Formulário Derivadas e Primitivas quase imediatas

$(u^p)' = p\,u^{p-1}\,u'$

$(\ln(u))' = \dfrac{u'}{u}$

$(\cos u)' = -u'\operatorname{sen} u$

$(\operatorname{sen} u)' = u'\ \cos u$

$(\operatorname{tg} u)' = u'\ \sec^2 u$

$(\operatorname{cotg} u)' = -u'\operatorname{cosec}^2 u$

$(\operatorname{senh}^{-1} u)' = \dfrac{u'}{\sqrt{1+u^2}}$

$(\operatorname{arcsen}(u))' = \dfrac{u'}{\sqrt{1-u^2}}$

$(\operatorname{arctg}(u))' = \dfrac{u'}{1+u^2}$

$(\sec u)' = u'\sec(u)\operatorname{tg}(u)$

$(\operatorname{cosec} u)' = -u'\operatorname{cosec}(u)\operatorname{cotg}(u)$

$(e^u)' = u'\,e^u$

$(a^u)' = \dfrac{u' a^u}{\ln(a)}\ ,\ a \in \mathbb{R}^+\setminus\{1\}$

$(uv)' = u'v + uv'$

$\int u'\,u^p\,dx = \frac{u^{p+1}}{p+1} + c,\ (p \neq -1)$

$\int \frac{u'}{u}\,\mathrm{d}x = \ln(|u|) + c$

$\int u'\ \sin u\,\mathrm{d}x = \cos u + c$

$\int u'\ \cos u dx = \sin u + c$

$\int u'\sec^2 u\,\mathrm{d}x = \tan u + c$

$\int u'\operatorname{cosec}^2 u\,\mathrm{d}x = -\mathrm{cotgu} + \mathrm{c}$

$\int \frac{u'}{\sqrt{1+u^2}}\,\mathrm{d}x = \operatorname{senh}^{-1} u + C$

$\int \frac{u'}{\sqrt{1-u^2}} dx = \arcsin(u) + c$

$\int \frac{u'}{1+u^2}\,\mathrm{d}x = \arctan(u) + c$

$\int u'\sec(u)\tan(u)\,\mathrm{d}x = \sec u + c$

$\int u'\operatorname{cosec}(u)\mathrm{cotg(u)dx} = -\mathrm{cosec\,u} + \mathrm{c}$

$\int u' e^u\,\mathrm{d}x = e^u + c$

$\int u' a^u\,\mathrm{d}x = \frac{a^u}{\ln(a)} + c,\quad a \in \mathbb{R}^+\setminus\{1\}$

$\int u'v + uv'\,\mathrm{d}x = uv + C$

Aula 9: Primitivação por partes

Sejam f e g funções reais definidas em $I \subset \mathbb{R}$ com f primitivável e com primitiva $F(x)$ e g derivável. Então fg é primitivável em I e temos:

$$\boxed{\int f\, g \,\mathrm{d}x \;=\; F\, g - \int F\, g' \,\mathrm{d}x.}$$

Dem: Fg é derivável e $Fg = \int (Fg)' \,\mathrm{d}x = \int F'g + Fg' \,\mathrm{d}x = \int fg + Fg' \,\mathrm{d}x.$

Exemplos: 5.1 $\int x \sec^2 x \,\mathrm{d}x =$ **5.2** $\int e^x \operatorname{sen} x \,\mathrm{d}x =$ **5.3** $\int \ln(x) \,\mathrm{d}x =$

Exercício 5.3: **1.** $\int \arctan x \,\mathrm{d}x$ **?.** $\int \sec x \,\mathrm{d}x$ **2.** $\int \sec^3 x \,\mathrm{d}x$

3. $\int \operatorname{sen}(2x)\operatorname{sen}(7x) \,\mathrm{d}x$ **8.** $\int \cos(\ln(x)) \,\mathrm{d}x$

Primitivas de produtos de $sen(\alpha x)$ e $cos(\beta x)$: $\quad \alpha \neq \beta$

- $\int sen(\alpha x)\, cos(\beta x)\ \mathrm{d}x$
- $\int sen(\alpha x)\, sen(\beta x)\ \mathrm{d}x$
- $\int cos(\alpha x)\, cos(\beta x)\ \mathrm{d}x$

Podemos usar a integração por partes duas vezes consecutivas, ou, em alternativa, usar as fórmulas trigonométricas

$$sen(A+B) = sen(A)cos(B) + cos(A)sen(B) \qquad cos(A+B) = cos(A)cos(B) - sen(A)sen(B)$$

$$sen(A-B) = sen(A)cos(B) - cos(A)sen(B) \qquad cos(A-B) = cos(A)cos(B) + sen(A)sen(B)$$

e deduzir:

- $sen(A)\, cos(B) \ = \ \frac{1}{2}\ (\, sen(A+B) + sen(A-B)\,)$
- $cos(A)\, cos(B) \ = \ \frac{1}{2}\ (\, cos(A+B) + cos(A-B)\,)$
- $sen(A)\, sen(B) \ = \ \frac{1}{2}\ (\, -cos(A+B) + cos(A-B)\,)$

Potências inteiras (positivas) de senos ou cosenos

Usar a fórmula do Binómio de Newton: $\quad (1+B)^n = \sum_{i=0}^{n} \binom{n}{i} B^i$

$$\int \operatorname{sen}^n x \, \mathrm{d}x =$$

$$\begin{cases} (n \text{ par}:) & \int (\operatorname{sen}^2 x)^{\frac{n}{2}} \, \mathrm{d}x = \int \left(\frac{1}{2}(1-\cos 2x)\right)^{\frac{n}{2}} \, \mathrm{d}x = \frac{1}{2^{\frac{n}{2}}} \int (1-\cos 2x)^{\frac{n}{2}} \, \mathrm{d}x \\ (n \text{ ímpar}:) & \int \operatorname{sen}^{(n-1)} x \operatorname{sen} x \, \mathrm{d}x = \int (\operatorname{sen}^2 x)^{\frac{n-1}{2}} \operatorname{sen} x \, \mathrm{d}x = \int (1-\cos^2 x)^{\frac{n-1}{2}} \operatorname{sen} x \, \mathrm{d}x \end{cases}$$

$$\int \cos^n x \, \mathrm{d}x =$$

$$\begin{cases} (n \text{ par}:) & \int (\cos^2 x)^{\frac{n}{2}} \, \mathrm{d}x = \int \left(\frac{1}{2}(1+\cos 2x)\right)^{\frac{n}{2}} \, \mathrm{d}x = \frac{1}{2^{\frac{n}{2}}} \int (1+\cos 2x)^{\frac{n}{2}} \, \mathrm{d}x \\ (n \text{ ímpar}:) & \int \cos^{(n-1)} x \cos x \, \mathrm{d}x = \int (\cos^2 x)^{\frac{n-1}{2}} \cos x \, \mathrm{d}x = \int (1-\operatorname{sen}^2 x)^{\frac{n-1}{2}} \cos x \, \mathrm{d}x \end{cases}$$

$\boxed{\operatorname{sen}^2 x = \frac{1}{2}(1-\cos 2x)}$ $\quad \boxed{\cos^2 x = \frac{1}{2}(1+\cos 2x)}$

Potências inteiras (positivas) da secante

$$\int sec\, x \, \mathrm{d}x = \int \sec x \frac{\sec x + \operatorname{tg} x}{\sec x + \operatorname{tg} x} \, \mathrm{d}x = \int \frac{\sec^2 x + \sec x \operatorname{tg} x}{\sec x + \operatorname{tg} x} \, \mathrm{d}x = \ln(|\sec x + \operatorname{tg} x|) + C$$

$$\int sec^2 x \, \mathrm{d}x = \operatorname{tg} x + C$$

$$\int sec^3 x \, \mathrm{d}x = \int \underset{(\mathrm{P})}{sec^2 x} \; \underset{(\mathrm{D})}{\sec x \, \mathrm{d}x} = \cdots = \frac{1}{2} \operatorname{tg} x \sec x + \frac{1}{2} \ln(|\sec x + \operatorname{tg} x|) + C$$

$$\int sec^n x \, \mathrm{d}x = \int \underset{(\mathrm{D})}{sec^{n-2} x} \; \underset{(\mathrm{P})}{\sec^2 x \, \mathrm{d}x} \qquad \text{(integração por partes)}$$

Depois da Int. por partes usar $\boxed{1 + \operatorname{tg}^2 x = \sec^2 x}$

Potências inteiras (positivas) da tangente

$$\int \operatorname{tg} x \;\mathrm{d}x \;=\; -\ln(|\cos x|) + C$$

$$\int \operatorname{tg}^2 x \;\mathrm{d}x \;=\; \int \sec^2 x - 1 \;\mathrm{d}x \;=\; \operatorname{tg} x + x + C$$

$$\int \operatorname{tg}^3 x \;\mathrm{d}x = \int \operatorname{tg}^2 x \operatorname{tg} x \;\mathrm{d}x = \int (\sec^2 x - 1) \operatorname{tg} x \;\mathrm{d}x = \int \underbrace{\sec^2 x}_{\text{u'}} \; \underbrace{\operatorname{tg} x}_{\text{u}} \;\mathrm{d}x - \int \operatorname{tg} x \;\mathrm{d}x = \frac{\operatorname{tg}^2 x}{2} + \ln(|\cos x|) + C$$

$$\int tg^n x \;\mathrm{d}x =$$

$$\begin{array}{rl} (n \text{ par : }) & \left\{ \begin{array}{l} \displaystyle\int (\operatorname{tg}^2 x)^{\frac{n}{2}} \;\mathrm{d}x \;=\; \int (\sec^2 x - 1)^{\frac{n}{2}} \;\mathrm{d}x \\[2em] \displaystyle\int \operatorname{tg}^{(n-1)} x \operatorname{tg} x \;\mathrm{d}x \;=\; \int (\operatorname{tg}^2 x)^{\frac{n-1}{2}} \operatorname{tg} x \;\mathrm{d}x \;=\; \int (\sec^2 x - 1)^{\frac{n-1}{2}} \operatorname{tg} x \;\mathrm{d}x \end{array} \right. \\ (n \text{ ímpar : }) & \end{array}$$

$$\int \sec^n x \operatorname{tg} x \;\mathrm{d}x = \int \sec^{n-1} \sec x \operatorname{tg} x \;\mathrm{d}x = \frac{\sec^n x}{n} + C$$

$$\boxed{1 + \operatorname{tg}^2 x = \sec^2 x}$$

Aula 9: Primitivação por mudança da variável

$$\int f(x)\,\mathrm{d}x$$

Substituição:

$x = u(t)$, u = função invertível e dif. nalgum int. J com $u(J) \subset D_f$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u'(t) \quad \Leftrightarrow \quad \mathrm{d}x = u'(t)\,\mathrm{d}t$$

$$\int f(x)\,\mathrm{d}x \;=\; \int f(u(t))\,u'(t)\,\mathrm{d}t \;=\; H(t) + C$$

Reverter a substituição:

$t = u^{-1}(x)$

$$\int f(x)\,\mathrm{d}x \;=\; H(u^{-1}(x)) + C$$

Aula 9: Regras de substituição

$$\int f(x)\,\mathrm{d}x$$

$f(x)$ contém		Substituição
$\sqrt[k]{a+bx}$	$\rightsquigarrow$	$\sqrt[k]{a+bx} = t \quad (t \geq 0 \text{ se } k \text{ par})$
$\sqrt{a^2+x^2}$	$\rightsquigarrow$	$x = a\,\mathrm{tg}\,t\,, \quad a>0 \text{ e } t \in]-\frac{\pi}{2}, \frac{\pi}{2}[$
$\sqrt{a^2-x^2}$	$\rightsquigarrow$	$x = a\,\mathrm{sen}\,t\,, \quad a>0 \text{ e } t \in]-\frac{\pi}{2}, \frac{\pi}{2}[\quad \left(\text{ou } x = a\,\cos t\,,\ t \in]0,\pi[\right)$
$\sqrt{x^2-a^2}$	$\rightsquigarrow$	$x = a\,\sec t\,, \quad a>0 \text{ e } t \in]0, \frac{\pi}{2}[$
$\sqrt{ax^2+bx+c}$	$\rightsquigarrow$	$x + \frac{b}{2a} = z$

$$\boxed{ax^2+bx+c = a\left[(x+\frac{b}{2a})^2 + K\right]}\,, \quad K = \frac{c}{a} - \frac{b^2}{4a^2}$$

Aula 9: Exercícios 1

Exemplo 5.5. Como calcular $\int \sqrt{9 - x^2}\, dx$, com $x \in]-3, 3[$?

Exemplo 5.6. $$\int \frac{1}{\sqrt{2x^2 + 8x - 24}}\, dx =$$

Exercício 5.8 Calcule:

1. $\displaystyle\int \frac{e^x}{\sqrt{4 - e^{2x}}}\, dx$;
2. $\displaystyle\int \frac{2x + 5}{\sqrt{9x^2 + 6x + 2}}\, dx$;
3. $\displaystyle\int \frac{1}{x(3 + \ln x)^3}\, dx$;
4. $\displaystyle\int \frac{1}{\sqrt{8 + 2x - x^2}}\, dx$;
5. $\displaystyle\int \frac{\operatorname{sen}^3 x}{\sqrt{\cos x}}\, dx$;
6. $\displaystyle\int \frac{1}{x^2\sqrt{5 - x^2}}\, dx$;