Distributed Dense Matrix Multiply

Introduction

1. Supercomputers are ranked using a matrix multiply benchmark
   - Matrix multiply is relatively easy to analyze

Matrix Multiply Basic Definitions

1. C <- C + A * B
   - C is m by n
   - A is m by k
   - B is k by n
   - Compute dot product between rows of A and columns of B and sum

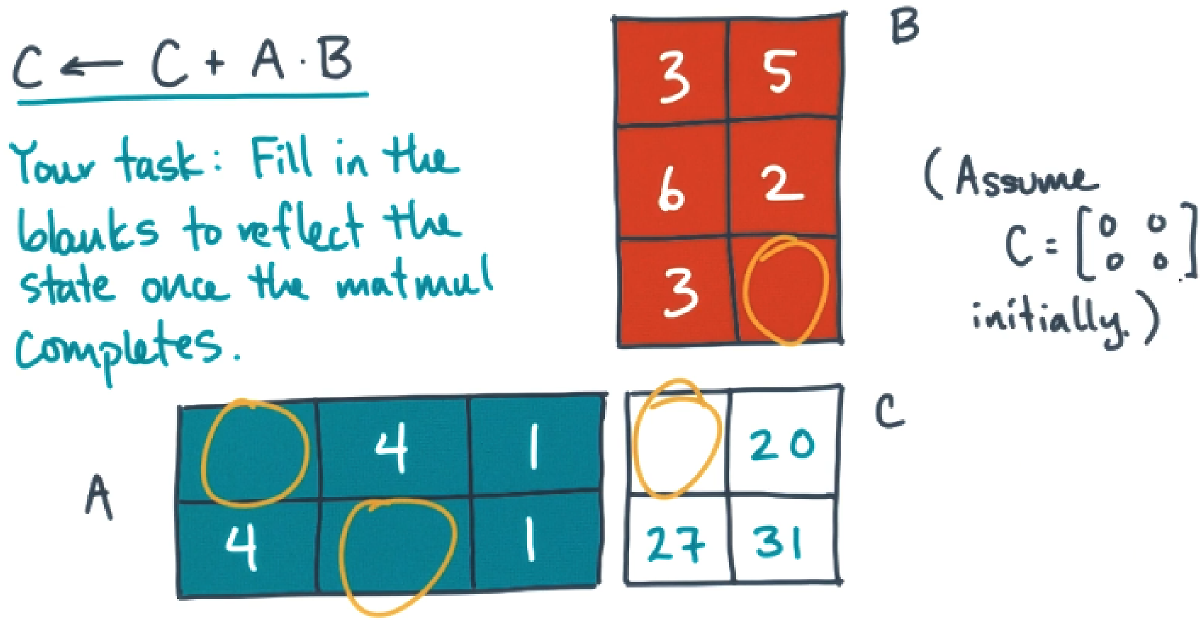
```
for i <- 1 to m do
    for j <- 1 to n do
        for l <- 1 to k do
            C[i,j] <- C[i,j] + A[i,l] * B[l,j]
```

2. Complexity
   - T(m,n,k) = O(mnk) = O(n^3)
   - W(n) = O(n^3)
   - D(n) = O(log(n))
   - The first two loops can be converted to parfor loops
   - You can block the computation to improve cache coherence

```
for i <- 1 to m do
    for j <- 1 to n do
        let T[1:k] = temp array
        for l <- 1 to k do
            T[l] <- A[i,l] * B[l,j]
        C[i,j] <- C[i,j] + reduce(T[:])
```

Definitions Check

1. Determine the state of matrices A, B, and C



Definitions Quiz

2. A = [[1 4 1]; [4 2 1]];
3. B = [[3 5]; [6 2]; [3 7]];
4. C = [[30 20]; [27 31]];

A Geometrical View

1. You can think of the matrix multiply computation as a cube that is m by n by k
2. For any integer cube of points, suppose I give you a subset of its surfaces Sa, Sb, Sc
   - There may be some volume of intersection in the interior, I
   - I <= sqrt(Sa * Sb * Sc)
   - Loomis and Whitney, 1949

Applying Loomis Whitney

1. Consider the following:
   - Sa: Some 3x5 piece of A
   - Sb: Some 5x4 piece of B
   - Sc: Some 2x2 piece of C
2. The number of multiplies is between X and Y inclusive
   - Maximum is sqrt(((3 * 5) * (5 * 4) * (2 * 2)) = 34
   - Minimum is 0 if slices don't intersect

1D Algorithms

1. Assume A,B,C are n x n and P | n

```
let Ahat[1:n/p][1:n] = local part of A
let Bhat[1:n/p][1:n] = local part of B
let Chat[1:n/p][1:n] = local part of C
let BO[1:n/p][1:n] = temporary storage
let rnext <- (RANK+1) % P
let rprev <- (RANK+P-1) % P
```

```
for L <- 0 to P-1 do
    Chat[:][:] += Ahat[:][L] * Bhat[L][:]
    sendAsync(Bhat -> rnext)
    recvAsync(B0 <- rprev)
    wait(*)
    swap(Bhat, B0)
```

1D Algorithm Cost Part 1

1. tau = time per FLOP (1 multiply or 1 add)
2. Tcomp(n;P) = 2 * tau * n^3 / P
3. What is Tnet(n;P)? Use a for alpha and b for beta
   - Each iteration sends n/p * n words = n^2 / P
   - There are P rounds of communication
   - Tnet(n;P) = a * P + b * n^2

1D Algorithm Cost Part 2

1. T1D(n;P) = 2 * tau * n^3 / P + a * P + B * n^2
2. How can we rearrange the statements in the body of the loop to get a factor of 2x improvement in the best case?
   - sendAsync(Bhat -> rnext)
   - recvAsync(B0 <- rprev)
   - Chat[:][:] += Ahat[:][L] * Bhat[L][:]
   - waitAll()
   - swap(Bhat, B0)
   - This overlaps the computation and communication
     - T1D(n;P) = max(2 * tau * n^3 / P, a * P + B * n^2)
     - a + b <= 2 * max(a,b)

Efficiency and the 1D Algorithm

1. Speedup: S1D(n;P) = T(n) / T1D(n;P)
   - S1D(n;P) = max(1, a * P^2 / (2 * tau * n^3) + B * P / (2 * tau * n))
   - n = omega(P) for this system to be efficient
     - This is called the isoefficiency function
2. Speedup / P = Parallel efficiency
   - A parallel system is efficient if its parallel efficiency is a constant
     - This means the systems scales well as P grows
     - Otherwise, we see diminishing returns as we increase the parallelism of the system
3. Temporary storage: M(n;P) = (3 + 1) * (n/p) * n = 4 * n^2 / P

Isoefficiency

1. Consider a tree-based all-to-one reduce
   - Ttree(n;P) = tau * n * log(P) + a * log(P) + B * n * log(P)
2. Which of the following best describes the isoefficiency function of a tree-based all-to-one reduce?
   - log(P)
   - O(P)
   - P * log(P)
   - P^2
   - none of these (true)
3. Effiency: E(n;P) = S(n;P) / P = T(n) / (P * T(n;P))
   - E(n;P) = (tau * n * P) / P * (tau * n * log(P) + a * log(P) + B * n * log(P))

- $E(n;P) = 1 / ((1 + P/tau) * log(P) + (a/tau) * (log(P)/n))$
  - Because the (1 + P/tau) * log(P) term scales with P, there is no value of n that causes this to tend to 0

A 2D Algorithm SUMMA

1. Intuitively, a 2D mesh or torus should be a better topology for matrix multiply
2. SUMMA gives each node a block of the output matrix C to update
   - Then, we provide a row of A of height s and a column of B of height s to each node (call this strip index l)
     - Owner of strip can simply broadcast
   - All processes execute the following for loop

```
for l <- 1 to n/s do
    broadcast(row, owner)
    broadcast(col, owner)
    matmul
```

3. SUMMA complexity
   - Tsumma(n;P,s) = n/s * (2 * tau * n^2 * s / P) + Tnet(n;P,s)
   - Tsumma(n;P,s) = 2 * tau * n^3/P + Tnet(n;P,s)

SUMMA Communication Time

1. Choose the best options for each term to satisfy the Tnet complexity
   - a * n / s
     - log(P)
     - sqrt(P)
     - P
   - B * n^2 / sqrt(P)
     - 1
     - log(P)
     - sqrt(P)
2. The answer depends on how the broadcast is implemented (bucket vs tree)
   - Ttree = O(a * log(P) + B * m * log(P))
     - log(P), log(P)
   - Tbucket = O(a * P + B * m)
     - P, 1

Efficiency of a 2D SUMMA

1. Is the 2D SUMMA scheme intrinsically more scalable than the 1D block-row scheme?
   - Quite possibly
2. Consider the efficiency with a tree-based broadcast
   - Etree = 1 / (1 + (aPlog(P))/(2tau * n^2) + (Bsqrt(P)log(P)/(2tau * n))
   - Isoefficiency function: ntree(P) = O(sqrt(P) * log(P))
     - n1D(P) = O(P)
     - nbucket(P) = O(P^(5/6)) (this trades lower communication volume for higher latency)

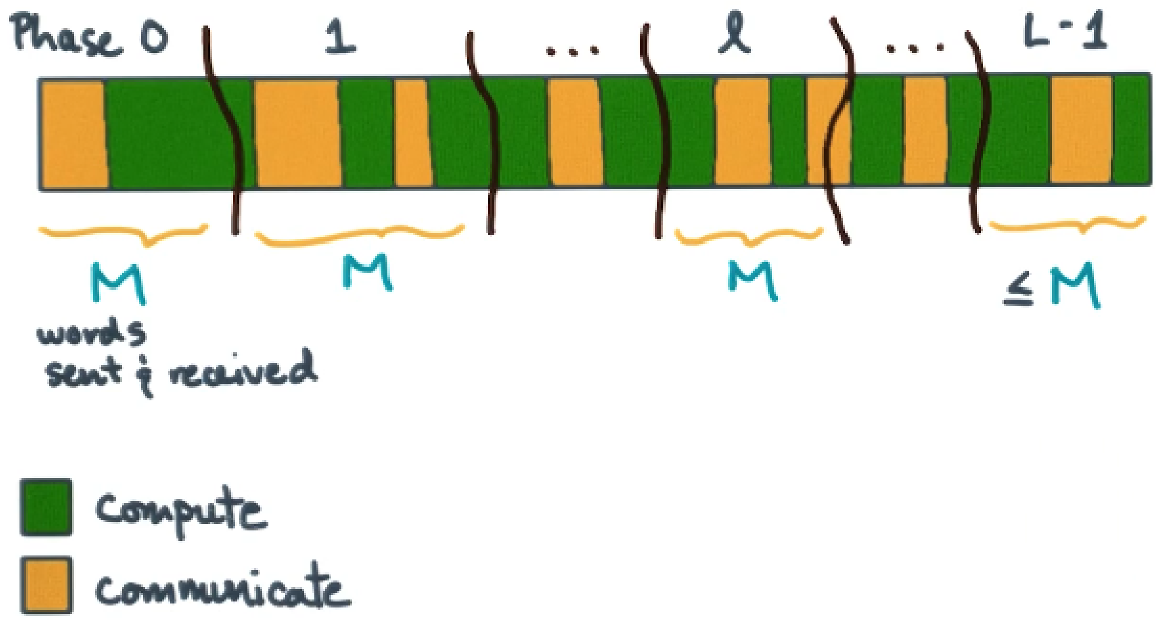
SUMMA Memory

1. How much memory does SUMMA need compared to the 1D scheme?
   - More than 1D
   - Less than 1D
   - Same as 1D
2. The size of additional memory depends on s, so it could be any of them

- Msumma = 3n^2/P + 2ns/sqrt(P)
  - Each node must store one block of A, B, C
  - Must also store a row and column of size ns/sqrt(P)
- SUMMA is the gold standard for dense matrix multiply due to the simplicity of the algorithm and tuning parameter that allows the tradeoff between time and storage

A Lower Bound on Communication

1. Consider a network of P nodes connected with some topology
   - For one specific node i, how many words must i communicate?
     - W multiplies, M words of memory
   - A node will alternate between periods of communication and computation
     - Sa, Sb, Sc: Set of elements of A seen in this phase
     - Sa <= 2 * M
     - Maximum multiplies per phase: sqrt(Sa * Sb * Sc) <= 2 * sqrt(2) * M^(3/2)
   - How many phases, L, does the computation take?
     - L >= # full phases
     - L >= floor(W / Max multiplies per phase) = W/(2 * sqrt(2) * M^(3/2))-1
   - Lower bound of transfers is number of phases times M
     - # Words communicated by 1 node >= W/(2 * sqrt(2) * sqrt(M)) - M
   - Number of multiplies W >= mnk / P
2. Lower bound on volume of communication by one node
   - # words >= n^3 / (2 * sqrt(2) * P * sqrt(M)) - M
   - M = O(n^2/P)
   - # words >= n^2 / sqrt(P)
   - Tnet(n;P) >= a + B * n^2/sqrt(P)
     - What is the factor for the alpha term?



Lower Bound on Communication

A Lower Bound on Communication Quiz

1. What is the lower bound on the number of messages a node must send?
   - n^2/sqrt(P) is the minimum volume sent by a node
   - M(n;P) = O(n^2/P) is the largest message a node can send
   -

     # messages = n^2/sqrt(P) / M(n;P) = O(sqrt(P))

Matching (Or Beating!) The Lower Bounds

1. SUMMA is off by a factor of log(P) in the alpha term
2. Cannon's algorithm beats SUMMA because it has a communication time that exactly matches the lower bound (1969)
   - Isn't practical to implement
3. The lower bound analysis assumes M = n^2/P
   - This assumption relates to distributing surfaces of the cube across nodes
   - If we distribute the volume instead of the surfaces (2D vs 3D), can we duplicate some data and reduce communication
4. 3D algorithm
   - M3D = M2D * P^(1/3)
   - T3Dnet = T2Dnet / P^(1/3)
   - Review this before the test (full vs partial replication)

$$T_{1D,net}(n;P) = \alpha \cdot P + \beta \cdot n^2$$

$$T_{SUMMA,net}(n;P,s) = \begin{cases} \alpha \frac{n}{s} \log P + \beta \frac{n^2}{\sqrt{P}} \log P & \text{(tree)} \\ \alpha \frac{n}{s} \sqrt{P} + \beta \frac{n^2}{\sqrt{P}} & \text{(bucket)} \end{cases}$$

$$1 \leq s \leq \frac{n}{\sqrt{P}}$$

$$\geq \alpha \sqrt{P} \log P + \beta \frac{n^2}{\sqrt{P}}$$

$$T_{lower}(n;P) = \Omega\left(\alpha \sqrt{P} + \beta \frac{n^2}{\sqrt{P}}\right) \qquad \text{assume: } M = \Theta\left(\frac{n^2}{P}\right)$$

Lower Bounds

Conclusion

1. Matrix multiply lends itself to different analysis techniques (1D vs 2D vs 3D) as well as analyzing the lower bound on communication
   - Other algorithms aren't necessarily as easy to analyze
2. Supercomputers are tuned to do problems that are computation-intensive
   - More communication-intensive algorithms might not scale as well