Math 6396 Riemannian Geometry, Metric, Connections, Curvature Tensors etc.

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1 Riemannian Metric

- A *Riemannian metric* on a differentiable manifold M is a symmetric, positive-definite, (smooth) $(0,2)$-tensor fields g on M, i.e., for any vector field $X, Y \in \Gamma(TM)$, $g(X,Y) = g(Y,X), g(X,X) \geq 0$ where the equality holds if and only if $X = 0$.

  In terms of the local coordinates, g can be expressed by

  $$g = g_{ij} dx^i \otimes dx^j, \qquad g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right),$$

  where $g_{ij} = g_{ji}$ and the matrix (g_{ij}) is positive definite everywhere.

- There exists a Riemannian metric on a differentiable manifold with countable basis (for the topology).

2 Affine Connections

- A affine (or linear) connection $\bigtriangledown$ is a map $\bigtriangledown : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ which satisfies, for all $X, Y, Z \in \Gamma(TM)$ and $f \in C^\infty(M)$,

  (a) $\bigtriangledown_{X+fY} Z = \bigtriangledown_X Z + f \bigtriangledown_Y Z$,

  (b) $\bigtriangledown_X (Y + Z) = \bigtriangledown_X Y + \bigtriangledown_X Z$,

  (c) $\bigtriangledown_X (fY) = (Xf)Y + f \bigtriangledown_X Y$.

- **Levi-Civita connection**. Let (M, g) be a Riemannian manifold. Then there exists a unique connection (called the *Levi-Civita connection*) such that

  (1) $\bigtriangledown_X Y - \bigtriangledown_Y X = [X, Y]$ (i.e. torsion free)

  (2) $X < Y, Z > = < \bigtriangledown_X Y, Z > + < Y, \bigtriangledown_X Z >$.

  Proof: We first make the following remark: Let $\alpha : \Gamma(TM) \to C^\infty(M)$ be a $C^\infty(M)$-linear, regarding $\Gamma(TM)$ as an $C^\infty(M)$-module. Then there exists a unique $U \in \Gamma(TM)$ such that $\alpha(Z) = < U, Z >$ for every $Z \in \Gamma(TM)$.

  Outline of the proof: First we can derive, from the conditions that the *Koszul formula* holds:

  $$\begin{aligned} 2 < \bigtriangledown_X Y, Z > &= X < Y, Z > + Y(X, Z > - Z < X, Y > \\ &\quad - < Y, [X, Z] > - < X, [Y, Z] > - < Z, [Y, X] > . \end{aligned}$$

  This shows the uniqueness. Also, from the above discussion, if we define $\bigtriangledown_X Y$ use Koszul formula, we can verify the properties are satisfied.

- **Christoffel symbols**. In terms of the coordinate chart (U, x^i), we write
  $$\bigtriangledown_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^m \Gamma_{ji}^k \frac{\partial}{\partial x^k},$$
  where the functions Γ_{ji}^k are called the *Christoffel symbols*.

- The connection $\bigtriangledown$ is torsion free if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$, and if $\bigtriangledown$ is torsion free and compatible with the Riemannian metric g, then its Christoffel symbols are given by, from the Koszul formula above,
  $$2\Gamma_{ij}^k g_{kl} = \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l},$$
  or equivalently,
  $$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^m g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

With coordinate charts (U, x^i) and (V, y^i). On $U \cap V$, the Christoffel symbols satisfy the following **transformation formula**

$$\Gamma'^{j}_{ik} = \sum_{p,q,r=1}^{m} \Gamma^{q}_{pr} \frac{\partial y^j}{\partial x^q} \frac{\partial x^p}{\partial y^i} \frac{\partial x^r}{\partial y^k} + \frac{\partial^2 x^p}{\partial y^i \partial y^k} \cdot \frac{\partial y^j}{\partial x^p}.$$

For every $X, Y \in \Gamma(TM)$, write

$$X = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial x^i},$$

$$Y = \sum_{j=1}^{m} \eta^j \frac{\partial}{\partial x^j},$$

then

$$\bigtriangledown_X Y = \sum_k \left(\sum_i \xi^i \frac{\partial \eta^k}{\partial x^i} + \sum_{i,j} \Gamma^k_{ji} \xi^i \eta^j \right) \frac{\partial}{\partial x^k}.$$

- In terms of the coordinate chart (U, x^i), we write

$$\bigtriangledown_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{m} \Gamma^k_{ji} \frac{\partial}{\partial x^k},$$

or

$$\bigtriangledown \frac{\partial}{\partial x^j} = \sum_{i=1}^{m} \sum_{k=1}^{m} \Gamma^k_{ji} dx^i \otimes \frac{\partial}{\partial x^k} = \sum_{k=1}^{m} \omega^k_j \frac{\partial}{\partial x^k},$$

where

$$\omega^k_j = \sum_{i=1}^{m} \Gamma^k_{ji} dx^i.$$

The matrix $\omega = (\omega^k_i)$ is called the **connection matrix** of $\bigtriangledown$ with respect to $\partial/\partial x^i, 1 \leq i \leq m$. So we see that we can regard $\bigtriangledown$ as the operator $\bigtriangledown : \Gamma(TM) \to \Gamma(T^*(M) \otimes T(M))$.

- **Parallel translation, geodesics**. Suppose $\gamma(t), 0 \leq t \leq b$, is a curve in M. A (tangent) vector field $X \in \Gamma(TM)$ (actually X can be the vector which is only defined along γ) is said to be **parallel** along the curve γ if and only if $\bigtriangledown_{\gamma'(t)}X = 0$ for $t \in [0, b]$. In terms of the local coordinate (U, x^i), write by $x^i(t) = x^i \circ \gamma(t),\ 1 \le i \le m$ and write

$$X(t) = \sum_{\alpha=1}^{m} X^i(t) \frac{\partial}{\partial x^i}|_{\gamma(t)}.$$

Then

$$\gamma'(t) = \sum_i \frac{dx^i(t)}{dt} \frac{\partial}{\partial x^i}.$$

Hence

$$\bigtriangledown_{\gamma'(t)}X = \sum_k \left(\frac{dX^k(t)}{dt} + \sum_{i,j} \Gamma^k_{ij} X^i(t) \frac{dx^j(t)}{dt} \right) \frac{\partial}{\partial x^k}|_{\gamma(t)}.$$

Hence $X(t)$ is parallel along the curve γ if and only if

$$\frac{dX^k(t)}{dt} + \sum_{i,j} \Gamma^k_{ij} X^i(t) \frac{dx^j(t)}{dt} = 0.$$

A curve γ is **geodesic** in M if and only if the tangent vector of γ is parallel along the curve γ. Suppose that γ is given in a coordinate chart (U, x^i) by $x^i(t) = x^i \circ \gamma(t)$ $1 \le i \le m$. Then γ is geodesic if and only if

$$\frac{d^2x^i}{dt^2} + \sum_{j,k=1}^{m} \Gamma^i_{kj} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \qquad 1 \le i \le m.$$

- **Covariant derivative(connection) for tensor fields**. Note that for any (fixed) tangent vector field X, according to the definition above, the operator $\bigtriangledown_X$ sends $(1, 0)$-tensor field Y (i.e. tangent vector field) to $(1, 0)$-tensor field $\bigtriangledown_X Y$. Here we want to show that such $\bigtriangledown$ induces, for any (fixed) tangent vector field X, a map (still denoted by $\bigtriangledown_X$ which sends $(0, s)$-tensor fields to $(0, s)$-tensor fields. We first consider $s = 0$ case. Let f be a function (i.e. a $(0, 0)$-tensor field, then define $\bigtriangledown_X(f) = X(f)$. When $s = 1$, for any $(0, 1)$-tensor field ω (which is a differential 1-form), $\bigtriangledown_X(\omega)$ is also a 1-form which is defined by, for any $Y \in \Gamma(T(M))$,

$$\bigtriangledown_X(\omega)(Y) = \bigtriangledown_X(\omega(Y)) - \omega(\bigtriangledown_X Y).$$

In general, let Φ be a $(0,s)$-tensor field (resp. a $(1,s)$-tensor field), and let $X \in \Gamma(TM)$. Then we define the *Covariant derivative of* Φ in the direction of X by the formula, for every $Y_1, \ldots, Y_s \in \Gamma(T(M))$,

$$\begin{aligned}(\bigtriangledown_X \Phi)(Y_1, \ldots, Y_s) : &= \bigtriangledown_X(\Phi(Y_1, \ldots, Y_s)) \\ &- \sum_{i=1}^{s} \Phi(Y_1, \ldots, Y_{i-1}, \bigtriangledown_X Y_i, Y_{i+1}, \ldots, Y_s).\end{aligned}$$

$\bigtriangledown_X \Phi$ is then also a $(0,s)$-tensor (resp. a $(1,s)$-tensor), and $\bigtriangledown \Phi$ is a $(0, s+1)$-tensor (resp. a $(1, s+1)$-tensor) by means of the formula

$$\bigtriangledown \Phi(X, Y_1, \ldots, Y_s) := (\bigtriangledown_X \Phi)(Y_1, \ldots, Y_s).$$

Let (U, x^i) be a local coordinate. Let Φ be a $(0,s)$-tensor field. Write

$$\Phi|_U = \Phi_{j_1 \cdots j_s} dx^{j_1} \otimes \cdots \otimes dx^{j_s},$$

then one can verify that

$$\bigtriangledown_{\frac{\partial}{\partial x^i}} \Phi = \Phi_{j_1 \cdots j_s, i} dx^{j_1} \otimes \cdots \otimes dx^{j_s},$$

where

$$\Phi_{j_1 \cdots j_s, i} = \frac{\partial \Phi_{j_1 \cdots j_s}}{\partial x^i} - \sum_{b=1}^{s} \Phi_{j_1 \cdots j_{b-1} k j_{b+1} \cdots j_s} \Gamma^k_{j_b i}.$$

- **Connections over vector bundles**. Let $\pi : E \to M$ be a vector bundle. Denote by $\Gamma(E)$ the set of smooth sections of E. $\Gamma(E)$ is a real vector space; it is also a $C^\infty(M)$-module. A connection $\bigtriangledown$ over E is a map $\bigtriangledown : \Gamma(T(M) \times \Gamma(E) \to \Gamma(E)$, $(X, s) \mapsto \bigtriangledown_X s$, which satisfies, for all $X, Y \in \Gamma(T(M))$, $s_1, s_2 \in \Gamma(E)$, c constant, and $f \in C^\infty(M)$:

  (a) $\bigtriangledown_{X+fY} s = \bigtriangledown_X s + f \bigtriangledown_Y s$,

  (b) $\bigtriangledown_X (s_1 + c s_2) = \bigtriangledown_X s_1 + c \bigtriangledown_X s_2$,

  (c) $\bigtriangledown_X (fs) = (Xf)s + f \bigtriangledown_X s$.

Note that we can also view $\bigtriangledown$ as the map $\Gamma(E) \to \Gamma(T^*(M) \otimes E)$ by $(\bigtriangledown s)(X) = \bigtriangledown_X s$ for every $s \in \Gamma(M), X \in \Gamma(TM)$.

Note that if $E = TM$, then $\bigtriangledown$ is the connection we introduced earlier. Let $\bigtriangledown$ be a connection on TM, according to the discussion earlier, it **induces a connection** on the cotangent bundle T^*M defined by, for every $\omega \in \Gamma(T^*M)$,

$$(\bigtriangledown_X \omega)(Y) = X(\omega(Y)) - \omega(\bigtriangledown_X Y),$$

for every $Y \in \Gamma(T(M))$.

There exists a connection on every vector bundle $\pi : E \to M$ if M is a differentiable manifold with countable basis (for the topology).

Let $\pi : E \to M$ be a vector bundle of rank k. Let $\bigtriangledown$ be a connection on E, i.e. $\bigtriangledown : \Gamma(E) \to \Gamma(T^*(M) \otimes E)$ Let (U, x^i) be a local coordinate of M. Let $\{s_\alpha, 1 \leq \alpha \leq k\}$ be a basis for $\Gamma(E|_U)$ (they are called the *local frame field*). Then, for every $s \in \Gamma(E)$, write

$$\bigtriangledown s_\alpha = \sum_{\beta=1}^{k} \omega_\alpha^\beta \otimes s_\beta.$$

So, the connection $\bigtriangledown$ is associated to a $k \times k$ matrix $\omega = (\omega_\alpha^\beta)$ of smooth differential one forms on U, with respect to the local frame $\{s_\alpha, 1 \leq \alpha \leq k\}$. It is easy to verify that

$$\omega_\alpha^\beta = \sum_{k=1}^{m} \Gamma_{\alpha k}^\beta dx^k,$$

where Γ_{ik}^j are the **Christoffel symbols**. The matrix ω is called the **connection matrix** of $\bigtriangledown$ with respect to $\{s_\alpha, 1 \leq \alpha \leq k\}$.

Let $s \in \Gamma(E)$ and write

$$s = \sum_{\alpha=1}^{k} \lambda^\alpha s_\alpha,$$

Then

$$\bigtriangledown s = \sum_{\alpha=1}^{k} \left(d\lambda^\alpha + \sum_{\beta=1}^{k} \lambda^\beta \omega_\beta^\alpha \right) \otimes s_\alpha.$$

For $X \in \Gamma(TM)$,

$$\nabla_X s = \sum_{\alpha=1}^{k} \left(X\lambda^\alpha + \sum_{\beta=1}^{k} \lambda^\beta \omega_\beta^\alpha(X) \right) \otimes s_\alpha.$$

- **Curvature tensor**. For every $X, Y \in \Gamma(TM)$, define $\mathcal{R}(X,Y)$ (called the curvature operator) as $\mathcal{R}(X,Y) : \Gamma(TM) \to \Gamma(TM)$ by, for every $Z \in \Gamma(TM)$,

  $$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

  It satisfies the following properties:

  (1) $\mathcal{R}(X,Y) = -\mathcal{R}(Y,X)$,

  (2) For every $f \in C^\infty(M)$, $\mathcal{R}(fX,Y) = \mathcal{R}(X,fY) = f\mathcal{R}(X,Y)$,

  (3) $\mathcal{R}(X,Y)(fZ) = f\mathcal{R}(X,Y)Z$,

  (4) When ∇ is torsion free, we have $\mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y = 0$ (it is called the first **Bianchi identity**).

  From property (3) above, we see that the map $\mathcal{R}(X,Y) : \Gamma(TM) \to \Gamma(TM)$ is a $C^\infty(M)$-linear map, so it is a smooth $(1,1)$-tensor field. It is thus called the **curvature tensor**.

  In addition, from (1) and (2) we see that $\mathcal{R}(X,Y)$ is also $C^\infty(M)$-linear in X and Y. So for every $v, w \in T_p(M)$, we can define the linear map $\mathcal{R}(v,w) : T_p(M) \to T_p(M)$. Further, we can define the map

  $$\mathcal{R} : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

  by $(Z,X,Y) \mapsto \mathcal{R}(X,Y)Z$ for every $X, Y, Z \in \Gamma(TM)$. From (1)-(4) above, we see that it is multi-$C^\infty(M)$-linear. So $\mathcal{R}$ is a $(1,3)$-tensor field.

  In terms of the local coordinates (U, x^i), write

  $$\mathcal{R}(\partial/\partial x^i, \partial/\partial x^j)\frac{\partial}{\partial x^k} = R^l_{kij}\frac{\partial}{\partial x^l},$$

or

$$\mathcal{R} = \sum_{i,j,k,l} R^l_{kij} dx^k \otimes \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j.$$

According to (1), we have $R^l_{kij} = -R^l_{kji}$.

Then R^l_{kij} satisfies the transformation rule for components of type (1,3) tensors, i.e. on $U \cap V \neq \emptyset$,

$$R'^l_{kij} = \sum_{p,q,r,s=1}^{m} R^l_{prs} \frac{\partial y^l}{\partial x^q} \frac{\partial x^p}{\partial y^k} \frac{\partial x^r}{\partial y^i} \frac{\partial x^s}{\partial y^j}.$$

Hence R^l_{kij} (thus $\mathcal{R}$) is an (1,3)-tensor field.

Write

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ji} \frac{\partial}{\partial x^k},$$

where Γ^k_{ji} are Christoffel symbols, then

$$\mathcal{R}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k} = \left(\frac{\partial \Gamma^l_{kj}}{\partial x^i} - \frac{\partial \Gamma^l_{ki}}{\partial x^j} + \Gamma^h_{kj}\Gamma^l_{hi} - \Gamma^h_{ki}\Gamma^l_{hj}\right) \frac{\partial}{\partial x^l}.$$

Hence

$$R^l_{kij} = \frac{\partial \Gamma^l_{kj}}{\partial x^i} - \frac{\partial \Gamma^l_{ki}}{\partial x^j} + \Gamma^h_{kj}\Gamma^l_{hi} - \Gamma^h_{ki}\Gamma^l_{hj}.$$

For $X, Y \in \Gamma(TM)$, if

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^j \frac{\partial}{\partial x^j},$$

then

$$\mathcal{R}(X, Y) = X^i Y^j R^l_{kij} dx^k \otimes \frac{\partial}{\partial x^l}.$$

In particular, if (M, g) is a Riemannian manifold. Then there is a unique connection (Levi-Civita connection) associated to g. Let, for $X, Y, Z, W \in \Gamma(TM)$,

$$R(X, Y, Z, W) = g(\mathcal{R}(Z, W)X, Y).$$

Then it is a map

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$$

which is multi-$C^\infty(M)$-linear. Hence R is a $(0, 4)$-tensor field. It is called the **Riemannian curvature tensor**. It satisfies the following property: for $X, Y, Z, W \in \Gamma(TM)$,

(1) $R(X, Y, Z, W) = -R(Y, X, Z, W)$,

(2) $R(X, Y, Z, W) = -R(X, Y, W, Z)$,

(3) First Bianchi identity:

$$R(X, Y, Z, W) + R(Z, Y, W, X) + R(W, Y, X, Z) = 0,$$

(4) $R(X, Y, Z, W) = R(Z, W, X, Y)$.

In terms of the local coordinates (U, x^i), write $R = R_{klij} dx^k \otimes dx^l \otimes dx^i \otimes dx^j$, then

$$R_{klij} = \frac{1}{2}\left(\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l}\right) + \Gamma^h_{ik}\Gamma^p_{jl} g_{ph} - \Gamma^p_{il}\Gamma^p_{jk} g_{ph}.$$

From (1)-(4), we get, (1) $R_{ijkl} = -R_{jikl} = -R_{ijlk}$,

(2) First Bianchi identity: $R_{ijkl} + R_{ljik} + R_{kjli} = 0$,

(3) $R_{ijkl} = R_{klij}$.

- **The connection matrix, and the curvature matrix**.

  The curvature tensor **can also be introduced** from the **connection matrix** as follows: Let (U, x^i) be a local coordinate of M. Then $\{\partial/\partial x^i, 1 \leq i \leq m\}$ is a local frame for $\Gamma(U, TM)$. Let $\{\omega^1, \ldots, \omega^m\}$ be its dual. Denote by $\omega = (\omega_i^j)$ be the connection matrix of $\bigtriangledown$ with respect to $\{\partial/\partial x^i, 1 \leq i \leq m\}$, i.e.

$$\bigtriangledown \frac{\partial}{\partial x^i} = \sum_{j=1}^m \omega_i^j \frac{\partial}{\partial x^j}.$$

Theorem. *Let R^j_{ikl} be components of $\mathcal{R}$ with respect to the local frame $\{\partial/\partial x^1, \ldots, \partial/\partial x^m\}$, i.e.*

$$\mathcal{R}(\partial/\partial x^k, \partial/\partial x^l)\frac{\partial}{\partial x^i} = R^j_{ikl}\frac{\partial}{\partial x^j}.$$

Then

$$d\omega^j_i = \omega^h_i \wedge \omega^j_h + \frac{1}{2}R^j_{ikl}\omega^k \wedge \omega^l.$$

For this reason, we let $\Omega = d\omega - \omega \wedge \omega$. We call Ω the **curvature matrix** with respect to the frame field $\{\partial/\partial x^i, 1 \leq i \leq m\}$.

Write $\Omega = (\Omega^j_i)$. Then, from the definition that $\Omega = d\omega - \omega \wedge \omega$, we have

$$\Omega^j_i = d\omega^j_i + \sum_{k=1}^m \omega^k_i \wedge \omega^j_k.$$

The above theorem implies that

$$\Omega^j_i = \frac{1}{2}\sum_{k,l=1}^m R^j_{ikl} dx^k \wedge dx^l.$$

This means we have **alternative way** to define the curvature tensor, $\mathcal{R}$, namely, from the **connection matrix** to get **the connection matrix**, then to get R^j_{ikl}, finally to get $\mathcal{R}$.

We have the **Bianchi identity**:

$$d\Omega = \omega \wedge \Omega - \Omega \wedge \omega.$$

Proof.

$$\begin{aligned} d\Omega &= -d\omega \wedge \omega + \omega \wedge d\omega \\ &= -(\Omega + \omega \wedge \omega) \wedge \omega + \omega \wedge (\Omega + \omega \wedge \omega) \\ &= -\Omega \wedge \omega + \omega \wedge \Omega. \end{aligned}$$

Recall operator $T(X,Y) = \bigtriangledown_X Y - \bigtriangledown_Y X - [X,Y]$ is called the *torsion operator*. We have the following theorem:

$$d\omega^i - \omega^j \wedge \omega^i_j = \frac{1}{2}\omega^i(T(\partial/\partial x^j, \partial/\partial x^k))\omega^j \wedge \omega^k.$$

The equations

$$d\omega^i = \omega^j \wedge \omega^i_j + \frac{1}{2}\omega^i(T(\partial/\partial x^j, \partial/\partial x^k))\omega^j \wedge \omega^k,$$

$$d\omega^j_i = \omega^k_i \wedge \omega^j_k + \frac{1}{2}R^j_{ikl}\omega^k \wedge \omega^l$$

are called the **structure** equations for $(M, \bigtriangledown)$. In particular, **if $\bigtriangledown$ is torsion free**, then the structure equations becomes:

$$d\omega^i = \omega^j \wedge \omega^i_j,$$

$$d\omega^j_i = \omega^k_i \wedge \omega^j_k + \frac{1}{2}R^j_{ikl}\omega^k \wedge \omega^l.$$

Note that if $\bigtriangledown$ is the **Levi-Civita connection** of (M,g), then we also have (from the torsion free condition and the metric compatiable condition) that

$$dg_{ij} = \sum_{k=1}^{m}(\omega^k_i g_{kj} + \omega^k_j g_{ik}),$$

where $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$.

- **Cartan's moving frame method**. The above results can be generalized to arbitrary local frames. On a smooth m-dimensional manifold M, in addtion to the (standard) local frame $\{\partial/\partial x^i, 1 \le i \le m\}$, we also often use an arbitrary local frame $e_1, \ldots, e_m$ for $\Gamma(U, T(M))$ (for example, we take the orthonormal basis after the Gram-Schmidt process). Let $\theta^1, \ldots, \theta^m$ be its dual. We still write $\bigtriangledown_{e_i} e_j = \Gamma^k_{ji} e_k$, where Γ^k_{ji} are the Christoffel symbol **with respect to the frame** $\{e_1, \ldots, e_m\}$. Let

  $$\theta^j_i = \Gamma^j_{ik}\theta^k.$$

  Then $\bigtriangledown e_i = \theta^j_i e_j$. The matrix $\theta = (\theta^j_i)$ is called the connection matrix of $\bigtriangledown$ with respect to the local frame $\{e_1, \ldots, e_m\}$.

Theorem. *Let* $\tilde{R}^j_{ikl}$ *be components of* $\mathcal{R}$ *with respect to the local frame* $\{e_1, \ldots, e_m\}$, *i.e.*

$$\mathcal{R}(e_k, e_l)e_i = \tilde{R}^j_{ikl} e_j,$$

or equivalently, $\tilde{R}^j_{ikl} = \theta^j(\mathcal{R}(e_k, e_l)e_i)$. *Then*

$$d\theta^j_i = \theta^h_i \wedge \theta^j_h + \frac{1}{2}\tilde{R}^j_{ikl}\theta^k \wedge \theta^l.$$

Proof.

$$\begin{aligned}
&(d\theta^j_i - \sum_{h=1}^m \theta^h_i \wedge \theta^j_h)(e_k, e_l) = e_k(\theta^j_i(e_l)) - e_l(\theta^j_i(e_k)) \\
&-\theta^j_i([e_k, e_l]) - \sum_{h=1}^m [\theta^h_i(e_k)\theta^j_h(e_l) - \theta^h_i(e_l)\theta^j_h(e_k)] \\
= \quad & e_k(\Gamma^j_{il}) - e_l(\Gamma^j_{ik}) - \sum_{h=1}^m \theta^h([e_k, e_l])\Gamma^j_{ih} \\
&- \sum_{h=1}^m [\Gamma^h_{ik}\Gamma^j_{hl} - \Gamma^h_{il}\Gamma^j_{hk}].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathcal{R}(e_k, e_l)e_i = \bigtriangledown_{e_k} \bigtriangledown_{e_l} e_i - \bigtriangledown_{e_l} \bigtriangledown_{e_k} e_i - \bigtriangledown_{[e_k, e_l]} e_i \\
= \quad & \sum_{j=1}^m \{\bigtriangledown_{e_k}(\Gamma^j_{il} e_j) - \bigtriangledown_{e_l}(\Gamma^j_{ik} e_j) - \sum_{h=1}^m \theta^h([e_k, e_l])\Gamma^j_{ih}\} e_j \\
= \quad & \sum_{j=1}^m \{\bigtriangledown_{e_k}(\Gamma^j_{il}) - \bigtriangledown_{e_l}(\Gamma^j_{ik}) + \sum_{h=1}^m (\Gamma^h_{il}\Gamma^j_{hk} - \Gamma^h_{ik}\Gamma^j_{hl} - \theta^h([e_k, e_l])\Gamma^j_{ih})\} e_j
\end{aligned}$$

Hence we have

$$\mathcal{R}(e_k, e_l)e_i = \sum_{j=1}^m [(d\theta^j_i - \sum_{h=1}^m \theta^h_i \wedge \theta^j_h)(e_k, e_l)] e_j.$$

Thus

$$\tilde{R}^j_{ikl} = \theta^j(\mathcal{R}(e_k, e_l)e_i) = (d\theta^j_i - \sum_{h=1}^m \theta^h_i \wedge \theta^j_h)(e_k, e_l).$$

The identity is thus proved.

By the same proof, we also have the following theorem:

$$d\theta^i = \theta^j \wedge \theta^i_j + \frac{1}{2}\theta^i(T(e_j, e_k))\theta^j \wedge \theta^k.$$

Proof.

$$\begin{aligned}
&(d\theta^i - \theta^j \wedge \theta^i_j)(e_k, e_l) = e_k(\theta^i(e_l)) - e_l(\theta^i(e_k)) \\
&-\theta^i([e_k, e_l]) - [\theta^j(e_k)\theta^i_j(e_l) - \theta^j(e_l)\theta^i_j(e_k)] \\
&= \theta^i_l(e_k) - \theta^i_k(e_l) - \theta^i([e_k, e_l]) \\
&= \Gamma^i_{lk} - \Gamma^i_{kl} - \theta^i([e_k, e_l]) \\
&= \theta^i(\bigtriangledown_{e_k} e_l - \bigtriangledown_{e_l} e_k - [[e_k, e_l]) \\
&= \theta^i(T(e_j, e_k).
\end{aligned}$$

This proves the theorem.

Let $\Omega = d\theta - \theta \wedge \theta$. Ω is called the **curvature matrix** with respect to the frame field $\{e_1, \ldots, e_m\}$. Write $\Omega = (\Omega^j_i)$. Then, from the definition that $\Omega = d\theta - \theta \wedge \theta$, we have

$$\Omega^j_i = d\theta^j_i + \sum_{k=1}^{m} \theta^k_i \wedge \theta^j_k.$$

The above theorem implies that

$$\Omega^j_i = \frac{1}{2}\sum_{k,l=1}^{m} \tilde{R}^j_{ikl}\theta^k \wedge \theta^l,$$

where $\mathcal{R}(e_k, e_l)e_i = \tilde{R}^j_{ikl}e_j$.

Similarily, we have the structure equations:

$$d\theta^i = \theta^j \wedge \theta^i_j + \frac{1}{2}\theta^i(T(e_j, e_k))\theta^j \wedge \theta^k,$$

$$d\theta_i^j - \sum_{k=1}^m \theta_i^k \wedge \theta_k^j = \Omega_i^j = \frac{1}{2}\sum_{k,l=1}^m \tilde{R}_{ikl}^j \theta^k \wedge \theta^l.$$

In particular, if $\bigtriangledown$ is torsion free, then have the structure equations:

$$d\theta^i = \theta^j \wedge \theta_j^i,$$

$$d\theta_i^j - \sum_{k=1}^m \theta_i^k \wedge \theta_k^j = \frac{1}{2}\sum_{k,l=1}^m \tilde{R}_{ikl}^j \omega^k \wedge \omega^l.$$

We also have the **Bianchi identity**:

$$d\Omega = \theta \wedge \Omega - \Omega \wedge \theta.$$

Note that if $\bigtriangledown$ is the **Levi-Civita connection** of (M, g), then we have the torsion free condition implies that

$$d\theta^i = \sum_{j=1}^m \theta^j \wedge \theta_j^i;$$

and the metric compatiable condition implies that

$$dg_{ij} = \sum_{k=1}^m (\theta_i^k g_{kj} + \theta_j^k g_{ik}),$$

where $g_{ij} = g(e_i, e_j)$. Now let $\theta_i = \theta^j g_{ji}, \theta_{ij} = \theta_i^k g_{kj}$, then the above two equations become

$$d\theta_i = \theta_i^j \wedge \theta_j, \qquad dg_{ij} = \theta_{ij} + \theta_{ji}.$$

We also let

$$\Omega_{ij} = \Omega_i^k g_{kj}.$$

Then we have

(1) $\Omega_{ij} = \frac{1}{2} R_{ijkl} \theta^k \wedge \theta^l$, where $R_{ijkl} =< \mathcal{R}(e_k, e_l)e_i, e_j >$,

(2) $\Omega_{ij} + \Omega_{ji} = 0$

(3) $\theta^i \wedge \Omega_{ij} = 0$,

(4) $d\Omega_{ij} = \theta_i^k \wedge \Omega_{kj} + \Omega_{ik} \wedge \theta_j^k = \theta_i^k \wedge \Omega_{kj} - \theta_j^k \wedge \Omega_{ik}$.

In particular, if $\{e_1, \ldots, e_m\}$ (or $\{\theta^i, 1 \le i \le m\}$) is an **orthonormal** (w.r.t. the Riemannian metric), then we have

$$d\theta^i = \sum_{j=1}^m \theta^j \wedge \theta_j^i;$$

$$\theta_j^i + \theta_i^j = 0.$$

- **The sectional curvature and its calculation**. Let (M, g) be a Riemannian metric and $\bigtriangledown$ is the Levi-Civita connection. Let $E \subset T_p(M)$ be a two-dimensional subspace, and X, Y be two linearly independent vector fields in E, then $K_p(E)$, the **sectional curvature** of M at p with respect to E, is defined by

$$K_p(E) = -\frac{R(X,Y,X,Y)}{g(X,Y,X,Y)}$$

where $g(X,Y,Z,W) = g(X,Z)g(Y,W) - g(X,W)g(Y,Z)$. Let $e_1, \ldots, e_m$ be an **orthonormal**(with respect to the Riemannian metric) local frame field for $T(M)$ on U with $e_1, e_2 \in E$. Let $\theta^1, \ldots, \theta^m$ be its dual (they are one-forms). Then $g = \sum_{i=1}^m \theta^i \otimes \theta^i$. So $g(e_1, e_2, e_1, e_2) = 1$. Thus

$$K_p(E) = -R(e_1, e_2, e_1, e_2).$$

where $R(e_1, e_2, e_1, e_2) = g(\mathcal{R}(e_1, e_2)e_1, e_2)$. Write

$$\mathcal{R}(e_1, e_2)e_1 = \sum_{j=1}^m \theta^j(\mathcal{R}(e_1, e_2)e_1)e_j.$$

Then $g(\mathcal{R}(e_1, e_2)e_1, e_2) = \theta^2(\mathcal{R}(e_1, e_2)e_1)$. Hence

$$K_p(E) = -\theta^2(\mathcal{R}(e_1, e_2)e_1).$$

Recall the (second) structure equation (taking $i = 1, j = 2$):

$$d\theta_1^2 = \sum_{h=1}^m \theta_1^h \wedge \theta_h^2 + \frac{1}{2} \sum_{k,l=1}^m \theta^2(\mathcal{R}(e_k, e_l)e_1)\theta^k \wedge \theta^l.$$

This means the sectional curvature $K_p(E)$ can calculated by calculating the connection forms θ_j^k, calculating $d\theta_1^2$ and using the (second) structure equation.

Example: On $\mathbf{R}^m$, define

$$g = \frac{4}{(1 + \sum_{i=1}^m (x^i)^2)^2} \sum_{i=1}^m dx^i \otimes dx^i.$$

Calculate its sectional curvature.

Solution: Let $A = 1 + \sum_{i=1}^m (x^i)^2$, then $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j) = 0$ if $i \neq j$ and $g_{ii} = \frac{2}{A}$. So let $e_i = (A/2)\partial/\partial x^i$. Then $\{e_1, \ldots, e_m\}$ are the orthonormal basis (with respect to this Riemannina metric). Let $\{\theta^1, \ldots, \theta^m\}$ be its dual basis. So have $\theta^i = \frac{2}{A} dx^i$. We can write

$$g = \sum_{i=1}^m \theta^i \otimes \theta^i.$$

Let $E \subset T_p\mathbf{R}^m$ be a subspace of dimension two, without loss of generality, we assume that $e_1, e_2 \in E$. From above, to calculate the sectional curvature, we only need to calculate $\theta^2(\mathcal{R}(e_1, e_2)e_1)$. To do this, we use the structure eqautions,

$$d\theta_1^2 = \sum_{h=1}^m \theta_1^h \wedge \theta_h^2 + \frac{1}{2} \sum_{k,l=1}^m \theta^2(\mathcal{R}(e_k, e_l)e_1)\theta^k \wedge \theta^l.$$

We **first** find the connection forms θ_i^j(using the fundamental theorem of Riemannian geometry). In fact,

$$d\theta^i = 2\frac{dx^i}{A^2} \wedge dA = \frac{4}{A^2} \sum_j x^j dx^i \wedge dx^j = \sum_j x^j \theta^i \wedge \theta^j.$$

On the other hand, by structure equation

$$d\theta^i = \sum_j \theta^j \wedge \theta_j^i, \quad \theta_j^i + \theta_i^j = 0.$$

To find θ^i_j, write (since $\theta^i, 1 \le i \le m$, is a basis)

$$\theta^i_j = \sum_{k=1}^m A^{i,j}_k \theta^k.$$

We need to determine $A^{i,j}_k$. First of all, from $\theta^i_j + \theta^j_i = 0$, We know that $\theta^i_i = 0$ and we know that $A^{i,j}_k = -A^{j,i}_k$. From the structure equation, we have

$$\sum_j \theta^j \wedge (-x^j \theta^i) = d\theta^i = \sum_j \theta^j \wedge \theta^i_j = \sum_j \theta^j \wedge (\sum_{k=1}^m A^{i,j}_k \theta^k).$$

By combining both sides, we get $A^{i,j}_i = -x^j$ for all $j \neq i$, and $A^{i,j}_k = 0$ if $k \neq i, k \neq j$. If $k = j$, then $A^{i,j}_j = -A^{j,i}_j = x^i$. Hence we get the connection forms

$$\theta^i_j = A^{ij}_i \theta^i + A^{ij}_j \theta^j = x^i \theta^j - x^j \theta^i.$$

In particular,

$$\theta^2_1 = x^2 \theta^1 - x^1 \theta^2.$$

To calculate the curvature, we use the second structure equation. To do so, we need to calculate $d\theta^2_1 - \sum_{h=1}^m \theta^h_1 \wedge \theta^2_h$. In fact,

$$\begin{aligned}
d\theta^2_1 - \sum_{h=1}^m \theta^h_1 \wedge \theta^2_h &= (dx^2 \wedge \theta^1 + x^2 d\theta^1 - dx^1 \wedge \theta^2 - x^1 d\theta^2) \\
&\quad - \sum_{h=1}^m (x^h \theta^1 - x^1 \theta^h) \wedge (x^2 \theta^h - x^h \theta^2) \\
&= \left(A\theta^2 \wedge \theta^1 + \sum_{k=1}^m x^2 x^k \theta^1 \wedge \theta^k - \sum_{k=1}^m x^1 x^k \theta^2 \wedge \theta^k \right) \\
&\quad - \sum_{h=1}^m (x^2 x^h \theta^1 \wedge \theta^h + x^1 x^h \theta^h \wedge \theta^2 + (x^h)^2 \theta^2 \wedge \theta^1) \\
&= \left(A - \sum_{k=1}^m (x^k)^2 \right) \theta^2 \wedge \theta^1 = -\theta^1 \wedge \theta^2
\end{aligned}$$

From the structure equation, we have

$$d\theta_1^2 - \sum_{h=1}^{m} \theta_1^h \wedge \theta_h^2 = \frac{1}{2} \sum_{k,l=1}^{m} \theta^2(\mathcal{R}(e_k, e_l)e_1)\theta^k \wedge \theta^l.$$

Hence

$$-\theta^1 \wedge \theta^2 = \frac{1}{2} \sum_{k,l=1}^{m} \theta^2(\mathcal{R}(e_k, e_l)e_1)\theta^k \wedge \theta^l.$$

Which means that $\theta^2(R(e_1, e_2)e_1) = -1$. we have

$$K_p(E) = -\theta^2(\mathcal{R}(e_1, e_2)e_1) = 1.$$

Hence its sectional curavture is constant. Q.E.D.

Remark: We can also use the above method to calculate the Christofell symbol. In fact, from above, $\frac{\partial}{\partial x^i} = \frac{2}{A}e_i$. Hence

$$\begin{aligned}
\nabla \frac{\partial}{\partial x^i} &= \nabla \left(\frac{2}{A}e_i\right) \\
&= \nabla \left(\frac{2}{1+\sum_{i=1}^m (x^i)^2} e_i\right) \\
&= d\left(\frac{2}{1+\sum_{i=1}^m (x^i)^2}\right) \otimes e_i + \frac{2}{1+\sum_{i=1}^m (x^i)^2}\theta_i^j \otimes e_j \\
&= \frac{4}{(1+\sum_{i=1}^m (x^i)^2)^2}\left(-\sum_j x^j dx^j \otimes e_i + \sum_j (x^j dx^i - x^i dx^j) \otimes e_j\right) \\
&= \frac{2}{1+\sum_{i=1}^m (x^i)^2} \sum_{j,k} (-x^j \delta_{ik} - x^i \delta_{ij}) dx^j \otimes \frac{\partial}{\partial x^k}.
\end{aligned}$$

Hence, we get

$$\Gamma_{ij}^k = \frac{2}{1+\sum_{i=1}^m (x^i)^2} \sum_{j,k} (-x^j \delta_{ik} - x^i \delta_{ij}).$$

F. Schur's Theorem. *Assuem that* $\dim M \geq 3$*. If for every* $p \in M$*,* $K_P(E)$ *is constant, i.e. it is independent of* E*. Then* K *is constant, i.e.* $_p$ *is independent of* p*.*

Proof. Let $\{e_i\}_{1\le i\le m}$ be a local frame, and let $\{\theta^1,\dots,\theta^m\}$ be its dual. By definition, for $E = span\{e_i, e_j\}$,

$$K_p(E) = -\frac{<\mathcal{R}(e_i,e_j)e_i, e_j>}{g_{ii}g_{jj} - g_{ij}^2}.$$

So, by the assumption, we have $<\mathcal{R}(e_i,e_j)e_i,e_j>|_p = -K_p(g_{ii}g_{jj}-g_{ij}^2)$ where K is a smooth function on M. This will implies that (the detial is omitted) that $R_{ijkl} =<\mathcal{R}(e_k,e_l)e_i,e_j>= -K(g_{ik}g_{jl}-g_{il}g_{jk})$. By the theorem above, we get

$$\Omega_{ij} = \frac{1}{2}R_{ijkl}\theta^k\wedge\theta^l = -\frac{K}{2}(g_{ik}g_{jl}-g_{il}g_{jk})\theta^k\wedge\theta^l = -K\theta_i\wedge\theta_j.$$

Our goal is to show that K is constant. To do so, we use Bianchi-identity

$$d\Omega = \theta\wedge\Omega - \Omega\wedge\theta.$$

So we first calculate $d\Omega$:

$$d\Omega_{ij} = -dK\wedge\theta_i\wedge\theta_j - Kd\theta_i\wedge\theta_j + K\theta_i\wedge d\theta_j.$$

By the structure equation, the above identity becomes

$$d\Omega_{ij} = -dK\wedge\theta_i\wedge\theta_j - K\theta_i^k\wedge\theta_k\wedge\theta_j + K\theta_i\wedge\theta_j^k\wedge\theta_k.$$

On the other hand,

$$\begin{aligned}\omega\wedge\Omega - \Omega\wedge\omega &= \theta_i^k\wedge\Omega_{kj} + \Omega_{ik}\wedge\theta_j^k\\ &= K\theta_i^k\wedge\theta_k\wedge\theta_j - K\theta_i\wedge\theta_k\wedge\theta_j^k.\end{aligned}$$

Hence the Bianchi-identity implies that

$$dK\wedge\theta_i\wedge\theta_j = 0.$$

If we let $dK = \sum_k K_k\theta_k$, then we get

$$\sum_k K_k\theta_k\wedge\theta_i\wedge\theta_j = 0.$$

Hence $K_k = 0$. This implies that K is a constant.

- **Ricci curvature and scalar curvature.** Denote by $K_p(E) = K_p(X \wedge Y)$. Let $X \in T_p(M)$ be a unit vector. We take an orthonormal basis $\{e_1 = X, e_2, \ldots, e_m\}$ in $T_p(M)$ and consider the following "averages":

$$Ric_p(X) := \sum_{j=2}^{m} K_p(X \wedge e_j),$$

$$\rho(P) := \sum_i Ric_p(e_i),$$

which are called the *Ricci curvature* in the direction X and the *scalar curvature* at p, respectively. It is easy to prove that they do not depend on the choice of the orthnormal basis.

In order to explain these facts,we define a bilinear form r on $T_p(M)$ as follows: let $X, Y \in \Gamma(TM)$ and put $r(X, Y) =$ the trace of the map: $Z \mapsto \mathcal{R}(Z, X)Y$, which is called the *Ricci tensor*. In the above orthonormal basis $\{e_1 = X, e_2, \ldots, e_m\}$, we have

$$r(X, Y) = \sum_j < \mathcal{R}(e_j, X)Y, e_j >= \sum_j < \mathcal{R}(e_j, Y)X, e_j >= r(Y, X),$$

i.e. r is symmetric and

$$r(X, X) = \sum_j R(X, e_j, X, e_j >= \sum_j K(X \wedge e_j) = Ric(X).$$

Also, the bilinear form r induces a linear self-adjoint map L_r such that $< L_r(X), Y >= r(X, Y)$. In an orthonormal basis $\{e_1, e_2, \ldots, e_m\}$,, we have

$$\rho = \sum_i Ric_p(e_i) = \sum_i r(e_i, e_i) = \sum_i < L_r(e_i), e_i >= \text{trace of } L_r.$$

In the local coordinates (U, x^i), these can be expressed as $r(X, Y) = R_{ij}X^iY^j$ with $R_{ij} = g^{kl}R_{ikjl}$,

$$\rho = g^{ik}R_{ik} = g^{ik}g^{jl}R_{ijkl}.$$

If M is of constant sectional curvature c, then $Ric(X) = (m-1)c$ and its scalar curvature is $\rho = m(m-1)c$.

A Riemannian manifold (M, g) is called an *Einstein* manifold if there is a constant λ on M such that $r = \lambda g$. In such casse, g is called the *Einstein* metric, and the scalar curvature $\rho = n\lambda$ is constant.