Math 401: Recitation Notes

Tim Farkas

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Def'n: Let $x \in \mathbb{R}$, and $\epsilon > 0$. Then a neighborhood of x is the set $N(x, \epsilon) = \{y \in \mathbb{R} : |y - x| < \epsilon$.

Def'n: If $S \subseteq \mathbb{R}$, then $x \in \mathbb{R}$ is an interior point of S if $\exists \epsilon > 0$, s.t. $N(x, \epsilon) \subseteq S$. A boundary point of S is $\forall \epsilon > 0 : N(x, \epsilon) \cap S \neq$ AND $N(x, \epsilon) \cap S^c \neq \emptyset$.

Def'n: Let $S \subseteq \mathbb{R}$, then S is

emphclosed if $\text{bd}(S) \subseteq S$

emphopen if $\text{bd}(S) \subseteq S^c$

Thm: Let $S \subseteq \mathbb{R}$, then

S is open iff $S = \text{int}(S)$.

Example: Prove that if $A \subseteq \mathbb{R}$, then $\text{int}(A) \subseteq A$.

Proof: Let $x \in \text{int}(A)$. Implies $\exists \epsilon > 0$ s.t. $N(x, \epsilon) \subseteq A$. Then $x \in N(x, \epsilon) \subseteq A$, thus $x \in A$.

Example: Prove that if $A \subseteq B$, then $\text{int}(A) \subseteq \text{int}(B)$.

Proof: Let $x \in A$. Then $\exists \epsilon > 0$ s.t. $N(x, \epsilon) \in A$. But then $x \in N(x, \epsilon) \subseteq A \subseteq B$. Hence $x \in B$.

Example: If $a < b$, prove that (a, b) is an open set.

Proof. We show that $(a, b) = \text{int}((a, b))$.

To show that $\text{int}(a, b) \subset (a, b)$, take $x \in \text{int}(a, b)$. Then if $\epsilon < \min(|b - x|, |a - x|)$. For the other direction, take $x \in (a, b)$ with ϵ as above, then $N(x, \epsilon) \subseteq (a, b)$, so $x \in \text{int}((a, b))$.

Example: If $a < b$, prove that $[a, b]$ is a closed set.

We show that $\text{bd}([a, b]) \subseteq S$.

$[a, b]^c = (-\infty, a) \cup (b, \infty)$. Each of $(-\infty, a)$ and (b, ∞) are open, so $[a, b]$ is closed.

Defn: open cover Defn: finite subcover Defn: $S \in \mathbb{R}$ is called *compact* if for *every* open cover of S there exists some finite subcover. Defn: x is an *limit point* of S if for every $\epsilon > 0$, $N(x, \epsilon)$ contains a point from S different from x.

Example: Let

$$S = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$$

Is S bounded? Yes, at 0 and 1. Is S closed? No, since 0 is a boundary point of S, but $0 \notin S$. Is S compact? No, since no finite subcover of S if U is scaled neighborhood of each point in S.

Example: Let

$$\bar{S} = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\} \cup 0$$

Is $\bar{S}$ bounded? Yes Is S closed? Yes! Because boundary 0 is in $\bar{S}$. The neighborhood approah now will achieve a finite subcover if you choose a neighborhood around 0, there is some finite number of other neighborhoods that will cover the rest of set.

Example: Let

$$S = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$$

Profve that 0 is a limite point of S.

WTS: $\forall \epsilon > 0$ the $N(0, \epsilon)$ contains at least one pt from S different.

Given $\epsilon > 0, \exists N$ s.t. $\frac{1}{N} < \epsilon$. So $\forall \epsilon > 0, \exists N$ s.t. $(-1/N, 1/N) \subseteq (-\epsilon, \epsilon)$. If $n \geq N$ then $1/n \leq 1/N$.

Proof. Let $\epsilon > 0$, then $\exists N \in \mathbb{N}$ s.t. $1/N < \epsilon$. Therefore $(-1/N, 1/N) \subseteq N(0, \epsilon)$. Now, let $n \geq N$, then $1/n \in S$ and $1/n \leq 1/N$. Therefore $1/n \in N(0, \epsilon)$.

5 Oct 2021

Def'n: A sequence is a fxn from $\mathbb{N}$ to $\mathbb{R}$. Instead of saysing $f : \mathbb{N} \to \mathbb{R}$, we say $a_n = \ldots$.

Def'n: We say $\lim_{n\to\infty} a_n = L$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, then $|a_n - L| < \epsilon$

Thm. Supose $a, b \in \mathbb{R}$. Then $a = b$ iff $\forall \epsilon > 0$, $|a - b| < \epsilon$.

Proof: Suppose $a = b$, and let $\epsilon > 0$. Then $a - b = 0$. So $|a - b| = 0 < \epsilon$. Conversely, suppose $\forall \epsilon > 0$, $|a - b| < \epsilon$. Then if $a - b > 0$, then $\exists \epsilon$ s.t. $|a - b| \geq \epsilon$. Then if $a - b < 0$, then $\exists \epsilon$ s.t. $|a - b| \geq \epsilon$.

Exercise: Suppose $a, b \in \mathbb{R}$. Prove that if $\forall \epsilon > 0$, $a \leq b + \epsilon$ then $a \leq b$.

Proof (by contrapositive): We show that $a > b$ implies $\exists \epsilon > 0$ s.t. $a > b + \epsilon$.

If $a > b$, then $a - b > 0$. Take $\epsilon = \frac{a-b}{2}$. Then $b + \epsilon = b + \frac{a-b}{2}$ ...

Exercise: Prove $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof: By Archimedian Property, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \epsilon$. Then, if $n \geq N$ then $\frac{1}{n} \leq \frac{1}{N}$, and $\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| \leq \frac{1}{N} < \epsilon$.

For proof by contradiction, negate definition of limit:

$$\exists \epsilon > 0, \forall N \in \mathbb{N} \text{we have} n \geq N \text{and} |a_n - L| \geq \epsilon$$

Exercise (Homework): Suppose $\lim a_n = L$. Define $b_n = a_{2n}$. Prove that $\lim b_n = L$.

Proof (by contradiction).

Suppose $\lim b_n \neq L$ and $\lim a_n = L$. Then $\exists \epsilon > 0$ s.t. $\forall N_1 \in \mathbb{N}$, if $n \geq N_1$ then $|b_n| \geq \epsilon$. But we also have $\exists N_2 \in \mathbb{N}$ s.t. if $n \geq N_2$ then $|a_n - L| < \epsilon$. Let $N* > \max\{N_1, N_2\}$ and suppose $n \geq N*$. Then $2n > n > N*$ and so $|a_{2n} - L| < \epsilon$, but this contradicts $|a_{2n} - L| = |b_n - L| < \epsilon$.

Alternate: Sine $\lim a_n = L$, we have $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$ then $|a_n - L| < \epsilon$. Suppose $n \geq N$, then $2n \geq N$ and $|b_n - L| < \epsilon$. $\square$.

Exercise: Suppose $S \subseteq \mathbb{R}$ is non-empty and bounded above. Prove there is a sequence $\{a_n\}$ s.t. $\forall n \in \mathbb{N}$ we have $a_n \in S$ and $\lim a_n = \sup(S)$.

1. $\forall a \in S$, $a \leq sup(S)$.

2. If $B < \sup(S)$, then B is not an upper bound, that is $\exists a \in S$ s.t. $B < a$.

For example, suppose $B = \sup(S) - \frac{1}{n}$. We know $\exists a_n \in S$ s.t. $\sup(S) - \frac{1}{n} < a_n \leq \sup(S)$. Since $\sup(S) + \frac{1}{n}$ is an upper bound. So $\sup(S) - \frac{1}{n} < a_n < \sup(S) + \frac{1}{n}$. Equivalently, $-\frac{1}{n} < a_n - \sup(S) < \frac{1}{n}$. Which is $|a_n - \sup(S)| < \frac{1}{n}$.

Formal proof:

Let $n \in \mathbb{N}$, then $\sup(S) - \frac{1}{n}$ is not an upper bound for S. Therefore, $\exists a_n \in S$ s.t. $\sup(S) - \frac{1}{n} < a_n$. But since $a_n \in S$, $a_n \leq \sup(S)$, and we always have $\sup(S) < \sup(S) + \frac{1}{n}$. Therefore, $|a_n - \sup(S)| < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Now, let $\epsilon > 0$. Then by Achimedean Property, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \epsilon$. So let $n \geq N$, then $|a_n - \sup(S)| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$.$\square$.

12 Oct 2021

Defn: A sequence (a_n) converges to L if $\forall \epsilon$, $\exists N \in \mathbb{N}$, s.t. $\forall n \in \mathbb{N}$, $n > N \Rightarrow |a_n - L| < \epsilon$.

Defn: A sequence (a_n) is Cauchy if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\forall n, m \in \mathbb{N}$, $m, n \geq N \Rightarrow |a_m - a_n| < \epsilon$.

Thm. A sequence converges iff it is Cauchy.

Corrolary: A sequence is not Cauchy if $\exists \epsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists m, n \in \mathbb{N}, m, n \geq N \Rightarrow |a_m - a_n| \geq \epsilon$.

Example: Let $a_n = sin(\frac{n\pi}{2})$, prove that (a_n) does not converge. $(a_n) = \{1, 0, -1, 0, 1...\}$. So even terms are 0, and odd terms alternate between -1 and 1.

Take $\epsilon = \frac{1}{2}$ and $N \in \mathbb{N}$. Then $2N+1, 2N \geq N$ and $\sin\left(2N\frac{\pi}{2}\right) = 0$ and $sin\left((2N+1)\frac{\pi}{2}\right)$ is 1 or -1. In either case $|a_{2N+1} - a_{2N}| = 1 \geq \frac{1}{2} = \epsilon$, so (a_n) is not Cauchy.

Example: Let $a_n = 1 + (-1)^n$. Prove (a_n) is not Cauchy.

Let $\epsilon = 1$ and $N \in \mathbb{N}$. Then $2N, 2N+1 \geq N$, and $1 + (-1)^{2N} = 2$ and $1 + (-1)^{2N+1} = 0$. Then $|a_{2N} - a_{2N+1}| = 2 > 1 = \epsilon$. Hence, (a_n) is not Cauchy.

Thm. If a sequence converges, then it is bounded.

Thm. If a sequence is bounded and monotone, then it converges.

Example: Let $a_1 = 2$ and $a_{n+1} = \frac{1}{5}(a_n + 7)$. Show (a_n) converges with the bounded monotone theorem.

Scratch: $a_n = \{2, 9/2...\}$. Solve for limit: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{L+7}{5} \Rightarrow L = \frac{7}{2} < 2$.

Proof. $a_1 \leq 2$. Assume $a_k \leq 2$ for some $k > 1$. Then $a_{k+1} = \frac{a_k+7}{5} \leq \frac{9}{5} \leq 2$. So $a_n \leq 2$ by induction.

Next, see that $a_1 = 2 > a_2 = \frac{9}{5}$. Now suppose $a_k > a_{k+1}$. Then $a_{k+1} = \frac{a_k+7}{5} \geq \frac{a_{k+1}+7}{5} = a_{k+2}$, hence (a_n) is monotone decreasing.

Example: Let $a_1 = 5$ and $a_{n+1} = \sqrt{4a_n + 1}$. Prove a_n converges.

HW Problem 4: $a_1 = 1$ and

$$a_n = \begin{cases} \sqrt{a_{n-1}} & \text{for } n \text{ odd} \\ a_{n-1} + 1 & \text{for } n \text{ even} \end{cases}$$

Show bounded: Let $b_n = a_{2n-1}$, then $b_1 = a_1 = 1$, and $b_n = a_{2n-1}$, and $b_{n+1} = ... = \sqrt{b_n + 1}$.

Use induction to show that both subsequences are bounded and monotone. also show for $c_n = a_{2n} = a_{2n-1} + 1 = b_n + 1$. Since $0 \leq b_n \leq 2$, we have $1 \leq c_n \leq 3$. So $0 \leq a_n \leq 3$.

26 Oct 2021

Defn: Let $D \subseteq R$, $c \in D$, and $f : D \to \mathbb{R}$. Then f is continuous at c if $\forall \epsilon > 0$, $\exists \delta > 0$, s.t. $\forall x \in \mathbb{R}$, if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.

Ex. Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Prove that f is continuous at $c = 2$.

Let ϵ be arbitrary, and find δ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

If $|x - 2| < 1$ then $|x - 2 + 2 + 2| \leq |x - 2| + |2 + 2| \leq 1 + 4 = 5$.

So PF

Let $\epsilon > 0$ and $\delta = \min\{1, \epsilon/5\}$. Then if $|x - 2| < \delta$, then $|x + 2| \leq |x - 2| + |2 + 2| = 5$, and $|f(x) - f(2)| = |x - 2\ ||x + 2| \leq 5\delta < \epsilon$.

Ex. Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Prove that f is continuous at $c = 2$.