Math 401: Homework 6

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Problem 1

Suppose a_n is real for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} a_n = L$. Define $b_1 = \pi$ and $b_n = a_{n-1}$ for all $n \geq 2$. Prove that (b_n) converges to L.

Proof: Since $\lim_{n\to\infty} a_n = L$, we have by definition that $\forall \epsilon > 0$, $\exists N_a$ s.t. $\forall n \in \mathbb{N}$, $n > N_a \to |a_n - L| < \epsilon$. For a given ϵ and corresponding N_a, take $N_b = N_a + 1$. Then if $n > N_b \geq 2$, we have $n > N_a + 1$, and $n - 1 > N_a$, which implies $|a_{n-1} - L| < \epsilon = |b_n - L| < \epsilon$. Proving the claim that (b_n) converges to L.

Problem 2

Suppose a_n is real for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} a_n = L$. Define $b_{2n} = a_n$ and $b_{2n+1} = a_n$ for all $n \in \mathbb{N}$. Prove that b_n converges to L.

Proof: We seek N_b such that for all $n \in \mathbb{N}$ if $n > N_b$ implies $|b_n - L| < \epsilon$. For any given ϵ, we have by definition of a limit a real number N_a such that if $n \in \mathbb{N} > N_a$ implies $|a_n - L| < \epsilon$.

Take $N_b = N_a + 1$.

Then when even $n > N_b$, we have $n > 2N_a$, and $\frac{n}{2} > N_a$. Hence $\left|a_{n/2} - L\right| < \epsilon$, and $|b_n - L| < \epsilon$.

And when odd $n > N_b$, we have $n > 2N_a + 1$, and $\frac{n-1}{2} > N_a$. Hence $\left|a_{(n-1)/2} - L\right| < \epsilon$, and $|b_n - L| < \epsilon$.

Problem 3

Suppose a_n and b_n are real for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$.

Define $c_{2n} = a_n$ an $c_{2n+1} = b_n$ for all $n \in \mathbb{N}$. Prove that c_n converges.

Proof: We seek N_c such that for all $n \in \mathbb{N}$ if $n > N_c$ implies $|c_n - L| < \epsilon$. Note we have by definition real numbers N_a and N_b to satisfy convergence for the same ϵ for a_n and b_n respectively. However, we do not know which is larger.

Therefore, take $N^* = \max\{N_a, N_b\}$ and $N_c = 2N^* + 1$.

If n is even, we have that $n > 2N^* + 1 > N_a$, so $\frac{n}{2} > N^* + \frac{1}{2} > N_a$. Hence $\left|a_{n/2} - L\right| < \epsilon$ and $|c_n - L| < \epsilon$.

If n is odd, we have that $n > 2N^* + 1 > N_b$, so $\frac{n-1}{2} > N^* \geq N_b$. Hence $\left|b_{(n-1)/2} - L\right| < \epsilon$ and $|c_n - L| < \epsilon$.