Math 401: Homework 5

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Problem 1

Prove that if $S \subseteq T \subseteq \mathbb{R}$, then every interior point of S is also an interior point of T.

Lemma. If $S \subseteq T$, then $T^c \subseteq S^c$.

Proof. If $S \subseteq T$ then $S \cup T = T$. Taking the complement yields $S^c \cap T^c = T^c$, indicating that $T^c \subseteq S^c$. $\square$

Proof of Problem 1. Take x an interior point of S. Then by definition $\exists \epsilon > 0$ s.t. $N(x, \epsilon) \cap (\mathbb{R}/S) = \emptyset$. Writing $\mathbb{R}/S$ as S^c, by the above lemma, we have that

$$\begin{aligned}
\emptyset &= N(x,\epsilon) \cap S^c \\
&= N(x,\epsilon) \cap (S^c \cup T^c) && \text{(applying the lemma)} \\
&= (N(x,\epsilon) \cap S^c) \cup (N(x,\epsilon) \cap T^c) \\
&= \emptyset \cup (N(x,\epsilon) \cap T^c) \\
&= N(x,\epsilon) \cap T^c
\end{aligned}$$

The above shows that if $N(x, \epsilon) \cap S^c = \emptyset$, we also have $N(x, \epsilon) \cap T^c = \emptyset$, proving that any interior point of S is also an interior point of T.

Problem 2

Define f: $\mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

a)

What are the interior points of $S = f((0, 1))$, and is S an open set?

Because f is monotonic increasing on (0, 1), we have that $S = (f(0), f(1)) = (0, 1)$. Hence the interior points are (0, 1). S is an open set because it's boundary points, 0 and 1, are not in S.

b)

What are the interior points of $S = f((-1, 2))$, and is S an open set?

f is monotic decreasing on $(-1, 0]$ and monotonic increasing on $[0, 2)$. Hence, we have that $S = [0, 4)$.

The interior points of S are (0, 4), and S is not open, since the boundary point at 0 is not in S.

Problem 3

Take f as defined above.

a)

What are the interior points of $S = f^{-1}((0, 1))$, and is S an open set?

$S = (-1, 0) \cup (0, 1)$. The interior points of S are equal to S, so yes, S is an open set.

b)

What are the interior points of $S = f^{-1}((-1, 4))$, and is S an open set?

The preimage $f^{-1}((-1, 0)) = \emptyset$, so $f^{-1}((-1, 4)) = f^{-1}([0, 4)) = (-2, 2)$. S is an open set, because the interior of S is equal to S.

Problem 4

Take f as defined above.

a)

What are the boundary points of $S = f^{-1}([0, 1])$, and is S a closed set?

$S = [-1, 1]$. The boudary points are -1 and 1. Yes, S is closed, because both boundary points are in S.

b)

What are the boundary points of $S = f^{-1}((-1, 4])$, and is S a closed set?

$S = [-2, 2]$, so the boundary points are -2 and 2. S is a closed set, because both boundary points are in S.

Problem 5

Let

$$S = \left\{ \frac{1}{n} \middle| n \in (\mathbb{Z}/\{-1, 0, 1\}) \right\}$$

Show that S is not compact by findind an open cover of S that has no finite subcover.

An open cover $\mathcal{F}$ of S is

$$\mathcal{F} = \left\{ \left(-1, -\frac{1}{n+1}\right) \cup \left(\frac{1}{n+1}, 1\right) \middle| n \in \mathbb{N}) \right\}$$

But $\mathcal{F}$ has no finite subcover of S.