Math 401: Homework 1

Tim Farkas

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Problem 1

1a

Statement: $\forall a \in A, \exists y \in B$ s.t. $y^2 = x$.

Negation: $\exists a \in A$ s.t. $\forall y \in B$, $y^2 \neq x$

1b

Statement: $\forall x \in A$, we have $\{x\} \subseteq B$.

Negation: $\exists x \in A$ s.t. $\{x\} \nsubseteq B$

1c

Statement: There is an element of A that is also an element of B.

Negation: $\forall a \in A, a \notin B$. Equivalently, $A \cap B = \emptyset$

Problem 2

Prove whether the following are true or false.

2a

Statement: $\forall x, y \in \mathbb{R}$ we have $x^2 = y^2$

Value: False.

Proof: Take $x = 0$, $y = 1$. Then $x^2 = 0 \neq 1 = y^2$ $\square$

2b

Statement: $\forall x \in \mathbb{Z}, \exists y \in \mathbb{R}$ s.t. $x^2 = y^2$

Value: True.

Proof: Because $\mathbb{Z} \subset \mathbb{R}$, we can always take $x = y$. Then $x^2 = y^2$. $\square$

2c

Statement: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy = 1$

Value: False.

Proof: Take $x = 0$. Then $\forall y$ we have $xy = 0$. $\square$

2d

Statement: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy = 0$

Value: True.

Proof: Take $y = 0$. Then $\forall x \in \mathbb{R}$ we have $xy = 0$

2e

Statement: $\exists y \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}$ we have $xy = 0$.

Value: True.

Proof: Statement 2e is equivalent to statement 2d. See proof to 2d.

Problem 3

Prove that if $x \in \mathbb{R}$, then $1 < x < 3$ implies $-10 < x^3 - x < 30$.

Proof: Assume $1 < x < 3$. This implies $1 < x^3 < 27$, and also that $-3 < -x < -1$. Adding the two implied statements yields $-2 < x^3 - x < 26$, and so we have $1 < x < 3$ implies $-2 < x^3 - x < 26$. Then, because $-10 < -2 < x^2 - x < 26 < 30$, we see that $1 < x < 3$ implies $-10 < x^3 - x < 30$. $\square$

Problem 4

For $n \in \mathbb{Z}$, prove that if n is even, then $n^2 = 4k + 4$ for some $k \in \mathbb{Z}$.

Proof: Assume n is even. Then $\exists q \in \mathbb{Z}$ s.t. $n = 2q$. To prove the statement, we substitute $2q$ for n in the consequent. Rearranging a bit first:

$k = (n^2 - 4)/4$

$k = ((2q)^2 - 4)/4$

$k = (4q^2 - 4)/4$

$k = q^2 - 1$

Hence, $n^2 = 4k + 4$ whenever $k = q^2 - 1$. $\square$

Problem 5

Prove that if $x \in \mathbb{R}$, then $-\frac{7}{16} < x^2 - x < -\frac{5}{16}$ implies that $x \notin \left(\frac{1}{10}, \frac{3}{10}\right)$.

Proof: To simplify calculations, we prove the contrapositive. If $x \in \mathbb{R}$, then $\frac{1}{10} < x < \frac{3}{10}$ implies $x^2 - x < -\frac{7}{16}$ or $-\frac{5}{16} < x^2 - x$.

Assume $\frac{1}{10} < x < \frac{3}{10}$. Then $\frac{1}{100} < x^2 < \frac{9}{100}$ and $-\frac{3}{10} < -x < -\frac{1}{10}$. Adding the last two inequalities yields $-\frac{29}{100} < x^2 - x < -\frac{1}{100}$.

Hence, if $\frac{1}{10} < x < \frac{3}{10}$, then $-\frac{29}{100} < x^2 - x$. To finish the proof, we show that $-\frac{5}{16} = -\frac{500}{1600} < -\frac{464}{1600} = -\frac{29}{100} < x^2 - x$, satisfying the condition. $\square$