1) Suppose $(a_n) \to L$ and $f: [1,\infty) \to \mathbb{R}$ by $f(x) = a_{\lfloor x \rfloor}$.
Prove that if $a_n \to L$ then $\lim_{x \to \infty} f(x) = L$

Since $(a_n) \to L$ we have $\exists N$ st. if $n > N$, then $|a_n - L| < \varepsilon$ for all $\varepsilon > 0$.

WTS that if $\exists N^*$ st. if $x > N^*$, then $|f(x) - L| < \varepsilon$.

Take $N^* = N+1$. Then we have that if $\lfloor x \rfloor > N^*$, $\lfloor x \rfloor > N$ which implies $|a_{\lfloor x \rfloor} - L| = |f(x) - L| < \varepsilon$ $\square$

12) If $f_-'(0) > 0$, then $\exists \delta_1$ s.t. if $0 < |x| < \delta_1$

then $\left| \frac{f(x) - f(c)}{x - c} - f_-'(0) \right| < \varepsilon$, $\forall \varepsilon > 0$

Take $\varepsilon = f_-'(0)$, Then we have

$$0 < \frac{f(x) - f(c)}{x - c} < 2f_-'(0)$$

$\Rightarrow$ $0 \geq f(x) - f(c) \Rightarrow f(c) \geq f(x)$

For the right limit, we have analagously

$$\left| \frac{f(x) - f(c)}{x - c} - f_+'(0) \right| < \varepsilon$$

Take $\varepsilon = -f_+'(0)$, then

$$2f_+'(0) < \frac{f(x) - f(c)}{x - c} < 0 \Rightarrow f(x) - f(c) < 0$$

$$\Rightarrow f(x) < f(c)$$

Hence we have $f(x) \leq f(c)$ in both cases.

Now, since δ_1 could be $>$ than $c - a$, we take $\delta_3 = \min\{c - a, \delta_1\}$, Likewise, δ_2 could be $>$ than $b - c$, so we take $\delta_4 = \min\{c + b, \delta_2\}$.

Lastly, take $\delta^* = \min\{\delta_3, \delta_4\}$ to ensure $f(x) \leq f(c)$ to the left and right of c.

P3) WTS that $\exists \delta$ st. if $0 < |x| < \delta$, then

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| < \varepsilon, \quad \forall \varepsilon > 0$$

Two cases:

1) if $x \neq a_n \ \forall n \in \mathbb{N}$, then

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| \frac{0}{x} \right| = 0 < \varepsilon, \quad \forall \varepsilon > 0.$$

2) if $x = a_n$, then

$$\left| \frac{f(x)}{x} \right| = \left| \frac{a_n/n}{a_n} \right| = \left| \frac{1}{n} \right| = \frac{1}{n}$$

Take $\delta = n a_n \varepsilon$, Then

$$|x| < n a_n \varepsilon \Rightarrow |a_n| < n a_n \varepsilon$$

$$\Rightarrow n \left| \frac{a_n}{n} \right| < n a_n \varepsilon \Rightarrow a_n \left| \frac{a_n/n}{a_n} \right| < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon$$

P4.) $f(x) = x^4 + x^3 + x^2 + 1$ $\quad f'''(x) = 24x + 6$

$f'(x) = 4x^3 + 3x^2 + 2x$

$f''(x) = 12x^2 + 6x + 2$

$p_2(x) = 1 + 0 + \frac{2}{2!}(x)^2 = x^2 + 1$

a) $f^{(3)}(x) = 24x + 6$ a line

So we have $x \in [-1, 1] \Rightarrow$ $C \geq |24(-1) + 6|$

$C \geq |-18| = 18$

and $C \geq |24(1) + 6|$

$C \geq |30|$

hence $C \geq 30 \geq 18$ and the smallest C in $\mathbb{R}$ is $\underline{\underline{30}}$

b) $|f(x) - P_n(x)| \leq D|x|^3 \Leftrightarrow n=2$

$\Rightarrow |f(x) - P_2(x)| \leq \frac{C}{3!}|x|^3$

$\Rightarrow |x^4 + x^3 + x^2 + 1 - (x^2+1)| = |x^4 + x^3| \leq \frac{C}{6}|x|^3$

$\Rightarrow \cancel{|x^3|}|x+1| \leq \frac{C}{6}\cancel{|x|^3} \Rightarrow |x+1| \leq \frac{C}{6}$

$\Rightarrow 6|x+1| \leq C$ $\quad 0 \leq 12 \leq C$

Hence $\underline{\underline{D \geq 2}}$