[Title of thesis]

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Abstract

Hier die Kurzfassung in ganzen Sätzen.

Contents

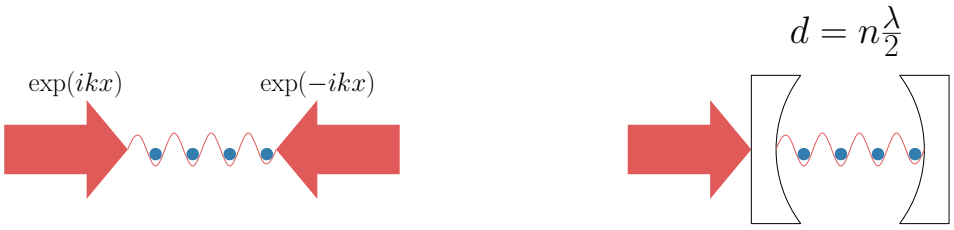
1 Introduction

We're constantly looking for materials with specific properties that can make our lives easier. To find a new material that fits our needs, we could look for it in nature. That means we'd go digging in the earth until we find something new, isolate it, take it to the lab, determine its properties and hope we found what we needed. This process is quite cumbersome and as technology gets more and more advanced, we'll find ourselves in need of materials that might not be naturally occurring in the first place. In that case we need to determine whether such materials could theoretically exist at all and if so, what are the requirements a compound needs to have in order to have a specific property? To find out, we conduct simulations which we're already doing very rigorously right now. However, when simulating quantum many-body systems, the Hilbert space scales dramatically and our conventional devices will fail.

One solution to this is to use a quantum computer. Algorithms with exponential time on classical devices would operate then on polynomial time. There's just one problem: We don't have a sophisticated quantum computer yet and will likely not have one in the next 10-20 years, at least for our specific purposes. So rather than utilizing a general-purpose quantum computer, for now there's another approach that's more fruitful: We'll restrict ourselves to a specific problem by constructing a quantum system that simulates another quantum system. Such a system won't be able to simulate anything, just what we designed it for. This is called an *analog quantum simulator*. With analog quantum simulators, we can not only simulate properties of compounds, but any quantum problem that we wish to solve. A nice introduction to quantum simulation can be found at [1].

2 Cold Atoms in Optical Cavities

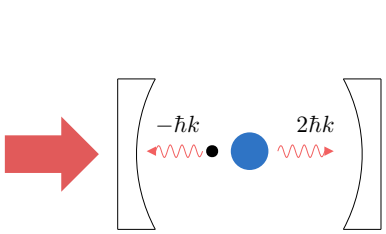
What we want to achieve is to order atoms in a lattice to simulate a solid. To prevent them from flying away, we have to fix them in space somehow. A great way to do this is to use a laser potential. Take a look at Figure 1. When two laser beams are opposing each other with contributions $\propto \exp(ikx)$ and $\propto \exp(-ikx)$, a cosine pattern will form. The idea is that the atoms will localize at the antinodes of the laser potential to minimize the energy, i.e. the atoms will be localized in the "valleys" of the potential. We'll take that approach one step further and put the atoms in a cavity. There are different types of cavities, such as the bow-tie, the ring and the Fabry-Perot cavity. We will utilize the Fabry-Perot cavity. There are two curved mirrors separated by a distance d. The laser light will enter the cavity through the partially transmissive mirrors. There are a couple advantages of using a cavity for us. If we choose d and λ such that $d = n\lambda/2$, $n \in \mathrm{N}$, then the light field in the cavity will be amplified and the atoms more tightly bounded. With this setup, the atoms will also take part in amplifying their own trapping potential.



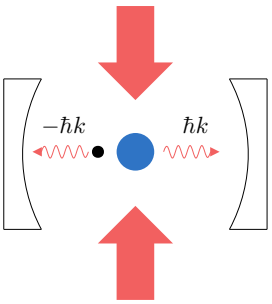
(a) Counter-propagating lasers. **(b)** Optical cavity.

Figure 1: Two counter-propagating lasers create a cosine-potential. If we put atoms in that potential, they will localize at the antinodes and form a lattice pattern. We'll take that approach even further and put the atoms in a cavity. That way the light field will be amplified and the atoms themselves take part in creating their own trapping potential.

We will work with a far red-detuned laser, that means the frequency of the laser ω_l is way smaller than the excitation frequency of atom ω_a. If we prevent the atom from being excited, a strong dipole force will build up and the atom will get strongly coupled to the cavity field. The lasers entering through the cavity walls into the cavity poses a problem: That way the motion of one atom affects the cooling of another. A solution is proposed in [2]: Pumping transversally to the cavity axis. Figure 2 shows a sketch of a longitudinally and transversally pumped cavity. These are now our two systems for which we'll derive the Hamiltonians. We'll now discuss some of the properties of these systems. There is a fundamental difference how atoms scatter light in a transversally pumped cavity and in a longitudinally pumped cavity. In the longitudinal case, if a photon with momentum $\hbar k$ in x-direction bumps into an atom, it will recoil backward, having now a momentum $-\hbar k$. Conservation of momentum thus requires the atom to have now a momentum of $2\hbar k$. In the transversal case, there are two counter-propagating laser beams as to prevent kicking the atoms out of the cavity. If an atom scatters a transversally incoming photon along the cavity axis, it will now have a momentum of $\hbar k$ and the photon $-\hbar k$. We see that the fundamental difference between longitudinal and transversal pump is the momenta the atoms will be able to acquire. In the longitudinal case, there will only be momenta of $2n\hbar k$, where $n \in \mathrm{N}$, whereas for the transversal pump there are momenta of $\hbar k$.



(a) Longitudinal pumping.



(b) Transversal pumping.

Figure 2: Longitudinal and transversal pumping. For longitudinal pumping, the atoms will acquire a momentum of $2\hbar k$ scattering a photon, whereas for transversal pumping there is momentum exchange of $\hbar k$.

To illustrate the discrete momenta, take a look what a wave function looks like inside the cavity:

$$\psi(k) = \frac{1}{N}\sum_l c_l \exp(likx) = \frac{1}{N}\Big(c_0 + c_{\pm 1}\exp(ikx) + c_{\pm 2}\exp(2ikx) + \dots\Big). \tag{1}$$

The variable N is a normalizing constant. As previously mentioned, an atom inside the cavity cannot have any arbitrary momentum, but only multiples of $\hbar k$ due to the way momentum is acquired. If we plot the wave function as a whole, we don't see the discreetness of the momenta. However, if we perform a Fourier transform, we can access the c_l's. For longitudinal pump, we thus only expect momenta of 0, $2\hbar k$, $4\hbar k$, ... corresponding to c_0, $c_{\pm 2}$, $c_{\pm 4}$ and so on. For transversal pump we expect momenta of 0, $\hbar k$, $2\hbar k$, $3\hbar k$, ... corresponding to c_0, $c_{\pm 1}$, $c_{\pm 2}$, $c_{\pm 3}$ and so on. In Table 2 there are the c_l's with the components of the wave function.

c_l	wave number	momentum
c_0	0	0
$c_{\pm 1}$	$k \to \exp(ikx)$	$\hbar k$
$c_{\pm 2}$	$2k \to \exp(2ikx)$	$2\hbar k$
$c_{\pm 3}$	$3k \to \exp(3ikx)$	$3\hbar k$
$\vdots$		

Table 1: Coefficients of the wave function. If we plot the wave function density in position space, longitudinal and transversal pump might look quite similar. We thus perform a Fourier transform so we can access the c_l's and gain further insight into the physical system. For longitudinal pump, we only expect c_0, $c_{\pm 2}$, $c_{\pm 4}$, ... while for transversal pump we expect c_0, $c_{\pm 1}$, $c_{\pm 2}$, $c_{\pm 3}$ and so on.

3 Derivation of the Hamiltonian

For those wishing to refresh their knowledge in quantum mechanics, the introductory chapters of Fox's quantum optics book will be a great help [3] (it's a great quantum optics book in general). In this section we'll derive the Hamiltonians being used for the simulation; one Hamiltonian for longitudinal pumping and one for transversal pumping. The Hamilton operator represents the total energy of a quantum system. We'll start with the Jaynes-Cummings Hamiltonian which describes the interaction of a two-level atom with a single mode of a cavity-field. We'll then modify the Hamiltonian according to our needs step by step. We'll tackle the crucial details and reference parts of the derivation which is not presented here.

3.1 The Jaynes-Cummings Hamiltonian

The Jaynes-Cummings model describes the interaction of a two-level atom with a single mode of a cavity field. The first appearance of the model was in [4]. Since we're dealing with both an atom and a light field at he same time, we have a composite system, i.e.

$$\psi_{\text{total}} = \psi_{\text{light}} \otimes \psi_{\text{atom}}. \tag{2}$$

The fact that we have two levels motivates a two-dimensional basis for the atom:

$$|\text{g}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |\text{e}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3}$$

where $|\text{g}\rangle$ is the ground state and $|\text{e}\rangle$ is the excited state. Both states respond to the operators

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

where σ^+ is the raising operator and σ^- is the lowering operator. They have the properties

$$\sigma^+|\text{g}\rangle = |\text{e}\rangle \qquad \sigma^-|\text{e}\rangle = |\text{g}\rangle. \tag{5}$$

The wave function for the atoms depends on the position, while this is not the case for the light field. Instead we will use photon number states or Fock states. A photon number state $|n\rangle$ thus represents a monochromatic quantized field with frequency ω containing n atoms. The ground state $|0\rangle$ corresponds to 0 photons. The creation and annihilation operators $a^\dagger$ and a correspond to creating and annihilating a photon. We'll restrict ourselves to one dimension and start with an atom (or a Bose-Einstein condensate) in an external potential:

$$H_0 = \frac{p^2}{2m} + V_{\text{ext}}(x). \tag{6}$$

Now we place that atom in a cavity and it will interact with the cavity mode, creating more terms in our Hamiltonian that we have to consider. First, there's the energy of the field:

$$H_{\text{field}} = -\hbar\omega_{\text{c}} a^\dagger a, \tag{7}$$

where ω_{c} is the resonance frequency of the cavity and $a^\dagger$ and a are the creation and annihilation operators. Next, we'll add a term describing the atomic transitions:

$$H_{\text{transition}} = -\frac{1}{2}\hbar\omega_{\text{a}}\sigma_z, \tag{8}$$

where ω_{a} is the resonance frequency of the atom and σ_z is the Pauli z-matrix which is defined as

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

The Hamiltonian $H_{\text{transition}}$ describes the atom being in the ground state or excited state, the transition energy is $1/2\hbar\omega_{\text{a}}$. The field-atom interaction we describe with following term:

$$H_{\text{interaction}} = \hbar g_0 \cos(kx)(\sigma^+ a + \sigma^- a^\dagger), \tag{10}$$

where g_0 is the coupling strength and σ^+ and σ^- are the raising and lowering operators. Finally, we'll add the term describing the pumping:

$$H_{\text{pump}} = \hbar\eta(ae^{i\omega_{\text{l}}t} + a^\dagger e^{-i\omega_{\text{l}}t}), \tag{11}$$

where η is the pumping strength and ω_{l} is the laser frequency. We now have the full Jaynes-Cummings Hamiltonian which is the sum of all terms above:

$$H_{\text{JC}} = \underbrace{p^2/2m}_{\text{atom}} + \underbrace{V_{\text{ext}}(x)}_{\text{external potential}} - \underbrace{1/2\hbar\omega_{\text{a}}\sigma_z}_{\text{atomic transitions}} - \underbrace{\hbar\omega_{\text{c}} a^\dagger a}_{\text{field}} + \underbrace{\hbar\eta(ae^{i\omega_{\text{l}}t} + a^\dagger e^{-i\omega_{\text{l}}t})}_{\text{pumping}} + \underbrace{\hbar g_0 \cos(kx)(\sigma^+ a + \sigma^- a^\dagger)}_{\text{field-atom interaction}}. \tag{12}$$

A more detailed derivation of the Jaynes-Cummings Hamiltonian (starting from Maxwell's equations and quantizing the cavity mode) can be found at [5]. In order

to get rid of the explicit time-dependence, we transform the Hamiltonian to a frame rotating with ω_l. The Hamiltonian now reads:

$$H_\mathrm{JC} = \frac{p^2}{2m} + V_\mathrm{ext}(x) - \frac{1}{2}\hbar\Delta_\mathrm{a}\sigma_z - \hbar\Delta_\mathrm{c}a^\dagger a + \hbar\eta(a + a^\dagger) + \\ + \hbar g_0 \cos(kx)(\sigma^+ a + \sigma^- a^\dagger), \tag{13}$$

where $\Delta_\mathrm{a} = \omega_\mathrm{l} - \omega_\mathrm{a}$ and $\Delta_\mathrm{c} = \omega_\mathrm{l} - \omega_\mathrm{c}$.

3.2 Detuning

The derivation for the Hamiltonians for the following sections is taken from [6]. Now we derive heuristically a modified Hamiltonian. Going to the Heisenberg picture, we get:

$$\dot{a} = \frac{i}{\hbar}[H, a] = i\Delta_\mathrm{c}a - i\eta - ig_0 \cos(kx)\sigma^-. \tag{14}$$

Obviously, the kinetic energy and potential term vanish under the commutator. For the other terms:

$$a^\dagger a = N, \qquad [N, a] = -a, \tag{15}$$

$$(a + a^\dagger)a - a(a + a^\dagger) = aa + a^\dagger a - aa - aa^\dagger = 1 \tag{16}$$

$$\text{because we know:} \quad aa^\dagger = a^\dagger a + 1,$$

$$[\sigma^+ a + \sigma^- a^\dagger, a] = \sigma^+ \underbrace{[a, a]}_{0} + \sigma^- \underbrace{[a^\dagger, a]}_{1} = \sigma^-. \tag{17}$$

The creation and annihilation operators ($a^\dagger$ and a) and the raising and lowering operators (σ^+ and σ^-) live in different Hilbert spaces and thus don't influence each other. A good reference for the commutator relation is [7]. The time-derivative for the raising operator reads:

$$\dot{\sigma}^+ = \frac{i}{\hbar}[H, \sigma^+] = \underbrace{-i\Delta_\mathrm{a}\sigma^+}_{(*)} + \underbrace{ig_0 \cos(kx)a^\dagger}_{(**)}. \tag{18}$$

For (*), we'll look at the matrix representation of the operators:

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19}$$

We calculate the commutator relation $[\sigma_z, \sigma^+]$ explicitly:

$$[\sigma_z, \sigma^+] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 2\sigma^+. \tag{20}$$

For (**), consider that $[\sigma^+, \sigma^-] = 1$. In our case, the pumping laser is far detuned from the atomic resonance frequency, i.e. $\Delta_\mathrm{a} = \omega_\mathrm{l} - \omega_\mathrm{a}$ is large. Thus the excitation probability of the atom is vanishing and we set $\dot{\sigma}^+ = 0$. We get:

$$\sigma^+ = \frac{g_0}{\Delta_\mathrm{a}} \cos(kx) a^\dagger, \qquad \sigma^- = \frac{g_0}{\Delta_\mathrm{a}} \cos(kx) a. \tag{21}$$

Putting the above relation in equation 14, we get:

$$\dot{a} = -i\Delta_\mathrm{c} a + \frac{ig_0}{\Delta_\mathrm{a}} \cos(kx) a - i\eta. \tag{22}$$

We can thus make a guess of the effective Hamiltonian:

$$H_\mathrm{long} = \frac{p^2}{2m} + V_\mathrm{ext}(x) - \hbar\Delta_\mathrm{c} a^\dagger a + \hbar\eta(a + a^\dagger) + \hbar U_0 \cos(kx)^2 a^\dagger a, \tag{23}$$

where we set $U_0 \coloneqq g_0^2/\Delta_\mathrm{a}$. Note that because $H_\mathrm{long} \propto \cos(kx)^2$, the Hamiltonian is $\lambda/2$-periodic. Later in the simulation program, we want to make sure all quantities are expressed in terms of the recoil energy $E_r = \hbar\omega_r$, where $\omega_r = \hbar k^2/2m$ is the recoil frequency. Therefore we factor our E_r to see what we have to type into the program:

$$\begin{aligned} H_\mathrm{long} = \hbar\omega_r \Bigg(\frac{1}{\hbar^2 k^2} p^2 + \frac{1}{\hbar\omega_r} V_\mathrm{ext}(x) - \frac{1}{\omega_r} \Delta_c a^\dagger a + \frac{1}{\omega_r} \eta(a + a^\dagger) + \\ + \frac{1}{\hbar\omega_r} U_0 \cos(kx)^2 a^\dagger a \Bigg). \end{aligned} \tag{24}$$

In the simulation program, we will thus set $\hbar = 1$ and multiply each quantity by the preceding factors.

3.3 Transversal Pump

Now we want to focus our attention at a different case where the laser is incident transversally relative to the axis of the mirrors. The cavity mode will thus only be populated by photons which were scattered off the atoms. The Hamiltonian now reads:

$$\begin{aligned} H_\mathrm{trans} = \frac{p^2}{2m} + V_\mathrm{ext}(x) - \hbar\Delta_\mathrm{c} a^\dagger a + \hbar\eta \cos(kx) \cos(kz)(a + a^\dagger) + \\ + \hbar \frac{\Omega^2}{\Delta_\mathrm{a}} \cos(kz)^2 + \hbar U_0 \cos(kx)^2 a^\dagger a, \end{aligned} \tag{25}$$

where Ω is the Rabi frequency. Let's look at the pump term a little more: When transversally pumping, there are two counter-propagating laser beams. We thus have to consider photons traveling in positive z-direction $\psi_\mathrm{photon} \propto \exp(ikz)$ and

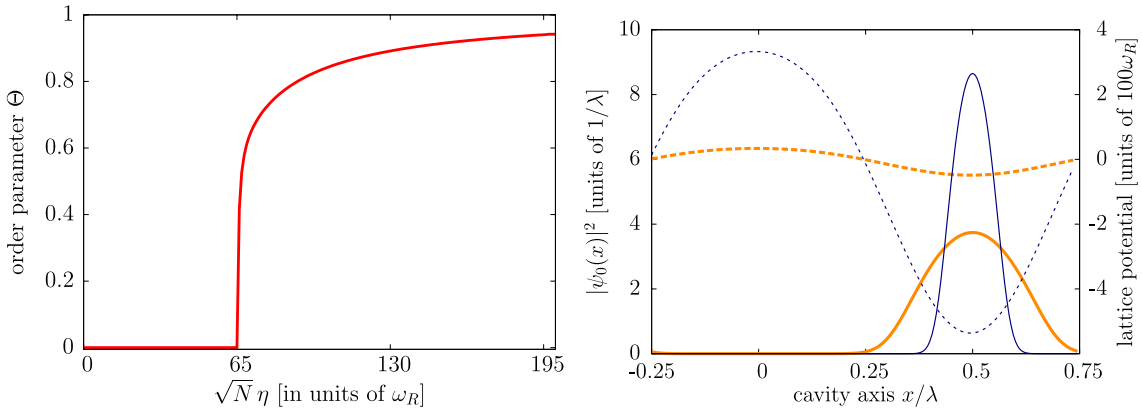
negative z-direction $\psi_{\text{photon}} \propto \exp(-ikz)$. Combined, we obtain the cosine term. For the x-direction, we can argue similarly: when an atom scatters a photon, we have to consider $\exp(ikx)$ and $\exp(-ikx)$. Here we only consider one dimension, so we set $z = 0$:

$$H_{\text{transv}} = \frac{p^2}{2m} + V_{\text{ext}}(x) - \hbar\Delta_c a^\dagger a + \hbar\eta\cos(kx)(a + a^\dagger) + \\ + \hbar U_0 \cos(kx)^2 a^\dagger a. \tag{26}$$

Note that because $H_{\text{trans}} \propto \cos(kx) + \cos(kx)^2$, the Hamiltonian is λ-periodic. As previously mentioned, keeping the right dimensionality in the simulation is very important. We define an *order parameter* Θ, which indicates whether the atoms are uniformly distributed or localized at potential minima:

$$\Theta \coloneqq \langle\psi|\cos(kx)|\psi\rangle. \tag{27}$$

At $\Theta = 0$, there's a uniform distribution and at $\Theta = \pm 1$, the atoms are localized at even or odd antinodes. An analytic solution of Θ and the lattice potential can be seen in Figure 3. For transversal pumping, there's a clear critical pumping strength η_{crit}, at which self-organization starts taking place. For longitudinal pumping, Θ would increase starting from 0.



(a) Order parameter. **(b)** Lattice potential.

Figure 3: Order parameter and lattice potential for transversal pumping. When we pump transversally, the atoms initially don't order themselves in a lattice. Only at a critical pump strength η_{crit}, self-organization takes place rapidly. The location of the atoms will be in the potential minima. Figures taken from [8].

4 Simulation

Having derived our Hamiltonians, we'll now set out to numerically simulate the systems with longitudinal and transversal pump each. In the following sections, we'll present the code necessary to simulate the quantum systems. However, any code to generate graphs will not be presented.

4.1 The Julia language and QuantumOptics.jl

Scientific computing requires high performance which low-level languages like C or Fortran can deliver. However, writing scripts in these languages can often be cumbersome. Julia is a programming language that combines the ease of use of high level languages and performance of low level languages [9]. Here we use Julia for our simulations with the framework QuantumOptics.jl [10]. The architecture of the functions of the package will not be discussed. A detailed documentation can be found at [11].

4.2 The Code

First, we'll add all the packages that we need:

```
using QuantumOptics, LinearAlgebra
```

The package `QuantumOptics` is the quantum simulation package mentioned earlier and `LinearAlgebra` is a package that comes with some useful functions like getting the diagonal entries of a matrix `diag()`. We'll set $k = 2\pi$, so that $\lambda = 1$. The recoil frequency we set $\omega_\mathrm{r} = 1$ and $\Delta_\mathrm{c} = -10\,\omega_\mathrm{r}$ and $U_0 = -1\,\omega_\mathrm{r}$:

```
k = 2*π
ωr = 1
Δc = -10 * ωr
U0 = -1 * ωr
```

For now, we'll allow a maximum of $N = 16$ photon states. Setting N_cutoff higher would increase the computational time. However, it might be necessary depending how much photons we have in the cavity which we control with the pumping strength η. The dangers of setting N_cutoff too low will be discussed in the Results. We'll confine the simulation spatially to $x_\mathrm{min} = 0$ and $x_\mathrm{max} = 1$. Setting a wider range would be redundant since the transversal Hamiltonian is λ-periodic. Usually, the step size is set to $N_\mathrm{steps} = 2^n$, where $n \in \mathbb{N}$. Here, we set $N_\mathrm{steps} = 64$ for a good compromise between simulation time and the look of the graphs.

```
N_cutoff = 16
xmin = 0
xmax = 1
Nsteps = 32
```

We define the bases, as well as the raising and lowering operators:

```
b_position = PositionBasis(xmin, xmax, Nsteps)
b_fock = FockBasis(N_cutoff)
p = momentum(b_position)
a = destroy(b_fock) ⊗ one(b_position)
ad = dagger(a)
```

The raising and lowering operators are a tensor product of the position and Fock basis. Note that in Julia it's possible to name variables with Greek symbols. In this case, the tensor product is defined with the symbol $\otimes$. We define the Hamiltonian and calculate the first three states with $\eta = 10\,\omega_\mathrm{r}$:

```
potential = x -> U0*cos(k*x)^2
H_int = (one(b_fock) ⊗ potentialoperator(b_position, potential))*ad*a
H_kin = (one(b_fock) ⊗ p^2) / k^2
H_cavity = -Δc*ad*a

function H(η)
    pump = x -> η*cos(k*x)
    H_pump = (one(b_fock) ⊗ potentialoperator(b_position, pump)) * (a + ad)
    return H_kin + dense(H_int) + H_pump + H_cavity
end

η = 10 * ωr
E, ψ_states = eigenstates((H(η) + dagger(H(η)))/2, 3)
```

If we want to plot the wave function, we'll have to extract the position part of the composite basis. We can do that with the command `ptrace()`. We thus obtain a matrix whose diagonal entries are the complex values of the wave function:

```
pos_dense = ptrace(ψ_states[1], 1)
density = diag(pos_dense.data)
```

Likewise, by changing the second argument of `ptrace()` to 2, we trace out the position basis. The diagonal entries of the obtained matrix represents the photon number distribution:

```
photon_dense = ptrace(ψ_states[1], 2)
probab = diag(photon_dense.data)
```

We can calculate the expected photon number as follows:

```
ada_exp = expect(ad*a, ψ_states[1])
```

We can also calculate the momentum distribution which is the Fourier transform of the position distribution. The function `transform()` performs a Fourier transform in the background:

```
b_momentum = MomentumBasis(b_position)
Tpx = transform(b_momentum, b_position)

pos_dense = ptrace(ψ_states[1], 1)
states_p = Tpx * pos_dense
density_p = diag(states_p.data)
```

Now let's tackle the transversal pump. The bases are the same as before. However, we have to define different Hamiltonians:

```
potential = x -> U0*cos(k*x)^2
H_int = (one(b_fock) ⊗ potentialoperator(b_position, potential))*ad*a
H_kin = (one(b_fock) ⊗ p^2) / k^2
H_cavity = -Δc*ad*a

function H(η)
    pump = x -> η*cos(k*x)
    H_pump = (one(b_fock) ⊗ potentialoperator(b_position, pump)) * (a + ad)
    return H_kin + dense(H_int) + H_pump + H_cavity
end

η = 10 * ωr
E, ψ_states = eigenstates((H(η) + dagger(H(η)))/2, 3)
```

To visualize the degree of self-organization, we'll take a look at the photon state. The Husimi Q representation is a way of visualizing a wave function. It's defined as follows:

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle, \tag{28}$$

where α is the state we want to visualize and ρ is the density operator:

$$\rho = |\psi\rangle\langle\psi|. \tag{29}$$

The Q-Function represents the state in phase space, i.e. the x-axis is the position and the y-axis is the momentum. Here we are dealing with a coherent state $|\alpha\rangle$, where $\alpha = X + iP$ and there's a quantum uncertainty of $1/2$ in each direction. Figure 4 shows a coherent state in phase space.

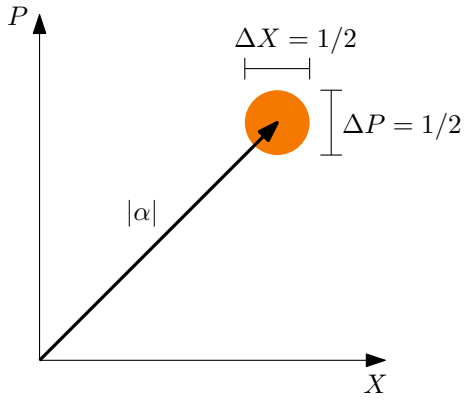


Figure 4: Coherent state in phase space. The x-axis represents the position and the y-axis represents the momentum. To each axis, there is a quantum uncertainty of $1/2$.

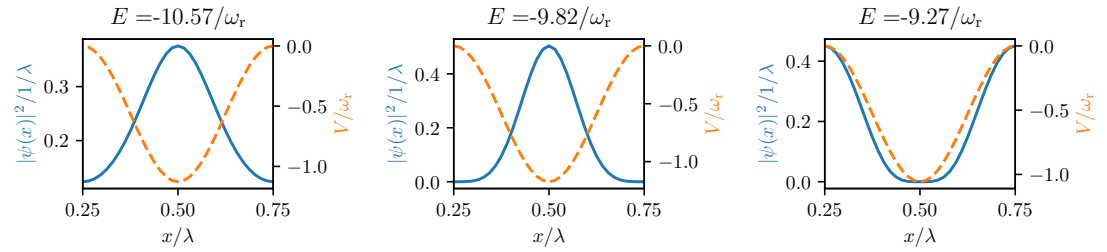
In Julia with QuantumOptics.jl, we can use the command `qfunc()` to get the phase space representation of the photon state:

```
bdr = 6
xvec = [-bdr:.1:bdr;]
yvec = [-bdr:.1:bdr;]
photon_dense = ptrace(ψ_states[1], 2)
grid = qfunc(photon_dense, xvec, yvec)
```

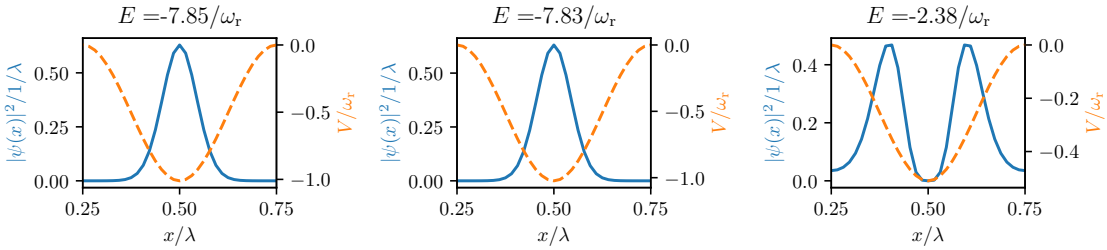
The variable `bdr` was set heuristically for plotting.

5 Results and Discussion

The longitudinal and transversal spatial wave function densities for $\eta = 10\,\omega_\mathrm{r}$ are depicted in Figure 5. The leftmost state is the first eigenstate, the second eigenstate is in the middle and the third on the right. Both for longitudinal and transversal pump, because of equal contributions of a and $a^\dagger$, the spatial densities are $\lambda/2$-periodic. The first eigenstate has the lowest energy. Each peak of the ground state density being located at the potential minima is in accordance with our expectations. However, these graphs don't give us enough insight into the physical processes of the system. For that purpose, looking at the momentum space is more fruitful.



(a) Longitudinal.

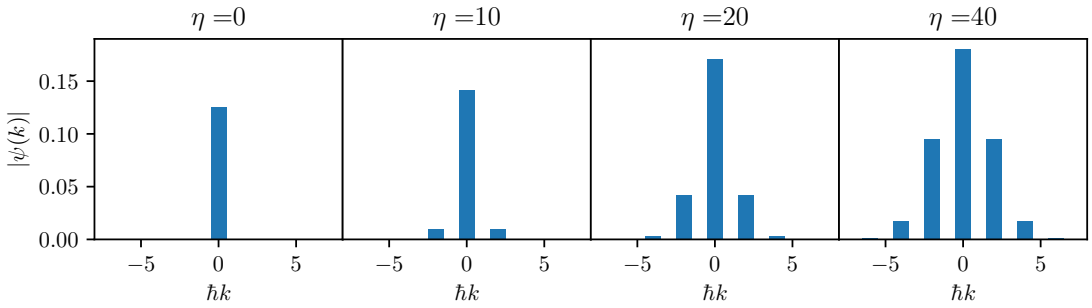


(b) Transversal.

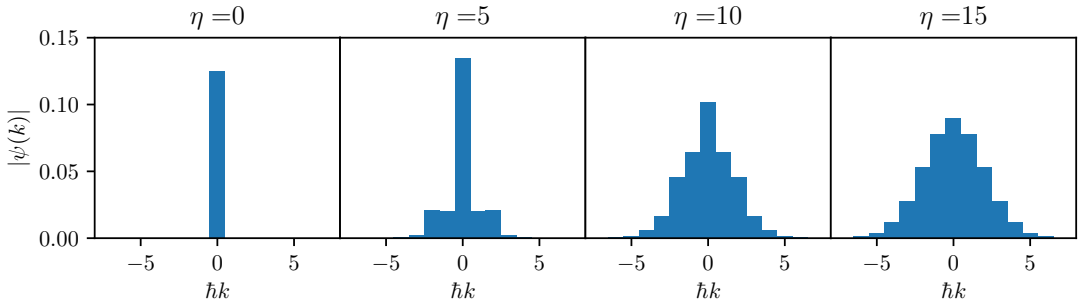
Figure 5: Longitudinal and transversal wave function densities for $\eta = 10\,\omega_\mathrm{r}$. The first eigenstate is on the left, the second in the middle and the third on the right. The wave function densities being located at potential minima meets our expectation, however much of the physics of the system remains hidden to us in this plot. The momentum plot gives us more insight.

The momentum distribution for different values of η can be seen in Figure 6. Here we can see much more of the actual physics of the system. At $\eta = 0$, i.e. when the laser is off, there's only a peak at 0, meaning the atoms have no momentum. When we start pumping, we get other peaks than 0. Now the atoms do have momentum. For longitudinal pumping, there's always a gap between each peak, which is not the case for transversal pumping. Take a look again at figure 2. When we pump longitudinally, a photon is only able to transfer a momentum of $2\hbar k$ because of momentum conservation. Thus we only observe peaks at $2n\hbar k$, where $n \in \mathbb{N}$. For transversal pumping, the same processes of photons transferring momenta of $2\hbar k$ are

happening, but now we also have a momentum transfer of $\hbar k$ when a transversally incoming photon is being scattered into the cavity. Naturally, the more we pump, the more outer momenta we will get.



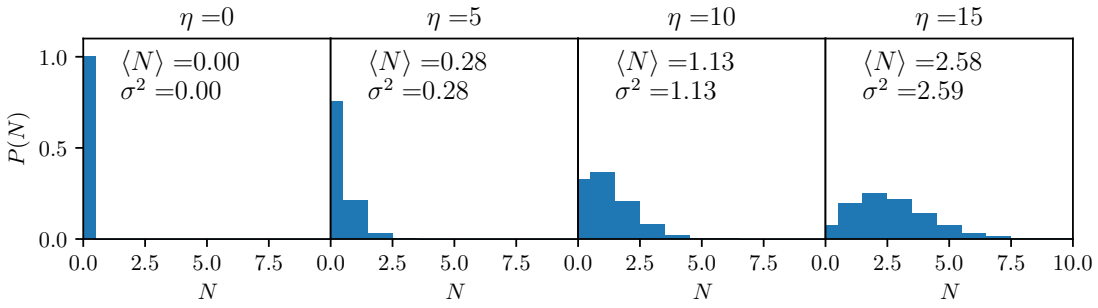
(a) Longitudinal.



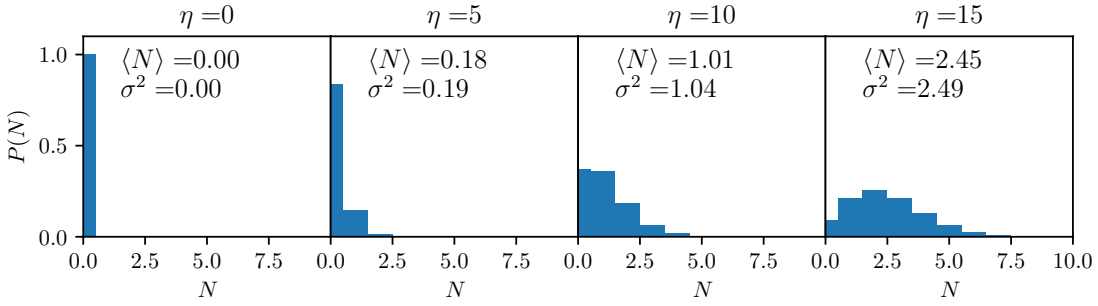
(b) Transversal.

Figure 6: Longitudinal and transversal momentum distributions. For longitudinal pump, there are only momenta of $2n\hbar k$ because longitudinal scattering processes only allow momenta transfer of $2\hbar k$. For transversal pump, there is no such restriction and we have momenta of $n\hbar k$.

Having looked at the atom part of the composite system, let's take a look at the photon part. The photon number distribution for different values of η can be seen in Figure 7. The mean and variance are pretty much the same, thus we have a Poisson distribution.



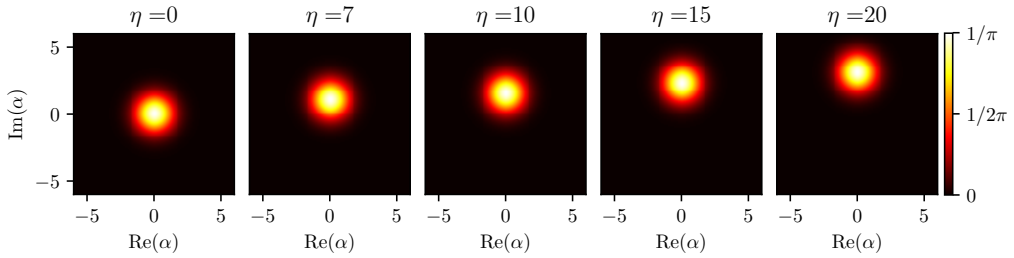
(a) Longitudinal.



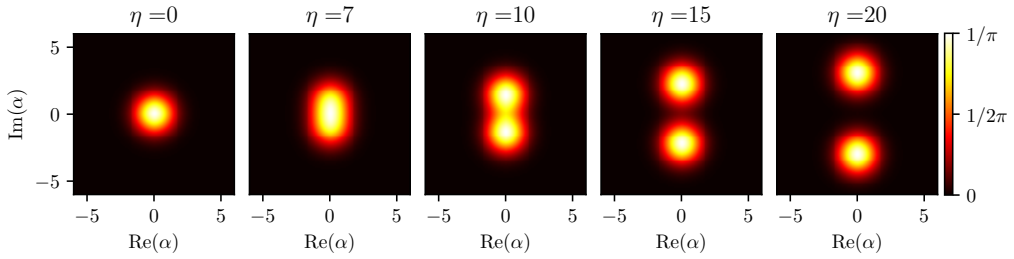
(b) Transversal.

Figure 7: Longitudinal and transversal photon distributions. Since the mean and the variance are the same, we have a Poisson distribution.

The Husimi Q representation of the photon states of both longitudinal and transversal pump can be seen in Figure 8. We remind ourselves that the x-axis represents the position while the y-axis represents the momentum of the state. To each state there's a quantum uncertainty of $1/2$, thus we don't see a point but a blob. At $\eta = 0$, there's no momentum since we don't pump at all and the blob is in the center. For longitudinal pumping, increasing η will directly result in more photon momentum and the blob rises. That's not the case for transversal pumping. Initially, increasing η will only result in more momentum uncertainty, hence the blob stretches. At a sufficient pump strength, the blob will separate into two. Two blobs means we're actually looking at the superposition of two states that are symmetric. Before the separation, the highest probability of the momentum is still at 0 meaning that when we pump transversally, the light field initially resists self-ordering.



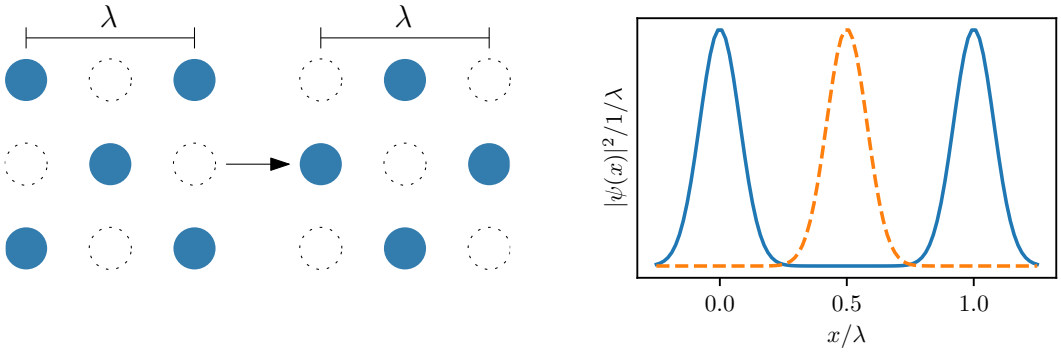
(a) Longitudinal.



(b) Transversal.

Figure 8: Husimi Q representation of photon states for longitudinal and transversal pumping. The y-axis represents the momentum of the state. For longitudinal pumping, increasing η directly results in more momentum. For transversal pumping, the state initially resists and the mean of the momentum stays at 0. Only at sufficient pump strengths, the momentum will rise which indicates self-ordering of the system. Observing two blobs for transversal pumping means we're actually looking at the superposition of two symmetric states.

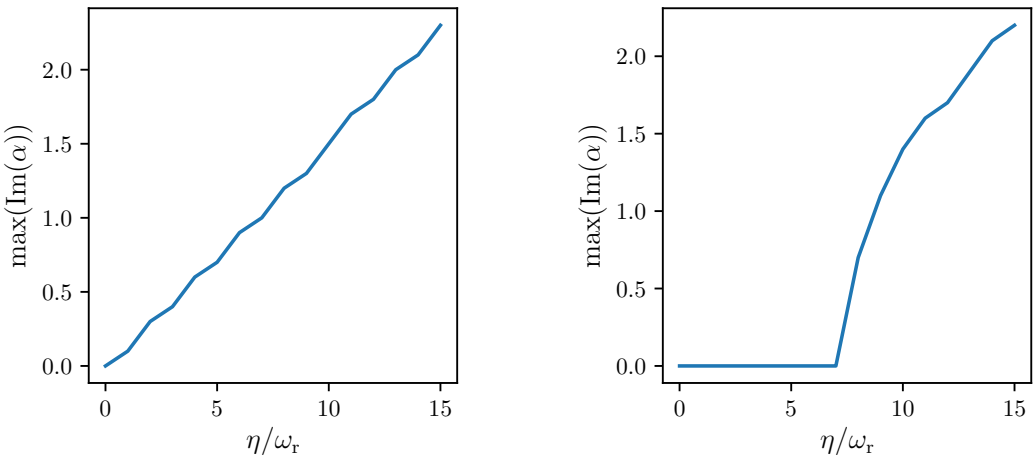
If we were to measure the system experimentally, we'd only obtain one blob since measuring means breaking the symmetry. The superposition is also reflected in the fact that the graph of Figure 5 with transversal pumping is $\lambda/2$-periodic. Actually, it should be λ-periodic. Figure 9 illustrates the superposition and the lattice of the atoms.



(a) Lattice. **(b)** Densities.

Figure 9: Lattice configuration and superposition of densities. If we were to measure the system experimentally, we'd only obtain one lattice pattern. In this simulation, however, we obtain the two configurations simultaneously, thus observing a $\lambda/2$-periodic wave function density.

We can break the symmetry artificially by only looking at one half of the graph. Figure 10 shows the momentum with the highest probability for longitudinal and transversal pumping. For longitudinal pumping, the momentum increases linearly. The light field gets stronger and stronger and the atoms will order gradually. For transversal pumping, the light field initially does not acquire any momentum which means that the atoms are resisting ordering. At a critical pump strength, we see that the momentum suddenly increases very rapidly and self-ordering of the atoms takes place.



(a) Longitudinal. **(b)** Transversal.

Figure 10: Most likely value of the photon state momentum looking at only the positive side of the phase space for longitudinal and transversal pumping. For longitudinal pumping, the momentum increases gradually. For transversal pumping, the light initially does not gain any momentum, meaning that the atoms resist being in order. At a critical pump strength, a light field suddenly builds up and the atoms self-order.

The more we pump, the more photons will appear. We have to take that into account by raising the maximum amount of allowed photon states N_{cutoff}. Raising N_{cutoff} results in longer simulation times, however if we don't do so, our results become faulty. Take a look at Figure 11 which depicts the photon number distributions for different values of N_{cutoff} at $\eta = 40\,\omega_{\text{r}}$. For our parameters, $40\,\omega_{\text{r}}$ is a relatively high value for η and we thus would expect a high average photon number which cannot be the case if limit N_{cutoff} to 8. To check the validity of our results, i.e. if N_{cutoff} is set high enough, we can look at the standard deviation. For a Poisson distribution, the mean has to be the same as the standard deviation which is not the case if we set the cutoff too low.

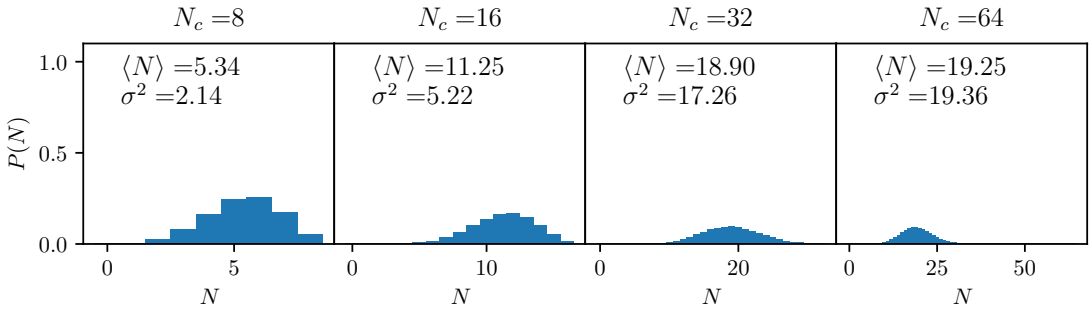


Figure 11: Photon number distributions for different values of N_{cutoff} at $\eta = 40\,\omega_{\text{r}}$. If we don't set N_{cutoff} sufficiently high, we get bogus results. A quick sanity check is to compare the mean with the variance. For a Poisson distribution, they have to be the same.

6 Outlook/Current research

Quantum simulation is a research field which is expanding rapidly. For those wishing to explore some of the recent advancements, [1] is part of a dossier with mini-reviews.

7 Conclusion

We demonstrated self-ordering in a longitudinal and transversally pumped cavity. Starting from the Jaynes-Cummings model, we modified the Hamiltonian for large detuning in a high-finesse cavity. We implemented simulations in QuantumOptics.jl, a package written in Julia. Our results meet our prior expectations and match with existing literature.

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