Math 401: Homework 4

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Problem 1

Suppose A and B are nonempty sets of negative numbers.

a)

Prove that $\sup(A \cup B)$ exists.

Proof: Because A and B both have an upper bound of 0, 0 is an upper bound of $A \cup B$. Furthermore, $A \cup B \neq \emptyset$, since $A \neq \emptyset$ and $B \neq \emptyset$. Therefore, $A \cup B$ is a nonempty subset of $\mathbb{R}$ and is bounded above, so by the completeness axiom, $A \cup B$ has a supremum.

b)

Prove that $\sup(A) \leq \sup(A \cup B)$.

Proof. Suppose $\sup(A) > \sup(A \cup B)$. This implies that $\forall c \in C = (A \cup B)$, $\exists a \in A$ such that $a > c$. This is because by definition of the supremum we have that for any number $\alpha' < \sup(A)$, there exists a number $a \in A$ such that $a > \alpha'$. Take $\sup(A \cup B) = \alpha'$ demonstrates the implication.

But we also have that $\forall a \in A$, $\exists c \in C$ such that $c = a$, a clear contradiction that $a > c$.

Therefore, $\sup(A) \not> \sup(A \cup B) \Leftrightarrow \sup(A) \leq \sup(A \cup B)$. $\square$

Problem 2

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Consider also the set

$$C = \left\{ \left(-\frac{1}{2}\right)^n \middle| n \in \mathbb{N} \right\}$$

Compute the following:

a)

$\inf((1,2)) = 1$

b)

$\inf(f((1,2))) = \inf((1,4))) = 1$, because f is monotonic increasing on the interval $(1,2)$.

c)

$\inf(C) = \inf([-1/2, 1/4]) = -1/2$, because the sequence defined by C alternates from negative to positive across increasing n, and decreases in absolute value. Hence $C(1) = -1/2$ is it's most extremely negative value, and $C(2) = 1/4$ is it's most extreme positive value, and the values of $C(n)$ are therefore contained in the closed interval $[-1/2, 1/4]$.

d)

$\inf((f(C))) = \inf((0, f(-1/2)]) = \inf((0, 1/4]) = 0$, because f maps all values to the positive reals, and $C(n)$ approaches but never equals zero as $n \to \infty$.

Problem 3

Consider two sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ with restrictions

$$0 \leq a_n \leq 1 \quad \text{and} \quad 0 \leq b_n \leq 1$$

for all n. We can form two different sets by adding these:

$$C = \{a_n + b_n | n \in \mathbb{N}\}$$

and

$$D = \{a_m + b_n | m, n \in \mathbb{N}\}$$

a)

Prove that $\sup(C) \leq \sup(D)$.

Proof: Suppose that $\sup(C) > \sup(D)$. This implies $\exists c \in C$ s.t. $\forall d \in D$ we have $c > d$. This is because by definition of the supremum we have that for any number $\alpha' < \sup(C)$, there exists a number $c \in C$ such that $c > \alpha'$. Take $\sup(D) = \alpha'$ demonstrates the implication. But we also have that $\forall c \in C, \exists d \in D$ s.t. $c = d$, a clear contradiction. This last statement is true because we have $\forall n \in \mathbb{N}, \exists m \in N$ s.t. $m = n$, yielding $c = d$. $\square$

b)

Find an example of two sequences for which $\sup(C) \neq \sup(D)$.

An example is $a_n = 1 - 1/n$, which increases monotonically with range = [0, 1), and $g_n = 1/n$, which decreases monotonically with range = (0, 1].

$C = \{1\}$ is a constant function of n, and hence has $\sup(C) = 1$. $D = (0, 2)$, and hence has $\sup(D) = 2$