1 Stability of equilibrium points & bifurcations

1.1 Simple population model

The population model has in general two solutions (and hence two fixed points) for $\dot{N} = 0$, namely

$$N_1 = 0 \quad \text{and} \quad N_2 = K\frac{\alpha - \beta}{\alpha}.$$

The stability of these fixed points in function of α and β can be summarised as follows:

Parameter region	Fixed points
$\alpha < \beta$	$N_1 = 0$: stable $N_2 < 0$: unstable
$\alpha = \beta$	$N_1 = N_2 = 0$: half-stable (unstable for $N < 0$, stable for $N > 0$)
$\alpha > \beta$	$N_1 = 0$: unstable $N_2 > 0$: stable

The system thus undergoes a transcritical bifurcation at $\alpha = \beta$. Note that the fixed point $N_2 < 0$ is not meaningful in this model, as N represents a non-negative population count.

For the given parameter values, $\alpha > \beta$. Using the above results, we therefore find an unstable fixed point $N_1 = 0$, and a stable fixed point $N_2 = K(\alpha - \beta)/\alpha = 4\,023\,913$. As the population starts at $N > 0$, it will evolve towards N_2. The difference between $N(t)$ and $N(\infty) = N_2$ decays exponentially, as a Taylor approximation of $\dot{N}$ around N_2 can show.

1.2 Gene control model

For $r = 0$, the system equations become decoupled:

$$\dot{x} = \frac{\alpha_1}{2} - x$$
$$\dot{y} = \frac{\alpha_2}{2} - y.$$

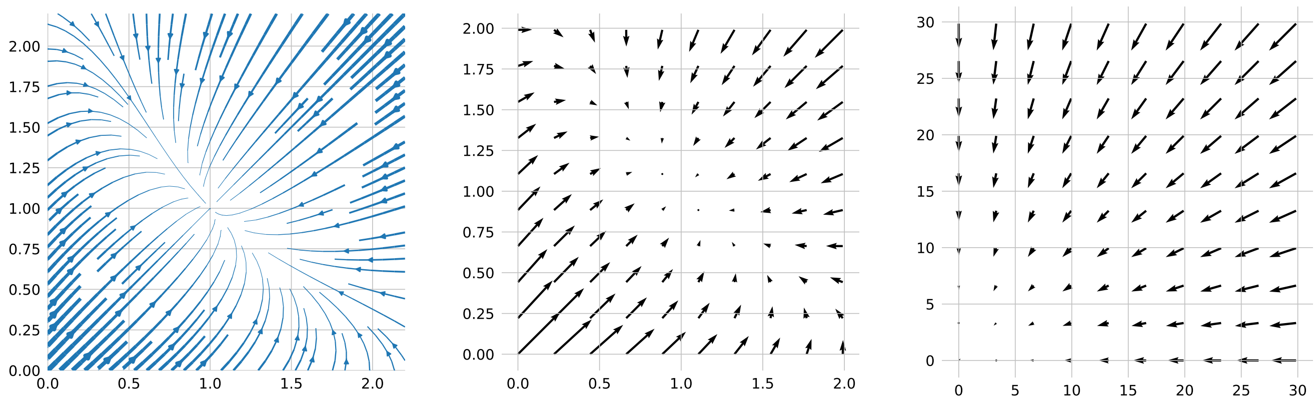

Figure 1: **There is only one fixed point for $0 \le r \le 2$.** Phase space plots of the gene control model, for $r = 1$ and $\alpha_1 = \alpha_2 = 2$. *Left*: some (partial) trajectories in phase space. Thicker lines represent a higher local speed. Note the attractor at $(1, 1)$. *Middle*: local velocities, evaluated on a grid. *Right*: same as middle, but for a larger region in phase space.

We can therefore analyse them separately. It is clear that there is one fixed point, at $x^* = \alpha_1/2$ and $y^* = \alpha_2/2$. It is a globally stable attractor, as $\forall x < x^*,\ \dot{x} > 0$ and $\forall x > x^*,\ \dot{x} < 0$ (and analogously for $\dot{y}$ and y^*). The fixed point is thus an attracting star.

For $r \ge 0$ and $\alpha_1 = \alpha_2 = 2$, the equilibrium equations become

$$x(1 + y^r) = 2$$
$$y(1 + x^r) = 2.$$

It is easily verified that $(1, 1)$ is a solution and hence a fixed point. We have already shown that it is the only fixed point for $r = 0$. Plotting the gradient in phase space for different $r \in (0, 2)$ strongly suggests that it is also the only fixed point for nonzero $r < 2$ (at least for $x \ge 0$ and $y \ge 0$). See e.g. fig. 1 for $r = 1$. As a final piece of evidence, different trajectories simulated back in time all either tend towards (∞, ∞) or cross into forbidden $x < 0,\ y < 0$ territory.

To analyse the stability of the $(1, 1)$ fixed point, we approximate $(\dot{x}, \dot{y})$ as a linear system around this point. The Jacobian of $(\dot{x}, \dot{y})$ evaluated in $(1, 1)$ is:

$$\begin{pmatrix} -1 & -\frac{r}{2} \\ -\frac{r}{2} & 1 \end{pmatrix}.$$

It has two distinct eigenvalue-eigenvector pairs:

$$\lambda_1 = -\frac{r}{2} - 1, \quad \mathbf{v}_1 = (1,\ 1)$$
$$\lambda_2 = \frac{r}{2} - 1, \quad \mathbf{v}_2 = (-1,\ 1).$$

For $0 \le r < 2$, $\lambda_1 \in (-2, -1]$ and $\lambda_2 \in [-1, 0)$. Both eigenvalues are negative, and $(1, 1)$ is therefore a stable node. Because $\lambda_1 < \lambda_2$, $\mathbf{v}_1 = (1,\ 1)$ is the fast eigendirection

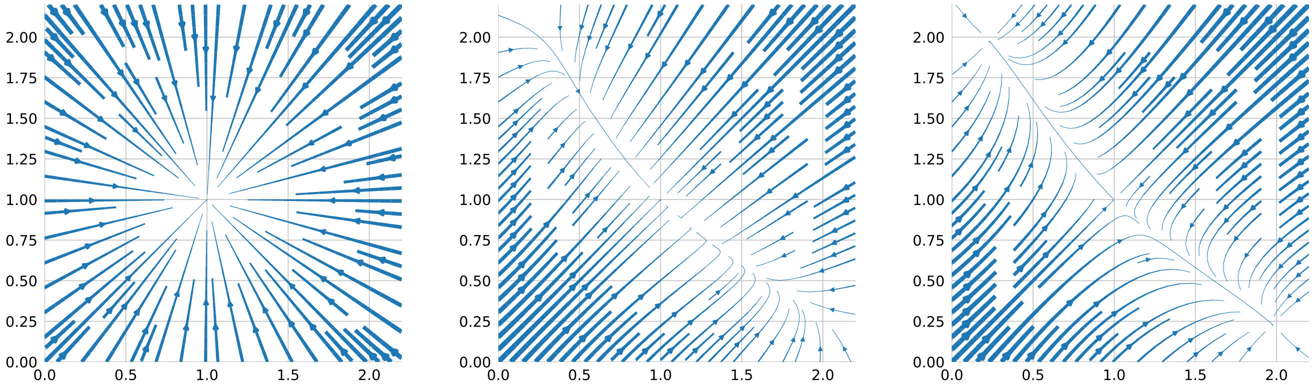

Figure 2: **Topologies of phase space**. *Left*: $r = 0$. The single fixed point is an attracting star node. (For the intermediate case between $r = 0$ and $r = 2$, see fig. 1).*Middle*: $r = 2$. There is still only one fixed point. Because we are exactly at the bifurcation point, the approach to the fixed point occurs very slowly (i.e. no longer exponentially fast). This is known as critical slowing down. *Right*: $r = 3$. There are three fixed points: a saddle point at $(1, 1)$, and two attracting nodes (mirror symmetric around $y = x$).

and $\mathbf{v}_2 = (-1,\ 1)$ is the slow eigendirection, as can be seen in fig. 1. (For $r = 0$, both eigenvalues are equal; i.e. $(1, 1)$ is then a star node).

Based on phase space plots for different values of r, it seems that a supercritical pitchfork bifurcation occurs at $r = 2$. As the problem is symmetrical for $\alpha_1 = \alpha_2$, this is expected (Strogatz 1994 [1]).

Figure 2 shows the different types of phase portrait that occur as $r \geq 0$ is varied. Figure 3 sketches the bifurcation diagram.

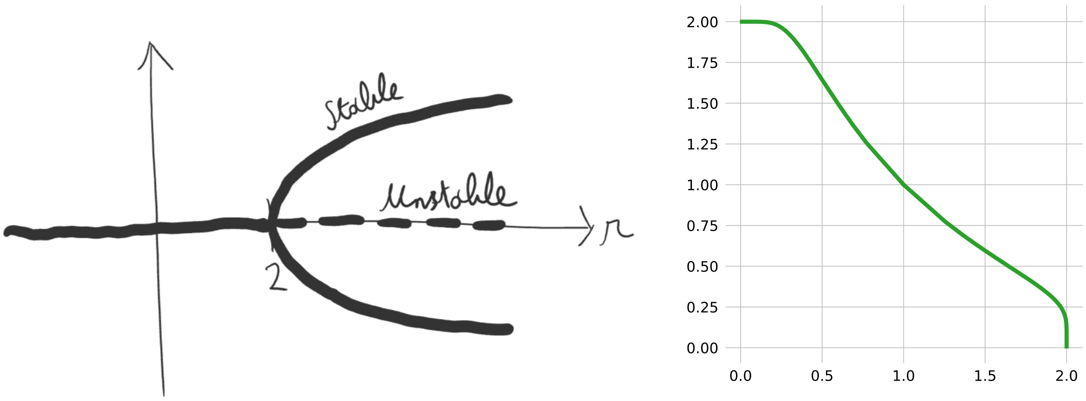

Figure 3: **The gene control model undergoes a supercritical pitchfork bifurcation**. *Left*: sketch of the bifurcation diagram. The vertical axis denotes distance in the bifurcation subspace from the $(1,1)$ point. *Right*: the one-dimensional bifurcation subspace embedded in two-dimensional phase space.

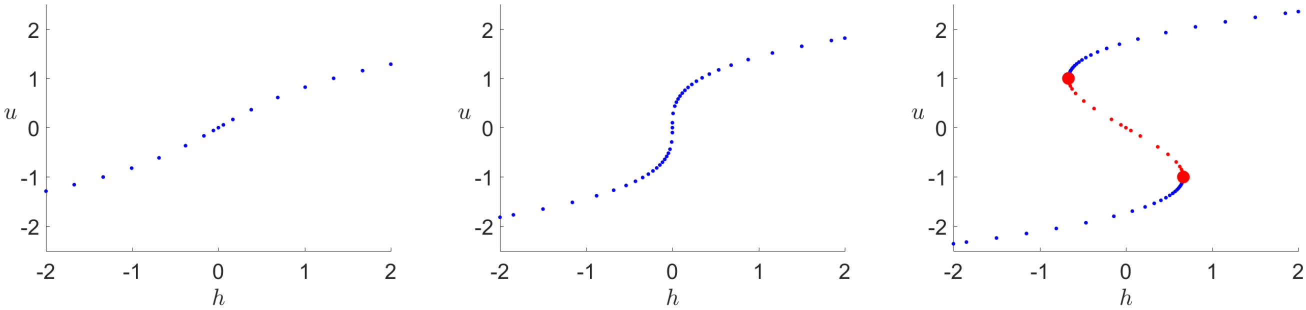

Figure 4: **(h, u)-bifurcation diagrams**. Blue dots are stable equilibria, red dots are unstable equilibria, and large red circles indicate saddle-node bifurcations. Left: $r = -1$. (Phase space topologies for other $r < 0$ are equivalent). Center: $r = 0$. Right: $r = 1$. (Phase space topologies for other $r > 0$ are equivalent).

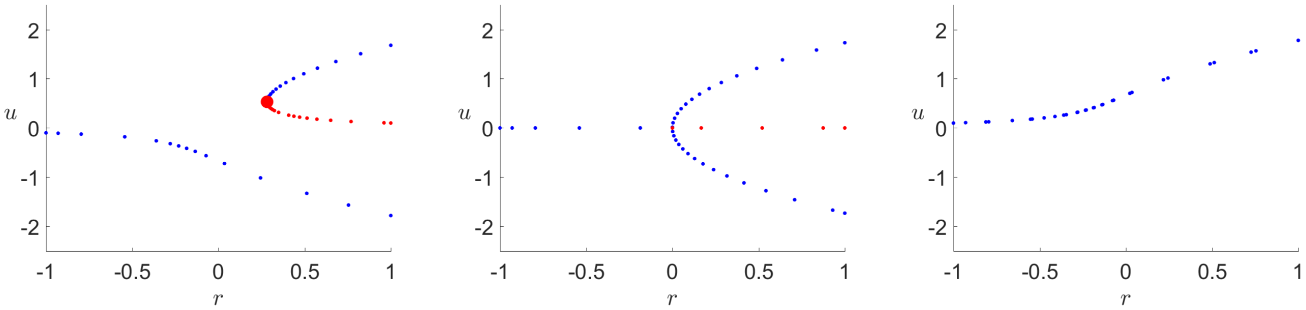

Figure 5: **(r, u)-bifurcation diagrams**. Colors as in fig. 4. Left: $h = -0.1$. Center: $h = 0$. Right: $h = 0.1$.

2 Imperfect bifurcations

When h is varied, we observe saddle-node bifurcations for $r > 0$ (see fig. 4). For $r = 0$, these two points coalesce into a single degenerate bifurcation point at $h = 0$, where u experiences critical slowing down as it approaches the origin.

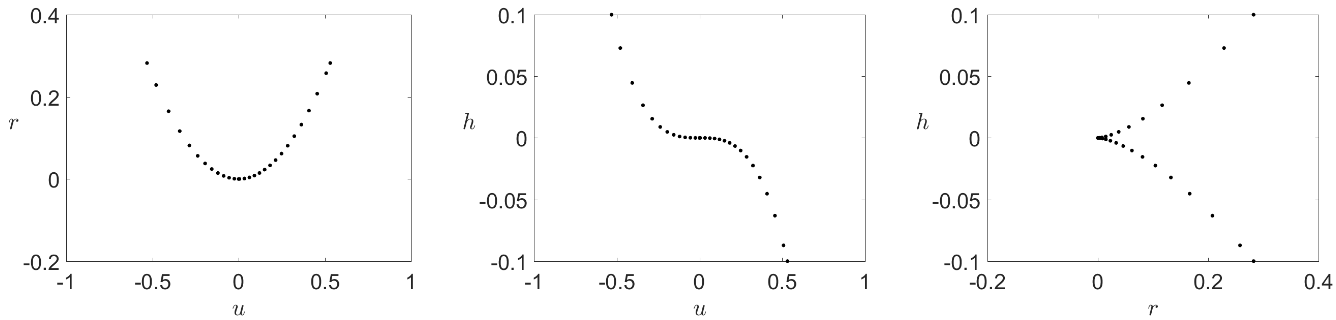

Figure 6: **The fold curve**. Continuation of the saddle-node bifurcations, seen in figs. 4 and 5 as large red dots, projected on the (u, r), (u, h), and (r, h) planes.

References

[1] Steven H. Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*. Perseus Books, 1994.