# Math 401: Homework 2

Tim Farkas

Aug 2021

## Problem 1

Define $f\colon (1,4) \to \mathbb{R}$ by

$$f(x) = \frac{1}{x^2 - 5x + 4}$$

We can analyze the behavior of $f$ by looking at the denominator only, which we define as $g(x)$.

$g(x) = x^2 - 5x + 4 = (x-4)(x-1)$, hence the function is indeed properly defined on the whole domain $(1,4)$.

$g'(x) = 2x - 5$, hence there is a single inflection point at $5/2$.

$g''(x) = 2 > 0$, hence the inflection point is a local minimum for $g(x)$, and a local maximum for $f(x)$.

### 1a

$f$ is not one to one, because there is an inflection point within the domain and the inflection point does not occur at either boundary of the domain.

### 1b

The inflection point occurs at $x = 5/2$. Because $5/2 > y$, $\forall y \in (1,2)$ and $f$ is concave, $f$ is monotonic increasing on $(1,2)$. Hence:

$f((1,2)) = (f(1)^+, f(2)^-) = (-\infty, -1/2)$

### 1c

Becuase the inflection point occurs at $x = 5/2 \in (1, 3)$ and $f$ is concave:

$f(1, 3) = (\min(f(1)^+, f(3)), f(5/2)) = (\min(-\infty, -1/2), -1/4) = (-\infty, -1/4)$

## Problem 2

Define $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) = 3\left|x - \frac{1}{3}\right|$$

We rewrite as a piecewise function:

$$f(x) = \begin{cases} 3x - 1 & x \geq 1/3 \leftrightarrow x \geq 1 \\ 1 - 3x & x < 1/3 \leftrightarrow x \leq 0 \end{cases}$$

### 2a

$f$ is one-to-one.

Proof:

Both piecewise elements of $f(x)$ are linear in $x$ and hence are monotonic, so both pieces are are themselves one-to-one.

We have left to show that the ranges of each piece have no common elements, which we prove by contradiction. Take $x < 0 < y$ and $x, y \in \mathbb{Z}$. The conditions under which two functions have the same outcome can be derived:

$$3x - 1 = 1 - 3y \tag{1}$$
$$3x = 2 - 3y \tag{2}$$
$$x = \frac{2}{3} - y \tag{3}$$

But then either $x \notin \mathbb{Z}$ or $y \notin \mathbb{Z}$ and the proof is complete. $\square$

### 2b

$f$ is not onto, because no negative integers are in the range, but all negative integers are in the codomain. To prove, we factor $f$ into 3 and $|x - 1/3|$ and note that both are non-negative for all

$x \in \mathbb{R} \supseteq \mathbb{Z}$.

# Problem 3

$f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \max(2x, 1)$$

For analysis, we rewrite piecewise:

$$f(x) = \begin{cases} 1 & x \leq 1/2 \\ 2x & x > 1/2 \end{cases}$$

## 3a

$f^{-1}((0,1)) = \emptyset$, because $f(x)$ is never less than 1.

## 3b

$f^{-1}((1,2)) = (1/2, 1)$. If $f(x) > 1$, as it is for the whole specified subset of the codomain, then $f(x) = 2x$ and $x = y/2 = f^{-1}(y)$. Then, because $f(x) = 2x$ is monotonic increasing, $f^{-1}((1,2)) = (f^{-1}(1)^+, f^{-1}(2)^-) = (1/2, 1)$. $\square$

## 3c

$f^{-1}((-\infty, 2)) = (-\infty, 1)$. Because the minimum value in the range is 1, $f^{-1}((-\infty, 1) = \emptyset$. The remainder of the codomain is $[1, 2)$, which differs from 3b only by the inclusion of 1. But $f^{-1}(\{1\}) = (-\infty, 1/2]$. Because $(-\infty, 1/2] \cup (1/2, 1) = (-\infty, 1)$, we see that $f^{-1}((-\infty, 2)) = (-\infty, 1)$.

# Problem 4

Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = y = \begin{cases} x^2 & x \leq 0 \\ x+1 & x > 0 \end{cases}$$

Therefore, we have that $f^{-1}(\{y\}) = \{-\sqrt{y} : y \geq 0\} \cup \{y - 1 : y > 1\}$. Note the functions defining both sets in this union are monotonic.

### 4a

$f^{-1}((0,1)) = (f^{-1}(1)^+, f^{-1}(0)^-) \cup \emptyset = (-\sqrt{1}, -\sqrt{0}) = (-1,0)$

### 4b

$$\begin{aligned} f^{-1}((1,2)) &= \{\{-\sqrt{y} : y \in (1,2) \cap [0,\infty)\}, \{y-1 : y \in (1,2) \cap (1,\infty)\}\} \\ &= \{(-\sqrt{2}, -1), (0,1)\} \\ &= (-\sqrt{2}, -1) \cup (0,1) \end{aligned}$$

### 4c

$f^{-1}((-\infty,2)) = (-\sqrt{2}, 1)$. Because the minimum value in the range is 0, $f^{-1}((-\infty,0) = \emptyset$. The remaining range is $[0,2)$, which differs from the union of ranges from 4a and 4b by inclusion of 0 and 1.

$$\begin{aligned} f^{-1}([0,2)) &= \{\{-\sqrt{y} : y \in [0,2) \cap [0,\infty) = [0,2)\}, \{y-1 : y \in [0,2) \cap (1,\infty) = (1,2)\}\} \\ &= \{(-\sqrt{2}, 0], (0,1)\} \\ &= (-\sqrt{2}, 0] \cup (0,1) \\ &= (-\sqrt{2}, 1) \end{aligned}$$

## Problem 5

If $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are both one-to-one, then the function $h\colon \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} f(x) & if x \leq 0 \\ g(x) & if x > 0 \end{cases}$$

is also one to one.

This statement is false. Take $f(x) = -x$ and $g(x) = x$. Each are one-to-one. But then $h(x) = |x|$, which is not one-to-one.