

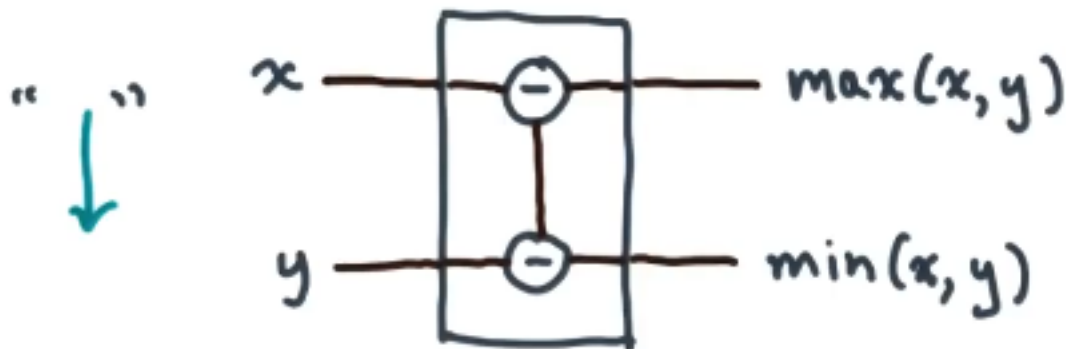
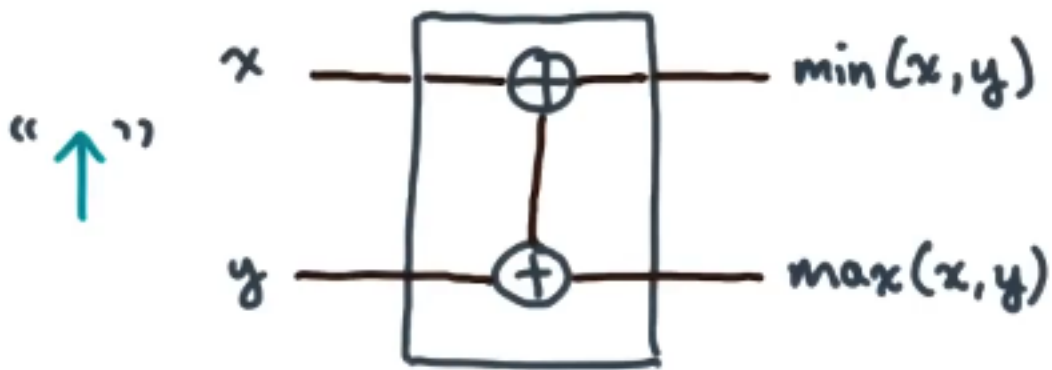
# Comparison-based Sorting

## Introduction

1. Covers parallel algorithms for sorting in the dynamic multi-threading model
  - Includes the idea of a sorting network

## Comparator Networks

1. Sorting network: Fixed circuit that sorts its inputs using a special gate called a comparator
  - Increasing/plus comparator puts the smaller of its inputs on the top wire
  - Decreasing/minus comparator puts the larger of its inputs on the top wire
2. Similar to work and span, we can analyze a circuit by the number of operations it performs and the critical path length



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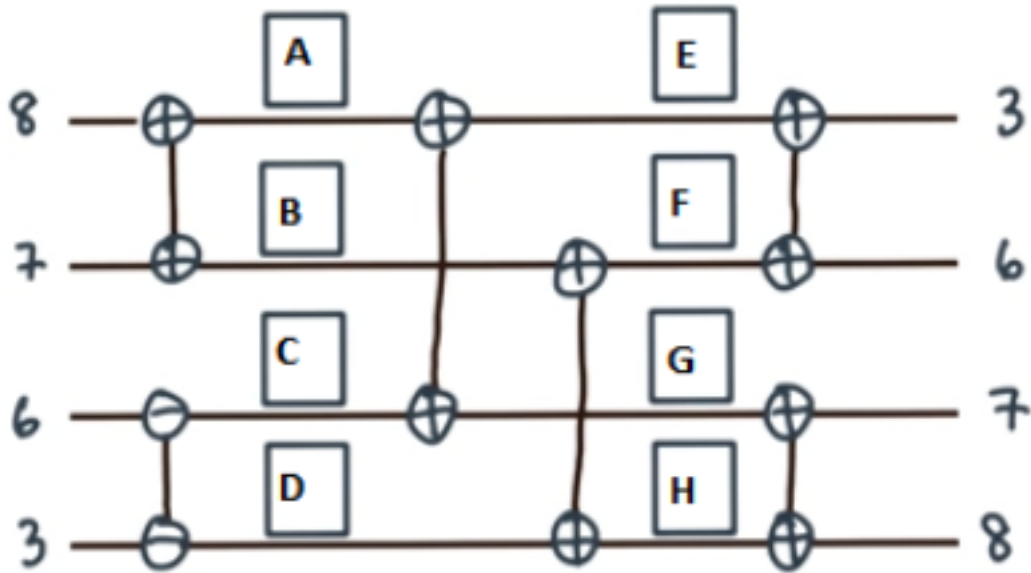
Comparator Networks

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## Sort 4 Values

1. Fill in the boxes with the values based on the wiring
  - A: 7
  - B: 8

- C: 6
- D: 3
- E: 6
- F: 3
- G: 7
- H: 8



### Comparator Quiz

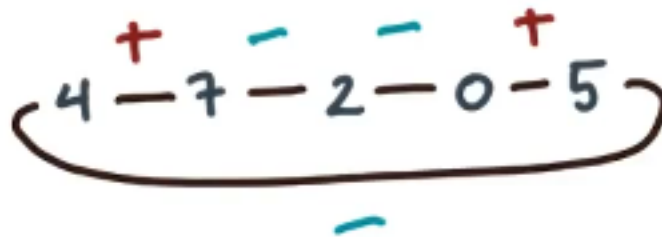
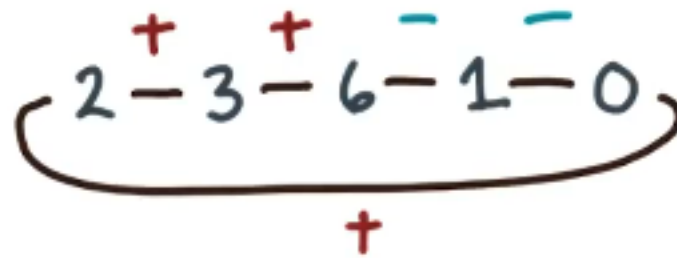
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## Bitonic Sequences

1. Bitonic sequence: Sequence of numbers which is monotonically increasing initially, then monotonically decreasing past some point
  - $(a_0, a_1, \dots, a_{n-1})$  is bitonic if:
    - $a_0 \leq a_1 \leq \dots \leq a_i$  AND
    - $a_{i+1} \geq \dots \geq a_{n-1}$
  - This is also true if the above condition holds after some circular shift

## Bitonic Sequences Quiz

1. Which sequence is bitonic?
  - 2, 3, 6, 1, 0 (yes)
  - 4, 7, 2, 0, 5 (no)



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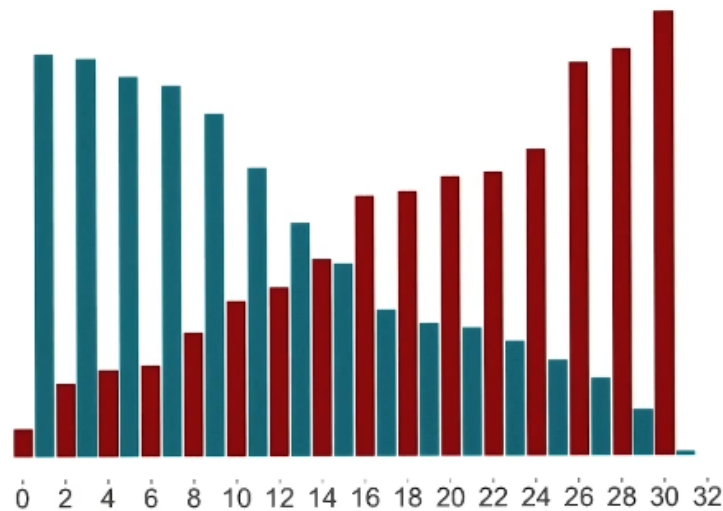
Bitonic Sequences Quiz

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### Bitonic Splits

1. Bitonic split: Pair the elements of a bitonic input sequence and then apply mins/maxes to pairs
  - Results in two bitonic subsequences
  - All elements of the max sequence are greater than or equal to all the elements of the minimum sequence

## Bitonic Splits



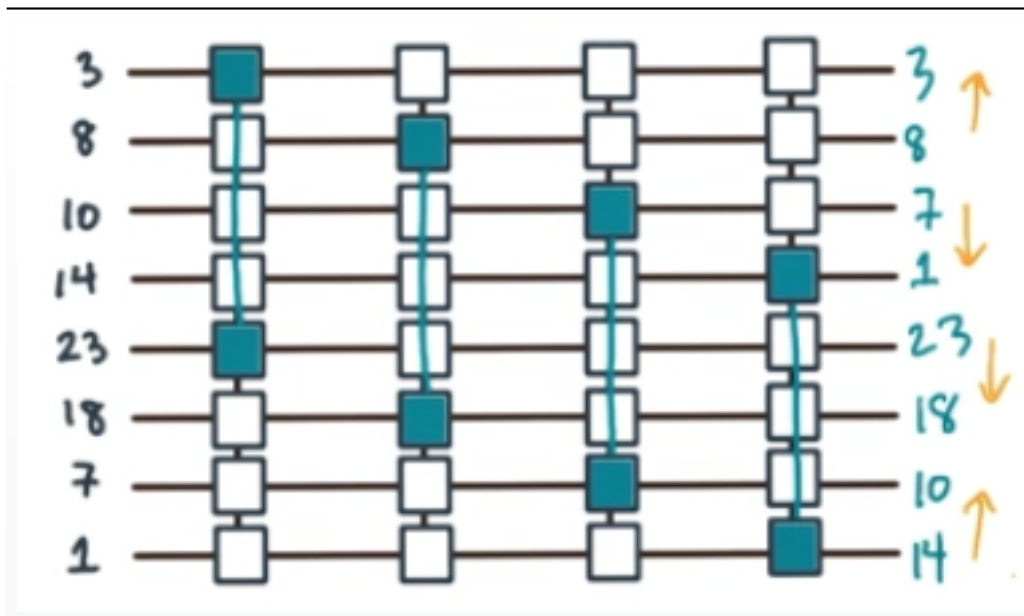
"Split"

$$\begin{aligned} & (a_0, a_{\frac{n}{2}}) \\ & (a_1, a_{\frac{n}{2}+1}) \\ & (a_2, a_{\frac{n}{2}+2}) \\ & \vdots \\ & (a_{\frac{n}{2}-1}, a_{n-1}) \end{aligned}$$

Bitonic Split

## Bitonic Split Quiz

1. Connect the comparators to implement a bitonic split



Bitonic Split Quiz

## Bitonic Splits: A Parallel Scheme

1. The following algorithm implements a parallel bitonic split:
  - Assume  $2 \mid n$

```
bitonicSplit(A[0:n-1])
{
    parfor i <- 0 to n/2-1 do
        a <- A[i]
        b <- A[i+n/2]
        A[i] <- min(a, b)
        A[i+n/2] <- max(a, b)
}
```

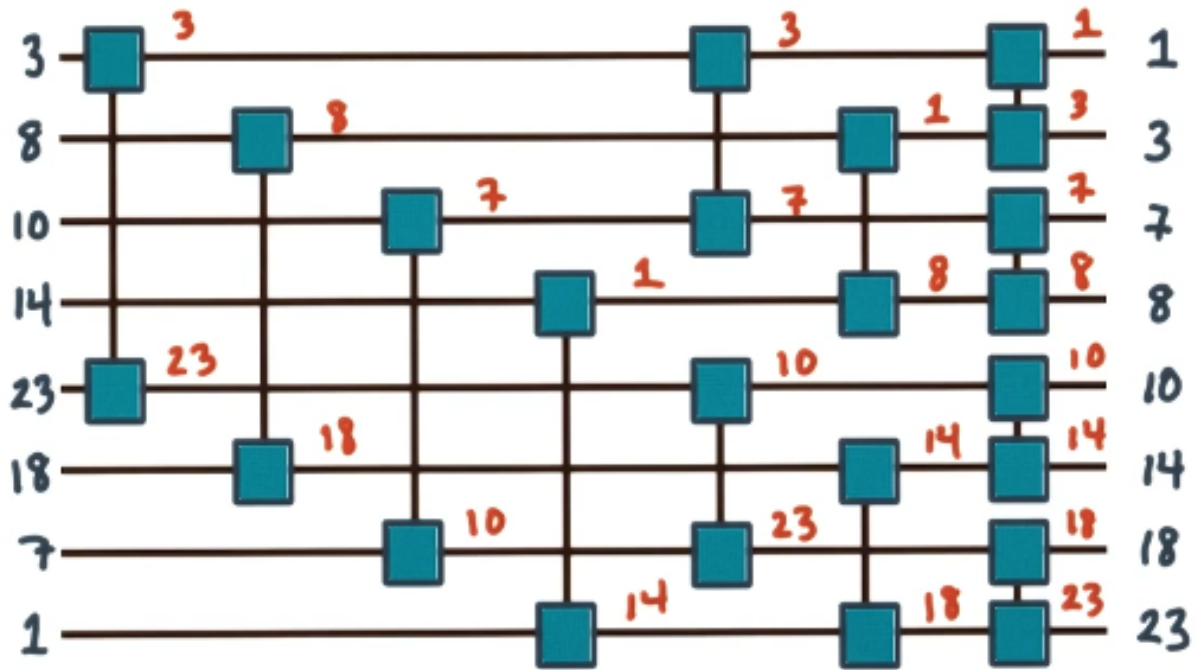
## Bitonic Merge

1. Bitonic split provides a simple divide-and-conquer framework for sorting
  - Recursively performing bitonic splits on each pair until no pairs remain will sort the list
  - Assume  $2 \mid n$
  - Because all the elements of one subsequence are less than or equal to all the elements of the other subsequence, we can spawn one of the merges

```
bitonicMerge(A[0:n-1])
{
    if n >= 2 then
        bitonicSplit(A[:])
        bitonicMerge(A[0:n/2-1]) // spawn
        bitonicMerge(A[n/2:n-1])
}
```

## Bitonic Merge Networks

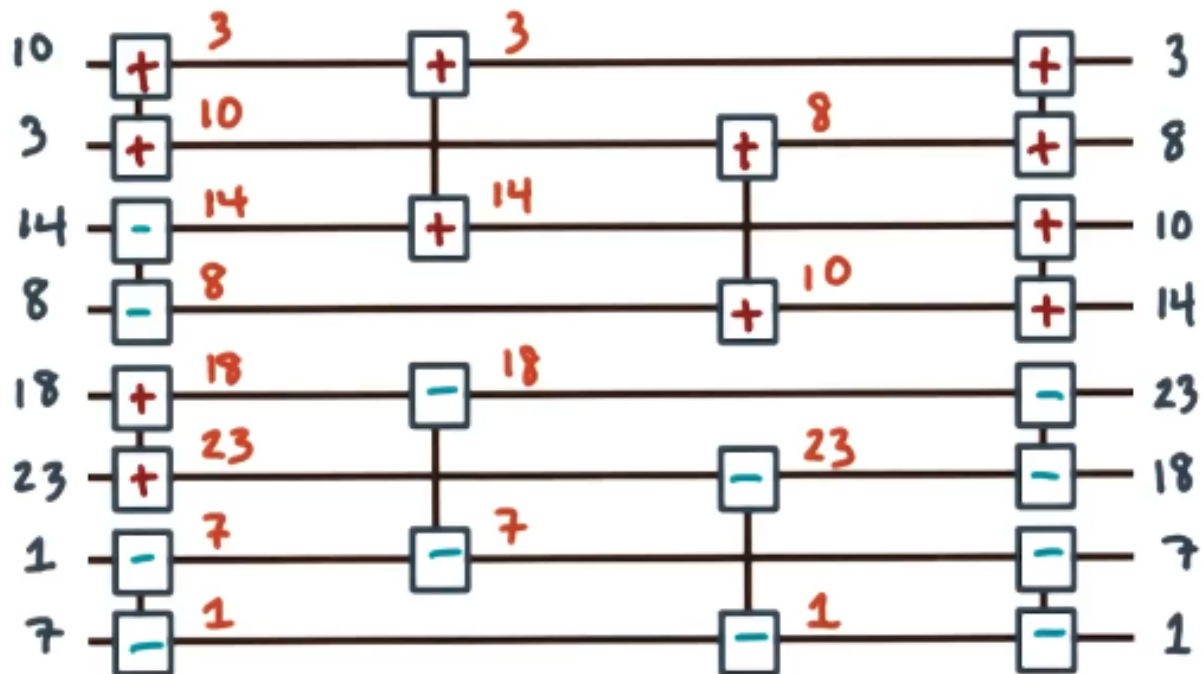
1. Connect the comparators to create a bitonic merge network



Bitonic Merge Quiz

## Generate a Bitonic Sequence

1. Connect the comparators to generate a bitonic sequence



Bitonic Sequence Quiz

## Bitonic Sort

1. First, generate a bitonic sequence
  - Assume  $2 \mid n$

```
genBitonic(A[0:n-1])
{
    if n >= 2 then
        spawn genBitonic(A[0:n/2-1])
        genBitonic(A[n/2:n-1])
        sync
        spawn bitonicMerge+(A[0:n/2-1])
        bitonicMerge-(A[n/2:n-1])
}
```

2. To sort, do the following:

```
bitonicSort(A[0:n-1])
{
    genBitonic(A[:])
    bitonicMerge+(A[:])
}
```

3. The work and span are:
  - $W(n) = O(n * (\log(n))^2)$
  - $D(n) = O(\log(n)^2)$
  - The span is polylogarithmic which is good, but this algorithm is not work-optimal
    - Comparison-based sorts can be completed in  $O(n * \log(n))$

## Conclusion

1. Bitonic sort has a fixed, regular parallel structure which lends itself to implementation on an FPGA
  - Also maps well to fixed data-parallel hardware (SIMD or graphics co-processors (GPUs))
2. Downside is that it's not work-optimal, even when restricted to the class of comparison-based algorithms
  - Have to make an engineering tradeoff based on the platform and scale you are interested in