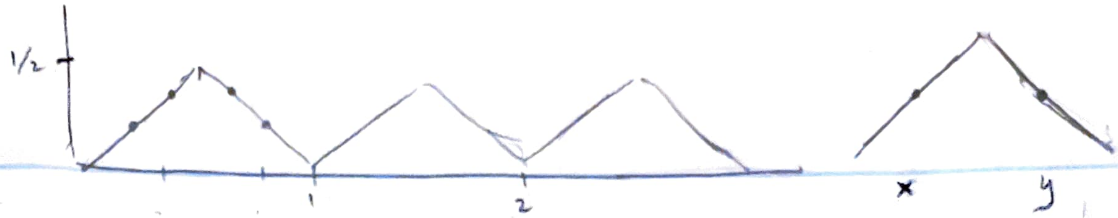


1)

a) $|x - c| < \delta$

Take $\delta \leq 1/2$ then there are 5 cases:

i) $n_x = n_c$; $r_x, r_c < \frac{1}{2}$

ii) $n_x = n_c$; $r_x < \frac{1}{2}$, $r_c > \frac{1}{2}$ (or vise versa)

iii) $n_x = n_c$; $r_x, r_c > \frac{1}{2}$

iv) $n_x = n_c + 1$, $r_x < \frac{1}{2}$, $r_c > \frac{1}{2}$

v) $n_x = n_c - 1$, $r_x > \frac{1}{2}$, $r_c < \frac{1}{2}$

Case I ; $|x - c| < \delta \leq 1/2$

$= |(n + r_x) - (n + r_c)| = |r_x - r_c| < \delta$

$|f(x) - f(c)| = |r_x - r_c| < \delta$, so take $\delta = \min\{\varepsilon, 1/2\}$

Case II ; $|x - c| = |r_x - r_c| < \delta \leq \frac{1}{2}$

$|f(x) - f(c)| = |r_x - (1 - r_c)| = |r_x + r_c - 1| <$

$= |r_x - r_c + 2r_c - 1| \leq |r_x - r_c| + |2r_c - 1| < \delta + |2r_c - 1|$

so take $\delta = \min\{1/2, \varepsilon - |2r_c - 1|\}$ (concerned δ could be < 0)

Case III $|x - c| = |r_x - r_c| < \delta$

$|f(x) - f(c)| = |(1 - r_x) - (1 - r_c)| = |r_c - r_x| = |r_x - r_c|$

So take $\delta = \min\{\varepsilon, 1/2\}$

Case IV $|x - c| = |(n_c + r_x + 1) - (n_c + r_c)| = |r_x - r_c + 1| < \delta$

$|f(x) - f(c)| = |r_x - (1 - r_c)| = |r_x + r_c - 1|$

$= |r_x - r_c + 1 + 2r_c - 2| \leq |r_x - r_c + 1| + 2|r_c - 1| < \delta + 2|r_c - 1|$

So take $\delta = \min\{1/2, \varepsilon - 2|r_c - 1|\}$

Case V $|x-c| = |(n_c + r_x - 1) - (n_c + r_c)| = |r_x - r_c - 1| < \partial$

$|f(x) - f(c)| = |(1-r_x) - r_c| = |r_c - r_x + 1| = |r_x - r_c - 1|$

so take $\partial = \min\{\varepsilon, \frac{1}{2}\}$

In total, take $\partial = \min\{\frac{1}{2}, \varepsilon, \varepsilon - |2r_c - 1|, \varepsilon - |2r_c - 2|\}$

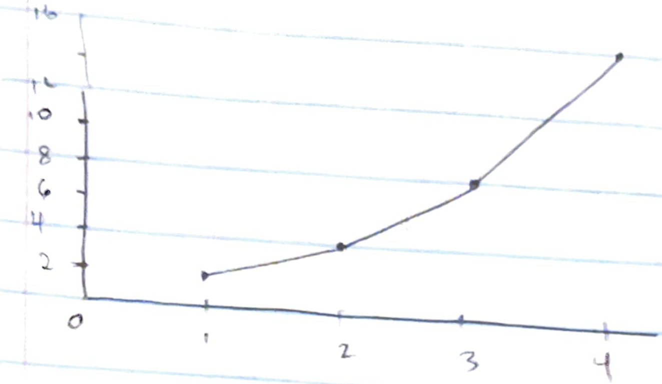
b) Yes, f is UC, because r_x, r_y are bounded on $(0,1)$. Hence, from the result in part a, take $\partial = \varepsilon - |2(1) - 2| = \varepsilon$.

by $f(n+r) = (1-r)2^n + r2^{n+1}$

$= 2^n[(1-r) + 2r]$

$= 2^n[r+1]$

$\Longrightarrow$



Not UC for sure

a) Take $|x-c| < 1 \Rightarrow$ 3 cases

Case I : $n_x = n_c$. $|x-c| = |r_x - r_c| < \delta \le 1$

$|f(x)-f(c)| = |2^n[r_x+1] - 2^n[r_c+1]| = 2^n|r_x - r_c| < 2^n\delta$

$\Rightarrow$ Take $\delta = \min\{1, \frac{\varepsilon}{2^n}\}$

Case II : $n_x = n_c + 1$. $|x-c| = |n_c+1+r_x - (n_c+r_c)| = |r_x - r_c + 1| < \delta$

$|f(x)-f(c)| = |2^{n_c+1}[r_x+1] - 2^{n_c}[r_c+1]| = 2^{n_c}|2(r_x+1) - (r_c+1)|$

$= 2^{n_c}|2r_x + 2 - r_c + 1| = 2^{n_c}|r_x - r_c + r_x + 3|$

$= 2^{n_c}|(r_x - r_x) + (r_x - r_c) + r_c + 3| = 2^{n_c}|2(r_x - r_c) + (r_c+3)|$

$\le 2^{n_c}[|2(r_x - r_c)| + |r_c+3|] < 2^{n_c}[2\delta + |r_c+3|]$

$\delta = \min\left\{\frac{\varepsilon}{2^{n_c+1}} - \frac{|r_c+3|}{2}, 1\right\}$

Case III $n_x = n_c - 1$. $|x-c| = |n_c - 1 + r_x - (n_c + r_c)| = |r_x - r_c - 1| < \delta$

$|f(x)-f(c)| = |2^{n_c-1}[r_x+1] - 2^{n_c}[r_c+1]| = 2^{n_c}[\frac{1}{2}(r_x+1) - (r_c+1)]$

$= 2^{n_c}[|\frac{1}{2}r_x + \frac{1}{2} - r_c - 1|] = 2^{n_c-1}[|(r_x - r_c) + (1-r_c)|] < 2^{n_c-1}|\delta + (1-r_c)|$

$\le 2^{n_c-1}[\delta + |1-r_c|] \Rightarrow \delta = \min\{1, \frac{\varepsilon}{2^{n_c-1}} - |1-r_c|\}$

So take $\delta = \min\left\{1, \frac{\varepsilon}{2^n}, \frac{\varepsilon}{2^{n_c+1}} - \frac{r_c+3}{2}, \frac{\varepsilon}{2^{n_c-1}} - r_c - 1\right\}$

b) Not UC, since δ depends on n_c, which is unbounded.

3) $f, g: \mathbb{R} \to \mathbb{R}$ are UC. Prove $f \circ g$ is UC

Let $\varepsilon > 0$. Then $\exists \partial_1 >$ st. $\forall x, y \in \mathbb{R}$
if $|x-y| < \partial_1$, then $|g(x) - g(y)| < \varepsilon_1$.

Similarly, $\exists \partial_2$ st $\forall x, y \in \mathbb{R}$, if $|x-y| < \partial_2$
then $|f(x) - f(y)| < \varepsilon_2$

Also, $\exists \partial_3$ st. $\forall x, y \in \mathbb{R}$, if $|x-y| < \partial_3$
we have $|g(x) - g(y)| < \partial_2$

Then, since $|g(x) - g(y)| < \partial_2$ we have that
$|f \circ g(x) - f \circ g(y)| = |f(g(x)) - f(g(y))| < \varepsilon_2$

... Not So sure about this one ...

4) $f: (0,2) \to \mathbb{R}$, $g: [0,1] \to \mathbb{R}$, $h: [1,2] \to \mathbb{R}$

$$f(x) = \begin{cases} g(x) & 0 < x \leq 1 \\ h(x) & 1 \leq x < 2 \end{cases}$$

g is UC $\Rightarrow \exists \partial_1 > 0$ st. $\forall \varepsilon > 0$, $\forall x, y \in [0,1]$
$|x - y| < \partial_1 \Rightarrow |g(x) - g(y)| < \varepsilon/2$

h is UC $\Rightarrow \exists \partial_2 > 0$ st $\forall \varepsilon > 0$, $\forall x, y \in [1,2]$
$|x - y| < \partial_2 \Rightarrow |h(x) - h(y)| < \varepsilon/2$

If $x, y \in [0,1]$ or $x, y \in [1,2)$ we are done, since then $f(x) = g(x)$ or $f(x) = h(x)$.

Take $\partial = \min\{1, \partial_1, \partial_2\}$
If $x \in [0,1]$ and $y \in [1,2]$, then we have

$|f(x) - f(y)| = |f(x) + f(1) - f(1) + f(y)|$
$\leq |f(x) - f(1)| + |f(y) - f(1)|$
$= |g(x) - g(1)| + |h(y) - h(1)|$
because $f(1) = g(1) = h(1)$ by definition.
Now, because $|x - 1| < 1 < \partial$ and $|y - 1| < 1 < \partial$
we have that $|g(x) - g(1)| < \varepsilon/2$ and $|h(x) - h(1)| < \varepsilon/2$
so $|g(x) - g(1)| + |h(x) - h(1)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$