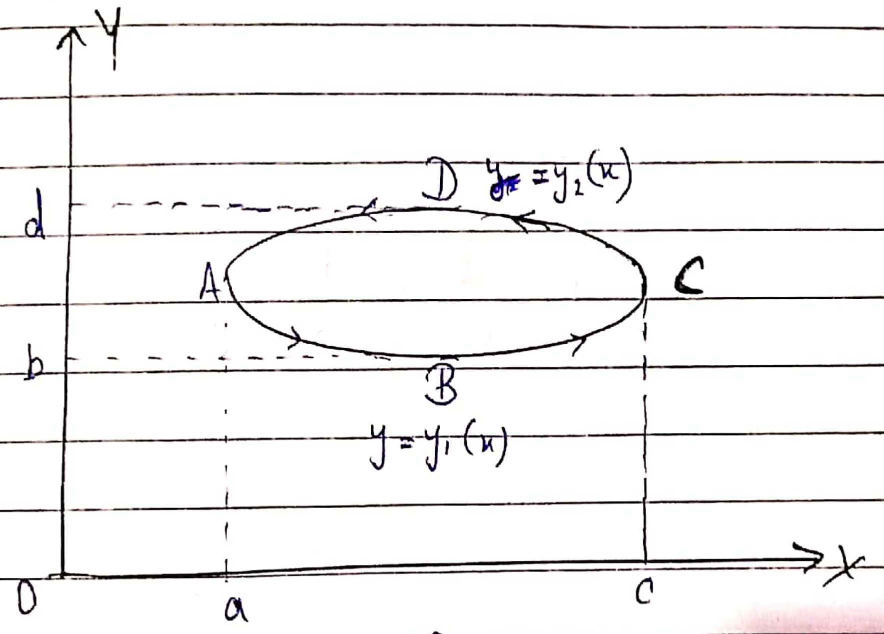
Green's Theorem:

Statement: If $\phi(x,y)$, $\psi(x,y)$, $\frac{\partial \phi}{\partial y}$ & $\frac{\partial \psi}{\partial x}$ be continous function over a region R bounded by simple curved surface C in x-y plane:



$\cancel{\text{Then, } \oint_c (\phi dx + \psi dy) = \iint_R (\phi dy}$

Then; $$\oint_c (\phi dx + \psi dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

Proof: Let the curve is divided into two curves (ABC) C_1 and (CDA) C_2.

Equation of Curve C_2 (CDA) $\ni$ $y = y_2(x)$

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Equation of Curve C_2 (ABC) : $y_1 = y(x)$

Starting from R.H.S

$$= \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$

$$= \iint_R \frac{\partial \psi}{\partial x} dx dy - \iint_R \frac{\partial \phi}{\partial y} \partial x \partial y \quad \text{--- (1)}$$

where ;

$$\iint_R \frac{\partial \phi}{\partial y} dx\, dy = \int_{x=a}^{x=c} \left[\int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial \phi}{\partial y} dy \right] dx$$

$$= \int_{x=a}^{x=c} \Big[\phi(x,y) \Big]_{y_1(x)}^{y_2(x)} dx$$

$$= \int_a^c \left[\phi(x,y_2) - \phi(x,y_1) \right] dx$$

$$= -\int_c^a \phi(x,y_2) dx - \int_a^c \phi(x,y_1) dx$$

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P.T.O →

$$= \left[\phi(c,y_2) - \phi(a,y_2) \right.$$

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$$= -\int_{C_2} \phi(x, y_2)\, dx - \int_{C_1} \phi(x, y_1)\, dx$$

$$= -\left[\int_{C_2} \phi(x, y_2)\, dx + \int_{C_1} \phi(x, y_1)\, dx\right]$$

$$= -\oint_{C} \phi(x, y)\, dx$$

Thus

$$\oint_{c} \phi(x, y)\, dx = -\iint_{R} \frac{\partial \phi}{\partial y}\, dx\, dy$$

and Similarly ;

$$\iint_{R} \frac{\partial \psi}{\partial x}\, dx\, dy = \oint_{c} \psi(x, y)\, dy$$

or Eqⁿ (1) can now also be written as :

$$= \oint_{c} \psi\, dy + \oint_{c} \phi\, dx$$

$$= \oint_{c} (\psi\, dy + \phi\, dx)$$

$$= \oint_{c} (\phi\, dx + \psi\, dy)$$

which is equal to our L.H.S

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Thus,

R.H.S = L.H.S

$$\oint_c (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$

Hence Proved.

(A)

In Vector form:

$$\int_c \vec{F} \cdot \vec{dr} = \iint_R (\nabla \times \vec{F})\, \hat{k}\, dR$$

where; $\vec{F} = \phi \hat{i} + \psi \hat{j}$

$\vec{r} = x\hat{i} + y\hat{j}$ $\Rightarrow$ $\vec{dr} = dx\,\hat{i} + dy\,\hat{j}$

$\hat{k}$ = unit vector along z-axis

$R = dx\, dy$

Proof:

L.H.S $\oint_c \vec{F} \cdot \vec{dr} = \oint_c (\phi \hat{i} + \psi \hat{j}) \cdot (dx\, \hat{i} + dy\, \hat{j})$

$$= \oint_c \phi\, dx + \psi\, dy$$

R.H.S

$$\iint_R (\nabla \times \vec{F}) \cdot \hat{k}\, dR$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi & \psi & 0 \end{vmatrix}$$

$$= -\frac{\partial \psi}{\partial z} \hat{i} + \frac{\partial \phi}{\partial z} \hat{j} + \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) \hat{k}$$

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Now,

$$(\nabla \times \vec{F}) \cdot \hat{k} = \left[\frac{\partial \psi}{\partial z}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)\hat{k}\right]\hat{k}$$

$$= \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}$$

$$(\nabla \times \vec{F}) \cdot \hat{k}\, dR = \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

$$\left\{ \because dR = dx\,dy \right\}$$

$$\iint_R (\nabla \times \vec{F}) \cdot \hat{k}\, dR = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

Now, from Eq^n A.
taking L.H.S = R.H.S

$$\oint_c \phi\, dx + \psi\, dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

From above Expression we ~~have~~ understand that on evaluating the vector form we will get scalar form.

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