Divide and Conquer: Solving Recurrences

Solving Recurrences

- 1. Divide and conquer examples:
 - MergeSort

$$- T(n) = 2T(n/2) + O(n) = O(n\log(n))$$

• Integer Multiplication

$$-T(n) = 4T(n/2) + O(n) = O(n^2)$$

• Improved Integer Multiplication

$$-T(n) = 3T(n/2) + O(n) = O(n^{\log_2 3})$$

• Median

$$- T(n) = T(3n/4) + O(n) = O(n)$$

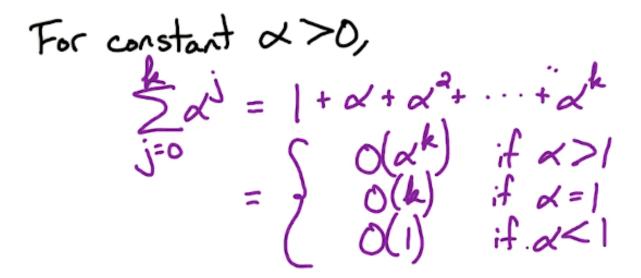
Example 1

- 1. T(n) = 4T(n/2) + O(n)
 - For some constant c > 0, T(n) = 4T(n/2) + cn, T(1) <= c
 - $T(n) \le cn + 4T(n/2)$
 - $T(n) \le cn + 4[4T(n/4) + cn/2]$
 - $-T(n) \le cn(1+4/2)+4^2T(n/4)$
 - $-T(n) \le cn(1+4/2) + 4^2[4T(n/8) + cn/4]$
 - $T(n) \le cn(1 + 4/2 + (4/2)^2) + 4^3T(n/2^3)$

Expanding Out

- 1. $T(n) \le cn(1 + 4/2 + (4/2)^2) + 4^3T(n/2^3)$
 - $T(n) \le cn(1 + (4/2) + (4/2)^2 + \dots + (4/2)^{i-1}) + 4^i T(n/2^i)$
 - let $i = log_2 n$ then $n/2^i = 1$
 - $T(n) \le cn(1 + (4/2) + (4/2)^2 + ... + (4/2)^{\log_2 n-1}) + 4^{\log_2 n}T(1) cn = O(n)$
 - $-((4/2)^{\log_2 n}) = O(n^2/n) = O(n)$
 - $-4^{\log_2 n} = O(n^2)$
 - Total is $O(n^2)$

Geometric Series



Geometric Series

Manipulating Polynomials

- 1. $4^{\log_2 n} = n^2$
- 2. $3^{\log_2 n} = n^c$
 - $3 = 2^{\log_2 3}$
 - $3^{\log_2 n} = (2^{\log_2 3})^{\log_2 n} = 2^{\log_2 3 * \log_2 n}$
 - $2^{\log_2 3 * \log_2 n} = (2^{\log_2 n})^{\log_2 3} = n^{\log_2 3}$
 - $c = log_2 3$

Example 2

- 1. T(n) = 3T(n/2) + O(n)
 - $T(n) \le cn + 3T(n/2)$
 - $T(n) \le cn(1 + (3/2) + (3/2)^2 + ... + (3/2)^{i-1}) + 3^i T(n/2^i)$ - let $i = log_2 n$
 - $T(n) \le cn(1 + (3/2) + (3/2)^2 + ... + (3/2)^{\log_2 n 1}) + 3^{\log_2 n} T(1)$ - cn = O(n)- $(1 + (3/2) + (3/2)^2 + ... + (3/2)^{\log_2 n - 1}) = O((3/2)^{\log_2 n}) = O(3^{\log_2 n})$ - $3^{\log_2 n} T(1) = O(3^{\log_2 n})$
 - Total is $O(n^{\log_2 3})$

General Recurrence

- 1. Constants a > 0, b > 1
 - T(n) = aT(n/b) + O(n)
 - $T(n) = cn(1 + (a/b) + (a/b)^2 + ... + (a/b)^{\log_b n-1}) + a^{\log_b n}T(1)$
 - If a > b: $O(n^{\log_b a})$
 - If a = b: $O(n\log(n))$
 - If a < b: O(n)
 - This is how the Master Theorem is derived

Some Properties of Logs

1.
$$\log_b(xy) = \log_b(x) + \log_b(y)$$

- $\begin{aligned} 2. & \log_b(x/y) = \log_b(x) \log_b(y) \\ 3. & \log_b(x^n) = nlog_b(x) \\ 4. & x^{log_b(y)} = y^{log_b(x)} \end{aligned}$

- 5. $\log_b(x) = \log_a(x) / \log_a(b)$ 6. We presume \log_2 unless specifically indicated or otherwise obvious