TOTAL DIFFERENTIAL AND DIFFERENTIABILITY

ONE VARIABLE:

Def. 1: We call a function $y = f(x)$ differentiable at a point (x, y) if

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

exists. The value of the above limit is called the derivative of f at x.

Def. 2: The function $y = f(x)$ is said to be differentiable at the point (x, y) if, at this point

$$\Delta y = f(x+\Delta x) - f(x) = a\,\Delta x + \varepsilon \cdot \Delta x$$

where a is independent of Δx and $\lim_{\Delta x \to 0} \varepsilon = 0$.
The value of a is the derivative of f at x.

REMARK: Not that Def 1 & 2 are equivalent as

$$f(x+\Delta x) - f(x) = a\,\Delta x + \varepsilon\,\Delta x$$

$$\Leftrightarrow \quad \frac{f(x+\Delta x) - f(x)}{\Delta x} = a + \varepsilon$$

$$\Leftrightarrow \quad \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(a)}{\Delta x} = a \qquad (\varepsilon \to 0 \text{ as } \Delta x \to 0)$$

Def 1 is more practical for verifying differentiability of a function.

DIFFERENTIAL: The differential of the dependent variable y, written as dy, is defined to be

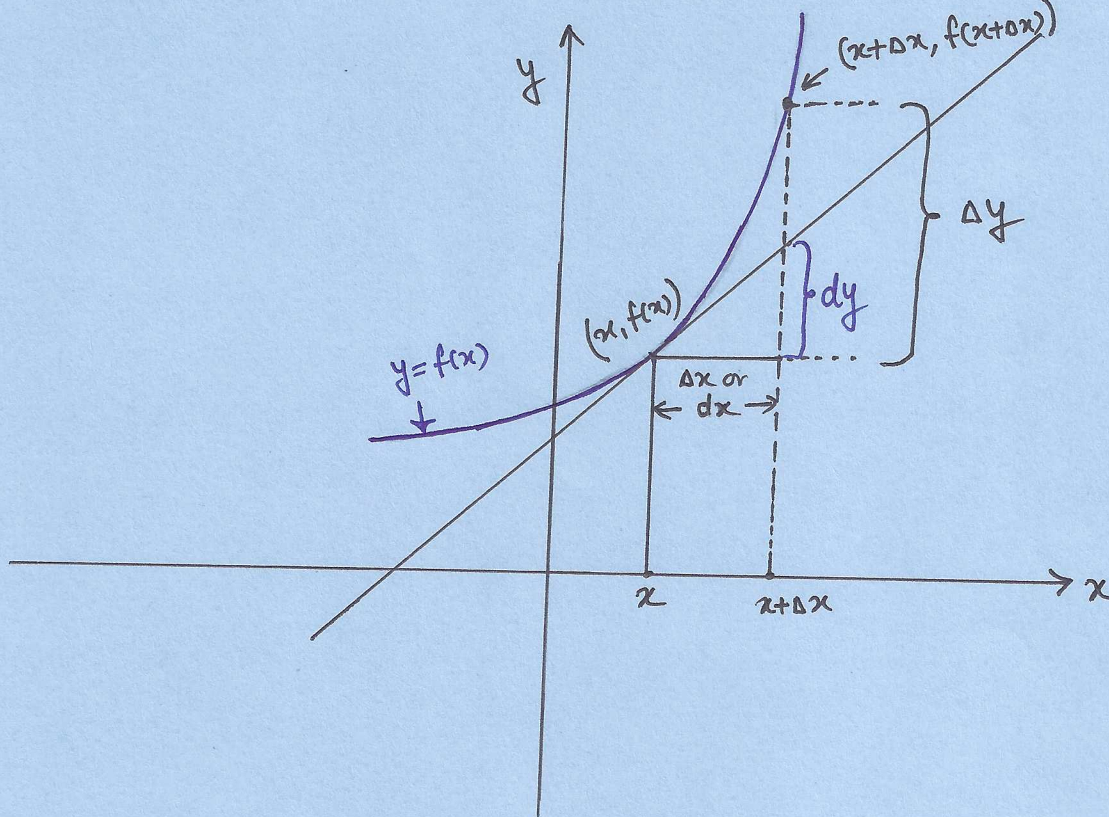
$$dy = f'(x)\,\Delta x, \quad \text{where } y = f(x)$$

$$\text{or } dy = f'(x)\,dx$$

$$\text{or } df = f'(x)\,dx$$

Differential of the independent variable x, written as dx, is same as Δx. one can also observed this by taking $y = x$ and using the above definition of differential as

$$dx = (x)' \Delta x \Rightarrow dx = \Delta x$$



Note that Δx (or dx) is an increment while dy is tolal differential. Δy is the change in y due to change in x by Δx or dx

Also Note that $$\Delta y = \underbrace{f'(x)\,\Delta x}_{\text{linear part}} + \varepsilon\,\Delta x$$

So the differential is a linear function of the increment Δx.

TWO VARIABLE :

The function $z = f(x, y)$ is said to be differentiable at the point (x, y) if, at this point

$$\Delta z = a\,\Delta x + b\,\Delta y + \epsilon_1 \cdot \Delta x + \epsilon_2 \cdot \Delta y$$

where a and b are independent of Δx, Δy and ϵ_1 and ϵ_2 are functions of Δx and Δy such that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_1 = 0 \quad \text{and} \quad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_2 = 0$$

The linear function of Δx and Δy $a\,\Delta x + b\,\Delta y$ is called the total differential of z at the point (x, y) and is denoted by dz

$$dz = a\,\Delta x + b\,\Delta y$$
$$= a\,dx + b\,dy$$

If Δx and Δy are sufficiently small, dz gives a close approximation to Δz.

EXAMPLE: Show that $z = x^2 + xy + xy^2$ is differentiable and write down its total differential.

SOLUTION:

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$= (x + \Delta x)^2 + (x + \Delta x)(y + \Delta y) + (x + \Delta x)(y + \Delta y)^2 - x^2 - xy - xy^2$$

$$= \Delta x\,(2x + y + y^2) + \Delta y\,(x + 2x$$

$$+ (\Delta x + \Delta y\,(1 + 2y))\,\Delta x$$

$$+ (x.\Delta y + \Delta x\,\Delta y)\,\Delta y$$

hence the function is differentiabl

Total differential

$$dz = (2x + y + y^2)\,dx + (x + 2xy)\,dy.$$

NECESSARY CONDITION FOR DIFFERENTIABILITY

THEOREM: If $z = f(x,y)$ is differentiabl then $f(x,y)$ is continuous and has partial derivatives with respe to x and y at the point (x,y) and that

$$a = f_x(x,y) = \frac{\partial z}{\partial x}, \quad b = f_y(x,y) = \frac{\partial z}{\partial y}$$

PROOF: Let f be differentiable, the

$$f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= a\cdot \Delta x + b\cdot \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

then

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f(x+\Delta x, y+\Delta y) = f(x,y)$$

Thus f is continuous.

Setting $\Delta y = 0$ and deviding by Δx yield the relation

$$\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = a + \epsilon_1$$

$$\Rightarrow \quad \boxed{f_x(x,y) = a}$$

Similarly $\boxed{f_y(x,y) = b}$

THEOREM: (SUFFICIENT CONDITION FOR DIFFERENTIABILITY)

If the function $z = f(x, y)$ has continuous first order partial derivatives at a point (x, y), then $f(x, y)$ is differentiable at (x, y).

PROOF:

$$\Delta z = f(x + \Delta x,\ y + \Delta y) - f(x, y)$$

$$= f(x + \Delta x, y + \Delta y) - f(x,\ y + \Delta y)$$
$$+ f(x, y + \Delta y) - f(x, y)$$

Using mean value theorem

$$\Delta z = \Delta x\ f_x\ (x + \theta_1 \Delta x,\ y + \Delta y)$$
$$+ \Delta y\ f_y\ (x,\ y + \theta_2\ \Delta y)$$

when $0 < \theta_1, \theta_2 < 1$

Since the partial derivatives f_x and f_y are continuous at the point (x, y), we can wri

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f_x(x + \theta_1 \Delta x,\ y + \Delta y) = f_x(x, y)$$

$$\Rightarrow f_x(x + \theta_1 \Delta x,\ y + \Delta y) = f_x(x, y) + \epsilon_1$$

[$\epsilon_1 \to 0$ as $\Delta x, \Delta$ $\to$

and

$$f_y(x,\ y + \theta_2 \Delta y) = f_y(x, y) + \epsilon_2$$

[$\epsilon_2 \to 0$ as $\Delta x, \Delta y$

Thus,

$$\Delta x = \Delta x\, f_x(x, y) + \Delta y\, f_y(x, y) + \epsilon_1 \Delta x + \epsilon_2$$

This implies differentiability of f.

▥ To test the differentiability at a point $P(x, y)$, we use either

$$\Delta z = \underbrace{\frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \cdot \Delta y}_{dz} + \epsilon_1 \Delta x + \epsilon_2$$

or

$$\frac{\Delta z - dz}{\Delta \rho} = \epsilon_1 \frac{\Delta x}{\Delta \rho} + \epsilon_2 \frac{\Delta y}{\Delta \rho},$$

where $\Delta \rho = \sqrt{\Delta x^2 + \Delta y^2}$

$$\Rightarrow \lim_{\Delta \rho \to 0} \frac{\Delta z - dz}{\Delta \rho}$$

$$= \lim_{\Delta \rho \to 0} \left[\epsilon_1 \left(\frac{\Delta x}{\Delta \rho} \right) + \epsilon_2 \left(\frac{\Delta y}{\Delta \rho} \right) \right]$$

$$= 0$$

Since $\left| \frac{\Delta x}{\Delta \rho} \right| \leq 1$ and $\left| \frac{\Delta y}{\Delta \rho} \right| \leq 1$

and ϵ_1 and ϵ_2 tends to zero as $\Delta \rho \to 0$.

To test the differentiability we show that

$$\boxed{\lim_{\Delta \rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = 0}$$

REMARKS:

① The function may not be differentiable at a point $P(x, y)$, even if the partial derivatives f_x and f_y exists at P.

(Because existence of partial derivatives is a necessary condition)

② A function may be differentiable even if f_x and f_y

are not continuous. (Because continuity of the f_x and f_y is a sufficient condition).

③ Sufficient conditions of continuity can be relaxed. It is sufficient that one of the partial derivative exist and the other is continuous.