Math 401: Homework 3

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Problem 1

Suppose $a_0 = 0$ and for every $n \in \mathbb{N}$ we have $a_n = a_{n-1} + 2^{-n}$.

Define closed intervals $I_n = [a_{n-1}, a_n]$ for all n.

Informally, we the sequence of I_n to be:

$$\langle [0, 1/2], [1/2, 3/4], [3/4, 7/8], [7/8, 15/16], \ldots, [\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n}] \rangle$$

a)

Calculate the set $\bigcap_{n=1}^{\infty} I_n$.

From the sequence above, we can see that the intersection of any three elements is $\emptyset$, hence the intersection of the entire sequence is also $\emptyset$.

b)

Calculate the set $\bigcup_{n=1}^{\infty} I_n$.

From the sequence above, we see that the union of any two adjacent elements is $[a, b]_n \cup [b, c]_{n+1} = [a, c]$. Hence, for any $n < \infty$ we have the union to equal $[0, \frac{2^n-1}{2^n}]$. Next, we see that $\lim_{n\to\infty} \frac{2^n-1}{2^n} = 1$, so this sequence has an asymptote at 1. Hence, the union over the whole sequence is [0, 1).

Problem 2

Define $f : \mathbb{N} \to \mathbb{R}$ by $f(1) = 0$ and

$$f(n) = \begin{cases} -f(n-1) - 1 & \text{if } n \text{ is even} \\ -f(n-1) & \text{if } n \text{ is odd} \end{cases}$$

The sequence defined by this function covers $\mathbb{Z}$ as follows: {1, -1, 2, -2, 3, -3, . . . , n, -n}.

a)

Following the above sequence, this function is one-to-one.

b)

Absolutely not. f takes $\mathbb{N}$ into $\mathbb{Z} \subset \mathbb{R}$, hence f is not onto.

Problem 3

Prove the following: If $f : A \to B$ is a function such that for all $C \subseteq A$ the sets C and $f(C)$ are equinumerous, then f is injective.

Proof:

This we can prove by the contrapositive: If f is not injective, then there exists some $C \subseteq A$ such that the sets C and $f(C)$ are not equinumerous.

Assume f is not injective. Then by definition of injection there exist two elements $a, b \in A$ for which $f(a) = f(b)$. Take $C = \{a, b\} \subseteq A$, as required. Then $f(C) = \{f(a), f(b)\} = \{f(a) = f(b)\}$, and $|C| = 2 \neq 1 = |f(C)|$. $\square$

Problem 3 (old)

Prove the following: If $f : A \to B$ is a function such that for all $C \subseteq A$ the sets A and $f(C)$ are equinumerous, then f is injective.

Proposterous! Take C a proper subset of A, such that $|C| < |A|$, and $|f(C)| = |A|$ as required. Then $|C| < |f(C)| = |A|$ and f cannot be 1-to-1.

Problem 4

Let $A = \{1, 2, 3\}$ and let ι and g denote the specific functions from A to A by

$$\iota(1) = 1, \quad \iota(2) = 2, \quad \iota(3) = 3$$

and

$$g(1) = 2, \quad g(2) = 1, \quad g(3) = 3$$

Now let Γ denote the set of *all* functions from A to A and define $\mathcal{F} : \Gamma \to \Gamma$ by

$$\mathcal{F}(f) = f \circ f$$

a)

Calculate $\mathcal{F}(g)$

$$\mathcal{F}(g) = g \circ g = \begin{cases} n & g & g(n) & g & g(g(n)) \\ 1 & \to & 2 & \to & 1 \\ 2 & \to & 1 & \to & 2 \\ 3 & \to & 3 & \to & 3 \end{cases}$$

Hence $\mathcal{F}(g) = g \circ g = \iota$.

b)

Calculate $\mathcal{F}^{-1}(\{\iota\})$.

From part a we see that $g \in \mathcal{F}^{-1}(\{\iota\})$, and it is easy to see also that $\iota \in \mathcal{F}^{-1}(\{\iota\})$. But we also see that any function q for which a pair of elements in ι are once permuted will have $\mathcal{F}(q) = \iota$. We also assert that no function from A to A that is not one-to-one can have a composition with itself that yields ι, and no function that is a full permutation of ι can have a function once composed with itself yielding ι (though full permutations *twice* composed will indeed be equal to ι). Hence,

$\mathcal{F}^{-1}(\{\iota\}) = \{\iota, \{q : A \to A \,|\, q(a) = a,\, q(b) = c,\, q(c) = b,\, \{a, b, c\} = A\}\}$

Note that $g \in q$.

c)

Calculate $\mathcal{F}^{-1}(\{g\})$.

$\mathcal{F}^{-1}(\{g\}) = \emptyset$. There is no non-injective function q for which $\mathcal{F}(q) = g$, since then q would also not be surjective, and g is surjective. Furthermore, we have exhausted above the consequences of permuting ι, and none of these cases yield $\mathcal{F}(q) = g$.