Notes on Spectral ~~Propertys~~ Properties of Random Graphs

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1 Introduction

In these notes, we explore the spectral properties of random graphs, with particular emphasis on the ~~eighenvalue~~ eigenvalue distribution of their adjacency matrices. The study of random graphs was initiated by Erdős and ~~Renyi~~ Rényi [1], and has since become a central topic in discrete mathematics and theoretical computer ~~scince.~~ science.

2 Preliminaries

Definition 1. *An ~~Erdős-Rényi~~ Erdős-Rényi random graph $G(n, p)$ is a graph on n vertices where each ~~posible~~ possible edge is included with probability p independent of all other edges.*

Let A be the adjacency matrix of a graph G. The eigenvalues of A are denoted $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

3 Spectral Gap in Random Regular Graphs

For a d-regular graph, it is well-known that $\lambda_1 = d$. The spectral gap, defined as $d - \lambda_2$, plays a crucial role in determining the expansion properties of the graph.

Theorem 1 (Alon-Bopanna)**.** *For any d-regular graph on n vertices, we have*

$$\lambda_2 \geq 2\sqrt{d-1} - o(1) \tag{1}$$

where the $o(1)$ ~~tends to zero as the diameter of the graph~~ term vanishes as $n \to \infty$ for sequences of graphs whose girth tends to infinity.

However, for random d-regular graphs, we can establish a stronger result.

Theorem 2 (Friedman's Theorem)**.** *For any $\epsilon > 0$, a random d-regular graph G satisfies*

$$\lambda_2(G) \leq 2\sqrt{d-1} + \epsilon \tag{2}$$

with probability approaching 1 as $n \to \infty$.

The proof relies on the trace method and careful analysis of ~~the momentsof~~ moments.

4 Concentration of Eigenvalues

For the ~~Erdős-Rényi~~ Erdős-Rényi random graph $G(n,p)$ with $p = \frac{d}{n}$ for some constant $d > 0$, the eigenvalues of the adjacency matrix exhibit interesting concentration phenomena.

Proposition 3. *Let A be the adjacency matrix of $G(n,p)$ with $p = \frac{d}{n}$. Then, as $n \to \infty$:*

1. $\lambda_1 = (1 + o(1))np = (1 + o(1))d$ *almost surely.*
2. *For $i \geq 2$,* ~~$|\lambda_i| \leq (2 + o(1))\sqrt{np(1-p)} = (2 + o(1))\sqrt{d}$ almost surely~~ $|\lambda_i| \leq (2 + o(1))\sqrt{np(1}$ *almost surely.*

This shows that the bulk of the spectrum is concentrated in a band of width $O(\sqrt{d})$ around the origin, while the largest eigenvalue is separated from this band.

References

[1] P. Erdős and A. ~~Rényi,~~ Rényi, *On the evolution of random graphs*, Publ. Math. Inst. Hung. Acad. Sci, 5(1):17–60, 1960.

[2] J. Friedman, *A proof of Alon's second eigenvalue conjecture and related problems*, Memoirs of the American Mathematical Society, 195(910), 2008.

[3] E. P. Wigner, *On the distribution of the roots of certain symmetric matrices*, Annals of Mathematics, 67(2):325–327, 1958.