Math 401: Homework 7

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Problem 1

Show that if (a_n) is Cauchy, that $c_n = \mu a_n + b$ is also Caucy, where $\mu, b \in \mathbb{R}$ and $\mu \neq 0$.

Proof (direct): Since (a_n) is Cauchy, we have for some ϵ that $\exists N_1 \in \mathbb{N}$ s.t. $m, n > N_1$ implies

$$\begin{aligned}
|a_m - a_n| &< \epsilon \\
\Leftrightarrow \left| \frac{c_m - b}{\mu} - \frac{c_n - b}{\mu} \right| &< \epsilon \\
\Leftrightarrow \left| \frac{c_m - c_n}{\mu} \right| &< \epsilon \\
\Leftrightarrow |c_m - c_n| &< |\mu| \, \epsilon
\end{aligned}$$

Hence, we can take $N_2 = |\mu| \, \epsilon$ to ensure $|c_m - c_n| < \epsilon$, proving (c_n) is a Cauchy sequence.

Problem 2

Prove that if (a_n) is a Cauchy sequence that $b_n = a_{n-1}$ is also a Cauchy sequence.

Proof: Since (a_n) is Cauchy, we have for some ϵ that $\exists N_1 \in \mathbb{N}$ s.t. $m, n > N_1$ implies $|a_m - a_n| < \epsilon$. Take $N_2 = N_1 + 1$. Then if $m > N_2$ and $n > N_2$, we have $m - 1 > N_1$ and $n - 1 > N_1$. Hence, $|a_{m-1} - a_{n-1}| < \epsilon \Leftrightarrow |b_m - b_n| < \epsilon$. $\square$.

Problem 3

Consider the sequence (a_n) defined recusively by $a_1 = 1$ and

$$a_n = \sqrt{6 + a_{n-1}}$$

Show that (a_n) is convergent and its limit is no greater than 10.

Proof: We demonstrate first that (a_n) is monotone, and then that its limit is no greater than 10, showing it is bounded and proving it is a convergent sequence.

1. (induction) Note that $a_1 < a_2 < a_3 \Leftrightarrow 1 < \sqrt{7} < \sqrt{6+\sqrt{7}}$. Then, for $n \geq 2$, suppose $a_{k+1} > a_k$ for some k. Then we have $a_{k+1} = \sqrt{6 + a_k} > \sqrt{6 + a_{k-1}} = a_k$, proving that $a_{k+1} > a_k$ for all $k \in \mathbb{N} \geq 2$.

2. (induction). Note that $a_1 < a_2 < 10 \Leftrightarrow 1 < \sqrt{7} < 10$. Suppose $a_k < 10$ for some $k \in \mathbb{N} > 2$. Then $a_{k+1} = \sqrt{6 + a_k} < \sqrt{6 + 10} = 4 < 10$. Hence $a_k < 10$ for all $k \in \mathbb{N} > 2$.

Hence (a_n) is a convergent sequence whose limit is at most 10.

Problem 4

Consider the sequence (a_n) defined recursively as $a_1 = 1$ and

$$a_n = \begin{cases} \sqrt{a_{n-1}} & \text{if } n \text{ is odd} \\ a_{n-1} + 1 & \text{if } n \text{ is even} \end{cases}$$

for $n \geq 2$.

(a) Show that (a_n) is bounded.

Proof (induction). We define two subsequences, one for each of even and odd $n \in \mathbb{N}$.

Define $b_n = a_{2n-1}$ to isolate the odd n. Then $b_n = \sqrt{a_{2n-2}}$ and $b_{n+1} = \sqrt{a_{2n-12} + 1}$. But then $b_{n+1} = \sqrt{b_n + 1}$.

We show by induction that $\lim_{n\to\infty} b_n \leq 2$. Note that $b_1 = 1 < 2$ and $b_2 = \sqrt{3} < 2$. Suppose $b_k \leq 2$. Then $b_{n+1} = \sqrt{b_n + 1} < \sqrt{2+1} < \sqrt{3} < 2$. Hence $b_n < 2$ for all n.

Next, define $c_n = a_{2n}$ to isolate the even n. Then $c_n = a_{2n} = a_{2n-1} + 1 = b_n + 1$. Since $\lim_{n\to\infty} b_n < 2$, we have that $\lim_{n\to\infty} c_n < 3$.

(b) Show that (a_n) is not Cauchy.

Since $a_n = a_{n-1} + 1$ when n is even, we can take $\epsilon = \frac{1}{2}$, and $m = n - 1$. Then $m > n$, but $|a_m - a_n| = 1 > \frac{1}{2}$ for all $m, n \geq 2$.

(c) Show that (a_n) is not convergent.

Proof: See Theorem 18.12: A sequence of real numbers is convergent iff it is a Cauchy sequence. By proof in part (b), (a_n) is not a Cauchy sequence, hence it is not convergent.

(d) Find a subsequence of (a_n) that converges.

The subsequence (b_n) of odd indices, defined as $b_n = a_{2k-1}$, for all $k \in \mathbb{N}$ converges. We showed in part (a) that it is bounded above by 2. To complete the proof we show that it is monotone increasing.

Notice $b_1 < b_2 < b_3 \Leftrightarrow 1 < \sqrt{2} < \sqrt{\sqrt{2}+1}$. Suppose $b_k < b_{k+1}$. Then $b_{k+1} = \sqrt{b_k + 1} < \sqrt{b_{k+1} + 1} = b_{k+2}$, completeing the induction step, and proving that (b_n) is bounded and monotone, and thus converges.

Problem 5

Suppose (a_n) is convergent and $n > 3 \Rightarrow a_{2n} \leq 1$. Show that $\lim_{n\to\infty} a_n \leq 1$.

Since (a_n) is convergent, it converges to a unique number s. Since $a_{2n}, n > 3$ is a subsequence of (a_n), it also converges to s (Thm. 19.4). Now, since $a_{2n} \leq 1$, we have that $\lim_{n\to\infty} a_{2n} \leq 1$, but $\lim_{n\to\infty} a_{2n} = s$, so $s \leq 1$ and $\lim_{n\to\infty} a_n = s \leq 1$.