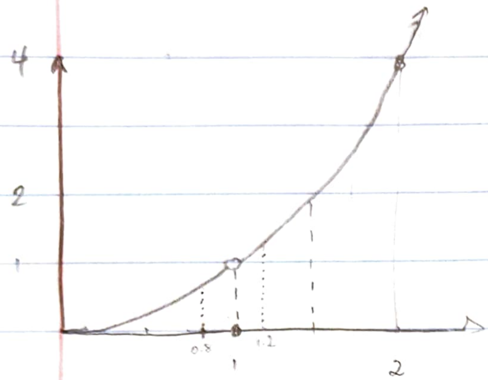
Hw 13

P1) $f : [0,2] \to \mathbb{R}$ by $f = \begin{cases} x^2 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$



a) $L(f,P)$ & $U(f,P)$

$L(f,P) = \sum_{i=1}^{3} m_i \Delta x_i$

$= 0(0.8) + 0(0.4) + 1.44(0.8)$

$= 1.152$

$U(f,P) = \sum M_i \Delta x_i = (0.64)(0.8) + (1.44)(0.4) + (4)(0.8)$

$= 4.288$

b) $L(f,Q) = 0(0.8) + \inf\{f(x) : x \in [1, 1.5]\} \Delta x = (0)(0.5) + (2.25)(.5) = \underline{1.125}$

$U(f,Q) = (1)(1) + (2.25)(0.5) + (2)(0.5) = \underline{3.125}$

sup!

0.32

c) $L(f, P \cup Q) = (0)(0.8) + (0.64)(0.2) + (0)(0.2) + (1.44)(0.3) + (2.25)(0.5) = \underline{1.877}$

$U(f, P \cup Q) = (0.64)(0.8) + (1)(0.2) + (1.44)(0.2) + (2.25)(0.3) + (4)(0.5) = \underline{3.675}$

P2. Suppose $f:[0,1]\to\mathbb{R}$ and $0\le f(x)\le 1$ for all $x\in[0,1]$. Show that if f is integrable on $[\frac{1}{n},1]$ for all $n\in\mathbb{N}$, then f is integrable on $[0,1]$.

Since f is integrable on $[\frac{1}{n},1]$, there exists a partition P_1 s.t. $U(f,P_1)-L(f,P_1)<\varepsilon^*=\varepsilon/2$.

Define $P_2=\{0\}\cup P_1=\{x_0=0,\ x_1=\frac{1}{n},\ x_2\ldots\ x_k=1\}$

Then $U(f,P_2)=\sum_{i=1}^{k}M_i\Delta x_i=M_1\Delta x_1+\sum_{i=2}^{k}M_i\Delta x_i$

$=M_1(\frac{1}{n})+U(f,P_1)$

and $L(f,P_2)=\ \ldots\ =m_1(\frac{1}{n})+L(f,P_1)$

Hence $U(f,P_2)-L(F,P_2)=M_1(\frac{1}{n})-m_1(\frac{1}{n})$

$+U(f,P_1)-L(f,P_1)$

$=\frac{1}{n}(M_1-m_1)+U(f,P_1)-L(f,P_1)$

Since $U(f,P_1)-L(f,P_1)<\varepsilon/2\Rightarrow$

we have $\frac{1}{n}(M_1-m_1)+U(f,P_1)-L(f,P_1)<\frac{1}{n}(M_1-m_1)+\varepsilon/2$

Since $0\le f(x)\le 1$, $M_1\le 1$, $m_1\ge 0$ and $(M_1-m_1)\le 1$

Hence $U(f,P_2)-L(f,P_2)<\frac{1}{n}+\varepsilon/2$

By Archimedes, $\exists N^*$ st $\frac{1}{N^*}<\varepsilon/2$ for any $\varepsilon>0$,

Hence, if we take $n>N^*$ for our partion P_1, we have

$U(f,P_2)-L(f,P_2)\le\varepsilon/2+\varepsilon/2=\varepsilon$ $\boxtimes$

3) $f: [0, \frac{3}{2}] \to \mathbb{R}$

$$f(x) = \begin{cases} \lfloor x \rfloor & x \text{ irrational} \\ 0 & x \text{ rational} \end{cases}$$

Calculate $L(f)$ and $U(f)$.

Since $f(x) = 0$ when $x < 1$ for irrational x, $f(x) = 0$ for all $x \geq 1$. Furthermore, $\exists x$ irrational $\in [x_{i-1}, x_i]$ for all i and any portion of $[0, 3/2]$. Therefore, $m_i = 0 \Delta x_i = 0$ for all i,
$L(f,P) = \sum_{i=1}^{k} m_i \Delta x_i = 0$ for any portion of $[0, 3/2]$, and
$\sup\{L(f,P) : P \text{ a partion}\} = 0 = L(f)$.

Since $f(x) = 0$ for all $x < 1$, $U(f) = 0 + U(f^*)$ where $f^* : [1, 3/2] \to \mathbb{R}$ by the same definition as for f. Since $\exists x$ irrational $\in [x_{i-1}, x_i]$ for all i and any partition P of $[1, 3/2]$, $M_i = (1)\Delta x_i$ for all i and any partition of $[1, 3/2]$. Then $U(f, P) = \sum_{i=1}^{k} (1) \Delta x_i$
$= (x_1 - 1) + (x_2 - x_1) + (x_3 - x_2) \ldots (x_{n-1} - x_{n-2}) + (3/2 - x_{n-1})$
$= 3/2 - 1 = 1/2$, we have $U(f, P) = 1/2$ for all portions of $[1, 3/2]$, and
$U(f) = \inf\{U(f,P) : P \text{ a part of } [1, 3/2]\} = 1/2$.