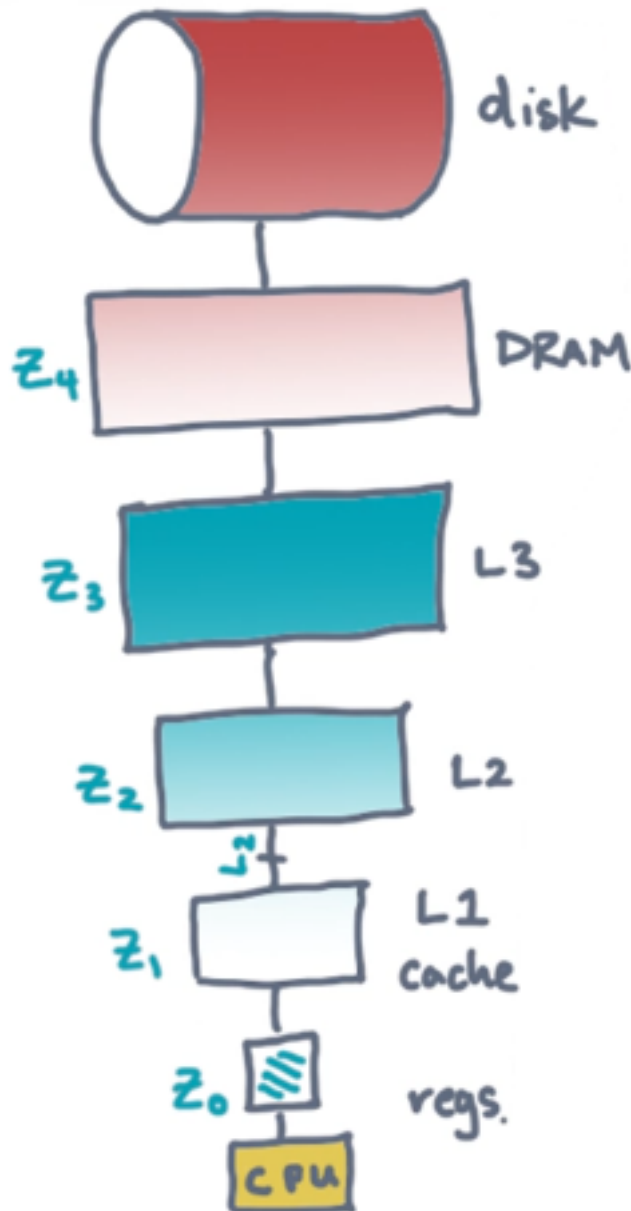


Basic Model of Locality

Introduction

1. Real machines have memory hierarchies
 - As we get closer to the processor, memory gets faster but also smaller
2. Usual way of analyzing algorithms doesn't consider memory hierarchy
 - Need to consider memory for best performance



Memory Hierarchy

A First, Basic Model

1. von Neumann architecture has a processor connected to main memory (slow)
 - In between processor and main memory, there's a faster memory
 - The size of this memory is Z words
2. Rules:
 - Local data rule: Processor may only compute on data in fast memory
 - Block transfer rule: Slow-fast transfers in blocks of size L (words)
 - Can't load a single address, need to consider how data is aligned
3. Costs:
 - Work: $W(n) = \#$ of computation operations
 - Transfers: $Q(n;Z,L) = \#$ of L -sized slow-fast transfers (loads/stores)
 - Consider this the I/O complexity
4. Example:
 - Reduction (sum all elements in an array)
 - $W(n)$ is greater than or equal to $n-1$ additions ($\Omega(n)$)
 - $Q(n;Z,L)$ is greater than or equal to $\text{ceil}(n/L)$ transfers ($\Omega(n/L)$)
 - I/O complexity doesn't depend on Z
 - Doesn't reuse data, which is bad

Two Level Memories

1. Which of the following pairs are examples of two-level (slow+fast) memories?
 - Hard disk + main memory (true)
 - L1 cache + CPU registers (true)
 - Tape store + hard disk (true)
 - Remote server RAM + local server RAM (true)
 - The Internet + your brain (true)

Alignment

1. How many transfers are necessary in the worst case, assuming nothing about alignment?
 - $Q(n;Z,L)$ is less than or equal to $\text{ceil}(n/L) + 1$
 - Suppose $n=4$, $L=2$
 - If n is aligned on a word boundary, we need two transfers ($\text{ceil}(n/L)$)
 - If n is not aligned on a word boundary, we need one extra transfer
 - Typically ignore this, especially when $n \gg L$

Minimum Transfers to Sort

1. Give a simple (trivial) lower bound on the asymptotic number of transfers when sorting an array of elements with a comparison-based algorithm
 - $W(n) = \Omega(n \log n)$
 - $Q(n;Z,L) = \Omega(\text{ceil}(n/L))$
 - Must read each element at least once, so the algorithm is bounded by $\text{ceil}(n/L)$
 - Actual answer is $(n/L) \log(n/L) / \log(Z/L)$

Minimum Transfers to Multiply Matrices

1. $C = \text{MATMUL}(A, B)$ where A, B, C are $n \times n$
2. Give a simple (trivial) lower bound on the asymptotic number of transfers
 - $W(n) = O(n^3)$ (non-Strassen)
 - $Q(n;Z,L) = \Omega(\text{ceil}(n^2/L))$
 - Must touch all n^2 elements divided by the block size
 - Tighter lower bound is $\Omega(n^3/L/\sqrt{Z})$

I/O Example Reduction

1. Consider the example of summing all elements in an array (reduction)

```
int s = 0; // local
for(int i = 0; i < n-1; i++)
    s = s + x[i];
```

2. Written more explicitly in terms of transfers:

```
int s = 0; // local
for(int i = 0; i < n-1; i++)
{
    int Lhat = min(n, i+L-1); // local, handle special case
    int y[0:Lhat-1] = X[i:(i+Lhat-1)]; // Lhat <= L
    for(int j = 0; j < Lhat - 1; j++)
        s = s + y[j]
}
```

2. Observations:
 - Painful, but clear
 - Yes, caches exist, but aren't sufficient to guarantee high performance

Matrix Vector Multiply

1. Consider a matrix-vector multiply algorithm $y = A * x$
 - A is arranged in column-major order ($A[i,j] = A[i + j * n]$)
 - i is number of rows, j is number of columns

2. Which of the following does fewer transfers?

- Algorithm A:

```
for(int i = 0; i < nrows; i++)
    for(int j = 0; j < ncols; j++)
        y[i] += A[i,j] * x[j]
```

- Algorithm B:

```
for(int j = 0; j < ncols; j++)
    for(int i = 0; i < nrows; i++)
        y[i] += A[i,j] * x[j]
```

3. Assumptions:

- $Z = 2n + O(L)$
- L divides n ($L \mid n$)
- x, y, and A are aligned on word boundaries (L)
 - Don't need to worry about alignment issues

4. Algorithm B does fewer transfers because it iterates over rows in the inner-most loop

- Algorithm A: $Q(n;Z,L) = 3 * n / L + n^2$
- Algorithm B: $Q(n;Z,L) = 3 * n / L + n^2 / L$

5. In the sequential model, these algorithms look identical

- Consider a fully-associative cache; will this solve the issue with algorithm A

Algorithmic Design Goals

1. Work optimality: Two-level algorithm should do the same asymptotic work as the RAM algorithm
 - $W(n) = \theta(W'(n))$
2. High computational intensity: Ratio of work to words transferred
 - Maximize $I(n;Z,L) = W(n) / (L * Q(n;Z,L))$
 - Intensity has units of operations per word
 - Measures data reuse (more operations per words in fast memory)
 - Can't sacrifice work optimality in favor of intensity

Which is Better?

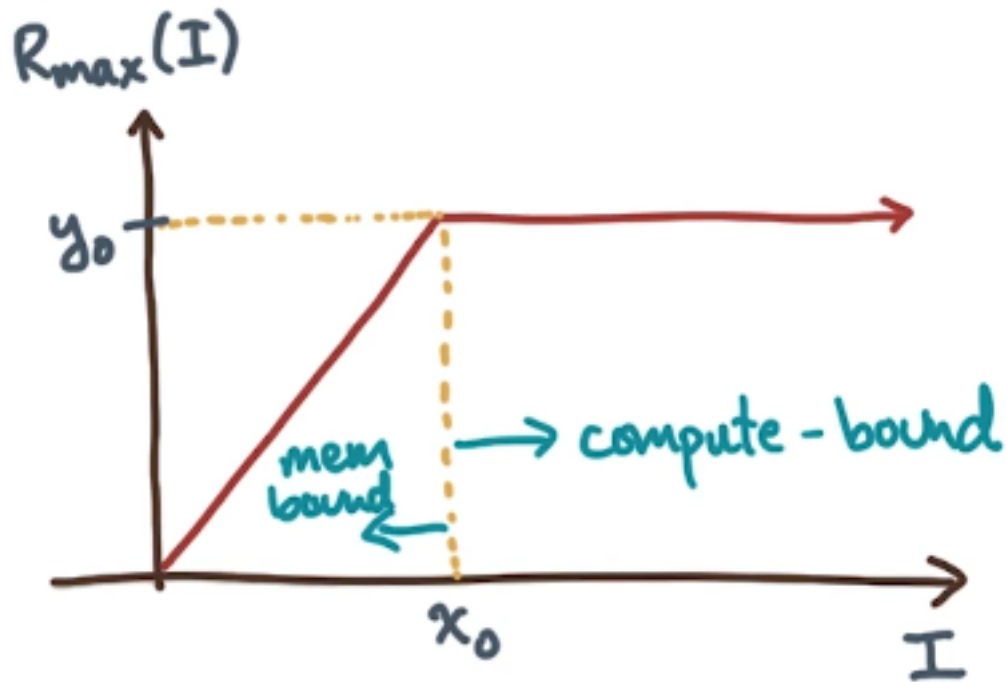
1. Consider the following algorithms:
 - Algorithm 1:
 - $W1(n) = \text{theta}(n)$
 - $Q1(n;Z,L) = \text{theta}(n/L)$
 - Algorithm 2:
 - $W2(n) = \text{theta}(n \log n)$
 - $Q2(n;Z,L) = \text{theta}(n / (L * \log Z))$
2. Which algorithm is better?
 - Insufficient information: We want low work and high intensity
 - $I1 = \text{theta}(1)$
 - $I2 = \text{theta}(\log n * \log Z)$
 - Algorithm 1 does lower work while algorithm 2 has higher intensity

Intensity - Balance and Time

1. Time to complete (T_{comp}) = $\tau * W$
 - τ = time to complete an operation = time / op
2. Time to execute Q transfers (T_{mem}) = $\alpha * L * Q$
 - α = time to move word between slow/fast memory = time / word
3. Minimum time to execute $T \geq \max(T_{\text{comp}}, T_{\text{mem}})$
 - Assumes perfect overlap
4. Refactor 3 such that $T \geq \tau * W * \max(1, (\alpha/\tau)/(W/(LQ)))$
 - $\tau * W$ is ideal computation time (assumes communication is free)
 - Second term is communication (transfer) penalty
 - Numerator (α/τ) is referred to as “machine balance”
 - How many operations can be executed in the time it takes to move a word of data?
 - Use B to refer to machine balance
 - Denominator is just intensity, has units operations/word
5. Minimum time to execute $T \geq \tau * W * \max(1, B/I)$
6. Maximum time to execute $T \leq \tau * W * (1 + B/I)$
7. Normalized performance $R = \tau * W' / T$
 - W' is work of best sequential algorithm
 - $R \leq W' / W * \min(1, I/B)$
 - Measure of performance is inversely proportional to time
 - Higher values are better

Roofline Plots

1. $R_{\text{max}} = W' / W * \min(1, I/B)$



Roofline Plot

2. What are the values of x_0 and y_0 ?
 - $x_0 = B$
 - $y_0 = W_{\text{star}} / W$
3. If an algorithm is to the right of x_0 , we say it's compute-bound
4. If an algorithm is to the left of x_0 , we say it's memory-bound

Intensity of Conventional Matrix Multiply Part 1

1. Consider a non-Strassen matrix multiply algorithm

```

for(int i = 0; i < n-1; i++)
{
    // read A[i,:]
    for(int j = 0; j < n-1; j++)
    {
        // read C[i,j] and B[:,j]
        for(int k = 0; k < n-1; k++)
        {
            C[i,j] += A[i,k] * B[k,j]
        }
        // store C[i,j]
    }
}

```

2. Assumptions:
 - $L = 1$ word
 - $Z = 2n + O(1)$
3. What is the intensity of this algorithm?
 - $W(n) = n^3$
 - $Q(n;Z) = n^2 + 2 * n^2 + n^3$

- n^2 reads for A
 - n^3 reads for B
 - $2 * n^2$ reads/writes for C
 - $I(n;Z) = W(n) / Q(n;Z) = 1$
4. Only n^2 data, which suggests there's a factor of n available for reuse

Intensity of Conventional Matrix Multiply Part 2

1. Consider a matrix multiply algorithm where we load blocks of data instead of single elements

```
for i <- 0 to n-1 by b do
  for j <- 0 to n-1 by b do
    let Chat = b x b block at C[i,j]
    for k <- 0 to n-1 by b do
      let Ahat = b x b block at A[i,k]
      let Bhat = b x b block at B[k,j]
      Chat = Chat + Ahat * Bhat
    C[i,j] block <- Chat
```

2. Assumptions:
 - $L = 1$
 - $b \mid n$
 - $n \mid Z$
 - $Z = 3 * b^2 + O(1)$
3. What is the intensity of this algorithm?
 - $W(n) = n^3$
 - $Q(n,Z) = n^3 / b$
 - Intensity = $W(n) / Q(n,Z) = b = \sqrt{Z}$

Informing the Architecture

1. Suppose you have an efficient machine for a matrix multiply at a particular problem size.
 - If the machine balance doubles, by how much should the size of fast memory increase?
2. Fast memory size must increase by a factor of 4 because the intensity of a matrix multiply is \sqrt{Z}
 - $R_{\max} = W_{\text{star}} / W * \min(1, \sqrt{Z}/B)$
 - If B doubles, Z must increase by a factor of 4

Conclusion

1. The two-level model captures the most important effects of real memories, capacity and transfer size
 - Lots of research on locality-sensitive algorithms based on this model
2. To exploit a memory hierarchy algorithmically, organize data accesses to maximize reuse
 - For an algorithm to scale well to future memory hierarchies, you want intensity to at least match, but preferably exceed, the machine balance