$4\pi a^3$

$\frac{4 \times 22}{7} \times 27$

$\frac{88}{7} \times 27 \quad = 108\pi$

Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ over entire space of region above xy plane bounded by cone

$z^2 = x^2 + y^2$ and plane $z = 4$ if

$\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$

Let V is the volume enclosed by surface S

then V is bounded by $z = 0$ to 4 & $z = x^2 + y^2$

By Gauss

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \cancel{\text{curl}}\ \text{div}\, \vec{F} \cdot dv$$

$$\iiint_V 4z + xz^2 + 3 \ dz\,dy\,dx$$

$$\int^4 \int^{\sqrt{z^2 - x^2}} \int^{z} 4z + xz^2 + 3 \ dy\, dx\, dz$$

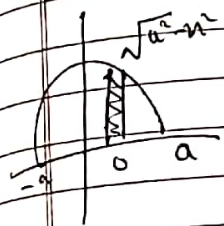
Q. Apply Green theorem to

Evaluate $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ where

C is the boundary of the area enclosed by the x-axis and upper half of the circle $x^2 + y^2 = a^2$

$$\iint \quad \oint_C \underset{\downarrow \atop M}{2x^2 - y^2}\, dx + \underset{\downarrow \atop N}{x^2 + y^2}\, dy =$$

$$\iint_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dx\,dy$$



$$\iint 2x - (-2y)\, dx\,dy$$

$$\iint 2x + 2y\, dx\,dy$$

$$\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} 2x + 2y\, dy\, dx$$

$$2\int^a \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$0 + 1 + \frac{2}{3} = \frac{5}{3}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ y^2 & x^2 & -x+z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

Q. Evaluate $\iint_S \vec{r} \cdot \hat{n}\, ds$ where S is closed surface and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

by gauss divergence theorem

$$\iint_S \vec{r} \cdot \hat{n}\, ds = \iiint_V \cancel{div} \cdot \nabla \cdot \vec{r}\, dv$$

$$= \iiint_V 3 \, dv$$

$$3 \iiint_V dv = 3V$$

Q. Find $\iint_S \vec{F} \cdot \hat{n}\, ds$, where $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$

and S is the surface of the sphere having centre $(3,-1,2)$ and radius 3.

By Gauss divergence theorem

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \text{div } \vec{F} \cdot dv$$

$$\text{div } F = 2 - 1 + 2 = 3$$

$$\iiint 3\, dV = 3 \iiint_V dv$$

$3 \times \frac{4}{3} \pi r^3$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$\text{curl}\,\vec{F} = -4y\hat{k}$

so given surface which is bound by in xy plane

$\therefore$ Normal on xy plane $\hat{n} = \hat{k}$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$\text{curl}\,\vec{F}\cdot\hat{n} = -4y\,\hat{k}\cdot\hat{k}$

$= -4y$

$$\iint_S \frac{\text{curl}\,\vec{F}\cdot\hat{n}\,dx\,dy}{|\hat{k}\cdot\hat{n}|}$$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dx\,dy$$

$$\int_{-a}^{a}\int_{0}^{b} -4y\,dx\,dy$$

$$\int_{-a}^{a}\left[-2y^2\right]_0^b dx$$

$$\int_{-a}^{a} -2b^2\,dx$$

$$2\int_0^a -2b^2\,dx$$

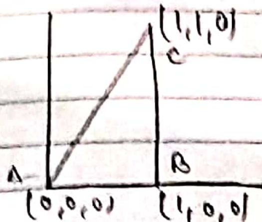
$-4b^2\,[x]_0^a = -4ab^2$ — (2)

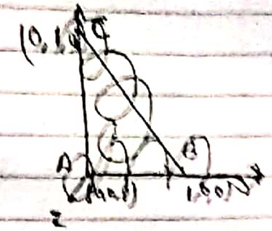
eq (1) = eq (2)

Hence Stoke theorem verified.

Q verify Stoke theorem

$\vec{F} = y^2\hat{i}\,x^2\hat{j} - (x+z)\hat{k}$ take around the triangle bonded by $(0,0,0)$, $(1,0,0)$, $(1,1,0)$





$$\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\hat{n}\,ds$$

$$\oint_C \vec{F}\cdot dr = \oint_{AB} F\cdot dr + \oint_{BC} F\cdot dr + \oint_{CA} \vec{F}\cdot dr$$

$$= \int_{AB} (y^2dx + x^2dy) + \oint_{BC} y^2dx + x^2dy$$

$y=0,\ dy=0$ $\qquad + \oint_{CA} y^2dx + x^2dy$

$$= \int_{AB} 0 + \int^1 dy + \int_0^1 2x^2\,dx$$

$0\,[y]_0^1 + \left(\frac{2x^3}{3}\right)_0^1$

according to given curve line is divided into 4 part AB, BC, CD, DA.

$$\oint_{AB} + \oint_{BC} + \oint_{CD} + \oint_{DA}$$

~~Line integral along AB~~ :

$\vec{r} = x\hat{i} + y\hat{j}$ $\quad d\vec{r} = dx\hat{i} + dy\hat{j}$

$\vec{F} \cdot d\vec{r} = (x^2+y^2)dx - 2xy\,dy$

along AB $\quad x$ varies $-a$ to a

$y = 0$

$dy = 0$

$$\oint_{AB} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} (x^2+\underset{0}{y^2})\,dx - 2xy\,\underset{0}{dy}$$

$$\int_{-a}^{a} x^2\,dx$$

$$\left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$\frac{a^3}{3} + \frac{a^3}{3} = \frac{2a^3}{3}$$

along BC $\quad x = a$, $y = 0$ to b

$dx = 0$

$$\oint_{BC} \vec{F}\cdot d\vec{r} = \int_0^b (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_0^b -2ay\,dy$$

$$= -ab^2$$

along CD $\quad y = b$ & x a to $-a$

$dy = 0$

$$\oint_{CD} \vec{F}\cdot d\vec{r} = \int_a^{-a} (x^2+b^2)\,dx$$

$$= -\frac{2a^3}{3} - 2ab^2$$

along DA $\quad x = -a$, $y = b$ to 0

$$\oint_{DA} \vec{F}\cdot d\vec{r} = -ab^2$$

$$\oint_{ABCD} \vec{F}\cdot d\vec{r} = \cancel{\frac{2a^3}{3}} + (-ab^2) - \cancel{\frac{2a^3}{3}} - 2ab^2 - ab^2$$

$$= -2ab^2 - 2ab^2$$

$$= -4ab^2 \quad \text{—} \; (1)$$

$$\int_0^2\int_0^6 \left(\frac{4y^3}{3} + \frac{y^3 x^2 \hat{k}}{3}\right)_0^6 dx$$

$$\int_0^2 96x - 6x^5\,\hat{i} + (-24x + 6x^3)\hat{j} + \frac{4}{3}\,288 + 72x^2\hat{k}\,dx$$

$$\left(\frac{96x^2}{2} - \frac{6x^6}{6}\right)_0^2\hat{i} + \left(\frac{-24x^2}{2} + \frac{6x^4}{4}\right)_0^2\hat{j} + \left(288x + \frac{72x^3}{3}\right)_0^2\hat{k}$$

$$128\hat{i} - 24\hat{j} + 384\hat{k}$$

STOKE THEOREM

Relation b/w line and surface

surface integral of the component of curl $\vec{F}$ along the normal to the surface S taken over the surface S bounded by the curve C is equal to the line integral of the vector pt function $\vec{F}$ taken along the closed curve C.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

The volume integral of divergence of a vector pt function $\vec{F}$ taken over the volume V enclosed by the surface S is equal to the surface integral of normal component of vector $\vec{F}$ taken over the closed surface S.

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dv}$$

GREEN THEOREM

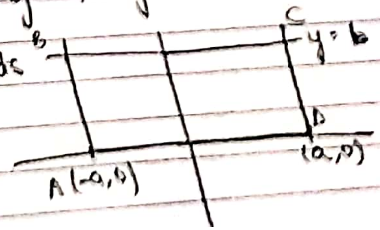
If C is regular curve in x-y plane and S be the region bounded by C

$$\oint_C (M\,dx + N\,dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

where M, N be the func^n of x & y resp.

Q. Verify Stoke theorem $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ take around the rectangle bounded by line $x = \pm a$, $y = 0$, & $y = b$.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$



x varies 0 to 4
~~y~~ varies 0 to 5

$= 90$

Volume integral

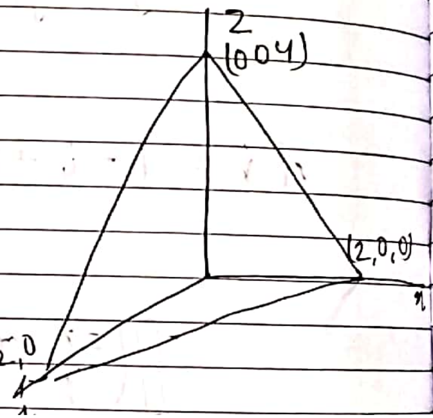
Q. If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla \cdot \vec{F}\, dv$; where V is bounded by the plane $x = 0, y = 0, z = 0$ & $2x + 2y + z = 4$.

$$\nabla \cdot F = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left((2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial y} - 4x$$

$$\iiint 2x \, \partial x \, dy \, dz$$

$$\int_{2}^{4}\int_{0}^{2}\int_{0}^{x} 2x \, dx \, dy \, dz$$



z varies 0 to $4 - 2x - 2y$
y varies 0 to $2 - x$

$$\int_0^{4}\int_0^{2-x}\int_0^{4-2x-2y} 2x \, dz \, dy \, dx$$

$$\int_0^{4}\int_0^{2-x} \left[2x^2\right]_0^{4-2x-2y} dy \, dz$$

$$\int_0^{4}\int_0^{2-x} \left[2y(4-2x)(2-x) + y^2 - 2y(4x - 2x \times 2y)\right] dy \, dz$$

$$\int_0^{4}\int_0^{2-x}$$

Q. If $\vec{A} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$ Evaluate $\iiint_V \vec{A} \cdot dv$ where V is region bounded by surface $x = 0, y = 0, y = 2, y = 6, z = x^2$ to 4.

$$\int_0^{2}\int_0^{6}\int_{x^2}^{4} 2xz\hat{i} - x\hat{j} + y^2\hat{k} \, dx \, dy \, dz$$

$dz \, dy \, dx$

$$\int^{2}\int^{6} \left[xz^2\hat{i} - xz\hat{j} + y^2 z\hat{k}\right]_{x^2}^{4} dy \, dx$$

$$\frac{1}{2}\iint 2x+2y^2-2x+4y\left(\frac{6-2x-y}{2}\right) dx\,dy$$

$$\frac{1}{2}\int \frac{6}{3}y(3-x)\,dy\,dx$$

Here x varies from 0 to 3

& y varies from 0 to $6-2x$

$$\frac{1}{2}\times\frac{6}{3}\int_0^3\int_0^{6-2x}(3y-yx)\,dy\,dx$$

$$\frac{4}{3}\int_0^3\left[\frac{3y^2}{2}-\frac{y^2}{2}x\right]_0^{6-2x} dx$$

$$\frac{4}{3}\int_0^3 \frac{3(6-2x)^2}{2}-\frac{(6-2x)^2}{2}x\,dx$$

81 ✓

Q. Evaluate $\iint \vec{A}\cdot\hat{n}\,ds$, where $\vec{A}=z\hat{i}+x\hat{j}-3y^2z\hat{k}$

and S is surface of cylinder $x^2+y^2=16$ include in the first octant b/w $z=0$, & $z=5$.

$$S=x^2+y^2-16$$

$$\hat{n}=\frac{\text{grad } S}{|\text{grad } S|}=\frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$$

$$=\frac{2x\hat{i}+2y\hat{j}}{2\sqrt{16}}$$

$$\hat{n}=\frac{x}{4}\hat{i}+\frac{y}{4}\hat{j}$$

Consider a projection of surface S on zx plane

$$\iint_S \vec{A}\cdot\hat{n}\,ds=\iint\frac{\vec{A}\cdot\hat{n}\,dx\,dz}{|\hat{j}\cdot\hat{n}|}$$

$$\hat{j}\cdot\hat{n}=\hat{j}\cdot\left(\frac{x}{4}\hat{i}+\frac{y}{4}\hat{j}\right)=\frac{y}{4}$$

$$\vec{A}\cdot\hat{n}=\cancel{(z\hat{i}+x\hat{j}+\dots)}\cdot(zi+xj-3y^2z\hat{k})\left(\frac{x}{4}i+\frac{y}{4}j\right)\,dx\,dz$$

$$=\iint\left(\frac{2x}{y}+x\right)dx\,dz$$

$$=\iint\frac{2x}{(16-x^2)^{1/2}}+x\,)\,dx\,dz$$

Q. Suppose $\vec{F} = 3xy\hat{i} - y^2\hat{j}$. Evaluate $\oint \vec{F}\cdot d\vec{r}$ where C is arc of the parabola $y = 2x^2$ from (0,0) to (1,2)

$\oint F\cdot dr$ $\qquad y = 2x^2$, $dy = 4x\,dx$

$r = x\hat{i} + y\hat{j}$

$dr = dx\hat{i} + dy\hat{j}$

$\oint (3xy\hat{i} - y^2\hat{j})\cdot(dx\hat{i} + dy\hat{j})$

$\oint (3x\,2x^2\,\hat{i} - 4x^4\hat{j})\,dx\hat{i} + 4x\,dx\hat{j}$

$\oint 6x^3 dx - (4x^4)\,4x\,dx$

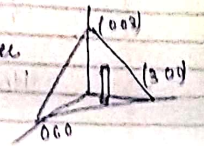
$\int_0^1 6x^3\,dx - \int_0^1 16x^5\,dx$

$\left[\frac{6x^4}{4}\right]_0^1 - \left[\frac{16x^6}{6}\right]_0^1$

$\frac{6}{4} - \frac{16}{6}$

$\frac{36-64}{24} = \frac{-7}{6}$

Q. Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of plane $2x + y + 2z = 6$ in first octate



$\hat{n}$ = unit normal of surface

$\hat{n} = \frac{\text{grad } s}{|\text{grad } s|} = \frac{\nabla s}{|\nabla s|}$

$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+y+2z-6)$

$\hat{n} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$

take projection of surface on the x-y plane

$\iint_S [(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}]$

$\iint_S \vec{A}\cdot\hat{n}\,ds = \int \vec{A}\cdot\hat{n}\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$

$\iint_S [(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}]\cdot\hat{n}\frac{dx\,dy}{\hat{k}\cdot\frac{(2\hat{i}+\hat{j}+2\hat{k})}{3}}$ $\} = \frac{2}{3}$

$\frac{3}{2}\int_0^6\int_0^3 (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\cdot\frac{(2\hat{i}+\hat{j}+2\hat{k})}{3}\,dx\,dy$

$\frac{3}{2\times3}\int\int 2x(x+y^2)\hat{i} - 2x\hat{j} + 2y\,\frac{6-2x-y}{2}\,dx\,dy$

$2\hat{i} - \hat{j} + 2\hat{k}$

$$\hat{n} = \frac{\text{grad}\, S}{|\text{grad}\, S|} = \frac{\vec{\nabla \phi}}{|\nabla \phi|}$$

Volume integral

Any integral evaluated over the volume

$$\iiint_V \vec{F} \cdot dv \text{ or } \iiint_V \phi \, dv$$

$$\hat{i} \iiint F_1 \, dx\,dy\,dz + \hat{j} \iiint F_2 \, dx\,dy\,dz$$

$$+ \hat{k} \iiint F_3 \, dx\,dy\,dz$$

Q. Suppose $\vec{F}(x, y, z) = x^3\hat{i} + 2y\hat{j} + z\hat{k}$ is a force field. Find the work done by $\vec{F}$ along the line for $(1,2,3)$ to $(3,5,7)$

eq. of line = ~~[illegible]~~ $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

$\vec{r} = x\hat{i} + y\hat{j}$

$d\vec{r} = \hat{i}dx + \hat{j}dy$

work done $= \int_{AB} \vec{F}\cdot d\vec{r}$

$= \int_{AB} (2x^2 y\,dx + 3xy\,dy)$

$y = 4x^2$

$dy = 8x\,dx$

$\int_0^1 2x^2(4x^2)\,dx + 3x(4x^2)(8x\,dx)$

$\int_0^1 8x^4\,dx + 96\int_0^1 x^4\,dx$

$8\left[\frac{x^5}{5}\right]_0^1 + 96\left[\frac{x^5}{5}\right]_0^1$

$= \frac{8}{5} + \frac{96}{5} = \frac{104}{5}$

Surface integral

An integral over a surface

$\iint_S F\,ds$

Note

Surface integral is evaluated in the particular direction

The normal surface integral of $\vec{S}$ in the direction of $\hat{n}$ where $\delta\vec{s} = \delta x\delta y$ in elementary area $\boxed{d\vec{s} = \hat{n}\cdot d\vec{s}}$

So $\iint_S \vec{F}\cdot\hat{n}\,d\vec{s}$

Q To evaluate surface integral, projection of $\vec{S}$ on the coordinate plane

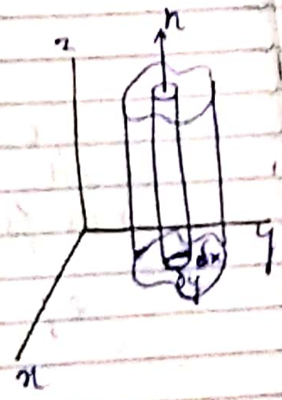
Suppose R be the projection of S on xy plane

Now $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ direction cosine of $\hat{n}$

$\therefore dx\,dy$ = projection of ds on x-y plane

$dx\,dy = ds\cos\gamma$

$ds = \frac{dx\,dy}{\cos\gamma} = \frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$



$\iint \vec{F}\cdot\hat{n}\,ds = \iint \vec{F}\cdot\hat{n}\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$

$= \iint \vec{F}\cdot\hat{n}\frac{dy\,dz}{|\hat{i}\cdot\hat{n}|}$

$= \iint \vec{F}\cdot\hat{n}\frac{dz\,dx}{|\hat{j}\cdot\hat{n}|}$

VECTOR INTEGRATION

- Line integral
- Surface integral
- Volume integral

Line integral
The integral which is evaluated along the curve is called line integral.

force W If $\vec{F}$ represents the variable force acting along a curve AB

total workdone = $\int_A^B \vec{F} \cdot d\vec{r}$

Q. If a force is $2x^2y\hat{i} + 3xy\hat{j}$ displace a particle in xy plane from (0,0) to (1,4) along curve $y = 4x^2$. workdone!

workdone = $\int_{(0,0)}^{(1,4)} \vec{F} \cdot d\vec{r}$ $y = 4x^2$

$d\vec{r} = $

$\int_{(0,0)}^{(1,4)} (2x^2y\hat{i} + 3xy\hat{j}) \cdot$

$\hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$2r\dfrac{\partial r}{\partial x} = 2x$

$\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ | $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ | $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$\dfrac{n r^{n-1}}{r}\left(\dfrac{xi}{r} + \dfrac{yj}{r} + \dfrac{zk}{r}\right)$

$\cancel{n r^{n-2} \cdot \vec{r}}$ $\quad n r^{n-2} \cdot \vec{r}$

put $n = -1$

$\nabla r^n = n r^{n-1} \cdot \vec{r}$

$\nabla\left(\dfrac{1}{r}\right) = -1 r^{-3} \cdot \vec{r}$

$= \dfrac{-1}{r^3} \cdot \vec{r}$

Q If $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

div $\vec{V} = ?$

$\left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right) \cdot \vec{V}$

$\dfrac{\partial}{\partial x}\dfrac{x}{\sqrt{x^2+y^2+z^2}}\,i + \dfrac{\partial}{\partial y}\cdot\dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\hat{j} + \dfrac{\partial}{\partial z}\cdot\dfrac{z}{\sqrt{x^2+y^2+z^2}}$

$\dfrac{(x^2+y^2+z^2)^{1/2}\cdot 1 - \frac{2x}{2}\,[x^2+y^2+z^2]^{1/2}}{x^2+y^2+z^2}\,\hat{i} + \dots$

$\dfrac{(x^2+y^2+z^2)^{1/2}[1 - x^2[x^2+y^2+z^2]^{-1}]}{x^2+y^2+z^2}$

$+$

$\dfrac{(x^2+y^2+z^2)^{1/2}(1 - y^2[x^2+y^2+z^2]^{-1}]}{x^2+y^2+z^2}$

$+$

$\dfrac{(x^2+y^2+z^2)^{1/2}[1 - z^2[x^2+y^2+z^2]^{-1}]}{x^2+y^2+z^2}$

$= \dfrac{1 - x^2(x^2+y^2+z^2]^{-1} + 1 - y^2(x^2+y^2+z^2]^{-1} + 1 - z^2[x^2+y^2+z^2]^{-1}}{[x^2+y^2+z^2]^{1/2}}$

$\dfrac{3 - [x^2+y^2+z^2]^{-1}(x^2+y^2+z^2)}{[x^2+y^2+z^2]^{1/2}}$

$\cancel{3-1}$ $\quad 3-1 = \dfrac{2}{\sqrt{x^2+y^2+z^2}}$

Divergence of a vector pt func.
differentiable pt funct vector pt funct. of vector $V = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is defined as

$$\text{div } \vec{V} = \vec{\nabla}\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$\text{div } \vec{V} = \frac{\partial (V_1)}{\partial x} + \frac{\partial}{\partial y}(V_2) + \frac{\partial}{\partial z}(V_3)$$

The divergence of vector is a scalar quantity.

CURL
The curl or rotation is a differential vector pt funct. of vector $V = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is defined as curl $\vec{V} = \vec{\nabla}\times\vec{V}$ =

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \left| \hat{i} \quad \hat{j} \quad \hat{k} \right|$$

Note
① if div $V = 0$
then $\vec{V}$ is called solenodial vector.

if curl of $\vec{V} = 0$
then $\vec{V}$ is called rotational vector.

Q. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then find
div $\vec{r}$
curl $\vec{r}$

div $\vec{r} = 3$

curl $r = 0$

Q. Show that $\nabla r^n = n r^{n-2}\vec{r}$ & hence evaluate $\nabla\left(\frac{1}{r}\right)$ when $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\nabla r^n = \hat{i}\frac{\partial}{\partial x}r^n + \hat{j}\frac{\partial}{\partial y}r^n + \hat{k}\frac{\partial r^n}{\partial z}$$

(2) If $\phi(x,y,z)$ = constant be a surface

$\boxed{\nabla\phi \cdot \delta\vec{r} = 0}$

i.e $\nabla\phi$ is $\perp$ to every $\delta\vec{r}$ lying in a surface

Hence $\nabla\phi$ is normal to the surface $\phi(x,y,z)=c$

(a) The gradient of a scalar field ϕ is a vector normal to the surface and has the magnitude of rate of change of ϕ along this normal.

Directional ~~derivative~~ derivative

The component $\nabla\phi$ in the direction of vector $\vec{d}$ is given by

$\boxed{\nabla\phi \cdot \vec{d}}$ directional derivative of ϕ in direction d.

$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ (n_1)

$\nabla\phi_2 =$ grad $\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$ (n_2)

Normal n_1 at $(2,-1,2)$

$n_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$

Normal n_2 at $(2,-1,2)$

$n_2 = 4\hat{i} - 2\hat{j} - \hat{k}$

Angle b/w two surface

$\cos\phi = \dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \dfrac{(16+4-4)}{\sqrt{16+4+16}\sqrt{16+4+1}}$

$= \dfrac{16}{\sqrt{36}\sqrt{21}} \Rightarrow \dfrac{16}{6\sqrt{21}} =$

velocity at $t = 1$ $\boxed{3\hat{i} + 2\hat{j} + 2\hat{k}}$

acc^n at $t = 1$ $\boxed{6\hat{i} + 2\hat{j} + 0\hat{k}}$

direction $\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$$\hat{a} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}}$$

$$= \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

velocity in direction $\hat{a}$

$$\frac{d\vec{r}}{dt} \cdot \hat{a} = (3\hat{i} + 2\hat{j} + 2\hat{k}) \cdot \left(\frac{2\hat{i}}{7} + \frac{3\hat{j}}{7} + \frac{6\hat{k}}{7}\right)$$

$$= \frac{6+6+12}{7}$$

$$= \frac{24}{7}$$

acc^n $\frac{d^2\vec{r}}{dt^2} \cdot \hat{a} = (6\hat{i} + 2\hat{j} + 0\hat{k}) \cdot \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$

$$= \frac{12+6}{7} = \frac{18}{7}$$

Q. At any pt of a curve $x = 3\cos t$; $y = 3\sin t$, $z = 4t$, find

① tangent vector

② unit tangent vector

$$\vec{r} = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 4t\,\hat{k}$$

$$\vec{a} \quad \frac{d\vec{r}}{dt} = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}}{\sqrt{9\sin^2 t + 9\cos^2 t + 16}}$$

$$= \frac{-3\sin t}{5}\hat{i} + \frac{3\cos t}{5}\hat{j} + \frac{4}{5}\hat{k}$$

Vector differential operator ∇ (Del)

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Gradient of a Scalar function :-

If $\phi(x,y,z)$ be a scalar function then

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$ is called grad ϕ

grad of function (scalar) is vector quantity

scalar pt a region of in a space at
if R pt of which there is a correspond
each pt of which
a unique scalar.

$$\phi = \phi(x, y, z)$$

Then ϕ is called scalar pt function.

Ex. temp during heating of a rod.

Vector pt function
Let R be a region of space, at each pt of which a vector is given.

$$\vec{v} = \vec{v}(x, y, z)$$

then v is called vector pt function

then and R is called field

Ex. Gravitational field

Ex A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$ where t is the time. find the component of its velocity and acceleration at time t=1

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5 Vector Calculus

If each value of scalar variable t corresponds a value of vector $\vec{r}$.

$$\vec{r} = \vec{r}(t) \text{ or } f(t)$$

Every vector can be uniquely express as three fixed non coplanar vector

$$\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

$f_1(t)$, $f_2(t)$, $f_3(t)$ component of vector $f(t)$ and
$\hat{i}, \hat{j}, \hat{k}$ are unit vector along x, y & z resp.

Derivative of vector function w.r.t to scalar is denoted by $\frac{d\vec{r}}{dt}$ is called derivative of vector r w.r.t t.

$\frac{d\vec{r}}{dt}$ is a vector in the direction of a tangent of pt.

GENERAL RULE OF DIFF^n

If $\vec{a}, \vec{b}$, & $\vec{c}$ are vector func^n of a scalar t & ϕ is scalar func^n of t

1) $\frac{d}{dt}(\vec{a} \pm \vec{b}) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$