Math 401: Midterm 2 Study Guide

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Chapter 3: The Real Numbers

Definitions & Theorems

Theorem 10.6(2) (Principle of Mathematical Induction) $P(n)$ is true for all $n > m$, $n, m \in \mathbb{N}$ provded

(a) $P(m)$ is true, and

(b) for each $k \in \mathbb{N} \geq m$, if $P(k)$ is true, then $P(k+1)$ is true.

Note: If $m = 1$, then $P(n)$ is true for all $n \in \mathbb{N}$.

Theorem 11.9d (Triangle Inequality) Let $x, y \in \mathbb{R}$. Then $|x + y| \leq |x| + |y|$.

Theorem 12.1 Let p be a prime number. Then $\sqrt{p}$ is not a rational number.

Definition 12.2 (Upper / Lower Bound) If there exists a real number m such that $m \geq s$ for all $s \in S$, then m is an *upper bound* of S, and we say S is "bounded above"

Flip for *lower bound*.

Definition 12.2 (Maximum / Minimum) If there exists an upper (lower) bound m of S that is a member of S, then m is the *maximum (minimum)* of S.

Definition 12.5 (Supremum / Infimum) An upper (lower) bound m of a non-empty subset of $\mathbb{R}$ is called the *supremum (infimum)* or *least upper (greatest lower) bound* of S iff

if for all $m' < m$, there exists an $s \in S$ such that $s > m'$.

Flip for infimum.

Axiom (Completeness of the Reals) Every non-empty subset of $\mathbb{R}$ that is bounded above (below) has a supremum (infimum).

Theorem 12.9(10) (The Archimedean Property) The following are equivalent:

(a) The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.

(b) For each $z \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > z$.

(c) For each $x > 0$ and for each $y \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $nx > y$.

(d) For each $x > 0$, there exists an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$.

Theorems 12.12 and 12.13 If x and y are real numbers and $x < y$, then there exists both an irrational number i such that $x < i < y$, and a rational number r such that $x < r < y$.

Definition 13.3 (Interior Point) A point x in $\mathbb{R}$ is an *interior point* of S if there exists a (non-deleted) neighborhood N of x such that $N \subseteq S$.

Definition 13.3 (Boundary Point) A point x in $\mathbb{R}$ is a *boundary point* of S if for every (non-deleted) neighborhood N of x, we have $N \cap S \neq \emptyset$ and $N \cap S^c \neq \emptyset$.

Definition 13.6 (Open / Closed Sets) Let $S \in \mathbb{R}$.

If $\operatorname{bd} S \subseteq S$, then S is *closed*.

If $\operatorname{bd} S \subseteq S^c$, then S is *open*.

Definition 13.14 (Accumulation / Isolated Points) A point $x \in \mathbb{R}$ is an *accumulation point* of S if every deleted neighborhood of x contains a point in S.

Otherwise, x is an *isolated point* of S.

Definition 14.1 (Open Cover) A family $\mathcal{F}$ of open sets is an *open cover* of the set S if $S \subseteq \bigcup \mathcal{F}$.

Definition 14.1 (Subcover) If $\mathcal{G} \subseteq \mathcal{F}$, and $\mathcal{G}$ is also an open cover of S, then $\mathcal{G}$ is a *subcover* of S.

Definition 14.1 (Compact Set) The set S is called *compact* if every open cover of S contains a finite subcover.

Theorem 14.5 (Heine-Borel) A subset S of $\mathbb{R}$ is compact iff S is closed and bounded.

Chapter 4: Sequences

Definitions & Theorems

Definition 16.2 (Convergence) A sequence s_n is said to *converge* to a number L if for every $\epsilon > 0$ there exists a number $N \in \mathbb{N}$ such that if $n > N$, then $|s_n - L| < \epsilon$.

Theorem 16.13 Every convergent sequence is bounded.

Theorem 16.14 If a sequence converges, it's limit is unique.

Theorem 17.1 (Fundamental Limit Theorems) Suppose that (s_n) and (t_n) are convergent sequences with limits s and t. Then

(a) $\lim_{n\to\infty}(s_n + t_n) = s + t$

(b) $\lim_{n\to\infty} ks_n = ks$ and $\lim_{n\to\infty}(k + s_n) = k + s$ for any $k \in \mathbb{R}$.

(c) $\lim_{n\to\infty} s_n t_n = st$.

(d) $\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{s}{t}$.

Definition 18.1 (Monotone) A sequence is *monotone* if either $s_n \leq s_{n+1}$ (increasing) or $s_n \geq s_{n+1}$ (decreasing) for all $n \in \mathbb{N}$.

Theorem 18.3 (Monotone Convergence Theorem) A monotone sequence is convergent iff it is bounded.

Definition 18.9 (Cauchy Sequence) A sequence (s_n) is a *Cauchy sequence* if for all $\epsilon > 0$, there exists a number $N \in \mathbb{N}$ such that if $n, m > N$, then $|s_n - s_m| < \epsilon$.

Theorem 18.12 (Cauchy Convergence Criterion A sequence of real numbers is convergent iff it is a Cauchy sequence.

Definition 19.1 (Subsequence) Let (n_k) be a sequence of natural numbers such that $n_1 < n_2 < ... < n_k$. Then s_{n_k} is a *subsequence* of s_n.

Theorem 19.4 If a sequence (s_n) converges to a real number s, then any subsequence of (s_n) converges to s.

Theorem 19.7 Every bounded sequence has at least one convergent subsequence.

Definition 19.9 (Limit Superior / Inferior) If S is the set of subsequential limits of a sequence (s_n), the *limit superior* lim sup $s_n = \sup(S)$. The *limit inferior* is defined likewise with the infimum of S.