# LMMSE

## 1 LMMSE

### 1.1 MMSE $\rightarrow$ LMMSE

#### 1.1.1 Mean

The zero forcing estimation is

$$\hat{x} = R_x H^H (H^H R_x H + \sigma^2 I)^{-1} H^H y = R_x (H^H H R_x + \sigma^2 I)^{-1} y \tag{1}$$

The proof is given. Suppose $\hat{h} = Cy$, we can minimize

$$\begin{aligned} J(C) &= E\{\|x - \hat{x}\|^2\} \\ &= E\{(x - Cy)^H (x - Cy)\} \\ &= E\{x^H x - x^H C y - y^H C^H x + y^H C^H C y\} \\ &= E\{tr(h^H h) - tr(h^H C y) - tr(y^H C^H h) + tr(y^H C^H C y)\} \end{aligned} \tag{2}$$

Therefore, we calculate the derivative,

$$\begin{aligned} \frac{\partial J(C)}{\partial C} &= \frac{\partial tr(x^H x)}{\partial C} - \frac{\partial tr(x^H C y)}{\partial C} - \frac{\partial tr(y^H C^H x)}{\partial C} + \frac{\partial tr(y^H C^H C y)}{\partial C} = 0 \\ &= 0 - x^* y^T - 0 + \frac{\partial tr(C y y^H C^H)}{\partial C} \\ 0 &= -x^* y^T + C^* (y y^H)^T \\ x^* y^T &= C^* (y y^H)^T \\ (x y^H)^* &= C^* (y y^H)^T \\ R_{xy}^* &= C^* R_y^T \end{aligned} \tag{3}$$

Here, $R_y$ is a Hermitian matrix, so $R_y^T = R_y^*$. (3) can be written as,

$$\begin{aligned} R_{xy}^* &= C^* R_y^* \\ R_{xy} &= C R_y \\ C &= R_{xy} R_y^{-1} \end{aligned} \tag{4}$$

Here,

$$\begin{aligned} R_{xy} &= E\{x y^H\} = E\{x (Hx + z)^H\} \\ &= E\{x x^H H^H\} + E\{x z^H\} \\ &= E\{x x^H\} H^H \\ &= R_x H^H \end{aligned} \tag{5}$$

and,

$$\begin{aligned} R_y &= E\{yy^H\} = E\{(Hx+z)(Hx+z)^H\} \\ &= E\{Hxx^H H^H\} + E\{zz^H\} \\ &= HR_xH^H + \sigma^2 I \end{aligned} \tag{6}$$

We load (5) and (6) into (4) to estimate symbols,

$$\begin{aligned} \hat{x} &= Cy = R_{xy}R_y^{-1}y \\ &= R_xH^H(HR_xH^H + \sigma^2 I)^{-1}y \\ &= R_x(H^HHR_x + \sigma^2 I)^{-1}H^Hy \end{aligned} \tag{7}$$

As with the result in (7),

$$\begin{aligned} (H^HHR_x + \sigma^2 I)H^H &= H^HHR_xH^H + \sigma^2 IH^H \\ &= H^HHR_xH^H + H^H\sigma^2 I \\ &= H^H(HR_xH^H + \sigma^2 I) \\ (H^HHR_x + \sigma^2 I)^{-1}(H^HHR_x + \sigma^2 I)H^H &= (H^HHR_x + \sigma^2 I)^{-1}H^H(HR_xH^H + \sigma^2 I) \\ H^H &= (H^HHR_x + \sigma^2 I)^{-1}H^H(HR_xH^H + \sigma^2 I) \\ H^H(HR_xH^H + \sigma^2 I)^{-1} &= (H^HHR_x + \sigma^2 I)^{-1}H^H(HR_xH^H + \sigma^2 I)(HR_xH^H + \sigma^2 I)^{-1} \\ H^H(HR_xH^H + \sigma^2 I)^{-1} &= (H^HHR_x + \sigma^2 I)^{-1}H^H \\ R_xH^H(HR_xH^H + \sigma^2 I)^{-1} &= R_x(H^HHR_x + \sigma^2 I)^{-1}H^H \end{aligned} \tag{8}$$

If we estimate $R_x$ as a power of $x$, $R_x$ is a diagonal matrix with the power of $x$. (7) can be written as

$$\hat{x} = diag(P_x) \cdot (H^HH \cdot diag(P_x) + \sigma^2 I)^{-1}H^Hy, \tag{9}$$

where $P_x = [P_{x_0}, P_{x_1}, \cdots P_{x_n}]$ is a vector of the signal power for each $x$.

### 1.1.2 Variance

The error of LMMSE is

$$\begin{aligned} D_e &= E\{(\hat{x}-x)(\hat{x}-x)^H\} \\ &= E\{\hat{x}\hat{x}^H\} - E\{x\hat{x}^H\} - E\{\hat{x}x^H\} + E\{xx^H\} \end{aligned} \tag{10}$$

Please note that the error is orthogonal to the estimation (that is, the correlation matrix is all zero), i.e.,

$$\begin{aligned} E\{e\hat{x}^H\} &= 0 \\ E\{(\hat{x}-x)\hat{x}^H\} &= 0 \\ E\{(\hat{x}\hat{x}^H\} - E\{x\hat{x}^H\} &= 0 \\ E\{\hat{x}\hat{x}^H\} &= E\{x\hat{x}^H\} \\ E\{(\hat{x}\hat{x}^H)^H\} &= E\{(x\hat{x}^H)^H\} \\ E\{\hat{x}\hat{x}^H\} &= E\{\hat{x}x^H\} \end{aligned} \tag{11}$$

We load (11) into (10),

$$
\begin{aligned}
cov(\hat{x}) &= E\{\hat{x}\hat{x}^H\} - E\{\hat{x}\hat{x}^H\} - E\{\hat{x}\hat{x}^H\} + E\{xx^H\} \\
&= E\{xx^H\} - E\{\hat{x}\hat{x}^H\} \\
&= R_x - R_x H^H R_y^{-1} y y^H {R_y^{-1}}^H H R_x^H \\
&= R_x - R_x H^H R_y^{-1} R_y R_y^{-1} H R_x \\
&= R_x - R_x H^H R_y^{-1} H R_x \\
&= R_x - R_x H^H (H R_x H^H + \sigma^2 I)^{-1} H R_x \\
&= R_x - R_x (H^H H R_x + \sigma^2 I)^{-1} H^H H R_x \\
&= R_x (I - (H^H H R_x + \sigma^2 I)^{-1} H^H H R_x) \\
&= R_x ((H^H H R_x + \sigma^2 I)^{-1} (H^H H R_x + \sigma^2 I) - (H^H H R_x + \sigma^2 I)^{-1} H^H H R_x) \\
&= R_x (H^H H R_x + \sigma^2 I)^{-1} (H^H H R_x + \sigma^2 I - H^H H R_x) \\
&= R_x (H^H H R_x + \sigma^2 I)^{-1} \sigma^2 I \\
&= \sigma^2 R_x (H^H H R_x + \sigma^2 I)^{-1}
\end{aligned}
\tag{12}
$$

Therefore, the variance is

$$
var(\hat{x}) = \sigma^2 \cdot diag(R_x (H^H H R_x + \sigma^2 I)^{-1}) \tag{13}
$$

## 1.2 Using the Estimated Channel Matrix

If we use the estimated channel matrix $H$, the system model can be updated as

$$
\begin{aligned}
y &= (\hat{H} + H - \hat{H})x + z \\
&= \hat{H}x + (H - \hat{H})x + z \\
&= \hat{H}x + \Delta H x + z \\
&= \hat{H}x + \tilde{z}
\end{aligned}
\tag{14}
$$

As $(H - \hat{H})x$ and $z$ are independent, the covariance of $\tilde{z}$ is

$$
\begin{aligned}
cov(\tilde{z}) &= E\{\Delta H x (\Delta H x)^H\} + E\{zz^H\} \\
&= E\{\Delta H x x^H \Delta H^H\} + \sigma^2 I
\end{aligned}
\tag{15}
$$

the error is

$$
\begin{aligned}
e &= \hat{x} - x = Cy - x \\
&= C(\hat{H}x + z + Hx - \hat{H}x) - x \\
&= C(\hat{H}x + z) + C(H - \hat{H})x \\
&= \hat{x} + C(H - \hat{H})x
\end{aligned}
\tag{16}
$$