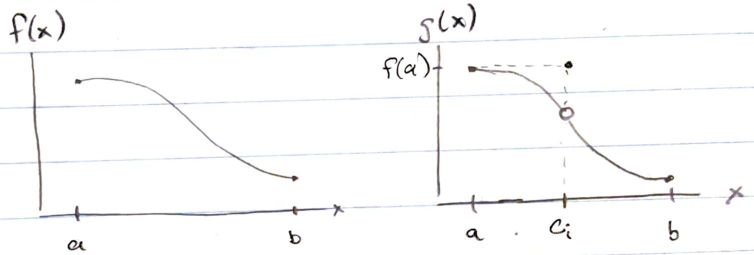
1) Suppose $f : [a,b] \to \mathbb{R}$ is integrable. Suppose $S = \{c_1, \dots, c_n\}$ is a finite set of points in (a,b), and define

$$g(x) = \begin{cases} f(a) & x \in S \\ f(x) & x \notin S \end{cases}$$

Prove that g is integrable and that

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$$



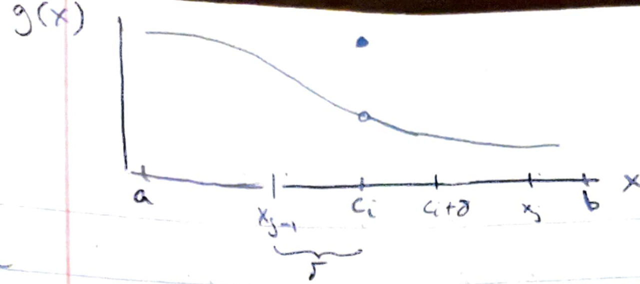
We show that g is integrable & $\int g = \int f$ for $S = \{c_i\}$ an arbitrary point, and by induction that g is integrable & $\int g = \int f$ for $S = \{c_1, \dots, c_n\}$.

Since f is integrable, then we have for $\forall \varepsilon > 0$ $\exists P_0 = \{x_0 = a, \dots, x_n = b\}$ st $U(f, P_0) - L(f, P_0) < \varepsilon$.

Since $c_i \in (a,b)$ we have some $x_j \in P_0$ st.

$x_{j-1} \le c_i \le x_j$.

P2



Let $\delta = \min\{c_i - x_{j-1},\ x_j - c_i\}$

Let $\delta^* = \min\{\delta,\ \varepsilon/8(H-J)\}$, where H is upper bound on f & J = lower bound on f on $[a,b]$.

Next define $g_1 = c_i - \delta^*$, $g_2 = c_i + \delta^*$ and

Let $P_1 = P_0 \cup \{g_1, g_2\} = \{x_0 = 0, x_1, \dots x_{j-1}, g_1, g_2, x_j, \dots x_n = b\}$

Then $|U(g,P_1) - U(f,P_1)| = \left|\sum_{i=1}^{n+2} (M_i(g) - M_i(f))\Delta x_i\right|$

$$\left.\begin{aligned} = & \left|\sum_{i=1}^{j-1} (M_i(g) - M_i(f))\Delta x_i\right| = 0 \\ + & \left|\sum_{i=g_1}^{g_2} (M_i(g) - M_i(f))\Delta x_i\right| \\ + & \left|\sum_{i=g_2}^{n+2} (M_i(g) - M_i(f))\Delta x_i\right| = 0 \end{aligned}\right\}\ \text{b/c } f = g \text{ except when } x = c_i$$

$= \left|\sum_{g_1}^{g_2} (M_i(g) - M_i(f))\Delta x_i\right| = |(M_i(g) - M_i(f))(g_2 - g_1)|$

$= |(M_i(g) - M_i(f))(2\delta^*)| \le (H-J)(2\delta^*)$

$\le (H-J)\dfrac{2\varepsilon}{8(H-J)} = \dfrac{\varepsilon}{4}$

the same argument yields $|L(g,P_1) - L(f,P_1)| \le \varepsilon/4$

Then $U(g,P_1) - L(g,P_1) \le U(f,P_1) + \varepsilon/4 - (L(f,P_1) - \varepsilon/4)$

b/c P_1 a refinement $\longrightarrow$ $\le U(f,P_0) + \varepsilon/4 - (L(f,P_0) - \varepsilon/4)$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

To show $\int_a^b f(x)\,dx = \int_c^b g(x)\,dx$

we have $\left| \int_a^b f(x)\,dx - \int_c^b g(x)\,dx \right| = \left| \int_c^b (f-g)(x)\,dx \right|$

Then $\left| \int_a^b f-g\,dx \right| = \left| \cancel{\int_a^{q_1} (f-g)\,dx}^{\,0} + \int_{q_1}^{q_2} (f-g)\,dx + \cancel{\int_{q_2}^{b} (f-g)\,dx}^{\,0} \right|$

b/c $f=g$ except at $c_1 \in (q_1, q_2)$.

$= \left| \int_{q_1}^{q_2} (f-g)\,dx \right| \leq \int_{q_1}^{q_2} |(f-g)|\,dx$. by triangle inequality

$\leq \sup\{|f-g| : x \in [q_1, q_2]\}(q_2 - q_1)$

$\leq (H-J)\,\frac{2\,\varepsilon}{8(H-J)} = \frac{1}{4}\varepsilon < \varepsilon$

Thus $\int f = \int g$.

P2 Suppose f is integrable on $[0, c]$ for all c.

Evaluate

$$\frac{d}{dx}\left(\int_0^{x^2} f(t)+t^3\,dt - \int_0^{x^2} f(t)-t^2\,dt\right)$$

Since $f(t)$ continuous

$$\Rightarrow \frac{\partial}{\partial x}\left(\int_0^{x^2} f(t)+t^3-f(t)+t^3\,dt\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\int_0^{x^2} 2t^3\,dt\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}t^4\Big|_0^{x^2}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{1}{2}x^8 - 0\right] = \frac{\partial}{\partial x}\frac{1}{2}x^8 = \underline{\underline{\frac{1}{4}x^7}}$$

P3) Suppose (a_n) is a sequence and define (b_n) by

$$b_n \begin{cases} a_{n+1} & n \text{ odd} \\ a_{n-1} & n \text{ even} \end{cases}$$

Show that if $\sum^{\infty} a_n$ is convergent, then $\sum b_n$ is convergent.

Let s_n be partial sums of $a_n = \sum^{n} a_n$
and $\tilde{s}_n$ be partial sums of $b_n = \hat{\sum} b_n$

Then we prove by induction that $\tilde{S} = \begin{cases} s_{n+1} - a_n, & n \text{ odd} \\ s_{n+1} - a_{n+1} & n \text{ even} \end{cases}$

We have that:

$$\tilde{S}_1 = b_1 = a_2 \quad = (a_2 + a_2) - a_1 = s_2 - a_1$$
$$\tilde{S}_2 = b_1 + b_2 = a_2 + a_1 \quad = s_3 - a_3$$
$$\tilde{S}_3 = b_1 + b_2 + b_3 = a_2 + a_1 + a_4 = s_4 - a_3$$
$$\hat{S}_4 = b_1 \ldots b_4 = a_1 \ldots a_4 = s_5 - a_4$$

Then we have

n even $\lim\limits_{n\to\infty} (s_{n+1} - a) = \lim\limits_{n\to\infty} s_{n+1} - \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} + 0$

since a_n is convergent.