Graph Partitioning

Introduction

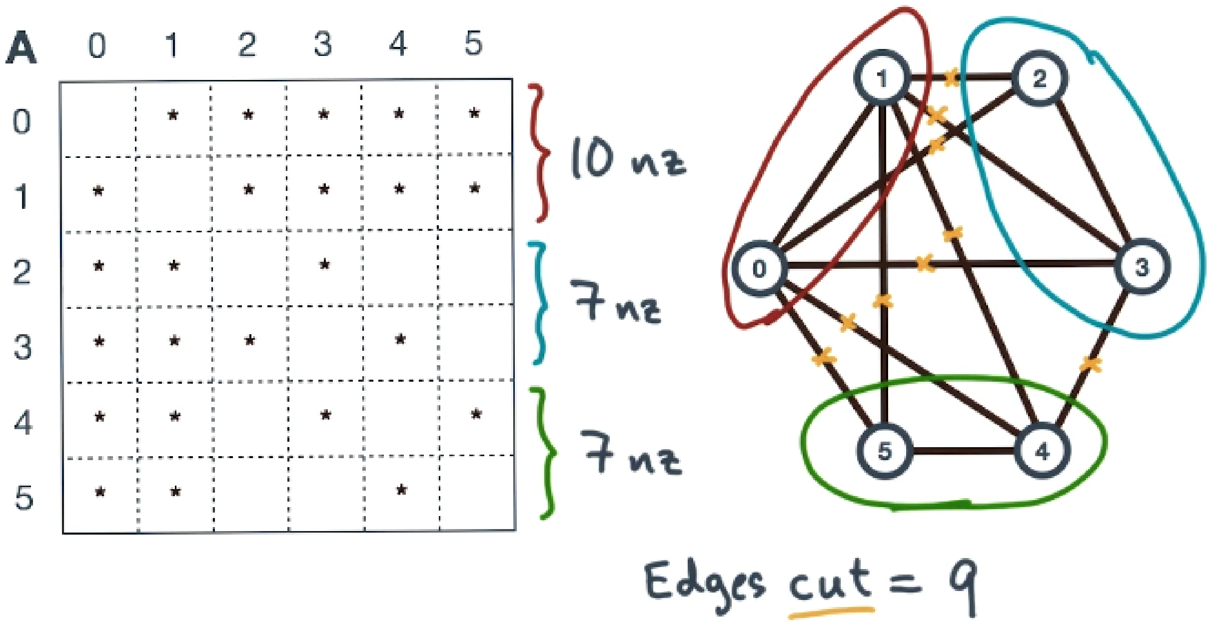
1. Common thread in a distributed memory algorithm: How do you distribute the data?
    - Spectral partitioning: Exploits the connection between graphs and linear algebra
        - Physics-based interpretation based on systems of springs

The Graph Partitioning Problem

1. Can represent a graph as an adjacency matrix
    - Can partition the matrix across rows, corresponding to a graph (vertex) partition (sparse matrix)
    - Can then do a BFS with a matrix-vector multiply
        - Implies partitioning the vector as well
    - Amount of work corresponds to the number of non-zero entries
        - Ideally, we should balance the work
    - Any time an edge crosses a process boundary, a communication exchange occurs
        - Reduce edge cuts to minimize communication volume
2. Two goals:
    - Balance work across processes
    - Minimize communication between processes
3. Graph partitioning problem:
    - Given G = (V, E) and number of partitions, P
    - Compute a (vertex) partition V = V0 U V1 U ... Vp-1 such that:
        - {Vi} are disjoint => intersect(Vi, Vj) = 0
        - {Vi} are roughly balanced => |Vi| ~ |Vj|
        - Let Ecut = {(u, v) | u in Vi, v in Vj, i != j} and minimize Ecut

Do You Really Want a Graph Partition?

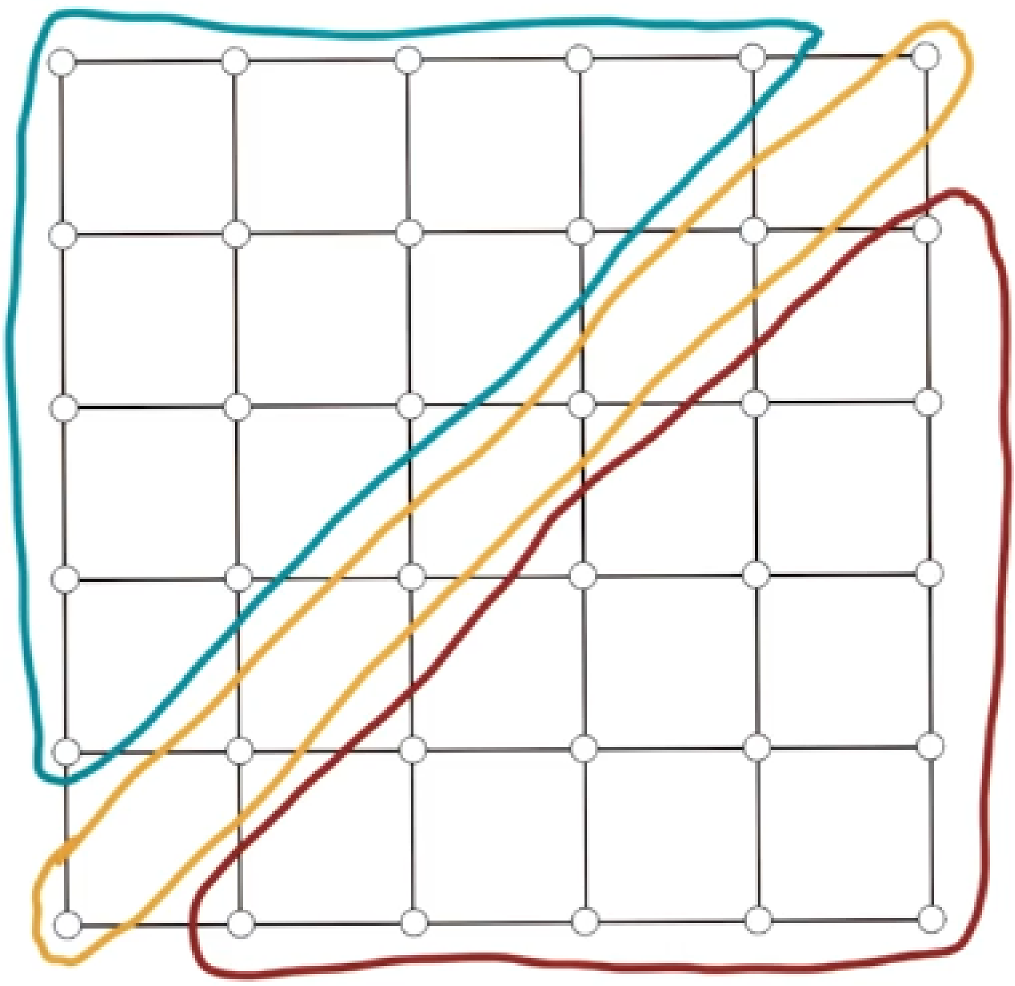
1. Does the graph partitioning problem as formalized above meet the intended goals?
    - No. Despite the number of vertices per partition being the same, the number of non-zero entries can vary, so the work is not evenly distributed



Uneven Distribution of Work

Graph Bisection and Planar Separators

1. Graph partitioning is NP-complete -> Need heuristics!
   - Heuristic: Bisection (divide and conquer)
     - Suppose you want P partitions
     - Start by using any algorithm to divide the graph into 2 partitions
     - Then, divide each half into two more partitions
     - Repeat until the P partitions are obtained
2. Planar graphs
   - Graphs that can be drawn in 2D with no edge crossings
3. Planar separators
   - Theorem: A planar graph G = (V, E) with |V| = n vertices has a disjoint partition V = A U S U B such that:
     - S separates A and B
     - |A|, |B| <= 2n / 3 (the larger of the two partitions is no more than twice the size of the smaller)
     - |S| <= 2 * sqrt(2n) = O(sqrt(N))
     - Lipton and Tarjan 1979
   - Lipton and Tarjan described a polynomial-time algorithm



Planar Graph

Partitioning via Breadth First Search

1. How can BFS be used to bisect a graph? Assume the graph is connected
   - Run BFS from any vertex
     - BFS will assign every vertex to a level
     - Levels separate subgraphs
   - Stop when about 1/2 of the vertices have been visited. Assign visited to one partition, unvisited to the other
     - Other criteria are possible
2. However, we wanted to use graph partitioning to distribute our BFS computation, so we can't use BFS to determine the graph partition

Kernighan Lin - Part 1: No Gain is Pain

1. Kernighan-Lin algorithm is the most well-known heuristic for graph partitioning
   - Given a graph, divide the vertices into two subsets of equal or nearly- equal size
     - V = V1 U V2, |V1| = |V2|
   - Cost of this partition is the number of edges between V1 and V2

        - Cost(V1, V2) = Number of edges between V1 and V2
    - Assume you have two evenly-sized subsets of V1 and V2
        - X1 from V1, X2 from V2 with |X1| = |X2|
        - If you swapped these two subsets, the cost will change, but by how much?
2. Formal definitions
    - Consider vertex a in V1 and vertex b in V2
    - External costs: Number of vertices in the opposite partition
        - E1(a in V1) = # edges(a, b in V2)
        - E2(b in V2) = # edges(b, a in V1)
    - Internal costs: Number of vertices in the same partition
        - I1(a in V1) = # edges(a, i in V1)
        - I2(b in V2) = # edges(b, i in V2)
    - Cost(V1, V2) = Cost(V1 - {a}, V2 - {b}) + E1(a) + E2(b) - c(a,b)
        - c is a constant to account for an edge between a and b. 1 if an edge exists, 0 otherwise
    - Then, consider the cost after swapping a and b
    - Cost(V1, V2) = Cost(V1 - {a}, V2 - {b}) + I1(a) + I2(b) + c(a,b)
    - Change in cost = E1(a) + E2(b) - I1(a) - I2(b) - 2c(a,b)
        - Larger change is better => Larger decrease in cost
    - Change in cost = gain(a in V1, b in V2)
        - Can be negative if cost increased

Kernighan Lin Algorithm Quiz

1. Consider the computation of the gain function a and b
    - Assume the following:
        - Every vertex has a partition label; O(1) access time
        - Maximum degree of any vertex is d
2. What is the sequential running time to compute gain(a, b) in terms of d, n1 = |V1|, n2 = |V2|
    - O(d)
    - Need to sweep over the adjacent vertices, which is at most d
        - Need to check partition label of each vertex, which is a constant-time operation

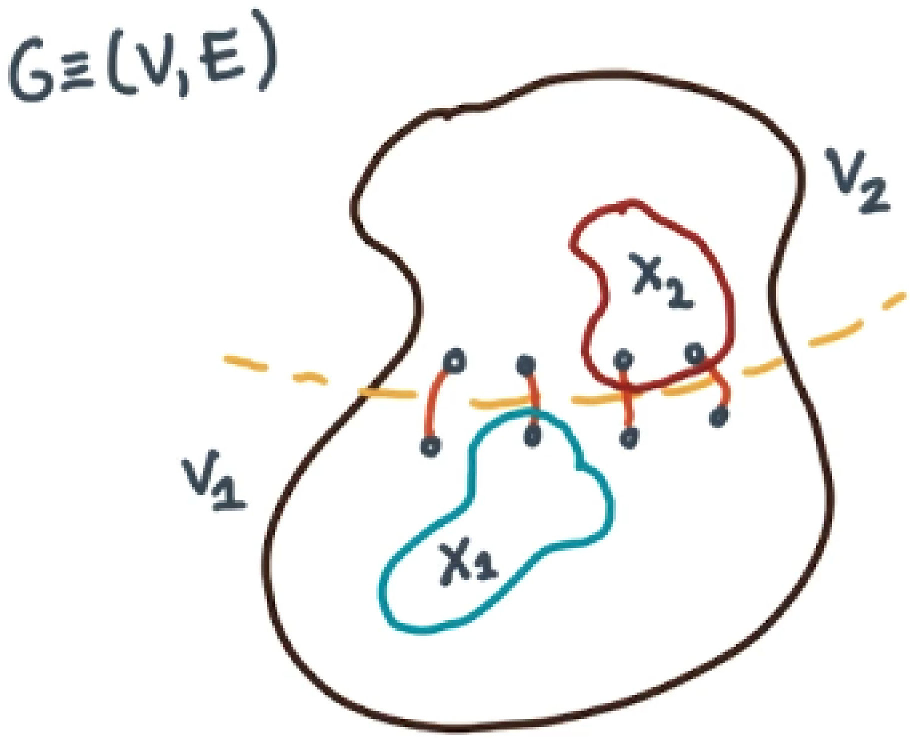
Kernighan-Lin Algorithm

1. How do we choose X1 and X2, the subsets of the partitions?

```
let V = V1 U V2
C = cost(V1, V2)
forall a in V1, do compute E1(a), I1(a)
forall b in V2, do compute E2(b), I2(b)
forall v in V do visited[v] <- false

while !all(visited) do
    choose unmark (a in V1, b in V2) with largest gain(a, b)
    visited[a], visited[b] <- true
    Update all E1, E2, I1, I2 // not actually swapping nodes, just updating costs

// this provides gain(a1, b1), gain(a2, b2), ...
let Gain(j) = sum(gain(a, b))
choose jmax = argmax(Gain(j))
if Gain(jmax) > 0 then
    X1 = {a1, a2, ... ajmax}
    X2 = {b1, b2, ... bjmax}
    Update C <- C - Gain(jmax)
    V1 <- (V1 - X1) U X2 // swap
```

```
    V2 <- (V2 - X2) U X1 // swap

// repeat until Gain(jmax) < 0
```

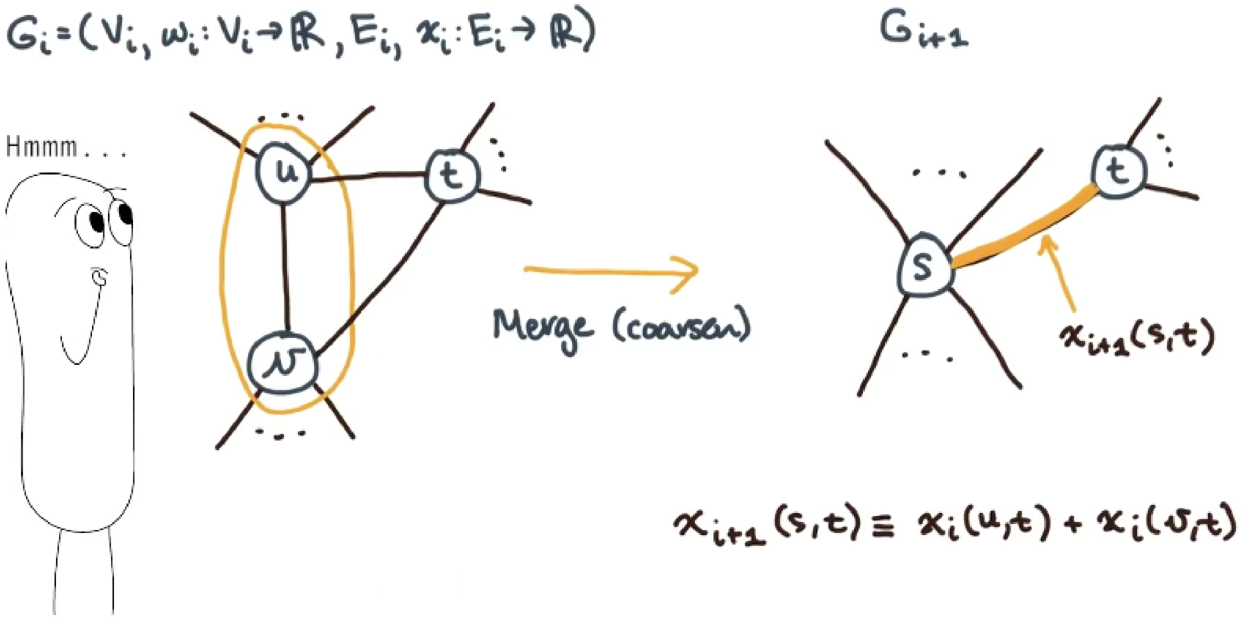
2. Cost:
    - Overall running time O(|v|^2 * d)
    - Can be reduced to O(|E|)
        - Fidducia and Matteyes 1982



Kernighan-Lin Algorithm

Graph Coarsening

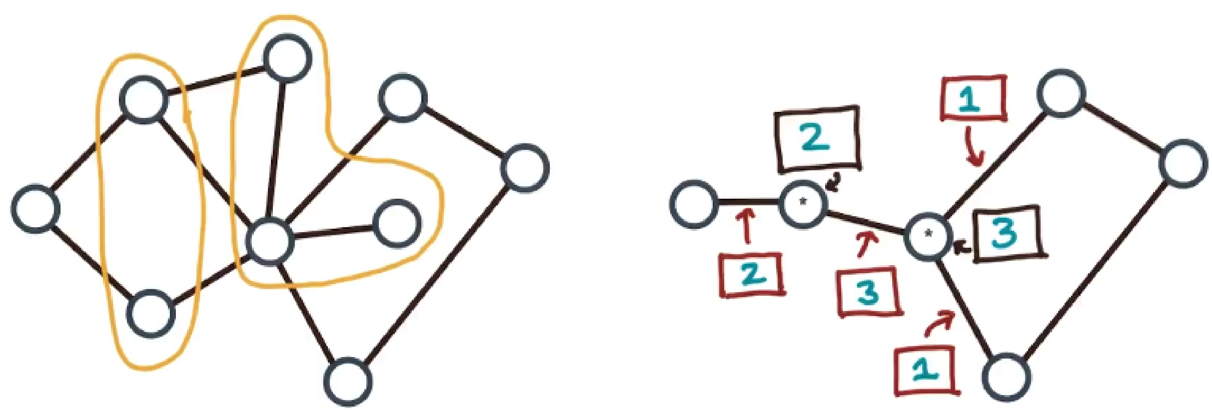
1. Multi-level graph coarsening: Different partitioning scheme
    - Divide-and-conquer
    - Continually replace graph with smaller graphs (fewer nodes and edges) but still somehow resembles original graph
        - Repeat until the graph is small enough to partition quickly
        - If we've done a good job of preserving the shape of the graph, the split will correspond to a roughly equivalent split in the larger graph
2. How do you actually coarsen a graph?
    - Identify at least one subset of the vertices to collapse
    - Replace subset with single "super" vertex
    - Track the fact that one vertex has replaced many by assigning a weight W to the vertex, where W is the number of replaced vertices
    - Track edge weights as well so we can cut edges accurately later on
    - Gi = (Vi, wi: Vi -> R, Ei, xi: Wi -> R)
        - wi and xi are functions that map edges and vertices to super edges and vertices
        - Simply sum the contributions from each vertex



Graph Coarsening Algorithm

Coarsen Me - Baby

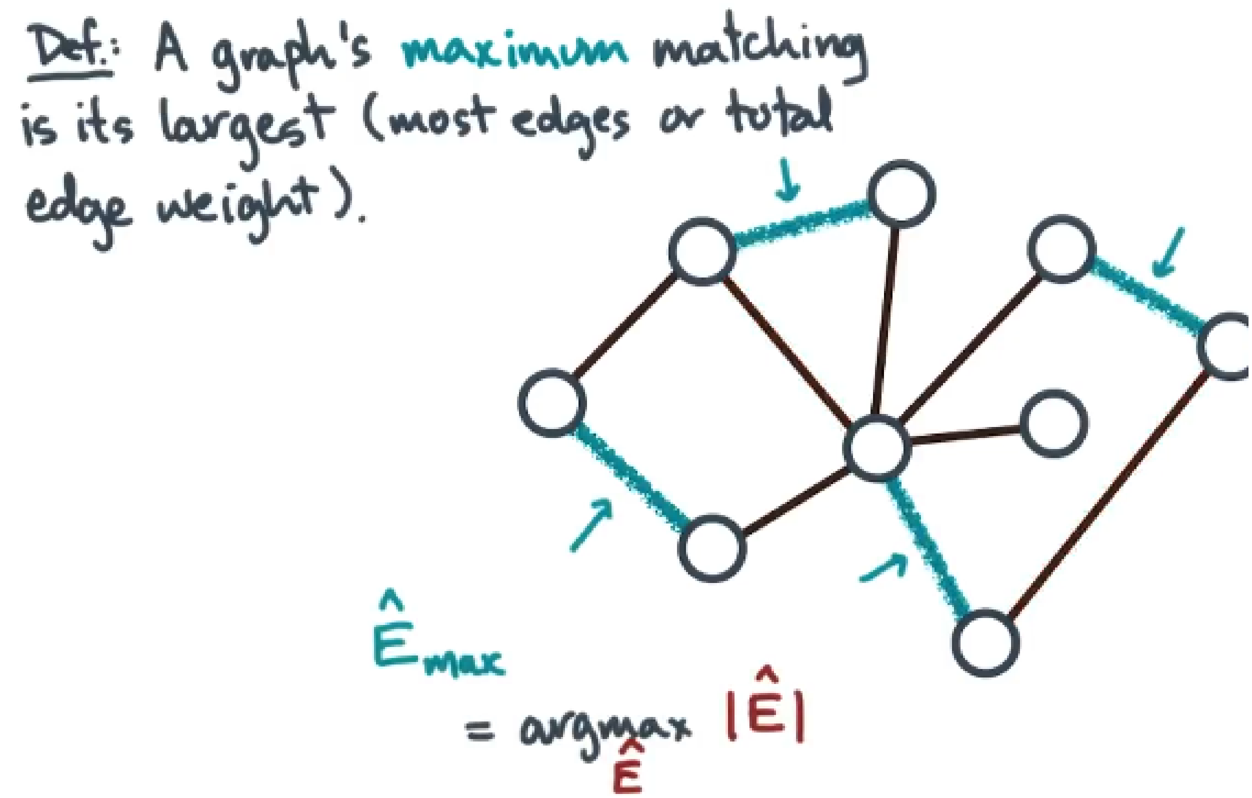
1. Determine the coarsen vertex and edge weights in the following graph:



Graph Coarsening Quiz

Maximal and Maximum Matchings

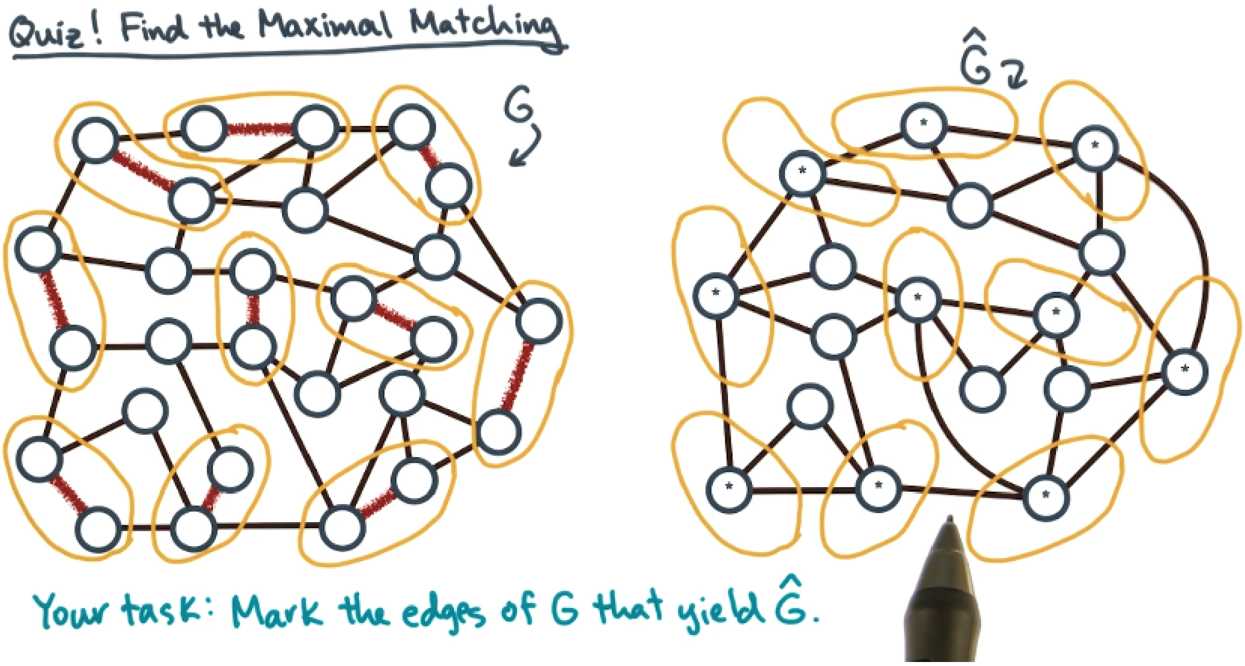
1. Matching: A subset of a graph G = (V, E) is a subset E' of E of edges with no common endpoints
   - Independent set, but for edges instead of vertices
2. Maximal matching: A matching is maximal if no more edges may be added
3. Maximum matching: A graph's maximum matching is its largest (most edges or total edge weight)



Maximal (and Maximum) Matching

Find the Maximal Matching

1. Mark the edges of G that yield Ghat



Maximal Matching Quiz

A Fact about Maximal Matchings

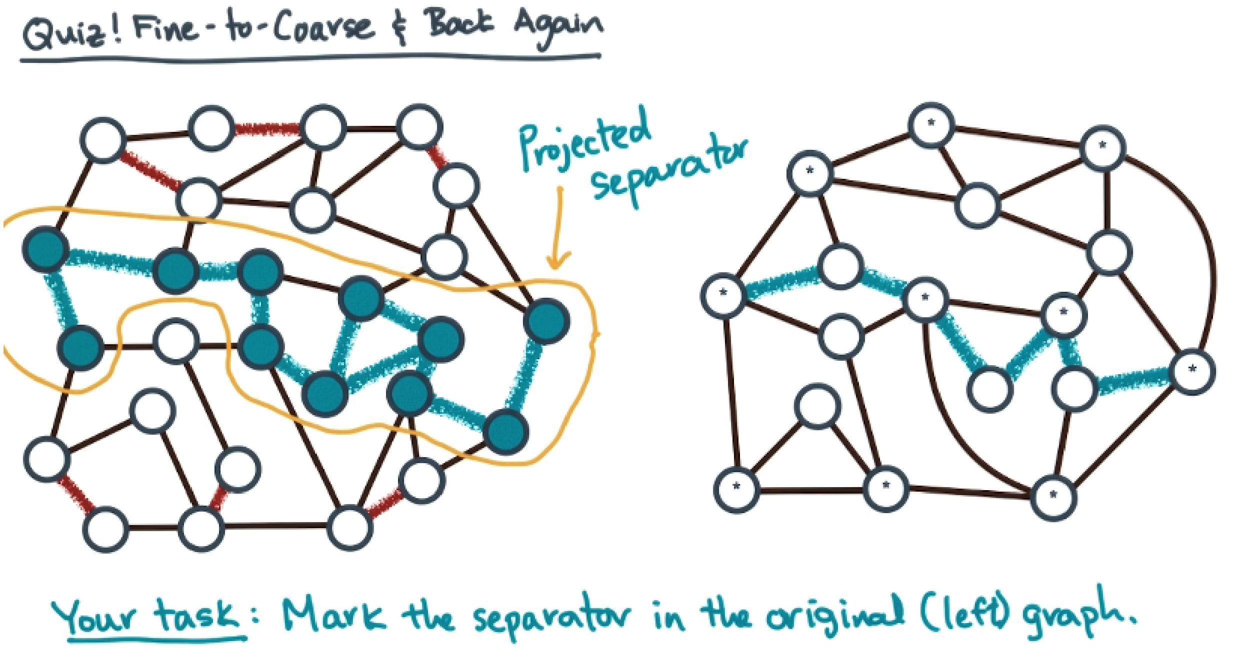
1. Consider an initial graph G. Suppose you find a maximal matching and use it to coarsen. Repeat this process producing a sequence of graphs from 1 to k. The original graph has n vertices and the final graph has s vertices.
2. How large must k be in terms of n and s?
   - log(n/s)
   - Each level must have at least half of the previous number of vertices, so |Vk| = n/2^k => k >= log2(n/s)

Computing a Maximal Matching

1. Pick an unmatched vertex at random
2. Match to one of its unmatched neighbors
   - Could pick randomly
   - Better strategy is to pick the heaviest edge
     - Not a lot of rigorous analysis, but lots of experimental evidence
     - The intuition comes from the fact that picking the heaviest edge should lead to the greatest reduction in edge weight when coarsening the graph

Fine-to-Coarse and Back Again

1. Mark the planar separator from the coarsened graph in the original graph
   - Some separator edges map ambiguously if they were merged in the coarsened graph



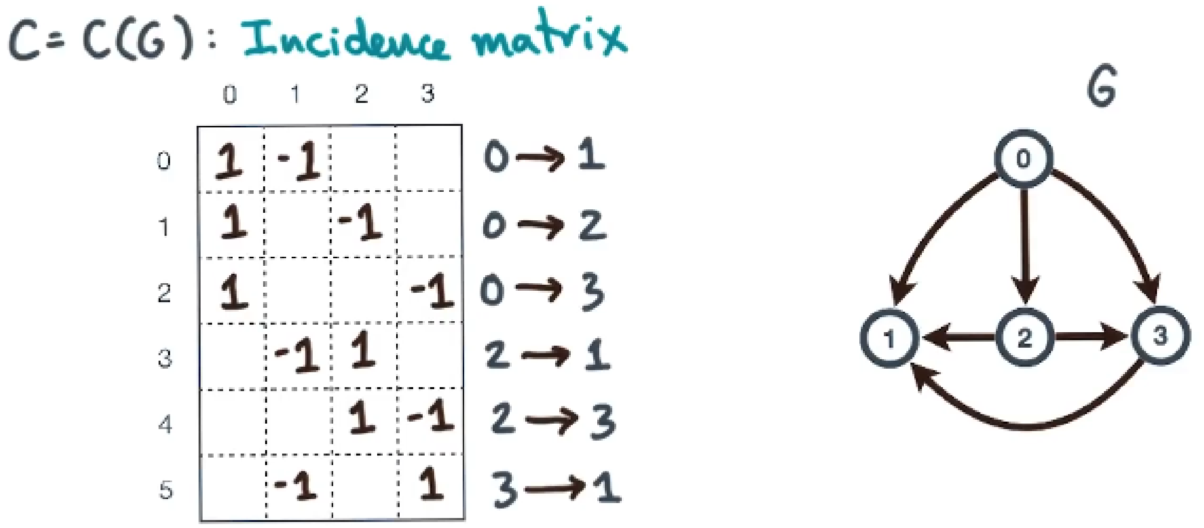
Planar Separator in Coarsened Graph

Partition Refinement

1. A minimum balanced edge cut in a coarsened graph minimizes the balanced edge cut in the next finer graph.
    - False; Coarsening is a heuristic, so it's possible that there's a better cut in the finer graph

Spectral Partitioning - Part 1: The Graph Laplacian

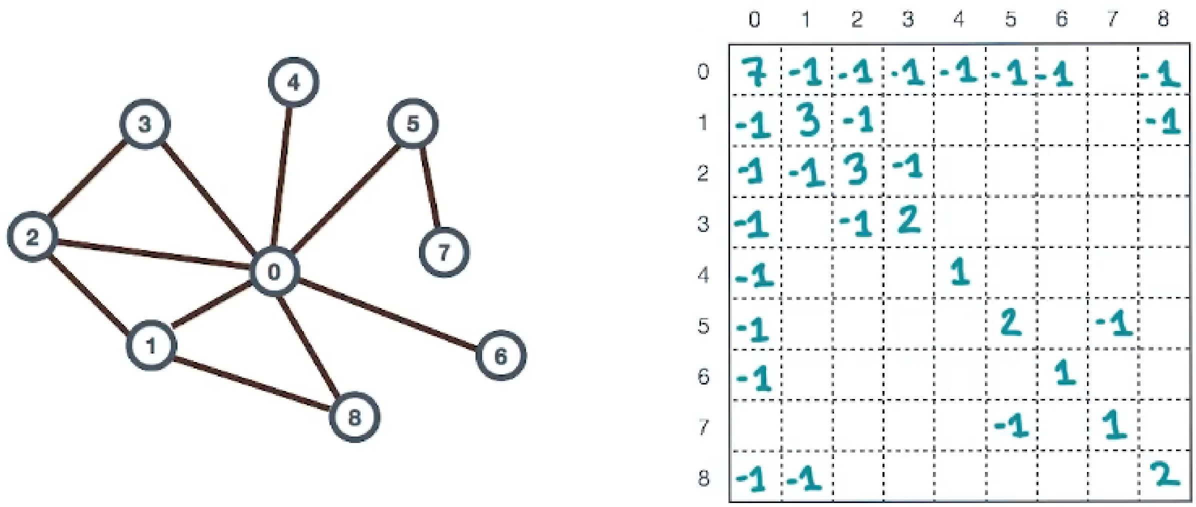
1. Instead of representing a graph as an adjacency matrix, represent it as an incidence matrix
    - Each row is an edge, each column is a vertex
    - Graph Laplacian, L(G) = C' * C
        - Diagonals: Count incident edges (always 1)
        - Off-diagonals: Says edges are adjacent
        - Tells us something about the undirected form of the original graph; we lose the direction information when calculating the Laplacian
    - Graph Laplacian, L(G) = C' * C = D - W
        - D is the degree of each vertex (along the diagonal)
        - W is the adjacency matrix marking all the edges



Incidence Matrix Representation

Graph Laplacian

1. Compute the graph Laplacian for the following graph
    - Matrix should be symmetric due to the definition of the Laplacian



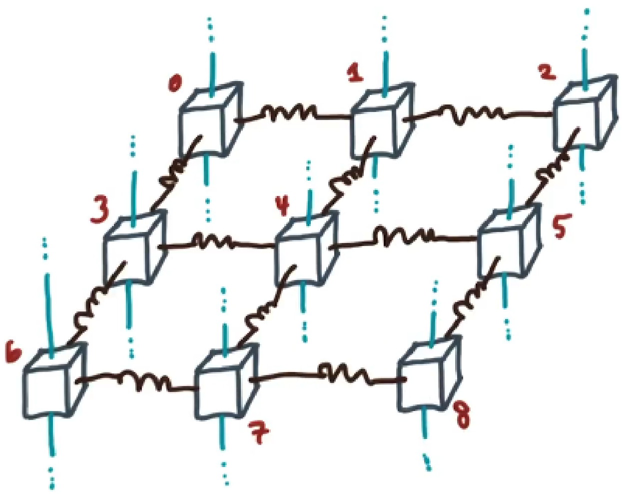
Laplacian Quiz 1

Spectral Partitioning - Part 2: Springs Fling

1. If we imagine a series of weights connected with springs displaced from a common point x, it provides some physical intuition for the Laplacian
    - Force on one weight is proportional to the displacement of the adjacent weights
    - This corresponds to the graph Laplacian for a line graph

A 2D Laplacian

1. What is the graph Laplacian for a group of 9 weights connected in the following way:

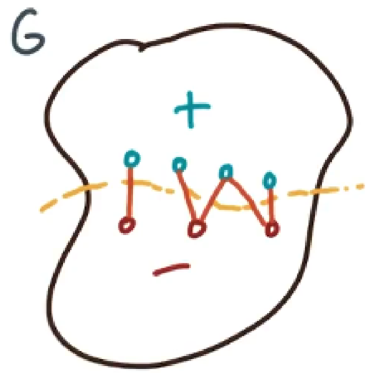


	0	1	2	3	4	5	6	7	8
0	2	-1		-1					
1	-1	3	-1		-1				
2		-1	2			-1			
3	-1			3	-1		-1		
4		-1		-1	4	-1		-1	
5			-1		-1	3			-1
6				-1			2	-1	
7					-1		-1	3	-1
8						-1		-1	2

Laplacian Quiz 2

Spectral Partitioning - Part 3: Algebraic Connectivity

1. Factoids
   - Laplacian L(G) is symmetric
   - L(G) has real-valued, non-negative eigenvalues and real-valued, orthogonal eigenvectors (not complex)
     - Eigenvalues and eigenvectors are pairs; multiplying L(G) by its eigenvector gives a scaled version of the eigenvector. The scaling factor is the eigenvalue
     - L(G) * Q = Q * Y where Q and Y are matrices
     - Columns of Q are the eigenvectors and diagonal entries of Y are the eigenvalues
     - Convention: Assume we can sort the eigenvalues
   - G has k connected components if and only if the k smallest eigenvalues are identically 0
     - Spectrum of L(G) tells us something about the connectivity of G
   - Let V+ be the number of vertices in the positive section and V- be the number of vertices in the negative section
     - Define x such that xi is 1 if i is in V+ and -1 if i is in V-
     - Then, the number of cut edges = 1/4 * x' * L(G) * x



let $V_+ \equiv \{\text{vertices in "+"}\}$

$V_- \equiv \{\text{vertices in "−"}\}$

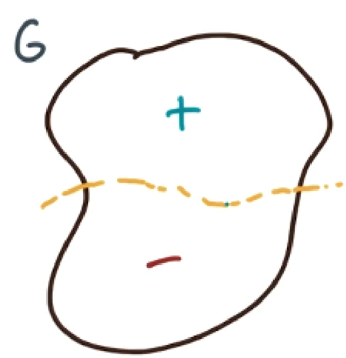
let $\vec{x} \equiv \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$ s.t. $x_i = \begin{cases} +1 & \text{if } i \in V_+ \\ -1 & \text{if } i \in V_- \end{cases}$

Fact 4) # of cut edges $= \frac{1}{4} x^T L(G) x$

Spectral Partitioning

Counting Edge Cuts

Quiz! Counting Edge Cuts



Your task:
Fill in the boxes!

$$x^T L(G) x = \sum_{i,j} l_{ij} x_i x_j$$

$$= \sum_{i=j} l_{ij} x_i x_j \quad = l_{ii} \, x_i^2 \quad \boxed{d_i} \; \boxed{+1}$$

$$+ \sum_{\substack{i,j \in V_+ \\ i \neq j}} l_{ij} x_i x_j \quad \boxed{-1} \; \boxed{1} \; \boxed{1} \quad + \sum_{\substack{i,j \in V_- \\ i \neq j}} l_{ij} x_i x_j \quad \boxed{-1} \; \boxed{1} \; \boxed{1}$$

$$+ \sum_{\substack{i \in V_+ \\ j \in V_-}} l_{ij} x_i x_j \quad \boxed{-1} \; \boxed{1} \; \boxed{-1} \quad + \sum_{\substack{i \in V_- \\ j \in V_+}} l_{ij} x_i x_j \quad \boxed{-1} \; \boxed{-1} \; \boxed{1}$$

Laplacian Quiz 3

Spectral Partitioning - Part 4

1. Start with a graph G = (V, E)
2. Construct its Laplacian L(G) = D - W
3. Suppose we have a partitioning of the vertices V = V+ U V-
4. Translate this into a partition vector xi = {+1 if i in V+, -1 if i in V-}

5. If we want to minimize the cut edges, we need to pick an x to minimize 1/4 * x' * L(G) * x
   - Want: min(1/4 * x' * L(G) * x)
     - sum(xi) = 0 (same number of vertices in each partition)
     - xi = +1 or -1
   - These constraints make the problem NP-complete
     - If we relax the constraint that we assign a +1 or -1 to every vertex, we can use the Courant-Fisher Minimax theorem
     - This says that the vector that minimizes this problem is q1, which is the eigenvector corresponding to the second smallest eigenvalue of L(G)
   - Choosing x to be q1 gives us a lower bound, but how do we take q1 and turn it into a partition vector?
     - Based on the spring model, the second smallest eigenvector has some sine-like shape where half the values are positive and half are negative
6. Spectral partitioning algorithm:
   - Create L(G)
   - Compute (Y,q1) eigenpair of L(G)
   - Choose x(i) = sign(q1(i))
7. Spectral partitioning works very well for planar graphs

Conclusion

1. Graph partitioning is NP-complete
   - Want algorithms with good heuristics
2. Heuristics:
   - Multilevel partitioning (divide and conquer)
   - Exploiting special structure (planarity)
   - Improvement techniques (Kernighan-Lin)
3. Spectral partitiong: Exploits relationship between graphs and matrices
   - K-L and spectral partitioning are very expensive relative to the motivation, which was a relatively cheap BFS