# Math 401: Homework 8

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## Problem 1

Suppose $f : \mathbb{R} \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ are functions. Suppose $c$ and $L$ are real numbers. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} f(x) & \text{if } x < c \\ g(g) & \text{if } x \geq c \end{cases}$$

Show that if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = L$, then $\lim_{x\to c} h(x) = L$.

Proof:

Since $\lim_{x\to c} f(x) = L$, we have that for a given $\epsilon$, there exists a $\delta_f$ such that if $|x - c| < \delta_f$, then $|f(x) - L| < \epsilon$.

Likewise, since $\lim_{x\to c} g(x) = L$, we have that for the same $\epsilon$, there exists a $\delta_g$ such that if $|x - c| < \delta_g$, then $|g(x) - L| < \epsilon$.

Take $\delta = \min\{\delta_f, \delta_g\}$. Then if $|x - c| < \delta$, we have that $|x - c| < \delta_f$ and $|x - c| < \delta_g$, and $|f(x) - L| < \epsilon$ and $|g(x) - L| < \epsilon$. Therefore we have that $|h(x) - L| < \epsilon$, and $\lim_{x\to c} h(x) = L$. $\square$.

## Problem 2

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is irrational} \\ 3x & \text{if } x \text{ is rational} \end{cases}$$

Show that $\lim_{x\to 0} f(x) = 0$.

Proof: Because between any two real numbers $x < y$ there exists an irrational number $r$, it suffices to show that $\lim_{x\to 0} 2x = 0$.

Take $\delta = \frac{\epsilon}{3}$. Then we have that $|f(x) - 0| = 3x < \epsilon$ whenever $|x - 0| = x < \delta = \frac{\epsilon}{3}$.

## Problem 3

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x-1) & \text{if } x < 5 \\ f(4) & \text{if } x \geq 5 \end{cases}$$

Show that if $\lim_{x\to 1} f(x) = L$, then $\lim_{x\to 2} g(x) = L$.

Since $\lim_{x\to 1} f(x) = L$ we have a $\delta > 0$ such that $|f(x) - L| < \epsilon$ for any $\epsilon$ whenever $|x - 1| < \delta$. Therefore, we also have that $|f(x-1) - L| < \epsilon$ whenever $|(x-1) - 1| = |x - 2| < \delta$. We are only concerned with cases when $x < 5$ so $g(x) = f(x-1)$, hence $|f(x-1) - L| = |g(x) - L| < \epsilon$, and $\lim_{x\to 2} g(x) = L$.