# 3F2 Lab Report — Pendulum Controller

## 1 Abstract

This lab focused on the feedback control of a carriage with a pendulum by using state feedback. The carriage was controlled in two different modes — one which had the pendulum pointing downwards and the other which inverted the pendulum and kept it upright. Four states were used to model the system, carriage position and velocity and pendulum angular position and angular velocity. Four feedback gain parameters (from the $K$ matrix) were modified, each relating to one of the states and the effects on the system were observed. Limit cycles and pole placement strategies were also investigated.

## 2 Crane control

### 2.1 Carriage Controller

The aim of this section was to only control the motion of the carriage and ensure that it followed the reference signal while compensating for the swinging pendulum to ensure a fast, accurate response. This was done by manually tuning $p_2$, $p_3$ and $p_4$.

Initially, $p_1 = 0.35$, $p_3 = p_4 = 0$ and $p_2$ was adjusted manually to achieve a critically damped response. I found setting $p_2 = 0.13$ gave it the fastest response without any overshoot or oscillation. For this value, the step response appeared critically damped and it took approximately 0.3 s to reach the desired position. The next step involved manually adjusting $p_3$ and $p_4$ to reduce the oscillations in the pendulum. By observing the oscillations on the data logger, I found the oscillations were most reduced by increasing $p_3$ to its maximum value of 1 and keeping $p_4 = 0$ as any increase in $p_4$ seemed to only increase the amplitude of oscillations. My best attempt is plotted in Figs 1 to 4 and are compared with the simulated values. The simulation provides a remarkably accurate prediction of the states except for not having the steady state error of the carriage position.

The pendulum took approximately 1.7 s for its oscillations to become small. The theoretical pole positions were $s_{1,2} = -10.27 \pm 38j$ and $s_{3,4} = -0.472 \pm 3.3j$. Both of these pairs were asymptotically stable. $s_{1,2}$ decayed faster but had more oscillation than $s_{3,4}$. This matched up with the results as $s_{3,4}$ would correspond to the carriage and $s_{1,2}$ to the pendulum. The pendulum clearly oscillates more but decays faster as can be seen in Fig 3.

### 2.2 Pole-placement

In order to control this system in a more sophisticated way than by manual experimentation, the closed loop poles were assigned to achieve a desired response and the gains were calculated from this. Initially, all poles were set to $-\omega_1 = -\sqrt{78.5}$ which gave the potentiometer settings $p_1 = 0.13$, $p_2 = 0.21$, $p_3 = 0.23$ and $p_4 = 0$. The step response of this system was very sluggish and reacted slowly to the desired signal. The theoretical poles obtained were $-12.5$, $-8.4 \pm 3j$ and $-6.2$.

To decrease the response time, I chose the poles to have values -15, -12 and $-11 \pm 11j$ as this was my best effort to make the real part of the poles more negative without requiring unobtainable potentiometer settings. This corresponded to potentiometer settings of $p_1 = 0.9$, $p_2 = 0.81$, $p_3 = 0.28$ and $p_4 = 0.67$. This gave the step response as shown in Figs 5 to 8. The response here was significantly faster in comparison to the previous case. The simulation predicts the carriage position and velocity well. The pendulum simulations are reasonable except that it underestimates the amount of oscillations that occur.

The final part of the crane control experiment was to vary $p_2$ till the system was bordering stability and instability. This occurred at $p_2 = 0.51$. The step response is shown in Figs 9 to 12. It can be seen that it is very oscillatory but decays after approximately 2 s. The model clearly fails in this situation to accurately predict the states. The fact that there are oscillations is predicted but the size and frequency are wrong. This may be because of the speed of the oscillations which causes Euler's method to rapidly diverge from the true system due to too large a time step in the simulation.

A comparison with the linear model derived in Appendix A of the handout will now be explored. The value of $k_2$ when stability is about to be lost is $0.51 \times 165 = 84.2$. The characteristic equation for the crane model is $d_c(s) = s^4 + (k_2 + k_4)s^3 + (k_1 + k_3 + \omega_0^2)s^2 + k_2\omega_1^2 s + k_1\omega_1^2$. Applying the Routh Hurwitz criteria gives the following inequalities to be satisfied for stability: $k_1 > 0$, $k_2 > 0$, $k_2 + k_4 > 0$, $k_1 + k_3 + \omega_0^2 > 0$ and $k_2^2(k_3 + \omega_0^2 - \omega_1^2) + k_2k_4(k_3 + \omega_0^2 - k_1) > k_1k_4^2$. Substituting $k_1 = 617p_1 = 555.3$, $k_3 = 1256p_3 = 352$, $k_4 = -126p_4 = -84.4$, $\omega_1^2 = 78.5$ and $\omega_0^2 = 103.3$ gives the inequalities $k_2 > 84.4$ and $376.8k_2^2 + 8440k_2 > 3955601$ which are dependent on $k_2$. Solving the quadratic inequality gives $k_2 > 91.87$ which absorbs the $k_2 > 84.4$ inequality. My observations suggested $k_2 > 84.2$ which matches up with the absorbed inequality as opposed to the quadratic inequality which might put doubt onto the theoretical values of $k_1$ and $k_3$. Both 91.87 and 84.2 are still close enough, however, to imply that the model is reasonable. From Fig 11, it can be seen that the observed frequency of oscillation is 4 Hz = 25 rad/s. The theoretical prediction uses the formula $\hat{\omega} = \sqrt{\frac{k_2}{k_2+k_4}}\omega_1 = 31.1$ rad/s in this case. This is within 25% of the actual value which is reasonable but not accurate enough for sophisticated applications.

# 3 Inverted Pendulum

## 3.1 No carriage feedback

The importance of carriage feedback was observed here. By setting $p_1$ and $p_2$ to zero and $p_3$ and $p_4$ to values appropriate for stabilising the pendulum, I observed that this led to an unstable system. A very small force was required on the pendulum to move the carriage. Without any feedback keeping the carriage in the same place, the carriage moved leading to the pendulum becoming unstable.

## 3.2 Pole Placement

The closed loop poles were placed at $-\omega_1 = -\sqrt{78.5}$ which gave potentiometer settings of $p_1 = 0.25$, $p_2 = 0.32$, $p_3 = 0.35$ and $p_4 = 0.28$. This led to the responses shown in Figs 13 to 16. It gave an oscillatory response which decayed slowly. As can be seen from the graphs, the simulation provides a very poor approximation to the model which may be due to the large oscilllations.

## 3.3 Limit Cycles and No Pendulum Feedback

For this section, the potentiometer settings were set to $p_1 = 0.23$, $p_2 = 0.50$, $p_3 = 0.63$ and $p_4 = 0.40$ which gave a stable and fast response. $p_2$ was then reduced until oscillations became large but constant in amplitude which occurred at $p_2 = 0.13$. This is known as a limit cycle — the system follows a periodic trajectory and does not deviate from it. The response is shown in Fig 17 and as can be seen, the simulation is fairly accurate in this case. The phase difference between the simulation and measurements occurred because the measurements were not started at a step input to the reference signal whereas the model is started at this point. The simulation did not capture the high frequency ripple occurring every half period in the pendulum velocity due to too large a time step.

The next step involved increasing $p_2$ until the system was almost unstable, then observing the step response. The system was found to be almost unstable at $p_2 = 0.81$. When a step input occurred, the system became unstable and operation was cut out. This occurred because the stability margins were too low with $p_2 = 0.81$ so a small disturbance led to a transition into the unstable region. The response is plotted in Figs 18 to 21. The simulation appeared to have provided a stable response which did not match up with the experiment. However, a simulation for a larger $p_2$ equal to 0.92 gave a closer match as shown in Fig 22.

The final test considered the effect on no pendulum feedback. In this setup, $p_1 = 0.25$, $p_2 = 0.32$, $p_3 = 0$ and $p_4 = 0$ which means the system is not adjusting according to the position of the pendulum. The observed motion was that the carriage was initially stable. However, as the pendulum dropped from its inverted position to pointing downwards, the carriage was disturbed and it jumped to the edge of the track and the operation was cut out. With no pendulum feedback, the carriage position is stable when the pendulum is stable but if the pendulum starts off inverted then the pendulum will be unstable and disturb the carriage out of its equilibrium position. The controller assumes linearised equations about equilibrium for the inverted pendulum configuration and this is no longer true when the pendulum drops, leading to instability.

## 4 Theoretical derivations

In this section, results from Appendix A will be verified. The first equation to verify is equation (18) on page 12. Writing this out as its comprising equations gives $\ddot{x} = (\omega_0^2 - \omega_1^2)L\phi + (u - f)$ and $L\ddot{\phi} = \omega_0^2 L\phi + (u - f)$ as the non trivial equations. We know equation (14) is true and can be linearised to equation (16) about the equilibrium giving $L\ddot{\phi} = \ddot{x} + g\phi$. By substituting in the equations from (18), (16) becomes $\omega_0^2 L\phi + (u - f) = (\omega_0^2 - \omega_1^2)L\phi + (u - f) + g\phi$. This can be simplified to $0 = -\omega_1^2 L\phi + g\phi$. But $\omega_1^2 = g/L$ so the equation becomes $0 = -(g/L)L + g$ which is $0 = 0$, hence the equation is consistent and (18) has been verified.

The examples paper question from last year is in Figure 3 in the handout. From this we can immediately see that d) matches with 5) as it has poles in the RHP so it must be unstable. Also, e) must correspond to 4) since the poles are very far to the left so the response must be very rapid. a) must match up to 2) since the large imaginary part leads to an oscillatory response which is seen in 2). Finally, b) would match to 3) and c) would match to 1) because c) has a pole closer to the RHP than the poles in b) so it would have a slower response than b).

Equation (28) in the handout is the closed loop characteristic polynomial $d_c(s)$ for the crane model and the roots of this equation correspond to the closed loop poles of the system. It is given by $\det(sI - A + \underline{BK})$ where $A$ and $\underline{B}$ are given in equation (12) in the handout and $\underline{K} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$. Hence, $sI - A + \underline{BK} = \begin{bmatrix} s & -1 & 0 & 0 \\ k_1 & s + k_2 & \omega_0^2 - \omega_1^2 + k_3 & k_4 \\ 0 & 0 & s & -1 \\ k_1 & k_2 & k_3 + \omega_0^2 & s + k_4 \end{bmatrix}$ which becomes $\begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -\omega_1^2 & -s \\ 0 & 0 & s & -1 \\ k_1 & k_2 & k_3 + \omega_0^2 & s + k_4 \end{bmatrix}$ by subtracting row 4 from row 2. The resulting matrix will have the same determinant. Evaluating the determinant from the third row gives $d_c(s) = s \begin{vmatrix} s & -1 & 0 \\ 0 & s & -s \\ k_1 & k_2 & s + k_4 \end{vmatrix} + \begin{vmatrix} s & -1 & 0 \\ 0 & s & -\omega_1^2 \\ k_1 & k_2 & k_3 + \omega_0^2 \end{vmatrix}$. Evaluating both $3 \times 3$ determinants from the first column gives $d_c(s) = s[s(s(s + k_4) + k_2 s) + k_1 s] + [s(s(k_3 + \omega_0^2) + \omega_1^2 k_2) + k_1 \omega_1^2] = s^4 + (k_2 + k_4)s^3 + (k_1 + k_3 + \omega_0^2)s^2 + k_2\omega_1^2 s + k_1\omega_1^2$ which is equation (28).

## 5 Conclusion

Overall, this lab has provided a good insight into how feedback gains relate to the operation of a control system. The importance of choosing appropriate poles to give a quick response was highlighted and the behaviour of systems near instability was observed. The Routh Hurwitz criteria for stability assuming a linear model matched reasonably well with the observed results on the non linear system. The non linear effects from friction such as limit cycles were observed where the system is periodic but stable and how a system with small stability margins may be stable initially but it could become unstable after being given a disturbance. The simulation seemed to predict the behaviour of stable systems well but failed to provide accurate predictions for unstable configurations.