\ dx = \frac{1}{2\pi}\int\limits_{0}^{\pi} x^2 \cos nx\ dx$ (even function)

$= \frac{1}{2\pi}\left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_0^{\pi} = \frac{1}{n^2}\cos n\pi = \frac{(-1)^n}{n^2}$

$b^n = \frac{1}{4\pi}\int\limits_{-\pi}^{\pi} x^2 \sin nx\ dx = 0$ (odd function)

Therefore,

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$$

6. $a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{1}{\pi}\left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_0^{\pi} = \frac{2}{n^2}\cos n\pi = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx\,dx = \frac{1}{\pi}\left[-\frac{x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos x\right]_0^{\pi} = \frac{2}{\pi n^3}\left[(-1)^n - 1\right] + \frac{\pi}{n}(-1)^{n+1}$$

Therefore,

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\left\{\frac{2}{\pi n^3}\left[(-1)^n - 1\right] + \frac{\pi}{n}(-1)^{n+1}\right\}\sin nx.$$

7. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right) = \frac{2}{\pi}\sinh \pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx = \frac{1}{\pi}\left[\frac{n^2 e^x}{1+n^2}\left(\frac{1}{n}\sin nx + \frac{1}{n^2}\cos nx\right)\right]_{-\pi}^{\pi} = \frac{1}{\pi}\,\frac{2\cos n\pi}{(1+n^2)}\left(\frac{e^{\pi} - e^{-\pi}}{2}\right) = \frac{2(-1)^n}{\pi(n^2+1)}\sinh \pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx = \frac{1}{\pi}\left[\frac{n^2 e^x}{1+n^2}\left(-\frac{1}{n}\cos nx + \frac{1}{n^2}\sin nx\right)\right]_{-\pi}^{\pi} = \frac{1}{\pi}\,\frac{2n\cos n\pi}{(1+n^2)}\left(\frac{e^{-\pi} - e^{\pi}}{2}\right) = \frac{2n(-1)^{n+1}}{\pi(n^2+1)}\sinh \pi$$

Therefore,

$$f(x) = \frac{1}{\pi}\sinh \pi + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi(n^2+1)}\sinh \pi \cos nx + \sum_{n=1}^{\infty}\frac{2n(-1)^{n+1}}{\pi(n^2+1)}\sinh \pi \sin nx$$

$$= \frac{\sinh \pi}{\pi}\left[1 + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2+1}\cos nx + \sum_{n=1}^{\infty}\frac{2n(-1)^{n+1}}{n^2+1}\sin nx\right]$$

$$= \frac{2\sinh \pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin nx)\right].$$

8. $a_0 = \frac{1}{\pi}\int_0^{\pi} e^x\,dx = \frac{1}{\pi}e^{\pi}$

$$a_n = \frac{1}{\pi}\int_0^{\pi} e^x \cos nx\,dx = \frac{1}{\pi}\left[\frac{n^2 e^x}{1+n^2}\left(\frac{1}{n}\sin nx + \frac{1}{n^2}\cos nx\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{n^2}{1+n^2}\right)\left(\frac{e^{\pi}}{n^2}\cos n\pi - \frac{1}{n^2}\right) = \frac{1}{\pi(n^2+1)}\left[e^{\pi}(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} e^x \sin nx\,dx = \frac{1}{\pi}\left[\frac{n^2 e^x}{1+n^2}\left(-\frac{1}{n}\cos nx + \frac{1}{n^2}\sin nx\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{n^2}{1+n^2}\right)\left(-\frac{e^\pi}{n}\cos n\pi + \frac{1}{n}\right) = \frac{n}{\pi\left(n^2+1\right)}\left[e^\pi(-1)^{n+1}+1\right]$$

Therefore,

$$f(x) = \frac{e^\pi}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi\left(n^2+1\right)}\left[e^\pi(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty} \frac{n}{\pi\left(n^2+1\right)}\left[e^\pi(-1)^{n+1}+1\right]\sin nx.$$

9. $a_0 = \frac{1}{\pi}\int_0^\pi \cos x\, dx = \frac{1}{\pi}\sin x\Big|_0^\pi = 0$

$$a_n = \frac{1}{\pi}\int_0^\pi \cos x \cos nx\, dx = \begin{cases} 0, & n \neq 1 \\ \frac{1}{2}, & n = 1 \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^\pi \cos x \sin nx\, dx = \begin{cases} \frac{1}{2\pi}\sin^2 x\Big|_0^\pi = 0, & n = 1 \\ \left(-\frac{\cos(n-1)x}{2\pi(n-1)} - \frac{\cos(n+1)x}{2\pi(n+1)}\right|_0^\pi, & n \neq 1 \end{cases} = \begin{cases} 0, & n = 1 \\ (1+(-1)^n)\,\frac{n}{\pi\left(n^2-1\right)}, & n \neq 1 \end{cases}$$

Therefore,

$$f(x) = \frac{1}{2}\cos x + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{n(1+(-1)^n)}{n^2-1}\sin nx.$$

10. $a_0 = \frac{1}{2}\int_{-2}^{0} -x\, dx + \frac{1}{2}\int_0^2 2\, dx = 3$

$$a_n = \frac{1}{2}\int_{-2}^{0} -x\cos\frac{n\pi x}{2}\, dx + \frac{1}{2}\int_0^2 2\cos\frac{n\pi x}{2}\, dx = \frac{1}{2}\left[-\frac{2x}{n\pi}\sin\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_{-2}^{0} + \left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] + 0$$

$$b_n = \frac{1}{2}\int_{-2}^{0} -x\sin\frac{n\pi x}{2}\, dx + \frac{1}{2}\int_0^2 2\sin\frac{n\pi x}{2}\, dx = \frac{1}{2}\left[\frac{2x}{n\pi}\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_{-2}^{0} - \left[\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{2}{n\pi}(-1)^n - \frac{2}{n\pi}\left[(-1)^n - 1\right] = \frac{2}{n\pi}$$

Therefore,

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{n^2\pi^2}\cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi x}{2}.$$

11. $a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dx = 1$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx\, dx = \frac{1}{n\pi}\sin nx\Big[_{-\pi/2}^{\pi/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nx\, dx = 0$$

Therefore,

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos nx.$$

Note: $\sin\frac{n\pi}{2} = \begin{cases} 0, & n = 2k \quad \text{(even)} \\ (-1)^k, & n = 2k+1 \quad \text{(odd)} \end{cases}$

Thus we can write f(x) in the form:

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\cos(2k+1)x.$$

12. $a_0 = \int_{-1}^{1} |x|\, dx = 2\int_0^1 x\, dx = 1$

$$a_n = \int_{-1}^{1} |x|\cos(n\pi x)\, dx = 2\int_0^1 x\cos(n\pi x)\, dx = 2\left[\frac{x}{n\pi}\sin n\pi x + \frac{1}{n^2\pi^2}\cos n\pi x\right]_0^1 = \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \int_{-1}^{1} |x|\sin(n\pi x)\, dx = 0 \qquad \text{(because } |x|\sin n\pi x \text{ is odd)}$$

Therefore,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}[(-1)^n - 1]\cos(n\pi x).$$

13. $a_0 = \int_{-1}^{1/2} -(2x-1)\, dx + \int_{1/2}^{1} (2x-1)\, dx = \left(x - x^2\right]_{-1}^{1/2} + \left(x^2 - x\right]_{1/2}^{1} = \frac{5}{2}$

$$a_n = \int_{-1}^{1/2} -(2x-1)\cos(n\pi x)\, dx + \int_{1/2}^{1} (2x-1)\cos(n\pi x)\, dx$$

$$= \left[\frac{(1-2x)}{n\pi}\sin(n\pi x) - \frac{2}{n^2\pi^2}\cos(n\pi x)\right]_{-1}^{1/2} + \left[\frac{(2x-1)}{n\pi}\sin(n\pi x) + \frac{2}{n^2\pi^2}\cos(n\pi x)\right]_{1/2}^{1}$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right] + \frac{2}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right] = \frac{4}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]$$

$$b_n = \int_{-1}^{1/2} -(2x-1)\sin(n\pi x)\, dx + \int_{1/2}^{1} (2x-1)\sin(n\pi x)\, dx$$

$$= \left[-\frac{(1-2x)}{n\pi}\cos(n\pi x) - \frac{2}{n^2\pi^2}\sin(n\pi x)\right]_{-1}^{1/2} + \left[\frac{(2x-1)}{n\pi}\cos(n\pi x) + \frac{2}{n^2\pi^2}\sin(n\pi x)\right]_{1/2}^{1}$$

$$= \left[\frac{3}{n\pi}\cos n\pi - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] + \left[-\frac{1}{n\pi}\cos n\pi - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{n\pi}(-1)^n - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{2}{n\pi}\left[(-1)^n - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right]$$

Therefore,

$$f(x) = \frac{5}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\cos(n\pi x) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[(-1)^n - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right]\sin(n\pi x).$$

14. $f(x) = x|x|$ is an odd function.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x|x|\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x|x|\cos nx\,dx = 0$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} x|x|\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\sin nx\,dx = \frac{2}{\pi}\left[-\frac{x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2\pi}{n^2}\sin n\pi + \frac{2}{n^3}\cos n\pi - \frac{2}{n^3}\right] = \frac{2}{\pi}\left[\frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^3}(-1)^n - \frac{2}{n^3}\right]$$

$$= \frac{2}{\pi}\left[\frac{(2-\pi^2 n^2)(-1)^n - 2}{n^3}\right]$$

Therefore,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(2-\pi^2 n^2)(-1)^n - 2}{n^3}\sin nx.$$

15. From exercise #5,

$$\frac{x^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$$

Setting $x = \pi$,

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi$$

$$\frac{3\pi^2}{12} - \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2},$$

or

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2} + \ldots.$$

16. From Exercise #6, $f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} \left\{ \frac{2[(-1)^n - 1]}{\pi n^3} + \frac{\pi}{n}(-1)^{n+1} \right\} \sin nx$. Setting $x = 0$ and multiplying both sides by $\frac{1}{2}$ gives $\frac{\pi^2}{12} = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$. Note: The Fourier series will converge to 0 at $x = 0$ because the discontinuity in f at $x = 0$ is removable.

17. $\int_{-L}^{L} \cos \frac{m\pi x}{L}\, dx = \frac{L}{m\pi} \sin \frac{m\pi x}{L} \Big|_{-L}^{L} = \frac{L}{m\pi}[\sin m\pi - \sin(-m\pi)] = \frac{L}{m\pi}(0 - 0) = 0.$

18. $\int_{-L}^{L} \sin \frac{m\pi x}{L}\, dx = -\frac{L}{m\pi} \cos \frac{m\pi x}{L} \Big|_{-L}^{L} = -\frac{L}{m\pi}[\cos m\pi - \cos(-m\pi)] = -\frac{L}{m\pi}(\cos m\pi - \cos m\pi) = 0.$

19. $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$

If $m \neq n$,

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx = \frac{1}{2} \int_{-L}^{L} \left[\cos \frac{(m+n)\pi x}{L} + \cos \frac{(n-m)\pi x}{L}\right] dx = 0, \text{ by exercise 17}$$

If $m = n$,

$$\begin{aligned}\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx &= \frac{1}{2} \int_{-L}^{L} \left(\cos \frac{2m\pi x}{L} + 1\right) dx = \frac{1}{2} \int_{-L}^{L} \cos \frac{2m\pi x}{L}\, dx + \frac{1}{2} \int_{-L}^{L} dx \\ &= 0 + L, \quad \text{by exercise 17} \\ &= L.\end{aligned}$$

20. $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

If $m \neq n$,

$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = \frac{1}{2} \int_{-L}^{L} \left[\cos \frac{(n-m)\pi x}{L} - \cos \frac{(n+m)\pi x}{L}\right] dx = 0, \text{ by exercise 17}$$

If $m = n$,

$$\begin{aligned}\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{mx\pi}{L}\, dx &= \frac{1}{2} \int_{-L}^{L} \left(1 - \cos \frac{2m\pi x}{L}\right) dx = \frac{1}{2} \int_{-L}^{L} dx - \frac{1}{2} \int_{-L}^{L} \cos \frac{2m\pi x}{L}\, dx \\ &= L - 0, \quad \text{by exercise 17} \\ &= L.\end{aligned}$$

21. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

If $m \neq n$,

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left[\sin\frac{(n+m)\pi x}{L} + \sin\frac{(n-m)\pi x}{L}\right]dx = 0, \text{ by exercise 18}$$

If $m = n$,

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(\sin\frac{2m\pi x}{L} + 0\right)dx = 0, \text{ by exercise 18}$$

22. If two functions, f and g, are piecewise continuous on an interval f, then so is $f+g$. This is true because of the properties of limits: $\lim_{x\to c^+}[f(x)+g(x)] = \lim_{x\to c^+} f(x) + \lim_{x\to c^+} g(x) = f(c^+) + g(c^+)$ and $\lim_{x\to c^-}[f(x)+g(x)]$ $= \lim_{x\to c^-} f(x) + \lim_{x\to c^-} g(x) = f(c^-) + g(c^-)$. Therefore, if f and g are piecewise continuous on I, then so is $f+g$. This result also applies to the functions f' and g', that is, if f' and g' are piecewise continuous on I, then so is $f'+g' = (f+g)'$. Consequently, Theorem 18 applies to $f+g$, and $f+g$ is equal to its Fourier series at all points of continuity, and at jump discontinuities in $f+g$, the Fourier series converges to the average $\frac{(f+g)(c^+)+(f+g)(c^-)}{2} = \frac{f(c^+)+f(c^-)}{2} + \frac{g(c^+)+g(c^-)}{2}$ where $f(c^+)$, $f(c^-)$, $g(c^+)$, and $g(c^-)$ denote the right and left limits of f and g at c.

23. (a) Since the function $f(x) = x$ and its derivative $f'(x) = 1$ are continuous on $-\pi < x < \pi$, the function f satisfies conditions of Theorem 18, and $f(x) = x = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin(nx)$.

(b) $\frac{d}{dx}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\frac{d}{dx}(\sin(nx)) = \sum_{n=1}^{\infty}(-1)^{n+1}\,2\cos(nx)$

This series diverges by the n^{th} term test because $\lim_{x\to\infty}\left((-1)^{n+1}\,2\cos(nx)\right) \neq 0$.

(c) We cannot be assured that term-by-term differentiation of the Fourier series of a piecewise continuous function gives a Fourier series that converges on the derivative of the function and, in fact, the series might not converge at all.

24. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ since f is piecewise continuous on $-\pi < x < \pi$. Therefore,

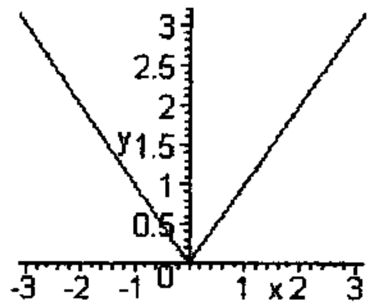
$$\int_{-\pi}^{x} f(s)\,ds = \int_{-\pi}^{x}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(ns) + b_n\sin(ns)]\right]ds = \int_{-\pi}^{x}\frac{a_0}{2}\,ds + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{x}\cos(ns)\,ds + b_n\int_{-\pi}^{x}\sin(ns)\,ds\right]$$

$$= \frac{a_0}{2}(x+\pi) + \sum_{n=1}^{\infty}\left[\frac{a_n}{n}(\sin(nx) - \sin(-n\pi)) - \frac{b_n}{n}(\cos(nx) - \cos(-n\pi)\right]$$

$$= \frac{a_0}{2}(x+\pi) + \sum_{n=1}^{\infty}\frac{1}{n}(a_n\sin(nx) - b_n(\cos(nx) - \cos(n\pi)))$$

8.10 FOURIER COSINE AND SINE SERIES

1. $a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}x^2\Big|_0^{\pi} = \pi$



Even extension of $f(x)$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi}$$

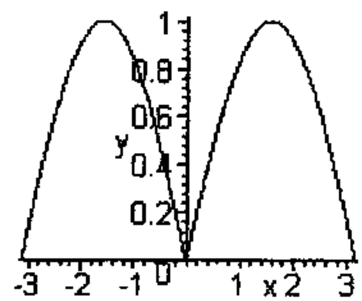
$$= \frac{2}{\pi}\left[\frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1]$$

$b_n = 0$

Therefore,

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos nx.$$

2. $a_0 = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx = \frac{2}{\pi}[-\cos x]_0^{\pi} = \frac{4}{\pi}$; $a_1 = \frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx = 0$;



Even extension of $f(x)$

For n = ≥ 2, $a_n = \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$

$$= \frac{2}{\pi}\left(\frac{n^2}{n^2-1}\right)\left[\frac{1}{n}\sin x\sin nx + \frac{1}{n^2}\cos x\cos nx\right]_0^{\pi}$$

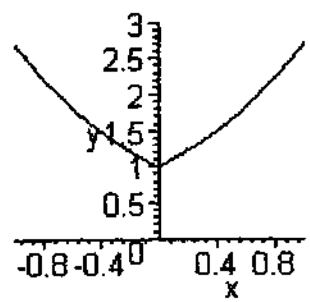
$$= \frac{2}{\pi}\left(\frac{n^2}{n^2-1}\right)\left(\frac{1}{n^2}\cos\pi\cos n\pi - +\frac{1}{n^2}\right) = \frac{2}{\pi(n^2-1)}[(-1)^{n+1} - 1]$$

$b_n = 0$

Therefore

$$f(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{[(-1)^n + 1]}{1 - n^2}\cos nx.$$

3. $a_0 = 2\int_0^{1} e^x\,dx = 2e^x\Big|_0^1 = 2(e-1)$



Even extension of $f(x)$

$$a_n = 2\int_0^{1} e^x\cos n\pi x\,dx$$

$$= 2\left(\frac{n^2\pi^2}{1+n^2\pi^2}\right)\left[e^x\left(\frac{1}{n\pi}\sin n\pi x + \frac{1}{n^2\pi^2}\cos n\pi x\right)\right]_0^1$$

$$= \frac{2}{1+n^2\pi^2}(e\cos n\pi - 1) = \frac{2[e(-1)^n - 1]}{1+n^2\pi^2}$$

$b_n = 0$

Therefore

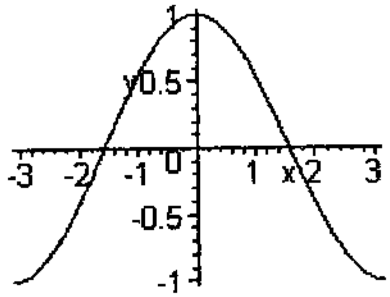
$$f(x) = (e-1) + 2\sum_{n=1}^{\infty}\frac{[e(-1)^n - 1]}{1+n^2\pi^2}\cos n\pi x.$$

4. $a_0 = \frac{2}{\pi}\int_0^{\pi} \cos x\, dx = \frac{2}{\pi}\sin x\Big|_0^{\pi} = 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos x\ \cos nx\, dx$$
$$= \frac{2}{\pi}\left(\frac{n^2}{n^2-1}\right)\left[\frac{1}{n}\cos x\sin nx - \frac{1}{n^2}\sin x\cos nx\right]_0^{\pi}$$
$$= 0,\ \text{if } n \neq 1.$$

For $n = 1$:
$$a_1 = \frac{2}{\pi}\int_0^{\pi}\cos^2 x\, dx = \frac{2}{\pi}\left[\frac{x}{2}+\frac{1}{4}\sin 2x\right]_0^{\pi} = 1$$

Therefore,
$$f(x) = a_1\cos\frac{\pi x}{\pi} = \cos x.$$

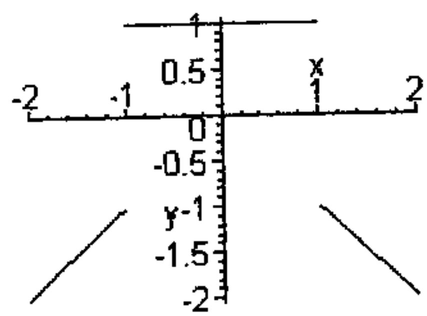


Even extension of $f(x)$

5. $a_0 = \frac{2}{2}\int_0^1 dx + \frac{2}{2}\int_1^2 -x\, dx = 1 + \left(-\frac{1}{2}x^2\right|_1^2 = -\frac{1}{2}$

$$a_n = \int_0^1 \cos\frac{n\pi x}{2}\, dx + \int_1^2 -x\cos\frac{n\pi x}{2}\, dx$$
$$= \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^1 - \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_1^2$$
$$= \frac{2}{n\pi}\sin\frac{n\pi}{2} - \frac{4}{n\pi}\sin n\pi - \frac{4}{n^2\pi^2}\cos n\pi + \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}$$
$$= \frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\left[(-1)^{n+1} + \cos\frac{n\pi}{2}\right]$$

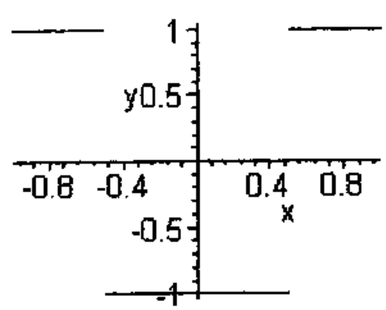
Therefore,
$$f(x) = -\frac{1}{4} + \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\frac{n\pi}{2} + \frac{1}{\pi n^2}\left((-1)^{n+1} + \cos\frac{n\pi}{2}\right)\right]\cos\frac{n\pi x}{2}.$$



Even extension of $f(x)$

6. $a_0 = \frac{2}{1}\int_0^{1/2} -dx + \frac{2}{1}\int_{1/2}^1 dx = 0$

$$a_n = 2\int_0^{1/2} -\cos n\pi x\, dx + 2\int_{1/2}^1 \cos n\pi x\, dx$$
$$= -\frac{2}{n\pi}\sin n\pi x\Big|_0^{1/2} + \frac{2}{n\pi}\sin n\pi x\Big|_{1/2}^1$$
$$= \frac{2}{n\pi}\left(-\sin\frac{n\pi}{2} + \sin n\pi - \sin\frac{n\pi}{2}\right)$$
$$= -\frac{4}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} -\frac{4}{n\pi}(-1)^k, & \text{if } n = 2k+1 \quad (\text{odd}) \\ 0, & \text{if } n = 2k \quad (\text{even}) \end{cases}$$

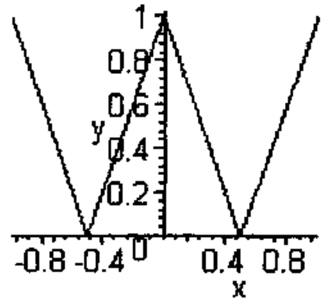


Even extension of $f(x)$

Thus,

$$f(x) = \sum_{k=0}^{\infty} \frac{4(-1)^{k+1}}{\pi(2k+1)} \cos(2k+1)\pi x.$$

7. $$a_0 = 2\int_0^{1/2} -(2x-1)\,dx + 2\int_{1/2}^{1} (2x-1)\,dx = 1$$



Even extension of $f(x)$

$$a_n = 2\int_0^{1/2} -(2x-1)\cos n\pi x\,dx + 2\int_{1/2}^{1} (2x-1)\cos n\pi x\,dx$$

$$= 2\left[\frac{(1-2x)}{n\pi}\sin n\pi x - \frac{2}{n^2\pi^2}\cos n\pi x\right]_0^{1/2}$$

$$+ 2\left[\frac{(2x-1)}{n\pi}\sin n\pi x + \frac{2}{n^2\pi^2}\cos n\pi x\right]_{1/2}^{1}$$

$$= 2\left[0 - \frac{2}{n^2\pi^2}\cos\frac{n\pi}{2} - 0 + \frac{2}{n^2\pi^2}\right]$$

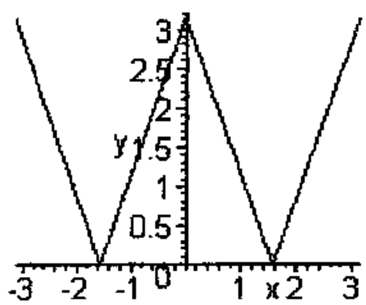
$$+ 2\left[0 + \frac{2}{n^2\pi^2}\cos n\pi - 0 - \frac{2}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{4}{n^2\pi^2}\left[1 - 2\cos\frac{n\pi}{2} + (-1)^n\right]$$

Therefore,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\right]\cos n\pi x.$$

8. $$a_0 = \frac{2}{\pi}\int_0^{\pi/2} -(2x-\pi)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (2x-\pi)\,dx = \pi$$



Even extension of $f(x)$

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} -(2x-\pi)\cos nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (2x-\pi)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{(\pi-2x)}{n}\sin nx - \frac{2}{n^2}\cos nx\right]_0^{\pi/2}$$

$$+ \frac{2}{\pi}\left[\frac{(2x-\pi)}{n}\sin nx + \frac{2}{n^2}\cos nx\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[0 - \frac{2}{n^2}\cos\frac{n\pi}{2} - 0 + \frac{2}{n^2}\right] + \frac{2}{\pi}\left[0 + \frac{2}{n^2}\cos n\pi - 0 - \frac{2}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{n^2}[1 + (-1)^n] - \frac{2}{n^2}\cos\frac{n\pi}{2}\right]$$

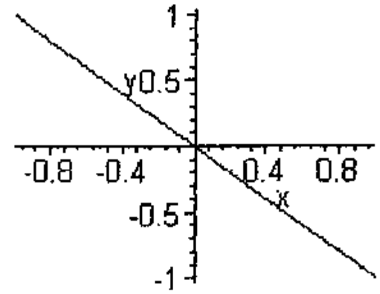
Therefore,

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\left[(-1)^n + 1 - 2\cos\frac{n\pi}{2}\right]\cos nx.$$

9. $$b_n = 2\int_0^1 -x\sin n\pi x\,dx = 2\left[\frac{x}{n\pi}\cos n\pi x - \frac{1}{n^2\pi^2}\sin n\pi x\right]_0^1 = \frac{2}{n\pi}\cos n\pi = \frac{2(-1)^n}{n\pi}$$

Therefore,

$$f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi x.$$



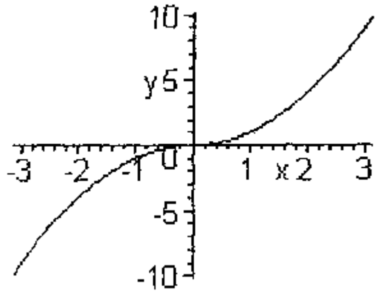
Odd extension of $f(x)$

10. $$b_n = \frac{2}{\pi}\int_0^{\pi} x^2\sin nx\,dx = \frac{2}{\pi}\left[-\frac{x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2}{n^3}\cos n\pi - \frac{2}{n^3}\right] = \frac{2}{\pi}\left[(-1)^n\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right)-\frac{2}{n^3}\right]$$

Therefore,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[(-1)^n\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right)-\frac{2}{n^3}\right]\sin nx.$$



Odd extension of $f(x)$

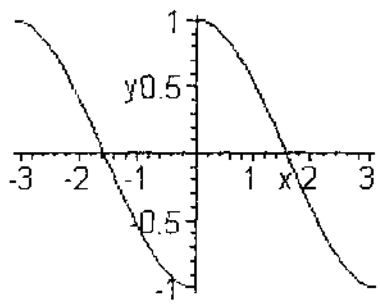
11. $$b_n = \frac{2}{\pi}\int_0^{\pi}\cos x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{n^2}{n^2-1}\left(-\frac{1}{n}\cos x\cos nx - \frac{1}{n^2}\sin x\sin nx\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{n}{n^2-1}(-\cos\pi\cos n\pi + 1)\right] = \frac{2n}{\pi(n^2-1)}[(-1)^n+1]$$

Therefore,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n\left[(-1)^n+1\right]}{n^2-1}\sin nx = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{k}{4k^2-1}\sin 2kx.$$



Odd extension of $f(x)$

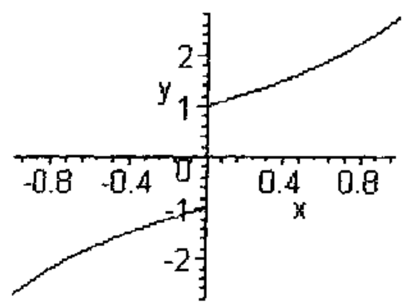
12. $$b_n = 2\int_0^1 e^x\sin n\pi x\,dx$$

$$= 2\left[\frac{n^2\pi^2}{n^2\pi^2+1}\cdot e^x\left(-\frac{1}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right)\right]_0^1$$

$$= \frac{2n^2\pi^2}{n^2\pi^2+1}\left(-\frac{e}{n\pi}\cos n\pi + \frac{1}{n\pi}\right) = \frac{2n\pi}{1+n^2\pi^2}\left[e(-1)^{n+1}+1\right]$$

Therefore,

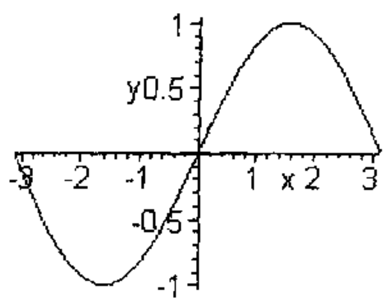
$$f(x) = 2\pi\sum_{n=1}^{\infty}\frac{\left[e(-1)^{n+1}+1\right]n}{1+n^2\pi^2}\sin n\pi x.$$



Odd extension of $f(x)$

13. $$b_n = \frac{2}{\pi}\int_0^{\pi}\sin x\sin nx\,dx$$

$$= 0, \text{ if } n \neq 1$$



Odd extension of $f(x)$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \, dx = \frac{2}{\pi}\left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1.$$

Therefore, $f(x) = b_1 \sin x = \sin x$.

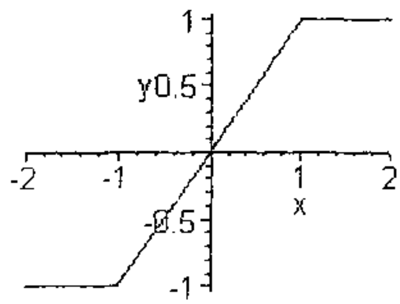
14. $$b_n = \int_0^1 x \sin\frac{n\pi x}{2}\, dx + \int_1^2 \sin\frac{n\pi x}{2}\, dx$$

$$= \left[-\frac{2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^1 - \left[\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_1^2$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{2}{n\pi}\cos n\pi + \frac{2}{n\pi}\cos\frac{n\pi}{2}$$

$$= \frac{2}{n\pi}(-1)^{n+1} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Therefore, $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n} + \frac{2}{n^2\pi}\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{2}$.



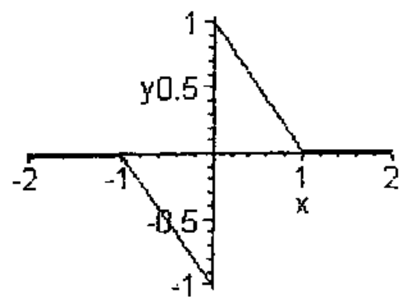
Odd extension of $f(x)$

15. $$b_n = \int_0^1 (1-x)\sin\frac{n\pi x}{2}\, dx$$

$$= \left[-\frac{2(1-x)}{n\pi}\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^1 = \frac{-4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}.$$

Therefore,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{2}{n^2\pi}\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{2}.$$



Odd extension of $f(x)$

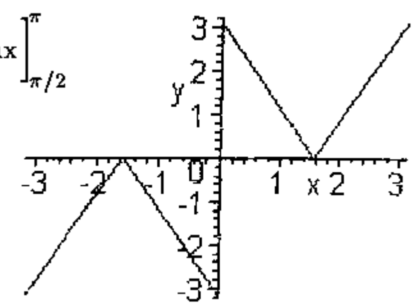
16. $$b_n = \frac{2}{\pi}\int_0^{\pi/2} -(2x-\pi)\sin nx\, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(2x-\pi)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[\frac{(2x-\pi)}{n}\cos nx - \frac{2}{n^2}\sin nx\right]_0^{\pi/2} + \frac{2}{\pi}\left[\frac{(\pi-2x)}{n}\cos nx + \frac{2}{n^2}\sin nx\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{n} - \frac{\pi}{n}\cos n\pi - \frac{2}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}[(-1)^{n+1}+1] - \frac{4}{n^2}\sin\frac{n\pi}{2}\right].$$

Therefore, $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\pi}{n}[(-1)^{n+1}+1] - \frac{4}{n^2}\sin\frac{n\pi}{2}\right]\sin nx$.



Odd extension of $f(x)$

17. (a) $b_n = \frac{2}{\pi}\int_0^{\pi}\sin nx\, dx = \frac{2}{n\pi}[-\cos nx]_0^{\pi} = \frac{2}{n\pi}(-\cos n\pi + 1) = \frac{2}{n\pi}[1-(-1)^n] \Rightarrow f(x) = \sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi}\sin nx$

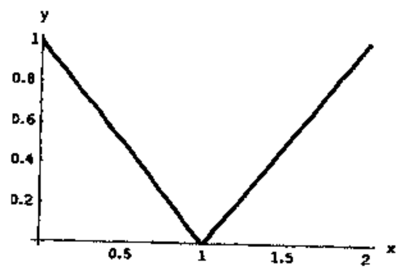
$$= \sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin[(2n-1)x] = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \ldots$$

$$\Rightarrow f(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots\right] \Rightarrow \frac{\pi}{4}f(x) = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots.$$

(b) Evaluate $f(x)$ at $x = \frac{\pi}{2} \Rightarrow \frac{\pi}{4}\cdot 1 = \sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \frac{1}{7}\sin\left(\frac{7\pi}{2}\right) + \ldots$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

18. (a)



(b) Use the even extension of f(x) over the interval $-2 < x < 2$.

$$a_0 = \frac{2}{2} \int_0^2 f(x)\,dx = \frac{2}{2} \cdot 1 = 1;$$

$$a_n = \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx = \int_0^1 (1-x) \cos\left(\frac{n\pi x}{2}\right) dx + \int_1^2 (x-1) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^1 \cos\left(\frac{n\pi x}{2}\right) dx - \int_1^2 \cos\left(\frac{n\pi x}{2}\right) dx - \int_0^1 x \cos\left(\frac{n\pi x}{2}\right) dx + \int_1^2 x \cos\left(\frac{n\pi x}{2}\right) dx$$

Evaluate the two integrals:

$$\int \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

$$\int x \cos\left(\frac{n\pi x}{2}\right) dx = \begin{bmatrix} u = x;\ dv = \cos\left(\frac{n\pi x}{2}\right) dx \\ du = dx;\ v = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \end{bmatrix} = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \frac{2}{n\pi} \int \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$$

Therefore, $a_n = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 - \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)\right]\Big|_0^1$

$$+ \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)\right]\Big|_1^2 = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right)$$

$$+ \frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2} \cos n\pi - \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) = \frac{4}{n^2\pi^2}\left[1 + \cos n\pi - 2\cos\left(\frac{n\pi}{2}\right)\right]$$

The b_n's are all zero since the Fourier series is for an even extension of f(x).

Table of coefficient values:

n	0	1	2	3	4	5	6	7	8	9	10	...
a_n	1	0	$\frac{4}{\pi^2}$	0	0	0	$\frac{4}{9\pi^2}$	0	0	0	$\frac{4}{25\pi^2}$	...

Therefore, the Fourier series representation of f(x) is:

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\frac{(4m-2)\pi x}{2} = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\big((2m-1)\pi x\big)$$

(c) Same answer as in part (b).

19. $f(x) = \sin x = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n+1\right]}{1-n^2}\cos nx = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2nx)$ for $0 < x < \pi$

Evaluate the function and its series representation at $x = \frac{\pi}{2}$.

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(n\pi) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{\pi}{4}\left(\frac{2}{\pi}-1\right) = \frac{1}{2} - \frac{\pi}{4}$$

20. Any piecewise continuous extension of f(x) over the interval $-2 < x < 2$ will give a Fourier series representation that will converge to f(x) in the interval $0 < x < 2$. For example, the function $g(x) = 2 - x$ for $-2 < x < 2$ will work.

CHAPTER 8 PRACTICE EXERCISES

1. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{(-1)^n}{n}\right) = 1$

2. converges to 0, since $0 \le a_n \le \frac{2}{\sqrt{n}}$, $\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty}\frac{2}{\sqrt{n}} = 0$ using the Sandwich Theorem for Sequences

3. converges to -1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1-2^n}{2^n}\right) = \lim_{n\to\infty}\left(\frac{1}{2^n} - 1\right) = -1$

4. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[1 + (0.9)^n\right] = 1 + 0 = 1$

5. diverges, since $\left\{\sin\frac{n\pi}{2}\right\} = \{0, 1, 0, -1, 0, 1, \ldots\}$

6. converges to 0, since $\{\sin n\pi\} = \{0, 0, 0, \ldots\}$

7. converges to 0, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln n^2}{n} = 2\lim_{n\to\infty}\frac{\left(\frac{1}{n}\right)}{1} = 0$

8. converges to 0, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln(2n+1)}{n} = \lim_{n\to\infty}\frac{\left(\frac{2}{2n+1}\right)}{1} = 0$

9. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+\ln n}{n}\right) = \lim_{n\to\infty}\frac{1+\left(\frac{1}{n}\right)}{1} = 1$

10. converges to 0, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln(2n^3+1)}{n} = \lim_{n\to\infty}\frac{\left(\frac{6n^2}{2n^3+1}\right)}{1} = \lim_{n\to\infty}\frac{12n}{6n^2} = \lim_{n\to\infty}\frac{2}{n} = 0$

11. converges to e^{-5}, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{n-5}{n}\right)^n = \lim\limits_{n\to\infty} \left(1+\frac{(-5)}{n}\right)^n = e^{-5}$ by Table 8.1

12. converges to $\frac{1}{e}$, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \lim\limits_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$ by Table 8.1

13. converges to 3, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{3^n}{n}\right)^{1/n} = \lim\limits_{n\to\infty} \frac{3}{n^{1/n}} = \frac{3}{1} = 3$ by Table 8.1

14. converges to 1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{3}{n}\right)^{1/n} = \lim\limits_{n\to\infty} \frac{3^{1/n}}{n^{1/n}} = \frac{1}{1} = 1$ by Table 8.1

15. converges to ln 2, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n\left(2^{1/n}-1\right) = \lim\limits_{n\to\infty} \frac{2^{1/n}-1}{\left(\frac{1}{n}\right)} = \lim\limits_{n\to\infty} \frac{\left[\frac{\left(-2^{1/n}\ln 2\right)}{n^2}\right]}{\left(\frac{-1}{n^2}\right)} = \lim\limits_{n\to\infty} 2^{1/n}\ln 2$

$= 2^0 \cdot \ln 2 = \ln 2$

16. converges to 1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt[n]{2n+1} = \lim\limits_{n\to\infty} \exp\left(\frac{\ln(2n+1)}{n}\right) = \lim\limits_{n\to\infty} \exp\left(\frac{\frac{2}{2n+1}}{1}\right) = e^0 = 1$

17. diverges, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(n+1)!}{n!} = \lim\limits_{n\to\infty} (n+1) = \infty$

18. converges to 0, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(-4)^n}{n!} = 0$ by Table 8.1

19. $\frac{1}{(2n-3)(2n-1)} = \frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1} \Rightarrow s_n = \left[\frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{5}\right] + \left[\frac{\left(\frac{1}{2}\right)}{5} - \frac{\left(\frac{1}{2}\right)}{7}\right] + \ldots + \left[\frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{2n-1}$

$\Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left[\frac{1}{6} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{1}{6}$

20. $\frac{-2}{n(n+1)} = \frac{-2}{n} + \frac{2}{n+1} \Rightarrow s_n = \left(\frac{-2}{2}+\frac{2}{3}\right) + \left(\frac{-2}{3}+\frac{2}{4}\right) + \ldots + \left(\frac{-2}{n}+\frac{2}{n+1}\right) = -\frac{2}{2} + \frac{2}{n+1} \Rightarrow \lim\limits_{n\to\infty} s_n$

$= \lim\limits_{n\to\infty} \left(-1+\frac{2}{n+1}\right) = -1$

21. $\frac{9}{(3n-1)(3n+2)} = \frac{3}{3n-1} - \frac{3}{3n+2} \Rightarrow s_n = \left(\frac{3}{2}-\frac{3}{5}\right) + \left(\frac{3}{5}-\frac{3}{8}\right) + \left(\frac{3}{8}-\frac{3}{11}\right) + \ldots + \left(\frac{3}{3n-1}-\frac{3}{3n+2}\right)$

$= \frac{3}{2} - \frac{3}{3n+2} \Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(\frac{3}{2}-\frac{3}{3n+2}\right) = \frac{3}{2}$

22. $\frac{-8}{(4n-3)(4n+1)} = \frac{-2}{4n-3} + \frac{2}{4n+1} \Rightarrow s_n = \left(\frac{-2}{9}+\frac{2}{13}\right) + \left(\frac{-2}{13}+\frac{2}{17}\right) + \left(\frac{-2}{17}+\frac{2}{21}\right) + \ldots + \left(\frac{-2}{4n-3}+\frac{2}{4n+1}\right)$

$= -\frac{2}{9} + \frac{2}{4n+1} \Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(-\frac{2}{9}+\frac{2}{4n+1}\right) = -\frac{2}{9}$

23. $\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \frac{1}{e^n}$, a convergent geometric series with $r = \frac{1}{e}$ and $a = 1 \Rightarrow$ the sum is $\frac{1}{1-\left(\frac{1}{e}\right)} = \frac{e}{e-1}$

24. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n} = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)\left(\frac{-1}{4}\right)^n$ a convergent geometric series with $r = -\frac{1}{4}$ and $a = \frac{-3}{4} \Rightarrow$ the sum is

$$\frac{\left(-\frac{3}{4}\right)}{1-\left(\frac{-1}{4}\right)} = -\frac{3}{5}$$

25. diverges, a p-series with $p = \frac{1}{2}$

26. $\sum_{n=1}^{\infty} \frac{-5}{n} = -5 \sum_{n=1}^{\infty} \frac{1}{n}$, diverges since it is a nonzero multiple of the divergent harmonic series

27. Since $f(x) = \frac{1}{x^{1/2}} \Rightarrow f'(x) = -\frac{1}{2x^{3/2}} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, the given series converges conditionally.

28. converges absolutely by the Direct Comparison Test since $\frac{1}{2n^3} < \frac{1}{n^3}$ for $n \ge 1$, which is the nth term of a convergent p-series

29. The given series does not converge absolutely by the Direct Comparison Test since $\frac{1}{\ln(n+1)} > \frac{1}{n+1}$, which is the nth term of a divergent series. Since $f(x) = \frac{1}{\ln(x+1)} \Rightarrow f'(x) = -\frac{1}{(\ln(x+1))^2(x+1)} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$, the given series converges conditionally by the Alternating Series Test.

30. $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^{1} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \left[-(\ln x)^{-1}\right]_2^b = -\lim_{b\to\infty} \left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Rightarrow$ the series converges absolutely by the Integral Test

31. converges absolutely by the Direct Comparison Test since $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$, the nth term of a convergent p-series

32. diverges by the Direct Comparison Test for $e^{n^n} > n \Rightarrow \ln\left(e^{n^n}\right) > \ln n \Rightarrow n^n > \ln n \Rightarrow \ln n^n > \ln(\ln n)$
$\Rightarrow n \ln n > \ln(\ln n) \Rightarrow \frac{\ln n}{\ln(\ln n)} > \frac{1}{n}$, the nth term of the divergent harmonic series

33. $\lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{n\sqrt{n^2+1}}\right)}{\left(\frac{1}{n^2}\right)} = \sqrt{\lim\limits_{n\to\infty} \dfrac{n^2}{n^2+1}} = \sqrt{1} = 1 \Rightarrow$ converges absolutely by the Limit Comparison Test

34. Since $f(x) = \dfrac{3x^2}{x^3+1} \Rightarrow f'(x) = \dfrac{3x(2-x^3)}{(x^3+1)^2} < 0$ when $x \geq 2 \Rightarrow a_{n+1} < a_n$ for $n \geq 2$ and $\lim\limits_{n\to\infty} \dfrac{3n^2}{n^3+1} = 0$, the series converges by the Alternating Series Test. The series does not converge absolutely: By the Limit Comparison Test, $\lim\limits_{n\to\infty} \dfrac{\left(\frac{3n^2}{n^3+1}\right)}{\left(\frac{1}{n}\right)} = \lim\limits_{n\to\infty} \dfrac{3n^3}{n^3+1} = 3$. Therefore the convergence is conditional.

35. converges absolutely by the Ratio Test since $\lim\limits_{n\to\infty} \left[\dfrac{n+2}{(n+1)!} \cdot \dfrac{n!}{n+1}\right] = \lim\limits_{n\to\infty} \dfrac{n+2}{(n+1)^2} = 0 < 1$

36. diverges since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{(-1)^n(n^2+1)}{2n^2+n-1}$ does not exist

37. converges absolutely by the Ratio Test since $\lim\limits_{n\to\infty} \left[\dfrac{3^{n+1}}{(n+1)!} \cdot \dfrac{n!}{3^n}\right] = \lim\limits_{n\to\infty} \dfrac{3}{n+1} = 0 < 1$

38. converges absolutely by the Root Test since $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{2^n 3^n}{n^n}} = \lim\limits_{n\to\infty} \frac{6}{n} = 0 < 1$

39. converges absolutely by the Limit Comparison Test since $\lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{\sqrt{n(n+1)(n+2)}}\right)}$

$= \sqrt{\lim\limits_{n\to\infty} \dfrac{n(n+1)(n+2)}{n^3}} = 1$

40. converges absolutely by the Limit Comparison Test since $\lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n\sqrt{n^2-1}}\right)} = \sqrt{\lim\limits_{n\to\infty} \dfrac{n^2(n^2-1)}{n^4}} = 1$

41. $\lim\limits_{n\to\infty} \left|\dfrac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\dfrac{(x+4)^{n+1}}{(n+1)3^{n+1}} \cdot \dfrac{n3^n}{(x+4)^n}\right| < 1 \Rightarrow \dfrac{|x+4|}{3} \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right) < 1 \Rightarrow \dfrac{|x+4|}{3} < 1$

$\Rightarrow |x+4| < 3 \Rightarrow -3 < x+4 < 3 \Rightarrow -7 < x < -1$; at $x = -7$ we have $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n 3^n}{n3^n} = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n}$, the alternating harmonic series, which converges conditionally; at $x = -1$ we have $\sum\limits_{n=1}^{\infty} \dfrac{3^n}{n3^n} = \sum\limits_{n=1}^{\infty} \frac{1}{n}$, the divergent harmonic series

(a) the radius is 3; the interval of convergence is $-7 \leq x < -1$

(b) the interval of absolute convergence is $-7 < x < -1$

(c) the series converges conditionally at $x = -7$

42. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{(x-1)^{2n}}{(2n+1)!} \cdot \frac{(2n-1)!}{(x-1)^{2n-2}}\right| < 1 \Rightarrow (x-1)^2 \lim\limits_{n\to\infty} \frac{1}{(2n)(2n+1)} = 0 < 1$, which holds for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

43. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{(3x-1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(3x-1)^n}\right| < 1 \Rightarrow |3x-1| \lim\limits_{n\to\infty} \frac{n^2}{(n+1)^2} < 1 \Rightarrow |3x-1| < 1$

$\Rightarrow -1 < 3x-1 < 1 \Rightarrow 0 < 3x < 2 \Rightarrow 0 < x < \frac{2}{3}$; at $x = 0$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n^2} = \sum\limits_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n^2}$

$= -\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, a nonzero constant multiple of a convergent p-series, which is absolutely convergent; at $x = \frac{2}{3}$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}(1)^n}{n^2} = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$, which converges absolutely

(a) the radius is $\frac{1}{3}$; the interval of convergence is $0 \le x \le \frac{2}{3}$

(b) the interval of absolute convergence is $0 \le x \le \frac{2}{3}$

(c) there are no values for which the series converges conditionally

44. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{n+2}{2n+3} \cdot \frac{(2x+1)^{n+1}}{2^{n+1}} \cdot \frac{2n+1}{n+1} \cdot \frac{2^n}{(2x+1)^n}\right| < 1 \Rightarrow \frac{|2x+1|}{2} \lim\limits_{n\to\infty} \left|\frac{n+2}{2n+3} \cdot \frac{2n+1}{n+1}\right| < 1$

$\Rightarrow \frac{|2x+1|}{2}(1) < 1 \Rightarrow |2x+1| < 2 \Rightarrow -2 < 2x+1 < 2 \Rightarrow -3 < 2x < 1 \Rightarrow -\frac{3}{2} < x < \frac{1}{2}$; at $x = -\frac{3}{2}$ we have

$\sum\limits_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{(-2)^n}{2^n} = \sum\limits_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n+1}$ which diverges by the nth-Term Test for Divergence since $\lim\limits_{n\to\infty} \left(\frac{n+1}{2n+1}\right) = \frac{1}{2} \ne 0$; at $x = \frac{1}{2}$ we have $\sum\limits_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{2^n}{2^n} = \sum\limits_{n=1}^{\infty} \frac{n+1}{2n+1}$, which diverges by the nth-Term Test

(a) the radius is 1; the interval of convergence is $-\frac{3}{2} < x < \frac{1}{2}$

(b) the interval of absolute convergence is $-\frac{3}{2} < x < \frac{1}{2}$

(c) there are no values for which the series converges conditionally

45. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{x^n}\right| < 1 \Rightarrow |x| \lim\limits_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right)\right| < 1 \Rightarrow \frac{|x|}{e} \lim\limits_{n\to\infty} \left(\frac{1}{n+1}\right) < 1$

$\Rightarrow \frac{|x|}{e} \cdot 0 < 1$, which holds for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

46. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} < 1 \Rightarrow |x| < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a divergent p-series

(a) the radius is 1; the interval of convergence is $-1 \le x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) the series converges conditionally at $x = -1$

47. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+2)x^{2n+1}}{3^{n+1}} \cdot \frac{3^n}{(n+1)x^{2n-1}}\right| < 1 \Rightarrow \frac{x^2}{3} \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right) < 1 \Rightarrow -\sqrt{3} < x < \sqrt{3}$; the series $\sum_{n=1}^{\infty} -\frac{n+1}{\sqrt{3}}$ and $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{3}}$, obtained with $x = \pm\sqrt{3}$, both diverge

(a) the radius is $\sqrt{3}$; the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$

(b) the interval of absolute convergence is $-\sqrt{3} < x < \sqrt{3}$

(c) there are no values for which the series converges conditionally

48. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-1)x^{2n+3}}{2n+3} \cdot \frac{2n+1}{(x-1)^{2n+1}}\right| < 1 \Rightarrow (x-1)^2 \lim_{n\to\infty} \left(\frac{2n+1}{2n+3}\right) < 1 \Rightarrow (x-1)^2(1) < 1$ $\Rightarrow (x-1)^2 < 1 \Rightarrow |x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$; at $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1}$ $= \sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$ which converges conditionally by the Alternating Series Test and the fact that $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges; at $x = 2$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n(1)^{2n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$, which also converges conditionally

(a) the radius is 1; the interval of convergence is $0 \le x \le 2$

(b) the interval of absolute convergence is $0 < x < 2$

(c) the series converges conditionally at $x = 0$ and $x = 2$

49. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\operatorname{csch}(n+1)x^{n+1}}{\operatorname{csch}(n)x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\frac{\left(\frac{2}{e^{n+1} - e^{-n-1}}\right)}{\left(\frac{2}{e^n - e^{-n}}\right)}\right| < 1$

$\Rightarrow |x| \lim_{n\to\infty} \left|\frac{e^{-1} - e^{-2n-1}}{1 - e^{-2n-2}}\right| < 1 \Rightarrow \frac{|x|}{e} < 1 \Rightarrow -e < x < e$; the series $\sum_{n=1}^{\infty} (\pm e)^n \operatorname{csch} n$, obtained with $x = \pm e$, both diverge since $\lim\limits_{n\to\infty} (\pm e)^n \operatorname{csch} n \neq 0$

(a) the radius is e; the interval of convergence is $-e < x < e$

(b) the interval of absolute convergence is $-e < x < e$

(c) there are no values for which the series converges conditionally

50. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{x^{n+1}\coth(n+1)}{x^n \coth(n)}\right| < 1 \Rightarrow |x| \lim\limits_{n\to\infty} \left|\frac{1+e^{-2n-2}}{1-e^{-2n-2}} \cdot \frac{1-e^{-2n}}{1+e^{-2n}}\right| < 1 \Rightarrow |x| < 1$

$\Rightarrow -1 < x < x$; the series $\sum\limits_{n=1}^{\infty} (\pm 1)^n \coth n$, obtained with $x = \pm 1$, both diverge since $\lim\limits_{n\to\infty} (\pm 1)^n \coth n \neq 0$

(a) the radius is 1; the interval of convergence is $-1 < x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) there are no values for which the series converges conditionally

51. The given series has the form $1 - x + x^2 - x^3 + \ldots + (-x)^n + \ldots = \frac{1}{1+x}$, where $x = \frac{1}{4}$; the sum is $\frac{1}{1+\left(\frac{1}{4}\right)} = \frac{4}{5}$

52. The given series has the form $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots = \ln(1+x)$, where $x = \frac{2}{3}$; the sum is $\ln\left(\frac{5}{3}\right) \approx 0.510825624$

53. The given series has the form $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sin x$, where $x = \pi$; the sum is $\sin \pi = 0$

54. The given series has the form $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots = \cos x$, where $x = \frac{\pi}{3}$; the sum is $\cos\frac{\pi}{3} = \frac{1}{2}$

55. The given series has the form $1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^n}{n!} + \ldots = e^x$, where $x = \ln 2$; the sum is $e^{\ln(2)} = 2$

56. The given series has the form $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^n \frac{x^{2n-1}}{(2n-1)} + \ldots = \tan^{-1} x$, where $x = \frac{1}{\sqrt{3}}$; the sum is $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

57. Consider $\frac{1}{1-2x}$ as the sum of a convergent geometric series with $a = 1$ and $r = 2x \Rightarrow \frac{1}{1-2x}$

$= 1 + (2x) + (2x)^2 + (2x)^3 + \ldots = \sum\limits_{n=0}^{\infty} (2x)^n = \sum\limits_{n=0}^{\infty} 2^n x^n$ where $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

58. Consider $\frac{1}{1+x^3}$ as the sum of a convergent geometric series with $a = 1$ and $r = -x^3 \Rightarrow \frac{1}{1+x^3} = \frac{1}{1-(-x^3)}$

$= 1 + (-x^3) + (-x^3)^2 + (-x^3)^3 + \ldots = \sum\limits_{n=0}^{\infty} (-1)^n x^{3n}$ where $|-x^3| < 1 \Rightarrow |x^3| < 1$

59. $\sin x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \pi x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{(2n+1)!}$

60. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \frac{2x}{3} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{2x}{3}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{3^{2n+1}(2n+1)!}$

61. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos\left(x^{5/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{5/2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n}}{(2n)!}$

62. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos \sqrt{5x} = \cos\left((5x)^{1/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left((5x)^{1/2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^n}{(2n)!}$

63. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{(\pi x/2)} = \sum_{n=0}^{\infty} \frac{\left(\frac{\pi x}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\pi^n x^n}{2^n n!}$

64. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

65. $f(x) = \sqrt{3+x^2} = \left(3+x^2\right)^{1/2} \Rightarrow f'(x) = x\left(3+x^2\right)^{-1/2} \Rightarrow f''(x) = -x^2\left(3+x^2\right)^{-3/2} + \left(3+x^2\right)^{-1/2}$
$\Rightarrow f'''(x) = 3x^3\left(3+x^2\right)^{-5/2} - 3x\left(3+x^2\right)^{-3/2}$; $f(-1) = 2$, $f'(-1) = -\frac{1}{2}$, $f''(-1) = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$,
$f'''(-1) = -\frac{3}{32} + \frac{3}{8} = \frac{9}{32} \Rightarrow \sqrt{3+x^2} = 2 - \frac{(x+1)}{2 \cdot 1!} + \frac{3(x+1)^2}{2^3 \cdot 2!} + \frac{9(x+1)^3}{2^5 \cdot 3!} + \ldots$

66. $f(x) = \frac{1}{1-x} = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f'''(x) = 6(1-x)^{-4}$; $f(2) = -1$, $f'(2) = 1$,
$f''(2) = -2$, $f'''(2) = 6 \Rightarrow \frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 - \ldots$

67. $f(x) = \frac{1}{x+1} = (x+1)^{-1} \Rightarrow f'(x) = -(x+1)^{-2} \Rightarrow f''(x) = 2(x+1)^{-3} \Rightarrow f'''(x) = -6(x+1)^{-4}$; $f(3) = \frac{1}{4}$,
$f'(3) = -\frac{1}{4^2}$, $f''(3) = \frac{2}{4^3}$, $f'''(2) = \frac{-6}{4^4} \Rightarrow \frac{1}{x+1} = \frac{1}{4} - \frac{1}{4^2}(x-3) + \frac{1}{4^3}(x-3)^2 - \frac{1}{4^4}(x-3)^3 + \ldots$

68. $f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4}$; $f(a) = \frac{1}{a}$, $f'(a) = -\frac{1}{a^2}$, $f''(a) = \frac{2}{a^3}$,
$f'''(a) = \frac{-6}{a^4} \Rightarrow \frac{1}{x} = \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + \ldots$

69. Assume the solution has the form $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1}x^{n-1} + a_n x^n + \ldots$
$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2 x + \ldots + na_n x^{n-1} + \ldots \Rightarrow \frac{dy}{dx} + y$
$= \left(a_1 + a_0\right) + \left(2a_2 + a_1\right)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = 0 \Rightarrow a_1 + a_0 = 0$, $2a_2 + a_1 = 0$,
$3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$. Since $y = -1$ when $x = 0$ we have $a_0 = -1$. Therefore $a_1 = 1$,
$a_2 = \frac{-a_1}{2 \cdot 1} = -\frac{1}{2}$, $a_3 = \frac{-a_2}{3} = \frac{1}{3 \cdot 2}$, $a_4 = \frac{-a_3}{4} = -\frac{1}{4 \cdot 3 \cdot 2}$, …, $a_n = \frac{-a_{n-1}}{n} = \frac{-1}{n} \frac{(-1)^n}{(n-1)!} = \frac{(-1)^{n+1}}{n!}$
$\Rightarrow y = -1 + x - \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 - \ldots + \frac{(-1)^{n+1}}{n!}x^n + \ldots = -\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = -e^{-x}$

70. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} - y$

$= (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = 0 \Rightarrow a_1 - a_0 = 0,\ 2a_2 - a_1 = 0,$ $3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$. Since $y = -3$ when $x = 0$ we have $a_0 = -3$. Therefore $a_1 = -3,$ $a_2 = \frac{a_1}{2} = \frac{-3}{2},\ a_3 = \frac{a_2}{3} = \frac{-3}{3\cdot 2},\ a_n = \frac{a_{n-1}}{n} = \frac{-3}{n!} \Rightarrow y = -3 - 3x - \frac{3}{2\cdot 1}x^2 - \frac{3}{3\cdot 2}x^3 - \ldots - \frac{-3}{n!}x^n + \ldots$

$= -3\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots\right) = -3\sum_{n=0}^{\infty} \frac{x^n}{n!} = -3e^x$

71. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} + 2y$

$= (a_1 + 2a_0) + (2a_2 + 2a_1)x + (3a_3 + 2a_2)x^2 + \ldots + (na_n + 2a_{n-1})x^{n-1} + \ldots = 0$. Since $y = 3$ when $x = 0$ we have $a_0 = 3$. Therefore $a_1 = -2a_0 = -2(3) = -3(2),\ a_2 = -\frac{2}{2}a_1 = -\frac{2}{2}(-2\cdot 3) = 3\left(\frac{2^2}{2}\right),\ a_3 = -\frac{2}{3}a_2$

$= -\frac{2}{3}\left[3\left(\frac{2^2}{2}\right)\right] = -3\left(\frac{2^3}{3\cdot 2}\right),\ \ldots,\ a_n = \left(-\frac{2}{n}\right)a_{n-1} = \left(-\frac{2}{n}\right)\left(3\left(\frac{(-1)^{n-1}2^{n-1}}{(n-1)!}\right)\right) = 3\left(\frac{(-1)^n2^n}{n!}\right)$

$\Rightarrow y = 3 - 3(2x) + 3\frac{(2)^2}{2}x^2 - 3\frac{(2)^3}{3\cdot 2}x^3 + \ldots + 3\frac{(-1)^n2^n}{n!}x^n + \ldots$

$= 3\left[1 - (2x) + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \ldots + \frac{(-1)^n(2x)^n}{n!} + \ldots\right] = 3\sum_{n=0}^{\infty} \frac{(-1)^n(2x)^n}{n!} = 3e^{-2x}$

72. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} + y$

$= (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = 1 \Rightarrow a_1 + a_0 = 1,\ 2a_2 + a_1 = 0,$ $3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$ for $n > 1$. Since $y = 0$ when $x = 0$ we have $a_0 = 0$. Therefore $a_1 = 1 - a_0 = 1,\ a_2 = \frac{-a_1}{2\cdot 1} = -\frac{1}{2},\ a_3 = \frac{-a_2}{3} = \frac{1}{3\cdot 2},\ a_4 = \frac{-a_3}{4} = -\frac{1}{4\cdot 3\cdot 2},\ \ldots,\ a_n$

$= \frac{-a_{n-1}}{n} = \left(\frac{-1}{n}\right)\frac{(-1)^n}{(n-1)!} = \frac{(-1)^{n+1}}{n!} \Rightarrow y = 0 + x - \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 - \ldots + \frac{(-1)^{n+1}}{n!}x^n + \ldots$

$= -1\left[1 - x + \frac{1}{2}x^2 - \frac{1}{3\cdot 2}x^3 - \ldots + \frac{(-1)^n}{n!}x^n + \ldots\right] + 1 = -\sum_{n=0}^{\infty} \frac{(-1)^nx^n}{n!} + 1 = 1 - e^{-x}$

73. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} - y$

$= (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = 3x \Rightarrow a_1 - a_0 = 0,\ 2a_2 - a_1 = 3,$ $3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$ for $n > 2$. Since $y = -1$ when $x = 0$ we have $a_0 = -1$. Therefore $a_1 = -1,\ a_2 = \frac{3 + a_1}{2} = \frac{2}{2},\ a_3 = \frac{a_2}{3} = \frac{2}{3\cdot 2},\ a_4 = \frac{a_3}{4} = \frac{2}{4\cdot 3\cdot 2},\ \ldots,\ a_n = \frac{a_{n-1}}{n} = \frac{2}{n!}$

$\Rightarrow y = -1 - x + \left(\frac{2}{2}\right)x^2 + \frac{3}{3 \cdot 2}x^3 + \frac{2}{4 \cdot 3 \cdot 2}x^4 + \ldots + \frac{2}{n!}x^n + \ldots$

$= 2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots\right) - 3 - 3x = 2\sum_{n=0}^{\infty} \frac{x^n}{n!} - 3 - 3x = 2e^x - 3x - 3$

74. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} + y$

$= (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = x \Rightarrow a_1 + a_0 = 0,\ 2a_2 + a_1 = 1,$

$3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$ for $n > 2$. Since $y = 0$ when $x = 0$ we have $a_0 = 0$. Therefore

$a_1 = 0,\ a_2 = \frac{1 - a_1}{2} = \frac{1}{2},\ a_3 = \frac{-a_2}{3} = -\frac{1}{3 \cdot 2},\ \ldots,\ a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!}$

$\Rightarrow y = 0 - 0x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots = \left(1 - x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots\right) - 1 + x$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} - 1 + x = e^{-x} + x - 1$

75. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} - y$

$= (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = x \Rightarrow a_1 - a_0 = 0,\ 2a_2 - a_1 = 1,$

$3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$ for $n > 2$. Since $y = 1$ when $x = 0$ we have $a_0 = 1$. Therefore

$a_1 = 1,\ a_2 = \frac{1 + a_1}{2} = \frac{2}{2},\ a_3 = \frac{a_2}{3} = \frac{2}{3 \cdot 2},\ a_4 = \frac{a_3}{4} = \frac{2}{4 \cdot 3 \cdot 2},\ \ldots,\ a_n = \frac{a_{n-1}}{n} = \frac{2}{n!}$

$\Rightarrow y = 1 + x + \left(\frac{2}{2}\right)x^2 + \frac{2}{3 \cdot 2}x^3 + \frac{2}{4 \cdot 2 \cdot 2}x^4 + \ldots + \frac{2}{n!}x^n + \ldots$

$= 2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots\right) - 1 - x = 2\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x = 2e^x - x - 1$

76. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} - y$

$= (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = -x \Rightarrow a_1 - a_0 = 0,\ 2a_2 - a_1 = -1,$

$3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$ for $n > 2$. Since $y = 2$ when $x = 0$ we have $a_0 = 2$. Therefore

$a_1 = 2,\ a_2 = \frac{-1 + a_1}{2} = \frac{1}{2},\ a_3 = \frac{a_2}{3} = \frac{1}{3 \cdot 2},\ a_4 = \frac{a_3}{4} = \frac{1}{4 \cdot 3 \cdot 2},\ \ldots,\ a_n = \frac{a_{n-1}}{n} = \frac{1}{n!}$

$\Rightarrow y = 2 + 2x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots$

$= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots\right) + 1 + x = \sum_{n=0}^{\infty} \frac{x^n}{n!} + 1 + x = e^x + x + 1$

77. $\lim_{x \to 0} \frac{7 \sin x}{e^{2x} - 1} = \lim_{x \to 0} \frac{7\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)}{\left(2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \ldots\right)} = \lim_{x \to 0} \frac{7\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right)}{\left(2 + \frac{2^2x}{2!} + \frac{2^3x^2}{3!} + \ldots\right)} = \frac{7}{2}$

78. $\lim\limits_{\theta\to 0} \frac{e^\theta - e^{-\theta} - 2\theta}{\theta - \sin\theta} = \lim\limits_{\theta\to 0} \frac{\left(1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\ldots\right)-\left(1-\theta+\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\ldots\right)-2\theta}{\theta-\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots\right)} = \lim\limits_{\theta\to 0} \frac{2\left(\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\ldots\right)}{\left(\frac{\theta^3}{3!}-\frac{\theta^5}{5!}+\ldots\right)}$

$= \lim\limits_{\theta\to 0} \frac{2\left(\frac{1}{3!}+\frac{\theta^2}{5!}+\ldots\right)}{\left(\frac{1}{3!}-\frac{\theta^2}{5!}+\ldots\right)} = 2$

79. $\lim\limits_{t\to 0} \left(\frac{1}{2-2\cos t}-\frac{1}{t^2}\right) = \lim\limits_{t\to 0} \frac{t^2-2+2\cos t}{2t^2(1-\cos t)} = \lim\limits_{t\to 0} \frac{t^2-2+2\left(1-\frac{t^2}{2}+\frac{t^4}{4!}-\ldots\right)}{2t^2\left(1-1+\frac{t^2}{2}-\frac{t^4}{4!}+\ldots\right)} = \lim\limits_{t\to 0} \frac{2\left(\frac{t^4}{4!}-\frac{t^6}{6!}+\ldots\right)}{\left(t^4-\frac{2t^6}{4!}+\ldots\right)}$

$= \lim\limits_{t\to 0} \frac{2\left(\frac{1}{4!}-\frac{t^2}{6!}+\ldots\right)}{\left(1-\frac{2t^2}{4!}+\ldots\right)} = \frac{1}{12}$

80. $\lim\limits_{h\to 0} \frac{\left(\frac{\sin h}{h}\right)-\cos h}{h^2} = \lim\limits_{h\to 0} \frac{\left(1-\frac{h^2}{3!}+\frac{h^4}{5!}-\ldots\right)-\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}-\ldots\right)}{h^2}$

$= \lim\limits_{h\to 0} \frac{\left(\frac{h^2}{2!}-\frac{h^2}{3!}+\frac{h^4}{5!}-\frac{h^4}{4!}+\frac{h^6}{6!}-\frac{h^6}{7!}+\ldots\right)}{h^2} = \lim\limits_{h\to 0} \left(\frac{1}{2!}-\frac{1}{3!}+\frac{h^2}{5!}-\frac{h^2}{4!}+\frac{h^4}{6!}-\frac{h^4}{7!}+\ldots\right) = \frac{1}{3}$

81. $\lim\limits_{z\to 0} \frac{1-\cos^2 z}{\ln(1-z)+\sin z} = \lim\limits_{z\to 0} \frac{1-\left(1-z^2+\frac{z^4}{3}-\ldots\right)}{\left(-z-\frac{z^2}{2}-\frac{z^3}{3}-\ldots\right)+\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\right)} = \lim\limits_{z\to 0} \frac{\left(z^2-\frac{z^4}{3}+\ldots\right)}{\left(-\frac{z^2}{2}-\frac{2z^3}{3}-\frac{z^4}{4}-\ldots\right)}$

$= \lim\limits_{z\to 0} \frac{\left(1-\frac{z^2}{3}+\ldots\right)}{\left(-\frac{1}{2}-\frac{2z}{3}-\frac{z^2}{4}-\ldots\right)} = -2$

82. $\lim\limits_{y\to 0} \frac{y^2}{\cos y-\cosh y} = \lim\limits_{y\to 0} \frac{y^2}{\left(1-\frac{y^2}{2}+\frac{y^4}{4!}-\frac{y^6}{6!}+\ldots\right)-\left(1+\frac{y^2}{2!}+\frac{y^4}{4!}+\frac{y^6}{6!}+\ldots\right)} = \lim\limits_{y\to 0} \frac{y^2}{\left(-\frac{2y^2}{2}-\frac{2y^6}{6!}-\ldots\right)}$

$= \lim\limits_{y\to 0} \frac{1}{\left(-1-\frac{2y^4}{6!}-\ldots\right)} = -1$

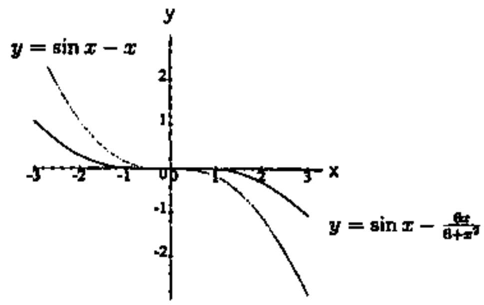
83. $\lim\limits_{x\to 0} \left(\frac{\sin 3x}{x^3}+\frac{r}{x^2}+s\right) = \lim\limits_{x\to 0} \left[\frac{\left(3x-\frac{(3x)^3}{6}+\frac{(3x)^5}{120}-\ldots\right)}{x^3}+\frac{r}{x^2}+s\right] = \lim\limits_{x\to 0} \left(\frac{3}{x^2}-\frac{9}{2}+\frac{81x^2}{40}+\ldots+\frac{r}{x^2}+s\right) = 0$

$\Rightarrow \frac{r}{x^2}+\frac{3}{x^2} = 0$ and $s-\frac{9}{2} = 0 \Rightarrow r = -3$ and $s = \frac{9}{2}$

84. (a) $\csc x \approx \frac{1}{x} + \frac{x}{6} \Rightarrow \csc x \approx \frac{6+x^2}{6x} \Rightarrow \sin x \approx \frac{6x}{6+x^2}$

(b) The approximation $\sin x \approx \frac{6x}{6+x^2}$ is better than

$\sin x \approx x$.



85. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 1$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\cos nx\,dx = -\frac{1}{n\pi}\sin nx\Big]_{-\pi}^{0} + \frac{2}{nx}\sin nx\Big]_{0}^{\pi} = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\sin nx\,dx = \frac{1}{n\pi}\cos nx\Big]_{-\pi}^{0} - \frac{2}{nx}\cos nx\Big]_{0}^{\pi}$

$= \frac{1}{n\pi}(1-\cos n\pi) - \frac{2}{n\pi}(\cos n\pi - 1) = \frac{3}{n\pi}(1-\cos n\pi) = \frac{3}{n\pi}(1-(-1)^n)$

Therefore, $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{3}{n\pi}(1-(-1)^n)\sin nx = \frac{1}{2} + \frac{6}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin[(2n-1)x]$

86. $a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 0\,dx + \int_{0}^{1} x\,dx = \frac{1}{2}$

$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{0}^{1} x\cos n\pi x\,dx = \left[\frac{\cos n\pi x}{n^2\pi^2} + \frac{x\sin n\pi x}{n\pi}\right]_{0}^{1} = \frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2} = -\frac{1}{n^2\pi^2}(1-(-1)^n)$

$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{0}^{1} x\sin n\pi x\,dx = \left[\frac{\sin n\pi x}{n^2\pi^2} - \frac{x\cos n\pi x}{n\pi}\right]_{0}^{1} = -\frac{\cos n\pi}{n\pi} - \frac{1}{n\pi}(-1)^n$

Therefore, $f(x) = \frac{1}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos n\pi x - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin n\pi x$

87. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+\pi)\,dx = \frac{1}{\pi}(2\pi^2) = 2\pi$

$a_n = 0$ because $f(x) - \pi$ is an odd function

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+\pi)\sin nx\,dx = \frac{1}{\pi}\left[\frac{\sin nx}{n^2} - \frac{(x+\pi)\cos nx}{n}\right]_{-\pi}^{\pi} = -\frac{2}{n}\cos n\pi = -\frac{2(-1)^n}{n}$

Therefore, $f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$

88. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}(-\cos x)\Big|_0^{\pi} = \frac{2}{\pi}$

$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x\,dx = 0 \qquad b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x\,dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]\Big|_0^{\pi} = \frac{1}{2}$

For $n \geq 2$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx = \frac{1}{\pi}\left[\frac{\cos[(n-1)x]}{2(n-1)} - \frac{\cos[(n+1)x]}{2(n+1)}\right]\Bigg|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{2(n-1)} + \frac{1}{2(n+1)} + \frac{\cos[(n-1)\pi]}{2(n-1)} - \frac{\cos[(n+1)\pi]}{2(n+1)}\right] = \frac{1+\cos n\pi}{(1-n^2)\pi} = \frac{1+(-1)^n}{(1-n^2)\pi}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\,dx = \frac{1}{\pi}\left[\frac{\sin[(n-1)x]}{2(n-1)} - \frac{\sin[(n+1)x]}{2(n+1)}\right]\Bigg|_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin[(n-1)\pi]}{2(n-1)} - \frac{\sin[(n+1)\pi]}{2(n+1)}\right] = 0$$

Therefore,

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{1-n^2}\cos nx = \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{1-(2n)^2}\cos[(2n)x]$$

89. $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}[2+4] = 3$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\int_{-2}^{0}\cos\left(\frac{n\pi x}{2}\right)dx + \int_0^2 (1+x)\cos\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \frac{1}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Bigg|_{-2}^{0} + \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{2(1+x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]\Bigg|_0^2 = \frac{2}{n^2\pi^2}[\cos n\pi - 1] = \frac{2\left((-1)^n - 1\right)}{n^2\pi^2}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\int_{-2}^{0}\sin\left(\frac{n\pi x}{2}\right)dx + \int_0^2 (1+x)\sin\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_{-2}^{0} + \left[-\frac{(1+x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right]\Bigg|_0^2 = \frac{1}{n\pi}(\cos n\pi - 1) + \frac{1}{n\pi}(1 - 3\cos n\pi)$$

$$= \frac{(-1)^n - 1}{n\pi} + \frac{1 - 3(-1)^n}{n\pi} = -\frac{2(-1)^n}{n\pi}$$

Therefore, $f(x) = \frac{3}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos\left(\frac{n\pi x}{2}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{2}\right)$

90. $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_0^1 x\,dx + \int_1^2 dx\right] = \frac{1}{2}\left[\frac{1}{2}+1\right] = \frac{3}{4}$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \left[\frac{2}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{1}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\right]\Bigg|_0^1 + \frac{1}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Bigg|_1^2 = \frac{2}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right)-1\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx + \int_1^2 \sin\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \left[\frac{2}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right) + \frac{1}{n\pi}x\cos\left(\frac{n\pi x}{2}\right)\right]\Bigg|_0^1 - \frac{1}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_1^2 = \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\cos n\pi$$

Therefore,

$$f(x) = \frac{3}{8} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right)-1}{n^2}\cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\cos n\pi\right)\sin\left(\frac{n\pi x}{2}\right)$$

91. (a) $a_0 = \frac{2}{1}\int_0^1 f(x)\,dx = 2\int_0^{1/2} dx = 1$; $a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\,dx = 2\int_0^{1/2}\cos n\pi x\,dx$

$$= \frac{2}{n\pi}\sin n\pi x\Big|_0^{1/2} = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right).\ \text{Therefore, } f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}\cos n\pi x$$

(b) $b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx = 2\int_0^{1/2}\sin n\pi x\,dx = -\frac{2}{n\pi}\cos n\pi x\Big|_0^{1/2} = \frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$

Therefore, $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)\sin n\pi x$

92. (a) $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_1^2 x\,dx = \frac{3}{2}$; $a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_1^2 x\cos\left(\frac{n\pi x}{2}\right)dx$

$$= \left[\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\right]\Bigg|_1^2 = \frac{4}{n^2\pi^2}\left[\cos n\pi - \cos\left(\frac{n\pi}{2}\right)\right] - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Therefore, $f(x) = \frac{3}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi}\left[\cos n\pi - \cos\left(\frac{n\pi}{2}\right)\right] - \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{n\pi x}{2}\right)$

(b) $b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_1^2 x\sin\left(\frac{n\pi x}{2}\right)dx = -\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi}\cos n\pi + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right).$

Therefore, $f(x) = \frac{2}{\pi}\sum\left(-\frac{2}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n}\cos n\pi + \frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi x}{2}\right)$

93. (a) $a_0 = \frac{2}{1}\int_0^1 f(x)\,dx = 2\int_0^1 \sin \pi x\,dx = -\frac{2}{\pi}\cos \pi x\Big|_0^1 = \frac{4}{\pi}$; $a_1 = \frac{2}{1}\int_0^1 f(x)\cos \pi x\,dx = 2\int_0^1 \sin \pi x \cos \pi x\,dx$

$= -\frac{1}{2\pi}\cos^2 \pi x\Big|_0^1 = 0$. For $n \geq 2$, $a_n = 2\int_0^1 f(x)\cos n\pi x\,dx = 2\int_0^1 \sin \pi x \cos n\pi x\,dx$

$$= \left[\frac{\cos[(n-1)\pi x]}{n-1} - \frac{\cos[(n+1)\pi x]}{n+1}\right]\Bigg|_0^1 = \frac{1}{\pi}\left[\frac{1}{n-1} + \frac{1}{n+1} + \frac{\cos[(n-1)\pi]}{n-1} - \frac{\cos[(n+1)\pi]}{n+1}\right]$$

Therefore, $f(x) = \frac{2}{\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{\cos[(n-1)\pi]}{n-1} - \frac{\cos[(n+1)\pi]}{n+1}\right]\cos n\pi x$

(b) $b_1 = \frac{2}{1}\int_0^1 f(x)\sin \pi x\,dx = 2\int_0^1 \sin^2 \pi x\,dx = \left[x - \frac{\sin 2\pi x}{2\pi}\right]\Big|_0^1 = 1$; $b_n = 0$ for $n \geq 2$.

Therefore, $f(x) = \sin \pi x$, as expected.

94. (a) $a_0 = \frac{4}{\pi}\int_0^{\pi/2} \cos x\,dx = \frac{4}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - 0\right] = \frac{4}{\pi}$; $a_n = \frac{4}{\pi}\int_0^{\pi/2} \cos x \cos 2nx\,dx$

$$= \frac{2}{\pi}\left[\frac{\sin[(2n-1)x]}{2n-1} + \frac{\sin[(2n+1)x]}{2n+1}\right]\Bigg|_0^{\pi/2} = \frac{2}{\pi}\left[\frac{\sin\left[\frac{(2n-1)\pi}{2}\right]}{2n-1} + \frac{\sin\left[\frac{(2n+1)\pi}{2}\right]}{2n+1}\right] = \frac{4(-1)^n}{\pi(1-4n^2)}$$

Therefore, $f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(1-4n^2)}\cos 2nx$

(b) $b_n = \frac{4}{\pi}\int_0^{\pi/2} f(x)\sin 2nx\,dx = \frac{4}{\pi}\int_0^{\pi/2} \cos x \sin 2nx\,dx = \ = \frac{2}{\pi}\left[-\frac{\cos[(2n-1)x]}{2n-1} - + \frac{\cos[(2n+1)x]}{2n+1}\right]\Bigg|_0^{\pi/2}$

$$= \frac{2}{\pi}\left[\frac{1}{2n-1} + \frac{1}{2n+1} - \frac{\cos\left[\frac{(2n-1)\pi}{2}\right]}{2n-1} - \frac{\cos\left[\frac{(2n+1)\pi}{2}\right]}{2n+1}\right] = \frac{8n}{(4n^2-1)\pi}$$

Therefore, $f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx$

95. (a) $a_0 = \frac{2}{3}\int_0^3 f(x)\,dx = \frac{2}{3}\int_0^3 (2x + x^2)\,dx = \frac{2}{3}\left(x^2 + \frac{x^3}{3}\right)\Big|_0^3 = \frac{2}{3}(18) = 12$

$a_n = \frac{2}{3}\int_0^3 f(x)\cos\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\int_0^3 (2x + x^2)\cos\left(\frac{n\pi x}{3}\right)dx =$ (using CAS)

$$= \frac{2}{n^3\pi^3}\left[6n\pi(1+x)\cos\left(\frac{n\pi x}{3}\right) + \left(n^2\pi^2 x(x+2) - 18\right)\sin\left(\frac{n\pi x}{3}\right)\right]\Bigg|_0^3 = \frac{12}{n^2\pi^2}\left[4(-1)^n - 1\right]$$

Therefore, $f(x) = 6 + \frac{12}{\pi^2}\sum_{n=1}^{\infty} \frac{4(-1)^n - 1}{n^2}\cos\left(\frac{n\pi x}{3}\right)$

(b) $b_n = \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\int_0^3 (2x + x^2)\sin\left(\frac{n\pi x}{3}\right)dx =$ (using CAS)

$= \frac{2}{n^3\pi^3}\left[-18n\pi(1+x)\sin\left(\frac{n\pi x}{3}\right) - \left(n^2\pi^2 x(x+2) - 18\right)\cos\left(\frac{n\pi x}{3}\right)\right]\Big|_0^3 = \frac{2\left[\left(18 - 15n^2\pi^2\right)(-1)^n - 18\right]}{n^3\pi^3}$

Therefore, $f(x) = \frac{6}{\pi^3}\sum_{n=1}^{\infty} \frac{\left(6 - 5n^2\pi^2\right)(-1)^n - 6}{n^3}\sin\left(\frac{n\pi x}{3}\right)$

96. (a) $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 e^{-x}\,dx = -e^{-x}\Big|_0^2 = 1 - \frac{1}{e^2} = \frac{e^2 - 1}{e^2}$

$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 e^{-x}\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2}{4 + n^2\pi^2}\left[e^{-x}\left(n\pi\sin\left(\frac{n\pi x}{2}\right) - 2\cos\left(\frac{n\pi x}{2}\right)\right)\right]\Big|_0^2$

$= \frac{4\left(e^2 - (-1)^n\right)}{\left(4 + n^2\pi^2\right)e^2}$. Therefore, $f(x) = \frac{e^2 - 1}{2e^2} + \frac{4}{e^2}\sum_{n=1}^{\infty} \frac{\left(e^2 - (-1)^n\right)}{\left(4 + n^2\pi^2\right)}\cos\left(\frac{n\pi x}{2}\right)$

(b) $a_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_0^2 e^{-x}\sin\left(\frac{n\pi x}{2}\right)dx = -\frac{2}{4 + n^2\pi^2}\left[e^{-x}\left(n\pi\cos\left(\frac{n\pi x}{2}\right) + 2\sin\left(\frac{n\pi x}{2}\right)\right)\right]\Big|_0^2$

$= \frac{2n\pi\left(e^2 - (-1)^n\right)}{\left(4 + n^2\pi^2\right)e^2}$. Therefore, $f(x) = \frac{2\pi}{e^2}\sum_{n=1}^{\infty} \frac{n\left(e^2 - (-1)^n\right)}{\left(4 + n^2\pi^2\right)}\sin\left(\frac{n\pi x}{2}\right)$

97. (a) $\sum_{n=1}^{\infty}\left(\sin\frac{1}{2n} - \sin\frac{1}{2n+1}\right) = \left(\sin\frac{1}{2} - \sin\frac{1}{3}\right) + \left(\sin\frac{1}{4} - \sin\frac{1}{5}\right) + \left(\sin\frac{1}{6} - \sin\frac{1}{7}\right) + \ldots + \left(\sin\frac{1}{2n} - \sin\frac{1}{2n+1}\right)$

$+ \ldots = \sum_{n=2}^{\infty} (-1)^n \sin\frac{1}{n}$; $f(x) = \sin\frac{1}{x} \Rightarrow f'(x) = \frac{-\cos\left(\frac{1}{x}\right)}{x^2} < 0$ if $x \ge 2 \Rightarrow \sin\frac{1}{n+1} < \sin\frac{1}{n}$, and

$\lim_{n\to\infty} \sin\frac{1}{n} = 0 \Rightarrow \sum_{n=2}^{\infty} (-1)^n \sin\frac{1}{n}$ converges by the Alternating Series Test

(b) $|\text{error}| < \left|\sin\frac{1}{42}\right| \approx 0.02381$ and the sum is an underestimate because the remainder is positive

98. (a) $\sum_{n=1}^{\infty}\left(\tan\frac{1}{2n} - \tan\frac{1}{2n+1}\right) = \sum_{n=2}^{\infty} (-1)^n \tan\frac{1}{n}$ (see Exercise 89); $f(x) = \tan\frac{1}{x} \Rightarrow f'(x) = \frac{-\sec^2\left(\frac{1}{x}\right)}{x^2} < 0$

$\Rightarrow \tan\frac{1}{n+1} < \tan\frac{1}{n}$, and $\lim_{n\to\infty} \tan\frac{1}{n} = 0 \Rightarrow \sum_{n=2}^{\infty} (-1)^n \tan\frac{1}{n}$ converges by the Alternating Series Test

(b) $|\text{error}| < \left|\tan\frac{1}{42}\right| \approx 0.02382$ and the sum is an underestimate because the remainder is positive

99. $\lim\limits_{n\to\infty} \left|\frac{2\cdot 5\cdot 8\cdots(3n-1)(3n+2)x^{n+1}}{2\cdot 4\cdot 6\cdots(2n)(2n+2)}\cdot\frac{2\cdot 4\cdot 6\cdots(2n)}{2\cdot 5\cdot 8\cdots(3n-1)x^n}\right| < 1 \Rightarrow |x| \lim\limits_{n\to\infty} \left|\frac{3n+2}{2n+2}\right| < 1 \Rightarrow |x| < \frac{2}{3}$

$\Rightarrow$ the radius of convergence is $\frac{2}{3}$

100. $\lim\limits_{n\to\infty} \left|\frac{3\cdot 5\cdot 7\cdots(2n+1)(2n+3)(x-1)^{n+1}}{4\cdot 9\cdot 14\cdots(5n-1)(5n+4)}\cdot\frac{4\cdot 9\cdot 14\cdots(5n-1)}{3\cdot 5\cdot 7\cdots(2n+1)x^n}\right| < 1 \Rightarrow |x| \lim\limits_{n\to\infty} \left|\frac{2n+3}{5n+4}\right| < 1 \Rightarrow |x| < \frac{5}{2}$

$\Rightarrow$ the radius of convergence is $\frac{5}{2}$

101. $\sum\limits_{k=2}^{n} \ln\left(1-\frac{1}{k^2}\right) = \sum\limits_{k=2}^{n} \left[\ln\left(1+\frac{1}{k}\right)+\ln\left(1-\frac{1}{k}\right)\right] = \sum\limits_{k=2}^{n} \left[\ln(k+1)-\ln k+\ln(k-1)-\ln k\right]$

$= [\ln 3-\ln 2+\ln 1-\ln 2]+[\ln 4-\ln 3+\ln 2-\ln 3]+[\ln 5-\ln 4+\ln 3-\ln 4]+[\ln 6-\ln 5+\ln 4-\ln 5]$

$+\ldots+[\ln(n+1)-\ln n+\ln(n-1)-\ln n] = [\ln 1-\ln 2]+[\ln(n+1)-\ln n]$ after cancellation

$\Rightarrow \sum\limits_{k=2}^{n} \ln\left(1-\frac{1}{k^2}\right) = \ln\left(\frac{n+1}{2n}\right) \Rightarrow \sum\limits_{k=2}^{\infty} \ln\left(1-\frac{1}{k^2}\right) = \lim\limits_{n\to\infty} \ln\left(\frac{n+1}{2n}\right) = \ln\frac{1}{2}$ is the sum

102. $\sum\limits_{k=2}^{n} \frac{1}{k^2-1} = \frac{1}{2}\sum\limits_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k+1}\right) = \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{n-2}-\frac{1}{n}\right)\right.$

$\left.+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\right] = \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right) = \frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right) = \frac{1}{2}\left[\frac{3n(n+1)-2(n+1)-2n}{2n(n+1)}\right] = \frac{3n^2-n-2}{4n(n+1)}$

$\Rightarrow \sum\limits_{k=2}^{\infty} \frac{1}{k^2-1} = \lim\limits_{n\to\infty}\left(\frac{3n^2-n-2}{4n^2+4n}\right) = \frac{3}{4}$

103. (a) $\lim\limits_{n\to\infty} \left|\frac{1\cdot 4\cdot 7\cdots(3n-2)(3n+1)x^{3n+3}}{(3n+3)!}\cdot\frac{(3n)!}{1\cdot 4\cdot 7\cdots(3n-2)x^{3n}}\right| < 1 \Rightarrow |x^3| \lim\limits_{n\to\infty} \frac{(3n+1)}{(3n+1)(3n+2)(3n+3)}$

$= |x^3|\cdot 0 < 1 \Rightarrow$ the radius of convergence is ∞

(b) $y = 1+\sum\limits_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3n-2)}{(3n)!}x^{3n} \Rightarrow \frac{dy}{dx} = \sum\limits_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3n-2)}{(3n-1)!}x^{3n-1}$

$\Rightarrow \frac{d^2y}{dx^2} = \sum\limits_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3n-2)}{(3n-2)!}x^{3n-2} = x+\sum\limits_{n=2}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3n-5)}{(3n-3)!}x^{3n-2}$

$= x\left(1+\sum\limits_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3n-2)}{(3n)!}x^{3n}\right) = xy+0 \Rightarrow a = 1$ and $b = 0$

104. (a) $\frac{x^2}{1+x} = \frac{x^2}{1-(-x)} = x^2+x^2(-x)+x^2(-x)^2+x^2(-x)^3+\ldots = x^2-x^3+x^4-x^5+\ldots = \sum\limits_{n=2}^{\infty}(-1)^n x^n$ which converges absolutely for $|x| < 1$

(b) $x = 1 \Rightarrow \sum\limits_{n=2}^{\infty}(-1)^n x^n = \sum\limits_{n=2}^{\infty}(-1)^n$ which diverges

105. Yes, the series $\sum_{n=1}^{\infty} a_n b_n$ converges as we now show. Since $\sum_{n=1}^{\infty} a_n$ converges it follows that $a_n \to 0 \Rightarrow a_n < 1$ for $n >$ some index $N \Rightarrow a_n b_n < b_n$ for $n > N \Rightarrow \sum_{n=1}^{\infty} a_n b_n$ converges by the Direct Comparison Test with $\sum_{n=1}^{\infty} b_n$

106. No, the series $\sum_{n=1}^{\infty} a_n b_n$ might diverge (as it would if a_n and b_n both equaled n) or it might converge (as it would if a_n and b_n both equaled $\frac{1}{n}$).

107. $\sum_{n=1}^{\infty} (x_{n+1} - x_n) = \lim_{n\to\infty} \sum_{k=1}^{\infty} (x_{k+1} - x_k) = \lim_{n\to\infty} (x_{n+1} - x_1) = \lim_{n\to\infty} (x_{n+1}) - x_1 \Rightarrow$ both the series and sequence must either converge or diverge.

108. It converges by the Limit Comparison Test since $\lim_{n\to\infty} \frac{\left(\frac{a_n}{1+a_n}\right)}{a_n} = \lim_{n\to\infty} \frac{1}{1+a_n} = 1$ because $\sum_{n=1}^{\infty} a_n$ converges

109. (a) $\sum_{n=1}^{\infty} \frac{a_n}{n} = a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \ldots \geq a_1 + \left(\frac{1}{2}\right)a_2 + \left(\frac{1}{3} + \frac{1}{4}\right)a_4 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)a_8$

$+\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \ldots + \frac{1}{16}\right)a_{16} + \ldots \geq \frac{1}{2}(a_2 + a_4 + a_8 + a_{16} + \ldots)$ which is a divergent series

(b) $a_n = \frac{1}{\ln n}$ for $n \geq 2 \Rightarrow a_2 \geq a_3 \geq a_4 \geq \ldots$, and $\frac{1}{\ln 2} + \frac{1}{\ln 4} + \frac{1}{\ln 8} + \ldots = \frac{1}{\ln 2} + \frac{1}{2 \ln 2} + \frac{1}{3 \ln 2} + \ldots$

$= \frac{1}{\ln 2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\right)$ which diverges so that $1 + \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by part (a)

110. (a) $T = \frac{\left(\frac{1}{2}\right)}{2}\left(0 + 2\left(\frac{1}{2}\right)^2 e^{1/2} + e\right) = \frac{1}{8}e^{1/2} + \frac{1}{4}e \approx 0.885660616$

(b) $x^2 e^x = x^2\left(1 + x + \frac{x^2}{2} + \ldots\right) = x^2 + x^3 + \frac{x^4}{2} + \ldots \Rightarrow \int_0^1 \left(x^2 + x^3 + \frac{x^4}{2}\right) dx = \left[\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{41}{60} = 0.6833\overline{3}$

(c) If the second derivative is positive, the curve is concave upward and the polygonal line segments used in the trapezoidal rule lie above the curve. The trapezoidal approximation is therefore greater than the actual area under the graph.

(d) All terms in the Maclaurin series are positive. If we truncate the series, we are omitting positive terms and hence the estimate is too small.

(e) $\int_0^1 x^2 e^x \, dx = \left[x^2 e^x - 2xe^x + 2e^x\right]_0^1 = e - 2e + 2e - 2 = e - 2 \approx 0.7182818285$

111. (a) $\int_0^x \frac{1}{1+t^2} dt = \int_0^x \left(1 - t^2 + t^4 - t^6 + \ldots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\right) dt$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^n}{2n+1}x^{2n+1} + \int_0^x \frac{(-1)^{n+1}t^{2n+2}}{1+t^2}dt$$

(b) By definition,

$$R_n(x) = f(x) - P_n(x) = \tan^{-1} x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^n}{2n+1}x^{2n+1}\right) = \int_0^x \frac{(-1)^{n+1}t^{2n+2}}{1+t^2}dt$$

If the integrand goes to zero in the limit, then so will the value of the integral.

$|x| < 1 \Rightarrow |t| < 1 \Rightarrow \lim_{n\to\infty} \frac{(-1)^{n+1}t^{2n+2}}{1+t^2} = \frac{1}{1+t^2} \lim_{n\to\infty} (-1)^{n+1}t^{2n+2} = 0$. If $|x| = 1$, then the value of the integrand will approach 0 for all values of t between 0 and x, while at $t = x$, it will oscillate between $\pm\frac{1}{1+t^2}$. However, the integral of a function will converge provided the function is piecewise continuous in the interval $0 < t < x$. Therefore, we would expect that convergence of $R_n(x)$ to zero would not be affected by the value of the integrand at the single value $t = x$ provided it is finite, which it is. Therefore, $\lim_{n\to\infty} R_n(x) = 0$ for $|x| \le 1$.

(c) For $|x| \le 1$, $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$.

(d) $\tan^{-1} 1 = \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(1)^{2n+1} = \sum_{n=0}^{\infty} 5\,\frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots + \frac{(-1)^n}{2n+1} + \ldots$

112. (a) Substituting x^2 for x in the Maclaurin series for sin x,

$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots + (-1)^n \frac{x^{4n+2}}{(2n+1)!}$.

Integrating term-by-term and observing that the constant term is 0,

$$\int_0^x \sin t^2\, dt = \frac{x^3}{3} - \frac{x^7}{7(3!)} + \frac{x^{11}}{11(5!)} - \ldots + (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + \ldots$$

(b) $\int_0^1 \sin x\, dx = \frac{1}{3} - \frac{1}{7(3!)} + \frac{1}{11(5!)} - \ldots + (-1)^n \frac{1}{(4n+3)(2n+1)!} + \ldots$.

Since the third term is $\frac{1}{11(5!)} = \frac{1}{1320} < 0.001$, it suffices to use the first two nonzero terms (through degree 7).

113. (a) $g(x) = 2x + 3 \Rightarrow g^{-1}(x) = \frac{x-3}{2}$ and when the iterative method is applied to $g^{-1}(x)$ we have $x_0 = 2$ $\Rightarrow -2.99999881$ in 23 iterations $\Rightarrow -3$ is the fixed point

(b) $g(x) = 1 - 4x \Rightarrow g^{-1}(x) = \frac{1-x}{4}$ and when the iterative method is applied to $g^{-1}(x)$ we have $x_0 = 2$ $\Rightarrow 0.199999571$ in 12 iterations $\Rightarrow 0.2$ is the fixed point

CHAPTER 8 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. converges since $\frac{1}{(3n-2)^{(2n+1)/2}} < \frac{1}{(3n-2)^{3/2}}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{(3n-2)^{3/2}}$ converges by the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{(3n-2)^{3/2}}\right)} = \lim_{n\to\infty} \left(\frac{3n-2}{n}\right)^{3/2} = 3^{3/2}$$

2. converges by the Integral Test: $\int\limits_1^{\infty} \left(\tan^{-1} x\right)^2 \frac{dx}{x^2+1} = \lim\limits_{b\to\infty} \left[\frac{\left(\tan^{-1} x\right)^3}{3}\right]_1^b = \lim\limits_{b\to\infty} \left[\frac{\left(\tan^{-1} b\right)^3}{3} - \frac{\pi^3}{192}\right]$

$= \left(\frac{\pi^3}{24} - \frac{\pi^3}{192}\right) = \frac{7\pi^3}{192}$

3. diverges by the nth-Term Test since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (-1)^n \tanh n = \lim\limits_{b\to\infty} (-1)^n\left(\frac{1-e^{-2n}}{1+e^{-2n}}\right) = \lim\limits_{n\to\infty} (-1)^n$ does not exist

4. converges by the Direct Comparison Test: $n! < n^n \Rightarrow \ln(n!) < n \ln(n) \Rightarrow \frac{\ln(n!)}{\ln(n)} < n$ $\Rightarrow \log_n(n!) < n \Rightarrow \frac{\log_n(n!)}{n^3} < \frac{1}{n^2}$, which is the nth-term of a convergent p-series

5. converges by the Direct Comparison Test: $a_1 = 1 = \frac{12}{(1)(3)(2)^2}$, $a_2 = \frac{1\cdot 2}{3\cdot 4} = \frac{12}{(2)(4)(3)^2}$, $a_3 = \left(\frac{2\cdot 3}{4\cdot 5}\right)\left(\frac{1\cdot 2}{3\cdot 4}\right)$ $= \frac{12}{(3)(5)(4)^2}$, $a_4 = \left(\frac{3\cdot 4}{5\cdot 6}\right)\left(\frac{2\cdot 3}{4\cdot 5}\right)\left(\frac{1\cdot 2}{3\cdot 4}\right) = \frac{12}{(4)(6)(5)^2}$, $\ldots \Rightarrow 1 + \sum\limits_{n=1}^{\infty} \frac{12}{(n+1)(n+3)(n+2)^2}$ represents the given series and $\frac{12}{(n+1)(n+3)(n+2)^2} < \frac{12}{n^4}$, which is the nth-term of a convergent p-series

6. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{n}{(n-1)(n+1)} = 0 < 1$

7. diverges by the nth-Term Test since if $a_n \to L$ as $n \to \infty$, then $L = \frac{1}{1+L} \Rightarrow L^2 + L - 1 = 0 \Rightarrow L = \frac{-1 \pm \sqrt{5}}{2}$ $\neq 0$

8. Split the given series into $\sum\limits_{n=1}^{\infty} \frac{1}{3^{2n+1}}$ and $\sum\limits_{n=1}^{\infty} \frac{2n}{3^{2n}}$; the first subseries is a convergent geometric series and the second converges by the Root Test: $\lim\limits_{n\to\infty} \sqrt[n]{\frac{2n}{3^{2n}}} = \lim\limits_{n\to\infty} \frac{\sqrt[n]{2}\,\sqrt[n]{n}}{9} = \frac{1\cdot 1}{9} = \frac{1}{9} < 1$

9. $f(x) = \cos x$ with $a = \frac{\pi}{3} \Rightarrow f\left(\frac{\pi}{3}\right) = 0.5$, $f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, $f''\left(\frac{\pi}{3}\right) = -0.5$, $f'''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $f^{(4)}\left(\frac{\pi}{3}\right) = 0.5$;

$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \ldots$

10. $f(x) = \sin x$ with $a = 2\pi \Rightarrow f(2\pi) = 0,\ f'(2\pi) = 1,\ f''(2\pi) = 0,\ f'''(2\pi) = -1,\ f^{(4)}(2\pi) = 0,\ f^{(5)}(2\pi) = 1,$
$f^{(6)}(2\pi) = 0,\ f^{(7)}(2\pi) = -1;\ \sin x = (x-2\pi) - \frac{(x-2\pi)^3}{3!} + \frac{(x-2\pi)^5}{5!} - \frac{(x-2\pi)^7}{7!} + \ldots$

11. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{2!} + \ldots$ with $a = 0$

12. $f(x) = \ln x$ with $a = 1 \Rightarrow f(1) = 0,\ f'(1) = 1,\ f''(1) = -1,\ f'''(1) = 2,\ f^{(4)}(1) = -6;$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$

13. $f(x) = \cos x$ with $a = 22\pi \Rightarrow f(22\pi) = 1,\ f'(22\pi) = 0,\ f''(22\pi) = -1,\ f'''(22\pi) = 0,\ f^{(4)}(22\pi) = 1,$
$f^{(5)}(22\pi) = 0,\ f^{(6)}(22\pi) = -1;\ \cos x = 1 - \frac{1}{2}(x-22\pi)^2 + \frac{1}{4!}(x-22\pi)^4 - \frac{1}{6!}(x-22\pi)^6 + \ldots$

14. $f(x) = \tan^{-1} x$ with $a = 1 \Rightarrow f(1) = \frac{\pi}{4},\ f'(1) = \frac{1}{2},\ f''(1) = -\frac{1}{2},\ f'''(1) = \frac{1}{2};$

$\tan^{-1} x = \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} + \ldots$

15. Yes, the sequence converges: $c_n = \left(a^n + b^n\right)^{1/n} \Rightarrow c_n = b\left(\left(\frac{a}{b}\right)^n + 1\right)^{1/n} \Rightarrow \lim_{n\to\infty} c_n = \lim_{n\to\infty} b\left(\left(\frac{a}{b}\right)^n + 1\right)^{1/n} = b$
since $0 < a < b$

16. $1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{2}{10^{3n-2}} + \sum_{n=1}^{\infty} \frac{3}{10^{3n-1}} + \sum_{n=1}^{\infty} \frac{7}{10^{3n}}$

$= 1 + \sum_{n=0}^{\infty} \frac{2}{10^{3n+1}} + \sum_{n=0}^{\infty} \frac{3}{10^{3n+2}} + \sum_{n=0}^{\infty} \frac{7}{10^{3n+3}} = 1 + \frac{\left(\frac{2}{10}\right)}{1-\left(\frac{1}{10}\right)^3} + \frac{\left(\frac{3}{10^2}\right)}{1-\left(\frac{1}{10}\right)^3} + \frac{\left(\frac{7}{10^3}\right)}{1-\left(\frac{1}{10}\right)^3}$

$= 1 + \frac{200}{999} + \frac{30}{999} + \frac{7}{999} = \frac{999+237}{999} = \frac{412}{333}$

17. $s_n = \sum_{k=0}^{n-1} \int_k^{k+1} \frac{dx}{1+x^2} \Rightarrow s_n = \int_0^1 \frac{dx}{1+x^2} + \int_1^2 \frac{dx}{1+x^2} + \ldots + \int_{n-1}^{n} \frac{dx}{1+x^2} \Rightarrow s_n = \int_0^n \frac{dx}{1+x^2}$
$\Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\tan^{-1} n - \tan^{-1} 0\right) = \frac{\pi}{2}$

18. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{(n+2)(2x+1)^{n+1}} \cdot \frac{(n+1)(2x+1)^n}{nx^n}\right| = \lim_{n\to\infty} \left|\frac{x}{2x+1} \cdot \frac{(n+1)^2}{n(n+2)}\right| = \left|\frac{x}{2x+1}\right| < 1$

$\Rightarrow |x| < |2x+1|$; if $x > 0$, $|x| < |2x+1| \Rightarrow x < 2x+1 \Rightarrow x > -1$; if $-\frac{1}{2} < x < 0$, $|x| < |2x+1|$

$\Rightarrow -x < 2x+1 \Rightarrow 3x > -1 \Rightarrow x > -\frac{1}{3}$; if $x < -\frac{1}{2}$, $|x| < |2x+1| \Rightarrow -x < -2x-1 \Rightarrow x < -1$. Therefore,

the series converges absolutely for $x < -1$ and $x > -\frac{1}{3}$.

19. (a) From Fig. 8.13 in the text with $f(x)=\frac{1}{x}$ and $a_k=\frac{1}{k}$, we have $\int\limits_1^{n+1} \frac{1}{x}\,dx \le 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$

$\le 1+\int\limits_1^{n} f(x)\,dx \Rightarrow \ln(n+1) \le 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} \le 1+\ln n \Rightarrow 0 \le \ln(n+1)-\ln n$

$\le \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)-\ln n \le 1$. Therefore the sequence $\left\{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)-\ln n\right\}$ is bounded above by 1 and below by 0.

(b) From the graph in Fig. 8.13(a) with $f(x)=\frac{1}{x}$, $\frac{1}{n+1} < \int\limits_n^{n+1} \frac{1}{x}\,dx = \ln(n+1)-\ln n$

$\Rightarrow 0 > \frac{1}{n+1}-\left[\ln(n+1)-\ln n\right] = \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+1}-\ln(n+1)\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln n\right)$.

If we define $a_n = 1+\frac{1}{2}=\frac{1}{3}+\frac{1}{n}-\ln n$, then $0 > a_{n+1}-a_n \Rightarrow a_{n+1} < a_n \Rightarrow \{a_n\}$ is a decreasing sequence of nonnegative terms.

20. (a) Each A_{n+1} fits into the corresponding upper triangular region, whose vertices are: $(n, f(n)-f(n+1))$, $(n+1, f(n+1))$ and $(n, f(n))$ along the line whose slope is $f(n+2)-f(n+1)$.

All the A_n's fit into the first upper triangular region whose area is $\frac{f(1)-f(2)}{2} \Rightarrow \sum\limits_{n=1}^{\infty} A_n < \frac{f(1)-f(2)}{2}$

(b) If $A_k = \frac{f(k+1)+f(k)}{2} - \int\limits_k^{k+1} f(x)\,dx$, then

$$\sum_{k=1}^{n-1} A_k = \frac{f(1)+f(2)+f(2)+f(3)+f(3)+\ldots+f(n-1)+f(n)}{2} - \int\limits_1^2 f(x)\,dx - \int\limits_2^3 f(x)\,dx - \ldots - \int\limits_{n-1}^{n} f(x)\,dx$$

$= \frac{f(1)+f(n)}{2} + \sum\limits_{k=2}^{n-1} f(k) - \int\limits_1^n f(x)\,dx \Rightarrow \sum\limits_{k=1}^{n-1} A_k = \sum\limits_{k=1}^{n} f(k) - \frac{f(1)+f(n)}{2} - \int\limits_1^n f(x)\,dx < \frac{f(1)-f(2)}{2}$, from part (a). The sequence $\left\{\sum\limits_{k=1}^{n-1} A_k\right\}$ is bounded above and increasing, so it converges and the limit in question must exist.

(c) From part (b) we have $\sum\limits_{k=1}^{\infty} f(k) - \int\limits_1^n f(x)\,dx < f(1) - \frac{f(2)}{2} + \frac{f(n)}{2}$

$\Rightarrow \lim\limits_{n\to\infty} \left[\sum\limits_{k=1}^{n} f(k) - \int\limits_1^n f(x)\,dx\right] < \lim\limits_{n\to\infty} \left[f(1)-\frac{f(2)}{2}+\frac{f(n)}{2}\right] = f(1)-\frac{f(2)}{2}$. The sequence $\left\{\sum\limits_{k=1}^{n} f(k) - \int\limits_1^n f(x)\,dx\right\}$ is bounded and increasing, so it converges and the limit in question must exist.

21. The number of triangles removed at stage n is 3^{n-1}; the side length at stage n is $\frac{b}{2^{n-1}}$; the area of a triangle at stage n is $\frac{\sqrt{3}}{4}\left(\frac{b}{2^{n-1}}\right)^2$.

(a) $\frac{\sqrt{3}}{4}b^2 + 3\frac{\sqrt{3}}{4}\left(\frac{b^2}{2^2}\right) + 3^2\frac{\sqrt{3}}{4}\left(\frac{b^2}{2^4}\right) + 3^3\frac{\sqrt{3}}{4}\left(\frac{b^2}{2^6}\right) + \ldots = \frac{\sqrt{3}}{4}b^2 \sum_{n=0}^{\infty} \frac{3^n}{2^{2n}} = \frac{\sqrt{3}}{4}b^2 \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$

(b) a geometric series with sum $\frac{\left(\frac{\sqrt{3}}{4}b^2\right)}{1-\left(\frac{3}{4}\right)} = \sqrt{3}b^2$

(c) No; for instance,the three vertices of the original triangle are not removed. However the total area removed is $\sqrt{3}b^2$ which equals the area of the original triangle. Thus the set of points not removed has area 0.

22. The sequence $\{x_n\}$ converges to $\frac{\pi}{2}$ from below so $\epsilon_n = \frac{\pi}{2} - x_n > 0$ for each n. By the Alternating Series Estimation Theorem $\epsilon_{n+1} \approx \frac{1}{3!}(\epsilon_n)^3$ with $|\text{error}| < \frac{1}{5!}(\epsilon_n)^5$, and since the remainder is negative this is an overestimate $\Rightarrow 0 < \epsilon_{n+1} < \frac{1}{6}(\epsilon_n)^3$.

23. (a) No, the limit does not appear to depend on the value of the constant a

(b) Yes, the limit depends on the value of b. The answer to part (c) shows how the limit depends on the value of (b).

(c) $s = \left(1 - \frac{\cos\left(\frac{a}{n}\right)}{n}\right)^n \Rightarrow \log s = \frac{\log\left(1 - \frac{\cos\left(\frac{a}{n}\right)}{n}\right)}{\left(\frac{1}{n}\right)} \Rightarrow \lim_{n\to\infty} \log s = \frac{\left(\frac{1}{1-\frac{\cos\left(\frac{a}{n}\right)}{n}}\right)\left(\frac{-\frac{a}{n}\sin\left(\frac{a}{n}\right)+\cos\left(\frac{a}{n}\right)}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}$

$= \lim_{n\to\infty} \frac{\frac{a}{n}\sin\left(\frac{a}{n}\right) - \cos\left(\frac{a}{n}\right)}{1 - \frac{\cos\left(\frac{a}{n}\right)}{n}} = \frac{0-1}{1-0} = -1 \Rightarrow \lim_{n\to\infty} s = e^{-1} \approx 0.3678794412$; similarly,

$$\lim_{n\to\infty} \left(1 - \frac{\cos\left(\frac{a}{n}\right)}{bn}\right)^n = e^{-1/b}$$

24. $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$; $\lim_{n\to\infty} \left[\left(\frac{1+\sin a_n}{2}\right)^n\right]^{1/n} = \lim_{n\to\infty} \left(\frac{1+\sin a_n}{2}\right) = \frac{1+\sin\left(\lim_{n\to\infty} a_n\right)}{2} = \frac{1+\sin 0}{2}$ $= \frac{1}{2} \Rightarrow$ the series converges by the nth-Root Test

25. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{b^{n+1}x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{b^n x^n}\right| < 1 \Rightarrow |bx| < 1 \Rightarrow -\frac{1}{b} < x < \frac{1}{b} = 5 \Rightarrow b = \pm\frac{1}{5}$

26. A polynomial has only a finite number of nonzero terms in its Taylor series, but the functions sin x, ln x and e^x have infinitely many nonzero terms in their Taylor expansions.

27. (a) $\frac{u_n}{u_{n+1}} = \frac{(n+1)^2}{n^2} = 1 + \frac{2}{n} + \frac{1}{n^2} \Rightarrow C = 2 > 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

(b) $\frac{u_n}{u_{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n} + \frac{0}{n^2} \Rightarrow C = 1 \le 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

28. $\frac{u_n}{u_{n+1}} = \frac{2n(2n+1)}{(2n-1)^2} = \frac{4n^2+2n}{4n^2-4n+1} = 1 + \frac{\left(\frac{6}{4}\right)}{n} + \frac{5}{4n^2-4n+1} = 1 + \frac{\left(\frac{3}{2}\right)}{n} + \frac{\left[\frac{5n^2}{(4n^2-4n+1)}\right]}{n^2}$ after long division

$\Rightarrow C = \frac{3}{2} > 1$ and $|f(n)| = \frac{5n^2}{4n^2-4n+1} = \frac{5}{\left(4 - \frac{4}{n} + \frac{1}{n^2}\right)} \le 5 \Rightarrow \sum_{n=1}^{\infty} u_n$ converges by Raabe's Test

29. (a) $\sum_{n=1}^{\infty} a_n = L \Rightarrow a_n^2 \le a_n \sum_{n=1}^{\infty} a_n = a_n L \Rightarrow \sum_{n=1}^{\infty} a_n^2$ converges by the Direct Comparison Test

(b) converges by the Limit Comparison Test: $\lim_{n\to\infty} \frac{\left(\frac{a_n}{1-a_n}\right)}{a_n} = \lim_{n\to\infty} \frac{1}{1-a_n} = 1$ since $\sum_{n=1}^{\infty} a_n$ converges and therefore $\lim_{x\to\infty} a_n = 0$

30. If $0 < a_n < 1$ then $|\ln(1-a_n)| = -\ln(1-a_n) = a_n + \frac{a_n^2}{2} + \frac{a_n^3}{3} + \ldots < a_n + a_n^2 + a_n^3 + \ldots = \frac{a_n}{1-a_n}$, a positive term of a convergent series, by the Limit Comparison Test and Exercise 29b

31. $(1-x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n$ where $|x| < 1 \Rightarrow \frac{1}{(1-x)^2} = \frac{d}{dx}(1-x)^{-1} = \sum_{n=1}^{\infty} nx^{n-1}$ and when $x = \frac{1}{2}$ we have $4 = 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} + \ldots$

32. (a) $\sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x} \Rightarrow \sum_{n=1}^{\infty} (n+1)x^n = \frac{2x-x^2}{(1-x)^2} \Rightarrow \sum_{n=1}^{\infty} n(n+1)x^{n-1} = \frac{2}{(1-x)^3} \Rightarrow \sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} = \frac{\frac{2}{x}}{\left(1-\frac{1}{x}\right)^3} = \frac{2x^2}{(x-1)^3}$, $|x| > 1$

(b) $x = \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} \Rightarrow x = \frac{2x^2}{(x-1)^3} \Rightarrow x^3 - 3x^2 + x - 1 = 0 \Rightarrow x = 1 + \left(1 + \frac{\sqrt{57}}{9}\right)^{1/3} + \left(1 - \frac{\sqrt{57}}{9}\right)^{1/3}$ ≈ 2.769292, using a CAS or calculator

33. $e^{-x^2} \le e^{-x}$ for $x \ge 1$, and $\int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b = \lim_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = e^{-1} \Rightarrow \int_1^{\infty} e^{-x^2}\,dx$ converges by the Comparison Test for improper integrals $\Rightarrow \sum_{n=0}^{\infty} e^{-n^2} = 1 + \sum_{n=1}^{\infty} e^{-n^2}$ converges by the Integral Test.

34. Yes, the series $\sum_{n=1}^{\infty} \ln(1+a_n)$ converges by the Direct Comparison Test: $1+a_n < 1+a_n+\frac{a_n^2}{2!}+\frac{a_n^3}{3!}+\ldots$ $\Rightarrow 1+a_n < e^{a_n} \Rightarrow \ln(1+a_n) < a_n$

35. (a) $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1+x+x^2+x^3+\ldots) = 1+2x+3x^2+4x^3+\ldots = \sum_{n=1}^{\infty} nx^{n-1}$

(b) from part (a) we have $\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right) = \left(\frac{1}{6}\right)\left[\frac{1}{1-\left(\frac{5}{6}\right)}\right] = 6$

(c) from part (a) we have $\sum_{n=1}^{\infty} np^{n-1}q = \frac{q}{(1-p)^2} = \frac{q}{q^2} = \frac{1}{q}$

36. (a) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} 2^{-k} = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k2^{-k} = \frac{1}{2}\sum_{k=1}^{\infty} k2^{1-k} = \left(\frac{1}{2}\right)\frac{1}{\left[1-\left(\frac{1}{2}\right)\right]^2} = 2$ by Exercise 35(a)

(b) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{5^{k-1}}{6^k} = \frac{1}{5}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^k = \left(\frac{1}{5}\right)\left[\frac{\left(\frac{5}{6}\right)}{1-\left(\frac{5}{6}\right)}\right] = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k\,\frac{5^{k-1}}{6^k} = \frac{1}{6}\sum_{k=1}^{\infty} k\left(\frac{5}{6}\right)^{k-1}$

$= \left(\frac{1}{6}\right)\frac{1}{\left[1-\left(\frac{5}{6}\right)\right]^2} = 6$

(c) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) = \lim_{k\to\infty}\left(1-\frac{1}{k+1}\right) = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k\left(\frac{1}{k(k+1)}\right)$

$= \sum_{k=1}^{\infty} \frac{1}{k+1}$, a divergent series so that $E(x)$ does not exist

37. (a) $R_n = C_0e^{-kt_0} + C_0e^{-2kt_0} + \ldots + C_0e^{-nkt_0} = \frac{C_0e^{-kt_0}\left(1-e^{-nkt_0}\right)}{1-e^{-kt_0}} \Rightarrow R = \lim_{n\to\infty} R_n = \frac{C_0e^{-kt_0}}{1-e^{-kt_0}} = \frac{C_0}{e^{kt_0}-1}$

(b) $R_n = \frac{e^{-1}\left(1-e^{-n}\right)}{1-e^{-1}} \Rightarrow R_1 = e^{-1} \approx 0.36787944$ and $R_{10} = \frac{e^{-1}\left(1-e^{-10}\right)}{1-e^{-1}} \approx 0.58195028$;

$R = \frac{1}{e-1} \approx 0.58197671$; $R - R_{10} \approx 0.00002643 \Rightarrow \frac{R-R_{10}}{R} < 0.0001$

(c) $R_n = \frac{e^{-.1}\left(1-e^{-.1n}\right)}{1-e^{-.1}}$, $\frac{R}{2} = \frac{1}{2}\left(\frac{1}{e^{.1}-1}\right) \approx 4.7541659$; $R_n > \frac{R}{2} \Rightarrow \frac{1-e^{-.1n}}{e^{.1}-1} > \left(\frac{1}{2}\right)\left(\frac{1}{e^{.1}-1}\right)$

$\Rightarrow 1-e^{-n/10} > \frac{1}{2} \Rightarrow e^{-n/10} < \frac{1}{2} \Rightarrow -\frac{n}{10} < \ln\left(\frac{1}{2}\right) \Rightarrow \frac{n}{10} > -\ln\left(\frac{1}{2}\right) \Rightarrow n > 6.93 \Rightarrow n = 7$

38. (a) $R = \frac{C_0}{e^{kt_0}-1} \Rightarrow Re^{kt_0} = R + C_0 = C_H \Rightarrow e^{kt_0} = \frac{C_H}{C_L} \Rightarrow t_0 = \frac{1}{k}\ln\left(\frac{C_H}{C_L}\right)$

(b) $t_0 = \frac{1}{0.05}\ln e = 20$ hrs

(c) Give an initial dose that produces a concentration of 2 mg/ml followed every $t_0 = \frac{1}{0.02}\ln\left(\frac{2}{0.5}\right) \approx 69.31$ hrs by a dose that raises the concentration by 1.5 mg/ml

(d) $t_0 = \frac{1}{0.2} \ln\left(\frac{0.1}{0.03}\right) = 5 \ln\left(\frac{10}{3}\right) \approx 6$ hrs

39. The convergence of $\sum_{n=1}^{\infty} |a_n|$ implies that $\lim_{n\to\infty} |a_n| = 0$. Let $N > 0$ be such that $|a_n| < \frac{1}{2} \Rightarrow 1 - |a_n| > \frac{1}{2}$ $\Rightarrow \frac{|a_n|}{1-|a_n|} < 2|a_n|$ for all $n > N$. Now $|\ln(1+a_n)| = \left|a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \frac{a_n^4}{4} + \ldots\right| \le |a_n| + \left|\frac{a_n^2}{2}\right| + \left|\frac{a_n^3}{3}\right| + \left|\frac{a_n^4}{4}\right| + \ldots$ $< |a_n| + |a_n|^2 + |a_n|^3 + |a_n|^4 + \ldots = \frac{|a_n|}{1-|a_n|} < 2|a_n|$. Therefore $\sum_{n=1}^{\infty} \ln(1+a_n)$ converges by the Direct Comparison Test since $\sum_{n=1}^{\infty} |a_n|$ converges.

40. $\sum_{n=3}^{\infty} \frac{1}{n \ln n(\ln(\ln n))^p}$ converges if $p > 1$ and diverges otherwise by the Integral Test: when $p = 1$ we have $\lim_{b\to\infty} \int_3^b \frac{dx}{x \ln x(\ln(\ln x))} = \lim_{b\to\infty} \left[\ln(\ln(\ln x))\right]_3^b = \infty$; when $p \ne 1$ we have $\lim_{b\to\infty} \int_3^b \frac{dx}{x \ln x(\ln(\ln x))^p}$

$$= \lim_{b\to\infty} \left[\frac{(\ln(\ln x))^{-p+1}}{1-p}\right]_3^b = \begin{cases} \frac{(\ln(\ln 3))^{-p+1}}{1-p} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}$$

NOTES:

CHAPTER 9 VECTORS IN THE PLANE AND POLAR FUNCTIONS

9.1 VECTORS IN THE PLANE

1. (a) $\langle 3(3), 3(-2)\rangle = \langle 9, -6\rangle$

   (b) $\sqrt{9^2 + (-6)^2} = \sqrt{117} = 3\sqrt{13}$

2. (a) $\langle -2(-2), -2(5)\rangle = \langle 4, -10\rangle$

   (b) $\sqrt{4^2 + (-10)^2} = \sqrt{116} = 2\sqrt{29}$

3. (a) $\langle 3 + (-2), -2 + 5\rangle = \langle 1, 3\rangle$

   (b) $\sqrt{1^2 + 3^2} = \sqrt{10}$

4. (a) $\langle 3 - (-2), -2 - 5\rangle = \langle 5, -7\rangle$

   (b) $\sqrt{5^2 + (-7)^2} = \sqrt{74}$

5. (a) $2\mathbf{u} = \langle 2(3), 2(-2)\rangle = \langle 6, -4\rangle$
   $3\mathbf{v} = \langle 3(-2), 3(5)\rangle = \langle -6, 15\rangle$
   $2\mathbf{u} - 3\mathbf{v} = \langle 6 - (-6), -4 - 15\rangle = \langle 12, -19\rangle$

   (b) $\sqrt{12^2 + (-19)^2} = \sqrt{505}$

6. (a) $-2\mathbf{u} = \langle -2(3), -2(-2)\rangle = \langle -6, 4\rangle$
   $5\mathbf{v} = \langle 5(-2), 5(5)\rangle = \langle -10, 25\rangle$
   $-2\mathbf{u} + 5\mathbf{v} = \langle -6 + (-10), 4 + 25\rangle = \langle -16, 29\rangle$

   (b) $\sqrt{(-16)^2 + 29^2} = \sqrt{1097}$

7. (a) $\frac{3}{5}\mathbf{u} = \left\langle \frac{3}{5}(3), \frac{3}{5}(-2)\right\rangle = \left\langle \frac{9}{5}, -\frac{6}{5}\right\rangle$
   $\frac{4}{5}\mathbf{v} = \left\langle \frac{4}{5}(-2), \frac{4}{5}(5)\right\rangle = \left\langle -\frac{8}{5}, 4\right\rangle$
   $\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v} = \left\langle \frac{9}{5} + \left(-\frac{8}{5}\right), -\frac{6}{5} + 4\right\rangle = \left\langle \frac{1}{5}, \frac{14}{5}\right\rangle$

   (b) $\sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{14}{5}\right)^2} = \frac{\sqrt{197}}{5}$

8. (a) $-\frac{5}{13}\mathbf{u} = \left\langle -\frac{5}{13}(3), -\frac{5}{14}(-2)\right\rangle = -\left\langle \frac{15}{13}, \frac{10}{13}\right\rangle$
   $\frac{12}{13}\mathbf{v} = \left\langle \frac{12}{13}(-2), \frac{12}{13}(5)\right\rangle = \left\langle -\frac{24}{13}, \frac{60}{13}\right\rangle$
   $-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v} = \left\langle -\frac{15}{13} + \left(-\frac{24}{13}\right), \frac{10}{13} + \frac{60}{13}\right\rangle = \left\langle -3, \frac{70}{13}\right\rangle$

   (b) $\sqrt{(-3)^2 + \left(\frac{70}{13}\right)^2} = \frac{\sqrt{6421}}{13}$

9. $\langle 2 - 1, -1 - 3\rangle = \langle 1, -4\rangle$

10. $\left\langle \frac{2 + (-4)}{2} - 0, \frac{-1 + 3}{2} - 0\right\rangle = \langle -1, 1\rangle$

11. $\langle 0 - 2, 0 - 3\rangle = \langle -2, -3\rangle$

12. $\overrightarrow{AB} = \langle 2 - 1, 0 - (-1)\rangle = \langle 1, 1\rangle$
    $\overrightarrow{CD} = \langle -2 - (-1), 2 - 3\rangle = \langle -1, -1\rangle$
    $\overrightarrow{AB} + \overrightarrow{CD} = \langle 1 + (-1), 1 + (-1)\rangle = \langle 0, 0\rangle$

13. $\left\langle \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$

14. $\left\langle \cos\left(-\frac{3\pi}{4}\right), \sin\left(-\frac{3\pi}{4}\right)\right\rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$

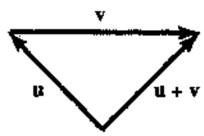
15. This is the unit vector which makes an angle of $120° + 90° = 210°$ with the positive x-axis;

    $\langle \cos 210°, \sin 210°\rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$

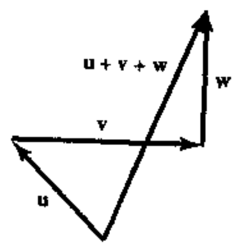
16. $\langle \cos 135°, \sin 135°\rangle = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$

17. The vector **v** is horizontal and 1 in. long. The vectors **u** and **w** are $\frac{11}{16}$ in. long. **w** is vertical and **u** makes a 45° angle with the horizontal. All vectors must be drawn to scale.

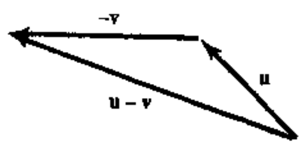
(a)



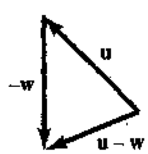
(b)



(c)

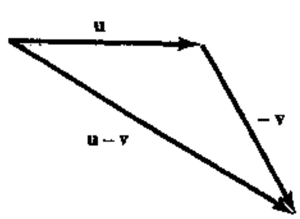


(d)

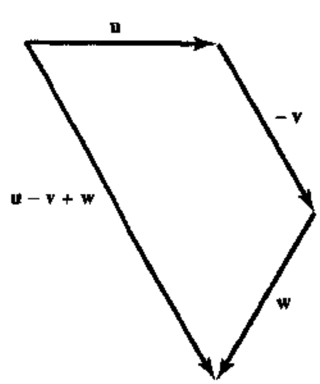


18. The angle between the vectors is 120° and vector **u** is horizontal. They are all 1 in. long. Draw to scale.

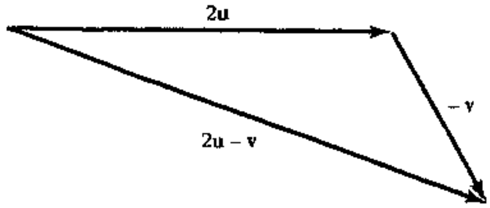
(a)



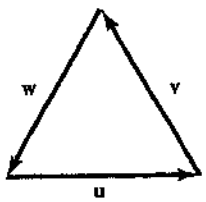
(b)



(c)

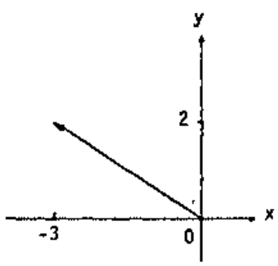


(d)

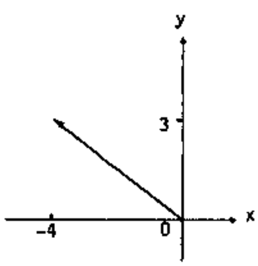


$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$

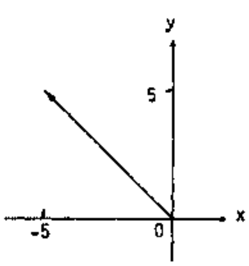
19. $\overrightarrow{P_1P_2} = (2-5)\mathbf{i} + (9-7)\mathbf{j} = -3\mathbf{i} + 2\mathbf{j}$



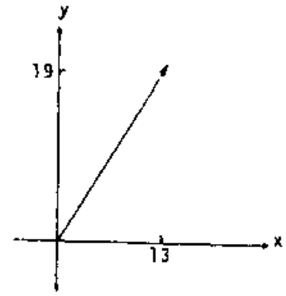
20. $\overrightarrow{P_1P_2} = (-3-1)\mathbf{i} + (5-2)\mathbf{j} = -4\mathbf{i} + 3\mathbf{j}$



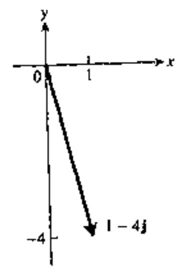
21. $\overrightarrow{AB} = (-10-(-5))\mathbf{i} + (8-3)\mathbf{j} = -5\mathbf{i} + 5\mathbf{j}$



22. $\overrightarrow{AB} = (6-(-7))\mathbf{i} + (11-(-8))\mathbf{j} = 13\mathbf{i} + 19\mathbf{j}$

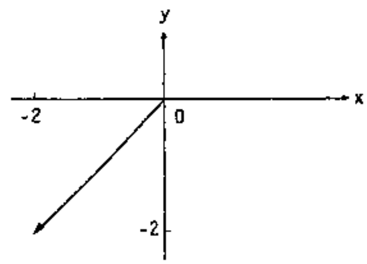


23. $\overrightarrow{P_1P_2} = (2-1)\mathbf{i} + (-1-3)\mathbf{j} = \mathbf{i} - 4\mathbf{j}$



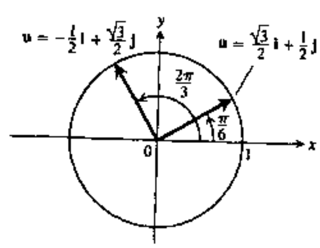
24. P_4 is $\left(\frac{2-4}{2}, \frac{-1+3}{2}\right) = (-1, 1)$

$\Rightarrow \overrightarrow{P_3P_4} = (-1-1)\mathbf{i} + (1-3)\mathbf{j} = -2\mathbf{i} - 2\mathbf{j}$



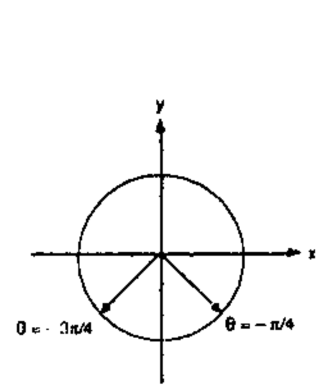
25. $\mathbf{u} = \left(\cos\frac{\pi}{6}\right)\mathbf{i} + \left(\sin\frac{\pi}{6}\right)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$;

$\mathbf{u} = \left(\cos\frac{2\pi}{3}\right)\mathbf{i} + \left(\sin\frac{2\pi}{3}\right)\mathbf{j} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$



26. $\mathbf{u} = \left(\cos\left(-\frac{\pi}{4}\right)\right)\mathbf{i} + \left(\sin\left(-\frac{\pi}{4}\right)\right)\mathbf{j} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$;

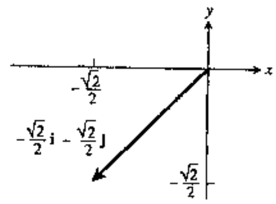
$\mathbf{u} = \left(\cos\left(-\frac{3\pi}{4}\right)\right)\mathbf{i} + \left(\sin\left(-\frac{3\pi}{4}\right)\right)\mathbf{j} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$



27. $\mathbf{u} = \left(\cos\left(\frac{\pi}{2} + \frac{3\pi}{4}\right)\right)\mathbf{i} + \left(\sin\left(\frac{\pi}{2} + \frac{3\pi}{4}\right)\right)\mathbf{j}$

$= \left(\cos\left(\frac{5\pi}{4}\right)\right)\mathbf{i} + \left(\sin\left(\frac{5\pi}{4}\right)\right)\mathbf{j}$

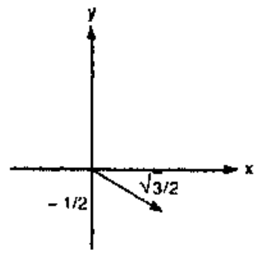
$= -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$



28. $\mathbf{u} = \left(\cos\left(\frac{\pi}{2} - \frac{2\pi}{3}\right)\right)\mathbf{i} + \left(\sin\left(\frac{\pi}{2} - \frac{2\pi}{3}\right)\right)\mathbf{j}$

$= \left(\cos\left(-\frac{\pi}{6}\right)\right)\mathbf{i} + \left(\sin\left(-\frac{\pi}{6}\right)\right)\mathbf{j}$

$= \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$



29. $\sqrt{3^2 + 4^2} = 5$; $\frac{1}{5}\langle 3, 4\rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

30. $\sqrt{4^2 + (-3)^2} = 5$; $\frac{1}{5}\langle 4, -3\rangle = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$

31. $\sqrt{(-15)^2 + 8^2} = 17$; $\frac{1}{17}\langle -15, 8\rangle = \left\langle -\frac{15}{17}, \frac{8}{17} \right\rangle$

32. $\sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$;

$\frac{1}{\sqrt{29}}\langle -5, -2\rangle = \left\langle -\frac{5}{\sqrt{29}}, -\frac{2}{\sqrt{29}} \right\rangle$

33. $|6\mathbf{i} - 8\mathbf{j}| = \sqrt{36 + 64} = 10 \Rightarrow \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{6}{10}\mathbf{i} - \frac{8}{10}\mathbf{j} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

34. $|-\mathbf{i} + 3\mathbf{j}| = \sqrt{1 + 9} = \sqrt{10} \Rightarrow \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$

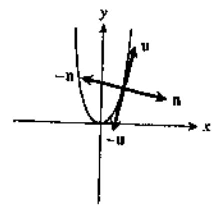
35. $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{25 + 144} = 13 \Rightarrow \mathbf{v} = |\mathbf{v}|\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = 13\left(\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}\right)$

36. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{4 + 9} = \sqrt{13} \Rightarrow \mathbf{v} = |\mathbf{v}|\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \sqrt{13}\left(\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}\right)$

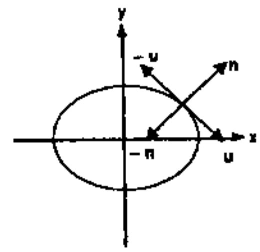
37. $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{9 + 16} = 5 \Rightarrow \mathbf{u} = \pm\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \pm\frac{1}{5}(3\mathbf{i} - 4\mathbf{j})$

38. $\mathbf{A} = -\mathbf{i} + 2\mathbf{j} \Rightarrow |\mathbf{A}| = \sqrt{1 + 4} = \sqrt{5} \Rightarrow \mathbf{v} = -2\frac{\mathbf{A}}{|\mathbf{A}|} = -2\left(-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}\right) = \frac{2}{\sqrt{5}}\mathbf{i} - \frac{4}{\sqrt{5}}\mathbf{j}$ is a vector of length 2 whose direction is opposite to $\mathbf{A}$; there is only one such vector

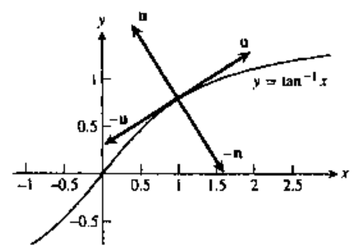
39. $\frac{dy}{dx} = 2x|_{x=2} = 4 \Rightarrow \mathbf{i} + 4\mathbf{j}$ is tangent to the curve at $(2,4)$ $\Rightarrow \mathbf{u} = \frac{1}{\sqrt{17}}\mathbf{i} + \frac{4}{\sqrt{17}}\mathbf{j}$ and $-\mathbf{u} = -\frac{1}{\sqrt{17}}\mathbf{i} - \frac{4}{\sqrt{17}}\mathbf{j}$ are unit tangent vectors; $\mathbf{n} = \frac{4}{\sqrt{17}}\mathbf{i} - \frac{1}{\sqrt{17}}\mathbf{j}$ and $-\mathbf{n} = -\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j}$ are unit normal vectors



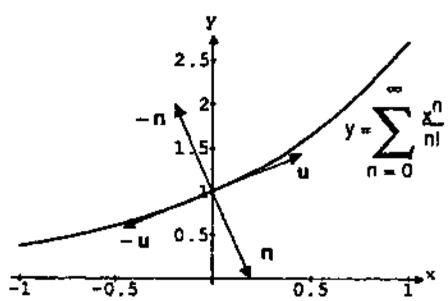
40. $2x + 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{4y}\Big|_{(2,1)} = -1 \Rightarrow \mathbf{i} - \mathbf{j}$ is tangent to the curve at $(2,1) \Rightarrow \mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ are unit tangent vectors; $\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ are unit normal vectors



41. $\frac{dy}{dx} = \frac{1}{1+x^2}\Big|_{x=1} = \frac{1}{2} \Rightarrow \mathbf{i} + \frac{1}{2}\mathbf{j}$ is tangent to the curve at $(1,1) \Rightarrow 2\mathbf{i} + \mathbf{j}$ is tangent $\Rightarrow \mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$ and $-\mathbf{u} = -\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}$ are unit tangent vectors; $\mathbf{n} = -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$ and $-\mathbf{n} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$ are unit normal vectors



42. $\frac{dy}{dx} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x|_{(0,1)} = 1 \Rightarrow \mathbf{i} + \mathbf{j}$ is tangent to the curve at $(0,1) \Rightarrow \mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ are unit tangent vectors; $\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ are unit normal vectors



43. $6x + 8y + 8x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{3x+4y}{4x+2y}\Big|_{(1,0)} = -\frac{3}{4} \Rightarrow 4\mathbf{i} - 3\mathbf{j}$ is tangent to the curve at $(1,0)$ $\Rightarrow \mathbf{u} = \pm\frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$ are unit tangent vectors and $\mathbf{v} = \pm\frac{1}{5}(3\mathbf{i} + 4\mathbf{j})$ are unit normal vectors

44. $2x - 6y - 6x\frac{dy}{dx} + 16y\frac{dy}{dx} - 2 = 0 \Rightarrow \frac{dy}{dx} = -\frac{x-3y-1}{8y-3x}\Big|_{(1,1)} = \frac{3}{5} \Rightarrow 5\mathbf{i} + 3\mathbf{j}$ is tangent to the curve at $(1,1)$ $\Rightarrow \mathbf{u} = \pm\frac{1}{\sqrt{34}}(5\mathbf{i} + 3\mathbf{j})$ are unit tangent vectors and $\mathbf{v} = \pm\frac{1}{\sqrt{34}}(-3\mathbf{i} + 5\mathbf{j})$ are unit normal vectors

45. $\frac{dy}{dx} = \sqrt{3+x^4}\Big|_{(0,0)} = \sqrt{3} \Rightarrow \mathbf{i} + \sqrt{3}\mathbf{j}$ is tangent to the curve at $(0,0) \Rightarrow \mathbf{u} = \pm\frac{1}{2}(\mathbf{i}+\sqrt{3}\mathbf{j})$ are unit tangent vectors and $\mathbf{v} = \pm\frac{1}{2}(-\sqrt{3}\mathbf{i}+\mathbf{j})$ are unit normal vectors

46. $\frac{dy}{dx} = \ln(\ln x)\Big|_{(e,0)} = \ln 1 = 0 \Rightarrow \mathbf{u} = \pm\mathbf{i}$ are unit tangent vectors and $\mathbf{v} = \pm\mathbf{j}$ are unit normal vectors

47. $2\mathbf{i}+\mathbf{j} = a(\mathbf{i}+\mathbf{j}) + b(\mathbf{i}-\mathbf{j}) = (a+b)\mathbf{i} + (a-b)\mathbf{j} \Rightarrow a+b = 2$ and $a-b = 1 \Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}$ and $b = a - 1 = \frac{1}{2}$

48. $\mathbf{i}-2\mathbf{j} = a(2\mathbf{i}+3\mathbf{j}) + b(\mathbf{i}+\mathbf{j}) = (2a+b)\mathbf{i} + (3a+b)\mathbf{j} \Rightarrow 2a+b = 1$ and $3a+b = -2 \Rightarrow a = -3$ and $b = 1-2a = 7 \Rightarrow \mathbf{u}_1 = a(2\mathbf{i}+3\mathbf{j}) = -6\mathbf{i}-9\mathbf{j}$ and $\mathbf{u}_2 = b(\mathbf{i}+\mathbf{j}) = 7\mathbf{i}+7\mathbf{j}$

49. If $|x|$ is the magnitude of the x-component, then $\cos 30° = \frac{|x|}{|\mathbf{F}|} \Rightarrow |x| = |\mathbf{F}|\cos 30° = (10)\left(\frac{\sqrt{3}}{2}\right) = 5\sqrt{3}$ lb
$\Rightarrow \mathbf{F}_x = 5\sqrt{3}\,\mathbf{i}$;
if $|y|$ is the magnitude of the y-component, then $\sin 30° = \frac{|y|}{|\mathbf{F}|} \Rightarrow |y| = |\mathbf{F}|\sin 30° = (10)\left(\frac{1}{2}\right) = 5$ lb $\Rightarrow \mathbf{F}_y = 5\mathbf{j}$.

50. If $|x|$ is the magnitude of the x-component, then $\cos 45° = \frac{|x|}{|\mathbf{F}|} \Rightarrow |x| = |\mathbf{F}|\cos 45° = (12)\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$ lb
$\Rightarrow \mathbf{F}_x = -6\sqrt{2}\,\mathbf{i}$ (the negative sign is indicated by the diagram);
if $|y|$ is the magnitude of the y-component, then $\sin 45° = \frac{|y|}{|\mathbf{F}|} \Rightarrow |y| = |\mathbf{F}|\sin 45° = (12)\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$ lb
$\Rightarrow \mathbf{F}_y = -6\sqrt{2}\,\mathbf{j}$ (the negative sign is indicated by the diagram).

51. 25° west of north is $90° + 25° = 115°$ north of east.
$800\langle\cos 115°, \sin 115°\rangle \approx \langle -338.095, 725.045\rangle$

52. 10° east of south is $270° + 10° = 280°$ "north" of east.
$600\langle\cos 280°, \sin 280°\rangle \approx \langle 104.189, -590.885\rangle$

53. (a) The tree is located at the tip of the vector $\overrightarrow{OP} = (5\cos 60°)\mathbf{i} + (5\sin 60°)\mathbf{j} = \frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j} \Rightarrow P = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$

(b) The telephone pole is located at the point Q, which is the tip of the vector $\overrightarrow{OP} + \overrightarrow{PQ}$
$$= \left(\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}\right) + (10\cos 315°)\mathbf{i} + (10\sin 315°)\mathbf{j} = \left(\frac{5}{2} + \frac{\sqrt{2}}{2}\right)\mathbf{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\mathbf{j}$$
$$\Rightarrow Q = \left(\frac{5+\sqrt{2}}{2}, \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)$$

54. (a) The tree is located at the tip of the vector $\overrightarrow{OP} = (7\cos 45°)\mathbf{i} + (7\sin 45°)\mathbf{j} = \frac{7\sqrt{2}}{2}\mathbf{i} + \frac{7\sqrt{2}}{2}\mathbf{j}$
$$\Rightarrow P = \left(\frac{7\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$$

(b) The telephone pole is located at the point Q which is the tip of the vector $\overrightarrow{OP} + \overrightarrow{PQ}$
$$= \left(\frac{7\sqrt{2}}{2}\mathbf{i} + \frac{7\sqrt{2}}{2}\mathbf{j}\right) + (8\cos 210°)\mathbf{i} + (8\sin 210°)\mathbf{j} = \left(\frac{7\sqrt{2}}{2} - \frac{8\sqrt{3}}{2}\right)\mathbf{i} + \left(\frac{7\sqrt{2}}{2} - \frac{8}{2}\right)\mathbf{j}$$

$$\Rightarrow Q = \left(\frac{7\sqrt{2}}{2} - 4\sqrt{3}, \frac{7\sqrt{2}}{2} - 4\right)$$

9.2 DOT PRODUCTS

NOTE: In Exercises 1-6 below we calculate $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ as the vector $\left(\frac{|\mathbf{u}|\cos\theta}{|\mathbf{v}|}\right)\mathbf{v}$, so the scalar multiplier of $\mathbf{v}$ is the number in column 5 divided by the number in column 2.

	$\mathbf{v}\cdot\mathbf{u}$	$\lvert\mathbf{v}\rvert$	$\lvert\mathbf{u}\rvert$	$\cos\theta$	$\lvert\mathbf{u}\rvert\cos\theta$	$\text{proj}_{\mathbf{v}}\,\mathbf{u}$
1.	-12	$2\sqrt{5}$	$2\sqrt{5}$	$-\frac{3}{5}$	$-\frac{6\sqrt{5}}{5}$	$-\frac{6}{5}\mathbf{i}+\frac{12}{5}\mathbf{j}$
2.	24	$2\sqrt{26}$	$2\sqrt{2}$	$\frac{3\sqrt{13}}{13}$	$\frac{6\sqrt{26}}{13}$	$\frac{3}{13}(2\mathbf{i}+10\mathbf{j})$
3.	$\sqrt{3}-\sqrt{2}$	$\sqrt{2}$	$\sqrt{5}$	$\frac{\sqrt{30}-\sqrt{20}}{10}$	$\frac{\sqrt{6}-2}{2}$	$\frac{\sqrt{3}-\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j})$
4.	$10+\sqrt{17}$	$\sqrt{26}$	$\sqrt{21}$	$\frac{10+\sqrt{17}}{\sqrt{546}}$	$\frac{10+\sqrt{17}}{\sqrt{26}}$	$\frac{10+\sqrt{17}}{26}(5\mathbf{i}+\mathbf{j})$
5.	$\frac{1}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{1}{5}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{5}\left\langle\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}}\right\rangle$
6.	-1	1	1	-1	-1	$-\left\langle\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\rangle$

7. $\theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{u}|}\right) = \cos^{-1}\left(\frac{(2)(1)+(1)(2)}{\sqrt{2^2+1^2}\,\sqrt{1^2+2^2}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{5}\,\sqrt{5}}\right) = \cos^{-1}\left(\frac{4}{5}\right) \approx 0.64$ rad

8. $\mathbf{v} = 2\mathbf{i}-2\mathbf{j},\ \mathbf{u} = 3\mathbf{i} \Rightarrow |\mathbf{v}| = \sqrt{2^2+(-2)^2} = 2\sqrt{2},\ |\mathbf{u}| = 3$, and $\mathbf{v}\cdot\mathbf{u} = 2(3)+(-2)0 = 6$

   $\Rightarrow \mathbf{v}\cdot\mathbf{u} = |\mathbf{v}||\mathbf{u}|\cos\theta$ gives $6 = \left(2\sqrt{2}\right)(3)\cos\theta \Rightarrow \cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4} \approx 0.79$

9. $\mathbf{v} = \sqrt{3}\mathbf{i}-7\mathbf{j},\ \mathbf{u} = \sqrt{3}\mathbf{i}+\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\left(\sqrt{3}\right)^2+(-7)^2} = 2\sqrt{13},\ |\mathbf{u}| = \sqrt{\left(\sqrt{3}\right)^2+1^2} = 2$, and

   $\mathbf{v}\cdot\mathbf{u} = \left(\sqrt{3}\right)\left(\sqrt{3}\right)+(-7)(1) = -4 \Rightarrow \mathbf{v}\cdot\mathbf{u} = |\mathbf{v}||\mathbf{u}|\cos\theta$ gives $-4 = \left(2\sqrt{13}\right)(2)\cos\theta$

   $\Rightarrow \theta = \cos^{-1}\left(-\frac{\sqrt{13}}{13}\right) \approx 1.85$

10. $\mathbf{v} = \mathbf{i}+\sqrt{2}\mathbf{j},\ \mathbf{u} = -\mathbf{i}+\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2+\left(\sqrt{2}\right)^2} = \sqrt{3},\ |\mathbf{u}| = \sqrt{(-1)^2+1^2} = \sqrt{2}$, and

    $\mathbf{v}\cdot\mathbf{u} = (1)(-1)+\left(\sqrt{2}\right)(1) = -1+\sqrt{2} \Rightarrow \mathbf{v}\cdot\mathbf{u} = |\mathbf{v}||\mathbf{u}|\cos\theta$ gives $-1+\sqrt{2} = \left(\sqrt{3}\right)\left(\sqrt{2}\right)\cos\theta$

$\Rightarrow \theta = \cos^{-1}\left(\frac{-1+\sqrt{2}}{\sqrt{6}}\right) \approx 1.40$

11. $\overrightarrow{AB} = \langle 3,1\rangle$, $\overrightarrow{BC} = \langle -1,-3\rangle$, and $\overrightarrow{AC} = \langle 2,-2\rangle$. $\overrightarrow{BA} = \langle -3,-1\rangle$, $\overrightarrow{CB} = \langle 1,3\rangle$, and $\overrightarrow{CA} = \langle -2,2\rangle$.

$|\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{10}$, $|\overrightarrow{BC}| = |\overrightarrow{CB}| = \sqrt{10}$, and $|\overrightarrow{AC}| = |\overrightarrow{CA}| = 2\sqrt{2}$.

Angle at A $= \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}\right) = \cos^{-1}\left(\frac{3(2)+1(-2)}{(\sqrt{10})(2\sqrt{2})}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$,

Angle at B $= \cos^{-1}\left(\frac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{|\overrightarrow{BC}||\overrightarrow{BA}|}\right) = \cos^{-1}\left(\frac{(-1)(-3)+(-3)(-1)}{(\sqrt{10})(\sqrt{10})}\right) = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°$, and

Angle at C $= \cos^{-1}\left(\frac{\overrightarrow{CB}\cdot\overrightarrow{CA}}{|\overrightarrow{CB}||\overrightarrow{CA}|}\right) = \cos^{-1}\left(\frac{1(-2)+3(2)}{(\sqrt{10})(2\sqrt{2})}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$,

12. $\overrightarrow{AC} = \langle 2,4\rangle$ and $\overrightarrow{BD} = \langle 4,-2\rangle$

$\overrightarrow{AC}\cdot\overrightarrow{BD} = 2(4)+4(-2) = 0$, so the angle measures 90°.

13. The sum of two vectors of equal length is *always* orthogonal to their difference, as we can see from the equation $(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_1-\mathbf{v}_2) = \mathbf{v}_1\cdot\mathbf{v}_1+\mathbf{v}_2\cdot\mathbf{v}_1-\mathbf{v}_1\cdot\mathbf{v}_2-\mathbf{v}_2\cdot\mathbf{v}_2 = |\mathbf{v}_1|^2-|\mathbf{v}_2|^2 = 0$

14. $\overrightarrow{CA}\cdot\overrightarrow{CB} = (-\mathbf{v}+(-\mathbf{u}))\cdot(-\mathbf{v}+\mathbf{u}) = \mathbf{v}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{u} = |\mathbf{v}|^2-|\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$ since both equal the radius of the circle. Therefore, $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

15. Let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rhombus $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u}+\mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u}+\mathbf{v}$ $\Rightarrow \mathbf{d}_1\cdot\mathbf{d}_2 = (\mathbf{u}+\mathbf{v})\cdot(-\mathbf{u}+\mathbf{v}) = -\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2-|\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$, since a rhombus has equal sides.

16. Let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rectangle $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u}+\mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u}+\mathbf{v}$. Since the diagonals are perpendicular we have $\mathbf{d}_1\cdot\mathbf{d}_2 = 0 \Leftrightarrow (\mathbf{u}+\mathbf{v})\cdot(-\mathbf{u}+\mathbf{v}) = -\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} = 0 \Leftrightarrow |\mathbf{v}|^2-|\mathbf{u}|^2 = 0$ $\Leftrightarrow (|\mathbf{v}|+|\mathbf{u}|)(|\mathbf{v}|-|\mathbf{u}|) = 0 \Leftrightarrow (|\mathbf{v}|+|\mathbf{u}|) = 0$ which is not possible, or $(|\mathbf{v}|-|\mathbf{u}|) = 0$ which is equivalent to $|\mathbf{v}| = |\mathbf{u}| \Rightarrow$ the rectangle is a square.

17. Clearly the diagonals of a rectangle are equal in length. What is not as obvious is the statement that equal diagonals happen only in a rectangle. We show this is true by letting the opposite sides of a parallelogram be the vectors $(v_1\mathbf{i}+v_2\mathbf{j})$ and $(u_1\mathbf{i}+u_2\mathbf{j})$. The equal diagonals of the parallelogram are

$d_1 = (v_1\mathbf{i}+v_2\mathbf{j})+(u_1\mathbf{i}+u_2\mathbf{j})$ and $d_2 = (v_1\mathbf{i}+v_2\mathbf{j})-(u_1\mathbf{i}+u_2\mathbf{j})$. Hence $|d_1| = |d_2| = |(v_1\mathbf{i}+v_2\mathbf{j})+(u_1\mathbf{i}+u_2\mathbf{j})|$

$= |(v_1\mathbf{i}+v_2\mathbf{j})-(u_1\mathbf{i}+u_2\mathbf{j})| \Rightarrow |(v_1+u_1)\mathbf{i}+(v_2+u_2)\mathbf{j}| = |(v_1-u_1)\mathbf{i}+(v_2-u_2)\mathbf{j}|$

$\Rightarrow \sqrt{(v_1+u_1)^2+(v_2+u_2)^2} = \sqrt{(v_1-u_1)^2+(v_2-u_2)^2} \Rightarrow v_1^2+2v_1u_1+u_1^2+v_2^2+2v_2u_2+u_2^2$

$= v_1^2-2v_1u_1+u_1^2+v_2^2-2v_2u_2+u_2^2 \Rightarrow 2(v_1u_1+v_2u_2) = -2(v_1u_1+v_2u_2) \Rightarrow v_1u_1+v_2u_2 = 0$

$\Rightarrow (v_1\mathbf{i}+v_2\mathbf{j})\cdot(u_1\mathbf{i}+u_2\mathbf{j}) = 0 \Rightarrow$ the vectors $(v_1\mathbf{i}+v_2\mathbf{j})$ and $(u_1\mathbf{i}+u_2\mathbf{j})$ are perpendicular and the parallelogram must be a rectangle.

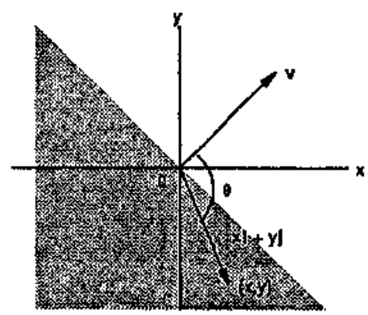
18. If $|\mathbf{u}| = |\mathbf{v}|$ and $\mathbf{u} + \mathbf{v}$ is the indicated diagonal, then $(\mathbf{u}+\mathbf{v}) \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} = |\mathbf{u}|^2 + \mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2$ $= \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = (\mathbf{u}+\mathbf{v}) \cdot \mathbf{v} \Rightarrow$ the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v}) \cdot \mathbf{u}}{|\mathbf{u}+\mathbf{v}||\mathbf{u}|}\right)$ between the diagonal and $\mathbf{u}$ and the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v}) \cdot \mathbf{v}}{|\mathbf{u}+\mathbf{v}||\mathbf{v}|}\right)$ between the diagonal and $\mathbf{v}$ are equal because the inverse cosine function is one-to-one. Therefore, the diagonal bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

19. horizontal component: $1200 \cos(8^\circ) \approx 1188$ ft/s; vertical component: $1200 \sin(8^\circ) \approx 167$ ft/s

20. $|\mathbf{w}| \cos(33^\circ - 15^\circ) = 2.5$ lb, so $|\mathbf{w}| = \frac{2.5 \text{ lb}}{\cos 18^\circ}$. Then $\mathbf{w} = \frac{2.5 \text{ lb}}{\cos 18^\circ}\langle \cos 33^\circ, \sin 33^\circ\rangle \approx \langle 2.205, 1.432\rangle$.

21. (a) Since $|\cos\theta| \le 1$, we have $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\cos\theta| \le |\mathbf{u}||\mathbf{v}|(1) = |\mathbf{u}||\mathbf{v}|$.
    (b) We have equality precisely when $|\cos\theta| = 1$ or when one or both of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{0}$. In the case of nonzero vectors, we have equality when $\theta = 0$ or π, i.e., when the vectors are parallel.

22. $(x\mathbf{i} + y\mathbf{j}) \cdot \mathbf{v} = |x\mathbf{i} + y\mathbf{j}||\mathbf{v}| \cos\theta \le 0$ when $\frac{\pi}{2} \le \theta \le \pi$. This means (x, y) has to be a point whose position vector makes an angle with $\mathbf{v}$ that is a right angle or bigger.



23. $\mathbf{v} \cdot \mathbf{u}_1 = (a\mathbf{u}_1 + b\mathbf{u}_2) \cdot \mathbf{u}_1 = a\mathbf{u}_1 \cdot \mathbf{u}_1 + b\mathbf{u}_2 \cdot \mathbf{u}_1 = a|\mathbf{u}_1|^2 + b(\mathbf{u}_2 \cdot \mathbf{u}_1) = a(1)^2 + b(0) = a$

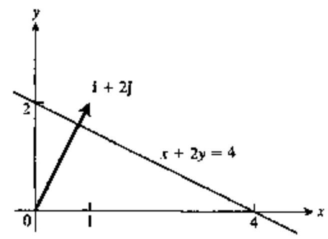
24. No, $\mathbf{v}_1$ need not equal $\mathbf{v}_2$. For example, $\mathbf{i} + \mathbf{j} \ne \mathbf{i} + 2\mathbf{j}$ but $\mathbf{i} \cdot (\mathbf{i} + \mathbf{j}) = \mathbf{i} \cdot \mathbf{i} + \mathbf{i} \cdot \mathbf{j} = 1 + 0 = 1$ and $\mathbf{i} \cdot (\mathbf{i} + 2\mathbf{j}) = \mathbf{i} \cdot \mathbf{i} + 2\mathbf{i} \cdot \mathbf{j} = 1 + 2 \cdot 0 = 1$.

25. $P(x_1, y_1) = P\left(x_1, \frac{c}{b} - \frac{a}{b}x_1\right)$ and $Q(x_2, y_2) = Q\left(x_2, \frac{c}{b} - \frac{a}{b}x_2\right)$ are any two points P and Q on the line with $b \ne 0$ $\Rightarrow \overrightarrow{PQ} = (x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_2 - x_1)\mathbf{j} \Rightarrow \overrightarrow{PQ} \cdot \mathbf{v} = \left[(x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_2 - x_1)\mathbf{j}\right] \cdot (a\mathbf{i} + b\mathbf{j}) = a(x_2 - x_1) + b\left(\frac{a}{b}\right)(x_2 - x_1)$ $= 0 \Rightarrow \mathbf{v}$ is perpendicular to $\overrightarrow{PQ}$ for $b \ne 0$. If $b = 0$, then $\mathbf{v} = a\mathbf{i}$ is perpendicular to the vertical line $ax = c$.

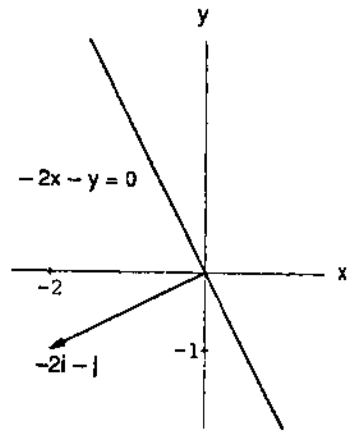
Alternatively, the slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of the line $ax + by = c$ is $-\frac{a}{b}$, so the slopes are negative reciprocals $\Rightarrow$ the vector $\mathbf{v}$ and the line are perpendicular.

26. The slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of $bx - ay = c$ is $\frac{b}{a}$, provided that $a \ne 0$. If $a = 0$, then $\mathbf{v} = b\mathbf{j}$ is parallel to the vertical line $bx = c$. In either case, the vector $\mathbf{v}$ is parallel to the line $ax - by = c$.

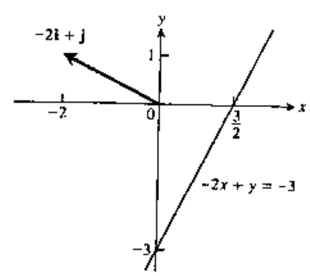
27. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ is perpendicular to the line $x + 2y = c$;
$P(2,1)$ on the line $\Rightarrow 2 + 2 = c \Rightarrow x + 2y = 4$



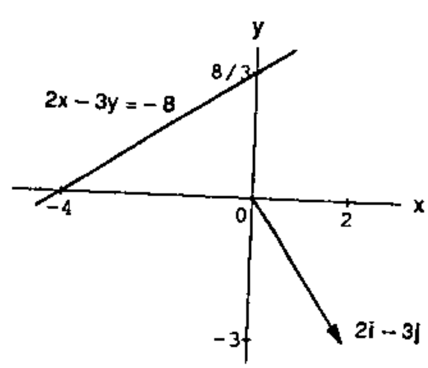
28. $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$ is perpendicular to the line $-2x - y = c$;
$P(-1,2)$ on the line $\Rightarrow (-2)(-1) - 2 = c \Rightarrow -2x - y = 0$



29. $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ is perpendicular to the line $-2x + y = c$;
$P(-2,-7)$ on the line $\Rightarrow (-2)(-2) - 7 = c \Rightarrow -2x + y = -3$

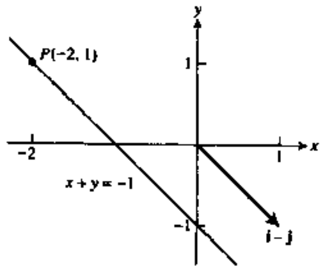


30. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ is perpendicular to the line $2x - 3y = c$;
$P(11,10)$ on the line $\Rightarrow (2)(11) - (3)(10) = c$
$\Rightarrow 2x - 3y = -8$



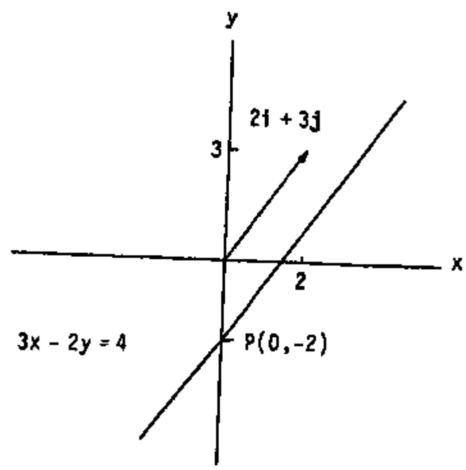
31. $\mathbf{v} = \mathbf{i} - \mathbf{j}$ is parallel to the line $x + y = c$;

$P(-2,1)$ on the line $\Rightarrow -2 + 1 = c \Rightarrow x + y = -1$



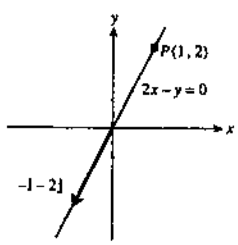
32. $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ is parallel to the line $3x - 2y = c$;

$P(0,-2)$ on the line $\Rightarrow 0 - 2(-2) = c \Rightarrow 3x - 2y = 4$

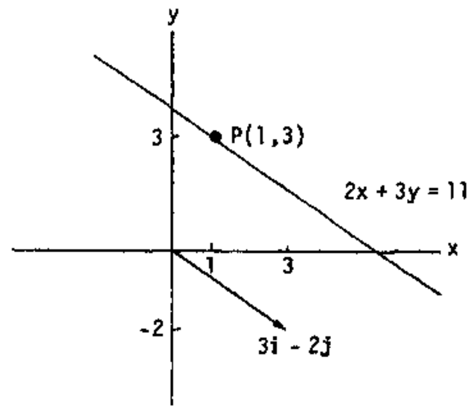


33. $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$ is parallel to the line $2x - y = c$;

$P(1,2)$ on the line $\Rightarrow (2)(1) - 2 = c \Rightarrow 2x - y = 0$



34. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ is parallel to the line $2x + 3y = c$;
$P(1,3)$ on the line $\Rightarrow (2)(1) + (3)(3) = c$
$\Rightarrow 2x + 3y = 11$



35. $P(0,0)$, $Q(1,1)$ and $\mathbf{F} = 5\mathbf{j} \Rightarrow \overrightarrow{PQ} = \mathbf{i} + \mathbf{j}$ and $\mathbf{W} = \mathbf{F} \cdot \overrightarrow{PQ} = (5\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) = 5\ \text{N} \cdot \text{m} = 5\ \text{J}$

36. $\mathbf{W} = |\mathbf{F}|\,(\text{distance}) \cos\theta = (602{,}148\ \text{N})(605\ \text{km})(\cos 0) = 364{,}299{,}540\ \text{N} \cdot \text{km} = (364{,}299{,}540)(1000)\ \text{N} \cdot \text{m}$
$= 3.6429954 \times 10^{11}\ \text{J}$

37. $\mathbf{W} = |\mathbf{F}|\left|\overrightarrow{PQ}\right| \cos\theta = (200)(20)(\cos 30^\circ) = 2000\sqrt{3} = 3464.10\ \text{N} \cdot \text{m} = 3464.10\ \text{J}$

38. $\mathbf{W} = |\mathbf{F}|\left|\overrightarrow{PQ}\right| \cos\theta = (1000)(5280)(\cos 60^\circ) = 2{,}640{,}000\ \text{ft} \cdot \text{lb}$

In Exercises 39-44 we use the fact that $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is normal to the line $ax + by = c$.

39. $\mathbf{n}_1 = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{6-1}{\sqrt{10}\,\sqrt{5}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

40. $\mathbf{n}_1 = -\sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \sqrt{3}\mathbf{i} + \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

41. $\mathbf{n}_1 = \sqrt{3}\mathbf{i} - \mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \sqrt{3}\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{\sqrt{3}+\sqrt{3}}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

42. $\mathbf{n}_1 = \mathbf{i} + \sqrt{3}\mathbf{j}$ and $\mathbf{n}_2 = \left(1 - \sqrt{3}\right)\mathbf{i} + \left(1 + \sqrt{3}\right)\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$

$$= \cos^{-1}\left(\frac{1-\sqrt{3}+\sqrt{3}+3}{\sqrt{1+3}\,\sqrt{1-2\sqrt{3}+3+1+2\sqrt{3}+3}}\right) = \cos^{-1}\left(\frac{4}{2\sqrt{8}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

43. $\mathbf{n}_1 = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{3+4}{\sqrt{25}\,\sqrt{2}}\right) = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right) \approx 0.14\ \text{rad}$

44. $\mathbf{n}_1 = 12\mathbf{i} + 5\mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{24-10}{\sqrt{169}\,\sqrt{8}}\right) = \cos^{-1}\left(\frac{14}{26\sqrt{2}}\right) \approx 1.18\ \text{rad}$

45. The angle between the corresponding normals is equal to the angle between the corresponding tangents. The points of intersection are $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$. At $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ the tangent line for $f(x) = x^2$ is $y - \frac{3}{4} = f'\left(-\frac{\sqrt{3}}{2}\right)\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right) \Rightarrow y = -\sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} \Rightarrow y = -\sqrt{3}x - \frac{3}{4}$, and the tangent line for $f(x) = \left(\frac{3}{2}\right) - x^2$ is $y - \frac{3}{4} = f'\left(-\frac{\sqrt{3}}{2}\right)\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right) \Rightarrow y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} = \sqrt{3}x + \frac{9}{4}$. The corresponding normals are $\mathbf{n}_1 = \sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = -\sqrt{3}\mathbf{i} + \mathbf{j}$. The angle at $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$ $= \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$; the angles are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. At $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ the tangent line for $f(x) = x^2$ is $y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} = \sqrt{3}x + \frac{9}{4}$ and the tangent line for $f(x) = \frac{3}{2} - x^2$ is $y = -\sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4}$ $= -\sqrt{3}x - \frac{3}{4}$. The corresponding normals are $\mathbf{n}_1 = -\sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \sqrt{3}\mathbf{i} + \mathbf{j}$. The angle at $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$; the angles are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

46. The points of intersection are $\left(0, \frac{\sqrt{3}}{2}\right)$ and $\left(0, -\frac{\sqrt{3}}{2}\right)$. The curve $x = \frac{3}{4} - y^2$ has derivative $\frac{dy}{dx} = -\frac{1}{2y} \Rightarrow$ the tangent line at $\left(0, \frac{\sqrt{3}}{2}\right)$ is $y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}(x - 0) \Rightarrow \mathbf{n}_1 = \frac{1}{\sqrt{3}}\mathbf{i} + \mathbf{j}$ is normal to the curve at that point. The curve $x = y^2 - \frac{3}{4}$ has derivative $\frac{dy}{dx} = \frac{1}{2y} \Rightarrow$ the tangent line at $\left(0, \frac{\sqrt{3}}{2}\right)$ is $y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x - 0)$ $\Rightarrow \mathbf{n}_2 = -\frac{1}{\sqrt{3}}\mathbf{i} + \mathbf{j}$ is normal to the curve. The angle between the curves is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$ $= \cos^{-1}\left(\frac{-\frac{1}{3} + 1}{\sqrt{\frac{1}{3} + 1}\,\sqrt{\frac{1}{3} + 1}}\right) = \cos^{-1}\left(\frac{\left(\frac{2}{3}\right)}{\left(\frac{4}{3}\right)}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ and $\frac{2\pi}{3}$. Because of symmetry the angles between the curves at the two points of intersection are the same.

47. The curves intersect when $y = x^3 = \left(y^2\right)^3 = y^6 \Rightarrow y = 0$ or $y = 1$. The points of intersection are $(0,0)$ and $(1,1)$. Note that $y \geq 0$ since $y = y^6$. At $(0,0)$ the tangent line for $y = x^3$ is $y = 0$ and the tangent line for $y = \sqrt{x}$ is $x = 0$. Therefore, the angle of intersection at $(0,0)$ is $\frac{\pi}{2}$. At $(1,1)$ the tangent line for $y = x^3$ is $y = 3x - 2$ and the tangent line for $y = \sqrt{x}$ is $y = \frac{1}{2}x + \frac{1}{2}$. The corresponding normal vectors are $\mathbf{n}_1 = -3\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = -\frac{1}{2}\mathbf{i} + \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$; the angles are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

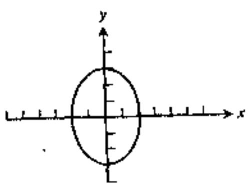
48. The points of intersection for the curves $y = -x^2$ and $y = \sqrt[3]{x}$ are $(0,0)$ and $(-1,-1)$. At $(0,0)$ the tangent line for $y = -x^2$ is $y = 0$ and the tangent line for $y = \sqrt[3]{x}$ is $x = 0$. Therefore, the angle of intersection at $(0,0)$

is $\frac{\pi}{2}$. At $(-1,-1)$ the tangent line for $y=-x^2$ is $y=2x+1$ and the tangent line for $y=\sqrt[3]{x}$ is $y=\frac{1}{3}x-\frac{2}{3}$. The corresponding normal vectors are $\mathbf{n}_1=2\mathbf{i}-\mathbf{j}$ and $\mathbf{n}_2=\frac{1}{3}\mathbf{i}-\mathbf{j} \Rightarrow \theta=\cos^{-1}\left(\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$

$$=\cos^{-1}\left(\frac{\frac{2}{3}+1}{\sqrt{5}\sqrt{\frac{1}{9}+1}}\right)=\cos^{-1}\left(\frac{\left(\frac{5}{3}\right)}{\frac{\sqrt{5}\sqrt{10}}{3}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4};$$ the angles are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

9.3 VECTOR-VALUED FUNCTIONS

1. (a)

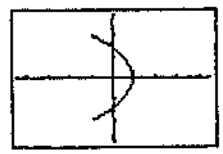


(b) $\mathbf{v}(t)=\frac{d}{dt}(2\cos t)\mathbf{i}+\frac{d}{dt}(3\sin t)\mathbf{j}$
$=(-2\sin t)\mathbf{i}+(3\cos t)\mathbf{j}$
$\mathbf{a}(t)=\frac{d}{dt}(-2\sin t)\mathbf{i}+\frac{d}{dt}(3\cos t)\mathbf{j}$
$=(-2\cos t)\mathbf{i}-(3\sin t)\mathbf{j}$

(c) $\mathbf{v}\left(\frac{\pi}{2}\right)=\langle -2,0\rangle$; speed $=\sqrt{(-2)^2+0^2}=2$,
direction $=\frac{1}{2}\langle -2,0\rangle=\langle -1,0\rangle$

(d) Velocity $=2\langle -1,0\rangle$

2. (a)



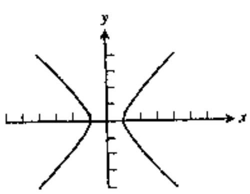
[−4.5, 4.5] by [−3, 3]

(b) $\mathbf{v}(t)=\frac{d}{dt}(\cos 2t)\mathbf{i}+\frac{d}{dt}(2\sin t)\mathbf{j}$
$=(-2\sin 2t)\mathbf{i}+(2\cos t)\mathbf{j}$
$\mathbf{a}(t)=\frac{d}{dt}(-2\sin 2t)\mathbf{i}+\frac{d}{dt}(2\cos t)\mathbf{j}$
$=(-4\cos 2t)\mathbf{i}-(2\sin t)\mathbf{j}$

(c) $\mathbf{v}(0)=\langle 0,2\rangle$; speed $=\sqrt{0^2+2^2}=2$,
direction $=\frac{1}{2}\langle 0,2\rangle=\langle 0,1\rangle$

(d) Velocity $=2\langle 0,1\rangle$

3. (a)



(b) $\mathbf{v}(t) = \frac{d}{dt}(\sec t)\mathbf{i} + \frac{d}{dt}(\tan t)\mathbf{i} = (\sec t \tan t)\mathbf{i} + (\sec^2 t)\mathbf{j}$

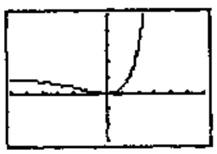
$\mathbf{a}(t) = \frac{d}{dt}(\sec t \tan t)\mathbf{i} + \frac{d}{dt}(\sec^2 t)\mathbf{j}$

$= (\sec t \tan^2 t + \sec^3 t)\mathbf{i} + (2 \sec^2 t \tan t)\mathbf{j}$

(c) $\mathbf{v}\left(\frac{\pi}{6}\right) = \left\langle \frac{2}{3}, \frac{4}{3} \right\rangle$; speed $= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2} = \frac{2\sqrt{5}}{3}$, direction $= \frac{3}{2\sqrt{5}}\left\langle \frac{2}{3}, \frac{4}{3} \right\rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

(d) Velocity $= \frac{2\sqrt{5}}{3}\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

4. (a)



$[-6, 6]$ by $[-3, 5]$

(b) $\mathbf{v}(t) = \frac{d}{dt}(2 \ln (t+1))\mathbf{i} + \frac{d}{dt}(t^2)\mathbf{j}$

$= \left(\frac{2}{t+1}\right)\mathbf{i} + (2t)\mathbf{j}$

$\mathbf{a}(t) = \frac{d}{dt}\left(\frac{2}{t+1}\right)\mathbf{i} + \frac{d}{dt}(2t)\mathbf{j} = \left(-\frac{2}{(t+1)^2}\right)\mathbf{i} + 2\mathbf{j}$

(c) $\mathbf{v}(1) = \langle 1, 2 \rangle$; speed $= \sqrt{1^2 + 2^2} = \sqrt{5}$, direction $= \frac{1}{\sqrt{5}}\langle 1, 2 \rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

(d) Velocity $= \sqrt{5}\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

5. $\mathbf{v}(t) = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$. Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $(\sin t - \sin t \cos t) + (\sin t \cos t) = 0$ implies $\sin t = 0$, which is true for $t = 0$, π, or 2π.

6. $\mathbf{v}(t) = (\cos t)\mathbf{i} + \mathbf{j}$, and $\mathbf{a}(t) = (-\sin t)\mathbf{i}$. Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $-\sin t \cos t = 0$, which is true for $t = \frac{k\pi}{2}$, k any nonnegative integer.

7. $\mathbf{v}(t) = (-3 \sin t)\mathbf{i} + (4 \cos t)\mathbf{j}$, and $\mathbf{a}(t) = (-3 \cos t)\mathbf{i} + (-4 \sin t)\mathbf{j}$. Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $(9 \sin t \cos t) - (16 \sin t \cos t) = 0$, which is true when $\sin t = 0$ or $\cos t = 0$, i.e., for $t = \frac{k\pi}{2}$, k any nonnegative integer.

8. $\mathbf{v}(t) = (-5 \sin t)\mathbf{i} + (5 \cos t)\mathbf{j}$, and $\mathbf{a}(t) = (-5 \cos t)\mathbf{i} + (-5 \sin t)\mathbf{j}$. Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $(25 \sin t \cos t) + (-25 \sin t \cos t) = 0$, which is true for all values of t.

9. $\mathbf{v}(t) = (-2 \sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and $\mathbf{a}(t) = (-2 \cos t)\mathbf{i} + (-\sin t)\mathbf{j}$. So $\mathbf{v}\left(\frac{\pi}{4}\right) = (-\sqrt{2})\mathbf{i} + \left(\frac{1}{\sqrt{2}}\right)\mathbf{j}$, and $\mathbf{a}\left(\frac{\pi}{4}\right) = (-\sqrt{2})\mathbf{i} + \left(-\frac{1}{\sqrt{2}}\right)\mathbf{j}$. Then $|\mathbf{v}| = |\mathbf{a}| = \sqrt{\frac{5}{2}}$. $\mathbf{v} \cdot \mathbf{a} = \frac{3}{2}$ and $\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}||\mathbf{a}|}\right) = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°$.

10. $\mathbf{v}(t) = 3\mathbf{i} + (2t)\mathbf{j}$, and $\mathbf{a}(t) = 2\mathbf{j}$. So $\mathbf{v}(0) = 3\mathbf{i}$, and $\mathbf{a}(0) = 2\mathbf{j}$. These are perpendicular, i.e., the angle between them measures 90°.

11. (a) Both components are continuous at $t = 3$, so the limit is $3\mathbf{i} + \left(\dfrac{3^2 - 9}{3^2 + 3(3)}\right)\mathbf{j} = 3\mathbf{i}$.

(b) Continuous so long as $t^2 + 3t \neq 0$, i.e., $t \neq 0, -3$

(c) Discontinuous when $t^2 + 3t = 0$, i.e., $t = 0$ or -3

12. (a) Use L'Hôpital's Rule for the i-component:

$$\lim_{t\to 0}\left(\frac{\sin 2t}{t}\right)\mathbf{i} + \lim_{t\to 0}(\ln(t+1))\mathbf{j} = \lim_{t\to 0}\left(\frac{2\cos 2t}{1}\right)\mathbf{i} + \lim_{t\to 0}(\ln(t+1))\mathbf{j} = 2\mathbf{i} + 0\mathbf{j} = 2\mathbf{i}.$$

(b) Continuous so long as $t \neq 0$ and $t + 1 > 0$, i.e., $(-1,0) \cup (0,\infty)$.

(c) Discontinuous when $t = 0$ or $t + 1 \leq 0$, i.e., $(-\infty,-1) \cup \{0\}$.

13. $\mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j}$, $\mathbf{r}(0) = -\mathbf{j}$ and $\mathbf{v}(0) = \mathbf{i}$. So the slope is zero (the velocity vector is horizontal).
(a) The horizontal line through $(0,-1)$: $y = -1$.
(b) The vertical line through $(0,-1)$: $x = 0$.

14. $\mathbf{v}(t) = (-2\sin t)\mathbf{i} + (3\cos t)\mathbf{j}$.

$\mathbf{r}\left(\frac{\pi}{4}\right) = \left(\sqrt{2-3}\right)\mathbf{i} + \left(\dfrac{3}{\sqrt{2}} + 1\right)\mathbf{j}$ and $\mathbf{v}\left(\frac{\pi}{4}\right) = \left(-\sqrt{2}\right)\mathbf{i} + \left(\dfrac{3}{\sqrt{2}}\right)\mathbf{j}$. So the slope is $\dfrac{3/\sqrt{2}}{-\sqrt{2}} = -\frac{3}{2}$.

(a) $y - \left(\dfrac{3}{\sqrt{2}} + 1\right) = -\frac{3}{2}\left[x - \left(\sqrt{2} - 3\right)\right]$ or $y = -\frac{3}{2}x + \dfrac{6\sqrt{2} - 7}{2}$

(b) $y - \left(\dfrac{3}{\sqrt{2}} + 1\right) = \frac{2}{3}\left[x - \left(\sqrt{2} - 3\right)\right]$ or $y = \frac{2}{3}x + \dfrac{5\sqrt{2} + 18}{6}$

15. $$\left(\int_1^2 (6 - 6t)\,dt\right)\mathbf{i} + \left(\int_1^2 3\sqrt{t}\,dt\right)\mathbf{j} = \left[6t - 3t^2\right]_1^2\mathbf{i} + \left[2t^{3/2}\right]_1^2\mathbf{j} = -3\mathbf{i} + \left(4\sqrt{2} - 2\right)\mathbf{j}$$

16. $$\left(\int_{-\pi/4}^{\pi/4} \sin t\,dt\right)\mathbf{i} + \left(\int_{-\pi/4}^{\pi/4} (1 + \cos t)\,dt\right)\mathbf{j} = \left[-\cos t\right]_{-\pi/4}^{\pi/4}\mathbf{i} + \left[t + \sin t\right]_{-\pi/4}^{\pi/4}\mathbf{j} = \left(\sqrt{2} + \frac{\pi}{2}\right)\mathbf{j}$$

17. $$\left(\int \sec t \tan t\,dt\right)\mathbf{i} + \left(\int \tan t\,dt\right)\mathbf{j} = (\sec t + C_1)\mathbf{i} + \left(\ln|\sec t| + C_2\right)\mathbf{j} = (\sec t)\mathbf{i} + \left(\ln|\sec t|\right)\mathbf{j} + \mathbf{C}$$

18. $$\left(\int \tfrac{1}{t}\,dt\right)\mathbf{i} + \left(\int \frac{1}{5-t}\,dt\right)\mathbf{j} = (\ln|t| + C_1)\mathbf{i} + (-\ln|5-t| + C_2)\mathbf{j} = (\ln|t|)\mathbf{i} - (\ln|5-t|)\mathbf{j} + \mathbf{C}$$

19. $\mathbf{r}(t) = (t+1)^{3/2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{C}$, and $\mathbf{r}(0) = \mathbf{i} - \mathbf{j} + \mathbf{C} = \mathbf{0}$, so $\mathbf{C} = -(\mathbf{i} - \mathbf{j}) = -\mathbf{i} + \mathbf{j}$
$\mathbf{r}(t) = \left((t+1)^{3/2} - 1\right)\mathbf{i} - (e^{-t} - 1)\mathbf{j}$

20. $\mathbf{r}(t) = \left(\frac{t^4}{4} + 2t^2\right)\mathbf{i} + \left(\frac{t^2}{2}\right)\mathbf{j} + \mathbf{C}$, and $\mathbf{r}(0) = \mathbf{C} = \mathbf{i} + \mathbf{j}$, so $\mathbf{r}(t) = \left(\frac{t^4}{4} + 2t^2 + 1\right)\mathbf{i} + \left(\frac{t^2}{2} + 1\right)\mathbf{j}$.

21. $\frac{d\mathbf{r}}{dt} = (-32t)\mathbf{j} + \mathbf{C}_1$ and $\mathbf{r}(t) = (-16t^2)\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$. $\mathbf{r}(0) = \mathbf{C}_2 = 100\mathbf{i}$ and $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j}$. So $\mathbf{r}(t) = (-16t^2)\mathbf{j} + (8\mathbf{i} + 8\mathbf{j})t + 100\mathbf{i} = (8t + 100)\mathbf{i} + (-16t^2 + 8t)\mathbf{j}$.

22. $\frac{d\mathbf{r}}{dt} = -t\mathbf{i} - t\mathbf{j} + \mathbf{C}_1$, and $\mathbf{r}(t) = \left(-\frac{t^2}{2}\right)\mathbf{i} + \left(-\frac{t^2}{2}\right)\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$, $\mathbf{r}(0) = \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j}$, and $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{C}_1 = \mathbf{0}$, so $\mathbf{r}(t) = \left(-\frac{t^2}{2}\right)\mathbf{i} + \left(-\frac{t^2}{2}\right)\mathbf{j} + (10\mathbf{i} + 10\mathbf{j}) = \left(-\frac{t^2}{2} + 10\right)\mathbf{i} + \left(-\frac{t^2}{2} + 10\right)\mathbf{j}$

23. $\mathbf{v}(t) = (\sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$; i.e., $\frac{dx}{dt} = \sin t$, and $\frac{dy}{dt} = 1 - \cos t$

$$\text{Distance} = \int_0^{2\pi/3} \sqrt{(\sin t)^2 + (1 - \cos t)^2}\, dt = \int_0^{2\pi/3} \sqrt{2 - 2\cos t}\, dt = \int_0^{2\pi/3} 2\sin\left(\frac{t}{2}\right) dt = \left[-4\cos\left(\frac{t}{2}\right)\right]_0^{2\pi/3} = 2$$

24. (a) $\mathbf{r}(0) = \left(\frac{1}{4}e^0 - 0\right)\mathbf{i} + (e^0)\mathbf{j} = \frac{1}{4}\mathbf{i} + \mathbf{j}$.

$\mathbf{r}(2) = \left(\frac{1}{4}e^8 - 2\right)\mathbf{i} + (e^4)\mathbf{j}$

Initial $= \left(\frac{1}{4}, 1\right)$, terminal $= \left(\frac{1}{4}e^8 - 2, e^4\right)$

(b) $\mathbf{v}(t) = (e^{4t} - 1)\mathbf{i} + (2e^{2t})\mathbf{j}$; $\frac{dx}{dt} = e^{4t} - 1$, and $\frac{dy}{dt} = 2e^{2t}$.

$$\text{Length} = \int_0^2 \sqrt{(e^{4t} - 1)^2 + (2e^{2t})^2}\, dt = \int_0^2 \sqrt{(e^{4t} + 1)^2}\, dt = \int_0^2 (e^{4t} + 1)\, dt = \left[\frac{1}{4}e^{4t} + t\right]_0^2 = \frac{e^8 + 7}{4} \approx 746.989$$

25. (a) $\mathbf{v}(t) = (\cos t)\mathbf{i} - (2\sin 2t)\mathbf{j}$

(b) $\mathbf{v}(t) = \mathbf{0}$ when both $\cos t = 0$ and $\sin 2t = 0$. $\cos t = 0$ at $t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$; $\sin 2t = 0$ at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π. So $\mathbf{v}(t) = \mathbf{0}$ at $t = \frac{\pi}{2}, \frac{3\pi}{2}$.

(c) $x = \sin t$, $y = \cos 2t$. Relate the two using the identity $\cos 2u = 1 - 2\sin^2 u$: $y = 1 - 2x^2$, where as x ranges over all possible values, $-1 \le x \le 1$. When t increases from 0 to 2π, the particle starts at $(0, 1)$, goes to $(1, -1)$, then goes to $(-1, -1)$, and then goes to $(0, 1)$, tracing the curve twice.

26. (a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 12}{6t^2 - 6t} = \frac{t^2 - 4}{2t^2 - 2t}$

(b) Horizontal tangents: $t^2 - 4 = 0$ for $t = \pm 2$.
Vertical tangents: $2t^2 - 2t = 0$ for $t = 0, 1$.
Plugging the t-values into $x = 2t^3 - 3t^2$ and $y = t^3 - 12t$ produces the x- and y-coordinates of the critical points.
$t = -2$; horizontal tangent at $(-28, 16)$
$t = 0$: vertical tangent at $(0, 0)$
$t = 1$: vertical tangent at $(-1, -11)$
$t = 2$: horizontal tangent at $(4, -16)$

27. $\mathbf{a}(t) = 3\mathbf{i} - \mathbf{j}$, so $\mathbf{v}(t) = (3t)\mathbf{i} - t\mathbf{j} + C_1$ and $\mathbf{r}(t) = \left(\frac{3}{2}t^2\right)\mathbf{i} - \left(\frac{1}{2}t^2\right)\mathbf{j} + C_1 t + C_2$. $\mathbf{r}(0) = C_2 = \mathbf{i} + 2\mathbf{j}$, and since $\mathbf{v}(0)$ must point directly from $(1,2)$ toward $(4,1)$ with magnitude 2,

$$\mathbf{v}(0) = C_1 = 2\left(\frac{(4-1)\mathbf{i} + (1-2)\mathbf{j}}{\sqrt{(4-1)^2 + (1-2)^2}}\right) = \frac{6}{\sqrt{10}}\mathbf{i} - \frac{2}{\sqrt{10}}\mathbf{j} = \frac{3\sqrt{10}}{5}\mathbf{i} - \frac{\sqrt{10}}{5}\mathbf{j}$$

So $\mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{3\sqrt{10}}{5}t + 1\right)\mathbf{i} + \left(-\frac{1}{2}t^2 - \frac{\sqrt{10}}{5}t + 2\right)\mathbf{j}$.

28. (a) $\frac{dx}{dt} = 1 - \frac{2}{t^2} = 0$ when $t = \sqrt{2}$. That corresponds to point $\left(\sqrt{2} + \frac{2}{\sqrt{2}}, 3\left(\sqrt{2}\right)^2\right) = \left(2\sqrt{2}, 6\right)$.

(b) $\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{6t}{1 - 2/t^2}$, which for $t = 1$ equals -6.

(c) When $y = 12$, $t = 2$. $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{(1 - 2/t^2)6 - (4/t^3)6t}{(1 - 2/t^2)^3}$, which for $t = 2$ equals -24.

29. (a) $\mathbf{v}(t) = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}$;

(i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow$ constant speed;

(ii) $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;

(iii) counterclockwise movement;

(iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

(b) $\mathbf{v}(t) = -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(4\cos 2t)\mathbf{i} - (4\sin 2t)\mathbf{j}$;

(i) $|\mathbf{v}(t)| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2 \Rightarrow$ constant speed;

(ii) $\mathbf{v} \cdot \mathbf{a} = 8\sin 2t\cos 2t - 8\cos 2t\sin 2t = 0 \Rightarrow$ yes, orthogonal;

(iii) counterclockwise movement;

(iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

(c) $\mathbf{v}(t) = -\sin\left(t - \frac{\pi}{2}\right)\mathbf{i} + \cos\left(t - \frac{\pi}{2}\right)\mathbf{j} \Rightarrow \mathbf{a}(t) = -\cos\left(t - \frac{\pi}{2}\right)\mathbf{i} - \sin\left(t - \frac{\pi}{2}\right)\mathbf{j}$;

(i) $|\mathbf{v}(t)| = \sqrt{\sin^2\left(t - \frac{\pi}{2}\right) + \cos^2\left(t - \frac{\pi}{2}\right)} = 1 \Rightarrow$ constant speed;

(ii) $\mathbf{v} \cdot \mathbf{a} = \sin\left(t - \frac{\pi}{2}\right)\cos\left(t - \frac{\pi}{2}\right) - \cos\left(t - \frac{\pi}{2}\right)\sin\left(t - \frac{\pi}{2}\right) = 0 \Rightarrow$ yes, orthogonal;

(iii) counterclockwise movement;

(iv) no, $\mathbf{r}(0) = 0\mathbf{i} - \mathbf{j}$ instead of $\mathbf{i} + 0\mathbf{j}$

(d) $\mathbf{v}(t) = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} + (\sin t)\mathbf{j}$;

(i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow$ constant speed;

(ii) $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;

(iii) clockwise movement;

(iv) yes, $\mathbf{r}(0) = \mathbf{i} - 0\mathbf{j}$

(e) $\mathbf{v}(t) = -2t\sin(t^2)\mathbf{i} + 2t\cos(t^2)\mathbf{j} \Rightarrow \mathbf{a}(t) = -\left(4t^2\cos(t^2) + 2\sin(t^2)\right)\mathbf{i} + \left(2\cos(t^2) - 4t^2\sin(t^2)\right)\mathbf{j}$;

(i) $|\mathbf{v}(t)| = \sqrt{\left(-2t\sin(t^2)\right)^2 + \left(2t\cos(t^2)\right)^2} = 2t \Rightarrow$ variable speed

(ii) $\mathbf{v}\cdot\mathbf{a} = 2t\cos(t^2)\left(2\cos(t^2) - 4t^2\sin(t^2)\right) + 2t\sin(t^2)\left(2\sin(t^2) + 4t^2\cos(t^2)\right)$

$= 4t\left(\left(\sin(t^2)\right)^2 + \left(\cos(t^2)\right)^2\right) = 4t \Rightarrow$ orthogonal only at $t = 0$

(iii) counterclockwise movement;

(iv) yes, $\mathbf{r}(0) = 1\mathbf{i} + 0\mathbf{j}$

30. The velocity vector is tangent to the graph of $y^2 = 2x$ at the point $(2, 2)$, has length 5, and a positive $\mathbf{i}$ component. Now, $y^2 = 2x \Rightarrow 2y\,\frac{dy}{dx} = 2 \Rightarrow \left.\frac{dy}{dx}\right|_{(2,2)} = \frac{2}{2\cdot 2} = \frac{1}{2} \Rightarrow$ the tangent vector lies in the direction of the vector $\mathbf{i} + \frac{1}{2}\mathbf{j} \Rightarrow$ the velocity vector is $\mathbf{v} = \frac{5}{\sqrt{1+\frac{1}{4}}}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \frac{5}{\left(\frac{\sqrt{5}}{2}\right)}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = 2\sqrt{5}\mathbf{i} + \sqrt{5}\mathbf{j}$

31. (a) The $\mathbf{j}$-component is zero at $t = 0$ and $t = 160$: 160 seconds.

(b) $-\frac{3}{64}(40)(40 - 160) = 225$ m

(c) $\frac{d}{dt}\left[-\frac{3}{64}t(t - 160)\right] = -\frac{3}{32}t + \frac{15}{2}$, which for $t = 40$ equals $\frac{15}{4}$ meters per second.

(d) $\mathbf{v}(t) = -\frac{3}{32}t + \frac{15}{2}$ equals 0 at $t = 80$ seconds (and is negative after that time).

32. (a) Solve $t - 3 = \frac{3t}{2} - 4$: $t = 2$. Then check that $(t-3)^2 = \frac{3t}{2} - 2$ for $t = 2$: it does.

(b) First particle: $\mathbf{v}_1(t) = \mathbf{i} + 2(t-3)\mathbf{j}$, so $\mathbf{v}_1(2) = \mathbf{i} - 2\mathbf{j}$ and the direction unit vector $\mathbf{v}_1$ is $\left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$.

Second particle: $\mathbf{v}_2(t) = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$, which is constant, and the direction unit vector is $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

33. (a) Referring to the figure, look at the circular arc from the point where $t = 0$ to the point "m." On one hand, this arc has length given by $r_0\theta$, but it also has length given by vt. Setting those two quantities equal gives the result.

(b) $\mathbf{v}(t) = \left(-v\sin\frac{vt}{r_0}\right)\mathbf{i} + \left(v\cos\frac{vt}{r_0}\right)\mathbf{j}$, and $\mathbf{a}(t) = \left(-\frac{v^2}{r_0}\cos\frac{vt}{r_0}\right)\mathbf{i} + \left(-\frac{v^2}{r_0}\sin\frac{vt}{r_0}\right)\mathbf{j} = -\frac{v^2}{r_0}\left[\left(\cos\frac{vt}{r_0}\right)\mathbf{i} + \left(\sin\frac{vt}{r_0}\right)\mathbf{j}\right]$

(c) From part (b) above, $\mathbf{a}(t) = -\left(\frac{v}{r_0}\right)^2\mathbf{r}(t)$. So, by Newton's second law, $\mathbf{F} = -m\left(\frac{v}{r_0}\right)^2\mathbf{r}$. Substituting for $\mathbf{F}$ in the law of gravitation gives the result.

(d) Set $\frac{vT}{r_0} = 2\pi$ and solve for vT.

(e) Substitute $\frac{2\pi r_0}{T}$ for v in $v^2 = \frac{GM}{r_0}$ and solve for T^2.

$$\left(\frac{2\pi r_0}{T}\right)^2 = \frac{GM}{r_0} \Rightarrow \frac{4\pi^2 r_0^2}{T^2} = \frac{GM}{r_0} \Rightarrow \frac{1}{T^2} = \frac{GM}{4\pi^2 r_0^3} \Rightarrow T^2 = \frac{4\pi^2}{GM}r_0^3$$

34. (a) The velocity of the boat at (x, y) relative to land is the sum of the velocity due to the rower and the velocity of the river, or $\mathbf{v} = \left[-\frac{1}{250}(y-50)^2 + 10\right]\mathbf{i} - 20\mathbf{j}$. Now, $\frac{dy}{dt} = -20 \Rightarrow y = -20t + c$; $y(0) = 100$

$\Rightarrow c = 100 \Rightarrow y = -20t + 100 \Rightarrow \mathbf{v} = \left[-\frac{1}{250}(-20t+50)^2 + 10\right]\mathbf{i} - 20\mathbf{j} = \left(-\frac{8}{5}t^2 + 8t\right)\mathbf{i} - 20\mathbf{j}$

$\Rightarrow \mathbf{r}(t) = \left(-\frac{8}{15}t^3 + 4t^2\right)\mathbf{i} - 20t\mathbf{j} + \mathbf{C}_1$; $\mathbf{r}(0) = 0\mathbf{i} + 100\mathbf{j} \Rightarrow 100\mathbf{j} = \mathbf{C}_1 \Rightarrow \mathbf{r}(t)$

$= \left(-\frac{8}{15}t^3 + 4t^2\right)\mathbf{i} + (100 - 20t)\mathbf{j}$

(b) The boat reaches the shore when $y = 0 \Rightarrow 0 = -20t + 100$ from part (a) $\Rightarrow t = 5$

$\Rightarrow \mathbf{r}(5) = \left(-\frac{8}{15}\cdot 125 + 4\cdot 25\right)\mathbf{i} + (100 - 20\cdot 5)\mathbf{j} = \left(-\frac{200}{3} + 100\right)\mathbf{i} = \frac{100}{3}\mathbf{i}$; the distance downstream is therefore $\frac{100}{3}$ m

35. (a) Apply Corollary 2 to each component separately. If the components all differ by scalar constants, the difference vector is a constant vector.

(b) Follows immediately from (a) since any two anti-derivatives of $\mathbf{r}(t)$ must have identical derivatives, namely $\mathbf{r}(t)$.

36. $\frac{d}{dv}|\mathbf{v}|^2 = \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \mathbf{v}'\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}' = 2\mathbf{v}\cdot\mathbf{v}' = 0$. Therefore, $|\mathbf{v}|$ is constant.

37. Let $\mathbf{u} = \mathbf{C} = \langle C_1, C_2\rangle$. $\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{C}}{dt} = \left\langle \frac{dC_1}{dt}, \frac{dC_2}{dt}\right\rangle = \langle 0, 0\rangle$.

38. (a) Suppose $\mathbf{u} = \langle u_1(t), u_2(t)\rangle$.

$\frac{d}{dt}(c\mathbf{u}) = \frac{d}{dt}\langle cu_1(t), cu_2(t)\rangle = \left\langle \frac{d}{dt}(cu_1(t)), \frac{d}{dt}(cu_2(t))\right\rangle = \left\langle c\frac{du_1}{dt}, c\frac{du_2}{dt}\right\rangle = c\left\langle \frac{du_1}{dt}, \frac{du_2}{dt}\right\rangle = c\frac{d\mathbf{u}}{dt}$

(b) $\frac{d}{dt}(f\mathbf{u}) = \frac{d}{dt}\langle fu_1, fu_2\rangle = \langle fu_1' + f'u_1, fu_2' + f'u_2\rangle = \langle fu_1', fu_2'\rangle + \langle f'u_1, f'u_2\rangle = f\mathbf{u}' + f'\mathbf{u}$

39. $\mathbf{u} = \langle u_1, u_2\rangle$, $\mathbf{v} = \langle v_1, v_2\rangle$

(a) $\frac{d}{dt}(\mathbf{u}+\mathbf{v}) = \frac{d}{dt}(\langle u_1 + v_1, u_2 + v_2\rangle) = \left\langle \frac{d}{dt}(u_1+v_1), \frac{d}{dt}(u_2+v_2)\right\rangle = \langle u_1' + v_1', u_2' + v_2'\rangle$

$= \langle u_1', u_2'\rangle + \langle v_1', v_2'\rangle = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$

(b) $\frac{d}{dt}(\mathbf{u}-\mathbf{v}) = \frac{d}{dt}(\langle u_1 - v_1, u_2 - v_2\rangle) = \left\langle \frac{d}{dt}(u_1-v_1), \frac{d}{dt}(u_2-v_2)\right\rangle = \langle u_1' - v_1', u_2' - v_2'\rangle$

$= \langle u_1', u_2'\rangle - \langle v_1', v_2'\rangle = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}$

40. Since $\mathbf{u}$ is a differentiable function of s, we can write $\mathbf{u}(s) = g(s)\mathbf{i} + h(s)\mathbf{j} = g(f(t))\mathbf{i} + h(f(t))\mathbf{j}$, where $g(s)$ and $h(s)$ are differentiable functions of s. Therefore, $\frac{d}{dt}\left[\mathbf{u}(f(t))\right] = \frac{d}{dt}\left[g(f(t))\mathbf{i} + h(f(t))\mathbf{j}\right] = \frac{d}{dt}\left[g(f(t))\right]\mathbf{i} + \frac{d}{dt}\left[h(f(t))\right]\mathbf{j}$

$= g'(f(t))f'(t)\mathbf{i} + h'(f(t))f'(t)\mathbf{j}$ (by the Chain Rule for scalar functions) $= f'(t)\left[g'(s)\mathbf{i} + h'(s)\mathbf{j}\right] = f'(t)\mathbf{u}'(s)$

$= f'(t)\mathbf{u}'(f(t))$.

41. $f(t)$ and $g(t)$ differentiable at $c \Rightarrow f(t)$ and $g(t)$ continuous at $c \Rightarrow \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ is continuous at c.

42. (a) Let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$.

$$\int_a^b k\mathbf{r}(t)\,dt = \int_a^b \langle kx(t), ky(t)\rangle\,dt = \left\langle \int_a^b kx(t)\,dt, \int_a^b ky(t)\,dt \right\rangle = \left\langle k\int_a^b x(t)\,dt, k\int_a^b y(t)\,dt\right\rangle$$

$$= k\left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt\right\rangle = k\int_a^b \langle x(t), y(t)\rangle\,dt \;= k\int_a^b \mathbf{r}(t)\,dt$$

(b) Let $\mathbf{r}_1(t) = \langle x_1(t), y_1(t)\rangle$ and $\mathbf{r}_2(t) = \langle x_2(t), y_2(t)\rangle$.

$$\int_a^b (\mathbf{r}_1(t) \pm \mathbf{r}_2(t))\,dt = \int_a^b (\langle x_1(t), y_1(t)\rangle \pm \langle x_2(t), y_2(t)\rangle)\,dt = \int_a^b \langle x_1(t) \pm x_2(t), y_1(t) \pm y_2(t)\rangle\,dt$$

$$= \left\langle \int_a^b (x_1(t) \pm x_2(t))\,dt, \int_a^b (y_1(t) \pm y_2(t))\,dt \right\rangle = \left\langle \int_a^b x_1(t)\,dt \pm \int_a^b x_2(t)\,dt, \int_a^b y_1(t)\,dt \pm \int_a^b y_2(t)\,dt\right\rangle$$

$$= \left\langle \int_a^b x_1(t)\,dt, \int_a^b y_1(t)\,dt\right\rangle \pm \left\langle \int_a^b x_2(t)\,dt, \int_a^b y_2(t)\,dt\right\rangle = \int_a^b \mathbf{r}_1(t)\,dt \pm \int_a^b \mathbf{r}_2(t)\,dt$$

(c) Let $\mathbf{C} = \langle C_1, C_2\rangle$, $\mathbf{r}(t) = \langle x(t), y(t)\rangle$

$$\int_a^b \mathbf{C}\cdot\mathbf{r}(t)\,dt = \int_a^b (C_1x(t) + C_2y(t)\,dt = C_1\int_a^b x(t)\,dt + C_2\int_a^b y(t)\,dt = \langle C_1, C_2\rangle \cdot \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt\right\rangle$$

$$= \mathbf{C}\cdot\int_a^b \mathbf{r}(t)\,dt$$

43. (a) Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$. Then

$$\frac{d}{dt}\int_a^t \mathbf{r}(q)\,dq = \frac{d}{dt}\int_a^t [f(q)\mathbf{i} + g(q)\mathbf{j}]\,dq \;= \frac{d}{dt}\left[\left(\int_a^t f(q)\,dq\right)\mathbf{i} + \left(\int_a^t g(q)\,dq\right)\mathbf{j}\right]$$

$$= \left(\frac{d}{dt}\int_a^t f(1)\,dq\right)\mathbf{i} + \left(\frac{d}{dt}\int_a^t g(q)\,dq\right)\mathbf{j} = f(t)\mathbf{i} + g(t)\mathbf{j} = \mathbf{r}(t).$$

(b) Let $S(t) = \int_a^t \mathbf{r}(q)\,dq$. Then part (a) shows that $\mathbf{S}(t)$ is an antiderivative of $\mathbf{r}(t)$. Let $\mathbf{R}(t)$ be any antiderivative of $\mathbf{r}(t)$. Then according to 35(b), $\mathbf{S}(t) = \mathbf{R}(t) + \mathbf{C}$. Letting $t = a$, we have $\mathbf{0} = \mathbf{S}(a) = \mathbf{R}(a) + \mathbf{C}$. Therefore, $\mathbf{C} = -\mathbf{R}(a)$ and $\mathbf{S}(t) = \mathbf{R}(t) - \mathbf{R}(a)$. The result follows by letting $t = b$.

9.4 MODELING PROJECTILE MOTION

1. $x = (v_0 \cos\alpha)t \Rightarrow (21\text{ km})\left(\frac{1000\text{ m}}{1\text{ km}}\right) = (840\text{ m/s})(\cos 60°)t \Rightarrow t = \frac{21{,}000\text{ m}}{(840\text{ m/s})(\cos 60°)} = 50$ seconds

2. $R = \frac{v_0^2}{g} \sin 2\alpha$ and maximum R occurs when $\alpha = 45^\circ \Rightarrow 24.5 \text{ km} = \left(\frac{v_0^2}{9.8 \text{ m/s}^2}\right)(\sin 90^\circ)$
$\Rightarrow v_0 = \sqrt{(9.8)(24{,}500) \text{ m}^2/\text{s}^2} = 490 \text{ m/s}$

3. (a) $t = \frac{2v_0 \sin \alpha}{g} = \frac{2(500 \text{ m/s})(\sin 45^\circ)}{9.8 \text{ m/s}^2} = 72.2$ seconds; $R = \frac{v_0^2}{g} \sin 2\alpha = \frac{(500 \text{ m/s})^2}{9.8 \text{ m/s}^2}(\sin 90^\circ) = 25{,}510.2 \text{ m}$

(b) $x = (v_0 \cos \alpha)t \Rightarrow 5000 \text{ m} = (500 \text{ m/s})(\cos 45^\circ)t \Rightarrow t = \frac{5000 \text{ m}}{(500 \text{ m/s})(\cos 45^\circ)} \approx 14.14$ s; thus,
$y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y \approx (500 \text{ m/s})(\sin 45^\circ)(14.14 \text{ s}) - \frac{1}{2}(9.8 \text{ m/s}^2)(14.14 \text{ s})^2 \approx 4020 \text{ m}$

(c) $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g} = \frac{((500 \text{ m/s})(\sin 45^\circ))^2}{2(9.8 \text{ m/s}^2)} = 6378 \text{ m}$

4. $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y = 32 \text{ ft} + (32 \text{ ft/sec})(\sin 30^\circ)t - \frac{1}{2}(32 \text{ ft/sec}^2)t^2 \Rightarrow y = 32 + 16t - 16t^2$;
the ball hits the ground when $y = 0 \Rightarrow 0 = 32 + 16t - 16t^2 \Rightarrow t = -1$ or $t = 2 \Rightarrow t = 2$ sec since $t > 0$; thus,
$x = (v_0 \cos \alpha)\, t \Rightarrow x = (32 \text{ ft/sec})(\cos 30^\circ)t = 32\left(\frac{\sqrt{3}}{2}\right)(2) \approx 55.4 \text{ ft}$

5. $x = x_0 + (v_0 \cos \alpha)t = 0 + (44 \cos 45^\circ)t = 22\sqrt{2}t$ and $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = 6.5 + (44 \sin 45^\circ)t - 16t^2$
$= 6.5 + 22\sqrt{2}t - 16t^2$; the shot lands when $y = 0 \Rightarrow t = \frac{22\sqrt{2} \pm \sqrt{968 + 416}}{32} \approx 2.135$ sec since $t > 0$; thus
$x = 22\sqrt{2}t \approx (22\sqrt{2})(2.134839) \approx 66.42 \text{ ft}$

6. $x = 0 + (44 \cos 40^\circ)t = 33.706t$ and $y = 6.5 + (44 \sin 40^\circ)t - 16t^2 \approx 6.5 + 28.283t - 16t^2$; $y = 0$
$\Rightarrow t \approx \frac{28.283 + \sqrt{(28.283)^2 + 416}}{32} \approx 1.9735$ sec since $t > 0$; thus $x = (33.706)(1.9735) \approx 66.51 \text{ ft} \Rightarrow$ the difference in distances is about $66.51 - 66.42 = 0.09$ ft or about 1 inch

7. (a) $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow 10 \text{ m} = \left(\frac{v_0^2}{9.8 \text{ m/s}^2}\right)(\sin 90^\circ) \Rightarrow v_0^2 = 98 \text{ m}^2\text{s}^2 \Rightarrow v_0 \approx 9.9 \text{ m/s}$;

(b) $6\text{m} \approx \frac{(9.9 \text{ m/s})^2}{9.8 \text{ m/s}^2}(\sin 2\alpha) \Rightarrow \sin 2\alpha \approx 0.59999 \Rightarrow 2\alpha \approx 36.87^\circ$ or $143.12^\circ \Rightarrow \alpha \approx 18.4^\circ$ or 71.6°

8. $v_0 = 5 \times 10^6$ m/s and $x = 40$ cm $= 0.4$ m; thus $x = (v_0 \cos \alpha)t \Rightarrow 0.4\text{m} = (5 \times 10^6 \text{ m/s})(\cos 0^\circ)t$
$\Rightarrow t = 0.08 \times 10^{-6} \text{ s} = 8 \times 10^{-8}$ s; also, $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2$
$\Rightarrow y = (5 \times 10^6 \text{ m/s})(\sin 0^\circ)(8 \times 10^{-8} \text{ s}) - \frac{1}{2}(9.8 \text{ m/s}^2)(8 \times 10^{-8} \text{ s})^2 = -3.136 \times 10^{-14}$ m or
-3.136×10^{-12} cm. Therefore, it drops 3.136×10^{-12} cm.

9. $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow 3(248.8) \text{ ft} = \left(\frac{v_0^2}{32 \text{ ft/sec}^2}\right)(\sin 18^\circ) \Rightarrow v_0^2 \approx 77{,}292.84 \text{ ft}^2/\text{sec}^2 \Rightarrow v_0 \approx 278.01 \text{ ft/sec} \approx 190 \text{ mph}$

10. $v_0 = \frac{80\sqrt{10}}{3}$ ft/sec and R = 200 ft $\Rightarrow 200 = \frac{\left(\frac{80\sqrt{10}}{3}\right)^2}{32}(\sin 2\alpha) \Rightarrow \sin 2\alpha = 0.9 \Rightarrow 2\alpha \approx 64.2^\circ \Rightarrow \alpha \approx 32.1^\circ$;

$y_{max} = \frac{\left[\left(\frac{80\sqrt{10}}{3}\right)(\sin 32.1^\circ)\right]^2}{2(32)} \approx 31.4$ ft. In order to reach the cushion, the angle of elevation will need to be about 32.1°. At this angle, the circus performer will go 31.4 ft into the air at maximum height and will not strike the 75 ft high ceiling.

11. $x = (v_0 \cos \alpha)t \Rightarrow 135 \text{ ft} = (90 \text{ ft/sec})(\cos 30^\circ)t \Rightarrow t \approx 1.732$ sec; $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$

$\Rightarrow y \approx (90 \text{ ft/sec})(\sin 30^\circ)(1.732 \text{ sec}) - \frac{1}{2}(32 \text{ ft/sec}^2)(1.732 \text{ sec})^2 \Rightarrow y \approx 29.94 \text{ ft} \Rightarrow$ the golf ball will clip the leaves at the top

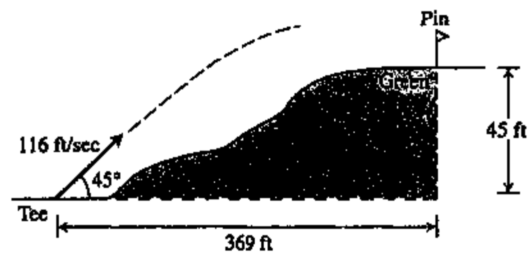
12. $v_0 = 116$ ft/sec, $\alpha = 45^\circ$, and $x = (v_0 \cos \alpha)t$

$\Rightarrow 369 = (116 \cos 45^\circ)t \Rightarrow t \approx 4.50$ sec;

also $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$

$\Rightarrow y = (116 \sin 45^\circ)(4.50) - \frac{1}{2}(32)(4.50)^2$

≈ 45.11 ft. It will take the ball 4.50 sec to travel 369 ft. At that time the ball will be 45.11 ft in the air and will hit the green just past the pin.



13. $x = (v_0 \cos \alpha)t \Rightarrow 315 \text{ ft} = (v_0 \cos 20^\circ)t \Rightarrow v_0 = \frac{315}{t \cos 20^\circ}$; also $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$

$\Rightarrow 34 \text{ ft} = \left(\frac{315}{t \cos 20^\circ}\right)(t \sin 20^\circ) - \frac{1}{2}(32)t^2 \Rightarrow 34 = 315 \tan 20^\circ - 16t^2 \Rightarrow t^2 \approx 5.04 \text{ sec}^2 \Rightarrow t \approx 2.25$ sec

$\Rightarrow v_0 = \frac{315}{(2.25)(\cos 20^\circ)} \approx 149$ ft/sec

14. $R = \frac{v_0^2}{g} \sin 2\alpha = \frac{v_0^2}{g}(2 \sin \alpha \cos \alpha) = \frac{v_0^2}{g}[2 \cos(90^\circ - \alpha) \sin(90^\circ - \alpha)] = \frac{v_0^2}{g}[\sin 2(90^\circ - \alpha)]$

15. $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow 16{,}000 \text{ m} = \frac{(400 \text{ m/s})^2}{9.8 \text{ m/s}^2} \sin 2\alpha \Rightarrow \sin 2\alpha = 0.98 \Rightarrow 2\alpha \approx 78.5^\circ$ or $2\alpha \approx 101.5^\circ \Rightarrow \alpha \approx 39.3^\circ$ or 50.7°

16. (a) $R = \frac{(2v_0)^2}{g} \sin 2\alpha = \frac{4v_0^2}{g} \sin 2\alpha = 4\left(\frac{v_0^2}{g} \sin \alpha\right)$ or 4 times the original range.

(b) Now, let the initial range be $R = \frac{v_0^2}{g} \sin 2\alpha$. Then we want the factor p so that pv_0 will double the range $\Rightarrow \frac{(pv_0)^2}{g} \sin 2\alpha = 2\left(\frac{v_0^2}{g} \sin 2\alpha\right) \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}$ or about 141%. The same percentage will approximately double the height.

17. $x = x_0 + (v_0 \cos \alpha)t = 0 + (v_0 \cos 40°)t \approx 0.766\, v_0 t$ and $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = 6.5 + (v_0 \sin 40°)t - 16t^2$ $\approx 6.5 + 0.643\, v_0 t - 16t^2$; now the shot went 73.833 ft $\Rightarrow 73.833 = 0.766\, v_0 t \Rightarrow t \approx \frac{96.383}{v_0}$ sec; the shot lands when $y = 0 \Rightarrow 0 = 6.5 + (0.643)(96.383) - 16\left(\frac{96.383}{v_0}\right)^2 \Rightarrow 0 \approx 68.474 - \frac{148{,}634}{v_0^2} \Rightarrow v_0 \approx \sqrt{\frac{148{,}634}{68.474}}$ ≈ 46.6 ft/sec, the shot's initial speed

18. $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g} \Rightarrow \frac{3}{4}y_{max} = \frac{3(v_0 \sin \alpha)^2}{8g}$ and $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow \frac{3(v_0 \sin \alpha)^2}{8g} = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$ $\Rightarrow 3(v_0 \sin \alpha)^2 = (8gv_0 \sin \alpha)t - 4g^2t^2 \Rightarrow 4g^2t^2 - (8gv_0 \sin \alpha)t + 3(v_0 \sin \alpha)^2 = 0 \Rightarrow 2gt - 3v_0 \sin \alpha = 0$ or $2gt - v_0 \sin \alpha = 0 \Rightarrow t = \frac{3v_0 \sin \alpha}{2g}$ or $t = \frac{v_0 \sin \alpha}{2g}$. Since the time it takes to reach y_{max} is $t_{max} = \frac{v_0 \sin \alpha}{g}$, then the time it takes the projectile to reach $\frac{3}{4}$ of y_{max} is the shorter time $t = \frac{v_0 \sin \alpha}{2g}$ or half the time it takes to reach the maximum height.

19. $\frac{d\mathbf{r}}{dt} = \int (-g\mathbf{j})\, dt = -gt\mathbf{j} + \mathbf{C}_1$ and $\frac{d\mathbf{r}}{dt}(0) = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \Rightarrow -g(0)\mathbf{j} + \mathbf{C}_1 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j}$ $\Rightarrow \mathbf{C}_1 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha - gt)\mathbf{j}$; $\mathbf{r} = \int [(v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha - gt)\mathbf{j}]\, dt$ $= (v_0 t \cos \alpha)\mathbf{i} + \left(v_0 t \sin \alpha - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{C}_2$ and $\mathbf{r}(0) = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow [v_0(0) \cos \alpha]\mathbf{i} + \left[v_0(0) \sin \alpha - \frac{1}{2}g(0)^2\right]\mathbf{j} + \mathbf{C}_2$ $= x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{C}_2 = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{r} = (x_0 + v_0 t \cos \alpha)\mathbf{i} + \left(y_0 + v_0 t \sin \alpha - \frac{1}{2}gt^2\right)\mathbf{j} \Rightarrow x = x_0 + v_0 t \cos \alpha$ and $y = y_0 + v_0 t \sin \alpha - \frac{1}{2}gt^2$

20. From Example 3(b) in the text, $v_0 \sin \alpha = \sqrt{(68)(64)} \Rightarrow v_0 \sin 57° \approx 65.97 \Rightarrow v_0 \approx 79$ ft/sec

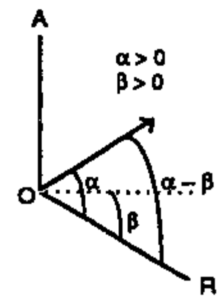
21. The horizontal distance from Rebollo to the center of the cauldron is 90 ft $\Rightarrow$ the horizontal distance to the nearest rim is $x = 90 - \frac{1}{2}(12) = 84 \Rightarrow 84 = x_0 + (v_0 \cos \alpha)t \approx 0 + \left(\frac{90g}{v_0 \sin \alpha}\right)t \Rightarrow 84 = \frac{(90)(32)}{\sqrt{(68)(64)}}t$ $\Rightarrow t = 1.92$ sec. The vertical distance at this time is $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2$ $\approx 6 + \sqrt{(68)(64)}\,(1.92) - 16(1.92)^2 \approx 73.7$ ft $\Rightarrow$ the arrow clears the rim by 3.7 ft

22. The projectile rises straight up and then falls straight down, returning to the firing point.

23. Flight time = 1 sec and the measure of the angle of elevation is about 64° (using a protractor) so that $t = \frac{2v_0 \sin \alpha}{g} \Rightarrow 1 = \frac{2v_0 \sin 64°}{32} \Rightarrow v_0 \approx 17.80$ ft/sec. Then $y_{max} = \frac{(17.80 \sin 64°)^2}{2(32)} \approx 4.00$ ft and $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow R = \frac{(17.80)^2}{32} \sin 128° \approx 7.80$ ft $\Rightarrow$ the engine traveled about 7.80 ft in 1 sec $\Rightarrow$ the engine velocity was about 7.80 ft/sec

24. When marble A is located R units downrange, we have $x = (v_0 \cos\alpha)t \Rightarrow R = (v_0 \cos\alpha)t \Rightarrow t = \frac{R}{v_0 \cos\alpha}$. At that time the height of marble A is $y = y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2 = (v_0 \sin\alpha)\left(\frac{R}{v_0 \cos\alpha}\right) - \frac{1}{2}g\left(\frac{R}{v_0 \cos\alpha}\right)^2$ $\Rightarrow y = R\tan\alpha - \frac{1}{2}g\left(\frac{R^2}{v_0^2 \cos^2\alpha}\right)$. The height of marble B at the same time $t = \frac{R}{v_0 \cos\alpha}$ seconds is $h = R\tan\alpha - \frac{1}{2}gt^2 = R\tan\alpha - \frac{1}{2}g\left(\frac{R^2}{v_0^2 \cos^2\alpha}\right)$. Since the heights are the same, the marbles collide regardless of the initial velocity v_0.

25. (a) At the time t when the projectile hits the line OR we have $\tan\beta = \frac{y}{x}$;
$x = [v_0 \cos(\alpha - \beta)]t$ and $y = [v_0 \sin(\alpha - \beta)]t - \frac{1}{2}gt^2 < 0$ since R is below level ground. Therefore let $|y| = \frac{1}{2}gt^2 - [v_0 \sin(\alpha - \beta)]t > 0$



so that $\tan\beta = \frac{\left[\frac{1}{2}gt^2 - (v_0 \sin(\alpha-\beta))t\right]}{[v_0 \cos(\alpha-\beta)]t} = \frac{\left[\frac{1}{2}gt - v_0 \sin(\alpha-\beta)\right]}{v_0 \cos(\alpha-\beta)}$

$\Rightarrow v_0 \cos(\alpha-\beta)\tan\beta = \frac{1}{2}gt - v_0 \sin(\alpha-\beta)$

$\Rightarrow t = \frac{2v_0 \sin(\alpha-\beta) + 2v_0 \cos(\alpha-\beta)\tan\beta}{g}$, which is the time when the projectile hits the downhill slope. Therefore,

$x = [v_0 \cos(\alpha-\beta)]\left[\frac{2v_0 \sin(\alpha-\beta) + 2v_0 \cos(\alpha-\beta)\tan\beta}{g}\right]$

$= \frac{2v_0^2}{g}\left[\cos^2(\alpha-\beta)\tan\beta + \sin(\alpha-\beta)\cos(\alpha-\beta)\right]$. If x is maximized, then OR is maximized:

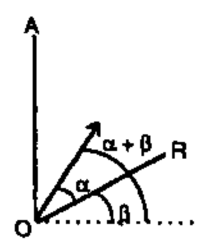
$\frac{dx}{d\alpha} = \frac{2v_0^2}{g}[-\sin 2(\alpha-\beta)\tan\beta + \cos 2(\alpha-\beta)] = 0 \Rightarrow -\sin 2(\alpha-\beta)\tan\beta + \cos 2(\alpha-\beta) = 0$

$\Rightarrow \tan\beta = \cot 2(\alpha-\beta) \Rightarrow 2(\alpha-\beta) = 90^\circ - \beta \Rightarrow \alpha - \beta = \frac{1}{2}(90^\circ - \beta) \Rightarrow \alpha = \frac{1}{2}(90^\circ + \beta) = \frac{1}{2}$ of $\angle AOR$.

(b) At the time t when the projectile hits OR we have $\tan\beta = \frac{y}{x}$;

$x = [v_0 \cos(\alpha+\beta)]t$ and $y = [v_0 \sin(\alpha+\beta)]t - \frac{1}{2}gt^2$

$\Rightarrow \tan\beta = \frac{[v_0 \sin(\alpha+\beta)]t - \frac{1}{2}gt^2}{[v_0 \cos(\alpha+\beta)]t} = \frac{\left[v_0 \sin(\alpha+\beta) - \frac{1}{2}gt\right]}{v_0 \cos(\alpha+\beta)}$



$\Rightarrow v_0 \cos(\alpha+\beta)\tan\beta = v_0 \sin(\alpha+\beta) - \frac{1}{2}gt$

$\Rightarrow t = \frac{2v_0 \sin(\alpha+\beta) - 2v_0 \cos(\alpha+\beta)\tan\beta}{g}$, which is the time when the projectile hits the uphill slope. Therefore,

$x = [v_0 \cos(\alpha+\beta)]\left[\frac{2v_0 \sin(\alpha+\beta) - 2v_0 \cos(\alpha+\beta)\tan\beta}{g}\right]$

$= \frac{2v_0^2}{g}\left[\sin(\alpha+\beta)\cos(\alpha+\beta) - \cos^2(\alpha+\beta)\tan\beta\right]$. If x is maximized, then OR is maximized:

$\frac{dx}{d\alpha} = \frac{2v_0^2}{g}[\cos 2(\alpha+\beta) + \sin 2(\alpha+\beta)\tan\beta] = 0 \Rightarrow \cos 2(\alpha+\beta) + \sin 2(\alpha+\beta)\tan\beta = 0$

$\Rightarrow \cot 2(\alpha+\beta)+\tan\beta=0 \Rightarrow \cot 2(\alpha+\beta)=-\tan\beta=\tan(-\beta) \Rightarrow 2(\alpha+\beta)=90^\circ-(-\beta)$

$=90^\circ+\beta \Rightarrow \alpha=\frac{1}{2}(90^\circ-\beta)=\frac{1}{2}$ of $\angle AOR$. Therefore v_0 would bisect $\angle AOR$ for maximum range uphill.

26. (a) $\mathbf{r}(t)=(x(t))\mathbf{i}+(y(t))\mathbf{j}$, where $x(t)=(145\cos 23^\circ-14)t$ and $y(t)=2.5+(145\sin 23^\circ)t-16t^2$.

(b) $y_{max}=\frac{(v_0\sin\alpha)^2}{2g}+2.5=\frac{(145\sin 23^\circ)^2}{64}+2.5\approx 52.655$ feet, which is reached at $t=\frac{v_0\sin\alpha}{g}$ $=\frac{145\sin 23^\circ}{32}\approx 1.771$ seconds.

(c) For the time, solve $y=2.5+(145\sin 23^\circ)t-16t^2=0$ for t, using the quadratic formula

$t=\frac{145\sin 23^\circ+\sqrt{(145\sin 23^\circ)^2+160}}{32}\approx 3.585$ sec. Then the range at $t\approx 3.585$ is about

$x=(145\cos 23^\circ-14)(3.585)\approx 428.262$ feet.

(d) For the time, solve $y=2.5+(145\sin 23^\circ)t-16t^2=20$ for t, using the quadratic formula

$t=\frac{145\sin 23^\circ\pm\sqrt{(145\sin 23^\circ)^2-1120}}{32}\approx 0.342$ and 3.199 seconds. At those times the ball is about $x(0.342)=(145\cos 23^\circ-14)(0.342)\approx 40.847$ feet from home plate and $x(3.199)=(145\cos 23^\circ-14)(3.199)$ ≈ 382.208 feet from home plate.

(e) Yes. According to part (d), the ball is still 20 feet above the ground when it is 382 feet from home plate.

27. (a) (Assuming that "x" is zero at the point of impact.)
$\mathbf{r}(t)=(x(t))\mathbf{i}+(y(t))\mathbf{j}$, where $x(t)=(35\cos 27^\circ)t$ and $y(t)=4+(35\sin 27^\circ)t-16t^2$.

(b) $y_{max}=\frac{(v_0\sin\alpha)^2}{2g}+4=\frac{(35\sin 27^\circ)^2}{64}+4\approx 7.945$ feet, which is reached at $t=\frac{v_0\sin\alpha}{g}$ $=\frac{35\sin 27^\circ}{32}\approx 0.497$ seconds.

(c) For the time, solve $y=4+(35\sin 27^\circ)t-16t^2=0$ for t, using the quadratic formula

$t=\frac{35\sin 27^\circ+\sqrt{(-35\sin 27^\circ)^2+256}}{32}\approx 1.201$ seconds. Then the range is about

$x(1.201)=(35\cos 27^\circ)(1.201)\approx 37.453$ feet.

(d) For the time, solve $y=4+(35\sin 27^\circ)t-16t^2=7$ for t, using the quadratic formula

$t=\frac{35\sin 27^\circ\pm\sqrt{(-35\sin 27^\circ)^2-192}}{32}\approx 0.254$ and 0.740 seconds. At those times the ball is about $x(0.254)=(35\cos 27^\circ)(0.254)\approx 7.906$ feet and $x(0.740)=(35\cos 27^\circ)(0.740)\approx 23.064$ feet from the impact point, or about $37.460-7.906\approx 29.554$ feet and $37.460-23.064\approx 14.396$ feet from the landing spot.

(e) Yes. It changes things because the ball won't clear the net ($y_{max}\approx 7.945$ ft).

28. The maximum height is $y=\frac{(v_0\sin\alpha)^2}{2g}$ and this occurs for $x=\frac{v_0^2}{2g}\sin 2\alpha=\frac{v_0^2\sin\alpha\cos\alpha}{g}$. These equations describe parametrically the points on a curve in the xy-plane associated with the maximum heights on the parabolic trajectories in terms of the parameter (launch angle) α. Eliminating the parameter α, we have

$$x^2 = \frac{v_0^4 \sin^2\alpha\cos^2\alpha}{g^2} = \frac{\left(v_0^4\sin^2\alpha\right)\left(1-\sin^2\alpha\right)}{g^2} = \frac{v_0^4\sin^2\alpha}{g^2} - \frac{v_0^4\sin^4\alpha}{g^2} = \frac{v_0^2}{g}(2y) - (2y)^2 \Rightarrow x^2 + 4y^2 - \left(\frac{2v_0^2}{g}\right)y = 0$$

$$\Rightarrow x^2 + 4\left[y^2 - \left(\frac{v_0^2}{2g}\right)y + \frac{v_0^4}{16g^2}\right] = \frac{v_0^4}{16g^2} \Rightarrow x^2 + 4\left(y - \frac{v_0^2}{4g}\right)^2 = \frac{v_0^4}{16g^2}, \text{ where } x \ge 0.$$

29. $\frac{d^2\mathbf{r}}{dt^2} + k\frac{d\mathbf{r}}{dt} = -g\mathbf{j} \Rightarrow P(t) = k$ and $\mathbf{Q}(t) = -g\mathbf{j} \Rightarrow \int P(t)\,dt = kt \Rightarrow v(t) = e^{\int P(t)\,dt} = e^{kt} \Rightarrow \frac{d\mathbf{r}}{dt}$

$= \frac{1}{v(t)}\int v(t)\mathbf{Q}(t)\,dt = -ge^{-kt}\int \mathbf{j}e^{kt}\,dt = -ge^{-kt}\left[\frac{e^{kt}}{k}\mathbf{j} + \mathbf{C}_1\right] = -\frac{g}{k}\mathbf{j} + \mathbf{C}e^{-kt}$, where $\mathbf{C} = -g\mathbf{C}_1$; apply the initial condition: $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j} = -\frac{g}{k}\mathbf{j} + \mathbf{C} \Rightarrow \mathbf{C} = (v_0\cos\alpha)\mathbf{i} + \left(\frac{g}{k} + v_0\sin\alpha\right)\mathbf{j}$

$\Rightarrow \frac{d\mathbf{r}}{dt} = \left(v_0e^{-kt}\cos\alpha\right)\mathbf{i} + \left[-\frac{g}{k} + e^{-kt}\left(\frac{g}{k} + v_0\sin\alpha\right)\right]\mathbf{j}$, $\mathbf{r} = \int \left\{\left(v_0e^{-kt}\cos\alpha\right)\mathbf{i} + \left[-\frac{g}{k} + e^{-kt}\left(\frac{g}{k} + v_0\sin\alpha\right)\right]\mathbf{j}\right\}dt$

$= \left(-\frac{v_0}{k}e^{-kt}\cos\alpha\right)\mathbf{i} + \left[-\frac{gt}{k} - \frac{e^{-kt}}{k}\left(\frac{g}{k} + v_0\sin\alpha\right)\right]\mathbf{j} + \mathbf{C}_2$; apply the initial condition:

$\mathbf{r}(0) = \mathbf{0} = \left(-\frac{v_0}{k}\cos\alpha\right)\mathbf{i} + \left(-\frac{g}{k^2} - \frac{v_0}{k}\sin\alpha\right)\mathbf{j} + \mathbf{C}_2 \Rightarrow \mathbf{C}_2 = \left(\frac{v_0}{k}\cos\alpha\right)\mathbf{i} + \left(\frac{g}{k^2} + \frac{v_0}{k}\sin\alpha\right)\mathbf{j}$

$\Rightarrow \mathbf{r}(t) = \left(\frac{v_0}{k}\left(1 - e^{-kt}\right)\cos\alpha\right)\mathbf{i} + \left[\frac{v_0}{k}\left(1 - e^{-kt}\right)\sin\alpha + \frac{g}{k^2}\left(1 - kt - e^{-kt}\right)\right]\mathbf{j}$

30. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$, where

$x(t) = \left(\frac{152}{0.12}\right)(1 - e^{-0.12t})(\cos 20°)$ and

$y(t) = 3 + \left(\frac{152}{0.12}\right)(1 - e^{-0.12t})(\sin 20°) + \left(\frac{32}{0.12^2}\right)(1 - 0.12t - e^{-0.12t})$

(b) Solve graphically using a calculator or CAS: At $t \approx 1.484$ seconds the ball reaches a maximum height of about 40.435 feet.

(c) Use a graphing calculator or CAS to find that $y = 0$ when the ball has traveled for ≈ 3.126 seconds. The range is about $x(3.126) = \left(\frac{152}{0.12}\right)(1 - e^{-0.12(3.126)})(\cos 20°) \approx 372.323$ feet.

(d) Use a graphing calculator or CAS to find that $y = 30$ for $t \approx 0.689$ and 2.305 seconds, at which times the ball is about $x(0.689) \approx 94.513$ feet and $x(2.305) \approx 287.628$ feet from home plate.

(e) Yes, the batter has hit a home run since a graph of the trajectory shows that the ball is more than 14 feet above the ground when it passes over the fence.

31. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$, where

$x(t) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08t})(152\cos 20° - 17.6)$ and

$y(t) = 3 + \left(\frac{152}{0.08}\right)(1 - e^{-0.08t})(\sin 20°) + \left(\frac{32}{0.08^2}\right)(1 - 0.08t - e^{-0.08t})$

(b) Solve graphically using a calculator or CAS: At $t \approx 1.527$ seconds the ball reaches a maximum height of about 41.893 feet.

(c) Use a graphing calculator or CAS to find that $y = 0$ when the ball has traveled for ≈ 3.181 seconds. The range is about $x(3.181) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08(3.181)})(152\cos 20° - 17.6) \approx 351.734$ feet.

(d) Use a graphing calculator or CAS to find that $y = 35$ for $t \approx 0.877$ and 2.190 seconds, at at which times the ball is about $x(0.877) \approx 106.028$ feet and $x(2.190) \approx 251.530$ feet from home plate.

(e) No; the range is less than 380 feet. To find the wind needed for a home run, first use the method of part (d) to find that $y = 20$ at $t \approx 0.376$ and 2.716 seconds. Then define

$x(w) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08(2.716)})(152\cos 20° + w)$,

and solve $x(w) = 380$ to find $w \approx 12.846$ ft/sec.

This is the speed of a wind gust needed in the direction of the hit for the ball to clear the fence for a home run.

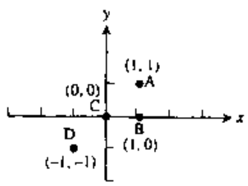
9.5 POLAR COORDINATES AND GRAPHS

For exercises 1 and 2, two pairs of polar coordinates label the same point if the r-coordinates are the same and the θ coordinates differ by an even multiple of π, or if the r-coordinates are opposites and the θ-coordinates differ by an odd multiple of π.

1. (a) and (e) are the same.
   (b) and (g) are the same.
   (c) and (h) are the same.
   (d) and (f) are the same.

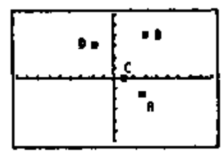
2. (a) and (f) are the same.
   (b) and (h) are the same.
   (c) and (g) are the same.
   (d) and (e) are the same.

3.



(a) $\left(\sqrt{2}\cos\frac{\pi}{4}, \sqrt{2}\sin\frac{\pi}{4}\right) = (1,1)$
(b) $(1\cos 0, 1\sin 0) = (1,0)$
(c) $\left(0\cos\frac{\pi}{2}, 0\sin\frac{\pi}{2}\right) = (0,0)$
(d) $\left(-\sqrt{2}\cos\frac{\pi}{4}, -\sqrt{2}\sin\frac{\pi}{4}\right) = (-1,-1)$

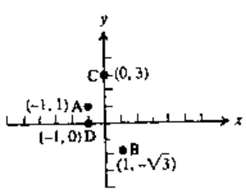
4.



[−9, 9] by [−6, 6]

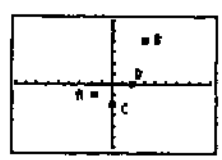
(a) $\left(-3\cos\frac{5\pi}{6}, -3\sin\frac{5\pi}{6}\right) = \left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$
(b) $\left(5\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right), 5\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right) = (3,4)$
(c) $(-1\cos 7\pi, -1\sin 7\pi) = (1,0)$
(d) $\left(2\sqrt{3}\cos\frac{2\pi}{3}, 2\sqrt{3}\sin\frac{2\pi}{3}\right) = (-\sqrt{3},3)$

5.



(a) $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, $\tan\theta = \frac{1}{-1} = -1$ with θ in quadrant II. The coordinates are $\left(\sqrt{2}, \frac{3\pi}{4}\right)$. $\left(\sqrt{2}, -\frac{5\pi}{4}\right)$ also works, since r is the same and θ differs by 2π.

6.



[−9, 9] by [−6, 6]

(a) $r = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = 2$, $\tan\theta = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$ with θ in quadrant III. The coordinates are $\left(2, \frac{7\pi}{6}\right)$. $\left(-2, \frac{\pi}{6}\right)$ also works, since r has the opposite sign and θ differs by π.

(b) $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$, $\tan\theta = -\frac{\sqrt{3}}{1} = -\sqrt{3}$ with θ in quadrant IV. The coordinates are $\left(2, -\frac{\pi}{3}\right)$. $\left(-2, \frac{2\pi}{3}\right)$ also works, since r has the opposite sign and θ differs by π.

(c) $r = \sqrt{0^2 + 3^2} = 3$, $\tan\theta = \frac{3}{0}$ is undefined with θ on the positive y-axis. The coordinates are $\left(3, \frac{\pi}{2}\right)$. $\left(3, \frac{5\pi}{2}\right)$ also works, since r is the same and θ differs by 2π.

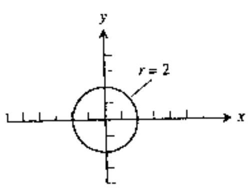
(d) $r = \sqrt{(-1)^2 + 0^2} = 1$, $\tan\theta = \frac{0}{-1} = 0$ with θ on the negative x-axis. The coordinates are $(1, \pi)$. $(-1, 0)$ also works, since r has the opposite sign and θ differs by π.

(b) $r = \sqrt{3^2 + 4^2} = 5$, $\tan\theta = \frac{4}{3}$ with θ in quadrant I. The coordinates are $\left(5, \tan^{-1}\frac{4}{3}\right)$. $\left(-5, \pi + \tan^{-1}\frac{4}{3}\right)$ also works, since r has the opposite sign and θ differs by π.

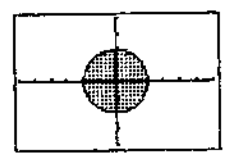
(c) $r = \sqrt{0 + (-2)^2} = 2$, $\tan\theta = -\frac{2}{0}$ is undefined with θ on the negative y-axis. The coordinates are $\left(2, \frac{3\pi}{2}\right)$. $\left(2, -\frac{\pi}{2}\right)$ also works, since r is the same and θ differs by 2π.

(d) $r = \sqrt{2^2 + 0^2} = 2$, $\tan\theta = \frac{0}{2} = 0$ with θ on the positive x-axis. The coordinates are $(2, 0)$. $(2, 2\pi)$ also works, since r is the same and θ differs by 2π.

7.

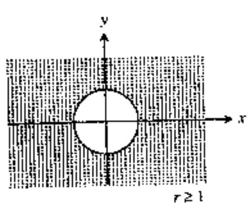


8.

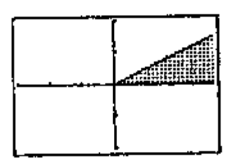


[−6, 6] by [−4, 4]

9.

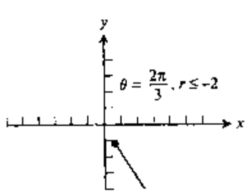


10.

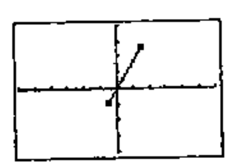


[−3, 3] by [−2, 2]

11.

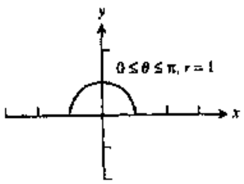


12.

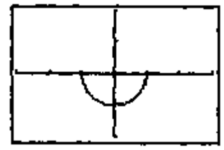


[−6, 6] by [−4, 4]

13.

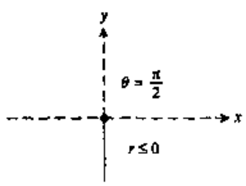


14.

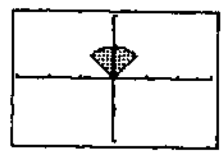


$[-3, 3]$ by $[-2, 2]$

15.

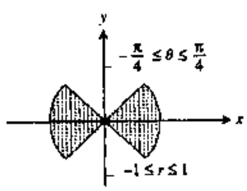


16.

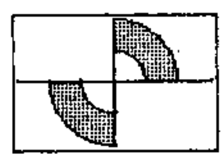


$[-3, 3]$ by $[-2, 2]$

17.



18.



$[-3, 3]$ by $[-2, 2]$

19. $y = r \sin \theta$, so the equation is $y = 0$, which is the x-axis.

20. $x = r \cos \theta$, so the equation is $x = 0$, which is the y-axis.

21. $r = 4 \csc \theta$
$r \sin \theta = 4$
$y = r \sin \theta$, so the equation is $y = 4$,
a horizontal line.

22. $r = -3 \sec \theta$
$r \cos \theta = -3$
$x = r \cos \theta$, so the equation is $x = -3$, a
vertical line.

23. $x = r \cos \theta$ and $y = r \sin \theta$, so the equation is $x + y = 1$, a line (slope $= -1$, y-intercept $= 1$).

24. $x^2 + y^2 = r^2$, so the equation is $x^2 + y^2 = 1$, a circle (center $= (0,0)$, radius $= 1$).

25. $x^2 + y^2 = r^2$ and $y = r \sin \theta$, so the equation is $x^2 + y^2 = 4y \Rightarrow x^2 + (y-2)^2 = 4$, a circle (center $= (0,2)$, radius $= 2$).

26. $r = \frac{5}{\sin\theta - 2\cos\theta} \Rightarrow r\sin\theta - 2r\cos\theta = 5$; $x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $y - 2x = 5$, a line (slope $= 2$, y-intercept $= 5$).

27. $r^2\sin 2\theta = 2 \Rightarrow 2r^2\sin\theta\cos\theta = 2 \Rightarrow (r\sin\theta)(r\cos\theta) = 1$; $x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $xy = 1 \left(\text{or, } y = \frac{1}{x}\right)$, a hyperbola.

28. $r = \cot\theta\csc\theta \Rightarrow r\sin\theta = \cot\theta$; $y = r\sin\theta$ and $\frac{x}{y} = \cot\theta$, so the equation is $y^2 = x$, a parabola.

29. $r = \csc\theta\, e^{r\cos\theta} \Rightarrow r\sin\theta = e^{r\cos\theta}$; $x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $y = e^x$, the exponential curve.

30. $\cos^2\theta = \sin^2\theta \Rightarrow (r\cos\theta)^2 = (r\sin\theta)^2$; $x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $x^2 = y^2$ or $y = \pm x$, the union of two lines.

31. $r\sin\theta = \ln r + \ln\cos\theta \Rightarrow r\sin\theta = \ln(r\cos\theta) \Rightarrow y = \ln x$, the logarithmic curve.

32. $r^2 + 2r^2\cos\theta\sin\theta = 1 \Rightarrow r^2 + 2(r\cos\theta)(r\sin\theta) = 1 \Rightarrow x^2 + y^2 + 2xy = 1 \Rightarrow (x+y)^2 = 1 \Rightarrow x + y = \pm 1$, the union of two lines.

33. $r^2 = -4r\sin\theta \Rightarrow x^2 + y^2 = -4x \Rightarrow x^2 + (y-4)^2 = 16$, a circle (center $= (0,4)$, radius $= 4$)

34. $r = 8\sin\theta \Rightarrow r^2 = 8r\sin\theta \Rightarrow x^2 + y^2 = 8y \Rightarrow (x+2)^2 + y^2 = 4$, a circle (center $= (-2,0)$, radius $= 2$).

35. $r = 2\cos\theta + 2\sin\theta \Rightarrow r^2 = 2r\cos\theta + 2r\sin\theta \Rightarrow x^2 + y^2 = 2x + 2y \Rightarrow (x-1)^2 + (y-1)^2 = 2$, a circle (center $= (1,1)$, radius $= \sqrt{2}$)

36. $r\sin\left(\theta + \frac{\pi}{6}\right) = 2 \Rightarrow r\left(\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right) = 2 \Rightarrow \frac{\sqrt{3}}{2}r\sin\theta + \frac{1}{2}r\cos\theta = 2 \Rightarrow \frac{\sqrt{3}}{2}y + \frac{1}{2}x = 2$
$\Rightarrow x + \sqrt{3}y = 4$, a line $\left(\text{slope} = -\frac{1}{\sqrt{3}}, \text{ y-intercept} = \frac{4}{\sqrt{3}}\right)$.

37. $x = 7 \Rightarrow r\cos\theta = 7$; The graph is a vertical line.

38. $y = 1 \Rightarrow r\sin\theta = 1$; The graph is a horizontal line.

39. $x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$. More generally, $\theta = \frac{\pi}{4} + 2k\pi$ for any integer k. The graph is a slanted line.

40. $x - y = 3 \Rightarrow r\cos\theta - r\sin\theta = 3$

41. $x^2 + y^2 = 4 \Rightarrow r^2 = 4$ or $r = 2$ (or $r = -2$)

42. $x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1$

43. $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{r^2\cos^2\theta}{9} + \frac{r^2\sin^2\theta}{4} = 1 \Rightarrow r^2(4\cos^2\theta + 9\sin^2\theta) = 36$

44. $xy = 2 \Rightarrow (r\cos\theta)(r\sin\theta) = 2 \Rightarrow r^2\cos\theta\sin\theta = 2 \Rightarrow r^2\, 2\cos\theta\sin\theta = 4 \Rightarrow r^2\sin 2\theta = 4$

45. $y^2 = 4x \Rightarrow r^2\sin^2\theta = 4r\cos\theta \Rightarrow r\sin^2\theta = 4\cos\theta$

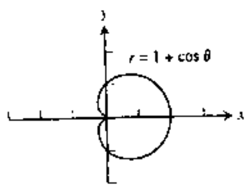
46. $x^2 + xy + y^2 = 1 \Rightarrow (r\cos\theta)^2 + (r\cos\theta)(r\sin\theta) + (r\sin\theta)^2 = 1 \Rightarrow r^2(1 + \cos\theta\sin\theta) = 1$

47. $x^2 + (y-2)^2 = 4 \Rightarrow r^2\cos^2\theta + (r\sin\theta - 2)^2 = 4 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta + 4 = 4$
$\Rightarrow r^2 - 4r\sin\theta = 0 \Rightarrow r = 4\sin\theta$. The graph is a circle centered at $(0, 2)$ with radius 2.

48. $(x-3)^2 + (y+1)^2 = 4 \Rightarrow (r\cos\theta - 3)^2 + (r\sin\theta + 1)^2 = 4$
$\Rightarrow r^2\cos^2\theta - 6r\cos\theta + 9 + r^2\sin^2\theta + 2r\sin\theta + 1 = 4 \Rightarrow r^2 - 6r\cos\theta + 2r\sin\theta + 6 = 0$
$\Rightarrow r = \dfrac{6\cos\theta - 2\sin\theta \pm \sqrt{(6\cos\theta - 2\sin\theta)^2 - 24}}{2} \Rightarrow r = 3\cos\theta - \sin\theta \pm \sqrt{(3\cos\theta - \sin\theta)^2 - 6}$

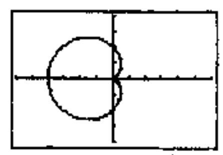
In Exercises 49-58, find the minimum θ-interval by trying different intervals on a graphing calculator.

49. (a)



(b) Length of interval $= 2\pi$

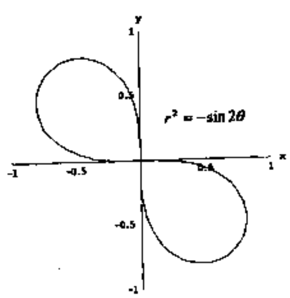
50. (a)



[−6, 6] by [−4, 4]

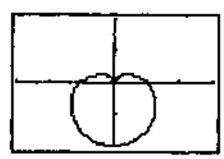
(b) Length of interval $= 2\pi$

51. (a)



(b) Length of interval $= \frac{\pi}{2}$

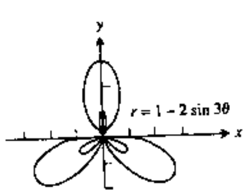
52. (a)



[−3, 3] by [−2, 2]

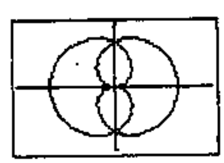
(b) Length of interval $= 2\pi$

53. (a)



(b) Length of interval $= 2\pi$

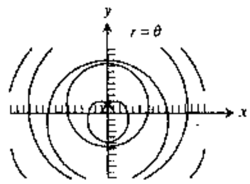
54. (a)



[−1.5, 1.5] by [−1, 1]

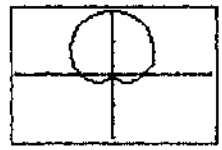
(b) Length of interval $= 4\pi$

55. (a)



(b) Required interval = $(-\infty, \infty)$

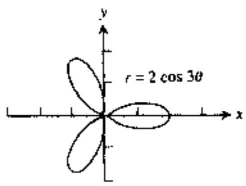
56. (a)



[−3, 3] by [−2, 2]

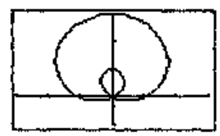
(b) Length of interval = 2π

57. (a)



(b) Length of interval = π

58. (a)



[−3, 3] by [−1, 3]

(b) Length of interval = 2π

59. If (r, θ) is a solution, so is $(-r, \theta)$. Therefore, the curve is symmetric about the origin. And if (r, θ) is a solution, so is $(r, -\theta)$. Therefore, the curve is symmetric about the x-axis. And since any curve with x-axis and origin symmetry also has y-axis symmetry, the curve is symmetric about the y-axis.

60. If (r, θ) is a solution, so is $(-r, \theta)$. Therefore, the curve is symmetric about the origin. The curve does not have x-axis or y-axis symmetry.

61. If (r, θ) is a solution, so is $(r, \pi - \theta)$. Therefore, the curve is symmetric about the y-axis. The curve does not have x-axis or origin symmetry.

62. If (r, θ) is a solution, so is $(-r, \theta)$. Therefore, the curve is symmetric about the origin. And if (r, θ) is a solution, so is $(r, -\theta)$. Therefore, the curve is symmetric about the x-axis. And since any curve with x-axis and origin symmetry also has y-axis symmetry, the curve is symmetric about the y-axis.

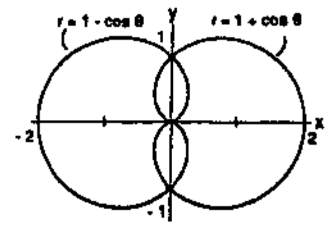
63. (a) Because $r = a \sec \theta$ is equivalent to $r \cos \theta = a$, which is equivalent to the Cartesian equation $x = a$.
(b) $r = a \csc \theta$ is equivalent to $y = a$.

64. (a) Let $r = f(\theta)$ be symmetric about the x-axis and the y-axis. Then (r, θ) on the graph $\Rightarrow (r, -\theta)$ is on the graph because of symmetry about the x-axis. Then $(-r, -(-\theta)) = (-r, \theta)$ is on the graph because of symmetry about the y-axis. Therefore $r = f(\theta)$ is symmetric about the origin.
(b) Let $r = f(\theta)$ be symmetric about the x-axis and the origin. Then (r, θ) on the graph $\Rightarrow (r, -\theta)$ is on the graph because of symmetry about the x-axis. Then $(-r, -\theta)$ is on the graph because of symmetry about the origin. Therefore $r = f(\theta)$ is symmetric about the y-axis.
(c) Let $r = f(\theta)$ be symmetric about the y-axis and the origin. Then (r, θ) on the graph $\Rightarrow (-r, -\theta)$ is on the graph because of symmetry about the y-axis. Then $(-(-r), -\theta) = (r, -\theta)$ is on the graph because of symmetry about the origin. Therefore $r = f(\theta)$ is symmetric about the x-axis.

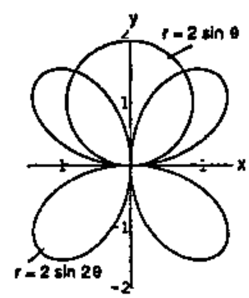
65. $\left(2, \frac{3\pi}{4}\right)$ is the same point as $\left(-2, -\frac{\pi}{4}\right)$; $r = 2 \sin 2\left(-\frac{\pi}{4}\right) = 2 \sin\left(-\frac{\pi}{2}\right) = -2 \Rightarrow \left(-2, -\frac{\pi}{4}\right)$ is on the graph $\Rightarrow \left(2, \frac{3\pi}{4}\right)$ is on the graph

66. $\left(\frac{1}{2}, \frac{3\pi}{2}\right)$ is the same point as $\left(-\frac{1}{2}, \frac{\pi}{2}\right)$; $r = -\sin\left(\frac{\left(\frac{\pi}{2}\right)}{3}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2} \Rightarrow \left(-\frac{1}{2}, \frac{\pi}{2}\right)$ is on the graph $\Rightarrow \left(\frac{1}{2}, \frac{3\pi}{2}\right)$ is on the graph

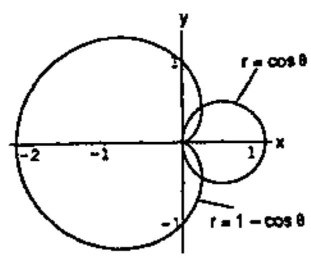
67. $1 + \cos\theta = 1 - \cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow r = 1$; points of intersection are $\left(1, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$. The point of intersection $(0,0)$ is found by graphing.



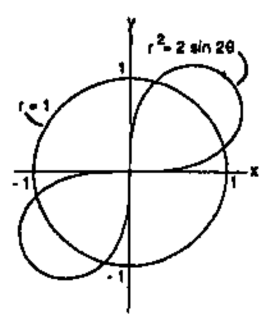
68. $2\sin\theta = 2\sin 2\theta \Rightarrow \sin\theta = \sin 2\theta \Rightarrow \sin\theta = 2\sin\theta\cos\theta \Rightarrow \sin\theta - 2\sin\theta\cos\theta = 0 \Rightarrow (\sin\theta)(1 - 2\cos\theta) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \Rightarrow \theta = 0, \frac{\pi}{3}$, or $-\frac{\pi}{3}$; $\theta = 0 \Rightarrow r = 0$, $\theta = \frac{\pi}{3} \Rightarrow r = \sqrt{3}$, and $\theta = -\frac{\pi}{3} \Rightarrow r = -\sqrt{3}$; points of intersection are $(0,0)$, $\left(\sqrt{3}, \frac{\pi}{3}\right)$, and $\left(-\sqrt{3}, -\frac{\pi}{3}\right)$



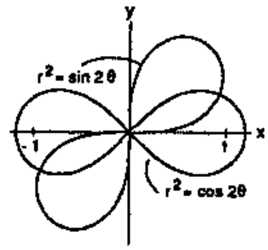
69. $\cos\theta = 1 - \cos\theta \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, -\frac{\pi}{3} \Rightarrow r = \frac{1}{2}$; points of intersection are $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, -\frac{\pi}{3}\right)$. The point $(0,0)$ is found by graphing.



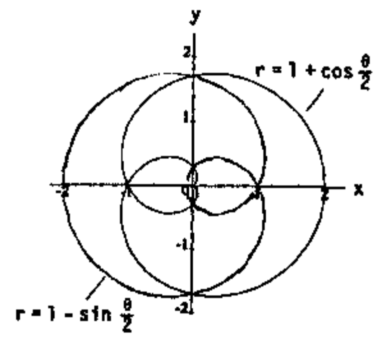
70. $1 = 2\sin 2\theta \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$; points of intersection are $\left(1, \frac{\pi}{12}\right)$, $\left(1, \frac{5\pi}{12}\right)$, $\left(1, \frac{13\pi}{12}\right)$, and $\left(1, \frac{17\pi}{12}\right)$. No other points are found by graphing.



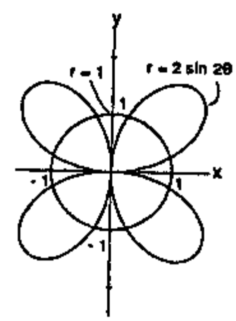
71. $r^2 = \sin 2\theta$ and $r^2 = \cos 2\theta$ are generated completely for $0 \le \theta \le \frac{\pi}{2}$. Then $\sin 2\theta = \cos 2\theta \Rightarrow 2\theta = \frac{\pi}{4}$ is the only solution on that interval $\Rightarrow \theta = \frac{\pi}{8} \Rightarrow r^2 = \sin 2\left(\frac{\pi}{8}\right) = \frac{1}{\sqrt{2}}$ $\Rightarrow r = \pm\frac{1}{\sqrt[4]{2}}$; points of intersection are $\left(\pm\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$. The point of intersection $(0,0)$ is found by graphing.



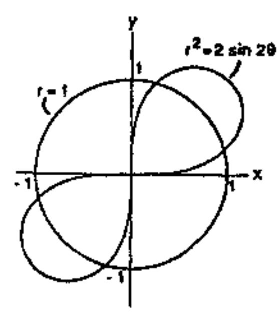
72. $1 - \sin\frac{\theta}{2} = 1 + \cos\frac{\theta}{2} \Rightarrow -\sin\frac{\theta}{2} = \cos\frac{\theta}{2} \Rightarrow \frac{\theta}{2} = \frac{3\pi}{4}, \frac{7\pi}{4}$ $\Rightarrow \theta = \frac{3\pi}{2}, \frac{7\pi}{2}$; $\theta = \frac{3\pi}{2} \Rightarrow r = 1 + \cos\frac{3\pi}{4} = 1 - \frac{\sqrt{2}}{2}$; $\theta = \frac{7\pi}{2} \Rightarrow r = 1 + \cos\frac{7\pi}{4} = 1 + \frac{\sqrt{2}}{2}$; points of intersection are $\left(1 - \frac{\sqrt{2}}{2}, \frac{3\pi}{2}\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, \frac{7\pi}{2}\right)$. The three points of intersection $(0,0)$ and $\left(1 \pm \frac{\sqrt{2}}{2}, \frac{\pi}{2}\right)$ are found by graphing and symmetry.



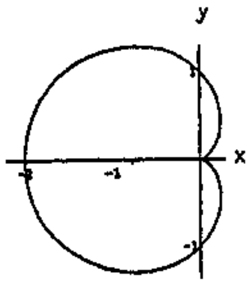
73. $1 = 2\sin 2\theta \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ $\Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$; points of intersection are $\left(1, \frac{\pi}{12}\right)$, $\left(1, \frac{5\pi}{12}\right)$, $\left(1, \frac{13\pi}{12}\right)$, and $\left(1, \frac{17\pi}{12}\right)$. The points of intersection $\left(1, \frac{7\pi}{12}\right)$, $\left(1, \frac{11\pi}{12}\right)$, $\left(1, \frac{19\pi}{12}\right)$ and $\left(1, \frac{23\pi}{12}\right)$ are found by graphing and symmetry.



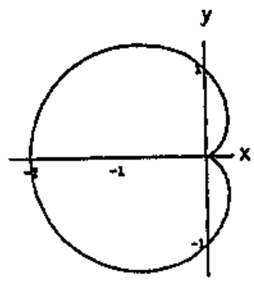
74. $r^2 = 2\sin 2\theta$ is completely generated on $0 \le \theta \le \frac{\pi}{2}$ so that $1 = 2\sin 2\theta \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$; points of intersection are $\left(1, \frac{\pi}{12}\right)$ and $\left(1, \frac{5\pi}{12}\right)$. The points of intersection $\left(-1, \frac{\pi}{12}\right)$ and $\left(-1, \frac{5\pi}{12}\right)$ are found by graphing.



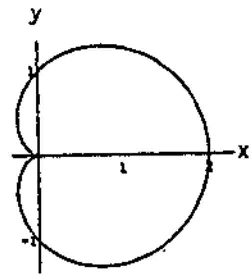
75. Note that (r, θ) and $(-r, \theta + \pi)$ describe the same point in the plane. Then $r = 1 - \cos\theta \Leftrightarrow -1 - \cos(\theta + \pi)$ $= -1 - (\cos\theta \cos\pi - \sin\theta \sin\pi) = -1 + \cos\theta = -(1 - \cos\theta) = -r$; therefore (r, θ) is on the graph of $r = 1 - \cos\theta \Leftrightarrow (-r, \theta + \pi)$ is on the graph of $r = -1 - \cos\theta \Rightarrow$ the answer is (a).



$r = 1 - \cos\theta$



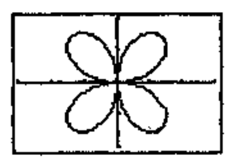
$r = -1 - \cos\theta$



$r = 1 + \cos\theta$

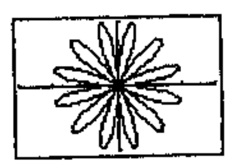
76. (a) The graph is the same for $n = 2$ and $n = -2$, and in general, it's the same for $n = 2k$ and $n = -2k$. The graphs for $n = 2$, 4, and 6 are roses with 4, 8, and 12 "petals" respectively. The graphs for $n = \pm 2$ and $n = \pm 6$ are shown below.

$n = \pm 2$



$[-3, 3]$ by $[-2, 2]$

$n = \pm 6$

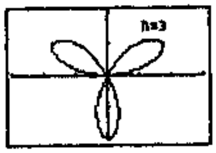


$[-3, 3]$ by $[-2, 2]$

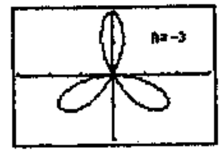
(b) 2π

(c) The graph is a rose with $2|n|$ "petals."

(d) The graphs are roses with 3, 5, and 7 "petals" respectively. The "center petal" points upward if $n = -3$, $+5$, or -7.
The graphs for $n = 3$ and $n = -3$ are shown below.



[−3, 3] by [−2, 2]

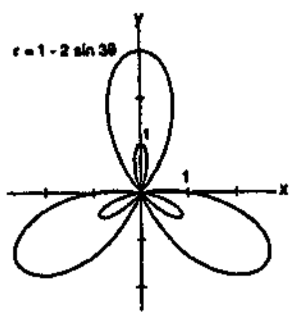


[−3, 3] by [−2, 2]

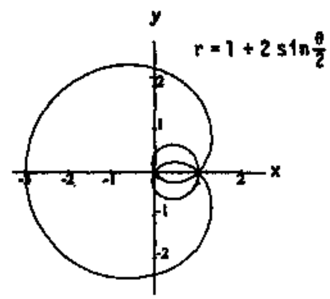
(e) π

(f) The graph is a rose with $|n|$ "petals."

77.



78.



79. (a) $r^2 = -4\cos\theta \Rightarrow \cos\theta = -\frac{r^2}{4}$; $r = 1 - \cos\theta \Rightarrow r = 1 - \left(-\frac{r^2}{4}\right) \Rightarrow 0 = r^2 - 4r + 4 \Rightarrow (r-2)^2 = 0$

$\Rightarrow r = 2$; therefore $\cos\theta = -\frac{2^2}{4} = -1 \Rightarrow \theta = \pi \Rightarrow (2, \pi)$ is a point of intersection

(b) $r = 0 \Rightarrow 0^2 = 4\cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow \left(0, \frac{\pi}{2}\right)$ or $\left(0, \frac{3\pi}{2}\right)$ is on the graph; $r = 0 \Rightarrow 0 = 1 - \cos\theta$

$\Rightarrow \cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow (0,0)$ is on the graph. Since $(0,0) = \left(0, \frac{\pi}{2}\right)$ for polar coordinates, the graphs intersect at the origin.

80. (a) We have $x = r\cos\theta$ and $y = r\sin\theta$. By taking $t = \theta$, we have $r = f(t)$, so $x = f(t)\cos t$ and $y = f(t)\sin t$.

(b) $x = 3\cos t$, $y = 3\sin t$

(c) $x = (1 - \cos t)\cos t$, $y = (1 - \cos t)\sin t$

(d) $x = (3\sin 2t)\cos t$, $y = (3\sin 2t)\sin t$

81. $d = \sqrt{(x_2 - x_1)^2 + (y_1 - y_2)^2}$

$$= \left[(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2\right]^{1/2}$$

$$= \left[r_2^2\cos^2\theta_2 - 2r_2r_1\cos\theta_2\cos\theta_1 + r_1^2\cos^2\theta_1 + r_2^2\sin^2\theta_2 - 2r_2r_1\sin\theta_2\sin\theta_1 + r_1^2\sin^2\theta_1\right]^{1/2}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

82. We wish to maximize $y = r \sin\theta = 2(1+\cos\theta)(\sin\theta) = 2\sin\theta + 2\sin\theta\cos\theta$. Then $\frac{dy}{d\theta} = 2\cos\theta + 2(\sin\theta)(-\sin\theta) + 2\cos\theta\cos\theta = 2\cos\theta - 2\sin^2\theta + 2\cos^2\theta = 2\cos\theta + 4\cos^2\theta - 2$; thus $\frac{dy}{d\theta} = 0 \Rightarrow 4\cos^2\theta + 2\cos\theta - 2 = 0 \Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0 \Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0 \Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -1 \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}, \pi$. From the graph, we can see that the maximum occurs in the first quadrant so we choose $\theta = \frac{\pi}{3}$. Then $y = 2\sin\frac{\pi}{3} + 2\sin\frac{\pi}{3}\cos\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$. The x-coordinate of this point is $x = r\cos\frac{\pi}{3}$ $= 2\left(1+\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3}\right) = \frac{3}{2}$. Thus the maximum height is $h = \frac{3\sqrt{3}}{2}$ occurring at $x = \frac{3}{2}$.

9.6 CALCULUS OF POLAR CURVES

1. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\cos\theta\sin\theta + (-1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (-1+\sin\theta)\sin\theta} = \frac{2\sin\theta\cos\theta - \cos\theta}{\cos^2\theta - \sin^2\theta + \sin\theta}$

$\left.\frac{dy}{dx}\right|_{\theta=0} = -\frac{1}{1} = -1$, $\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{1}{1} = 1$

2. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-2\sin 2\theta\sin\theta + \cos 2\theta\cos\theta}{-2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}$

$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{1}{0}$, which is undefined; $\left.\frac{dy}{dx}\right|_{\theta=\pm\pi/2} = \pm\frac{0}{1} = 0$; and $\left.\frac{dy}{dx}\right|_{\theta=\pi} = -\frac{1}{0}$, which is undefined.

3. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-3\cos\theta\sin\theta + (2-3\sin\theta)\cos\theta}{-3\cos\theta\cos\theta - (2-3\sin\theta)\sin\theta} = \frac{2\cos\theta - 6\sin\theta\cos\theta}{-2\sin\theta - 3(\cos^2\theta - \sin^2\theta)}$

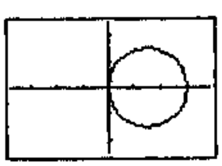
$\left.\frac{dy}{dx}\right|_{(2,0)} = \left.\frac{dy}{dx}\right|_{\theta=0} = \frac{2}{-3} = -\frac{2}{3}$, $\left.\frac{dy}{dx}\right|_{(-1,\pi/2)} = \left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{0}{-1} = 0$, $\left.\frac{dy}{dx}\right|_{(2,\pi)} = \left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{2}{3}$, and $\left.\frac{dy}{dx}\right|_{(5,3\pi/2)} = \left.\frac{dy}{dx}\right|_{\theta=3\pi/2} = \frac{0}{-5} = 0$.

4. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{3\sin^2\theta + 3\cos\theta(1-\cos\theta)}{3\sin\theta\cos\theta - 3\sin\theta(1-\cos\theta)} = \frac{3\cos\theta - 3(\cos^2\theta - \sin^2\theta)}{6\sin\theta\cos\theta - 3\sin\theta}$

$\left.\frac{dy}{dx}\right|_{(1.5,\pi/3)} = \frac{\frac{1}{2} - \left(-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}$, which is undefined; $\left.\frac{dy}{dx}\right|_{(4.5,2\pi/3)} = \frac{-\frac{1}{2} - \left(-\frac{1}{2}\right)}{-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}} = 0$;

$\left.\frac{dy}{dx}\right|_{(6,\pi)} = \frac{-1-1}{0-0}$, which is undefined; and $\left.\frac{dy}{dx}\right|_{(3,3\pi/2)} = \frac{0-(-1)}{0-(-1)} = 1$.

5.



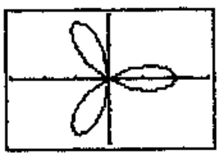
[−3.8, 3.8] by [−2.5, 2.5]

The graph passes through the pole when $r = 3\cos\theta = 0$, which occurs when $\theta = \frac{\pi}{2}$ and when $\theta = \frac{3\pi}{2}$. Since the θ-interval $0 \le \theta \le \pi$ produce the entire graph, we need only consider $\theta = \frac{\pi}{2}$. At this point, there appears to be a vertical tangent line with equation $\theta = \frac{\pi}{2}$ (or $x = 0$). Confirm analytically:

$x = (3\cos\theta)\cos\theta = 3\cos^2\theta$ and $y = (3\cos\theta)\sin\theta$;

$\frac{dy}{d\theta} = (-3\sin\theta)\sin\theta + (3\cos\theta)\cos\theta = 3(\cos^2\theta - \sin^2\theta)$ and $\frac{dx}{d\theta} = 6\cos\theta(-\sin\theta)$. At $\left(0, \frac{\pi}{2}\right)$, $\left.\frac{dx}{d\theta}\right|_{\theta=\pi/2} = 0$, and $\left.\frac{dy}{d\theta}\right|_{\theta=\pi/2} = 3(0^2 - 1^2) = -3$. So at $\left(0, \frac{\pi}{2}\right)$, $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \ne 0$, so $\frac{dy}{dx}$ is undefined and the tangent line is vertical.

6.



[−3, 3] by [−2, 2]

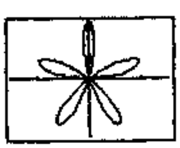
A trace of the graph suggests three tangent lines, one with positive slope for $\theta = \frac{\pi}{6}$, a vertical one for $\theta = \frac{\pi}{2}$, and one with negative slope for $\theta = \frac{5\pi}{6}$. Confirm analytically:

$\frac{dy}{d\theta} = -6\sin 3\theta\sin\theta + 2\cos 3\theta\cos\theta$ and $\frac{dx}{dt} = -6\sin 3\theta\cos\theta - 2\cos 3\theta\sin\theta$. $\left(0, \frac{\pi}{6}\right)$, $\left(0, \frac{\pi}{2}\right)$, and $\left(0, \frac{5\pi}{6}\right)$ are all solutions. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$, and so $\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{-6(1)(1/2) + 2(0)\left(\sqrt{3}/2\right)}{-6(1)\left(\sqrt{3}/2\right) - 2(0)(1/2)} = \frac{1}{\sqrt{3}}$;

$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{-6(-1)(1) + 2(0)(0)}{-6(-1)(0) - 2(0)(1)}$, which is undefined; and $\left.\frac{dy}{dx}\right|_{\theta=5\pi/6} = \frac{-6(1)(1/2) + 2(0)\left(-\sqrt{3}/2\right)}{-6(1)\left(-\sqrt{3}/2\right) - 2(0)(1/2)} = -\frac{1}{\sqrt{3}}$.

The tangent lines have equations $\theta = \frac{\pi}{6}\left[y = \frac{1}{\sqrt{3}}x\right]$, $\theta = \frac{\pi}{2}[x = 0]$, and $\theta = \frac{5\pi}{6}\left[y = -\frac{1}{\sqrt{3}}x\right]$.

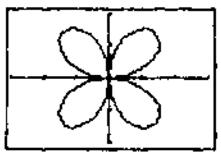
7.



[−1.5, 1.5] by [−1, 1]

The polar solutions are $\left(0, \frac{k\pi}{5}\right)$ for k = 0, 1, 2, 3, 4, and for a given k, the line $\theta = \frac{k\pi}{5}$ appears to be tangent to the curve at $\left(0, \frac{k\pi}{5}\right)$. This can be confirmed analytically by noting that the slope of the curve, $\frac{dy}{dx}$, equals the slope of the line, $\tan \frac{k\pi}{5}$. so the tangent lines are $\theta = 0$ [$y = 0$], $\theta = \frac{\pi}{5}\left[y = \left(\tan \frac{\pi}{5}\right)x\right]$, $\theta = \frac{2\pi}{5}\left[y = \left(\tan \frac{2\pi}{5}\right)x\right]$, $\theta = \frac{3\pi}{5}\left[y = \left(\tan \frac{3\pi}{5}\right)x\right]$, and $\theta = \frac{4\pi}{5}\left[y = \left(\tan \frac{4\pi}{5}\right)x\right]$.

8.



[−3, 3] by [−2, 2]

The polar solutions are $\left(0, \frac{k\pi}{2}\right)$ for k = 0, 1, 2, 3, 4, and for a given k, the line $\theta = \frac{k\pi}{2}$ appears to be tangent to the curve at $\left(0, \frac{k\pi}{2}\right)$. This can be confirmed analytically by noting that the slope of the curve, $\frac{dy}{dx}$, equals the slope of the line, $\tan \frac{k\pi}{2}$. So the tangent lines are $\theta = 0$ [$y = 0$] and $\theta = \frac{\pi}{2}$ [$x = 0$]. ($\theta = \pi$, $\theta = \frac{3\pi}{2}$ and $\theta = 2\pi$ are duplicate solutions.)

9. $\frac{dy}{d\theta} = \cos\theta \sin\theta + (-1 + \sin\theta)\cos\theta = \cos\theta(2\sin\theta - 1) = \sin 2\theta - \cos\theta$

$\frac{dx}{d\theta} = \cos^2\theta - (-1 + \sin\theta)\sin\theta = \cos^2\theta + \sin\theta - \sin^2\theta = -2\sin^2\theta + \sin\theta + 1$

$\frac{dy}{d\theta} = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ ($\cos\theta = 0$) or when $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ ($2\sin\theta - 1 = 0$). $\frac{dx}{d\theta} = 0$ when $\sin\theta = \frac{-1 \pm \sqrt{9}}{-4}$ $= -\frac{1}{2}$ or 1, i.e., when $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$, or $\frac{\pi}{2}$. So there is a horizontal tangent line for $\theta = \frac{3\pi}{2}$, $r = -2$ [the line $y = -2\sin\frac{3\pi}{2} = 2$], for $\theta = \frac{\pi}{6}$, $r = -\frac{1}{2}$ [the line $y = -\frac{1}{2}\sin\frac{\pi}{6} = -\frac{1}{4}$] and for $\theta = \frac{5\pi}{6}$, $r = -\frac{1}{2}$ [again, the line $y = -\frac{1}{2}\sin\frac{5\pi}{6} = -\frac{1}{4}$]. There is a vertical tangent line for $\theta = \frac{7\pi}{6}$, $r = -\frac{3}{2}$ [the line $x = -\frac{3}{2}\cos\frac{7\pi}{6} = \frac{3\sqrt{3}}{4}$] and for $\theta = \frac{11\pi}{6}$, $r = -\frac{3}{2}$ [the line $x = -\frac{3}{2}\cos\frac{11\pi}{6} = -\frac{3\sqrt{3}}{4}$]. For $\theta = \frac{\pi}{2}$, $\frac{dy}{d\theta} = \frac{dx}{d\theta} = 0$, but $\frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = 2\cos 2\theta + \sin\theta = -1$ for $\theta = \frac{\pi}{2}$ and $\frac{d}{d\theta}\left(\frac{dx}{d\theta}\right) = -4\sin\theta\cos\theta + \cos\theta = 0$ for $\theta = \frac{\pi}{2}$, so by L'Hôpital's rule $\frac{dy}{dx}$ is undefined and the tangent line is vertical at $\theta = \frac{\pi}{2}$, $r = 0$ [the line $x = 0$]. This information can be summarized as follows.

Horizontal at: $\left(-\frac{1}{2}, \frac{\pi}{6}\right)\left[y = -\frac{1}{4}\right]$, $\left(-\frac{1}{2}, \frac{5\pi}{6}\right)\left[y = -\frac{1}{4}\right]$, $\left(-2, \frac{3\pi}{2}\right)$ [$y = 2$]

Vertical at: $\left(0, \frac{\pi}{2}\right)$ [$x = 0$], $\left(-\frac{3}{2}, \frac{7\pi}{6}\right)\left[x = \frac{3\sqrt{3}}{4}\right]$, $\left(-\frac{3}{5}, \frac{11\pi}{6}\right)\left[x = -\frac{3\sqrt{3}}{4}\right]$

10. $\frac{dy}{d\theta} = -\sin^2\theta + (1+\cos\theta)\cos\theta = \cos^2\theta + \cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1$

$\frac{dx}{d\theta} = -\sin\theta\cos\theta - (1+\cos\theta)\sin\theta = -\sin\theta(1+2\cos\theta) = -\sin 2\theta - \sin\theta$

$\frac{dy}{d\theta} = 0$ when $\cos\theta = \frac{-1\pm\sqrt{9}}{4} = -1$ or $\frac{1}{2}$, i.e., when $\theta = \pi, \frac{\pi}{3}$ or $\frac{5\pi}{3}$. $\frac{dx}{d\theta} = 0$ when $\theta = 0, \pi, 2\pi$ (then $\sin\theta = 0$) or when $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ (then $1 + 2\cos\theta = 0$). So there is a horizontal tangent line for $\theta = \frac{\pi}{3}$, $r = \frac{3}{2}$ [the line $y = \frac{3}{2}\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{4}$] and for $\theta = \frac{5\pi}{3}$, $r = \frac{3}{2}$ [the line $y = \frac{3}{2}\sin\frac{5\pi}{3} = -\frac{3\sqrt{3}}{4}$]. There is a vertical tangent line for $\theta = 0$, $r = 2$ [the line $x = 2\cos 0 = 2$], for $\theta = \frac{2\pi}{3}$, $r = \frac{1}{2}$ [the line $x = \frac{1}{2}\cos\frac{2\pi}{3} = -\frac{1}{4}$] and for $\theta = \frac{4\pi}{3}$, $r = \frac{1}{2}$ [again, the line $x = \frac{1}{2}\cos\frac{2\pi}{3} = -\frac{1}{4}$]. For $\theta = \pi$, $\frac{dy}{d\theta} = \frac{dx}{d\theta} = 0$, but $\frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = -4\cos\theta\sin\theta - \sin\theta = 0$ for $\theta = \pi$, and $\frac{d}{d\theta}\left(\frac{dx}{d\theta}\right) = -2\cos 2\theta - \cos\theta = -1$ for $\theta = \pi$, so by L'Hôpital's rule $\frac{dy}{dx} = 0$ and the tangent line is horizontal at $\theta = \pi$, $r = 0$ [the line $y = 0$]. This information can be summarized as follows.

Horizontal at: $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ [$y = \frac{3\sqrt{3}}{4}$], $(0, \pi)$ [$y = 0$], $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$ [$y = -\frac{3\sqrt{3}}{4}$]

Vertical at: $(2, 0)$ [$x = 2$], $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$ [$x = -\frac{1}{4}$], $\left(\frac{1}{2}, \frac{4\pi}{3}\right)$ [$x = -\frac{1}{4}$], $(2, 2\pi)$ [$x = 2$]

11. $y = 2\sin^2\theta \Rightarrow \frac{dy}{d\theta} = 4\sin\theta\cos\theta = 2\sin 2\theta$

$x = 2\sin\theta\cos\theta = \sin 2\theta \Rightarrow \frac{dx}{d\theta} = 2\cos 2\theta$

$\frac{dy}{d\theta} = 0$ when $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{dx}{d\theta} = 0$ when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. They are never both zero. For $\theta = 0, \frac{\pi}{2}, \pi$ the curve has horizontal asymptotes at $(0,0)$ [$y = 0\sin 0 = 0$], $\left(2, \frac{\pi}{2}\right)$ [$y = 2\sin\frac{\pi}{2} = 2$], and $(0, \pi)$ [$y = 0\sin\pi = 0$]. For $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ the curve has vertical asymptotes at $\left(\sqrt{2}, \frac{\pi}{4}\right)$ [$x = \sqrt{2}\cos\frac{\pi}{4} = 1$] and $\left(\sqrt{2}, \frac{3\pi}{4}\right)$ [$x = \sqrt{2}\cos\frac{3\pi}{4} = -1$]. This information can be summarized as follows.

Horizontal at: $(0,0)$ [$y = 0$], $\left(2, \frac{\pi}{2}\right)$ [$y = 2$], $(0, \pi)$ [$y = 0$]

Vertical at: $\left(\sqrt{2}, \frac{\pi}{4}\right)$ [$x = 1$], $\left(\sqrt{2}, \frac{3\pi}{4}\right)$ [$x = -1$]

12. $\frac{dy}{d\theta} = 4\sin^2\theta + (3 - 4\cos\theta)\cos\theta = 4(\sin^2\theta - \cos^2\theta) + 3\cos\theta = -8\cos^2\theta + 3\cos\theta + 4$

$\frac{dx}{d\theta} = 4\sin\theta\cos\theta - (3 - 4\cos\theta)\sin\theta = \sin\theta(8\cos\theta - 3) = 4\sin 2\theta - 3\sin\theta$

$\frac{dy}{d\theta} = 0$ when $\cos\theta = \frac{-3\pm\sqrt{137}}{-16}$, i.e., when $\theta \approx 0.405, 2.146, 4.137$, or 5.878 (values solved for with a graphing calculator). $\frac{dx}{d\theta} = 0$ when $\theta = 0, \pi$ or 2π (then $\sin\theta = 0$) or when $\theta = \cos^{-1}\left(\frac{3}{8}\right) \approx 1.186$ or $2\pi - \cos^{-1}\left(\frac{3}{8}\right) \approx 5.097$ (then $8\cos\theta - 3 = 0$). So there is a horizontal tangent line for $\theta \approx 0.405$, $r \approx -0.676$ [the line $y \approx -0.676\sin 0.405 \approx -0.267$], for $\theta \approx 2.146$, $r \approx 5.176$ [the line $y \approx 5.176\sin 2.146 \approx 4.343$], for $\theta \approx 4.137$, $r \approx 5.176$ [the line $y \approx 5.176\sin 4.137 \approx -4.343$], and for $\theta \approx 5.878$, $r \approx -0.676$ [the line

$y \approx -0.676 \sin 5.878 \approx 0.267$]. There is a vertical tangent for $\theta = 0$, $r = -1$ [the line $x = -1 \cos 0 = -1$], for $\theta = \pi$, $r = 7$ [the line $x = 7 \cos \pi = -7$], for $\theta = 2\pi$, $r = -1$ [again, the line $x = -1 \cos 2\pi = -1$], for $\theta = \cos^{-1}\left(\frac{3}{8}\right)$, $r = \frac{3}{2}$ [the line $x = \frac{9}{16}$], and for $\theta = 2\pi - \cos^{-1}\left(\frac{3}{8}\right)$, $r = \frac{3}{2}$ [again, the line $x = \frac{9}{16}$]. This information can be summarized as follows.

Horizontal at: $(-0.676, 0.405)$ [$y \approx -0.267$], $(5.176, 2.146)$ [$y \approx 4.343$], $(5.176, 4.137)$ [$y \approx -4.343$], $(-0.676, 5.878)$ [$y \approx 0.267$]

Vertical at: $(-1, 0)$ [$x = -1$], $(1.5, 1.186)$ [$x = \frac{9}{16}$], $(7, \pi)$ [$x = -7$], $(1.5, 5.097)$ [$x = \frac{9}{16}$], $(-1, 2\pi)$, [$x = -1$]

13. $$A = \int_0^{2\pi} \frac{1}{2}(4 + 2\cos\theta)^2\,d\theta = \int_0^{2\pi} \frac{1}{2}\left(16 + 16\cos\theta + 4\cos^2\theta\right)d\theta = \int_0^{2\pi} \left[8 + 8\cos\theta + 2\left(\frac{1+\cos 2\theta}{2}\right)\right]d\theta$$
$$= \int_0^{2\pi} (9 + 8\cos\theta + \cos 2\theta)\,d\theta = \left[9\theta + 8\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 18\pi$$

14. $$A = \int_0^{2\pi} \frac{1}{2}[a(1+\cos\theta)]^2\,d\theta = \int_0^{2\pi} \frac{1}{2}a^2\left(1 + 2\cos\theta + \cos^2\theta\right)d\theta = \frac{1}{2}a^2 \int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= \frac{1}{2}a^2 \int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{2}a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3}{2}\pi a^2$$

15. $$A = 2\int_0^{\pi/4} \frac{1}{2}\cos^2 2\theta\,d\theta = \int_0^{\pi/4} \frac{1+\cos 4\theta}{2}\,d\theta = \frac{1}{2}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4} = \frac{\pi}{8}$$

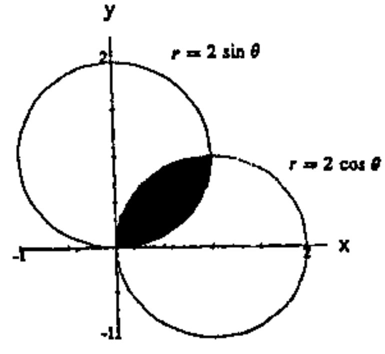
16. $$A = 2\int_{-\pi/4}^{\pi/4} \frac{1}{2}\left(2a^2\cos 2\theta\right)d\theta = 2a^2\int_{-\pi/4}^{\pi/4} \cos 2\theta\,d\theta = 2a^2\left[\frac{\sin 2\theta}{2}\right]_{-\pi/4}^{\pi/4} = 2a^2$$

17. $$A = 2\int_0^{\pi/2} \frac{1}{2}(4\sin 2\theta)\,d\theta = \int_0^{\pi/2} 2\sin 2\theta\,d\theta = [-\cos 2\theta]_0^{\pi/2} = 2$$

18. $$A = (6)(2)\int_0^{\pi/6} \frac{1}{2}(2\sin 3\theta)\,d\theta = 12\int_0^{\pi/6} \sin 3\theta\,d\theta = 12\left[-\frac{\cos 3\theta}{3}\right]_0^{\pi/6} = 4$$

19. $r = 2\cos\theta$ and $r = 2\sin\theta \Rightarrow 2\cos\theta = 2\sin\theta$ $\Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$; therefore

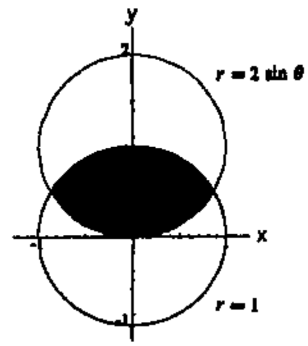
$$A = 2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2\,d\theta = \int_0^{\pi/4} 4\sin^2\theta\,d\theta$$
$$= \int_0^{\pi/4} 4\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \int_0^{\pi/4} (2 - 2\cos 2\theta)\,d\theta$$



$= [2\theta - \sin 2\theta]_0^{\pi/4} = \frac{\pi}{2} - 1$

20. $r = 1$ and $r = 2\sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$; therefore $A = \pi(1)^2 - \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left[(2\sin\theta)^2 - 1^2\right] d\theta$



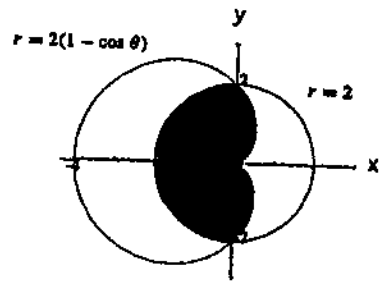
$= \pi - \int_{\pi/6}^{5\pi/6} \left(2\sin^2\theta - \frac{1}{2}\right) d\theta = \pi - \int_{\pi/6}^{5\pi/6} \left(1 - \cos 2\theta - \frac{1}{2}\right) d\theta$

$= \pi - \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2} - \cos 2\theta\right) d\theta = \pi - \left[\frac{1}{2}\theta - \frac{\sin 2\theta}{2}\right]_{\pi/6}^{5\pi/6}$

$= \pi - \left(\frac{5\pi}{12} - \frac{1}{2}\sin\frac{5\pi}{3}\right) + \left(\frac{\pi}{12} - \frac{1}{2}\sin\frac{\pi}{3}\right) = \frac{4\pi - 3\sqrt{3}}{6}$

21. $r = 2$ and $r = 2(1 - \cos\theta) \Rightarrow 2 = 2(1 - \cos\theta) \Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \pm\frac{\pi}{2}$; therefore $A = 2\int_0^{\pi/2} \frac{1}{2}[2(1 - \cos\theta)]^2\, d\theta$



$+ \frac{1}{2}$ area of the circle $= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \cos^2\theta\right) d\theta + \left(\frac{1}{2}\pi\right)(2)^2$

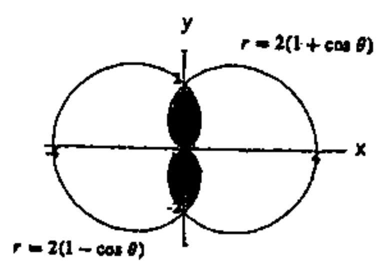
$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta + 2\pi$

$= \int_0^{\pi/2} (4 - 8\cos\theta + 2 + 2\cos 2\theta)\, d\theta + 2\pi$

$= \left[6\theta - 8\sin\theta + \sin 2\theta\right]_0^{\pi/2} + 2\pi = 5\pi - 8$

22. $r = 2(1 - \cos\theta)$ and $r = 2(1 + \cos\theta) \Rightarrow 1 - \cos\theta = 1 + \cos\theta$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$; the graph also gives the point of intersection $(0,0)$; therefore



$A = 2\int_0^{\pi/2} \frac{1}{2}[2(1 - \cos\theta)]^2\, d\theta + 2\int_{\pi/2}^{\pi} \frac{1}{2}[2(1 + \cos\theta)]^2\, d\theta$

$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \cos^2\theta\right) d\theta + \int_{\pi/2}^{\pi} 4\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$

$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta + \int_{\pi/2}^{\pi} 4\left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$

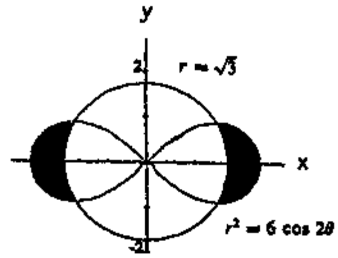
$= \int_0^{\pi/2} (6 - 8\cos\theta + 2\cos 2\theta)\, d\theta + \int_{\pi/2}^{\pi} (6 + 8\cos\theta + 2\cos 2\theta)\, d\theta$

$= \left[6\theta - 8 \sin \theta + \sin 2\theta\right]_0^{\pi/2} + \left[6\theta + 8 \sin \theta + \sin 2\theta\right]_{\pi/2}^{\pi} = 6\pi - 16$

23. $r = \sqrt{3}$ and $r^2 = 6 \cos 2\theta \Rightarrow 3 = 6 \cos 2\theta \Rightarrow \cos 2\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6}$ (in the 1st quadrant); we use symmetry of the graph to find the area, so $A = 4 \int_0^{\pi/6} \left[\frac{1}{2}(6 \cos 2\theta) - \frac{1}{2}\left(\sqrt{3}\right)^2\right] d\theta$

$= 2 \int_0^{\pi/6} (6 \cos 2\theta - 3)\, d\theta = 2[3 \sin 2\theta - 3\theta]_0^{\pi/6} = 3\sqrt{3} - \pi$



24. $r = 3a \cos \theta$ and $r = a(1 + \cos \theta) \Rightarrow 3a \cos \theta = a(1 + \cos \theta)$

$\Rightarrow 3 \cos \theta = 1 + \cos \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$; the graph also gives the point of intersection $(0,0)$; therefore

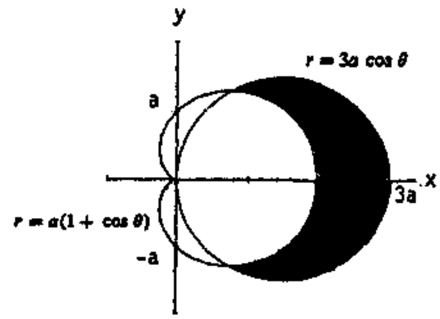
$A = 2 \int_0^{\pi/3} \frac{1}{2}\left[(3a \cos \theta)^2 - a^2(1 + \cos \theta)^2\right] d\theta$

$= \int_0^{\pi/3} \left(9a^2 \cos^2 \theta - a^2 - 2a^2 \cos \theta - a^2 \cos^2 \theta\right) d\theta$

$= \int_0^{\pi/3} \left(8a^2 \cos^2 \theta - 2a^2 \cos \theta - a^2\right) d\theta = \int_0^{\pi/3} \left[4a^2(1 + \cos 2\theta) - 2a^2 \cos \theta - a^2\right] d\theta$

$= \int_0^{\pi/3} \left(3a^2 + 4a^2 \cos 2\theta - 2a^2 \cos \theta\right) d\theta = \left[3a^2\theta + 2a^2 \sin 2\theta - 2a^2 \sin \theta\right]_0^{\pi/3} = \pi a^2 + 2a^2\left(\frac{1}{2}\right) - 2a^2\left(\frac{\sqrt{3}}{2}\right)$

$= a^2\left(\pi + 1 - \sqrt{3}\right)$



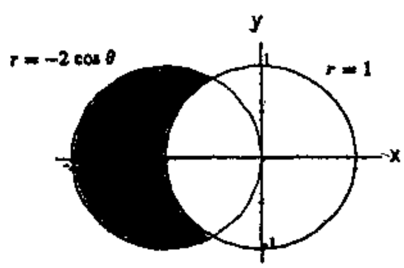
25. $r = 1$ and $r = -2 \cos \theta \Rightarrow 1 = -2 \cos \theta \Rightarrow \cos \theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}$ in quadrant II; therefore

$A = 2 \int_{2\pi/3}^{\pi} \frac{1}{2}\left[(-2 \cos \theta)^2 - 1^2\right] d\theta = \int_{2\pi/3}^{\pi} \left(4 \cos^2 \theta - 1\right) d\theta$

$= \int_{2\pi/3}^{\pi} [2(1 + \cos 2\theta) - 1]\, d\theta = \int_{2\pi/3}^{\pi} (1 + 2 \cos 2\theta)\, d\theta$

$= \left[\theta + \sin 2\theta\right]_{2\pi/3}^{\pi} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$



26. (a) $A = 2\int_0^{2\pi/3} \frac{1}{2}(2\cos\theta + 1)^2\, d\theta = \int_0^{2\pi/3} \left(4\cos^2\theta + 4\cos\theta + 1\right) d\theta = \int_0^{2\pi/3} [2(1+\cos 2\theta) + 4\cos\theta + 1]\, d\theta$

$= \int_0^{2\pi/3} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta = \left[3\theta + \sin 2\theta + 4\sin\theta\right]_0^{2\pi/3} = 2\pi - \frac{\sqrt{3}}{2} + \frac{4\sqrt{3}}{2} = 2\pi + \frac{3\sqrt{3}}{2}$

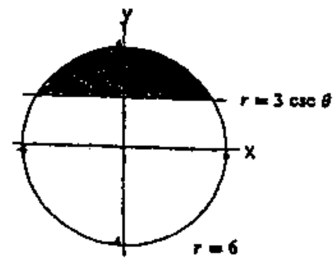
(b) $A = \left(2\pi + \frac{3\sqrt{3}}{2}\right) - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}$ (from 26(a) above and Example 4 in the text)

27. $r = 6$ and $r = 3\csc\theta \Rightarrow 6\sin\theta = 3 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

or $\frac{5\pi}{6}$; therefore $A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left(6^2 - 9\csc^2\theta\right) d\theta$

$= \int_{\pi/6}^{5\pi/6} \left(18 - \frac{9}{2}\csc^2\theta\right) d\theta = \left[18\theta + \frac{9}{2}\cot\theta\right]_{\pi/6}^{5\pi/6}$

$= \left(15\pi - \frac{9}{2}\sqrt{3}\right) - \left(3\pi + \frac{9}{2}\sqrt{3}\right) = 12\pi - 9\sqrt{3}$



28. $r^2 = 6\cos 2\theta$ and $r = \frac{3}{2}\sec\theta \Rightarrow \frac{9}{4}\sec^2\theta = 6\cos 2\theta \Rightarrow \frac{9}{24} = \cos^2\theta\cos 2\theta \Rightarrow \frac{3}{8} = \left(\cos^2\theta\right)\left(2\cos^2\theta - 1\right)$

$\Rightarrow \frac{3}{8} = 2\cos^4\theta - \cos^2\theta \Rightarrow 2\cos^4\theta - \cos^2\theta - \frac{3}{8} = 0 \Rightarrow 16\cos^4\theta - 8\cos^2\theta - 3 = 0 \Rightarrow \left(4\cos^2\theta - 1\right)\left(4\cos^2\theta - 3\right)$

$= 0 \Rightarrow \cos^2\theta = \frac{3}{4}$ or $\cos^2\theta = -\frac{1}{4} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$ (the second equation has no real roots) $\Rightarrow \theta = \frac{\pi}{6}$ (in the first quadrant); thus $A = 2\int_0^{\pi/6} \frac{1}{2}\left(6\cos 2\theta - \frac{9}{4}\sec^2\theta\right) d\theta = \int_0^{\pi/6}\left(6\cos 2\theta - \frac{9}{4}\sec^2\theta\right) d\theta = \left[3\sin 2\theta - \frac{9}{4}\tan\theta\right]_0^{\pi/6}$

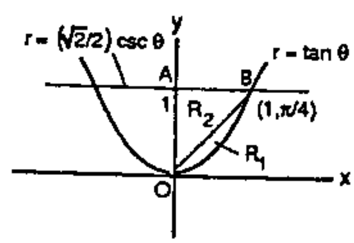
$= 3\left(\frac{\sqrt{3}}{2}\right) - \frac{9}{4\sqrt{3}} = \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}$

29. (a) $r = \tan\theta$ and $r = \left(\frac{\sqrt{2}}{2}\right)\csc\theta \Rightarrow \tan\theta = \left(\frac{\sqrt{2}}{2}\right)\csc\theta$

$\Rightarrow \sin^2\theta = \left(\frac{\sqrt{2}}{2}\right)\cos\theta \Rightarrow 1 - \cos^2\theta = \left(\frac{\sqrt{2}}{2}\right)\cos\theta$

$\Rightarrow \cos^2\theta + \left(\frac{\sqrt{2}}{2}\right)\cos\theta - 1 = 0 \Rightarrow \cos\theta = -\sqrt{2}$ or

$\frac{\sqrt{2}}{2}$ (use the quadratic formula) $\Rightarrow \theta = \frac{\pi}{4}$ (the solution



in the first quadrant); therefore the area of R_1 is $A_1 = \int_0^{\pi/4} \frac{1}{2}\tan^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/4}\left(\sec^2\theta - 1\right) d\theta$

$= \frac{1}{2}[\tan\theta - \theta]_0^{\pi/4} = \frac{1}{2}\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{1}{2} - \frac{\pi}{8}$; $AO = \left(\frac{\sqrt{2}}{2}\right)\csc\frac{\pi}{2} = \frac{\sqrt{2}}{2}$ and $OB = \left(\frac{\sqrt{2}}{2}\right)\csc\frac{\pi}{4} = 1$

$\Rightarrow AB = \sqrt{1^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{2}}{2} \Rightarrow$ the area of R_2 is $A_2 = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{4}$; therefore the area of the region shaded in the text is $2\left(\frac{1}{2}-\frac{\pi}{8}+\frac{1}{4}\right)=\frac{3}{2}-\frac{\pi}{4}$. Note: The area must be found this way since no common interval generates the region. For example, the interval $0 \le \theta \le \frac{\pi}{4}$ generates the arc OB of $r = \tan\theta$ but does not generate the segment AB of the line $r = \frac{\sqrt{2}}{2}\csc\theta$. Instead the interval generates the half-line from B to $+\infty$ on the line $r = \frac{\sqrt{2}}{2}\csc\theta$.

(b) $\lim\limits_{\theta\to\pi/2^-} \tan\theta = \infty$ and the line $x = 1$ is $r = \sec\theta$ in polar coordinates; then $\lim\limits_{\theta\to\pi/2^-} (\tan\theta - \sec\theta)$ $= \lim\limits_{\theta\to\pi/2^-} \left(\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}\right) = \lim\limits_{\theta\to\pi/2^-} \left(\frac{\sin\theta - 1}{\cos\theta}\right) = \lim\limits_{\theta\to\pi/2^-} \left(\frac{\cos\theta}{-\sin\theta}\right) = 0 \Rightarrow r = \tan\theta$ approaches $r = \sec\theta$ as $\theta \to \frac{\pi}{2}^- \Rightarrow r = \sec\theta$ (or $x = 1$) is a vertical asymptote of $r = \tan\theta$. Similarly, $r = -\sec\theta$ (or $x = -1$) is a vertical asymptote of $r = \tan\theta$.

30. It is not because the circle is generated twice from $\theta = 0$ to 2π. The area of the cardioid is

$$A = 2\int_0^{\pi} \tfrac{1}{2}(\cos\theta + 1)^2\, d\theta = \int_0^{\pi} \left(\cos^2\theta + 2\cos\theta + 1\right) d\theta = \int_0^{\pi} \left(\frac{1+\cos 2\theta}{2} + 2\cos\theta + 1\right) d\theta$$

$= \left[\frac{3\theta}{2} + \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi} = \frac{3\pi}{2}$. The area of the circle is $A = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4} \Rightarrow$ the area requested is actually $\frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$

31. $r = \theta^2,\ 0 \le \theta \le \sqrt{5} \Rightarrow \frac{dr}{d\theta} = 2\theta$; therefore Length $= \int_0^{\sqrt{5}} \sqrt{\left(\theta^2\right)^2 + (2\theta)^2}\, d\theta = \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta$

$= \int_0^{\sqrt{5}} |\theta|\sqrt{\theta^2+4}\, d\theta =$ (since $\theta \ge 0$) $\int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\, d\theta$; $\left[u = \theta^2 + 4 \Rightarrow \frac{1}{2}\,du = \theta\, d\theta;\ \theta = 0 \Rightarrow u = 4,\right.$

$\left.\theta = \sqrt{5} \Rightarrow u = 9\right] \to \int_4^9 \frac{1}{2}\sqrt{u}\, du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_4^9 = \frac{19}{3}$

32. $r = \frac{e^\theta}{\sqrt{2}},\ 0 \le \theta \le \pi \Rightarrow \frac{dr}{d\theta} = \frac{e^\theta}{\sqrt{2}}$; therefore Length $= \int_0^{\pi} \sqrt{\left(\frac{e^\theta}{\sqrt{2}}\right)^2 + \left(\frac{e^\theta}{\sqrt{2}}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{2\left(\frac{e^{2\theta}}{2}\right)}\, d\theta$

$= \int_0^{\pi} e^\theta\, d\theta = \left[e^\theta\right]_0^{\pi} = e^\pi - 1$

33. $r = 1 + \cos\theta \Rightarrow \frac{dr}{d\theta} = -\sin\theta$; therefore Length $= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$

$= 2\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta = 2\int_0^{\pi} \sqrt{\frac{4(1+\cos\theta)}{2}}\, d\theta = 4\int_0^{\pi} \sqrt{\frac{1+\cos\theta}{2}}\, d\theta = 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta = 4\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8$

34. $r = a\sin^2\frac{\theta}{2},\ 0 \le \theta \le \pi,\ a > 0 \Rightarrow \frac{dr}{d\theta} = a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$; therefore Length $= \int_0^{\pi} \sqrt{\left(a\sin^2\frac{\theta}{2}\right)^2 + \left(a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}\, d\theta$

$$= \int_0^{\pi} \sqrt{a^2\sin^4\frac{\theta}{2} + a^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\, d\theta = \int_0^{\pi} a\left|\sin\frac{\theta}{2}\right|\sqrt{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}}\, d\theta = (\text{since } 0 \le \theta \le \pi)\ a\int_0^{\pi} \sin\left(\frac{\theta}{2}\right) d\theta$$

$$= \left[-2a\cos\frac{\theta}{2}\right]_0^{\pi} = 2a$$

35. $r = \frac{6}{1+\cos\theta},\ 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} = \frac{6\sin\theta}{(1+\cos\theta)^2}$; therefore Length $= \int_0^{\pi/2} \sqrt{\left(\frac{6}{1+\cos\theta}\right)^2 + \left(\frac{6\sin\theta}{(1+\cos\theta)^2}\right)^2}\, d\theta$

$$= \int_0^{\pi/2} \sqrt{\frac{36}{(1+\cos\theta)^2} + \frac{36\sin^2\theta}{(1+\cos^2\theta)^4}}\, d\theta = 6\int_0^{\pi/2} \left|\frac{1}{1+\cos\theta}\right|\sqrt{1+\frac{\sin^2\theta}{(1+\cos\theta)^2}}\, d\theta$$

$$= \left(\text{since } \frac{1}{1+\cos\theta} > 0 \text{ on } 0 \le \theta \le \frac{\pi}{2}\right)\ 6\int_0^{\pi/2} \left(\frac{1}{1+\cos\theta}\right)\sqrt{\frac{1+2\cos\theta+\cos^2\theta+\sin^2\theta}{(1+\cos\theta)^2}}\, d\theta$$

$$= 6\int_0^{\pi/2} \left(\frac{1}{1+\cos\theta}\right)\sqrt{\frac{2+2\cos\theta}{(1+\cos\theta)^2}}\, d\theta = 6\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{(1+\cos\theta)^{3/2}} = 6\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\left(2\cos^2\frac{\theta}{2}\right)^{3/2}} = 6\int_0^{\pi/2} \left|\sec^3\frac{\theta}{2}\right| d\theta$$

$$= 6\int_0^{\pi/2} \sec^3\frac{\theta}{2}\, d\theta = 12\int_0^{\pi/4} \sec^3 u\, du = (\text{use tables})\ 6\left(\left[\frac{\sec u\tan u}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec u\, du\right)$$

$$= 6\left(\frac{1}{\sqrt{2}} + \left[\frac{1}{2}\ln|\sec u + \tan u|\right]_0^{\pi/4}\right) = 3\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$

36. $r = \frac{2}{1-\cos\theta},\ \frac{\pi}{2} \le \theta \le \pi \Rightarrow \frac{dr}{d\theta} = \frac{-2\sin\theta}{(1-\cos\theta)^2}$; therefore Length $= \int_{\pi/2}^{\pi} \sqrt{\left(\frac{2}{1-\cos\theta}\right)^2 + \left(\frac{-2\sin\theta}{(1-\cos\theta)^2}\right)^2}\, d\theta$

$$= \int_{\pi/2}^{\pi} \sqrt{\frac{4}{(1-\cos\theta)^2}\left(1+\frac{\sin^2\theta}{(1-\cos^2\theta)^2}\right)}\, d\theta = 6\int_{\pi/2}^{\pi} \left|\frac{2}{1-\cos\theta}\right|\sqrt{\frac{(1-\cos\theta)^2+\sin^2\theta}{(1-\cos\theta)^2}}\, d\theta$$

$$= \left(\text{since } 1-\cos\theta \ge 0 \text{ on } \frac{\pi}{2} \le \theta \le \pi\right)\ 2\int_{\pi/2}^{\pi} \left(\frac{1}{1-\cos\theta}\right)\sqrt{\frac{1-2\cos\theta+\cos^2\theta+\sin^2\theta}{(1-\cos\theta)^2}}\, d\theta$$

$$= 2\int_{\pi/2}^{\pi} \left(\frac{1}{1-\cos\theta}\right)\sqrt{\frac{2-2\cos\theta}{(1-\cos\theta)^2}}\, d\theta = 2\sqrt{2}\int_{\pi/2}^{\pi} \frac{d\theta}{(1-\cos\theta)^{3/2}} = 2\sqrt{2}\int_{\pi/2}^{\pi} \frac{d\theta}{\left(2\sin^2\frac{\theta}{2}\right)^{3/2}} = \int_{\pi/2}^{\pi} \left|\csc^3\frac{\theta}{2}\right| d\theta$$

$$= 6\int_{\pi/2}^{\pi} \csc^3\left(\frac{\theta}{2}\right) d\theta = \left(\text{since } \csc\frac{\theta}{2} \ge 0 \text{ on } \frac{\pi}{2} \le \theta \le \pi\right)\ 2\int_{\pi/4}^{\pi/2} \csc^3 u\, du = (\text{use tables})$$

$$6\left(\left[-\frac{\csc u \cot u}{2}\right]_{\pi/4}^{\pi/2}+\frac{1}{2}\int_{\pi/4}^{\pi/2} \csc u \, du\right)=2\left(\frac{1}{\sqrt{2}}-\left[\frac{1}{2}\ln|\csc u+\cot u|\right]_{\pi/4}^{\pi/2}\right)=2\left[\frac{1}{\sqrt{2}}+\frac{1}{2}\ln\left(\sqrt{2}+1\right)\right]$$

$$=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$$

37. $r=\cos^3\frac{\theta}{3} \Rightarrow \frac{dr}{d\theta}=-\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}$; therefore Length $=\int_0^{\pi/4}\sqrt{\left(\cos^3\frac{\theta}{3}\right)^2+\left(-\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}\right)^2}\,d\theta$

$$=\int_0^{\pi/4}\sqrt{\cos^6\left(\frac{\theta}{3}\right)+\sin^2\left(\frac{\theta}{3}\right)\cos^4\left(\frac{\theta}{3}\right)}\,d\theta=\int_0^{\pi/4}\left(\cos^2\frac{\theta}{3}\right)\sqrt{\cos^2\left(\frac{\theta}{3}\right)+\sin^2\left(\frac{\theta}{3}\right)}\,d\theta=\int_0^{\pi/4}\cos^2\left(\frac{\theta}{3}\right)d\theta$$

$$=\int_0^{\pi/4}\frac{1+\cos\left(\frac{2\theta}{3}\right)}{2}\,d\theta=\frac{1}{2}\left[\theta+\frac{3}{2}\sin\frac{2\theta}{3}\right]_0^{\pi/4}=\frac{\pi}{8}+\frac{3}{8}$$

38. $r=\sqrt{1+\sin 2\theta}$, $0\le\theta\le\pi\sqrt{2} \Rightarrow \frac{dr}{d\theta}=\frac{1}{2}(1+\sin 2\theta)^{-1/2}(2\cos 2\theta)=(\cos 2\theta)(1+\sin 2\theta)^{-1/2}$; therefore

$$\text{Length}=\int_0^{\pi\sqrt{2}}\sqrt{(1+\sin 2\theta)+\frac{\cos^2 2\theta}{(1+\sin 2\theta)}}\,d\theta=\int_0^{\pi\sqrt{2}}\sqrt{\frac{1+2\sin 2\theta+\sin^2 2\theta+\cos^2 2\theta}{1+\sin 2\theta}}\,d\theta$$

$$=\int_0^{\pi\sqrt{2}}\sqrt{\frac{2+2\sin 2\theta}{1+\sin 2\theta}}\,d\theta=\int_0^{\pi\sqrt{2}}\sqrt{2}\,d\theta=\left[\sqrt{2}\,\theta\right]_0^{\pi\sqrt{2}}=2\pi$$

39. $r=\sqrt{1+\cos 2\theta} \Rightarrow \frac{dr}{d\theta}=\frac{1}{2}(1+\cos 2\theta)^{-1/2}(-2\sin 2\theta)$; therefore Length $=\int_0^{\pi\sqrt{2}}\sqrt{(1+\cos 2\theta)+\frac{\sin^2 2\theta}{(1+\cos 2\theta)}}\,d\theta$

$$=\int_0^{\pi\sqrt{2}}\sqrt{\frac{1+2\cos 2\theta+\cos^2 2\theta+\sin^2 2\theta}{1+\cos 2\theta}}\,d\theta=\int_0^{\pi\sqrt{2}}\sqrt{\frac{2+2\cos 2\theta}{1+\cos 2\theta}}\,d\theta=\int_0^{\pi\sqrt{2}}\sqrt{2}\,d\theta=\left[\sqrt{2}\,\theta\right]_0^{\pi\sqrt{2}}=2\pi$$

40. (a) $r=a \Rightarrow \frac{dr}{d\theta}=0$; Length $=\int_0^{2\pi}\sqrt{a^2+0^2}\,d\theta=\int_0^{2\pi}|a|\,d\theta=[a\theta]_0^{2\pi}=2\pi a$

(b) $r=a\cos\theta \Rightarrow \frac{dr}{d\theta}=-a\sin\theta$; Length $=\int_0^{\pi}\sqrt{(a\cos\theta)^2+(-a\sin\theta)^2}\,d\theta=\int_0^{\pi}\sqrt{a^2\left(\cos^2\theta+\sin^2\theta\right)}\,d\theta$

$$=\int_0^{\pi}|a|\,d\theta=[a\theta]_0^{\pi}=\pi a$$

(c) $r=a\sin\theta \Rightarrow \frac{dr}{d\theta}=a\cos\theta$; Length $=\int_0^{\pi}\sqrt{(a\cos\theta)^2+(a\sin\theta)^2}\,d\theta=\int_0^{\pi}\sqrt{a^2\left(\cos^2\theta+\sin^2\theta\right)}\,d\theta$

$$= \int_0^{\pi} |a| \, d\theta = [a\theta]_0^{\pi} = \pi a$$

41. Let $r = f(\theta)$. Then $x = f(\theta) \cos \theta \Rightarrow \frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta \Rightarrow \left(\frac{dx}{d\theta}\right)^2 = [f'(\theta) \cos \theta - f(\theta) \sin \theta]^2$
$= [f'(\theta)]^2 \cos^2\theta - 2f'(\theta) f(\theta) \sin \theta \cos \theta + [f(\theta)]^2 \sin^2\theta$; $y = f(\theta) \sin \theta \Rightarrow \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta$
$\Rightarrow \left(\frac{dy}{d\theta}\right)^2 = [f'(\theta) \sin \theta + f(\theta) \cos \theta]^2 = [f'(\theta)]^2 \sin^2\theta + 2f'(\theta)f(\theta) \sin \theta \cos \theta + [f(\theta)]^2 \cos^2\theta$. Therefore
$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f'(\theta)]^2(\cos^2\theta + \sin^2\theta) + [f(\theta)]^2(\cos^2\theta + \sin^2\theta) = [f'(\theta)]^2 + [f(\theta)]^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$.
Thus, $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$.

42. (a) $r_{av} = \frac{1}{2\pi - 0} \int_0^{2\pi} a(1 - \cos \theta) \, d\theta = \frac{a}{2\pi}[\theta - \sin \theta]_0^{2\pi} = a$

(b) $r_{av} = \frac{1}{2\pi - 0} \int_0^{2\pi} a \, d\theta = \frac{1}{2\pi}[a\theta]_0^{2\pi} = a$

(c) $r_{av} = \frac{1}{\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)} \int_{-\pi/2}^{\pi/2} a \cos \theta \, d\theta = \frac{1}{\pi}[a \sin \theta]_{-\pi/2}^{\pi/2} = \frac{2a}{\pi}$

43. $r = 2f(\theta)$, $\alpha \le \theta \le \beta \Rightarrow \frac{dr}{d\theta} = 2f'(\theta) \Rightarrow r^2 + \left(\frac{dr}{d\theta}\right)^2 = [2f(\theta)]^2 + [2f'(\theta)]^2 \Rightarrow \text{Length} = \int_{\alpha}^{\beta} \sqrt{4[f(\theta)]^2 + 4[f'(\theta)]^2} \, d\theta$
$= 2 \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta$ which is twice the length of the curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$.

44. (a) Let $r = 1.75 + \frac{0.06\theta}{2\pi}$.

(b) Since $\frac{dr}{d\theta} = \frac{b}{2\pi}$, this is just Equation 4 for the length of the curve.

(c) Using the integral function on a calculator or CAS, $\int_0^{80\pi} \sqrt{\left(1.75 + \frac{0.06\theta}{2\pi}\right)^2 + \left(\frac{0.06}{2\pi}\right)^2} \, d\theta$ evaluates to ≈ 741.420 cm ≈ 7.414 m.

(d) $\left(r^2 + \left(\frac{b}{2\pi}\right)^2\right)^{1/2} = r\left(1 + \left(\frac{b}{2\pi r}\right)^2\right)^{1/2} \approx r$ since $\left(\frac{b}{2\pi r}\right)^2$ is a very small quantity squared.

(e) $L \approx 741.420$ cm (from part (c)), $L_a = \int_0^{80\pi} \left(1.75 + \frac{0.06\theta}{2\pi}\right) d\theta = \left[1.750 + \frac{0.03\theta^2}{2\pi}\right]_0^{80\pi} = 236\pi \approx 741.416$ cm

45. (a) Use the approximation, L_a, from Exercise #45(e). If the reel has made n complete turns, then the angle is $2\pi n$. So from the integral, $L_a = \pi b n^2 + 2\pi r_0 n$. Solving for n gives $n = \left(\frac{r_0}{b}\right)\left(\sqrt{\frac{bL}{r_0^2\pi} + 1} - 1\right)$.

(b) The take up reel slows down as time progresses.

(c) Since L is proportional to time, the formula in part (a) shows that n will grow roughly as the square root of time.

CHAPTER 9 PRACTICE EXERCISES

1. (a) $3\langle -3,4\rangle - 4\langle 2,-5\rangle = \langle -9-8, 12+20\rangle = \langle -17, 32\rangle$
   (b) $\sqrt{17^2+32^2} = \sqrt{1313}$

2. (a) $\langle -3+2, 4-5\rangle = \langle -1,-1\rangle$
   (b) $\sqrt{1^2+1^2} = \sqrt{2}$

3. (a) $\langle -2(-3), -2(4)\rangle = \langle 6,-8\rangle$
   (b) $\sqrt{6^2+8^2} = 10$

4. (a) $\langle 5(2), 5(-5)\rangle = \langle 10,-25\rangle$
   (b) $\sqrt{10^2+25^2} = \sqrt{725} = 5\sqrt{29}$

5. $\frac{\pi}{6}$ radians below the negative x-axis: $\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$ [assuming counterclockwise].

6. $\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$

7. $2\left(\frac{1}{\sqrt{4^2+1^2}}\right)(4\mathbf{i}-\mathbf{j}) = \left(\frac{8}{\sqrt{17}}\mathbf{i} - \frac{2}{\sqrt{17}}\mathbf{j}\right)$

8. $-5\left(\frac{1}{\sqrt{(3/5)^2+(4/5)^2}}\right)\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) = (-3\mathbf{i} - 4\mathbf{j})$

9. length $= \left|\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}\right| = \sqrt{2+2} = 2$, $\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} = 2\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

10. length $= |-\mathbf{i}-\mathbf{j}| = \sqrt{1+1} = \sqrt{2}$, $-\mathbf{i}-\mathbf{j} = \sqrt{2}\left(-\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\right) \Rightarrow$ the direction is $-\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$

11. $\frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j}$; at the point $(0,2)$, $t = \frac{\pi}{2} \Rightarrow \left.\frac{d\mathbf{r}}{dt}\right|_{t=\pi/2} = -2\mathbf{i}$; length $= |-2\mathbf{i}| = 2$;
direction $= -\mathbf{i} \Rightarrow \left.\frac{d\mathbf{r}}{dt}\right|_{t=\pi/2} = 2(-\mathbf{i})$

12. $\frac{d\mathbf{r}}{dt} = \left[e^t(\cos t - \sin t)\right]\mathbf{i} + \left[e^t(\sin t + \cos t)\right]\mathbf{j} \Rightarrow \left.\frac{d\mathbf{r}}{dt}\right|_{t=\ln 2} = 2[\cos(\ln 2) - \sin(\ln 2)]\mathbf{i} + 2[\sin(\ln 2) + \cos(\ln 2)]\mathbf{j}$

$\Rightarrow$ length $= 2\sqrt{[\cos(\ln 2) - \sin(\ln 2)]^2 + [\sin(\ln 2) + \cos(\ln 2)]^2}$

$= 2\sqrt{[1 - 2\sin(\ln 2)\cos(\ln 2)] + [1 + 2\sin(\ln 2)\cos(\ln 2)]} = 2\sqrt{2}$;

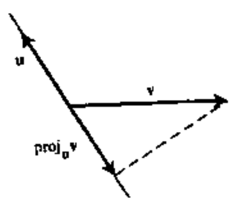
direction $= \frac{[\cos(\ln 2) - \sin(\ln 2)]}{\sqrt{2}}\mathbf{i} + \frac{[\sin(\ln 2) + \cos(\ln 2)]}{\sqrt{2}}\mathbf{j}$

$\Rightarrow \frac{d\mathbf{r}}{dt}\Big|_{t=\ln 2} = 2\sqrt{2}\left(\frac{[\cos(\ln 2) - \sin(\ln 2)]}{\sqrt{2}}\mathbf{i} + \frac{[\sin(\ln 2) + \cos(\ln 2)]}{\sqrt{2}}\mathbf{j}\right)$

13. $y = \tan x \Rightarrow [y']_{\pi/4} = [\sec^2 x]_{\pi/4} = 2 = \frac{2}{1} \Rightarrow \mathbf{T} = \mathbf{i} + 2\mathbf{j} \Rightarrow$ the unit tangents are $\pm\left(\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}\right)$ and the unit normals are $\pm\left(-\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}\right)$

14. $x^2 + y^2 = 25 \Rightarrow [y']_{(3,4)} = \left[-\frac{x}{y}\right]_{(3,4)} = \frac{-3}{4} \Rightarrow \mathbf{T} = 4\mathbf{i} - 3\mathbf{j} \Rightarrow$ the unit tangents are $\pm\frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$ and the unit normals are $\pm\frac{1}{5}(3\mathbf{i} + 4\mathbf{j})$

15.



16. $\mathbf{a} = \text{proj}_\mathbf{v}\, \mathbf{u}$, $\mathbf{b} = \text{proj}_\mathbf{u}\, \mathbf{v}$, $\mathbf{c} = \mathbf{v} - \mathbf{b} = \mathbf{v} - \text{proj}_\mathbf{u}\, \mathbf{v}$

17. $|\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$, $|\mathbf{u}| = \sqrt{2^2 + 1^2} = \sqrt{5}$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = 1(2) + 1(1) = 3$, $\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$

$= \cos^{-1}\left(\frac{3}{\sqrt{10}}\right) \approx 0.32$ rad, $|\mathbf{u}| \cos\theta = \sqrt{5}\left(\frac{3}{\sqrt{10}}\right) = \frac{3\sqrt{2}}{2}$, $\text{proj}_\mathbf{v}\, \mathbf{u} = (|\mathbf{u}| \cos\theta)\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)$

$= \left(\frac{3\sqrt{2}}{2}\right)\left(\frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}\right) = \frac{3}{2}(\mathbf{i} + \mathbf{j})$

18. $|\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$, $|\mathbf{u}| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = (1)(-1) + (1)(-3) = -4$,

$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-4}{2\sqrt{5}}\right) \approx 2.68$ rad, $|\mathbf{u}| \cos\theta = (\sqrt{10})\left(\frac{-2}{\sqrt{5}}\right) = -2\sqrt{2}$,

$\text{proj}_\mathbf{v}\, \mathbf{u} = (|\mathbf{u}| \cos\theta)\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = (-2\sqrt{2})\left(\frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}\right) = -2(\mathbf{i} + \mathbf{j})$

19. Vector component of **u** parallel to **v**: $\text{proj}_\mathbf{v}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \frac{(1)(2) + (-1)(1)}{2^2 + 1^2}(2\mathbf{i} - \mathbf{j}) = \frac{2}{5}\mathbf{i} - \frac{1}{5}\mathbf{j}$

Vector component of **u** orthogonal to **v**: $\mathbf{u} - \text{proj}_\mathbf{v}\mathbf{u} = (\mathbf{i} + \mathbf{j}) - \left(\frac{2}{5}\mathbf{i} - \frac{1}{5}\mathbf{j}\right) = \frac{3}{5}\mathbf{i} + \frac{6}{5}\mathbf{j}$

Therefore, $\mathbf{u} = \text{proj}_\mathbf{v}\mathbf{u} + (\mathbf{u} - \text{proj}_\mathbf{v}\mathbf{u}) = \left(\frac{2}{5}\mathbf{i} - \frac{1}{5}\mathbf{j}\right) + \left(\frac{3}{5}\mathbf{i} + \frac{6}{5}\mathbf{j}\right)$.

20. Vector component of $\mathbf{u}$ parallel to $\mathbf{v}$:

$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \frac{(-1)(1)+(1)(-2)}{1^2+2^2}(\mathbf{i}-2\mathbf{j}) = -\frac{3}{5}\mathbf{i}+\frac{6}{5}\mathbf{j}$

Vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$: $\mathbf{u}-\text{proj}_{\mathbf{v}}\,\mathbf{u} = (-\mathbf{i}+\mathbf{j}) - \left(-\frac{3}{5}\mathbf{i}+\frac{6}{5}\mathbf{j}\right) = -\frac{2}{5}\mathbf{i}-\frac{1}{5}\mathbf{j}$

Therefore, $\mathbf{u} = \text{proj}_{\mathbf{v}}\,\mathbf{u} + (\mathbf{u}-\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(-\frac{3}{5}\mathbf{i}+\frac{6}{5}\mathbf{j}\right)+\left(-\frac{2}{5}\mathbf{i}-\frac{1}{5}\mathbf{j}\right).$

21. (a) $\mathbf{v}(t) = \frac{d}{dt}\left[(4\cos t)\mathbf{i}+\left(\sqrt{2}\sin t\right)\mathbf{j}\right]$

$= (-4\sin t)\mathbf{i}+\left(\sqrt{2}\cos t\right)\mathbf{j}$

$\mathbf{a}(t) = \frac{d}{dt}\left[(-4\sin t)\mathbf{i}+\left(\sqrt{2}\cos t\right)\mathbf{j}\right]$

$= (-4\cos t)\mathbf{i}+\left(-\sqrt{2}\sin t\right)\mathbf{j}$

(b) $\left|\mathbf{v}\left(\frac{\pi}{4}\right)\right| = \sqrt{\left(-4\sin\frac{\pi}{4}\right)^2+\left(\sqrt{2}\cos\frac{\pi}{4}\right)^2} = \sqrt{8+1} = 3$

(c) At $t=\frac{\pi}{4}$, $\mathbf{v} = -2\sqrt{2}\mathbf{i}+\mathbf{j}$, $\mathbf{a} = -2\sqrt{2}\mathbf{i}-\mathbf{j}$, and

$\theta = \cos^{-1}\frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}||\mathbf{a}|} = \cos^{-1}\frac{8-1}{(3)(3)} = \cos^{-1}\frac{7}{9} \approx 38.94^\circ.$

22. (a) $\mathbf{v}(t) = \frac{d}{dt}[(\sqrt{3}\sec t)\mathbf{i}+(\sqrt{3}\tan t)\mathbf{j}]$

$= (\sqrt{3}\sec t\tan t)\mathbf{i}+(\sqrt{3}\sec^2 t)\mathbf{j}$

$\mathbf{a}(t) = \frac{d}{dt}[(\sqrt{3}\sec t\tan t)\mathbf{i}+(\sqrt{3}\sec^2 t)\mathbf{j}]$

$= \sqrt{3}(\sec t\tan^2 t+\sec^3 t)\mathbf{i}+(2\sqrt{3}\sec^2 t\tan t)\mathbf{j}$

(b) $|\mathbf{v}(0)| = \sqrt{3\sec^2 0\tan^2 0+3\sec^4 0} = \sqrt{0+3} = \sqrt{3}$

(c) At $t=0$, $\mathbf{v} = \sqrt{3}\mathbf{j}$, $\mathbf{a} = \sqrt{3}\mathbf{i}$

$\theta = \cos^{-1}\frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}||\mathbf{a}|} = \frac{0+0}{(\sqrt{3})(\sqrt{3})} = \cos^{-1}0 = 90^\circ.$

23. $\mathbf{v}(t) = -\frac{t}{(1+t^2)^{3/2}}\mathbf{i}+\frac{1}{(1+t^2)^{3/2}}\mathbf{j}$

$\left|\frac{d\mathbf{r}}{dt}\right| = |\mathbf{v}(t)| = \sqrt{\left(-\frac{t}{(1+t^2)^{3/2}}\right)^2+\left(\frac{1}{(1+t^2)^{3/2}}\right)^2} = \frac{1}{1+t^2}$ which is at a maximum of 1 when $t=0$.

24. Minimizing $\left|\frac{d\mathbf{r}}{dt}\right|^2$ will minimize $\left|\frac{d\mathbf{r}}{dt}\right|$.

$\frac{d\mathbf{r}}{dt} = \left[e^t(\cos t-\sin t)\right]\mathbf{i}+\left[e^t(\sin t+\cos t\right]\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|^2 = \left[e^t(\cos t-\sin t)\right]^2+\left[e^t(\sin t+\cos t)\right]^2$

$= e^{2t}\left[(1-2\sin t\cos t)+(1+2\sin t\cos t)\right] = 2e^{2t}$. For $t\ge 0$, the minimum value of $2e^{2t}$ is 2 at $t=0$, and it has no maximum value. Therefore, the minimum speed is $\sqrt{2}$ and there is no maximum speed.

25. $\left(\int_0^t (3+6t)\,dt\right)\mathbf{i}+\left(\int_0^1 6\pi\cos\pi t\,dt\right)\mathbf{j}$

$=\left[3t+3t^2\right]_0^1\mathbf{i}+\left[6\sin\pi t\right]_0^1\mathbf{j}=6\mathbf{i}$

26. $\left(\int_e^{e^2}\frac{2\ln t}{t}\,dt\right)\mathbf{i}+\left(\int_e^{e^2}\frac{1}{t\ln t}\,dt\right)\mathbf{j}$

$=\left[\ln^2 t\right]_e^{e^2}\mathbf{i}+\left[\ln(\ln t)\right]_e^{e^2}\mathbf{j}=3\mathbf{i}+(\ln 2)\mathbf{j}$

27. $\mathbf{r}(t)=\int\frac{d\mathbf{r}}{dt}\,dt=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}+\mathbf{C}$

$\mathbf{r}(0)=\mathbf{i}+\mathbf{C}=\mathbf{j}$, so $\mathbf{C}=-\mathbf{i}+\mathbf{j}$, and

$\mathbf{r}(t)=(\cos t-1)\mathbf{i}+(\sin t+1)\mathbf{j}$

28. $\mathbf{r}(t)=\int\frac{d\mathbf{r}}{dt}\,dt=(\tan^{-1}t)\mathbf{i}+\sqrt{t^2+1}\,\mathbf{j}+\mathbf{C}$

$\mathbf{r}(0)=\mathbf{j}+\mathbf{C}=\mathbf{i}+\mathbf{j}$, so $\mathbf{C}=\mathbf{i}$, and

$\mathbf{r}(t)=(\tan^{-1}t+1)\mathbf{i}+\sqrt{t^2+1}\,\mathbf{j}$

29. $\frac{d\mathbf{r}}{dt}=\int\frac{d^2\mathbf{r}}{dt^2}\,dt=2t\mathbf{j}+\mathbf{C}_1$, $\mathbf{r}(t)=\int\frac{d\mathbf{r}}{dt}\,dt=t^2\mathbf{j}+\mathbf{C}_1t+\mathbf{C}_2$

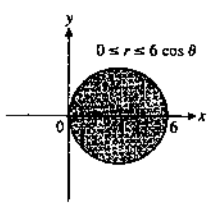
$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0}=\mathbf{C}_1=\mathbf{0}$, so $\mathbf{r}(t)=t^2\mathbf{j}+\mathbf{C}_2$. And $\mathbf{r}(0)=\mathbf{C}_2=\mathbf{i}$, so $\mathbf{r}(t)=\mathbf{i}+t^2\mathbf{j}$

30. $\frac{d\mathbf{r}}{dt}=\int\frac{d^2\mathbf{r}}{dt^2}\,dt=(-2t)\mathbf{i}+(-2t)\mathbf{j}+\mathbf{C}_1$, $\mathbf{r}(t)=\int\frac{d\mathbf{r}}{dt}\,dt=-t^2\mathbf{j}-t^2\mathbf{j}+\mathbf{C}_1t+\mathbf{C}_2$

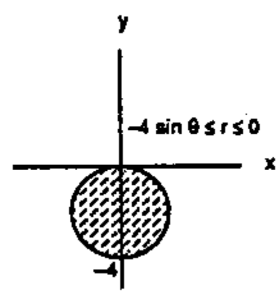
$\left.\frac{d\mathbf{r}}{dt}\right|_{t=1}=-2\mathbf{i}-2\mathbf{j}+\mathbf{C}_1=4\mathbf{i}$, so $\mathbf{C}_1=6\mathbf{i}+2\mathbf{j}$ and $\mathbf{r}(t)=(-t^2+6t)\mathbf{i}+(-t^2+2t)\mathbf{j}+\mathbf{C}_2$

$\mathbf{r}(1)=5\mathbf{i}+\mathbf{j}+\mathbf{C}_2=3\mathbf{i}+3\mathbf{j}$, so $\mathbf{C}_2=-2\mathbf{i}+2\mathbf{j}$, and $\mathbf{r}(t)=(-t^2+6t-2)\mathbf{i}+(-t^2+2t+2)\mathbf{j}$

31.



32.



33. d

34. e

35. l

36. f

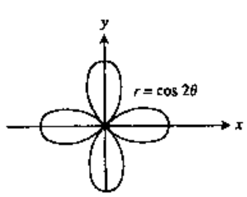
37. k

38. h

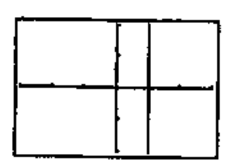
39. i

40. j

41. (a)



42. (a)

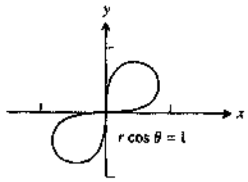


[−3, 3] by [−2, 2]

(b) 2π

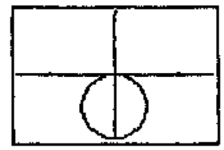
(b) π

43. (a)



(b) $\frac{\pi}{2}$

44. (a)



$[-1.5, 1.5]$ by $[-1, 1]$

(b) π

45. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-2\sin 2\theta\sin\theta + \cos 2\theta\cos\theta}{-2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}$

$\left(0, \frac{\pi}{4}\right)$, $\left(0, \frac{3\pi}{4}\right)$, $\left(0, \frac{5\pi}{4}\right)$ and $\left(0, \frac{7\pi}{4}\right)$ are polar solutions.

$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{-2/\sqrt{2}}{-2\sqrt{2}} = 1$, $\left.\frac{dy}{dx}\right|_{\theta=3\pi/4} = \frac{2/\sqrt{2}}{-2\sqrt{2}} = -1$, $\left.\frac{dy}{dx}\right|_{\theta=5\pi/4} = \frac{2/\sqrt{2}}{2\sqrt{2}} = 1$, $\left.\frac{dy}{dx}\right|_{\theta=7\pi/4} = \frac{-2/\sqrt{2}}{2\sqrt{2}} = -1.$

The Cartesian equations are $y = \pm x$.

46. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-2\sin 2\theta\sin\theta + (1+\cos 2\theta)\cos\theta}{-2\sin 2\theta\cos\theta - (1+\cos 2\theta)\sin\theta} = \frac{-4\sin^2\theta\cos\theta + \cos\theta + 2\cos^3\theta - \cos\theta}{-4\cos^2\theta\sin\theta - \sin\theta - 2\cos^2\theta\sin\theta + \sin\theta}$

$= \frac{-4\sin^2\theta + 2\cos^2\theta}{-6\cos\theta\sin\theta} = \frac{4\sin^2\theta - 2\cos^2\theta}{3\sin 2\theta}.$

$\left(0, \frac{\pi}{2}\right)$ and $\left(0, \frac{3\pi}{2}\right)$ are polar solutions.

$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy}{dx}\right|_{\theta=3\pi/2} = \frac{4}{0}$ is undefined, so the tangent lines are vertical with equation $x = 0$.

47. $\frac{dy}{d\theta} = \frac{d}{d\theta}\left[\left(1 - \cos\left(\frac{\theta}{2}\right)\right)\sin\theta\right] = \frac{1}{2}\sin\left(\frac{\theta}{2}\right)\sin\theta + \cos\theta - \cos\left(\frac{\theta}{2}\right)\cos\theta$

$\frac{dx}{d\theta} = \frac{d}{d\theta}\left[\left(1 - \cos\left(\frac{\theta}{2}\right)\right)\cos\theta\right] = \frac{1}{2}\sin\left(\frac{\theta}{2}\right)\cos\theta - \sin\theta + \cos\left(\frac{\theta}{2}\right)\sin\theta$

Solve $\frac{dy}{d\theta} = 0$ for θ with a graphing calculator: the solutions are 0, ≈ 2.243, ≈ 4.892, ≈ 7.675, ≈ 10.323, and 4π. Using the middle four solutions to find $y = r\sin\theta$ reveals horizontal tangent lines at $y \approx \pm 0.443$ and $y \approx \pm 1.739$. Solve $\frac{dx}{d\theta} = 0$ for θ with a graphing calculator: the solutions are 0, ≈ 1.070, ≈ 3.531, 2π, ≈ 9.035, ≈ 11.497, and 4π. Using the middle five solutions to find $x = r\cos\theta$ reveals vertical tangent lines at $x = 2$, $x \approx 0.067$, and $x \approx -1.104$. Where $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both equal zero ($\theta = 0, 4\pi$), close inspection of the plot shows that the tangent lines are horizontal, with equation $y = 0$. (This can be confirmed using L'Hôpital's rule.)

48. $\frac{dy}{d\theta} = \frac{d}{d\theta}[2(1 - \sin\theta)\sin\theta] = -4\sin\theta\cos\theta + 2\cos\theta$

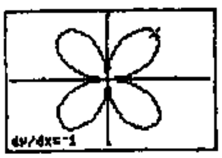
$\frac{dx}{d\theta} = \frac{d}{d\theta}[2(1 - \sin\theta)\cos\theta] = -2\cos^2\theta - 2\sin\theta + 2\sin^2\theta = 4\sin^2\theta - 2\sin\theta - 2$

Solve $\frac{dy}{d\theta} = 0$ for θ: the solutions are $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$.

Using the first, third, and fourth solutions to find $y = r\sin\theta$ reveals horizontal tangent lines at $y = \frac{1}{2}$ and $y = -4$.

Solve $\frac{dx}{d\theta} = 0$ for θ (by first using the quadratic formula to find $\sin\theta$): the solutions are $\frac{\pi}{2}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Using the last two solutions to find $x = r\cos\theta$ reveals vertical tangent lines at $x = \pm\frac{3\sqrt{3}}{2} \approx \pm 2.598$. Where $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both equal zero $(\theta = \frac{\pi}{2})$, inspection of the plot shows that the tangent line is vertical, with equation $x = 0$. (This can be confirmed using L'Hôpital's rule.)

49.



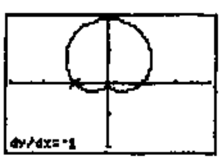
$[-1.5, 1.5]$ by $[-1, 1]$

The tips have Cartesian coordinates $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. From the curve's symmetries, it is evident that the tangent lines at those points have slopes of -1, 1, -1, and 1, respectively. So the equations of the tangent lines are

$y - \frac{1}{\sqrt{2}} = -\left(x - \frac{1}{\sqrt{2}}\right)$ or $y = -x + \sqrt{2}$, $y - \frac{1}{\sqrt{2}} = x + \frac{1}{\sqrt{2}}$ or $y = x + \sqrt{2}$,

$y + \frac{1}{\sqrt{2}} = -\left(x + \frac{1}{\sqrt{2}}\right)$ or $y = -x - \sqrt{2}$, and $y + \frac{1}{\sqrt{2}} = x - \frac{1}{\sqrt{2}}$ or $y = x - \sqrt{2}$.

50.



$[-3, 3]$ by $[-2, 2]$

As the plot shows, the curve crosses the x-axis at (x, y)-coordinates $(-1, 0)$ and $(1, 0)$, with slope -1 and 1, respectively. (This can be confirmed analytically.) So the equations of the tangent lines are

$y - 0 = -(x + 1)$

$y = -x - 1$ and

$y - 0 = x - 1$

$y = x - 1$.

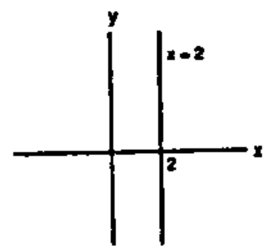
51. $r \cos \theta = r \sin \theta \Rightarrow x = y$, a line

52. $r = 3 \cos \theta \Rightarrow r^2 = 3r \cos \theta \Rightarrow x^2 + y^2 = 3x \Rightarrow x^2 - 3x + \frac{9}{4} + y^2 = \frac{9}{4} \Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$, a circle (center $= \left(\frac{3}{2}, 0\right)$, radius $= \frac{3}{2}$)

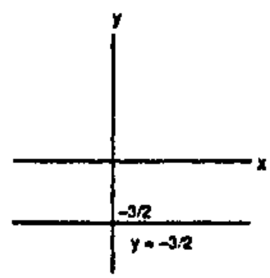
53. $r = 4 \tan \theta \sec \theta \Rightarrow r \cos \theta = 4 \frac{r \sin \theta}{r \cos \theta} \Rightarrow x = 4\frac{y}{x}$ or $x^2 = 4y$, a parabola

54. $r \cos\left(\theta + \frac{\pi}{3}\right) = 2\sqrt{3} \Rightarrow r \cos \theta \cos\left(\frac{\pi}{3}\right) - r \sin \theta \sin\left(\frac{\pi}{3}\right) = 2\sqrt{3} \Rightarrow \frac{1}{2} r \cos \theta - \frac{\sqrt{3}}{2} r \sin \theta = 2\sqrt{3}$
$\Rightarrow \frac{1}{2}x - \frac{\sqrt{3}}{2}y = 2\sqrt{3} \Rightarrow x - \sqrt{3}y = 4\sqrt{3}$ or $y = \frac{x}{\sqrt{3}} - 4$, a line

55. $r = 2 \sec \theta \Rightarrow r = \frac{2}{\cos \theta} \Rightarrow r \cos \theta = 2 \Rightarrow x = 2$



56. $r = -\frac{3}{2} \csc \theta \Rightarrow r \sin \theta = -\frac{3}{2} \Rightarrow y = -\frac{3}{2}$



57. $x^2 + y^2 + 5y = 0$
$r^2 + 5r \sin \theta = 0$
$r = -5 \sin \theta$

58. $x^2 + y^2 - 2y = 0$
$r^2 - 2r \sin \theta = 0$
$r = 2 \sin \theta$

59. $x^2 + 4y^2 = 16$
$(r \cos \theta)^2 + 4(r \sin \theta)^2 = 16$
$r^2 \cos^2 \theta + 4r^2 \sin^2 \theta = 16$, or $r^2 = \frac{16}{\cos^2 \theta + 4 \sin^2 \theta}$

60. $(x + 2)^2 + (y - 5)^2 = 16$
$(r \cos \theta + 2)^2 + (r \sin \theta - 5)^2 = 16$

61. $A = 2 \int_0^{\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi} (2 - \cos \theta)^2 \, d\theta = \int_0^{\pi} \left(4 - 2 \cos \theta + \cos^2 \theta\right) d\theta = \int_0^{\pi} \left(4 - 2 \cos \theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$

$$= \int_0^{\pi} \left(\frac{9}{2} - 2\cos\theta + \frac{\cos 2\theta}{2}\right) d\theta = \left[\frac{9}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{9}{2}\pi$$

62. $A = \int_0^{\pi/3} \frac{1}{2}(\sin^2 3\theta)\, d\theta = \int_0^{\pi/3} \frac{1}{2}\left(\frac{1-\cos 6\theta}{2}\right) d\theta = \frac{1}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{\pi}{12}$

63. $r = 1 + \cos 2\theta$ and $r = 1 \Rightarrow 1 = 1 + \cos 2\theta \Rightarrow 0 = \cos 2\theta \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$; therefore

$$A = 4\int_0^{\pi/4} \frac{1}{2}\left[(1+\cos 2\theta)^2 - 1^2\right] d\theta = 2\int_0^{\pi/4} \left(1 + 2\cos 2\theta + \cos^2 2\theta - 1\right) d\theta$$

$$= 2\int_0^{\pi/4} \left(2\cos 2\theta + \frac{1}{2} + \frac{\cos 4\theta}{2}\right) d\theta = 2\left[\sin 2\theta + \frac{1}{2}\theta + \frac{\sin 4\theta}{8}\right]_0^{\pi/4} = 2\left(1 + \frac{\pi}{8} + 0\right) = 2 + \frac{\pi}{4}$$

64. The circle lies interior to the cardiod. Thus,

$$A = 2\int_{-\pi/2}^{\pi/2} \frac{1}{2}[2(1+\sin\theta)]^2\, d\theta - \pi \text{ (the integral is the area of the cardiod minus the area of the circle)}$$

$$= \int_{-\pi/2}^{\pi/2} 4\left(1 + 2\sin\theta + \sin^2\theta\right) d\theta - \pi = \int_{-\pi/2}^{\pi/2} (6 + 8\sin\theta - 2\cos 2\theta)\, d\theta - \pi = [6\theta - 8\cos\theta - \sin 2\theta]_{-\pi/2}^{\pi/2} - \pi$$

$$= [3\pi - (-3\pi)] - \pi = 5\pi$$

65. $r = -1 + \cos\theta \Rightarrow \frac{dr}{d\theta} = -\sin\theta$; Length $= \int_0^{2\pi} \sqrt{(-1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$

$$= \int_0^{2\pi} \sqrt{\frac{4(1-\cos\theta)}{2}}\, d\theta = \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta = \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = (-4)(-1) - (-4)(1) = 8$$

66. $r = 2\sin\theta + 2\cos\theta,\ 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} = 2\cos\theta - 2\sin\theta$; $r^2 + \left(\frac{dr}{d\theta}\right)^2 = (2\sin\theta + 2\cos\theta)^2 + (2\cos\theta - 2\sin\theta)^2$

$$= 8\left(\sin^2\theta + \cos^2\theta\right) = 8 \Rightarrow L = \int_0^{\pi/2} \sqrt{8}\, d\theta = \left[2\sqrt{2}\,\theta\right]_0^{\pi/2} = 2\sqrt{2}\left(\frac{\pi}{2}\right) = \pi\sqrt{2}$$

67. $r = 8\sin^3\left(\frac{\theta}{3}\right),\ 0 \le \theta \le \frac{\pi}{4} \Rightarrow \frac{dr}{d\theta} = 8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)$; $r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left[8\sin^3\left(\frac{\theta}{3}\right)\right]^2 + \left[8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)\right]^2$

$$= 64\sin^4\left(\frac{\theta}{3}\right) \Rightarrow L = \int_0^{\pi/4} \sqrt{64\sin^4\left(\frac{\theta}{3}\right)}\, d\theta = \int_0^{\pi/4} 8\sin^2\left(\frac{\theta}{3}\right) d\theta = \int_0^{\pi/4} 8\left[\frac{1-\cos\left(\frac{2\theta}{3}\right)}{2}\right] d\theta$$

$$= \int_0^{\pi/4} \left[4 - 4\cos\left(\frac{2\theta}{3}\right)\right] d\theta = \left[4\theta - 6\sin\left(\frac{2\theta}{3}\right)\right]_0^{\pi/4} = 4\left(\frac{\pi}{4}\right) - 6\sin\left(\frac{\pi}{6}\right) - 0 = \pi - 3$$

68. $r = \sqrt{1 + \cos 2\theta} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(1 + \cos 2\theta)^{-1/2}(-2 \sin 2\theta) = \frac{-\sin 2\theta}{\sqrt{1 + \cos 2\theta}} \Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{1 + \cos 2\theta}$

$\Rightarrow r^2 + \left(\frac{dr}{d\theta}\right)^2 = 1 + \cos 2\theta + \frac{\sin^2 2\theta}{1 + \cos 2\theta} = \frac{(1 + \cos 2\theta)^2 + \sin^2 2\theta}{1 + \cos 2\theta} = \frac{1 + 2 \cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}{1 + \cos 2\theta}$

$= \frac{2 + 2 \cos 2\theta}{1 + \cos 2\theta} = 2 \Rightarrow L = \int_{-\pi/2}^{\pi/2} \sqrt{2}\, d\theta = \sqrt{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \sqrt{2}\,\pi$

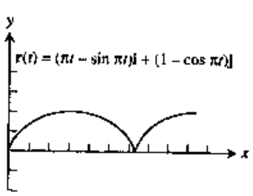
69. x degrees east of north is $(90 - x)$ degrees north of east.
Add the vectors:
$\langle 540 \cos 10^\circ, 540 \sin 10^\circ\rangle + \langle 55 \cos(-10^\circ), 55 \sin(-10^\circ)\rangle = \langle 595 \cos 10^\circ, 485 \sin 10^\circ\rangle \approx \langle 585.961, 84.219\rangle$.
Speed $\approx \sqrt{585.961^2 + 84.219^2} \approx 591.982$ mph.
Direction $\approx \tan^{-1}\left(\frac{585.961}{84.219}\right) \approx 81.821^\circ$ east of north

70. Add the vectors:
$\langle 120 \cos 20^\circ, 120 \sin 20^\circ\rangle + \langle 300 \cos(-5^\circ), 300 \sin(-5^\circ)\rangle \approx \langle 411.6220, 14.896\rangle$.
Direction $\approx \tan^{-1}\left(\frac{14.896}{411.622}\right) \approx 2.073^\circ$
Length $\approx \sqrt{411.622^2 + 14.896^2} \approx 411.891$ lbs

71. Taking the launch point as the origin, $y = (44 \sin 45^\circ)t - 16t^2$ equals -6.5 when $t \approx 2.135$ sec (as can be determined graphically or using the quadratic formula). Then $x \approx (44 \cos 45^\circ)(2.135) \approx 66.421$ horizontal feet from where it left the thrower's hand. Assuming it doesn't bounce or roll, it will still be there 3 seconds after it was thrown.

72. $y_{max} = \frac{(80 \sin 45^\circ)^2}{2(32)} + 7 = 57$ feet

73. (a)



(b) $\mathbf{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle \pi - \pi \cos \pi t, \pi \sin \pi t\rangle$ and $\mathbf{a}(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t\rangle$

$\mathbf{v}(0) = \langle 0, 0\rangle$ $\mathbf{v}(1) = \langle 2\pi, 0\rangle$ $\mathbf{v}(2) = \langle 0, 0\rangle$ $\mathbf{v}(3) = \langle 2\pi, 0\rangle$

$\mathbf{a}(0) = \langle 0, \pi^2\rangle$ $\mathbf{a}(1) = \langle 0, -\pi^2\rangle$ $\mathbf{a}(2) = \langle 0, \pi^2\rangle$ $\mathbf{a}(3) = \langle 0, -\pi^2\rangle$

(c) Topmost point: 2π ft/sec; center of wheel: π ft/sec
Reasons: Since the wheel rolls half a circumference, or π feet every second, the center of the wheel will move π feet every second. Since the rim of the wheel is turning at a rate of π ft/sec about the center, the velocity of the topmost point relative to the center is π ft/sec, giving it a total velocity of 2π ft/sec.

74. $v_0 = \sqrt{\frac{Rg}{\sin 2\alpha}}$, where $\alpha = 45°$, $g = 32$, and R = range

for 4325 yds = 12,975 ft; $v_0 \approx 644.360$ ft/sec
for 4752 yds = 14,256 ft; $v_0 \approx 675.420$ ft/sec

75. (a) $v_0 = \sqrt{\frac{Rg}{\sin 2\alpha}} = \sqrt{(109.5)(32)} \approx 59.195$ ft/sec

(b) The cork lands at $y = -4$, $x = 177.75$.

Solve $y = -\left(\frac{g}{2v_0^2 \cos^2 \alpha}\right)x^2 + (\tan \alpha)x$ for v_0, with $\alpha = 45°$; $v_0 = \sqrt{-\frac{gx^2}{y-x}} \approx 74.584$ ft/sec

76. (a) The javelin lands at $y = -6.5$, $x = 262\frac{5}{12}$.

Solve $y = -\left(\frac{g}{2v_0^2 \cos^2 \alpha}\right)x^2 + (\tan \alpha)x$ for v_0, with $\alpha = 40°$:

$$v_0 = \sqrt{-\frac{gx^2}{(2\cos^2 40°)(y - x\tan 40°)}} \approx 91.008 \text{ ft/sec}$$

(b) $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g} + 6.5 \approx \frac{(91.008 \sin 40°)^2}{64} + 6.5 \approx 59.970$ ft

77. We have $x = (v_0 t)\cos \alpha$ and $y + \frac{gt^2}{2} = (v_0 t)\sin \alpha$. Squaring and adding gives

$$x^2 + \left(y + \frac{gt^2}{2}\right)^2 = (v_0 t)^2(\cos^2 \alpha + \sin^2 \alpha) = v_0^2 t^2.$$

78. (a) $\mathbf{r}(t) = (155 \cos 18° - 11.7)t\mathbf{i} + (4 + 155 \sin 18°t - 16t^2)\mathbf{j}$
$x(t) = (155 \cos 18° - 11.7)t$
$y(t) = 4 + 155 \sin 18°t - 16t^2$

(b) $y_{max} = \frac{(155 \sin 18°)^2}{2(32)} + 4 \approx 39.847$ feet, reached at $t_{max} = \frac{155 \sin 18°}{32} \approx 1.497$ sec

(c) $y(t) = 0$ when $t \approx 3.075$ sec (found using the quadratic formula), and then

$x \approx (155 \cos 18° - 11.7)(3.075) \approx 417.307$ ft.

(d) Solve $y(t) = 25$ using the quadratic formula: $t = \frac{-155 \sin 18° \pm \sqrt{155^2 \sin^2 18° - 4(16)(21)}}{-32}$

≈ 0.534 and 2.460 seconds.

At those times, $x = (155 \cos 18° - 11.7)t$ equals ≈ 72.406 and ≈ 333.867 feet from home plate.

(e) Yes, the batter has hit a home run. When the ball is 380 feet from home plate (at $t \approx 2.800$ seconds), it is approximately 12.673 feet off the ground and therefore clears the fence by at least two feet.

79. (a) $\mathbf{r}(t) = \left[(155 \cos 18° - 11.7)\frac{1}{0.09}(1 - e^{-0.09t})\right]\mathbf{i}$

$$+\left[4 + \left(\frac{155 \sin 18°}{0.09}\right)(1 - e^{-0.09t}) + \frac{32}{0.09^2}(1 - 0.09t - e^{-0.09t}\right]\mathbf{j}$$

$x(t) = (155 \cos 18° - 11.7)\frac{1}{0.09}(1 - e^{-0.09t})$

$y(t) = 4 + \left(\frac{155 \sin 18^\circ}{0.09}\right)(1 - e^{-0.09t}) + \frac{32}{0.09^2}(1 - 0.09t - e^{-0.09t})$

(b) Plot y(t) and use the maximum function fo find $y \approx 36.921$ feet at $t \approx 1.404$ seconds.

(c) Plot y(t) and find that $y(t) = 0$ at $t \approx 2.959$ sec, then plug this into the expression for x(t) to find $x(2.959) \approx 352.520$ ft.

(d) Plot y(t) and find that $y(t) = 30$ at $t \approx 0.753$ and 2.068 seconds. At those times, $x \approx 98.799$ and 256.138 feet (from home plate).

(e) No, the batter has not hit a home run. If the drag coefficient k is less than ≈ 0.011, the hit will be a home run. (This result can be found by trying different k-values until the parametrically plotted curve has $y \geq 10$ for $x = 380$.)

80. (a) $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$

(b) $\overrightarrow{AP} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BD} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$

(c) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$, so by part (b), $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AC}$.

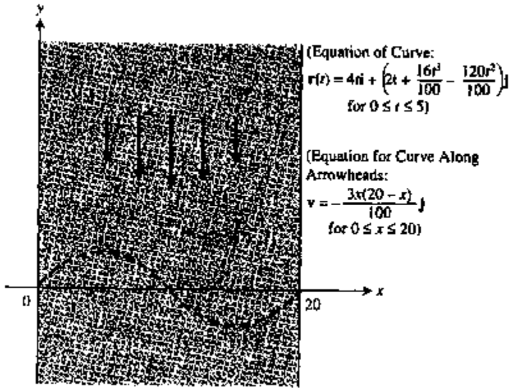
81. The widths between the successive turns are constant and are given by $2\pi a$.

CHAPTER 9 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. (a) Let $a\mathbf{i} + b\mathbf{j}$ be the velocity of the boat. The velocity of the boat relative to an observer on the bank of the river is $\mathbf{v} = a\mathbf{i} + \left[b - \frac{3x(20 - x)}{100}\right]\mathbf{j}$. The distance x of the boat as it crosses the river is related to time by $x = at \Rightarrow \mathbf{v} = a\mathbf{i} + \left[b - \frac{3at(20 - at)}{100}\right]\mathbf{j} = a\mathbf{i} + \left(b + \frac{3a^2t^2 - 60at}{100}\right)\mathbf{j} \Rightarrow \mathbf{r}(t) = at\mathbf{i} + \left(bt + \frac{a^2t^3}{100} - \frac{30at^2}{100}\right)\mathbf{j} + \mathbf{C}$; $\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j} \Rightarrow \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{r}(t) = at\mathbf{i} + \left(bt + \frac{a^2t^3 - 30at^2}{100}\right)\mathbf{j}$. The boat reaches the shore when $x = 20$ $\Rightarrow 20 = at \Rightarrow t = \frac{20}{a}$ and $y = 0 \Rightarrow 0 = b\left(\frac{20}{a}\right) + \frac{a^2\left(\frac{20}{a}\right)^3 - 30a\left(\frac{20}{a}\right)^2}{100} = \frac{20b}{a} + \frac{(20)^3 - 30(20)^2}{100a}$ $= \frac{2000b + 8000 - 12{,}000}{100a} \Rightarrow b = 2$; the speed of the boat is $\sqrt{20} = |\mathbf{v}| = \sqrt{a^2 + b^2} = \sqrt{a^2 + 4} \Rightarrow a^2 = 16$ $\Rightarrow a = 4$; thus, $\mathbf{v} = 4\mathbf{i} + 2\mathbf{j}$ is the velocity of the boat

(b) $\mathbf{r}(t) = at\mathbf{i} + \left(bt + \frac{a^2t^3 - 30at^2}{100}\right)\mathbf{j} = 4t\mathbf{i} + \left(2t + \frac{16t^3}{100} - \frac{120t^2}{100}\right)\mathbf{j}$ by part (a), where $0 \leq t \leq 5$

(c) $x = 4t$ and $y = 2t + \frac{16t^3}{100} - \frac{120t^2}{100}$
$= \frac{4}{25}t^3 - \frac{6}{5}t^2 + 2t = \frac{2}{25}t\left(2t^2 - 15t + 25\right)$
$= \frac{2}{25}t(2t - 5)(t - 5)$, which is the graph of the cubic displayed here



2. $\frac{d\mathbf{r}}{dt}$ orthogonal to $\mathbf{r} \Rightarrow 0 = \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = \frac{1}{2}\frac{d\mathbf{r}}{dt} \cdot \mathbf{r} + \frac{1}{2}\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{2}\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) \Rightarrow \mathbf{r} \cdot \mathbf{r} = K$, a constant. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t, then $\mathbf{r} \cdot \mathbf{r} = x^2 + y^2 \Rightarrow x^2 + y^2 = K$, which is the equation of a circle centered at the origin.

3. $\mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j}$
$\Rightarrow \mathbf{a} = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t)\mathbf{j}$
$= (-2e^t \sin t)\mathbf{i} + (2e^t \cos t)\mathbf{j}$. Let θ be the angle between $\mathbf{r}$ and $\mathbf{a}$. Then $\theta = \cos^{-1}\left(\frac{\mathbf{r} \cdot \mathbf{a}}{|\mathbf{r}||\mathbf{a}|}\right)$
$$= \cos^{-1}\left(\frac{-2e^{2t} \sin t \cos t + 2e^{2t} \sin t \cos t}{\sqrt{(e^t \cos t)^2 + (e^t \sin t)^2}\sqrt{(-2e^t \sin t)^2 + (2e^t \cos t)^2}}\right) = \cos^{-1}\left(\frac{0}{2e^{2t}}\right) = \cos^{-1} 0 = \frac{\pi}{2} \text{ for all } t$$

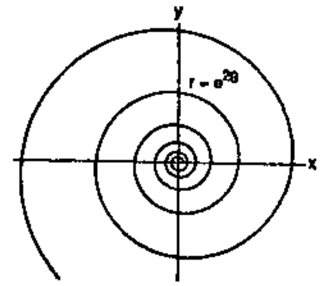
4. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ and $\mathbf{v} \cdot \mathbf{i} = y \Rightarrow \frac{dx}{dt} = y$. Since the particle moves around the unit circle $x^2 + y^2 = 1$, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{x}{y}(y) = -x$. Since $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -x$, we have $\mathbf{v} = y\mathbf{i} - x\mathbf{j} \Rightarrow$ at $(1, 0)$, $\mathbf{v} = -\mathbf{j}$ and the motion is clockwise.

5. $9y = x^3 \Rightarrow 9\frac{dy}{dt} = 3x^2\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{3}x^2\frac{dx}{dt}$. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t, then $\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$. Hence $\mathbf{v} \cdot \mathbf{i} = 4 \Rightarrow \frac{dx}{dt} = 4$ and $\mathbf{v} \cdot \mathbf{j} = \frac{dy}{dt} = \frac{1}{3}x^2\frac{dx}{dt} = \frac{1}{3}(3)^2(4) = 12$ at $(3, 3)$. Also, $\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$ and $\frac{d^2y}{dt^2} = \left(\frac{2}{3}x\right)\left(\frac{dx}{dt}\right)^2 + \left(\frac{1}{3}x^2\right)\frac{d^2x}{dt^2}$. Hence $\mathbf{a} \cdot \mathbf{i} = -2 \Rightarrow \frac{d^2x}{dt^2} = -2$ and $\mathbf{a} \cdot \mathbf{j} = \frac{d^2y}{dt^2} = \frac{2}{3}(3)(4)^2 + \frac{1}{3}(3)^2(-2) = 26$ at the point $(x, y) = (3, 3)$.

6. The two vectors $|\mathbf{v}|\mathbf{u}$ and $|\mathbf{u}|\mathbf{v}$ have the same magnitude, which is $|\mathbf{u}||\mathbf{v}|$. Therefore, using the result from Exercise 18, Section 9.2, the vector $\mathbf{w} = |\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v}$ bisects the angle between $|\mathbf{v}|\mathbf{u}$ and $|\mathbf{u}|\mathbf{v}$. The vector $\mathbf{w}$ also bisects the angle between $\mathbf{u}$ and $\mathbf{v}$ because $\mathbf{u}$ is in the same direction as $|\mathbf{v}|\mathbf{u}$ and $\mathbf{v}$ is in the same direction as $|\mathbf{u}|\mathbf{v}$.

7. (a) $x = e^{2t}\cos t$ and $y = e^{2t}\sin t \Rightarrow x^2 + y^2 = e^{4t}\cos^2 t + e^{4t}\sin^2 t = e^{4t}$. Also $\frac{y}{x} = \frac{e^{2t}\sin t}{e^{2t}\cos t} = \tan t$

$\Rightarrow t = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow x^2 + y^2 = e^{4\tan^{-1}(y/x)}$ is the Cartesian equation. Since $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, the polar equation is $r^2 = e^{4\theta}$ or $r = e^{2\theta}$ for $r > 0$



(b) $ds^2 = r^2\,d\theta^2 + dr^2$; $r = e^{2\theta} \Rightarrow dr = 2e^{2\theta}\,d\theta$

$\Rightarrow ds^2 = r^2\,d\theta^2 + \left(2e^{2\theta}\,d\theta\right)^2 = \left(e^{2\theta}\right)^2 d\theta^2 + 4e^{4\theta}\,d\theta^2$

$= 5e^{4\theta}\,d\theta^2 \Rightarrow ds = \sqrt{5}\,e^{2\theta}\,d\theta \Rightarrow L = \int_0^{2\pi} \sqrt{5}\,e^{2\theta}\,d\theta$

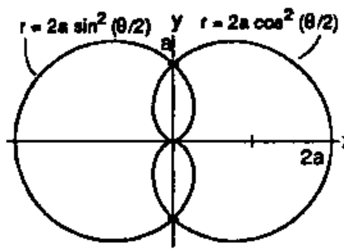
$= \left[\frac{\sqrt{5}\,e^{2\theta}}{2}\right]_0^{2\pi} = \frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right)$

8. $r = 2\sin^3\left(\frac{\theta}{3}\right) \Rightarrow dr = 2\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)d\theta \Rightarrow ds^2 = r^2\,d\theta^2 + dr^2 = \left[2\sin^3\left(\frac{\theta}{3}\right)\right]^2 d\theta^2 + \left[2\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)d\theta\right]^2$

$= 4\sin^6\left(\frac{\theta}{3}\right)d\theta^2 + 4\sin^4\left(\frac{\theta}{3}\right)\cos^2\left(\frac{\theta}{3}\right)d\theta^2 = \left[4\sin^4\left(\frac{\theta}{3}\right)\right]\left[\sin^2\left(\frac{\theta}{3}\right) + \cos^2\left(\frac{\theta}{3}\right)\right]d\theta^2 = 4\sin^4\left(\frac{\theta}{3}\right)d\theta^2$

$\Rightarrow ds = 2\sin^2\left(\frac{\theta}{3}\right)d\theta$. Then $L = \int_0^{3\pi} 2\sin^2\left(\frac{\theta}{3}\right)d\theta = \int_0^{3\pi}\left[1 - \cos\left(\frac{2\theta}{3}\right)\right]d\theta = \left[\theta - \frac{3}{2}\sin\left(\frac{2\theta}{3}\right)\right]_0^{3\pi} = 3\pi$

9. The region in question is the figure eight in the middle. The arc of $r = 2a\sin^2\left(\frac{\theta}{2}\right)$ in the first quadrant gives $\frac{1}{4}$ of that region. Therefore the area is $A = 4\int_0^{\pi/2} \frac{1}{2}r^2\,d\theta$



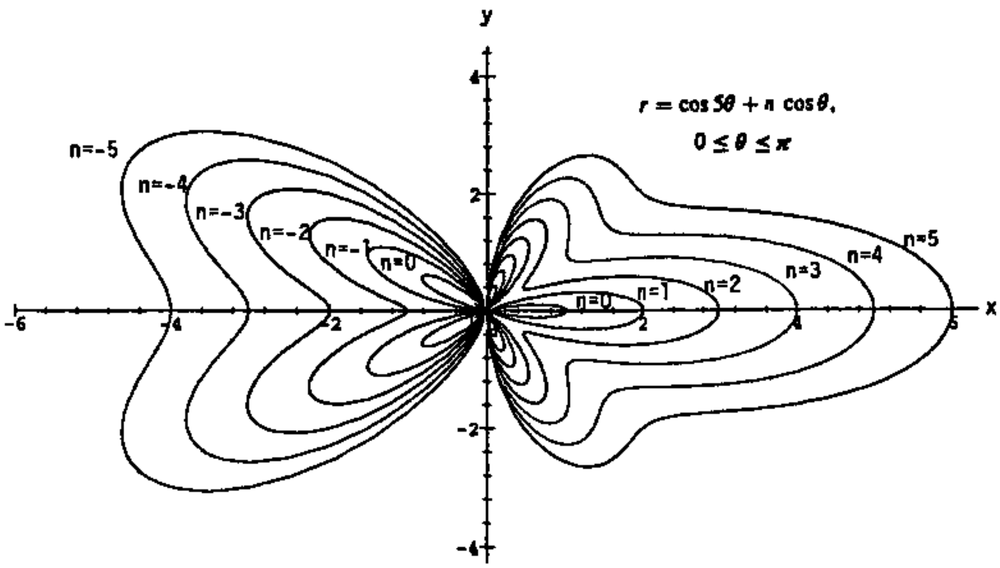
$= 4\int_0^{\pi/2} \frac{1}{2}\left[2a\sin^2\left(\frac{\theta}{2}\right)\right]^2 d\theta = 8a^2\int_0^{\pi/2}\sin^4\left(\frac{\theta}{2}\right)d\theta$

$= 8a^2\int_0^{\pi/2}\sin^2\left(\frac{\theta}{2}\right)\left[1 - \cos^2\left(\frac{\theta}{2}\right)\right]d\theta = 8a^2\int_0^{\pi/2}\left[\sin^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\right]d\theta = 8a^2\int_0^{\pi/2}\left(\frac{1-\cos\theta}{2} - \frac{\sin^2\theta}{4}\right)d\theta$

$= 2a^2\int_0^{\pi/2}\left(2 - 2\cos\theta - \frac{1-\cos 2\theta}{2}\right)d\theta = a^2\int_0^{\pi/2}(3 - 4\cos\theta + \cos 2\theta)\,d\theta = a^2\left[3\theta - 4\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$

$= a^2\left(\frac{3\pi}{2} - 4\right)$

10.


y
x
$r = \cos 5\theta + n\cos\theta,$
$0 \le \theta \le \pi$
n=-5
n=-4
n=-3
n=-2
n=-1
n=0
n=0
n=1
n=2
n=3
n=4
n=5
4
2
-2
-4
-6
-4
-2
2
4
5

NOTES: