Introduction to Digital Systems

Part I (4 lectures)

2023/2024

Introduction

Number Systems and Codes

Combinational Logic Design Principles

Arnaldo Oliveira, Augusto Silva, Iouliia Skliarova

Universidade de Aveiro

Lecture 2 contents

- Addition and subtraction of unsigned nondecimal numbers
- Representation of negative numbers
- Two's-complement addition and subtraction
- Codes
  - Character codes
  - Binary-coded decimal
  - Gray code

Addition of Binary Numbers

- Addition and subtraction of nondecimal numbers by hand uses the same technique that you know from school for decimal numbers.
- The only catch is that the addition and subtraction tables are different.
- To add two **unsigned binary numbers** *X* and *Y*, we add together the least significant bits with an initial carry (c_{in}) of 0, producing carry (c_{out}) and sum (s) bits according to the table. We continue processing bits from right to left, adding the carry out of each column into the next column's sum.

Example:

```
    1 1 0 0 0 0 1
    0 0 1 0 1 1 0 1
  + 0 1 1 0 0 0 0 1
  -----------------
    1 0 0 0 1 1 1 0
```

c_{in}	x	y	c_{out}	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Universidade de Aveiro

Subtraction of Binary Numbers

- Binary subtraction is performed similarly, using borrows (b_{in} and b_{out}) instead of carries between steps, and producing a difference bit d.

b_{in}	x	y	b_{out}	d
0	0	0	0	0
0	0	1	1	1
0	1	0	0	1
0	1	1	0	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	0
1	1	1	1	1

Examples:

```
    0 1 1 1 1 0 0
    1 1 1 0 0 0 0 1
 -  1 0 1 0 1 1 0 1
 ------------------
    0 0 1 1 0 1 0 0
```

```
    1 1 1
    1 0 0 0
 -  0 0 1 1
 ----------
    0 1 0 1
```

Overflow

- With n bits it is possible to represent **unsigned integer numbers** ranging from 0 to 2^n-1.
- If an arithmetic operation produces a result that exceeds the range of the number system, **overflow** is said to occur.
- Overflows can easily be detected by analyzing **a carry or borrow from the most significant bit**.
  - the carry bit c_{out} or the borrow bit b_{out} out of the MSB = 1

Examples:

n=8: [0..255]

$173_{10} + 97_{10} = 270_{10}$

1	1	1	0	0	0	0	1	
	1	0	1	0	1	1	0	1
+	0	1	1	0	0	0	0	1
	0	0	0	0	1	1	1	0

overflow

n=4: [0..15]

$4_{10} – 11_{10} = -7_{10}$

1	0	1	1	
	0	1	0	0
-	1	0	1	1
	1	0	0	1

overflow

Addition of Octal Numbers

- To add two **octal numbers** *X* and *Y*, we add together the least significant digits with an initial carry (c_{in}) of 0. If the *intermediate result* is less than or equal to 7, then c_{out} = 0 and sum (s) digit = *intermediate result*. If the *intermediate result* is greater than 7, then c_{out}= 1 and sum (s) digit = *intermediate result* – 8.
- We continue processing digits from right to left, adding the carry out of each column into the next column's sum.

Examples (radix 8):

$$\begin{array}{rcccc} & 0 & 0 & 0 & \\ & 3 & 0 & 4 & 1 \\ + & 1 & 7 & 3 & 2 \\ \hline & 4 & 7 & 7 & 3 \end{array} \qquad \begin{array}{rcccc} & 1 & 1 & 1 & \\ & 3 & 4 & 5 & 6 \\ + & 1 & 7 & 3 & 4 \\ \hline & 5 & 4 & 1 & 2 \end{array}$$

Addition of Hexadecimal Numbers

- To add two **hexadecimal numbers** *X* and *Y*, we add together the least significant digits with an initial carry (c_{in}) of 0. If the *intermediate result* is less than or equal to 15, then c_{out} = 0 and sum (s) digit = *intermediate result*. If the *intermediate result* is greater than 15, then c_{out}= 1 and sum (s) digit = *intermediate result* – 16.
- We continue processing digits from right to left, adding the carry out of each column into the next column's sum.

Examples (radix 16):

	1	0	0	
	3	A	4	1
+	1	7	8	2
	5	1	C	3

	1	1	0	
	3	F	A	A
+	B	7	E	4
	F	7	8	E

Universidade de Aveiro

Subtraction of Octal and Hexadecimal Numbers

- When subtracting **octal numbers**, a borrow brings the value 8.
- When subtracting **hexadecimal numbers**, a borrow brings the value 16.

Examples:

radix 8

$$
\begin{array}{ccccc}
 & 1 & 0 & 1 & \\
 & 3 & 0 & 4 & 1 \\
- & 1 & 7 & 3 & 2 \\
\hline
 & 1 & 1 & 0 & 7
\end{array}
\qquad
\begin{array}{ccccc}
 & 1 & 1 & 1 & \\
 & 6 & 0 & 0 & 0 \\
- & 1 & 5 & 7 & 7 \\
\hline
 & 4 & 2 & 0 & 1
\end{array}
$$

radix 16

$$
\begin{array}{ccccc}
 & 0 & 1 & 1 & \\
 & 3 & A & 4 & 1 \\
- & 1 & 7 & 8 & 2 \\
\hline
 & 2 & 2 & B & F
\end{array}
\qquad
\begin{array}{ccccc}
 & 1 & 1 & 1 & \\
 & B & 0 & 0 & 0 \\
- & A & 7 & E & 4 \\
\hline
 & 0 & 8 & 1 & C
\end{array}
$$

Universidade de Aveiro

Representation of Negative Numbers

- There are many ways to represent negative numbers.
- In everyday business we use the **signed-magnitude system** (i.e. reserve a special symbol to indicate whether a number is negative).
- However, most computers use **two's-complement representation**:
  - The **most significant bit** (**MSB**) of a number in this system serves as the sign bit; a number is negative if and only if its MSB is 1.
  - The weight of the MSB is negative: for an *n*-bit number the weight is -2^{n-1}.
  - The decimal equivalent for a two's-complement binary number is computed the same way as for an unsigned number, except that the weight of the MSB is negative:
    - D= $d_{n-1}d_{n-2}\dots d_1 d_0 = -2^{n-1} + \sum_{i=0}^{n-2} d_i \times 2^i$

Examples:

$1010_2 = ???_{10}$ $\quad$ $1010_2 = -2^3 + 2^1 = -8 + 2 = -6_{10}$

$1111_2 = ???_{10}$ $\quad$ $1111_2 = -2^3 + 2^2 + 2^1 + 2^0 = -8 + 4 + 2 + 1 = -1_{10}$

$0111_2 = ???_{10}$ $\quad$ $0111_2 = 2^2 + 2^1 + 2^0 = 4 + 2 + 1 = 7_{10}$

Universidade de Aveiro

Two's Complement Representation

- For *n* bits, the range of representable numbers is $[-2^{n-1}, 2^{n-1}-1]$.
- For *n*=4, the range is [-8, 7]:

0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
-8	1	0	0	0
-7	1	0	0	1
-6	1	0	1	0
-5	1	0	1	1
-4	1	1	0	0
-3	1	1	0	1
-2	1	1	1	0
-1	1	1	1	1

Universidade de Aveiro

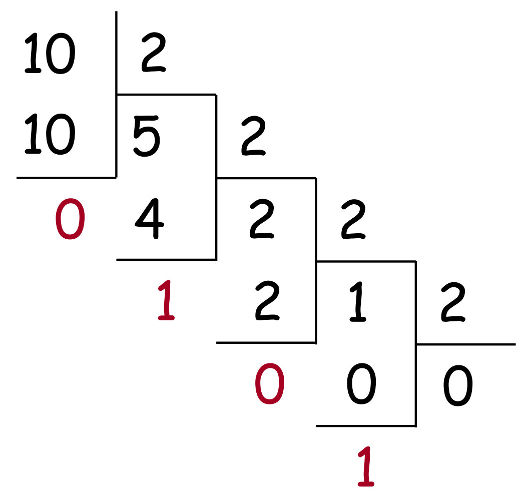
Conversion between Decimal and Two's Complement

- The decimal value of the number expressed in two's complement can be found by expanding the formula (D= $d_{n-1}d_{n-2}\ldots d_1 d_0 = -2^{n-1} + \sum_{i=0}^{n-2} d_i \times 2^i$) using radix-10 arithmetic.
- The integer number *D* expressed in decimal can be converted to *n*-bit two's complement by **successful division of *D* by 2** (using radix-10 arithmetic, until the result is 0) with **reverse recording** of all the obtained **remainders**.
  - If there are **empty bit positions** left, **fill** them with **0s**.
  - **Do not exceed** the allowed **range** of representable numbers: [-2^{n-1} , $2^{n-1}-1$].
  - If the number is negative, the result must be **negated**:
    - Invert all the bits individually and add 1 or
    - Copy all the bits starting from the least significant until the first 1 is copied, then invert all the remaining bits.

Examples (with n=8):

10_{10} = $???_2$ $\qquad$ 10_{10}= 00001010_2

-10_{10} = $???_2$ $\qquad$ -10_{10}= 11110110_2



Changing the Number of Bits

- We can convert an n-bit two's-complement number into an m-bit one.
- If $m > n$, perform **sign extension**:
  - append $m - n$ copies of the sign bit to the left
- If $m < n$, **discard** $n - m$ leftmost bits; however, **the result is valid only if all of the discarded bits are the same as the sign bit of the result**.

Examples:

n = 5
m = 8

00101 = **000**00101
11110 = **111**11110

n = 5
m = 3

00101 = 101 – result is <u>not</u> valid
11110 = 110 – result is valid

Two's-Complement Addition

- Addition is performed in the same way as for nonnegative numbers.
- Carries beyond the MSB are **ignored**.
- The result will always be the correct sum as long as the range of the number system is not exceeded.
- If an addition operation produces a result that exceeds the range of the number system, **overflow** is said to occur.
- Addition of two numbers with different signs can never produce overflow.
- Addition of two numbers of like sign can produce overflow if
  - the addends' signs are the same but the sum's sign is different from the addends'
  - the carry bits c_{in} into and c_{out} out of the sign position are different

Examples (n=4):

$$
\begin{array}{cccccc}
1 & 1 & 0 & 0 & & \\
 & 0 & 1 & 0 & 0 \\
+ & 1 & 1 & 0 & 1 \\
\hline
 & 0 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{ccccc}
 & 0 & 1 & 0 & \\
 & 0 & 0 & 1 & 0 \\
+ & 0 & 0 & 1 & 1 \\
\hline
 & 0 & 1 & 0 & 1
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{1} & \mathbf{0} & 0 & 0 & \\
 & 1 & 0 & 0 & 1 \\
+ & 1 & 1 & 1 & 0 \\
\hline
 & 0 & 1 & 1 & 1
\end{array}
$$

overflow

Two's-Complement Subtraction

- Two's-complement numbers may be subtracted as if they were ordinary unsigned binary numbers.
- However, **most subtraction circuits for two's-complement numbers do not perform subtraction directly**.
- Rather, they **negate the subtrahend** by taking its two's complement, and then **add** it to the minuend using the normal rules for addition (X-Y=X+(-Y)).
- Overflow in subtraction can be detected using the same rule as in addition.
- Negating the subtrahend and adding the minuend can be accomplished with only one addition operation:
  - Perform a bit-by-bit complement of the subtrahend and add the complemented subtrahend to the minuend with an initial carry (c_{in}) of 1 instead of 0.

Examples (n=4):

0010 - 0011:

$$
\begin{array}{rccccl}
 & 0 & 0 & 0 & 1 & \\
 & 0 & 0 & 1 & 0 & \text{Minuend} \\
+ & 1 & 1 & 0 & 0 & \text{Subtrahend} \\
\hline
 & 1 & 1 & 1 & 1 &
\end{array}
$$

1011 - 0110:

$$
\begin{array}{rccccc}
 & 1 & 0 & 1 & 1 & 1 \\
 & & 1 & 0 & 1 & 1 \\
+ & & 1 & 0 & 0 & 1 \\
\hline
 & & 0 & 1 & 0 & 1
\end{array}
$$

overflow

Information Encoding

- Digital systems are built from circuits that process binary digits
- Very few real-life problems are based on binary numbers or any numbers at all
- Some correspondence must be established between the binary digits processed by digital circuits and real-life numbers, events, and conditions
  - How to represent familiar numeric quantities? ✓
    - number systems: binary, octal, and hexadecimal
  - How to represent nonnumeric data?

Codes

- A **code** is a set of *n*-bit strings in which different bit strings represent different numbers or other things.
- A **code word** is a particular combination of *n* bit-values.
- To code *m* values, the code length *n* must respect the following equation: $n \geq \lceil log_2 m \rceil$.

floor	encoding	encoding	encoding
basement	000	000	000001
ground floor	001	001	000010
1^{st} floor	010	011	000100
2^{nd} floor	011	010	001000
3^{rd} floor	100	110	010000
4^{th} floor	101	111	100000

Character Codes

- The most common type of nonnumeric data is text, strings of characters from some character set.
- Each character is represented in the digital system by a bit string according to an established convention.
- The most commonly used character code is **ASCII** (American Standard Code for Information Interchange).
  - ASCII represents each character with a 7-bit string, yielding a total of 128 different characters.

		$b_6b_5b_4$ (column)							
$b_3b_2b_1b_0$	Row (hex)	000 0	001 1	010 2	011 3	100 4	101 5	110 6	111 7
0000	0	NUL	DLE	SP	0	@	P	‘	p
0001	1	SOH	DC1	!	1	A	Q	a	q
0010	2	STX	DC2	"	2	B	R	b	r
0011	3	ETX	DC3	#	3	C	S	c	s
0100	4	EOT	DC4	$	4	D	T	d	t
0101	5	ENQ	NAK	%	5	E	U	e	u
0110	6	ACK	SYN	&	6	F	V	f	v
0111	7	BEL	ETB	’	7	G	W	g	w
1000	8	BS	CAN	(	8	H	X	h	x
1001	9	HT	EM	)	9	I	Y	i	y
1010	A	LF	SUB	*	:	J	Z	j	z
1011	B	VT	ESC	+	;	K	[	k	{
1100	C	FF	FS	,	<	L	\	l	|
1101	D	CR	GS	−	=	M	]	m	}
1110	E	SO	RS	.	>	N	^	n	~
1111	F	SI	US	/	?	O	_	o	DEL

Binary Codes for Decimal Numbers

- Even though binary numbers are the most appropriate for the internal computations of a digital system, most people still prefer to deal with decimal numbers.
- As a result, the external interfaces of a digital system may read or display decimal numbers, and some digital devices actually process decimal numbers directly.
- A decimal number is represented in a digital system by a string of bits, where different combinations of bit values in the string represent different decimal numbers.
- To code *m* = 10 decimal digits, at least $\lceil log_2 10 \rceil = 4$ bits are required.
- Is the maximum number of bits limited?
- Is the number of possible codes limited?

Universidade de Aveiro

Binary-Coded Decimal (BCD)

- Perhaps the most "natural" decimal code is **binary-coded decimal** (BCD), which encodes the digits 0 through 9 by their 4-bit unsigned binary representations, 0000 through 1001.
- The code words 1010 through 1111 are not used.
- Conversions between BCD and decimal representations are trivial, a direct substitution of four bits for each decimal digit.

decimal digit	BCD (8421)
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

Example:

$25_{10} = 11001_2$

$25_{10} = 00100101_{BCD}$

Universidade de Aveiro

Gray Code

- Sometimes, it is required to code values so that only **one bit changes** between each pair of successive code words.
- Such a code is called a **Gray code**.
- There are two convenient ways to construct a Gray code with any desired number of bits.

1 bit	2 bits	3 bits	4 bits
0	00	000	0000
1	01	001	0001
	11	011	0011
	10	010	0010
		110	0110
		111	0111
		101	0101
		100	0100
			1100
			1101
			1111
			1110
			1010
			1011
			1001
			1000

Universidade de Aveiro

Constructing Gray Code

- The first method is based on the fact that Gray code is a reflected code; it can be defined (and constructed) recursively using the following rules:
  - A 1-bit Gray code has two code words, 0 and 1.
  - The first 2^n code words of an $(n + 1)$-bit Gray code equal the code words of an n-bit Gray code, written in order with a leading 0 appended.
  - The last 2^n code words of an $(n + 1)$-bit Gray code equal the code words of an n-bit Gray code, but written in reverse order with a leading 1 appended.

Constructing Gray Code (cont.)

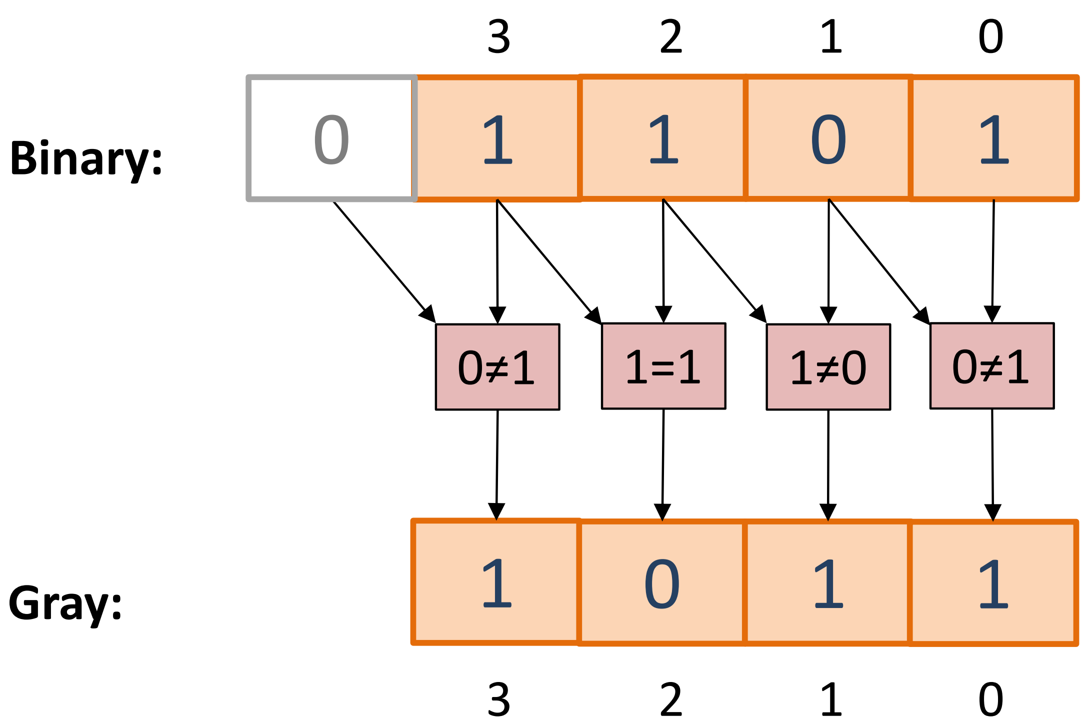
- The second method allows us to derive an *n*-bit Gray-code code word directly from the corresponding *n*-bit binary code word:
  - The bits of an *n*-bit binary or Gray-code code word are numbered from right to left, from 0 to *n* - 1.
  - Bit *i* of a Gray-code code word is 0 if bits *i* and *i* + 1 of the corresponding binary code word are the same, else bit *i* is 1.
  - When *i* + 1 = *n*, bit *n* of the binary code word is considered to be 0
- Similarly, an *n*-bit Gray-code code word can be converted to the corresponding *n*-bit binary code word:
  - The bits of an *n*-bit Gray-code code word are numbered from right to left, from 0 to *n* - 1.
  - Bit *n* – 1 of a binary code word is equal to bit *n* - 1 of a Gray-code code word.
  - Bit *i* (*i* = *n*-2, *n*-3,…, 1, 0) of a binary code word is 0 if bits *i* of the corresponding Gray-code code word and *i* + 1 of the corresponding binary code word are the same, else bit *i* is 1.

Example: $11001_2 = 10101_{GRAY}$

Universidade de Aveiro

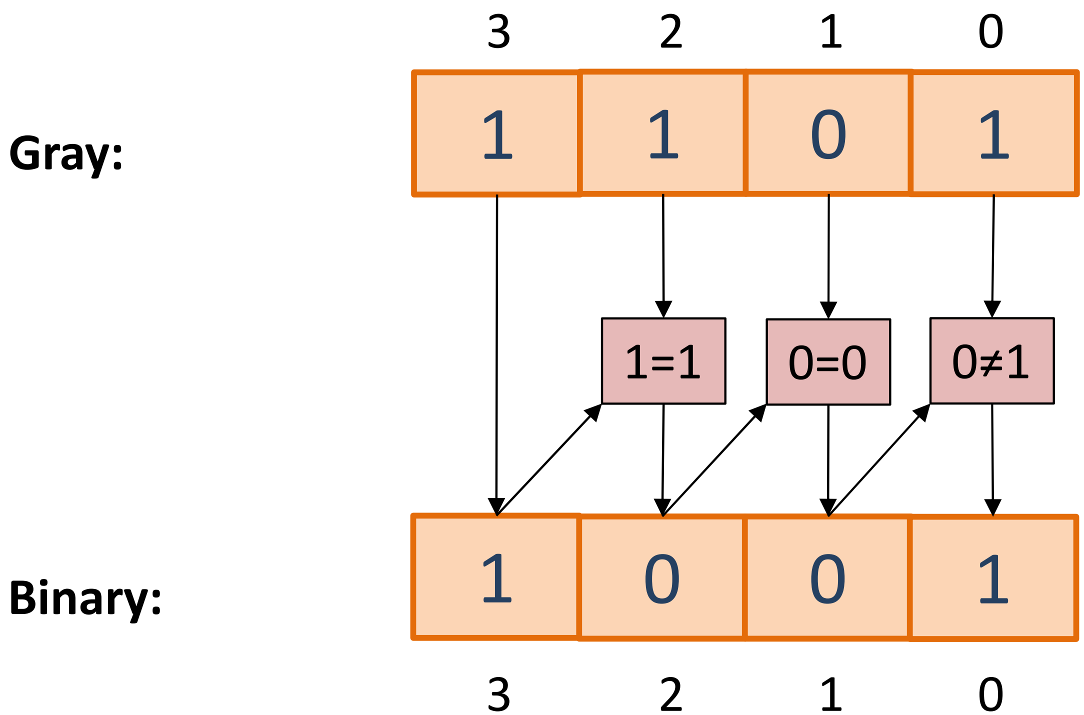
Converting Binary to Gray Code

- The bits of an *n*-bit binary or Gray-code code word are numbered from right to left, from 0 to *n* - 1.
- Bit *i* of a Gray-code code word is 0 if bits *i* and *i* + 1 of the corresponding binary code word are the same, else bit *i* is 1.
- When *i* + 1 = *n*, bit *n* of the binary code word is considered to be 0

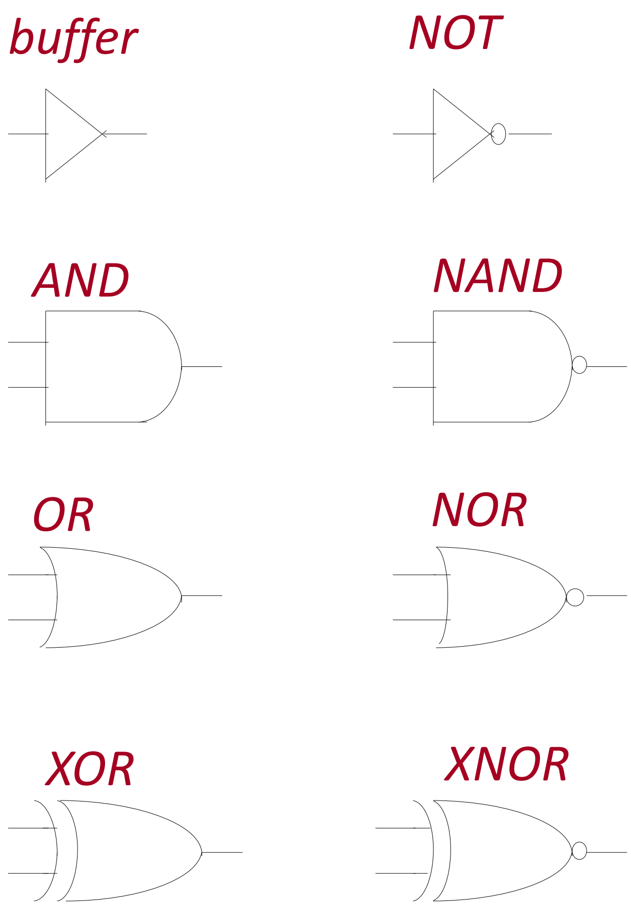


Converting Gray Code to Binary

- The bits of an *n*-bit Gray-code code word are numbered from right to left, from 0 to *n* - 1.
- Bit *n* – 1 of a binary code word is equal to bit *n* - 1 of a Gray-code code word.
- Bit *i* (*i* = *n*-2, *n*-3,..., 1, 0) of a binary code word is 0 if bits *i* of the corresponding Gray-code code word and *i* + 1 of the corresponding binary code word are the same, else bit *i* is 1.



XOR and XNOR Gates



$$x \oplus y$$

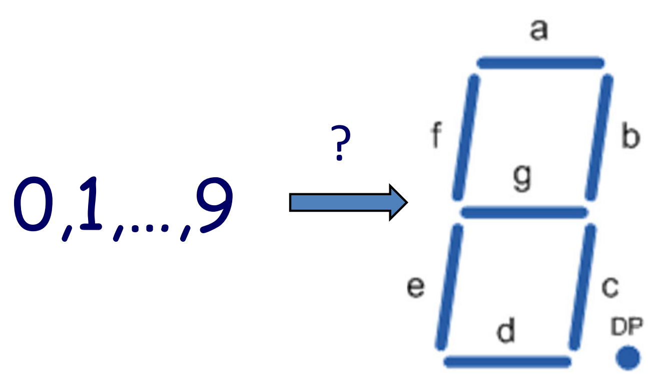
x	y	x XOR y
0	0	0
0	1	1
1	0	1
1	1	0

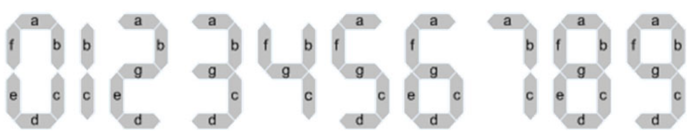
$$\overline{x \oplus y}$$

x	y	x XNOR y
0	0	1
0	1	0
1	0	0
1	1	1

7-segment Display Codes

- 7-segment displays are used in watches, calculators, and instruments to display decimal data.
- A digit is displayed by illuminating a subset of the seven line segments.





BCD	digit	individual segments						
		a	b	c	d	e	f	g
0000	0	1	1	1	1	1	1	0
0001	1	0	1	1	0	0	0	0
0010	2	1	1	0	1	1	0	1
0011	3	1	1	1	1	0	0	1
0100	4	0	1	1	0	0	1	1
0101	5	1	0	1	1	0	1	1
0110	6	1	0	1	1	1	1	1
0111	7	1	1	1	0	0	0	0
1000	8	1	1	1	1	1	1	1
1001	9	1	1	1	1	0	1	1

Universidade de Aveiro

Hamming Distance

- The **Hamming distance** between two *n*-bit strings is the number of bit positions in which they differ.
- In the Gray code, the Hamming distance between each pair of successive code words is 1.

Example:

1	0	1	0	1	0	1	0
1	1	0	1	1	1	1	0

Hamming distance = 4

Bits, Bytes, Words, etc.

- The prefixes K (kilo-), M (mega-), G (giga-), and T (tera-) mean 10^3, 10^6, 10^9, and 10^{12}, respectively, when referring to bps, hertz, ohms, watts, and most other engineering quantities.
- However, when referring to memory sizes, the prefixes mean 2^{10}, 2^{20}, 2^{30}, and 2^{40}.

Bit	*b*	0 or 1
Byte	*B*	8 bits
Nibble		4 bits
Word		8, 16, 32, 64 ... bits (depends on the context)

1 K/k	$10^3 \approx 2^{10}$	(*kilo*)
1 M	$10^6 \approx 2^{20}$	(*mega*)
1 G	$10^9 \approx 2^{30}$	(*giga*)
1 T	$10^{12} \approx 2^{40}$	(*tera*)

IEEE 1541-2002:

Ki	2^{10} = 1 024	(*kibi*)
Mi	2^{20} = 1 048 576	(*mebi*)
Gi	2^{30} = 1 073 741 824	(*gibi*)
Ti	2^{40} = 1 099 511 627 776	(*tebi*)
Pi	2^{50} = 1 125 899 906 842 624	(*pebi*)
Ei	2^{60} = 1 152 921 504 606 846 976	(*exbi*)

Exercises

- Represent the following numbers in two's complement with 8 bits: 39_{10}, -22_{10}.

- Calculate the results of the following operations in two's complement with 8 bits. Detect overflows if any.

	1	0	1	0	1	0	1	0
+	0	1	0	1	0	1	0	1

	0	0	0	0	0	0	0	
	1	0	1	0	1	0	1	0
+	0	1	0	1	0	1	0	1
	1	1	1	1	1	1	1	1

	0	1	1	1	1	1	1	1
+	0	0	1	1	1	1	1	1

0	1	1	1	1	1	1	1	
	0	1	1	1	1	1	1	1
+	0	0	1	1	1	1	1	1
	1	0	1	1	1	1	1	0

	1	1	0	1	0	0	1	0
-	0	1	1	0	1	1	0	1

1	0	0	1	0	0	1	0	1
	1	1	0	1	0	0	1	0
+	1	0	0	1	0	0	1	0
	0	1	1	0	0	1	0	1

Exercises (cont.)

- Add the following pairs of octal numbers:

$$\begin{array}{r} 1\ 7\ 7\ 6 \\ +\ 1\ 4\ 3\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 3\ 7\ 7\ 7 \\ +\ 1\ 7\ 7\ 7 \\ \hline \end{array}$$

- Add the following pairs of hexadecimal numbers:

$$\begin{array}{r} 1\ 7\ 7\ 6 \\ +\ 1\ 4\ 3\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 3\ F\ F\ F \\ +\ A\ B\ C\ D \\ \hline \end{array}$$

- Each of the following arithmetic operations is correct in at least one number system. Determine possible radices of the numbers in each operation.
  - 1234 + 5432 = 6666
  - $\sqrt[2]{41} = 5$

Universidade de Aveiro

Exercises (cont.)

- How many bits of information can be stored on a 16 GB pen?
- How many digital photos is it be possible to store on an 8 GiB pen assuming that each photo has 4000 x 3000 pixels and each pixel is coded with 24 bits?
- Assuming that the following quantity is represented in two's complement, indicate its decimal value: 11111111111111111111111111111111111111111001
- Express in decimal, binary, and hexadecimal systems the value of the largest non-negative integer you can represent in a register with a storage capacity of 2 octal digits.

Exercises (cont.)

- How many bits are required to code in BCD the number 123456_{10}?
- Represent the following values in binary and in BCD and Gray codes.

108_{10} $= 000100001000_{BCD}$
$= 1101100_2$
$= 1011010_{GRAY}$

33_8 $= 00100111_{BCD}$
$= 011011_2$
$= 010110_{GRAY}$

- Prove that a two's-complement number can be converted to a representation with more bits by *sign extension.*

- Determine the Hamming distance between the following code words:

011010101011
000010101011

= 2

Universidade de Aveiro

Exercises (cont.)

- Airport names are encoded by sequences of three capital letters of English alphabet (having 26 letters).
- How many airports can be coded this way?
- How many bits will be required in ASCII code to binary encode the airport codes?
- And if you use the most efficient code possible to encode only uppercase letters?

Universidade de Aveiro