Zero Forcing

1 Zero Forcing

1.1 Hermitian Matrix

If the Hermitian transpose of a matrix is itself, this matrix is called the Hermitian matrix, i.e,

$$A^H = A \tag{1}$$

Also, the inverse of a Hermitian matrix is Hermitian, i.e,

$$\begin{aligned} I &= AA^{-1} \\ I = I^H &= (AA^{-1})^H = (A^{-1})^H A^H = (A^{-1})^H A \\ I &= (A^{-1})^H A \\ A^{-1} &= (A^{-1})^H AA^{-1} \\ A^{-1} &= (A^{-1})^H \end{aligned} \tag{2}$$

1.2 The Implementation of the Channel Matrix

We implement the Hermitian matrix knowledge into the channel matrix H: $H^H H$ or HH^H is a Hermitian matrix, i.e,

$$\begin{aligned} (H^H H)^H &= H^H (H^H)^H = H^H H \\ (HH^H)^H &= (H^H)^H H^H = HH^H \end{aligned} \tag{3}$$

Here, we need to know that $(AB)^H = B^H A^H$. Also,

$$\begin{aligned} ((H^H H)^{-1})^H &= (H^H H)^{-1} \\ ((HH^H)^{-1})^H &= (HH^H)^{-1} \end{aligned} \tag{4}$$

1.3 Least Square

1.3.1 Mean

The zero forcing estimation is

$$\hat{x} = (H^H H)^{-1} H^H y \tag{5}$$

1.3.2 Variance

The error of zero forcing is

$$
\begin{aligned}
e &= \hat{x} - x \\
&= (H^H H)^{-1} H^H y - x \\
&= (H^H H)^{-1} H^H (Hx + n) - x \\
&= (H^H H)^{-1} H^H H x - x + (H^H H)^{-1} H^H n \\
&= H^{-1} H x - x + (H^H H)^{-1} H^H n \\
&= (H^H H)^{-1} H^H n
\end{aligned} \tag{6}
$$

Therefore, the covariance is

$$
\begin{aligned}
cov(\hat{x}) &= E(ee^H) \\
&= E[(H^H H)^{-1} H^H n((H^H H)^{-1} H^H n)^H] \\
&= E[(H^H H)^{-1} H^H n n^H H (H^H H)^{-1}] \\
&= (H^H H)^{-1} H^H E[nn^H] H (H^H H)^{-1} \\
&= (H^H H)^{-1} H^H \sigma^2 H (H^H H)^{-1} \\
&= \sigma^2 (H^H H)^{-1} H^H H (H^H H)^{-1} \\
&= \sigma^2 (H^H H)^{-1} [H^H H (H^H H)^{-1}] \\
&= \sigma^2 (H^H H)^{-1} I \\
&= \sigma^2 (H^H H)^{-1}
\end{aligned} \tag{7}
$$

The variance is the diagonal, i.e.,

$$var(\hat{x}) = diag(\sigma^2 (H^H H)^{-1}) \tag{8}$$

1.4 Using the Estimated Channel Matrix

If we use the estimated channel matrix H, the error is

$$
\begin{aligned}
e = \hat{x} - x &= (\hat{H}^H \hat{H})^{-1} \hat{H}^H y - x \\
&= (\hat{H}^H \hat{H})^{-1} \hat{H}^H (Hx + n) - x \\
&= (\hat{H}^H \hat{H})^{-1} \hat{H}^H H x - x + (\hat{H}^H \hat{H})^{-1} \hat{H}^H n \\
&= (\hat{H}^H \hat{H})^{-1} \hat{H}^H (\hat{H} + H - \hat{H}) x - x + (\hat{H}^H \hat{H})^{-1} \hat{H}^H n \\
&= (\hat{H}^H \hat{H})^{-1} \hat{H}^H \hat{H} x - x + (\hat{H}^H \hat{H})^{-1} \hat{H}^H (H - \hat{H}) x + (\hat{H}^H \hat{H})^{-1} \hat{H}^H n \\
&= (\hat{H}^H \hat{H})^{-1} \hat{H}^H (H - \hat{H}) x + (\hat{H}^H \hat{H})^{-1} \hat{H}^H n
\end{aligned} \tag{9}
$$

For simplification, we assume $W = (\hat{H}^H \hat{H})^{-1} \hat{H}^H$ and $\Delta H = \hat{H} - H$. Therefore, (9) can be written as

$$e = -W \Delta H x + W n \tag{10}$$

Therefore, assuming the noise $n \sim \mathcal{CN}(0, \sigma^2)$, we can write the covariance as

$$
\begin{aligned}
cov(\hat{x}) &= E(ee^H) \\
&= E[W\Delta Hx(W\Delta Hx)^H] + E(Wnn^HW^H) \\
&= E(W\Delta Hxx^H\Delta H^HW^H) + E(Wnn^HW^H) \\
&= \sigma_x^2 E(\Delta H\Delta H^H)WW^H + \sigma^2 WW^H
\end{aligned} \tag{11}
$$

The WW^H in (11) can be simplified as,

$$
\begin{aligned}
WW^H &= (\hat{H}^H\hat{H})^{-1}\hat{H}^H((\hat{H}^H\hat{H})^{-1}\hat{H}^H)^H \\
&= (\hat{H}^H\hat{H})^{-1}\hat{H}^H\hat{H}(\hat{H}^H\hat{H})^{-1} \\
&= (\hat{H}^H\hat{H})^{-1}
\end{aligned} \tag{12}
$$

Therefore, (11) can be written as

$$
cov(\hat{x}) = \sigma_x^2 W E(\Delta H\Delta H^H)W^H + \sigma^2(\hat{H}^H\hat{H})^{-1} \tag{13}
$$

Herefore, the variance is

$$
\begin{aligned}
var(\hat{x}) &= diag(\sigma_x^2 W E(\Delta H\Delta H^H)W^H + \sigma^2(\hat{H}^H\hat{H})^{-1}) \\
&= diag(\sigma_x^2 W E(\Delta H\Delta H^H)W^H) + diag(\sigma^2(\hat{H}^H\hat{H})^{-1}) \\
&= \sigma_x^2 diag(W E(\Delta H\Delta H^H)W^H) + \sigma^2 diag((\hat{H}^H\hat{H})^{-1})
\end{aligned} \tag{14}
$$

Here, $E(\Delta H\Delta H^H)$ is a diagonal matrix because each value in ΔH is independent, i.e.,

$$
E(\Delta H\Delta H^H) = D_e = diag(\sum_{j=0}^{N_t} \Sigma_{\hat{H}}[:, j]), \tag{15}
$$

where $D_e \in \mathbb{R}^{N_r \times N_r}$ and $\Sigma_{\hat{H}} \in \mathbb{R}^{N_r \times N_t}$. $\sum_{j=0}^{N_t} \Sigma_{\hat{H}}[:, j]$ means we sum all columns of $\Sigma_{\hat{H}}$ together to build a column vector.

1.5 Variance

The variance can be expressed in an explicit form,

$$
var(\hat{x}_i) = \underbrace{E(|[Wn]_i|^2)}_{Noise\,Part} + \underbrace{E(|[W\Delta Hx]_i|^2)}_{Channel\,Estimation\,Error\,Part} \tag{16}
$$

where $E(|[Wn]_i|^2)$ comes from the noise and $E(|[W\Delta Hx]_i|^2)$ comes from the channel estimation error. After simplification, (16) can be written as

$$
var(\hat{x}_i) = \underbrace{\sigma^2 \sum_{j=1}^{N_r} |W_{i,j}|^2}_{Noise\,Part} + \underbrace{\sigma_x^2 \sum_{k=1}^{N_t}\sum_{j=1}^{N_r} |W_{ij}|^2 \sigma_{e,jk}^2}_{Channel\,Estimation\,Error\,Part} \tag{17}
$$

1.5.1 Noise Part

$$
E(|[Wn]_i|^2) = \sigma^2 ||W_{i,:}||^2 = \sigma^2 \sum_{j=1}^{N_r} |W_{i,j}|^2 \tag{18}
$$

where $W_{i,:}$ means we take ith row from W

1.5.2 Channel Estimation Error Part

$$\begin{aligned}
[W\Delta Hx]_i &= \sum_{j=1}^{N_r} W_{ij} \cdot (\sum_{k=1}^{N_t} \Delta H_{jk} x_k) \\
&= \sum_{j=1}^{N_r} \sum_{k=1}^{N_t} W_{ij} \Delta H_{jk} x_k \\
&= \sum_{k=1}^{N_t} \sum_{j=1}^{N_r} W_{ij} \Delta H_{jk} x_k \\
&= \sum_{k=1}^{N_t} x_k (\sum_{j=1}^{N_r} W_{ij} \Delta H_{jk})
\end{aligned} \tag{19}$$

Here, x_k and ΔH_{jk} are independent with zero means, i.e.,

$$\begin{aligned}
E[|[W\Delta Hx]_i|^2] &= \sum_{k=1}^{N_t} \sigma_x^2 E\left[\left|\sum_{j=1}^{N_r} W_{ij} \Delta H_{jk}\right|^2\right] \\
&= \sigma_x^2 \sum_{k=1}^{N_t} E\left[\left|\sum_{j=1}^{N_r} W_{ij} \Delta H_{jk}\right|^2\right] \\
&= \sigma_x^2 \sum_{k=1}^{N_t} \sum_{j=1}^{N_r} |W_{ij}|^2 \sigma_{e,jk}^2
\end{aligned} \tag{20}$$

where $\sigma_{e,jk}$ is (j, k) entry of the estimated channel variance matrix $\Sigma_{\Delta H}$.