# 3F3 Full Technical Report
# Digital Filters

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# 1 Abstract

This report provides a detailed analysis on experiments performed with digital filters. It covers experiments performed in DIGSIG which are compared with theoretical derivations. The phase responses of FIR and IIR filters are analysed in terms of pole-zero locations and windowing functions for a low pass filter their advantages and disadvantages are compared. The impulse response of IIR filters is theoretically derived and compared with graphical results for various pole locations. Finally, a notch filter is created and its performance in removing a sine wave from Gaussian noise is observed and explained.

# 2 Phase Responses

The phase response of a digital filter can be explained in terms of the pole-zero positions of the filters transfer function. We know that if we have the frequency response of a system $H(z)$ then $H(e^{j\omega T})$ provides us with the frequency response. Any transfer function can be decomposed into its poles and zeros such that $H(z) = G\frac{\prod_{q=1}^{N}(z-c_q)}{\prod_{q=1}^{M}(z-d_q)}$ where $c_q$ are the zeros $d_q$ are the poles, This gives the frequency response $H(e^{j\omega T}) = Ge^{j\omega(M-N)T}\frac{\prod_{q=1}^{N}(e^{j\omega T}-c_q)}{\prod_{q=1}^{M}(e^{j\omega T}-d_q)}$. An important point to note is that the phase of the product of two complex numbers is the sum of their phases, i.e. $\angle(AB) = \angle(A) + \angle(B)$. Thus, $\angle(H(e^{j\omega T})) = \omega(M-N)T + \sum_{q=1}^{N}\angle(e^{j\omega T} - c_q) - \sum_{q=1}^{M}\angle(e^{j\omega T} - d_q)$. This has an important geometrical interpretation which allows the phase response to be calculated more intuitively and can be summarised as follows:

1. Take the point $e^{j\Omega}$ and vary $\Omega$ from 0 to $\pi$.
2. Add up all the phases of the vectors going from the zeros to $e^{j\Omega}$ and subtract the phases of all the vectors going from the poles to $e^{j\Omega}$. Be certain to include the $M-N$ zeros at 0 if $M > N$ or the $N-M$ poles at 0 if $M < N$.
3. The total phase response at an angular frequency $\omega$ is given by the sum of phases as described above for $\Omega = \omega T$ where $T$ is the sampling period.

When the magnitude response is considered, poles and zeros at the origin can be ignored as they always have a distance of 1 to $e^{j\Omega}$ whereas with the phase response they are important and need to be considered.

## 2.1 First Order FIR

For a filter with a transfer function $H(z) = 1 - 0.9z^{-1}$ there is one zero at $z = 0.9$ and one pole at $z = 0$. As $\Omega$ increases from 0 to $\pi$, the pole adds a negative component and the zero adds a positive component. When $\Omega$ is small, the zero is closer to $e^{j\Omega}$ so it adds phase faster than the rate at which the pole subtracts phase. Thus, the overall phase response is positive and increasing. When $\Omega$ becomes large, the phase response is still positive because the angle contributed from the zero is larger in magnitude than that from the pole but the pole is subtracting phase faster, thus the phase response decreases. The phase response reaches its maximum when the angle from the zero to $e^{j\Omega} = \pi/2$ and at this point, using basic trigonometry, $\Omega = \arccos(0.9)$.

Using the cosine rule and the sine rule provides an expression for the phase response as follows:

$$\angle(H(e^{j\Omega})) = \begin{cases} -\Omega + \arcsin\left(\frac{\sin(\Omega)}{\sqrt{1+0.9^2-2\times0.9\cos(\Omega)}}\right) & \text{if } 0 \le \Omega \le \arccos(0.9) \\ -\Omega + \pi - \arcsin\left(\frac{\sin(\Omega)}{\sqrt{1+0.9^2-2\times0.9\cos(\Omega)}}\right) & \text{if } \arccos(0.9) < \Omega \le \pi \end{cases} \tag{1}$$

This phase response is in figure 1 in the short lab report.

## 2.2 Second Order FIR

With the second order FIR filter, we have a transfer function given by $H(z) = b_0+b_1z^{-1}+b_2z^{-2}$. To find the zeros of this transfer function, consider the quadratic equation $b_0z^2 + b_1z + b_2 = 0$. Using the quadratic formula gives $z = (-b_1 \pm \sqrt{b_1^2 - 4b_0b_2})/(2b_0)$. For $b_0 = 1.0$, $b_1 = -1.0$ and $b_2 = 1.0$, the zeros occur at $c_{0,1} = e^{\pm j\frac{\pi}{3}}$ and because $M - N = -2$, there are two poles on the origin.

Using basic geometry, while $\Omega < \pi/3$, the angle from $e^{j\frac{\pi}{3}}$ to $e^{j\Omega}$ is $\Omega/2 - \pi/3$ and the angle from $e^{-j\frac{\pi}{3}}$ to $e^{j\Omega}$ is $\Omega/2 + \pi/3$. Adding this to the $-2\Omega$ from the poles gives $\angle(H(e^{j\Omega})) = -\Omega$ for $0 \leq \Omega < \pi/3$.

There is a discontinuity at $\Omega = \pi/3$ because there is a zero exactly on the unit circle. As $\Omega$ approaches $\pi/3$, the angle from $e^{-j\frac{\pi}{3}}$ to $e^{j\Omega}$ tends to $-\pi/6$. When $\Omega$ is infinitesimally larger than $\pi/3$, this angle then becomes $5\pi/6$. In general, when $\Omega$ is larger than $\pi/3$, the angle from $e^{-j\frac{\pi}{3}}$ to $e^{j\Omega}$ is $\Omega/2 + 2\pi/3$. This leads to a step phase increase of $\pi$ at $\Omega = \pi/3$. The total phase response for $\pi/3 < \Omega \leq \pi$ is thus given by $\angle(H(e^{j\Omega})) = \pi - \Omega$. It will equal $2\pi/3$ when $\Omega$ is slightly larger than $\pi/3$ and will become 0 when $\Omega = \pi$. This phase response is in figure 3 of the short lab report and matches the predictions.

For the case with $b_0 = 1.0$, $b_1 = -0.9$, $b_2 = 0.81$, we have two zeros at $c_{0,1} = 0.9e^{\pm j\frac{\pi}{3}}$. This case is very similar to the previous case except that the zeros have been drawn inwards and are not exactly on the unit circle. As a result, there should be no discontinuity in the phase response since $e^{j\Omega}$ does not pass through a zero. We can expect to see the same phase characteristics when $\Omega$ is far away from $\pi/3$ but with the discontinuity smoothed out when $\Omega$ is near $\pi/3$. The observed phase response is in figure 5 in the short report.

## 2.3 IIR Filter

This section will cover the phase response of a 2nd order IIR filter of the form $H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$. This filter has two poles at $d_{0,1} = re^{\pm j\theta}$ and two zeros at the origin.

### 2.3.1 $r = 0.1$, $\theta = 0$

This case has two poles at $d_{0,1} = 0.1$ and two zeros at the origin. The phase response of this is composed of the two zeros at the origin adding phase while the two poles subtract phase from the response. When $\Omega = 0$, all of the phase contributions are zero leading to a 0 phase response. When $\Omega = \pi$, the poles add $-2\pi$ and the zeros add $2\pi$ leading to a 0 phase response as well. When $\Omega$ is small, the 2 poles are closer than the 2 zeros and thus their phase contributions change faster, thus the phase decreases. When $\Omega$ is closer to $\pi$ then the phase contributions from the zeros change faster so the phase increases. Using the sine and cosine rule provides an overall expression for the phase response as $\angle(H(e^{j\Omega})) = -2\arctan\left(\frac{\sin(\Omega)}{10-\cos(\Omega)}\right)$. The minimum occurs when $e^{j\Omega}$ is directly above the two poles at 0.1 which is when $\Omega = \arccos(0.1)$. At this frequency, $\angle(H(e^{j\Omega})) = -0.2$ rad $= -11.5\,^\circ$. This phase response is in figure 13 in the short report.

### 2.3.2 $r = 0.9$, $\theta = 0$

In this example, there are two poles at 0.9 and two zeros at the origin which provide a similar phase response characteristic to the previous example. However, in this case, the minimum is reached much faster and is of a greater magnitude than previously. The phase response is zero at $\Omega = 0$ and $\Omega = \pi$ for the same reasons as described above. An expression for the phase response, derived using the sine and cosine rules, is:

$$\angle(H(e^{j\Omega})) = \begin{cases} 2\Omega - 2\arctan\left(\frac{\sin(\Omega)}{\cos(\Omega)-0.9}\right) & \text{if } 0 \leq \Omega \leq \arccos(0.9) \\ 2\Omega - 2\arctan\left(\frac{\sin(\Omega)}{\cos(\Omega)-0.9}\right) + 2\pi & \text{if } \arccos(0.9) < \Omega \leq \pi \end{cases} \tag{2}$$

The minimum in this case occurs when $e^{j\Omega}$ is directly above 0.9, i.e. $\Omega \to \arccos(0.9) = 0.45$ rad. Here, $\angle(H(e^{j\Omega})) = -2.24$ rad.

### 2.3.3 $r = 0.5$, $\theta = \pi/3$

This filter has two poles at $d_{0,1} = 0.5e^{\pm j\frac{\pi}{3}}$ and two zeros at the origin. When $\Omega = 0$, the phase from both zeros is 0, the $0.5e^{j\frac{\pi}{3}}$ pole contributes a phase of $\pi/6$ and the $0.5e^{-j\frac{\pi}{3}}$ pole contributes a phase of $-\pi/6$ leading to a total phase of 0. For small positive values of $\Omega$, the curvature of the unit circle can be taken to be small. In this case, the zeros contribute $2\Omega$ to the phase. The $0.5e^{j\frac{\pi}{3}}$ pole contributes a phase of $\pi/6 - \Omega$ and the $0.5e^{-j\frac{\pi}{3}}$ pole contributes a phase of $-(\pi/6 + \Omega)$. This leads to an overall phase of 0 for small $\Omega$ and as a result, there is only a very small phase lag for small $\Omega$. However, as $\Omega$ becomes larger, the positive phase contribution from the $0.5e^{j\frac{\pi}{3}}$ pole decreases and the negative phase contribution from the $0.5e^{-j\frac{\pi}{3}}$ pole increases. This happens faster than the positive phase contribution from the zeros increases, largely because the $0.5e^{j\frac{\pi}{3}}$ pole is close to $e^{j\Omega}$ This leads to decreasing and negative phase response. When $\Omega$ increases past $\pi/2$, the phase from the zeros increase faster then the phase from the poles are decreasing. Thus, the phase response rises. When $\Omega = 2\pi$, the zeros contribute $2\pi = 0$ rad, the $0.5e^{j\frac{\pi}{3}}$ pole contributes 2.81 rad and the $0.5e^{-j\frac{\pi}{3}}$ pole contributes -2.81 rad which gives an overall response of 0. This response is shown in figure 14 in the short report.

### 2.3.4 $r = 0.5$, $\theta = 2\pi/3$

This is very similar to the previous case, except that the poles have been reflected in the imaginary axis. This means that the phase response will be the same as before but will be positive instead of negative and will be back to front. All the points discussed previously will apply in this case but $\Omega$ will correspond to $\pi - \Omega$ and the phase response will be multiplied by -1. The response is in figure 17 in the short report.

### 2.3.5 $r = 0.9$, $\theta = \pi/2$

To work out what the phase response for this case will be, we will consider the phase response of the case where $r = 1$ and $\theta = \pi/2$. This has two poles on the unit circle at $\theta = \pm\pi/2$. By considering the geometry of the poles and their angles to $e^{j\Omega}$ while $0 \leq \Omega < \pi$, it is clear that the angle from $j\pi/2$ to $e^{j\Omega}$ is $\Omega/2 - \pi/4$ meaning that its phase contribution is $\pi/4 - \Omega/2$. The phase contribution from $-j\pi/2$ to $e^{j\Omega}$ is $-\Omega/2 - \pi/4$. The two zeros contribute $2\Omega$ to the phase. Adding these contributions together gives a phase response of $-\Omega$ for $0 \leq \Omega \leq \pi$.

At $\Omega = \pi/2$, there is a discontinuity because the $j\pi/2$ pole is exactly on the unit circle. When $\Omega$ is slightly smaller than $\pi$, the $j\pi/2$ pole contributes 0 to the phase response. When $\Omega$ is slightly larger than $\pi$, it contributes $-\pi$ to the phase response. For $\pi \geq \Omega > \pi/2$, the

$j\pi/2$ pole contributes $-\pi/4 - \Omega/2$ to the phase response leading to an overall phase response of $\Omega - \pi$ in this region.

With $r = 0.9$, the phase response is similar to the response described above, except that the discontinuity at $\Omega = \pi/2$ is smoothed out as the pole is not exactly on the unit circle. The phase response does not reach the extremes of the case above. The obtained response is in figure 17 in the short report.

# 3 Window functions

The creation of a digital low pass filter can be created by considering the ideal frequency response and converting this to an impulse response. The impulse response can then be implemented with a digital FIR filter. A low pass filter with a cut off frequency of $\omega_c$ has the frequency response:

$$H_d(\omega) = \begin{cases} 1 & \text{if } |\omega| \leq \omega_c \\ 0 & \text{if } \omega_c < \omega \leq \pi \end{cases} \tag{3}$$

Taking the fourier transform of this gives us the impulse response to be:

$$h_d(n) = \begin{cases} \frac{\omega_c}{\pi} & \text{if } n = 0 \\ \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n} & \text{if } n \neq 0 \end{cases} \tag{4}$$

This impulse response clearly extends to infinity in both directions. This is impossible to implement in practice as it is non causal and requires and infinite number of coefficients if implemented with an FIR filter. There are many techniques to achieve a close frequency response to the ideal filter in practice and the one that will be discussed here is windowing.

## 3.1 Rectangular Window

The windowing method multiplies the ideal impulse response with a window function which is non zero for a finite duration. First, the desired frequency response must be made causal such that:

$$H_d(\omega) = \begin{cases} 1e^{-j\omega(M-1)/2} & \text{if } |\omega| \leq \omega_c \\ 0 & \text{if } \omega_c < \omega \leq \pi \end{cases} \tag{5}$$

The delay of $(M-1)/2$ is added for causality and because the final filter will be of length $M$. To calculate the impulse response of this, we take the inverse fourier transform of $H_d(\omega)$ which is:

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega\left(n-\frac{M-1}{2}\right)}\, d\omega = \frac{\sin\omega_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} \text{ for } n \neq \frac{M-1}{2} \tag{6}$$

If we multiply this impulse response by a rectangular window function $w(n)$ defined as

$$w(n) = \begin{cases} 1 & \text{if } n = 0, 1, \ldots, M-1 \\ 0 & \text{if otherwise} \end{cases} \tag{7}$$

we achieve a causal and finite impulse response of

$$h(n) = \frac{\sin\omega_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} \text{ for } 0 \leq n \leq M-1 \text{ if } n \neq \frac{M-1}{2} \tag{8}$$

This is how a rectangular window function is used to achieve a realisable low pass filter. To examine the properties of this filter we must consider what the frequency response looks like once the impulse response has been multiplied by the window function. Multiplication in the time domain is equivalent to convolution in the frequency. The fourier transform of the rectangular window is

$$W(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} = e^{-j\omega(M-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)} \tag{9}$$

The magnitude of the frequency response of this window function is

$$|W(\omega)| = \frac{|\sin(\omega M/2)|}{|\sin(M/2)|} \tag{10}$$

This equation explains many of the characteristics of the frequency response of the rectangular window function low pass filter. The magnitude of the window function consists of slowly decaying sidelobes which become narrower with increasing $M$ but the area under each sidelobe stays constant. This explains the large sidelobe magnitude with a rectangular window. When the window function is convolved with the ideal frequency response the large sidelobes lead to Gibb's phenomenon as sidelobes convolve with the rectangular pulse to give large passband ripples. Another way to understand Gibb's phenomenon is that the frequency response is obtained via the inverse fourier transform of the impulse response. We have truncated the infinite impulse response and have thus truncated an infinite fourier series which leads to the characteristic frequency response with large ripples. The impulse response can be seen in figure 10 in the short report and it is clear that it has been abruptly truncated. The frequency response is in figure 8 in the short report and shows the large passband ripple which is characteristic of a rectangular window.

Another interesting observation is that as $M$ increases, the lobes of $|W(\omega)|$ become narrower. This is precisely why the transition band slope with a rectangular window becomes steeper with increasing filter order as can be seen in figure 7 in the short report which shows a positive linear relationship between transition band slope and filter order. The maximum passband ripple, as predicted, is mostly unchanged by filter order as expected from the above function. In figure 6 in the short report, we can see it only varies between 1.04 and 1.11. Reducing this passband ripple is one of the main advantages of using a Hamming window, as discussed below.

## 3.2 Hamming Window

The Hamming window alleviates the problem of large passband ripples by *tapering* the impulse response coefficients to zero. This reduces the effects of Gibb's phenomenon and figure 11 in the short report clearly shows the tapered impulse response. It employs a window function of the form:

$$w(n) = \begin{cases} 0.54 - 0.46\cos(\frac{2\pi}{N}r) & \text{if } 0 < r < N \\ 0 & \text{if otherwise} \end{cases} \tag{11}$$

The Hamming filter employs a clever method to reduce the ripples in the filter. The inverse fourier transform of the window actually produces three sinc functions. The 0.54 constant term creates one sinc function in the middle and the cosine term two sinc functions either side of the central sinc function. They are aligned is such a way that the sidelobes of the two sinc functions on the side cancel the sidelobes of the main sinc function with their own sidelobes, as can be seen in figure 12 in the short report. However, these extra sinc functions add to the width of the original sinc function which cause a shallower slope in the transition band. As a result, the Hamming window transition band is generally half as steep as a rectangular window transition band, as can be seen in figure 7 in the short report. The passband ripple is relatively unchanged with filter order but it is a significant improvement over the rectangular window, as shown in figure 6 in the short report. In conclusion, the Hamming window trades a lower pass band ripple with a broader transition region.

# 4 Impulse response of IIR Filters

## 4.1 Theory

The poles and zeros can be used to determine the impulse response of a system. By considering the general transfer function of a system with the denominator separated into poles:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})} \tag{12}$$

Performing a partial fraction expansion on the transfer function gives:

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N_1} \frac{A_k}{1 - d_k z^{-1}} + \sum_{k=N_1}^{N_2} \frac{C_k d_k z^{-1}}{(1 - d_k z^{-1})^2} + \cdots \tag{13}$$

The first term corresponds to terms which appear if the order of the numerator is higher than the order of the denominator. The second term corresponds to poles of multiplicity 1 and the third term corresponds to poles of multiplicity 2. Higher multiplicity terms may also be present but have been excluded above. Taking the inverse $z$-transform of this expression gives us the impulse response as follows:

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n - r] + \sum_{k=1}^{N_1} A_k (d_k)^n u[n] + \sum_{n=N_1}^{N_2} C_k n (d_k)^n u[n] + \cdots \tag{14}$$

where $u[n]$ is the discrete step function. The excess $M - N$ zeros simply become delta functions and do not contribute to the long term property of the impulse response. We can draw some very important conclusions from this result. The poles of the transfer function add to the response exponentially. Thus, if a pole has a magnitude of less than 1, it will add an exponentially decaying component to the impulse response. If the magnitude of any pole is greater than 1, it will add an exponentially growing component and therefore the system will be unstable. If the magnitude equals 1 and the pole has multiplicity 1, it will add a constant response which is marginally stable. However, if it is of multiplicity 2 or greater, the pole will add a term which grows polynomially (linearly if multiplicity 1, quadratic if multiplicity 2 etc.).

As long as the coefficients $a_k$ and $b_k$ of the filter are real, all complex poles will occur in conjugate pairs. Thus, complex poles will cancel each others imaginary component and will create a sinusoidal component with a frequency given by the phase of the poles. This sinusoid will be enveloped by an exponential function based on the magnitude of the pole. If it is less than 1, the sinusoid will decay, if it is greater than one, the sinusoid will grow and if it equals one, the sinusoid will be of constant amplitude. An important general comment on the impulse response is that the closer the poles are to the origin, the faster the impulse response decays which implies a responsive system. If the poles are close to the unit circle, the response will be sluggish and the impulse response will decay slowly.

The final case to consider is if the pole has a negative real part. As the impulse response at $n$ is given by the poles raised to the power $n$, we can expect these to have the same response as described above but the response will be positive for even $n$ but negative for odd $n$.

## 4.2 Experimental observations

For the IIR filter considered in the lab where

$$H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})} = \frac{1}{1-e^{-2j\theta}} \times \frac{1}{1-re^{j\theta}z^{-1}} + \frac{1}{1-e^{2j\theta}} \times \frac{1}{1-re^{-j\theta}z^{-1}} \quad (15)$$

we have two poles at $z = re^{\pm j\theta}$. Taking the inverse $z$-transform gives the impulse response:

$$h(k) = r^k \left( \frac{1}{1-e^{-2j\theta}} e^{j\theta k} + \frac{1}{1-e^{2j\theta}} e^{-j\theta k} \right) = r^k \frac{\sin(\theta(k+1))}{\sin(\theta)} \quad (16)$$

This equation only applies when $\theta \neq 0$. If $\theta = 0$, $\sin(\theta) \to \theta$ so the impulse response becomes $h(k) = kr^k + r^k$. These results can be applied to the IIR filters considered in the lab.

The first case to consider is when $r = 0.1$ and $\theta = 0$. There are two poles at $r = 0.1$. The impulse response is $h(k) = 0.1^k(k+1) = (1, 0.2, 0.03, 0.004, \ldots)$. This shows rapid exponential decay as can be seen in figure 1 on page i as expected as the poles are close to the origin.

The second case is when $r = 0.5$ and $\theta = \pi/3$. This has a slower response (but is still quick) and has a sinusoidal response because of the non zero $\theta$. Analytically, the response is given by $h(k) = \frac{2}{\sqrt{3}} \times 0.5^k \sin(\pi/3(k+1)) = (1, 0.5, 0, -0.125, \ldots)$. This can be seen in figure 1 on page i.

The third case is when $r = 0.5$ and $\theta = 2\pi/3$. This is very similar to the second case except the poles are reflected in the imaginary axis. The impulse response is the same as before except when $k$ is odd, the sign of the response at that point in the sequence is reversed. The response is therefore given by:

$$h(k) = \frac{2}{\sqrt{3}} \times 0.5^k \sin((2\pi/3)(k+1)) = \begin{cases} h(k) = \frac{2}{\sqrt{3}} \times 0.5^k \sin((\pi/3)(k+1)) & \text{if } k \text{ is even} \\ h(k) = -\frac{2}{\sqrt{3}} \times 0.5^k \sin(\pi/3(k+1)) & \text{if } k \text{ is odd} \end{cases} \quad (17)$$

so $h(k) = (1, -0.5, 0, 0.125, \ldots)$ which can be seen in figure 3 on page ii.

The fourth case is when $r = 0.9$ and $\theta = 0$. This is similar to the first case except that the pole is much closer to the unit circle. The impulse response is $h(k) = 0.9^k(k+1) = (1, 1.8, 2.43, \ldots)$. This decays to 0 as $k$ becomes large but when $k$ is small, $k$ increases faster than $0.9^k$ decreases. Thus, the impulse response increases initially but then decays to zero. The slower response is due to the poles being closer to the unit circle. The response is in figure 3 on page ii.

The fifth case is when $r = 0.9$ and $\theta = \pi/2$. This has a damped sinusoidal response of $h(k) = 0.9^k \sin((k+1)\pi/2) = (1, 0, -0.81, 0, 0.6561, 0, -0.5314, \ldots)$ as $\theta$ is non zero. The poles are also close to the unit circle so the sinusoid decays slowly as seen in figure 5 on page iii.

The sixth case has $r = 1.2$ and $\theta = \pi/3$. This has a magnitude greater than one which means it will be unstable. It has an exponentially growing sinusoidal response due to the complex component. The impulse response is given by $h(k) = \frac{2}{\sqrt{3}} 1.2^k \sin(\frac{\pi}{3}(k+1))$ and is in figure 6 on page iii.

The final case to consider is when $r = 1$ and $\theta = 0$. In general, poles of multiplicity one and a magnitude of one lead to a marginally stable system. However, in this case, the magnitude is one but there is a multiplicity of two, meaning there are two poles a 1. This gives an impulse response of $h(k) = k + 1 = (1, 2, 3, 4, \ldots)$ and is unstable with a linearly growing impulse response as shown in figure 7 on page iv.

# 5 Notch Filter

An IIR filter of the form:

$$H(z) = \frac{1 - 2\cos(\theta)z^{-1} + z^{-2}}{1 - 2r\cos(\theta)z^{-1} + r^2 z^{-2}} \tag{18}$$

with $0 < r < 1.0$ produces a basic *notch* filter. This has two zeros on the unit circle at a phase $\pm\theta$ and two poles with a magnitude $r$ and at a phase $\pm\theta$. The frequency $\omega$ which will be removed corresponds to $\omega = \theta/T$. The zeros ensure that at the notch frequency the magnitude response is 0. The poles dictate how sharp the notch produced in the magnitude response. If the poles are close to the unit circle, the notch will be sharp and the passband gain will tend to 1. If they are closer to the origin, the notch will be shallow and the passband gain will tend to 4. The two frequency responses for $r = 0$ and $r = 0.99$ are shown in figures 8 and 9 respectively. Applications of this filter include removing 50 Hz mains hum. It can also be on the output of a radio receiver to remove an interfering frequencies from a local transmitter which may be swamping the receiver.

## 5.1 Removing sine wave from Gaussian noise

To measure the performance of this filter, a sin wave of angular frequency $0.01\pi$ was added to Gaussian noise of variance 0.3 and the filter was used to remove the sine wave. The value of $\theta$ was set to $0.01\pi$ and $r$ was varied from 0 to 1 and the performance analysed. The two signals are shown in figure 10 on page v.

With $r = 0.3$, the output is shown in figure 11. The sin wave has clearly been removed, however, a large part of the low frequency components of the Gaussian noise seem to have been removed as well and only the high frequency components are present with a significant magnitude. This is to be expected as the notch will be very spread out for $r = 0.3$ meaning a large amount of the noise has been removed.

For $r = 0.9$, the output has significantly improved as shown in figure 12. The output looks much closer to the original noise than with $r = 0.3$. The magnitude response is far more selective and does not remove a significant amount of the noise but still completely removes the sin wave. It may be expected that as the filter is narrowed, the output will only improve. However, there comes a point where the filter is too selective. In this case, the generated sin wave is not actually exactly at a frequency of $0.01\pi$ due to quantisation errors. In real life, the frequency being blocked may vary slightly and increasing $r$ too much decreases the robustness of the filter. If $r = 0.99$, it is clear that the filter does not stop any of the sin wave and the signal passes through largely unchanged as can be seen in figure 13. To summarise, there is a clear performance vs robustness trade-off. The output gain is also decreased for a larger $r$.

# 6 Conclusion

Overall, it can be seen that many properties of filters can be intuitively understood and theoretically derived from pole-zero locations such as the phase response. The Hamming window was found to have eliminated the large passband ripples associated with a simple rectangular window which came at the expense of a wider transition region. The transition region width was reduced with increasing filter order for both window functions whereas the passband ripple was largely unaffected.

The impulse response of the given IIR filter was found to depend directly on the pole locations. If a pole was outside the unit circle, the system would be unstable and poles with imaginary components were found to have an oscillatory response. Finally, the notch filter was found to have a clear performance vs robustness trade-off. Better performance involved the poles being closer to the unit circle as this kept the low frequency component of the Gaussian noise. However, it the poles were too close, the filter was found to be too selective and the relatively small effect of quantisation noise affecting the frequency of the sine wave actually led to a large degradation in the filters performance as sine wave was not removed from the noise.

# Appendix

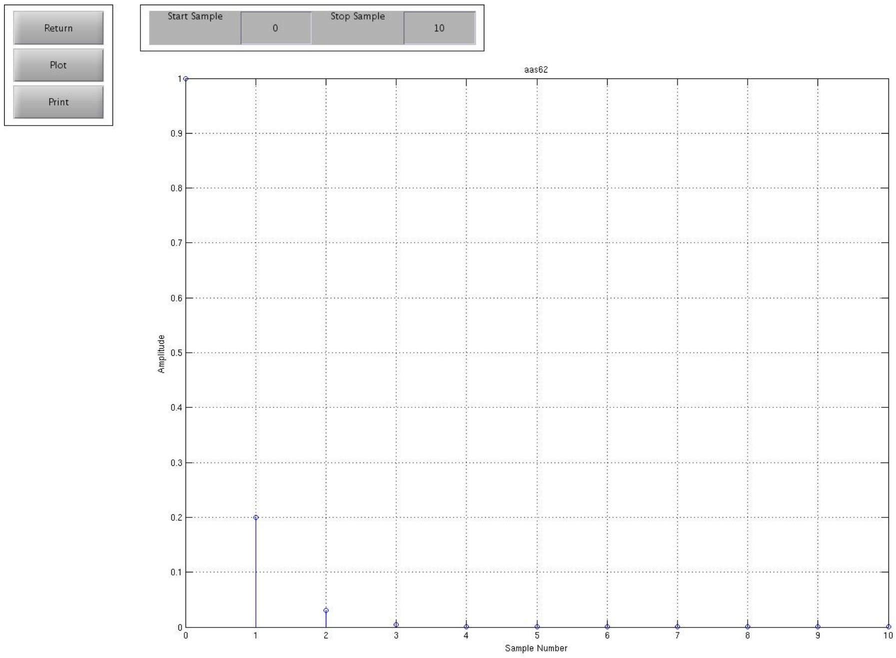


Figure 1: Impulse Response of IIR filter with $r = 0.1$ and $\theta = 0$

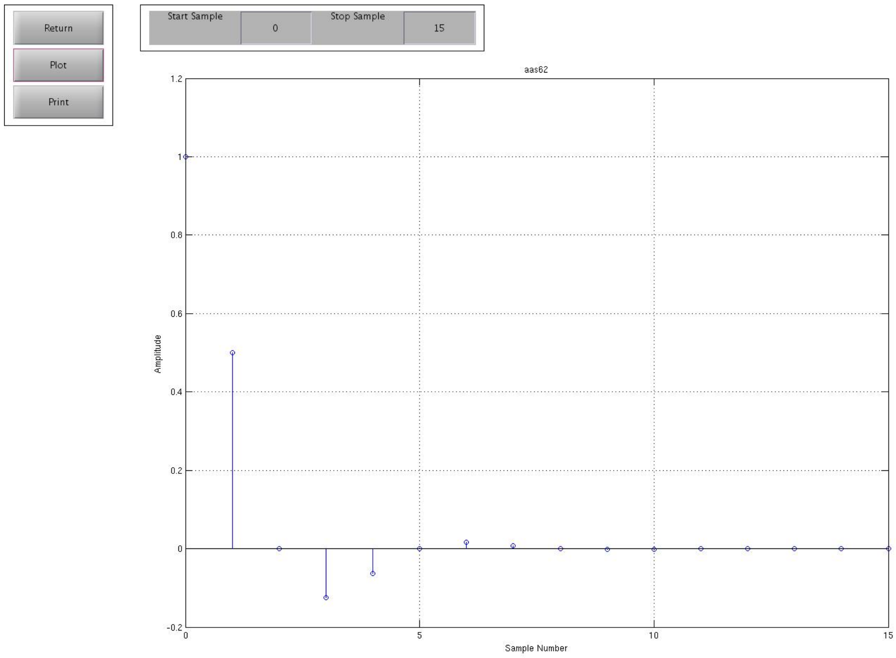


Figure 2: Impulse Response of IIR filter with $r = 0.5$ and $\theta = \pi/3$

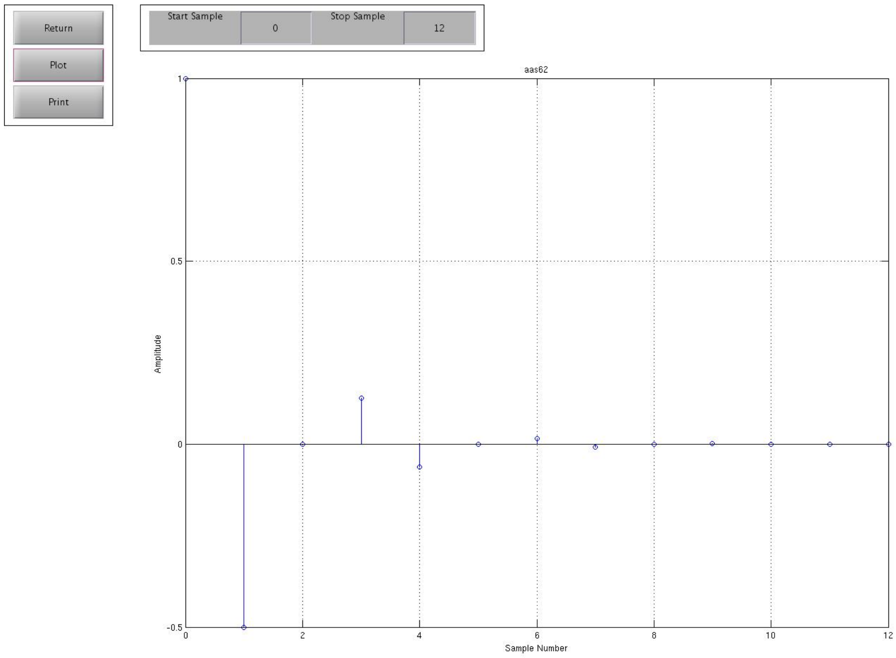


Figure 3: Impulse Response of IIR filter with $r = 0.5$ and $\theta = 2\pi/3$

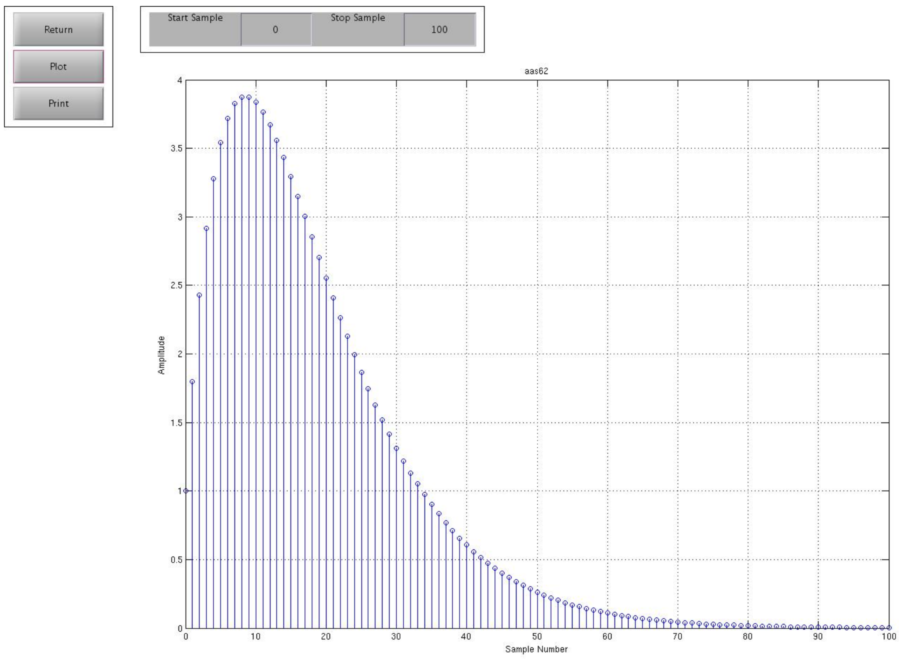


Figure 4: Impulse Response of IIR filter with $r = 0.9$ and $\theta = 0$

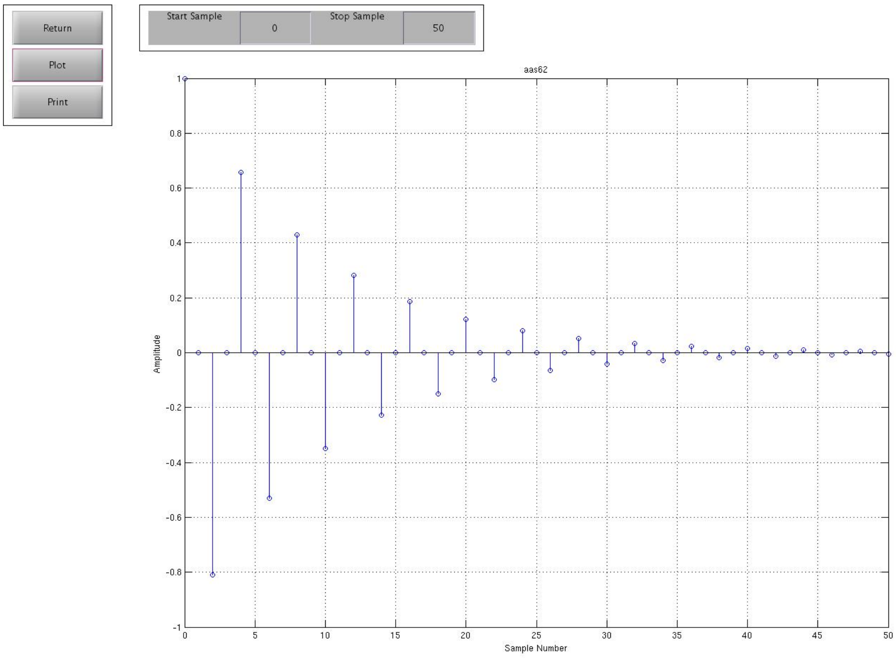


Figure 5: Impulse Response of IIR filter with $r = 0.9$ and $\theta = \pi/2$

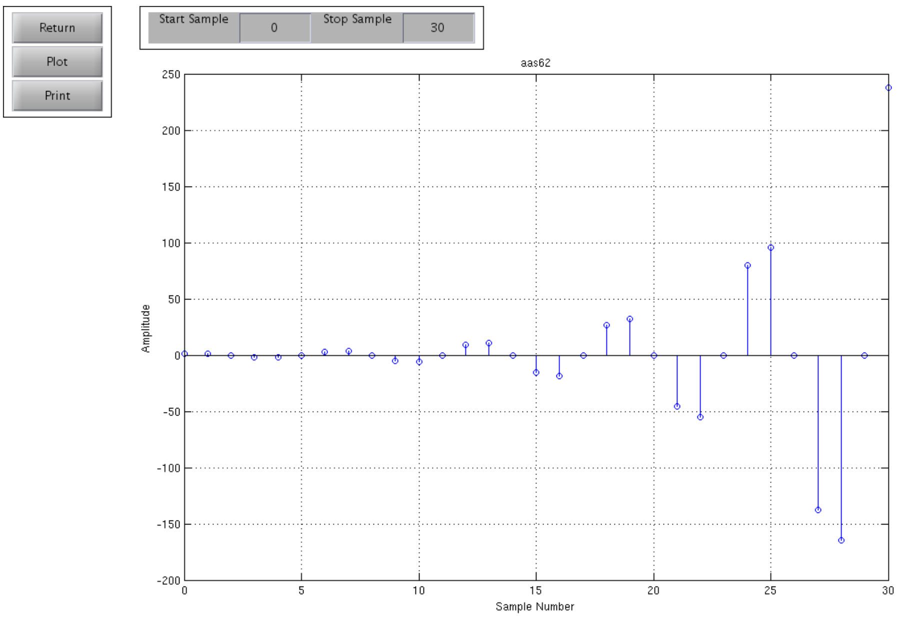


Figure 6: Impulse Response of IIR filter with $r = 1.2$ and $\theta = \pi/3$

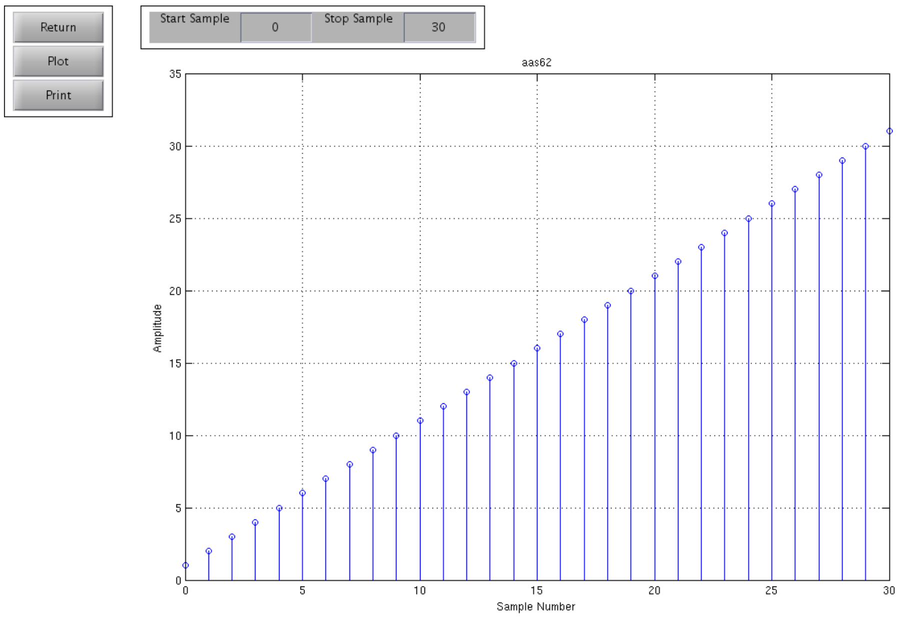


Figure 7: Impulse Response of IIR filter with $r = 1$ and $\theta = 0$

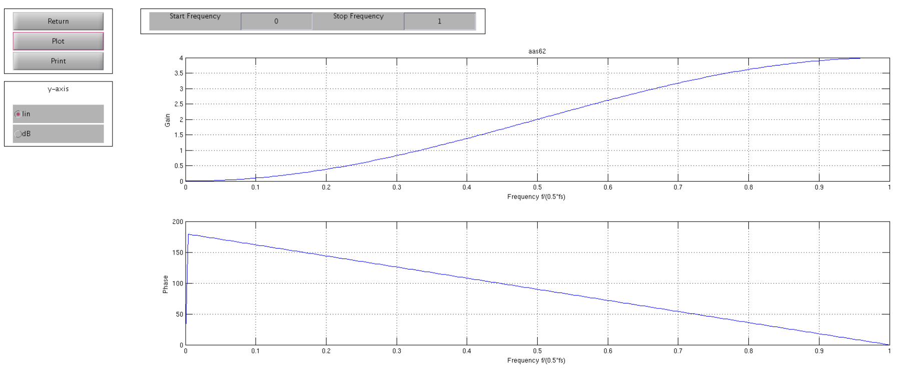

Figure 8: Frequency response of notch filter with $r = 0$

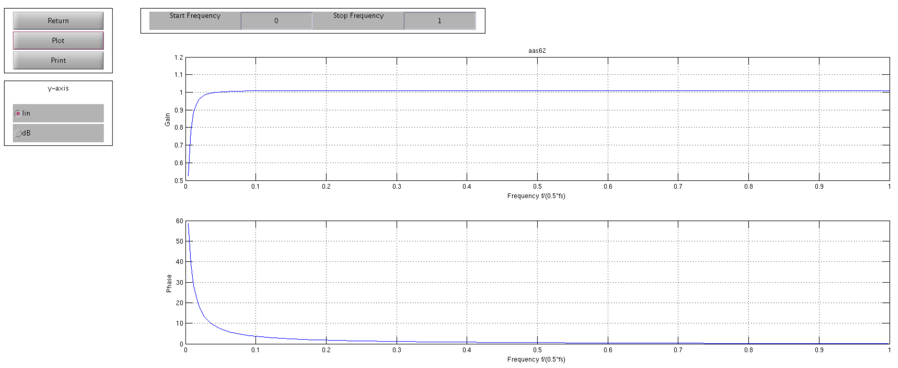

Figure 9: Frequency response of notch filter with $r = 0.99$

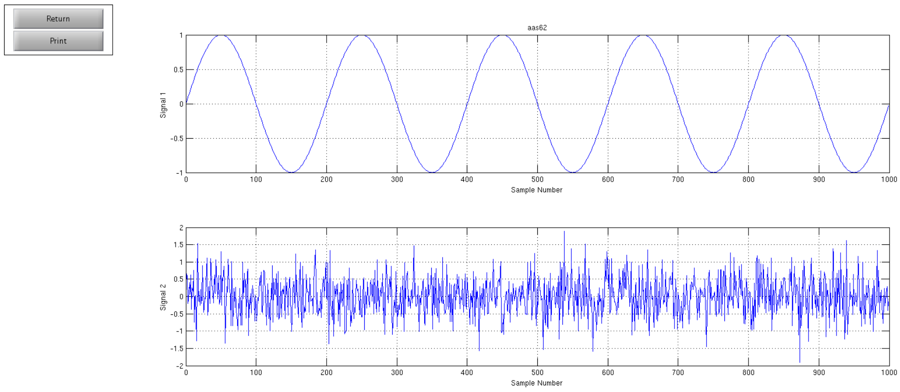


Figure 10: Sin wave and Gaussian noise used to test the notch filter

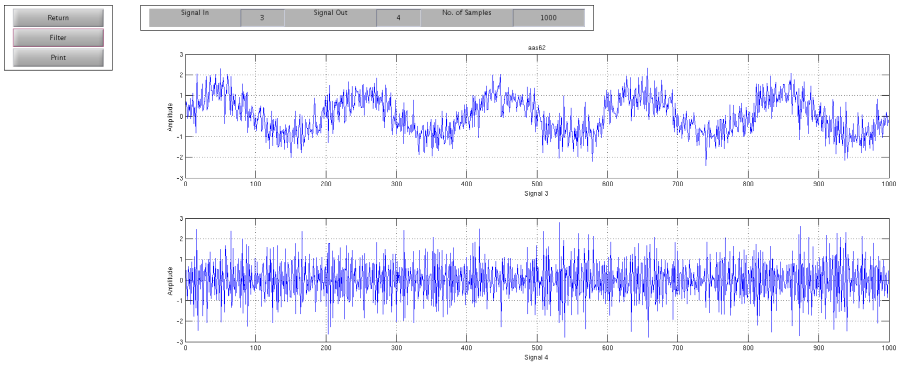


Figure 11: Output of notch filter with $r = 0.3$

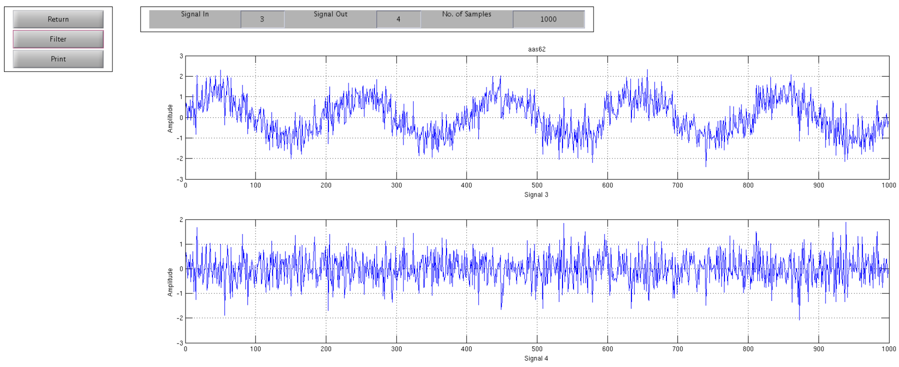


Figure 12: Output of notch filter with $r = 0.9$

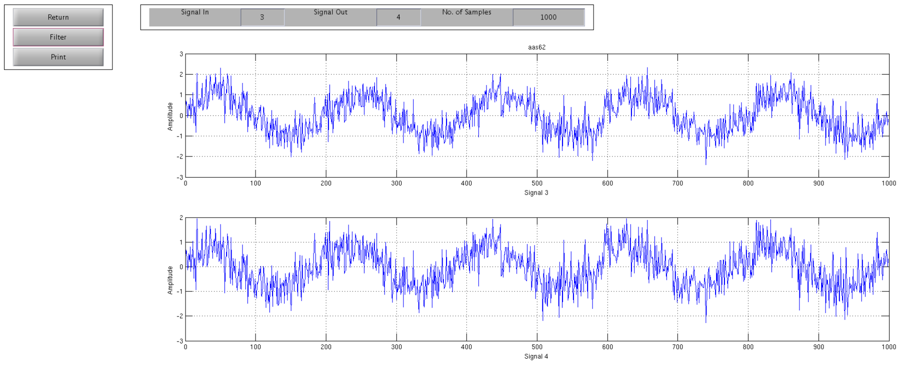


Figure 13: Output of notch filter with $r = 0.99$