Statistics

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1 Asymptotic analysis

1.1 Limits

We can get as close to our goals as we want by making δ smaller.

Limit from above

Setting: A function $f : I \mapsto \mathbb{R}^n$ with $(0, a) \subset I \subset \mathbb{R}$ for some a and some n

We define the limit of $\mathbf{f}(x)$ as x approaches 0 from above as:

$$\begin{aligned} & \lim_{x \searrow 0} \mathbf{f}(x) = \mathbf{0} && (1) \\ \leftrightarrow \quad & \text{for every } \epsilon \in \mathbb{R}_{>0} \text{ there is a } \delta \in \mathbb{R}_{>0} \text{ s.t. } |\mathbf{f}(x)| < \epsilon \text{ for all } 0 < x < \delta\,. \end{aligned}$$

Interpretation: $\mathbf{f}$ becomes arbitrarily close to zero for all small enough x $\leftrightarrow$ You can make $\mathbf{f}$ however small you want it to be by picking a small enough x.

Exercise Why don't we make the definition $\left[\lim_{x \searrow 0} f(x) = 0\right] \leftrightarrow \left[\text{for every } \epsilon \in \mathbb{R}_{>0} \text{ there is an } x > 0 \text{ s.t. } |f(x)| < \epsilon\right]$? For what kind of functions does this give a different answer? Give an example.

Limit along a path

$$\begin{aligned} & \lim_{\epsilon \searrow 0} E_\delta = 0 && (2) \\ \leftrightarrow \quad & \text{E becomes arbitrarily close to zero for all small enough } \delta \in \mathbb{R}_{>0} && (3) \\ \leftrightarrow \quad & \text{for every } \epsilon \in \mathbb{R}_{>0} \text{ there is a } \delta \in \mathbb{R}_{>0} \text{ s.t. } |E_\delta| < \epsilon\,. && (4) \end{aligned}$$

Limit

$$\lim_{\delta \to 0} E_\delta = 0 \tag{5}$$

$$\leftrightarrow \quad \text{E becomes arbitrarily close to zero for all small enough } \delta \tag{6}$$

$$\leftrightarrow \quad \text{for every } \epsilon \in \mathbb{R} \text{ there is a } \delta \in \mathbb{R}_{>0} \text{ s.t. } |f(x)| < \epsilon\,. \tag{7}$$

This is nice, but we also want to know how fast our estimate approach 0 for small δ, to 1. see if ϵ is small enough to reach our required precision, and 2. to compare different expressions (does one ‘expression approach zero fast than the other?).

1.2 Big O notation

Big-O notation

$$f_\epsilon = O(g_\epsilon) \text{ as } \epsilon \to 0 \tag{8}$$

$$\leftrightarrow \tag{9}$$

$$\leftrightarrow \quad \text{there is a } C \in \mathbb{R}_{>0} \text{ s.t. there is an } \epsilon* \text{ s.t. } |\frac{\epsilon}{\epsilon}| < C \text{ for all } \epsilon < \epsilon*\,. \tag{10}$$

Little-O notation

$$f_\delta = (g_\delta) \text{ as } \delta \to 0 \tag{11}$$

$$\leftrightarrow \tag{12}$$

$$\leftrightarrow \quad \lim_{\delta \to 0} f(x)/g(x) = 0\,. \tag{13}$$

Asymptotic equivalence

f_δ is asymptotically equivalent to g_δ (as $\delta \to 0$ if:

$$f_\delta \sim g_\delta \text{ as } \delta \to 0 \tag{14}$$

$$\leftrightarrow \tag{15}$$

$$\leftrightarrow \quad \lim_{\delta \to 0} f(x)/g(x) = 1\,. \tag{16}$$

2 Taylor series

Derivative is the best linear approximation $\rightarrow f(x) \approx f(0) + \frac{\mathrm{d}f(x)}{\mathrm{d}x}\big|_{x=0} \cdot x$ What is the error? Consider by the fundamental theorem of calculus that $f(x) = f(0) + \int_0^x f'(x)\mathrm{d}x$.

3 Saddle-point approximation

Consider

$$\int \exp(-ax^2 + \epsilon x^4) \tag{17}$$