Math 401: Homework 11

Tim Farkas

Nov 2021

Problem 1

Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable at 0 and that $f(0) = g(0) = 0$. Define

$$k(x) = \begin{cases} f \circ g(x) & x < 0 \\ 0 & x = 0 \\ g \circ f(x) & x > 0 \end{cases}$$

Show that k is differentiable at 0 and find $k'(0)$.

Since $f \circ g(x)$ and $g \circ f(x)$ are differentiable at 0, we have that $(f \circ g)'(x) = f'(g(x))g'(x)$ and $(g \circ f)'(x) = g'(f(x))f'(x)$. Evaluating each derivative at $x = 0$ yields

$$\begin{aligned} (f \circ g)'(0) &= f'(g(0))g'(0) \\ &= f'(0)g'(0) \end{aligned}$$

and

$$\begin{aligned} (g \circ f)'(0) &= g'(f(0))f'(0) \\ &= f'(0)g'(0) \end{aligned}$$

Hence we can show that both the left and right limits of k at $x \to 0$ are equal:

$$
\begin{aligned}
\lim_{x \to^- 0} \frac{k(x) - k(0)}{x - 0} &= \frac{(f \circ g)(x) - 0}{x} \\
&= \frac{(f \circ g)(x) - (f \circ g)(0)}{x - 0} \\
&= (f \circ g)'(x) \\
&= f'(0)g'(0)
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{x \to^+ 0} \frac{k(x) - k(0)}{x - 0} &= \frac{(g \circ f)(x) - 0}{x} \\
&= \frac{(g \circ f)(x) - (g \circ f)(0)}{x - 0} \\
&= (g \circ f)'(x) \\
&= f'(0)g'(0)
\end{aligned}
$$

Hence k is differentiable at 0 and $k'(0) = f'(0)g'(0)$.

Problem 2

Suppose $f, g : \mathbb{R} \to \mathbb{R}$, S is a finite set where $S \subseteq (0, \infty)$ and $\forall x \notin S$, $f(x) = g(x)$.

Show that if f is differentiable at 0, then g is differentiable at 0 and $f'(0) = g'(0)$.

Since S is a finite set, S has a minimum and $\min\{S\} > 0$.

Hence, take $\delta = \min\{S\}$. Then there exists a neighborhood $N(0, \delta)$ wherein $f(x) = g(x)$, and hence $f'(0) = g'(0)$.

Problem 3

Prove that if a polynomial function $p : \mathbb{R} \to \mathbb{R}$ is divisible by $(x - 2)^3$, then the polynomial $p'(x)$ is divisible by $(x - 2)^2$.

By definition we can take $p = (x - 2)^3 p^*$, where p^* is some polynomial factor of p.

Then we have

$$\begin{aligned}
p' &= \frac{d}{dx}(x-2)^3 p^* \\
&= 3(x-2)^2 p^* + (x-2)^3 p'^* && \text{(product rule)} \\
&= (x-2)^2 \left[3p^* + (x-2)p'^*\right] \\
&= (x-2)p^{**}
\end{aligned}$$

Where p^{**} is a polynomial, since the derivative of a polynomial is a polynomial, and the sum of polynomials is polynomial.

Problem 4

Prove that if $f(x)$ is an even function, then $f'(0) = 0$.

If $f(x)$ is even, then $f'(x)$ is odd, by proof in recitation.

If $f'(x)$ is odd, then we have that $f'(0) = f'(-0) = -f(0)$, so $f'(0) = -f'(0)$, and therefore we have that $2f'(0) = 0$, and $f'(0) = 0$.