Math 401: Homework 9

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Problem 1

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & x \text{ is rational} \\ 2\,|x| & x \text{ is irrational} \end{cases}$$

Prove that f is continuous at $c = 0$.

Proof. Take $\delta = \epsilon/2$. There are two cases.

If x is rational, then we have $|x - c| = |x - 0| = |x| < \delta = \epsilon/2$. This implies that $|f(x) - f(0)| = |x - 0| = |x| < \delta = \epsilon/2 < \epsilon$.

If x is irrational, then we have $|f(x) - f(0)| = |2\,|x|| = 2\,|x| < 2\delta = \epsilon$.

Hence in both cases, $|x - 0| < \delta = \epsilon/2$ implies $|f(x) - 0| < \epsilon$, proving f is continous at $c = 0$.

Problem 2

Given f as in Problem 1, show that f is not continuous at $c = 2$.

We prove by showing that there exists a sequence (a_n) that converges to $c = 2$, but which $f((a_n))$ does not converge to $f(2)$.

Proof. Take $(a_n) = 2 - \frac{1}{\sqrt{n}}$, where n is a prime number. Then we have that a_n is irrational for all n, and hence $f((a_n)) = 2\left|2 - \frac{1}{\sqrt{n}}\right| = \left|4 - \frac{2}{\sqrt{n}}\right|$. Then (a_n) converges to 2, but $f((a_n))$ converges to 4, but $f(2) = 2$ because 2 is rational, so $\lim_{n\to\infty} f((a_n)) \neq f(2)$, and f is not continous at $c = 2$.

Problem 3

Suppose $f, g, h : \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$, and $f(x) \leq g(x) \leq h(x)$ for all $x \in \mathbb{R}$.

Show that if $f(c) = h(c)$ and f and h are continuous at c, that g is also continous at c.

Proof: Since $f(x) \leq g(x) \leq h(x)$ for all x (including c), we have that $f(c) = h(c)$ implies that $f(c) = g(c) = h(c)$.

Now, since f is continous at c, we have that $\forall \epsilon > 0$, $\exists \delta_1 > 0$, such that $|x - c| < \delta_1$ implies that $|f(x) - f(c)| < \epsilon \Leftrightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$.

Likewise for h, since h is continous at c, we have that $\forall \epsilon > 0$, $\exists \delta_2 > 0$, such that $|x - c| < \delta_1$ implies that $|h(x) - h(c)| < \epsilon \Leftrightarrow h(c) - \epsilon < h(x) < h(c) + \epsilon$.

Take $\delta^* = \min\{\delta_1, \delta_2\}$. First, if $|x - c| < \delta^*$, we have that both f and h are continuous, since $\delta * \leq \delta_1$ and $\delta^* \leq \delta_2$.

Then, since $f(x) \leq g(x) \leq h(x)$, we have that $f(c) - \epsilon < f(x) < g(x) < h(x) < h(c) + \epsilon$, and since $f(c) = g(c) = h(c)$, we have that $g(c) - \epsilon < g(x) < g(c) + \epsilon \Leftrightarrow |g(x) - g(c)| < \epsilon$, proving that g is continuous at c.

Problem 4

Let

$$D = \{0\} \cup \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$$

and suppose $f : D \to \mathbb{R}$ is a function. Let (a_n) be the sequence defined

$$a_n = f\left(\frac{1}{n}\right)$$

Show that if $\lim_{n\to\infty} a_n = f(0)$, then f is continuous at 0.

Proof. Take $x_n = \frac{1}{n}$. Then we have that (x_n) converges to 0, and we have been given that $a_n = f((x_n))$ converges to $f(0)$. Hence, by Theorem 21.2b, f is continuous at 0.