Tree Computations

Introduction

1. This lesson covers how to apply parallelism to algorithms based on trees
    - Basic principle is to split trees into lines and break into segments
    - Requires applying randomization

Tree Warm Up

1. Can store a tree as an array pool, just like we did with linked lists
    - Number the nodes of the tree
    - Store the parent of each node i at index i
2. Serial algorithm for finding the root of the tree:
    - Running time = O(n)

```
root(P[1:n])
{
    if n < 1 then return 0
    node <- (any node, 1..n)
    while P[node] == 0 do
        node <- P[node]
    return node
}
```

3. How do we make this parallel?
    - Idea: Explore from all nodes simultaneously. At each node, change parent to grandparent
    - Eventually, each node points to the root

```
hasGrandparent(k, P[1:n])
{
    return (k > 0) and (P[k] > 0) and (P[P[k]] > 0)
}

adopt(P[1:n], G[1:n])
{
    parfor i <- i to n do
        if hasGrandparent(i, P[:])
            G[i] <- P[P[i]]
        else
            G[i] <- P[i]
}

findRoots(P[1:n], R[1:n])
{
    let Pcur[1:n] <- P[:]
    let Pnext[1:n] <- (tempbuf)
    for l <- 1 to ceil(log(n)) do
        adopt(Pcur[:], Pnext[:])
        Pcur[:] <- Pnext[:]
    R[:] <- Pcur[:]
}
```

Parallel Root Finder

1. Which of these claims about findRoots are true?

- Uses pointer jumping (true)
- Is work-optimal
- Has polylogarithmic span (true)
- Works on a forest, not just one tree (true)

Work-Optimal List Scan/Prefix-Sum - Part 1

1. Wyllie's algorithm uses prefix-sums to compute list ranks, but is not work- optimal
   - W(n) = O(n * log(n))
   - D(n) = O(log(n))
2. Consider a trick where we shrink the list to size m < n
   - Then, run Wyllie O(mlog(m)) and restore full list and ranks
3. Assume step 2 dominates overall work. What choice of 'm' leads to work- optimality?
   - O(log(n))
   - O(n / log(n)) (true)
   - O(n * log(n))
   - O(sqrt(n))
   - O(n)
   - O(n ^ 2)
4. 1 and 4 are not possible because O(m * log(m)) would be asymptotically less than n
5. 3, 5, and 6 are not good choices because they give sub-optimal algorithms

Parallel Independent Sets - Part 1

1. An independent set is a subset I of vertices such that any vertex in the set does not also have its successor in the set
2. Consider the following linked list
   - 4 -> 2 -> 7 -> 1 -> 3 -> 5 -> 6 -> 8
   - N[i] = [3, 7, 5, 2, 6, 8, 1, 0]
   - I = {3, 7, 8} is an independent set
   - I = {3, 4, 6, 8} is not an independent set because 8 is a successor to 6
3. Computing an independent set in serial is easy; just skip nodes
4. Computing an independent set in parallel is more difficult due to the problem of symmetry (all nodes look the same)
   - Need a scheme to break the symmetry
     - At every node, flip an unbiased coin (Pr[heads] == Pr[tails] == 1/2)
     - If the coin is heads, the node is a candidate to join the set
     - If a node sees its successor has a head, it changes to a tail

```
ParIndSet(N[1:n], I[:])
{
    let C[1:n], Chat[1:n] = space for coins
    parfor i <- 1 to n do
        C[i] <- flipcoin(H or T)
        Chat[i] <- C[i] // make a copy
    parfor i <- 1 to n do
        if(Chat[i] == H) and (N[i] > 0) and (Chat[N[i]] == H) then
            C[i] <- T
    I[:] <- gatherIf(1:n, C[1:n])
}
```

Parallel Independent Sets - Part 2

1. Consider the following linked list and initial coin flips
   - 4 -> 2 -> 7 -> 1 -> 3 -> 5 -> 6 -> 8

    - N[i] = [3, 7, 5, 2, 6, 8, 1, 0]
    - C[i] = [H, T, T, H, T, H, H, H]
2. After breaking symmetry
    - C[i] = [H, T, T, H, T, T, T, H]
    - Independent set: [1, 4, 8]

Parallel Independent Sets - Part 3

1. What are the work and span of the parallel independent set algorithm?
    - W(n) = O(n)
        - Loops go from 1 to n, work per iteration is constant
    - D(n) = O(log(n))
        - A parallel for-loop with constant work per iteration is O(log(n))

Parallel Independent Sets - Part 4

1. What is the average number of vertices that end up in the independent set?
    - 1n / 8
    - 3n / 4
    - 1n / 4
    - 7n / 8
    - 1n / 2
2. The answer is n / 4. Initially, it's n / 2, but after applying the correction in the parallel independent sets algorithm for when the next node is also heads, only one of the four options remain

Work-Optimal List Scan/Prefix-Sum - Part 2

1. To shrink the list for Wyllie's algorithm, we can use the parallel independent set algorithm
    - Similar to pointer-jumping, but only over the elements of the indpendent set
    - After we've calculated the independent set, we remove it from the list
        - This requires updating the next pointers and pushing each nodes' rank to its neighbor
        - We might need to repeat this process until we reach our desired size of n / log(n)
    - Once we reach the desired size, we can do the list scan to get the ranks
        - To get the ranks for the other nodes, we need to run the process for shrinking the list in reverse
        - This isn't necessarily difficult, but requires lots of bookkeeping. Psuedocode is available in instructor notes

Work-Optimal List Scan/Prefix-Sum - Part 3

1. How many times do you need to run the independent set to shrink the list to O(n / log(n)) in length?
    - O(1)
    - O(log(n))
    - O(log(log(n))
    - O(sqrt(n))
2. The answer is O(log(log(n))
    - After each iteration, the list is roughly 3/4 the size of the previous list
    - After k calls, the list length is (3/4)^k * n <= n /log(n)
    - Solving this relation gives O(log(log(n))
    - Work and span:
        - W(n) = O(n * log(log(n)))
        - D(n) = O(log(n) * log(log(n)))
3. Additional considerations:
    - This is the average case, but there's some distribution around this case and we care what the distribution is

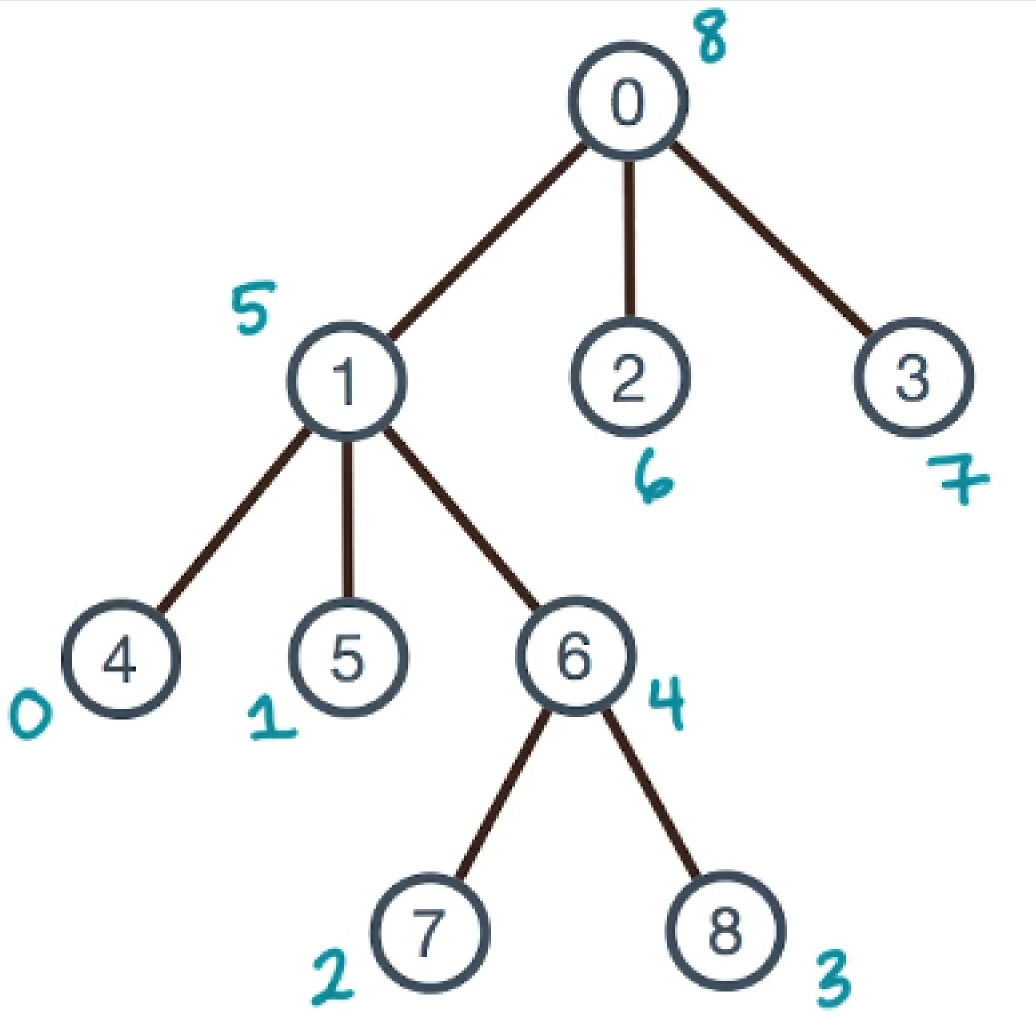
  - Is the distribution skinny or fat? How much weight is in the tails?
- Also, how much bookkeeping is required to shrink and restore the list?

A Seemingly Sequential Tree Algorithm

1. Suppose you want to compute a postorder numbering of a tree

```
postorder(root, V[1:n], v0)
{
    v <- v0
    foreach C in children(root) do
        v = postorder(C, V[:], v) + 1
    V[root] = v
    return v
}
```

2. How do we do this in parallel?
   - Looks inherently sequential, but looks similar to list ranking
   - Need a way to convert a tree to a list

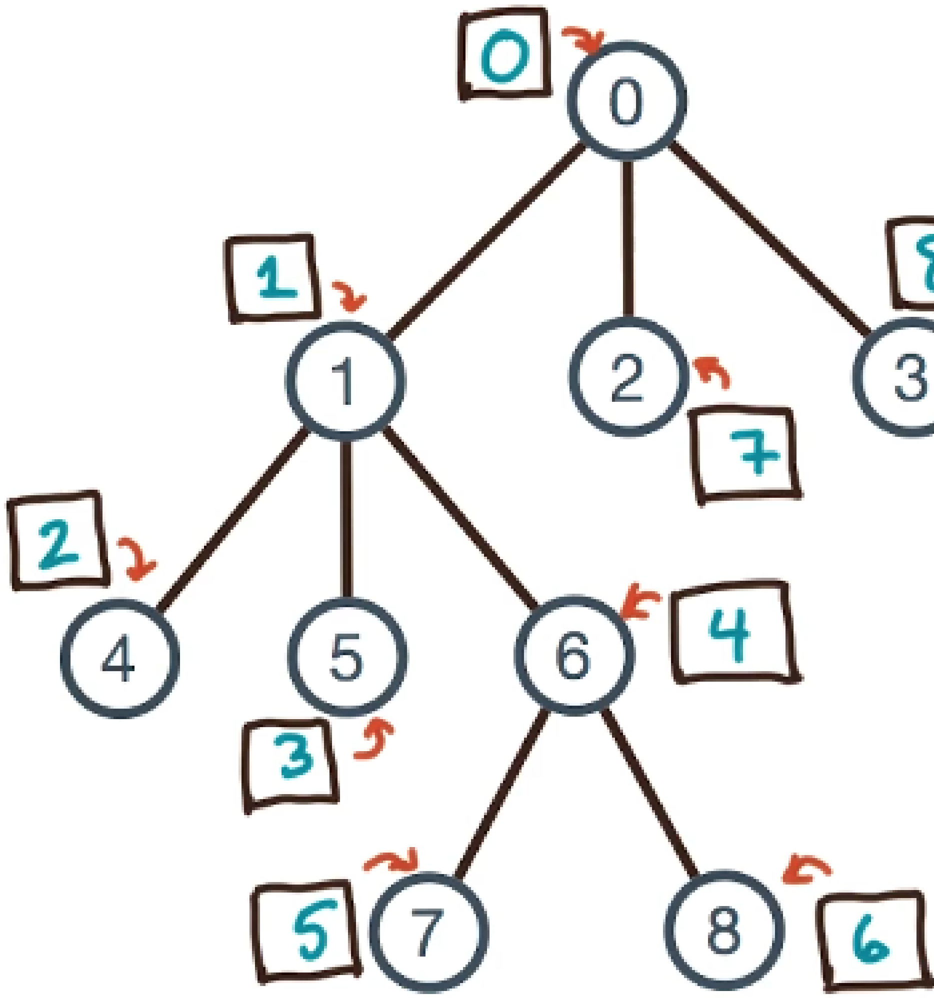


Postorder Notation

Another Tree Traversal

1. Consider the same tree from the previous side. What would a preorder notation look like for this tree?

```
preorder(root, V[1:n], v0)
{
    v <- v0
    V[root] = v
    foreach C in children(root) do
        v = preorder(C, V[:], v) + 1
    return v
}
```

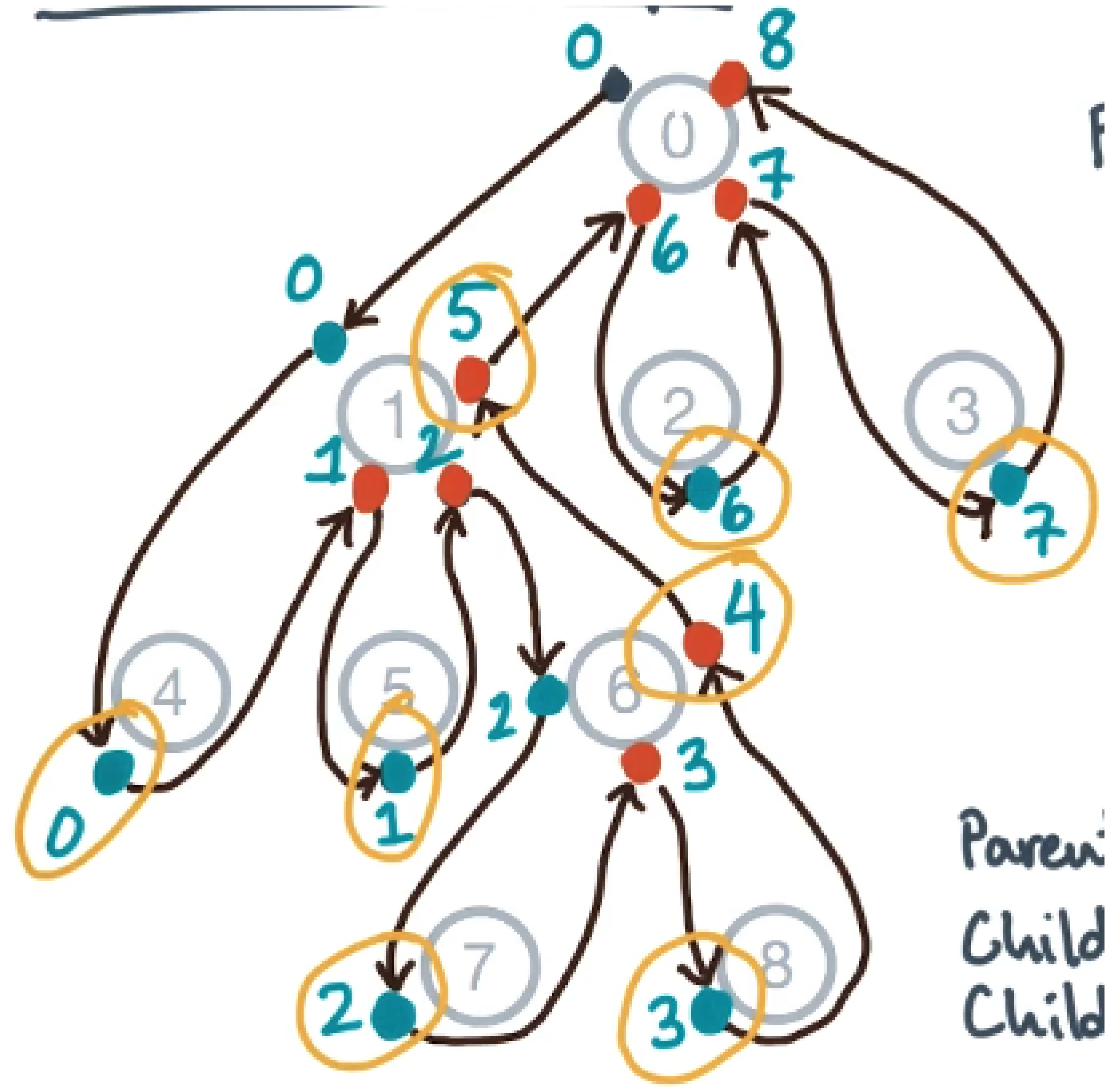


Preorder Notation

The Euler Tour Technique

1. Take every undirected edge of the tree and represent it as a pair of directed edges
    - At each node, the number of incoming and outgoing edges are the same
    - This fact makes the graph Eulerian
2. Euler circuit: A closed path that uses every edge once

3. If we use this notation and come up with a clever choice of initial values, prefix-summing over the tree will give the postorder values
    - Set the parent-to-child sinks to 0
    - Set the child-to-parent sinks to 1
4. Summary:
    - Convert the tree to a list (Euler circuit)
    - Apply the numbering to each node
    - Prefix sum over the list



Euler Tour of a Tree

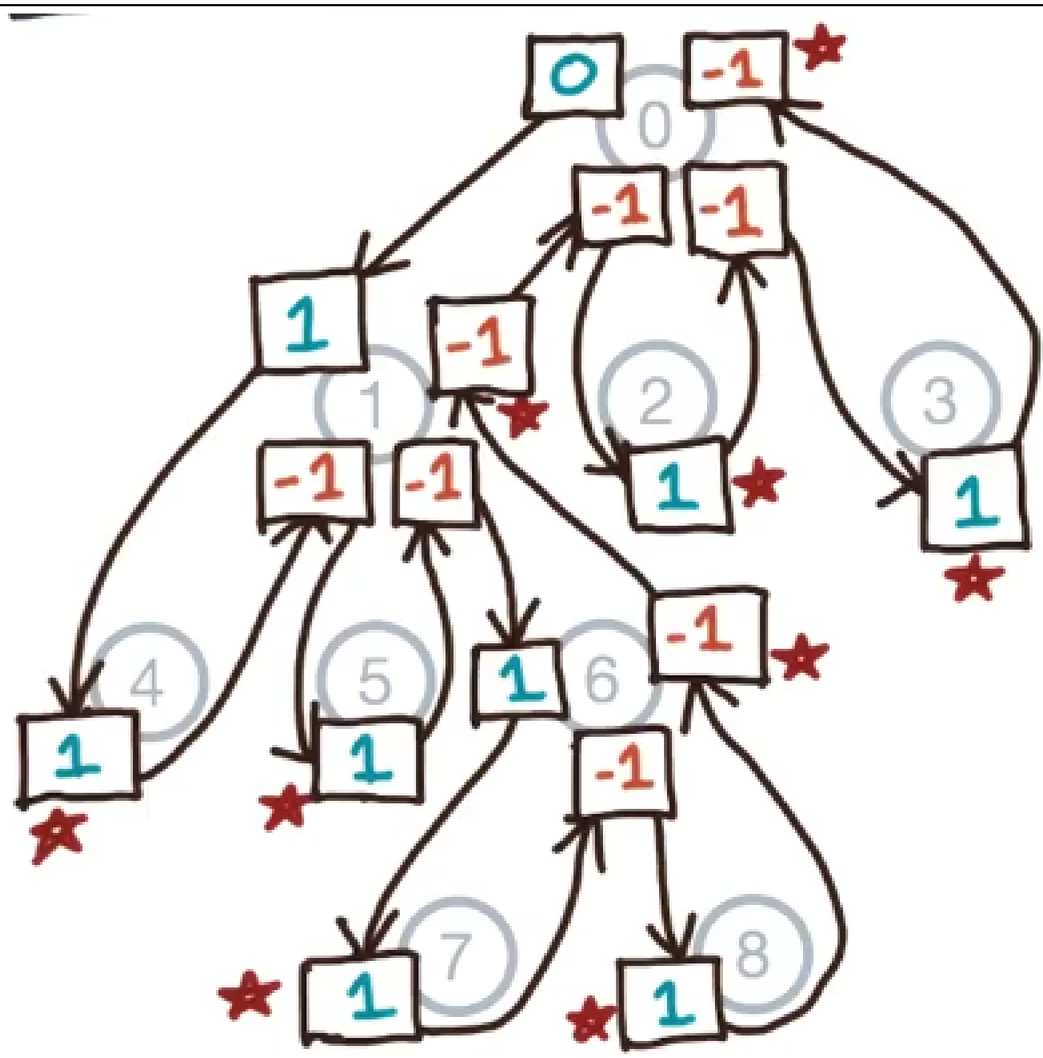
The Span of an Euler Tour

1. The work of the Euler tour is W(n) = O(n)
2. The span of the prefix sum is O(log(n))
3. Should the overall span of the Euler tour algorithm depend on the overall depth of the tree, or is it just O(log(n))?
    - D(n) = O(log(n))
    - After we've converted the tree into a list, the span no longer depends on the shape of the tree

- This hinges on being able to convert the computation into an equivalent Euler tour computation, which is not always possible

Computing Levels

1. Suppose we want to compute the level or depth of each node in a tree
   - The level (depth) of a tree node is the minimum number of edges from the root to the node
2. Suppose you apply the Euler tour techinque. Choose the initial list node values so that the levels appear at each child-to-parent source nodes (stars)
3. At each sink node:
   - Parent-to-child: +1
   - Child-to-parent: -1
4. Then, the prefix sum gives the depth



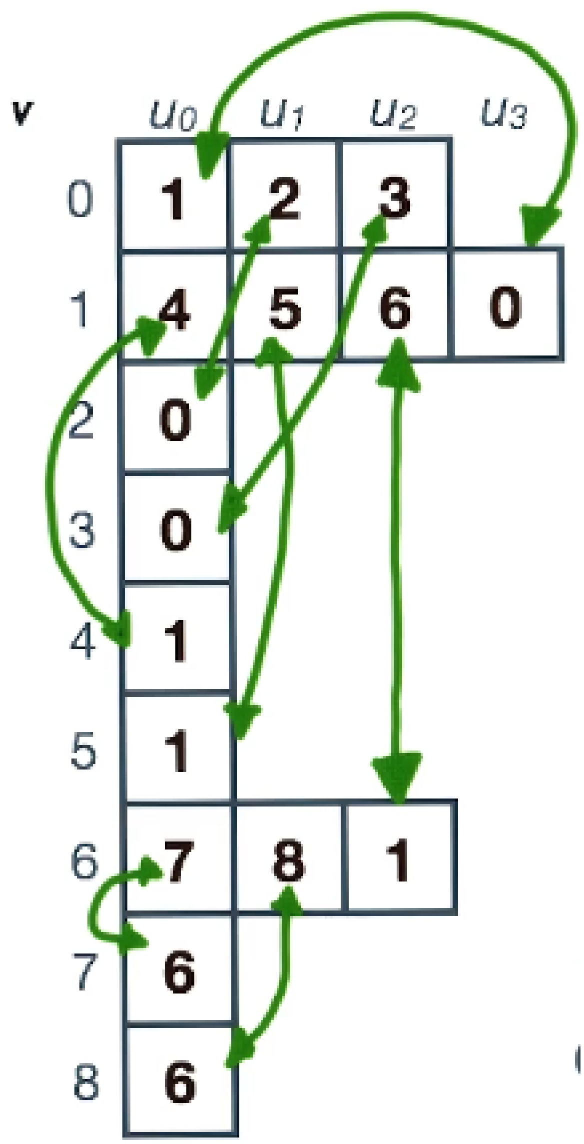
Depth of Nodes in a Tree

Implementing Euler Tours

1. How do we compute an Euler tour and store the tree?

- Start with a version of the tree in which each undirected edge is represented by a pair of directed edges
- For each node v, we define its adjacency list as the set of its outgoing neighbors

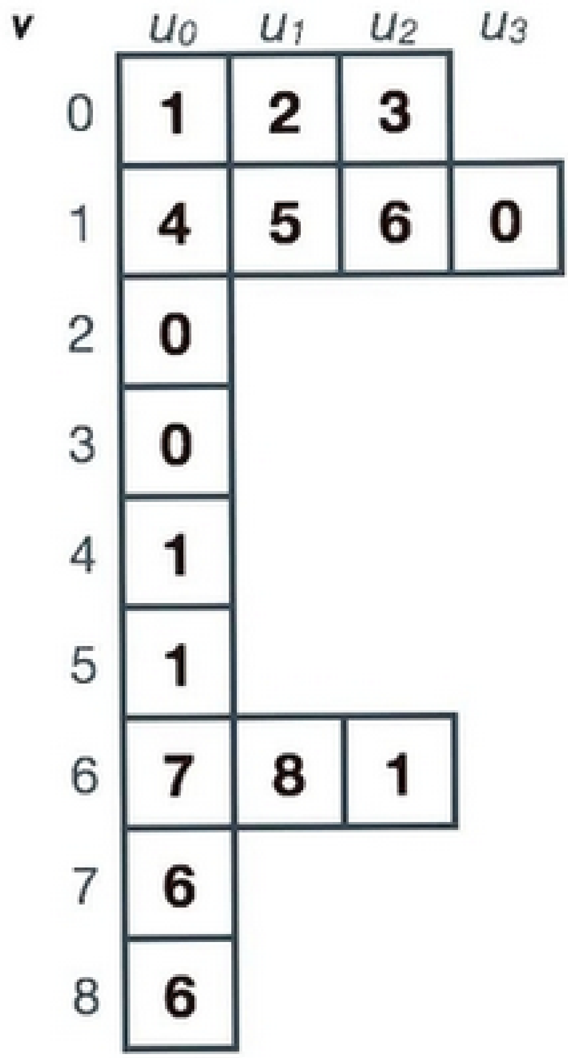
2. Successor function:
   - s(Ui, v) = (v, U(i+1) % dv)
   - Given an edge that goes from Ui to v, the successor function returns the next neighbor in v's adjacency list
     - Modulo makes the list circular
3. Is the cost of applying the successor function constant?
   - Yes, but we need to add the cross-edges to the adjacency list
     - Cross-edge: What next node does the end of a list point to?



Cross Edges

Successor or Failure-or

1. What is s(s(s(6,8))) for the following adjacency list?
   - s(6,8) = (8,6)
   - s(8,6) = (6,1)
   - s(6,1) = (1,0)



Adjacency List

Conclusion

1. Two frameworks for performing parallel operations on trees
   - One is built on top of work-optimal lists
   - The other is on the rank-compress framework for evaluating expressiong trees
2. Linearizing a tree is an important concept for achieving load balance

- Load balancing is required for performing parallel algorithms at scale