## Topics

- Elementary Boolean operators
- Duality
- Basic theorems
- Algebraic minimization

## Problems

**1** Recall the truth table of the elementary Boolean operators. Solve the following system of equations for the variables A, B, C and D

$$\begin{cases} A' + A.B & = & 0 \\ A.C & = & A.B \\ A.B + A.C' + C.D & = & C'.D \end{cases}$$

**2** Verify, by perfect induction, the following simplification theorems. Write the dual counterpart of each theorem.

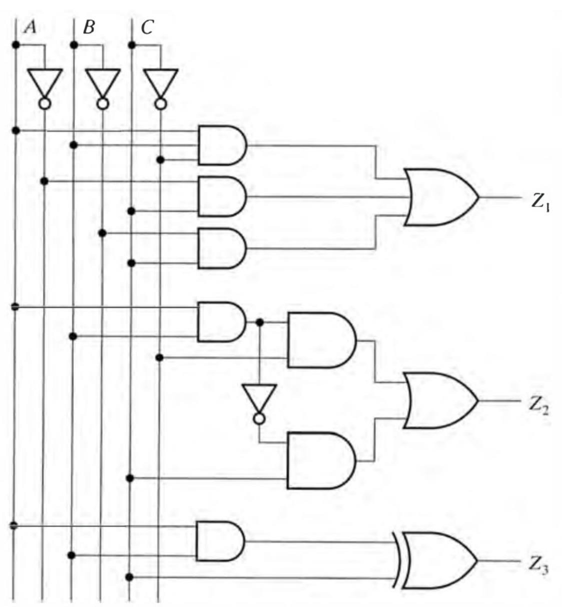
a) $x + x.y = x$ b) $x + x'.y = x + y$ c) $x.y + x.y' = x$

**3** Show that $x'.y'.z' + x'.y'.z + x'.y.z' + x'.y.z + x.y.z' + x.y.z = x' + y$

**4** Show that $x.y + x'.z + y.z = x.y + x'.z$. Write the dual version of the previous expression.

**5** Write the truth table of the XOR operation $x \oplus y$. Express this operator as an elementary sum of logic products.

**6** Consider the logic circuits in the figure and show, by algebraic methods, that $Z_1 = Z_2 = Z_3$.



**7** Use DeMorgan's laws to obtain the complement of:

a) $(x.y' + x'.y)$ b) $(x.y + z.(x + y') + z.y)$

**8** Show that $(a'.b + a.c).(a + b').(a' + c') = 0$

**9** Show that the dual of an XOR is an XNOR, that is $(x \oplus y)^D = (x \oplus y)'$.

**10** Implement the XOR operation with NAND gates. Assume that both uncomplemented and complemented inputs are available.

**11** Consider the following Boolean functions:

$$S = x \oplus y \oplus c_i$$
$$C_o = x.y + c_i.(x + y)$$

a) Draw the logic circuit.

b) Redraw the circuit using only NAND gates.

**12** The Majority function $M(x, y, z)$, is 1 whenever there are at least two inputs equal to 1.

a) Write the truth table for $M(x, y, z)$.

b) From the truth table propose a Boolean expression for $M(x, y, z)$.

c) Draw the corresponding logic circuit.

d) Show that using the set $S = \{M(x, y, z), NOT, "0"\}$ we can express any logic function. Suggestion: show how to implement the fundamental Boolean operators {"+","."} using the elements of *S*.

# ISD - P3

## Problems:

① Tabela de verdade das operações booleanas

AND

| $x$ | $y$ | $x \cdot y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

NAND

| $x$ | $y$ | $\overline{x \cdot y}$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

OR

| $x$ | $y$ | $x + y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

NOT

| $x$ | $\overline{x}$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

NOR

| $x$ | $y$ | $\overline{x + y}$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 0 |

XOR

| $x$ | $y$ | $x \oplus y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

XNOR

| $x$ | $y$ | $\overline{x \oplus y}$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

BUFFER

| $x$ | $x$ |
|---|---|
| 0 | 0 |
| 1 | 1 |

↳ $x \oplus y = \overline{x} \cdot y + x \, \overline{y}$

Resolver equação:

→ equivalente à negação → $\overline{A}$

$$\begin{cases} A' + A \cdot B = 0 \\ A \cdot C = A \cdot B \\ A \cdot B + A \cdot C' + C \cdot D = C' \cdot D \end{cases} \iff$$

$$\iff \begin{cases} \overline{A} = 0 \quad e \quad A \cdot B = 0 \\ A \cdot C = A \cdot B \\ A\,C + A \cdot \overline{C} + C\,D = \overline{C}\,D \end{cases} \iff \begin{cases} A = 1 \quad e \quad B = 0 \\ 1 \cdot C = 1 \cdot 0 \\ 1 \cdot C + 1 \cdot \overline{C} + C \cdot D = \overline{C} \cdot D \end{cases} \iff \begin{cases} A = 1 \ e \ B = 0 \\ C = 0 \\ 0 + 1 + 0 = 1 \cdot D \end{cases} \iff$$

$$\iff \begin{cases} A = 1 \quad e \quad B = 0 \\ C = 0 \\ D = 1 \end{cases}$$

Solução: $A = 1$ ; $B = 0$ ; $C = 0$ ; $D = 1$

② indução perfeita → substituir $x$ e $y$ por todos os valores possíveis

a) $x + x \cdot y = x$

Propriedade de absorção
→ $A + (A \cdot B) = A$
→ $A \cdot (A + B) = A$

Se $x = 0$ → $y = 0 \Rightarrow 0 + 0 \cdot 0 = 0 \Leftrightarrow 0 = 0$
↳ $y = 1 \Rightarrow 0 + 0 \cdot 1 = 0 \Leftrightarrow 0 = 0$

Se $x = 1$ → $y = 0 \Rightarrow 1 + 1 \cdot 0 = 1 \Leftrightarrow 1 = 1$
↳ $y = 1 \Rightarrow 1 + 1 \cdot 1 = 1 \Leftrightarrow 1 = 1$

dual version: $x \cdot (x + y) = x$

b) $x + \bar{x} \cdot y = x + y$

outras identidades
→ $A + \bar{A} B = A + B$
→ $(A + B) \cdot (A + C) = A + B \cdot C$

se $x = 0$ → $y = 0 \Rightarrow 0 + 1 \cdot 0 = 0 + 0 \Leftrightarrow 0 = 0$
↳ $y = 1 \Rightarrow 0 + 1 \cdot 1 = 0 + 1 \Leftrightarrow 1 = 1$

se $x = 1$ → $y = 0 \Rightarrow 1 + 0 \cdot 0 = 1 + 0 \Leftrightarrow 1 = 1$
↳ $y = 1 \Rightarrow 1 + 0 \cdot 1 = 1 + 1 \Leftrightarrow 1 = 1$

dual version: $x \cdot (\bar{x} + y) = x \cdot y$

c) $x \cdot y + x \cdot \bar{y} = x$

Se $x = 0$ → $y = 0 \Rightarrow 0 \cdot 0 + 0 \cdot 1 = 0 \Leftrightarrow 0 = 0$
↳ $y = 1 \Rightarrow 0 \cdot 1 + 0 \cdot 0 = 0 \Leftrightarrow 0 = 0$

Se $x = 1$ → $y = 0 \Rightarrow 1 \cdot 0 + 1 \cdot 1 = 1 \Leftrightarrow 1 = 1$
↳ $y = 1 \Rightarrow 1 \cdot 1 + 1 \cdot 0 = 1 \Leftrightarrow 1 = 1$

dual version $(x + y) \cdot (x + \bar{y}) = x$

$+ x \cdot y \cdot \bar{z}$

③ $\bar{x} \cdot \bar{y} \cdot \bar{z} + x \cdot \bar{y} \cdot z + \bar{x} \cdot y \cdot \bar{z} + \bar{x} \cdot y \cdot z + x \cdot y \cdot z = \bar{x} + y \Leftrightarrow$

negação, 1 deles = 1 e outro = 0

$\Leftrightarrow \bar{x} (\bar{y} \cdot \bar{z} + \bar{y} \cdot z + y \cdot \bar{z} + y \cdot z) + x \cdot (y \cdot z + y \cdot \bar{z}) = \bar{x} + y \Leftrightarrow$

$\Leftrightarrow \bar{x} \cdot 1 + (x \cdot y) \cdot (z + \bar{z}) = \bar{x} + y \Leftrightarrow$

$\Leftrightarrow \bar{x} + x \cdot y = \bar{x} + y \Leftrightarrow$

pelo teorema $a + \bar{a} \cdot b = a + b$

$\Leftrightarrow \bar{x} + y = \bar{x} + y$

by Staples

④ $x \cdot y + \bar{x} \cdot z + y \cdot z = x \cdot y + x' \cdot z \; (=)$

Dual version
↳ transformar os + em · e vice-versa
→ é apenas a troca de + por · e vice-versa

$(=) \; x \cdot y + \bar{x} \cdot z + y \cdot z \times 1 = x \cdot y + \bar{x} \cdot z \; (=)$

adicionar as variáveis

$(=) \; x \cdot y + \bar{x} \cdot z + y \cdot z \cdot (\bar{x} + x) = x \cdot y + \bar{x} \cdot z \; (=)$

$(=) \; x \cdot y + \bar{x} \cdot z + \bar{x} \cdot z \cdot y + x \cdot y \cdot z = x \cdot y + \bar{x} \cdot z \; (=)$

$(=) \; x \cdot y \, (1 + z) + \bar{x} \cdot z \, (1 + y) = x \cdot y + \bar{x} \cdot z \; (=)$

$(=) \; x \cdot y + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$

dual version: a negação → $(x + y) \cdot (\bar{x} + z) \cdot (y + z) = (x + y) \cdot (\bar{x} + z)$

⑤ Tabela de verdade XOR e Sum of products (SOP) → quando é = 1

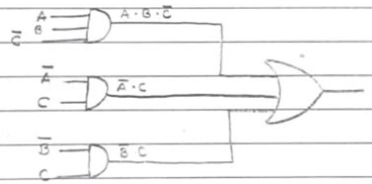
XOR ⇒D—

| $x$ | $y$ | $x \oplus y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

SOP: $f(x) = \bar{x} \cdot y + x \cdot \bar{y}$

⑥ mostrar que $z_1 = z_2 = z_3$

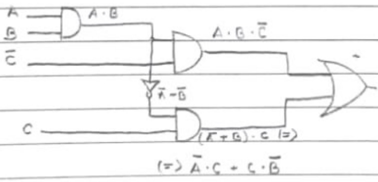
$z_1$




$z_1 = A \cdot B \cdot \bar{C} + \bar{A} \cdot C + \bar{B} \cdot C =$
$= A \cdot B \cdot \bar{C} + (\bar{A} \cdot \bar{B}) \cdot C =$
$= A \cdot B \cdot \bar{C} + \overline{A \cdot B} \cdot C$

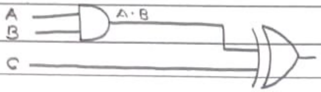
não necessario

$z_2$




$z_2 = A \cdot B \cdot \bar{C} + \bar{A} \cdot C + C \cdot \bar{B}$

$z_3$




$z_3 = \overline{(A\cdot B)}\cdot C + (A\cdot B)\cdot\overline{C} =$

$= (\overline{A}+\overline{B})\cdot C + A\cdot B\cdot\overline{C} =$

$= A\cdot B\cdot\overline{C} + \overline{A}\cdot C + \overline{B}\cdot C$

Concluimos que $z_1 = z_2 = z_3$

2) Usar a lei de Morgan: apenas negar

a) $(x\cdot\overline{y} + \overline{x}\cdot y)$

→ a negação /complemento:

$\overline{(x\cdot\overline{y}) + \overline{x}\cdot y)} =$

$= \overline{(x\cdot\overline{y})}\cdot\overline{(\overline{x}\cdot y)}$ (basta fazer isto)

b) $(x\cdot y + z\cdot(x+\overline{y}) + z\cdot y)$

→ a negação / complemento: $\overline{(x\cdot y + z\cdot(x+\overline{y}) + z\cdot y)} =$

$= \overline{x\cdot y}\cdot\overline{z(x+\overline{y})}\cdot\overline{z\, y}$

⑧ $(\bar a\cdot b + a\cdot c)\cdot(a+\bar b)\cdot(\bar a+\bar c)=0 \Leftrightarrow$

$\Leftrightarrow (\overbrace{\bar a\cdot b\cdot a}^{=0} + \overbrace{\bar a\cdot b\cdot \bar b}^{=0} + a\cdot c\cdot a + a\cdot c\cdot \bar b)\cdot(\bar a\cdot \bar c)=0 \Leftrightarrow$

$\Leftrightarrow (a\cdot c + a\cdot c\cdot \bar b)\cdot(\bar a\cdot \bar c)=0 \Leftrightarrow$

$\Leftrightarrow (\overbrace{a\cdot c\cdot \bar a}^{=0} + \overbrace{a\cdot c\cdot \bar c}^{=0} + \overbrace{a\cdot c\cdot \bar b\cdot \bar a}^{=0} + \overbrace{a\cdot c\cdot \bar b\cdot \bar c}^{=0})=0 \Leftrightarrow$

$\Leftrightarrow 0=0$

* Resolução com a Lei de Morgan no final

⑨ Mostrar que $(x\oplus y)^D = (x\oplus y)'$

→ dual de XOR é XNOR

Dual: trocar + por · e vice-versa

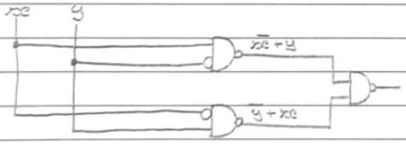
$(x\oplus y)^D$ = dual de $\bar x\cdot y + x\cdot\bar y \Rightarrow (\bar x + y)\cdot(x+\bar y)$

$(x\oplus y)'$ = negação de $\bar x\cdot y + x\cdot \bar y \Rightarrow (x+\bar y)\cdot(\bar x + y)$

iguais

⑩ Fazer XOR com vários NAND

Xor → $\bar x\cdot y + x\cdot\bar y$ NAND → $\overline{x\cdot y} \to \bar x + \bar y$




$\overline{(\bar x + y)\cdot(\bar y + x)} = x\cdot\bar y + y\cdot\bar x$

⑪ $S = x\oplus y\oplus c_i$

$C_o = x\cdot y + c_i\cdot(x+y)$

↳ É um 1 bit full Adder

Como pensar no que usar:

→ XOR $\bar x\cdot y + x\cdot\bar y$

O que falta para a NAND: negação dos termos

→ $\overline{\bar x\cdot y}$ e $\overline{x\cdot\bar y}$

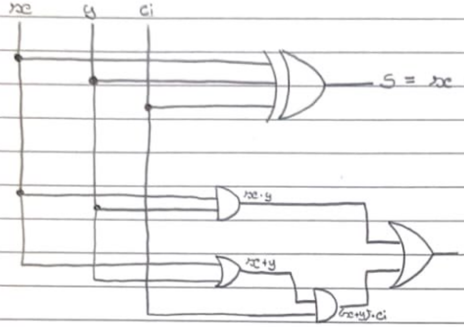
Para uma Nand ligar os 2 termos que temos: $(\overline{\bar x\cdot y})$ e $(\overline{x\cdot \bar y})$

$\underset{A}{(\overline{\bar x\cdot y})}$ $\underset{B}{(\overline{x\cdot\bar y})}$

→ $\overline{\bar A\cdot\bar B}$ =

= A + B

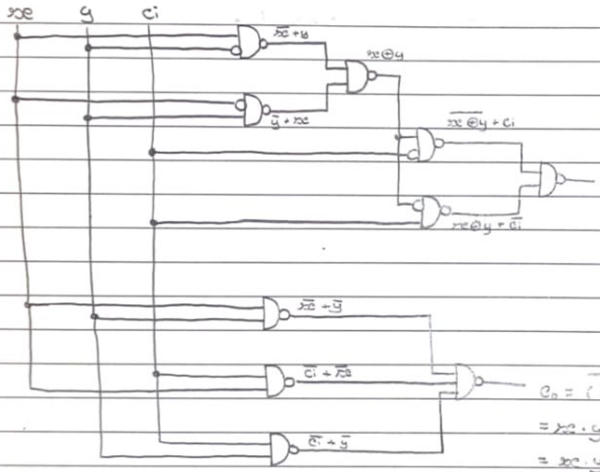
a) Desenhar o circuito




$S = x \oplus y \oplus c_i$

$C_o = x \cdot y + (x + y) \cdot c_i$

b) Desenhar com portas NAND




$S = x \oplus y \oplus c_i$

$C_o = \overline{(\overline{x} + \overline{y}) \cdot (\overline{c_i} + \overline{x}) \cdot (\overline{c_i} + \overline{y})} =$

$= x \cdot y + c_i \cdot x + c_i \cdot y =$

$= x \cdot y + c_i (x + y)$

(12)

$M(x, y, z) = 1$ quando pelo menos 2 input $= 1$

## a) Tabela de verdade

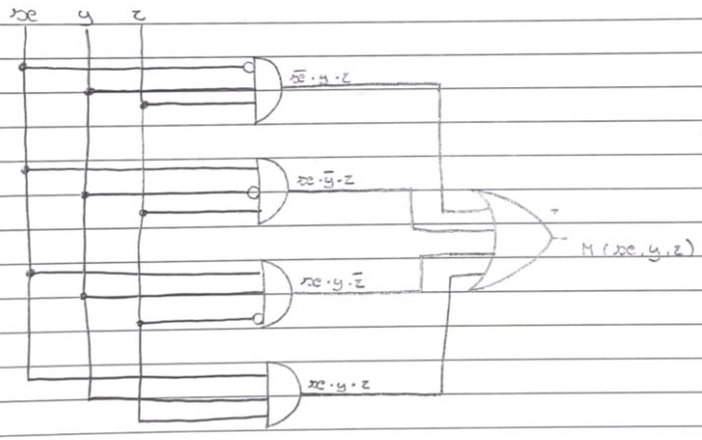
| $x$ | $y$ | $z$ | $M(x,y,z)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

## b) Expressão booleana para $M(x,y,z)$

Podemos fazer SOP:

$$M(x,y,z) = \bar{x} \cdot y \cdot z + x \cdot \bar{y} \cdot z + x \cdot y \cdot \bar{z} + x \cdot y \cdot z$$

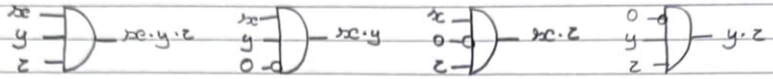
## c) circuito lógico




## d)

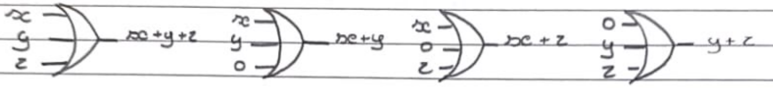
d) $S = \{ M(x, y, z), NOT, "0" \}$

→ Significa que temos de provar que, ao tornarmos uma das variaveis o complemento de zero → que é um termos um output valido




AND

Para as OR quando temos o complemento vai dar sempre 1, por isso não usarmos o complemento




O mesmo acontece com outras portas logicas

⚠
(Não sei se o exercicio pretendia que fizessemos isso, pode estar errado)

⑧ Com a lei de Morgan

$(\bar{a} \cdot b + a \cdot c) \cdot (a + \bar{b}) \cdot (\bar{a} + \bar{c}) = 0 \iff$

$\iff \overline{(\bar{a} \cdot b + a \cdot c) \cdot (a + \bar{b}) \cdot (\bar{a} + \bar{c})} = 1 \iff$

$\iff \overline{\bar{a} \cdot b + a \cdot c} + \overline{a + \bar{b}} + \overline{\bar{a} + \bar{c}} = 1 \iff$

$\iff (a + \bar{b}) \cdot (\bar{a} + \bar{c}) + a \cdot \bar{b} + a \cdot c = 1 \iff$

$\iff \overbrace{a \cdot \bar{a}}^{=0} + a \cdot \bar{c} + \bar{b} \cdot \bar{a} + \bar{b} \cdot \bar{c} + \bar{a} \cdot b + a \cdot c = 1 \iff$

$\iff a\,\overbrace{(\bar{c} + c)}^{=1} + \bar{a}\,\overbrace{(\bar{b} + b)}^{=1} + \bar{b} \cdot \bar{c} = 1 \iff$

$\iff \overbrace{a + \bar{a}}^{=1} + \bar{b} \cdot \bar{c} = 1 \iff 1 = 1$

by Staples