DIFFERENTIATION AND INTEGRATION FORMULAS

$\frac{d}{dx} f(x)$

$\tan x = \sec^2 x$

$\cot x = -\operatorname{cosec}^2 x$

$\sec x = \sec x \tan x$

$\operatorname{cosec} x = -\operatorname{cosec} x \cot x$

$\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

$\tan^{-1} x = \frac{1}{1+x^2}$

$\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

$\cot^{-1} x = \frac{-1}{1+x^2}$

$\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$

$\operatorname{cosec}^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$

$a^x = a^x \ln a$

$\int f(x)dx$

$\cos x = \sin x$

$\sin x = -\cos x$

$\sec^2 x = \tan x$

$\operatorname{cosec}^2 x = -\cot x$

$\sec x \tan x = \sec x$

$\operatorname{cosec} x \cot x = -\operatorname{cosec} x$

$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x$ $\quad \int \frac{-1}{\sqrt{1-x^2}} = \cos^{-1} x$

$\int \frac{1}{1+x^2} = \tan^{-1} x$ $\quad \int \frac{-1}{1+x^2} = \cot^{-1} x$

$\int \frac{1}{x\sqrt{x^2-1}} = \sec^{-1} x$ $\quad \int \frac{-1}{x\sqrt{x^2-1}} = \operatorname{cosec}^{-1} x$

$\int e^x dx = e^x$ $\quad \int \frac{1}{x} = \ln x$

$\int a^x = \frac{a^x}{\ln a}$

$\int \frac{dx}{x^2 a^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right|$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \left|\frac{a+x}{a-x}\right|$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$

$\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}|$

$\int \frac{d}{dx} I \cdot II = I \int II - \int \left[\frac{dI}{dx} \int II\right]$

$\int e^x (f(x)+f'(x)) = e^x f(x)$

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots - \frac{(-1)^n x^{2n-1}}{2n-1}$

$\frac{\pi}{4} \to 45°$ $\quad \frac{\pi}{6} \to 30°$ $\quad \frac{\pi}{3} \to 60°$ $\quad \frac{\pi}{2} \to 90°$ $\quad \pi \to 18$

$\frac{2\pi}{3} \to 120°$ $\quad \frac{5\pi}{6} \to 150°$ $\quad \frac{3\pi}{4} \to 135°$ $\quad \frac{3\pi}{2} \to 2$

$$\int \sqrt{x^2-a^2} = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|$$

$$\int \sqrt{a^2-x^2} = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$\int \sqrt{x^2+a^2} = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}$$

$$\sinh x = \left(\frac{e^x - e^{-x}}{2}\right) \qquad \cosh x = \left(\frac{e^x + e^{-x}}{2}\right)$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\boxed{e^{\pi} > \pi^{e}}$$

<u>SOME LIMITS</u>

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0 \qquad \lim_{n\to\infty} x^{1/n} = 1 \ (x>0)$$

$$\lim_{n\to\infty} n^{1/n} = 1 \qquad \lim_{n\to\infty} x^n = 0 \ (|x|<1)$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \ (\text{any } x) \qquad \lim_{n\to\infty}\frac{x^n}{n!} = 0$$

$\pi = 3.14$

$\frac{\pi}{2} = 1.57$

$\frac{\pi}{4} = 0.785$

$\frac{\pi}{6} = 0.523$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \qquad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$$

$$\sin x = \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

SEQ and SERIES

=) A convergent seq has a unique limit

=) Algebra rules

=) SANDWICH TH : $a_n \le b_n \le c_n$ $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = L$

=) $a_n \to L$ f is cont at L for all a_n $f(a_n) \to f(L)$

=) $f(x)$ is defined for all ~~x~~ $x \ge n_0$ $a_n = f(n)$ $n \ge n_0$ then

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$$

=) ~~lim~~ => Upper bound $a_n \le M$ for all n

BOUNDED + MONOTONOUS → CONVERGENT

=) Const. seq is both non-inc and non-dec

=> n^{th} partial sum converge => series converge

=) GP

=) If the series $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. Converse is true

$A \pm B$. A conv. B conv => $A \pm B$ conv

A div B conv => $A \pm B$ div

INTEGRAL TEST: a_n = seq $a_n > 0$ for all n

Let $a_n = f(n)$ $f(x) \to$ cont, +ve, ↓ing f^n of x for $x \ge N$

Then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge / diverge

COMPARISION TEST: $\sum a_n, \sum c_n, \sum d_n$ be series with +ve terms. For N

$d_n \le a_n \le c_n$ for all $n > N$

If $\sum c_n$ converges → $\sum a_n$ converges

If $\sum d_n$ diverges → $\sum a_n$ diverges

$\frac{S}{S_{n-1}}$

LIMIT COMP TEST: Let $a_n, b_n > 0$ for $n \ge N$ $N \in N$

$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ $\sum a_n$ and $\sum b_n$ both conv/div

$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and if $\sum b_n$ con => $\sum a_n$ con

$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and if $\sum b_n$ div => $\sum a_n$ div

RATIO TEST $a_n > 0$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = p$

$p < 1$ converge
$p > 1$ diverge
$p = 1$ inconclusive

$\boxed{\frac{1}{n^p} \quad \begin{matrix} p \le 1 \text{ diverges} \\ p > 1 \text{ converge} \end{matrix}}$

$\lim_{n\to\infty} C_n = L$

ROOT TEST: $\lim_{n\to\infty} (a_n)^{1/n} = p$

$p < 1$ converge
$p > 1$ diverge
$p = 1$ inconclusive

ALTERNATIN SERIES: of the form $\sum_{n=1}^{\infty} (-1)^n a_n$ $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ $a_n \geq 0$

$\$$ $\lim_{n\to\infty} a_n = 0$ and a_n is ↓ ing then alt. series converges

TEST (lebny test) $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$ converges if

all a_n's are +ve

The +ve a_n's are non-increasing i.e $a_n \geq a_{n+1}$ for $n \geq N$ / as $n \to \infty$ $a_n \to 0$

* If $p > 0$ then series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \dots$ converges

ABS. CONVERGENCE:

If $\sum_{n=1}^{\infty} |a_n|$ converges | If a_n is abs conv $\Rightarrow$ a_n is conv.

CONDITIONAL CONVERGENCE

a_n is convergent but not absolutely convergent

POWER SERIES:

$\Rightarrow$ A power series about $x = 0$ is of the form

$C_0, C_1 \to$ constants

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

$\Rightarrow$ A power series about $x = a$ is a series of form

$a \to$ centre
$C_0, C_1 \to$ coeff

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1 (x-a) + C_2 (x-a)^2 + \dots$$

* The geometric series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ converges to $\frac{1}{1-x}$ if $|x| < 1$

diverges at $x = \pm 1$ and $|x| > 1$

CONVERGENCE OF POWER SERIES:

If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x=c$, $c \neq 0$, then it converges absolutely for all x; $|x|<|c|$

If $\sum_{n=0}^{\infty} a_n x^n$ diverges at $x=d$, then it diverges for all x with $|x|>|d|$

$\Rightarrow$ Applicable to $\sum_{n=0}^{\infty} a_n x^n$. BUT if series is of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$

then $x-a \to t$

$\Rightarrow$ IF $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for some $x=c$, then it is absolutely convergent for $|x-a|<|x-c|$

$\Rightarrow$ " $\sum_{n=0}^{\infty} a_n(x-a)^n$ diverges for some $x=d$, " " " divergent for $|x-a|>|x-d|$

RADIUS OF CONVERGENCE: (R)

3 cases for $\sum a_n(x-a)^n$

① There is a +ve no R st series diverges for x if $|x-a|>R$
" converges " " " $|x-a|<R$

* SERIES may or may not converge at endpoints $x=a-R$; $x=a+R$

② Series is ABS CONVERGENT for every $x \Rightarrow R=\infty$

③ " Converges only at $x=a$ and diverges elsewhere $\Rightarrow R=0$

Interval of R at $x=a$ is INTERVAL OF CONVERGENCE

To find $R \to$ Use Ratio / n^{th} Root test to find interval where series converges absolutely. USUALLY $|x-a|<R$

If R = finite $\to$ test at $x=a \pm R$. Use Comparison / Integral test

If interval is $|x-a|>R \to$ it does not imply conditional convergence cos n^{th} term does not tend to zero for these x

OPERATION: $A(x)=\sum a_n x^n$, $B(x)=\sum b_n x^n \to$ converge

$C_n = a_n b_0 + a_{n-1}b_1 + a_{n-2}b_2 + \cdots a_0 b_n = \sum_{k=0}^{n} a_{n-k} b_k$

$\sum C_n x^n$ converges to $A(x)B(x)$ for $|x|<R$

TERM BY TERM DIFF.

$\sum a_n(x-a)^n$ has $R>0$ defines a f^n $f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ on $a-R<x<a+R$
This $f(x)$ has all order derivative in range and each derivative converges in range.

INTEGRATION:

$f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for $|x-a|<R$, then $\sum_{n=0}^{\infty} \frac{a_n(x-a)^{n+1}}{n+1}$ converges for $|x-a|<R$

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n(x-a)^{n+1}}{n+1} + C \text{ for } |x-a|<R$$

If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<R$ then $\sum_{n=0}^{\infty} a_n[f(x)]^n$ converges abs for cont f on $|f(x)|<R$

If p-series $\sum_{n=1}^{\infty} a_n(x-a)^n$ has $R>0$ then $g(x)=\sum_{n=1}^{\infty} a_n(x-a)^n$ is diff. ∞ many times

BUT if a f^n has derivatives of all orders on an interval, can it be Xpressed as p-series in some interval

TAYLOR SERIES: Let f be a f^n with derivatives of all orders throughout some interval containing 'a' as its interior pt. Then taylor series is given by

$$\sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\cdots \frac{f^{(n)}(a)}{n!}(x-a)^n \ldots$$

MC'LAUREN SERIES: Taylor series generated by f at $x=0$

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!}x^k = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2 \cdots \frac{f^n(0)}{n!}x^n \cdots$$

E.g. $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

TAYLOR POLYNOMIAL: Let $f(x)$ be a f^n with derivates of order k for $k=1,2,3,\ldots N$ in some interval containing $a\in R$ as an interior point. Then for any integer n from $0\to N$, Taylor poly of order 'n' generated by f at $x=a$ is

$$P_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots \frac{f^n(a)}{n!}(x-a)^n$$

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POLAR COORDINATES

$x = r\cos\theta \quad y = r\sin\theta \quad r^2 = x^2+y^2 \quad \theta = \tan^{-1}\frac{y}{x}$

LIMACONS • $r = a+b\cos\theta \quad r = a+b\sin\theta$

a, b +ve

$a>b$ + without inner loops

$a<b$ + with " "

LEMNISCATES (RIBBON): $r^2 = a^2\cos 2\theta \quad r^2 = a^2\sin 2\theta$

SPIRALS : $r = \pm\theta \quad r = e^{2} \quad r = \frac{a}{\theta}$

CARDIOIDS $r = a \pm a\cos\theta \quad r = a \pm a\sin\theta$ Heart shape

ROSES: $r = a\cos(n\theta) \quad r = a\sin(n\theta)$

n petals if n = odd 2n petals if n = even

AREA

$$A = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 \, d\theta$$

LENGTH OF ARC

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

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PARTIAL DERIVATIVES
DOUBLE VARIABLE f^n

DEF ⇒ Let 'D' be a set of n-tuples of real nos. $(x_1, x_2, x_3 \ldots x_n)$. A real valued f^n f on D is a rule that assigns a unique (single) real no $W = f(x_1, x_2, x_3 \ldots x_n)$ to each element in D.
D → Domain W → Range Bounded by xy-plane → adisc

LEVEL CURVE: Set of points in a plane where a f^n has a const value
~~f(x,y,z)~~ $f(x,y) = c$

CONTOUR CURVE: $z = c$ cuts $f(x,y) = c$

LEVEL SURFACE: Set of pts (x, y, z) in space where $f(x,y,z) = c$

LIMIT $f(x,y) \to L$ as $(x,y) \to (x_0, y_0) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

If for every $\varepsilon > 0$ there exists a $\delta > 0$ ST $|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

CONTINUITY: $f(x,y)$ is cont at (x_0, y_0) if

① f^n is defined at (x_0, y_0)
② $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
③ $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

⇒ If a f^n $f(x,y)$ has 2 different limits along 2 different paths in the domain of f as $(x,y) \to (x_0, y_0)$, then $\lim_{x,y \to x_0,y_0} f(x,y)$ does not exist

⇒ f is cont at (x_0, y_0) and g is a single variable f^n $h = g(f(x))$
⇒ $h(x,y) = g(f(x,y))$ is cont at (x_0, y_0)

SANDWICH TH: $g < f < h$ if $g, h \to c$ then $f \to c$
$f(x,y) = f(r\cos\theta, r\sin\theta)$

PARTIAL DERI (PD)

PD of f wrt x at (x_0, y_0) is $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial}{\partial x} f(x, y_0)\right|_{x=x_0}$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

For y $\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$

SECOND ORDER PD

$$f \to \frac{\partial}{\partial x} \to \frac{\partial f}{\partial x} \to \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial f}{\partial x} \to \frac{\partial}{\partial y} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

$$f \to \frac{\partial}{\partial y} \to \frac{\partial f}{\partial y} \to \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

$$\frac{\partial f}{\partial y} \to \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

If all f_x, f_y, f_{xy}, f_{yx} are defined in open (a,b) and cont in (a,b) then $\boxed{f_{xy} = f_{yx}}$

CHAIN RULE: $z = f(x,y)$ $x = x(t), y = y(t)$

$$z(t) = f(x(t), y(t))$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

For 3 variables $w = f(x,y,z)$ $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$

$\Rightarrow$ When x, y, z are itself PD of $2f^n$ $w = f(x,y,z)$

i.e. $x = x(s,t)$ $y = y(s,t)$ $z = z(s,t)$ then we have 2PD of w

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds} + \frac{\partial w}{\partial z}\frac{dz}{ds} \; ; \; \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

JACOBIAN MATRICES $J_1 = \begin{bmatrix} \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{bmatrix}$ $J_2 = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$ $W = \begin{bmatrix} \frac{\partial w}{\partial s} \\ \frac{\partial w}{\partial t} \end{bmatrix}$

$\boxed{W = J_1 J_2}$

IMPLICIT F^n: $F(x,y) = 0$ y as a differentiable f^n of x

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

UNIT TANGENT, TANGENT LINE, ETC :

$\Rightarrow r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. If x, y, z are cont f^n, then r = curve

TH-1 $r(t)$ defined on I and $t_0 \in I$. Let $L = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$

$\boxed{\lim_{t\to t_0} r(t) = L}$ if for every $\epsilon > 0$ $\forall \delta > 0$ $|r(t) - L| < \epsilon$ when $0 < |t - t_0| < \delta$

TH-2 $r(t)$ has a limit at $t = t_0$ if and only if $x(t), y(t), z(t)$ has limit at $t = t_0$

TH-3 $r(t)$ is cont at $t = t_0$ if $\lim_{t\to t_0} r(t) = r(t_0)$

Differentiation: $\Delta r = r(t+\Delta t) - r(t)$

$r'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$

$r'(t)$ at pt $P(t=t_0)$ is VECTOR TANGENT at P

TANGENT LINE : To a curve $r(t)$ at pt $P(x(t_0), y(t_0), z(t_0))$ is a line thru P and $\parallel$ to vector $r'(t_0) = x'(t_0)\hat{i} + y'(t_0)\hat{j} + z'(t_0)\hat{k}$

$\therefore$ $\boxed{\gamma(\lambda) = \lambda r'(t_0) + r(t_0)}$

* **SMOOTH CURVE** $\rightarrow$ $r'(t)$ is cont. and $r'(t) \neq 0$

VELOCITY $(\vec{v})$: $\vec{v}(t) = \frac{d\vec{r}}{dt}$ direc^n of v = direc^n of motion

$|v|$ = speed $\quad \vec{a} = \frac{d\vec{v}}{dt}$ = acc = $\frac{d^2\vec{r}}{dt^2}$ $\rightarrow$ Acc $(\vec{a})$

UNIT TANGENT : Direc^n of motion at a time 't'
(T)

$\boxed{\vec{T} = \frac{\vec{v}}{|v|}}$ $\quad |T| = 1 \quad$ $\boxed{T = \frac{dr}{ds}}$ $\rightarrow$ T changes its direc^n as curve bends

* $\frac{d}{dt}(U\cdot V) = U'(t)\cdot V(t) + U(t)\cdot V'(t) \qquad \frac{d}{dt}(U\times V) = U'\times V + U\times V'$

$\frac{d}{dt}(Uf(v)) = f'(v)(U'f(v))$

VECTOR F^n OF CONST LENGTH

$\frac{d}{dt}(U\cdot U) = \frac{d}{dt}\text{const} = 0$

$\frac{dU}{dt}\cdot U + U\frac{dU}{dt} = 0$

$2U\frac{dU}{dt} = 0 \Rightarrow U\frac{dU}{dt} = 0$

$\left(r\cdot\frac{dr}{dt}\right) = 0$!!

INTEGRALS :

TH-1 $\int r(t)dt = R(t) + C \to$ const vector

TH-2 $\int_a^b r(t)dt = R(t)\Big|_a^b = R(b) - R(a)$

ARC LENGTH : $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ $\quad t \in [a,b]$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad \boxed{L = \int_a^b |\vec{v}(t)|dt}$$

A FROM A BASE POINT $t = t_0$

$L = \int_{t_0}^{t} |\vec{v}(t)|dt$ = ARC LENGTH PARAMETER $\underset{\text{speed}}{(S)}$ $\Rightarrow S = \int_{t_0}^{t} |\vec{v}(\tau)|d\tau$

$$\boxed{S = \int_{t_0}^{t} |\vec{v}(\tau)|d\tau} \qquad \boxed{\frac{ds}{dt} = |\vec{v}(t)| > 0} \to \text{SPEED}$$

TORSION (τ) $\quad \tau, T, \vec{v}, \vec{a}, \vec{s}$, speed, B, N, k

$$\boxed{\tau = \frac{(\vec{v}\times\vec{a})\cdot\vec{a}'(t)}{|\vec{v}\times\vec{a}|^2}}$$

$\tau \to$ Rate at which osculating plane turns direct

$T = \frac{\vec{v}}{|v|}$ $\quad |T| = 1$ $\quad T = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} \Rightarrow \boxed{T = \frac{dr}{ds}} \to$ T changes its direcⁿ as curve bends

↳ only direcⁿ changes length remains const

CURVATURE : (k) $\to$ Rate at which 'T' changes per unit length

$$\boxed{k = \left|\frac{dT}{ds}\right|}$$

k for st line $= 0$

k " circle $= 1$

$$k = \left|\frac{dT}{ds}\right|\left|\frac{dt}{dt}\right| \Rightarrow k\left(\frac{dT}{dt}\right)\frac{1}{\left(\frac{ds}{dt}\right)} \Rightarrow \boxed{k = \frac{1}{|v|}\frac{dT}{dt}} \to \text{IMP}$$

PRINCIPLE UNIT VECTOR [N] : where $k \neq 0$

$$N = \frac{\left(\frac{dT}{ds}\right)}{\left|\frac{dT}{ds}\right|} \Rightarrow \boxed{N = \frac{1}{k}\frac{dT}{ds}} \text{ IMP}$$

CIRCLE OF CURVATURE / OSCULATING CIRCLE:

It is a circle at a point P, where $\kappa \neq 0$, in the plane of the curve that is tangent to given curve at P, has same curvature as the curve has at P, has centre that lies on inner side of curve

RADIUS OF CURVATURE (ρ) $\boxed{\rho = \frac{1}{\kappa}}$

BINORMAL VECTOR (B) $\boxed{\vec{B} = \vec{T} \times \vec{N}}$.

Together, ($\bar{B}, \bar{N}, \bar{T}$ define a moving (frenet frame) → rt handed vector frame

TORSION (τ) → Rate at which osculating plane turns about T

$$\tau = -\frac{dB}{ds} \cdot N = \boxed{\frac{-1}{|v|}\left(\frac{dB}{dt} \cdot N\right) = \tau}$$

$$\vec{v} \times \vec{a} = \kappa \left(\frac{ds}{dt}\right)^3 B \quad (T \times T = 0 \; T \times N = B) \Rightarrow |v \times a| = \kappa \left|\frac{ds}{dt}\right|^3 |B| = \kappa |v|^3$$

$$\Rightarrow \boxed{|v \times a| = \kappa |v|^3}$$

$$\boxed{\kappa = \frac{|v \times a|}{|v|^3}} \quad \boxed{\tau = \frac{(v \times a) \cdot a'(t)}{|v \times a|^2}} \quad \boxed{v = T\frac{ds}{dt}}$$

$$\frac{dB}{ds} = \frac{d(T \times N)}{ds} = \left(\frac{dT}{ds} \times N\right) + \left(T \times \frac{dN}{ds}\right) = 0 + T \times \frac{dN}{ds}$$

$$\Rightarrow \boxed{\frac{dB}{ds} = T \times \frac{dN}{ds}} \Rightarrow \boxed{\frac{dB}{ds} = -\tau N}$$

OSCULATING → T N RECTIFYING → T B NORMAL → N B

If $\vec{a} = a_T \vec{T} + a_n \vec{N}$

$$a_T = \frac{d^2 s}{dt^2} = \frac{d|v|}{dt} \quad a_N = \kappa \left(\frac{ds}{dt}\right)^2 = \kappa |v|^2 \quad \boxed{a_T = \frac{d|v|}{dt}} \quad \boxed{a_N = \kappa |v|^2}$$

$T \cdot N = 0$ $|a|^2 = |a_T|^2 + |a_N|^2$

DIRECTIONAL DERIVATIVE

DEF Derivative of $f(x,y)$ in the direction of unit vector $U = U_1\hat{i} + U_2\hat{j}$ at $P_0 = (x_0, y_0)$ is $\left(\frac{\partial f}{\partial s}\right)_{U,P_0}$

$$\left(\frac{\partial f}{\partial s}\right)_{U,P_0} = \lim_{h \to 0} \frac{f(x_0 + hU_1, y_0 + hU_2) - f(x_0, y_0)}{h} = (D_U f)_{P_0}$$

GRADIENT VECTOR $[\nabla]$: ∇ is a f^n $f(x,y)$ at $P_0 = (x_0, y_0)$

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}}$$

DIRECTIONAL DERIVATIVES: The directional derivative of a function $f(x,y)$ at a pt P_0 in the direction of a unit vector $\vec{U}$ is given

$$\boxed{\left(\frac{\partial f}{\partial s}\right)_{U,P_0} = (\nabla f)_{P_0} \cdot U}$$

* For 3 variables $\vec{U} = U_1\hat{i} + U_2\hat{j} + U_3\hat{k}$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \qquad \left(\frac{\partial f}{\partial s}\right)_U = D_U f = \nabla f \cdot U = U_1\frac{\partial f}{\partial x} + U_2\frac{\partial f}{\partial y} + U_3\frac{\partial f}{\partial z}$$

* **MOST RAPID CHANGE**: f^n ↑es in direct^n of +ve ∇f
  f^n ↓es " " -ve ∇f
  If orthogonal to direc^n of ∇f, $\frac{\partial f}{\partial s} = 0$

* $\nabla(f \pm g) = \nabla f \pm \nabla g$ | $\nabla(af) = a\nabla f$ | $\nabla(fg) = f\nabla g + g\nabla f$ | $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

* Gradient at (x_0, y_0) is ⊥^{r} to tangent at (x_0, y_0) of the level curve

TANGENT LINE: Tangent line at (x_0, y_0) of level curve $f(x,y) = f(x_0, y_0)$ ~~and~~ of $z = f(x,y)$ is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0) = 0$$

$$\Rightarrow \boxed{f_x(x - x_0) + f_y(y - y_0) = 0}$$

TANGENT PLANE : $f(x,y,z)$ be diff $r = x\hat{i} + y\hat{j} + z\hat{k}$ is a smooth curve on LEVEL SURFACE of $f(x,y,z) = c$ passing thru $P_0(x_0, y_0, z_0) = r(t_0)$ $f(x(t), y(t), z(t)) = c \to$ Diff both sides wrt $t = t_0$

$$\boxed{\nabla f|_{P_0} \cdot r'(t_0) = 0}$$

मजाक

EQ^n of TANGENT PLANE $= f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0$

EQ^n of PLANE TANGENT $= f_x(x-x_0) + f_y(y-y_0) + z - z_0 = 0$

* Tangents to all smooth curves on level surface, passing thru P_0 lie in plane through P_0 and normal to $\nabla f|_{P_0}$. Plane thru P_0, normal to $\nabla f|_{P_0}$

NORMAL LINE → Line passing thru P_0 and $\parallel$ to $\nabla f|_{P_0}$

$$N = P_0 + \lambda \nabla f|_{P_0}$$

If 2 curves are given, then for Tangent line $\nabla f_1 \times \nabla f_2$ → line $\parallel$ to pt $P + \lambda L$

* The change df at a small dist ds from P_0 in $\vec{U}$ direc^n is

$$df = (\nabla f|_{P_0} \cdot u)\, ds$$

LINEARIZATION: $L(x,y) \sim f(x,y) = \boxed{f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0)}$

ERROR IN SLA: M is upper bound of $|f_{xx}|, |f_{yy}|, |f_{xy}|$

$$|E(x,y)| \le \frac{M}{2}\left[|(x-x_0)| + |(y-y_0)|\right]^2$$

TOTAL DIFFERENTIAL: $df = f_x dx + f_y dy$

LOCAL MAX $f(a,b) \ge f(x,y)$ LOCAL MIN $f(a,b) \le f(x,y)$

MAX MIN $f_x = f_y = 0$

CRITICAL PTS → $f_x = f_y = 0$ OR one of them doesn't exist

INFLECTION POINT ⟺ SADDLE POINT

SADDLE POINT : $f(x,y)$ has a saddle point at $P(a,b)$ if every disk centered at P $f(x,y) > f(a,b)$ and $f(x,y) < f(a,b)$

$[(a,b)\ f(a,b)]$ $z = f(x,y)$

II DERI TEST AND LAGRANGE:

II DERI TEST: $f_x = f_y = 0$

$f_{xx} > 0$ $D > 0$ local min upward curve

$f_{xx} < 0$ $D > 0$ " max Downward "

$D < 0$ saddle pt

$D = 0$ Test inconclusive

$D = f_{xx} f_{yy} (f_{xy})^2$

$D \to$ hessian

$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

* ABS MAX/MIN CP + Boundary pts

LAGRANGE MULTIPLIER

At critical pt (CP), $\frac{df}{dt} = 0$ or doesnt exist. Also consider boundary pt

METHOD: constraint $g(x,y) = 0$

$f(x,y,z)$ be diff in R $C: r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ P_0 is pt on

f has a local max/min on P_0, ∇f is $\perp^r$ to velocity vector at P_0

$r(t) = a(t)\hat{i} + b(t)\hat{j}$ $f(x,y,z)$ $g(x,y,z)$ $\nabla g \neq 0$ $g(x,y,z) = 0$ constraint

$\boxed{\nabla f \cdot v = 0}$ $v = \frac{dr}{dt}$ $\boxed{\nabla f = \lambda \nabla g}$

when x, y, z, λ satisfy then f has a local max/min at that (x,y,z)

* FOR 2 CONSTRAINTS

$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ $g_1 = 0 = g_2$

ORTHOGONAL GRADIENT:

$f(x,y,z) \to$ diff in R. Interior of R contains smooth curve $C = r(t)$

$r(t) = a(t)\hat{i} + b(t)\hat{j} + c(t)\hat{k}$. If P_0 is a pt on C

At P_0, f has its local max/min relating to its value on C, then ∇f is orthogonal to $r'(t_0)$

$r(t) = a(t)\hat{i} + b(t)\hat{j}$ $f(x,y)$ has its local max/min $\boxed{\nabla f \cdot v = 0}$

$v = \frac{dr}{dt}$ and $g = 0$ $f(x,y,z)$ $g_1 = 0$, $g_2 = 0$

$\nabla f = \lambda \nabla g$ $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ ∇g_1 and ∇g_2 are $\perp^r$ to ∇

EXTREME VALUE ON PARAMETERIZED CURVES:

$f(x,y)$ where $x = x(t)$ $y = y(t)$

$\Rightarrow$ pts where $\frac{df}{dt} = 0$ or $\frac{df}{dt}$ doesnt exist

$\Rightarrow$ End pts

INTEGRATION

SINGLE : $f \to [a,b]$ $f(x) \geq 0$ on $[a,b]$

n small rectangles $i = 1, 2, 3, 4 \dots n$ $\Delta x = \frac{b-a}{n}$ $\}\to$ Single variable

$A \approx \sum_{i=1}^{n} f(x_i)\Delta x \Rightarrow A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)dx \Rightarrow A = \int_a^b f(x)dx$

* For **DOUBLE** VARIABLE

$\iint_R f(x,y)\,dA = \lim_{||P||\to 0} \sum_{i=1}^{n} f(u_i v_i)\Delta A_i$ $\quad ||P|| \to$ length of largest diagonal amongst the rectangles in R

$||P|| \to$ NORM OF PARTITION $\to$ largest of all Δx_i's

* **VOLUME BY SLICING** $V = \int_a^b A(x)dx$

DOUBLE : $z = f(x,y) \to$ cont on $R = \{(x,y)\ x \in [a,b]\ y \in [c,d]\}$ $f(x,y) \geq 0$ on R

FUBINI'S TH (I FORM) $f(x,y) \to$ cont on R $x \in [a,b]$ $y \in [c,d]$

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

* FOR NON-RECTANGULAR GENERAL REGIONS

FUBINI'S TH (II FORM) $f(x,y) \to$ cont on R

$R : x \in [a,b]$ $y \in [g_1(x), g_2(x)]$ $g_1 g_2$ cont on $[a,b]$

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \quad \text{or VICE VERSA}$$

DOMINATION : ① $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R

② $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on R

ADDITIVITY: $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$ $\quad R_1, R_2$ - non overlapping

AREA BY DOUBLE INTEGRAL $A = \iint_R dA$

* AVG VALUE $\boxed{R = \frac{1}{\text{area of } R} \iint_R f\,dA}$

DOUBLE POLAR

$$A = \iint_R f(r,\theta) r\,dr\,d\theta \qquad A = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta) r\,dr\,d\theta$$

TRIPLE RECT

$|m| \to$ longest diagonal of cube

$$\iiint_Q f(x,y,z)\,dV = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

* In general $\displaystyle\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z)\,dx\,dy\,dz$

* VOLUME $\quad V = \iiint_D dV$

TRIPLE CYLINDRICAL:

$(r,\theta,z) \quad x = r\cos\theta \quad y = r\sin\theta \quad z = z \qquad \theta = \tan^{-1}\frac{y}{x}$

$x^2+y^2 = r^2 \quad r \geq 0 \quad \theta \in [0, 2\pi]$

$r = a \to$ Cylinder of radius 'a' on z axis

$\theta = \theta_0$ Plane containing z-axis

$z = z_0$ Plane $\perp$ to z axis $\parallel$ to xy

$\Delta V = \Delta z\, r\, \Delta r\, \Delta\theta$

$$\iiint_D f\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} dz\,r\,dr\,d\theta$$

SPHERICAL TRIPLE:

$(\rho,\phi,\theta) \quad r = \rho\sin\phi$

$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi \quad \rho^2 = \cancel{\sqrt{}}x^2+y^2+z^2 = r^2+z^2$

$\rho \geq 0 \quad \phi \in [0,\pi] \quad \theta \in [0, 2\pi]$

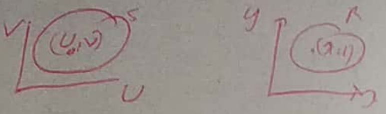
$\rho = a \to$ sphere of radius 'a'

$\phi = \phi_0 \to$ single cone (upper/lower)

$\theta = \theta_0 \to$ half plane

$$\iiint_D dV = \int_{\alpha}^{\beta} \int_{\phi_{min}}^{\phi_{max}} \int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} f(\rho,\phi,\theta)\, \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

SUBSTITUTION TRANSFORMATION



let $T : S(\subseteq \mathbb{R}^2) \to R(\subseteq \mathbb{R}^2)$ is $T(u,v) = (x,y)$

$x = g(u,v)$ $y = h(u,v)$

FOR DOUBLE INTEGRALS $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$

$$\iint_R f(x,y)\, dA = \iint_S f(g(u,v))\, h(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

JACOBIAN

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

A FOR TRIPLE INTEGRALS

(1) CYLINDRICAL

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = r$$

(2) spherical

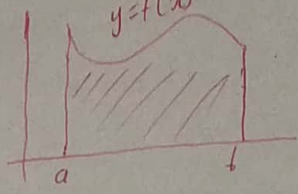
$$\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2 \sin\phi$$

VECTOR FIELDS - INTEGRATION

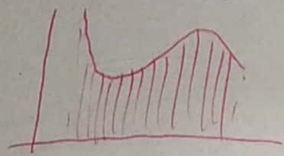
16.1 LINE INTEGRALS

mass of wire from $x=a$ to $x=b$

Linear mass density $= f(x)$ at each pt



$\Rightarrow$



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

MASS OF CURVED WIRE

Wire $r(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$ $\quad t \in [a,b]$

Density per unit arc length $= f(x,y,z)$

<u>DEF</u> If f is defined on a curve C given parametrically by $r(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$ $a \le t \le b$ then <u>LINE INTEGRAL</u> of f over C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k$$

wkt $\quad s = \int_a^t |v(\tau)|\,d\tau$

$$\frac{ds}{dt} = |v| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\Rightarrow ds = \int_a^b |v(t)|\,dt$$

$$\therefore \int_C f(x,y,z)\,ds = \int_a^b f(g(t), h(t), k(t))\,|v(t)|\,dt$$

<u>How</u>

1. Find smooth parametrization of C Eg $r(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$
2. $\int_C f(x,y,z)\,ds = \int_a^b f(g(t), h(t), k(t))\,|v(t)|\,dt$

Line integral of $f(x,y)$ along C

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

ADDITIVITY $C_1, C_2, C_3 \ldots C_n$

$$\int_C f ds = \int_{C_1} f ds + \int_{C_2} f ds + \ldots \int_{C_n} f ds$$

16.2 WORK CIRCULATION AND FLUX

SCALAR FIELD $f: D \subseteq R^n \to R \quad n \geq 1$

Associates a unique scalar f for each pt of its domain D

VECTOR FIELD: $F: D \subseteq R^n \to R^m \quad n \geq 1 \; m \geq 1$ Eg Gravitation, velocity, force

Associates a unique scalar F for each pt of D

E.g $F: D \subseteq R^3 \to R^3$ is a vector field

$F(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$

vector field $\quad M, N, P \to$ scalar field

LINE INTEGRAL OF VECTOR FIELD

Let F be a vector field, cont components C parametrized by $r(t)$ $t \in [a,b]$

$$\int_C F \cdot T ds = \int_C F \cdot \frac{dr}{ds} ds = \int_C F \cdot d\vec{r} = \int_a^b F(r(t)) \cdot \frac{dr}{dt} dt$$

LINE INTEGRALS WRT xyz COORDINATES

$\int_C M dx \Rightarrow$ Let $F(x,y,z) = M(x,y,z)\hat{i}$

C $\to$ parametrized by $r(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$

$$\int_C F dr = \int_C F \cdot \frac{dr}{dt} dt = \int_C M(x,y,z)\, g'(t) dt$$

$$\int_C M(x,y,z) dx = \int_a^b M(g(t),h(t),k(t))\, g'(t)\, dt$$

$$\int_C N(x,y,z) dy = \int_a^b N(g(t),h(t),k(t))\, h'(t) dt$$

$$\int_C P(x,y,z) dz = \int_a^b P(g(t),h(t),k(t))\, k'(t) dt$$

WORK DONE

$C: r(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$ $\quad F \rightarrow$ cont. force fld

$$\int_C F \cdot T\, ds = \int_C F \cdot \frac{dr}{ds} \cdot ds = \int_C F \cdot dr = \int_C M dx + N dy + P dz$$

$$= \int_C F \cdot \frac{dr}{dt} dt = \boxed{\int_a^b F(r(t)) \cdot \frac{dr}{dt} dt} = W$$

FLOW INTEGRALS FOR VELOCITY FIELDS:

$C: r(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$ $\quad F \rightarrow$ velo fld

$$\int_a^b F(r(t)) \cdot \frac{dr}{dt} dt \Rightarrow \text{FLOW INTEGRAL}$$

CIRCULATION ~~OR~~ FOR VELOCITY FLD

$\Rightarrow$ If the curve starts and ends at the same point $A = B$ the flow is called CIRCULATION AROUND CURVE

SIMPLE AND CLOSED CURVES:

A curve in X-Y plane is SIMPLE if it does not CROSS ITSELF

When curve starts and ends at same pt $\rightarrow$ CLOSED CURVE

FLUX:

$F = M(x,y)\hat{i} + N(x,y)\hat{j}$ $\quad C \quad$ $\vec{n} \Rightarrow$ outward pointing unit normal vector on C

Flux of F across $C = \int_C F \cdot n\, ds$

Clockwise ~~n outwards~~ n inwards

Anticlockwise n outwards.

$$n = T \times \hat{k} = \left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}\right) \times \hat{k} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$$

$$F = M(x,y)\hat{i} + N(x,y)\hat{j}$$

$$F \cdot n = M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}$$

$$\int_C F \cdot n\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds = \boxed{\oint_C M dy - N dx} \rightarrow \text{Flux}$$

Parametrizations $x = g(t)$ $y = h(t)$

16.3

PATH INDEPENDENCE:

$F \to$ vector field in D   A, B lie in D

$\int_A^B F \cdot dr$ along C is same for all paths from $A \to B$

$\Rightarrow \int_C F \cdot dr$ is path independent and field is <u>conservative</u> on D

$\Rightarrow \int_C F \cdot dr = \int_A^B F \cdot dr$

POTENTIAL FUNCTION:

F = vector field on D

If $\boxed{F = \nabla f}$ for a scalar f^n f on D, then $f \to$ <u>potential f^n for F</u>

ASSUMPTIONS ON CURVE, DOMAIN, FIELD:

1. Curves are piecewise smooth
2. Components of F (vector field) are cont.
3. D is open region in space
4. D is <u>connected</u>. Any 2 pts in D can be joined by a curve lying in D.
5. D is <u>simply connected</u>: Every loop in D can be contracted to a pt in D without ever leaving D (no loop holes)

<u>TH-1</u> FUNDAMENTAL TH. FOR LINE INTEGRALS

$C \to$ smooth curve <u>parameterized</u> by $r(t)$. A and B on C are joined

$f \to$ diff f^n $\boxed{F = \nabla f}$ F = <u>gradient vector</u> $F \to$ cont on D ($C \subseteq D$)

$$\boxed{\int_C F \cdot dr = f(B) - f(A)}$$

<u>PROOF</u> $C: r(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$ $t \in [a,b]$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$\Rightarrow \frac{df}{dt} = \nabla f \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) \Rightarrow \frac{df}{dt} = \nabla f \cdot \frac{dr}{dt} \Rightarrow \frac{df}{dt} = F \cdot \frac{dr}{dt}$$

$$\int_C F \cdot dr = \int_{t=a}^{t=b} F \cdot \frac{dr}{dt} dt = \int_a^b \frac{df}{dt} dt = f(B) - f(A)$$

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TH-2 CONSERVATIVE FIELDS ARE GRADIENT FIELDS

Let $F = M\hat{i} + N\hat{j} + P\hat{k}$ $\boxed{F \text{ is conservative} \Leftrightarrow F \text{ is a gradient fld } \nabla f}$

$f \to$ diff f^n

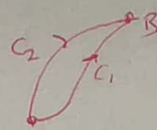
* **PROOF** $F = \nabla f$ $\int_C F \cdot dr = f(B) - f(A)$

i.e. value of line integral does not depend on C but depends only on A and B $\Rightarrow$ line integral is path independent

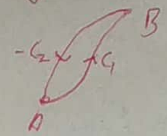
TH-3 LOOP PROPERTY OF CONSERVATIVE FLDS

$\oint_C F \cdot dr = 0$ around every close curve in D $\Leftrightarrow$ F is conservative on D

PROOF



i.e.



$\underbrace{\oint_{C_1} F dr - \int_{C_2} F dr}_{\text{loop}} = \int_{C_1} F dr + \int_{-C_2} F dr$

$= \int_C F \cdot dr = 0$

To Find ∇f / f : COMPOUND TEST

Let $F = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ be on D

F is conservative $\Leftrightarrow$ $\boxed{\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$

PROOF $F = \nabla f$

i.e. $M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$\Rightarrow M = \frac{\partial f}{\partial x} \quad N = \frac{\partial f}{\partial y} \quad P = \frac{\partial f}{\partial z}$

$P = \frac{\partial f}{\partial z},\ \frac{\partial P}{\partial y} = \frac{\partial f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial z}$ and similarly others can be proved

DIFFERENTIAL FORM:

Any Xpression $M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$ is a diff form

* **EXACT** $\Rightarrow$ if $Mdx + Ndy + Pdz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$

for some F

COMPONENT TEST FOR EXACTNESS OF $Mdx + Ndy + Pdz$

$Mdx + Ndy + Pdz$ is exact on D (connected + simply connected)

$\Updownarrow$

$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$\Updownarrow$

F is conservative

16.4

GREEN'S THEORM IN PLANE

CIRCULATION DENSITY / k-COMPONENT OF THE CURL / (CURL F).k

curl F.k for a vector field $F = M\hat{i} + N\hat{j}$ at a pt (x, y) is the scalar expression $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

⇒ It measures the rate of fluid rotaⁿ at a point

⇒ It is +ve → anticlockwise rotaⁿ −ve → clockwise 0 → no rotation

DIVERGENCE / FLUX DENSITY / div F

div F of $F = M\hat{i} + N\hat{j}$ at x, y = $\boxed{\text{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}}$

⇒ div F measures to what extent a gas is expanding or compressing at each point

GREEN'S THEORM I [~~FLUX D~~ CIRCULATION CURL / TANGENTIAL FORM]

C: a piecewise smooth, simple, closed curve enclosing 'R' in a plane

$F = M\hat{i} + N\hat{j}$: M, N have cont I PD in R

Then, the anticlockwise circulation of F across C equals the double integral of ~~div~~ (curl F.k) over R

$$\oint_C F.T\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Anticlockwise circulation Curl integral

GREEN'S THEORM II [FLUX DIVERGENCE / NORMAL FORM]

C: Same as in form I ~~&~~ F: Same as in form I

Then, the outward flux ~~of~~ of F across C equals the double integral of div F over R

$$\oint F.n\,ds = \oint M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

Outward flux Divergence integral

16.5

PARAMETRIZATION OF SURFACES

Let $r : G(\subseteq R^2) \to R^3$ $\quad r(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$

$r \to$ cont vector f^n one to one on interior of G

then range of R is called the surface S traced by R

For simplicity $G = (u,v)$ $u \in [a,b]$ $v \in [c,d]$

DEF → A parametrized surface $r(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$ is SMOOTH if ① r_u and r_v are continuous ② $r_u \times r_v \neq 0$ on interior

SMOOTH CURVE ↑

AREA OF SMOOTH CURVE: Curve is $r(u,v)$ $\quad u \in [a,b]$ $\quad v \in [a,b]$

$$A = \iint_S d\sigma = \iint_R |r_u \times r_v| \, dA = \int_c^d \int_a^b |r_u \times r_v| \, du\, dv$$

$$d\sigma = |r_u \times r_v| \, du\, dv$$

~~If $x = f(u,v)$ $y = g(u,v)$ $z = h(u,v)$~~

App ①

TANGENT PLANE

Curve is r

Let P_T be tangent plane of r

Normal vector to P_T is $\vec{r_u} \times \vec{r_v} = N = a\hat{i} + b\hat{j} + c\hat{k}$ at a pt (x_0, y_0, z_0)

Eq^n of P_T is $\quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

where a, b, c are parametrized values of (x_0, y_0, z_0)

App ① SA ✓

FORMULAS FOR SA of AN IMPLICIT SURFACE:

Area of surface $F(x,y,z) = c$ over closed bound R is

$$A = \iint_R \frac{|\nabla F|}{|\nabla F \cdot p|} \, dA$$

where $p = \hat{i}, \hat{j}$ or $\hat{k}$ i.e whichever is normal to R and $|\nabla F \cdot p| \neq 0$

$$d\sigma = |r_u \times r_v| \, du\, dv = \frac{|\nabla F|}{|\nabla F \cdot p|} \, dx\, dy$$

FORMULA FOR S A of $z=f(x,y)$

$z=f(x,y)$ over R

$$\boxed{A=\iint_R \sqrt{f_x^2+f_y^2+1}\,dx\,dy}$$

PROOF ① $x=u$ $y=v$ $z=f(x,y)$ $r(u,v)=u\hat{i}+v\hat{j}+f(u,v)\hat{k}$

$r_u\times r_v=-f_u\hat{i}-f_v\hat{j}+\hat{k}$

$|r_u\times r_v|\,du\,dv=\sqrt{f_u^2+f_v^2+1}\Rightarrow\sqrt{f_x^2+f_y^2+1}$

② $F(x,y,z)=f(x,y)-z$ $\quad p=\hat{k}$ $\quad \nabla F=f_x\hat{i}+f_y\hat{j}-\hat{k}$

$|\nabla F\cdot p|=|-1|=1.$ $\quad \nabla F=\sqrt{f_x^2+f_y^2+1}$ ...

16.6 SURFACE INTEGRALS

$$\iint_S G(x,y,z)\,d\sigma=\lim_{n\to\infty}\sum_{k=1}^{n}G(x_k,y_k,z_k)\,\Delta\sigma_k$$

APPLICATIONS : FORMULAS for Surface integral of a Scalar f^n

① S: smooth surface $r(u,v)=f(u,v)\hat{i}+g(u,v)\hat{j}+h(u,v)\hat{k}$ $u,v\in R$

$G(x,y,z)\to$ cont f^n on S

$$\iint_S G(x,y,z)\,d\sigma=\iint_R G[f(u,v),g(u,v),h(u,v)]\,|r_u\times r_v|\,du\,dv$$

② S: Implicit surface by $F(x,y,z)=c$ F→cont diff

4 lying above its closed and bound region R

$$\iint_S G(x,y,z)\,d\sigma=\iint_R G(x,y,z)\,\frac{|\nabla F|}{|\nabla F\cdot p|}\,dA$$

p → unit normal vector to R and ∇F $\quad p\neq 0$

③ S: explicit surface $z=f(x,y)$ f→cont diff over R

$$\iint_G (x,y,z)\,d\sigma=\iint_R G(x,y,f(x,y))\sqrt{f_x^2+f_y^2+1}\,dx\,dy$$

ADDITIVITY

$$\iint_S G\,d\sigma=\iint_{S_1}G\,d\sigma+\iint_{S_2}G\,d\sigma+\cdots\iint_{S_n}G\,d\sigma$$

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ORIENTATION OF A SURFACE

⇒ Surface S is orientable / 2-sided if it is possible to define a unit normal vector $\vec{n}$ at each pt (x,y,z) not on the boundary of S on if $\vec{n}$ is a cont f^n of x, y, z

⇒ In this case, S has 2 identifiable sides (top/bottom or in/out)

⇒ Spheres / other smooth surfaces are orientable

⇒ Any portion of orientable surface is orientable

★ THE +ve orientation of a sphere → outward normal vector

By convention : $\vec{n}$ points outward

NON-ORIENTABLE / SINGLE SIDED SURFACE

E.g. MOBIUS BAND (Single sided)

SURFACE INTEGRALS OF VECTOR FIELD:

F: vector field in 3-D space with cont components over S

$\vec{n}$ = unit normal

$\iint_S F \cdot \vec{n} \, d\sigma$ = surface integral of F over S

⇒ If F is velocity field of 3-D fluid flow, then FLUX of F across S is the net rate at which fluid is crossing S per unit time in the chosen +ve $\vec{n}$ dire^n

$$\text{Flux} = \iint_S F \cdot n \, d\sigma$$

$$= \iint_R F \cdot \frac{r_u \times r_v}{|r_u \times r_v|} |r_u \times r_v| \, du\,dv$$

$$= \iint_R F \cdot (r_u \times r_v) \, du\,dv$$

★ If S is part of $g(x,y,z) = c$ $\quad n = \pm \frac{\nabla g}{|\nabla g|}$

$$\text{Flux} = \iint_S F \cdot n \, d\sigma = \iint_R F \cdot \frac{\pm \nabla g}{|\nabla g|} \frac{|\nabla g|}{|\nabla g \cdot p|} \, dA$$

$$\Rightarrow \boxed{\text{Flux} = \iint_R F \cdot \frac{\pm \nabla g}{|\nabla g \cdot p|} \, dA}$$

16.7

CURL VECTOR

Particles ... an axis

★ + Length

STOKE

S: piece

C: a

F: + ve

Then, ...

wrt

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16.7

CURL VECTOR F: → Velocity vector fld of fluid
Particles near the pt (x,y,z) in the fluid tend to rotate around an axis through (x,y,z) that is $\parallel$ to curl vector

$\boxed{\text{CURL} F = \nabla \times F}$ → direction by rt hand thumb rule

* → Length of vector measures rate of rotation

STOKE'S THEORM

S: piecewise smooth oriented surface
C: a " " boundary of S
F: → vector fld on S $\quad F = M\hat{i} + N\hat{j} + P\hat{k}$

Then, circulation of F around C in the direcn anticlockwise wrt surface's unit normal vector $\vec{n}$ equals $\int \nabla \times F$ over S

$$\oint F \cdot dr = \iint_S \nabla \times F \cdot n \, d\sigma$$

Anticlockwise circulation $\quad$ Curl integral

① If 2 different surfaces S_1 and S_2 have same boundry C, their curl integrals are equal

$$\iint_{S_1} \nabla \times F \cdot n_1 \, d\sigma = \iint_{S_2} \nabla \times F \cdot n_2 \, d\sigma$$

② C → Curve in xy + anticlockwise orientation ⇒ $d\sigma = dx\,dy$

$$(\nabla \times F) \cdot \vec{n} = (\nabla \times F) \cdot \hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$\therefore \oint_C F \, dr = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\Rightarrow \oint_C F\,dr = \iint_R \nabla \times F \cdot \hat{k} \, dA$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\underline{\nabla \times F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

STOKE'S TH FOR SURFACES WITH HOLES:

IDENTITY $\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$

$= (f_{zy} - f_{yz})\hat{i} - (f_{zx} - f_{xz})\hat{j} + (f_{yx} - f_{xy})\hat{k}$

$\boxed{\nabla \times \nabla f = 0}$

TH: If $\nabla \times F = 0$ at every pt on D, then on C $\oint F \cdot dr = 0$

PROOF $\oint_C F \cdot dr = \iint_S \nabla \times F \cdot n \, d\sigma = 0$

F conservative on D $\Longleftrightarrow$ $F = \nabla f$ on D

$\Updownarrow$ $\Downarrow$

$\oint_C F \cdot dr = 0$ on D $\Longleftarrow$ $\nabla \times F = 0$ on D

16.8

DIVERGENCE THEORM:

F: → Vector fld S: → piecewise smooth oriented closed surface

$$\iint_S F \cdot n \, d\sigma = \iiint \nabla \cdot F \, dV$$

* The outward flux across S is zero for any F if $\text{div } F = 0$

TH: $F = M\hat{i} + N\hat{j} + P\hat{k}$

$$\boxed{\text{div}(\text{curl } F) = \nabla \cdot (\nabla \times F) = 0}$$

Tang form of green th: $\oint_C F \cdot T \, ds = \iint_R (\nabla \times F) \cdot \hat{k} \, dA$

Normal " " " " " " " : ~~$\oint F \cdot T \, ds = \iint \nabla \times F$~~ $\oint F \cdot n \, ds = \iint_R \nabla \cdot F \, dA$

STOKE'S TH $\oint_C F \cdot T \, ds = \iint_S \nabla \times F \cdot n \, d\sigma$

DIVERGENCE

$$\iint_S F \cdot n \, d\sigma = \iiint_D \nabla \cdot F \, dV$$

DIFF OF FUNC^n WITH 2 variable

$f(x,y)$ on R^2

① f_x, f_y exists at (x_0, y_0)

② ~~$\Delta z = f(a+\Delta x, b+\Delta y$~~ $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$

Satisfies linearization property

i.e. $\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

where $\varepsilon_1, \varepsilon_2 \to 0$ when $\Delta x, \Delta y \to 0$

③ $$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - hf_x(x_0,y_0) - kf_y(x_0,y_0)}{\sqrt{h^2+k^2}}$$

$R^2 \to$ BOTH OPEN AND CLOSED