

# Non-linear systems

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# 1 Stability of equilibrium points & bifurcations

## 1.1 Simple population model

The population model has in general two solutions (and hence two fixed points) for  $\dot{N} = 0$ , namely

$$N_1 = 0 \quad \text{and} \quad N_2 = K \frac{\alpha - \beta}{\alpha}.$$

The stability of these fixed points in function of  $\alpha$  and  $\beta$  can be summarised as follows:

Parameter region	Fixed points
$\alpha < \beta$	$N_1 = 0$ : stable $N_2 < 0$ : unstable
$\alpha = \beta$	$N_1 = N_2 = 0$ : half-stable (unstable for $N < 0$ , stable for $N > 0$ )
$\alpha > \beta$	$N_1 = 0$ : unstable $N_2 > 0$ : stable

The system thus undergoes a transcritical bifurcation at  $\alpha = \beta$ . Note that the fixed point  $N_2 < 0$  is not meaningful in this model, as  $N$  represents a non-negative population count.

For the given parameter values,  $\alpha > \beta$ . Using the above results, we therefore find an unstable fixed point  $N_1 = 0$ , and a stable fixed point  $N_2 = K(\alpha - \beta)/\alpha = 4\,023\,913$ . As the population starts at  $N > 0$ , it will evolve towards  $N_2$ . The difference between  $N(t)$  and  $N(\infty) = N_2$  decays exponentially, as a Taylor approximation of  $\dot{N}$  around  $N_2$  can show.

## 1.2 Gene control model

For  $r = 0$ , the system equations become decoupled:

$$\begin{aligned}\dot{x} &= \frac{\alpha_1}{2} - x \\ \dot{y} &= \frac{\alpha_2}{2} - y.\end{aligned}$$

We can therefore analyse them separately. It is clear that there is one fixed point, at  $x^* = \alpha_1/2$  and  $y^* = \alpha_2/2$ . It is a globally stable attractor, as  $\forall x < x^*$ ,  $\dot{x} > 0$  and  $\forall x > x^*$ ,  $\dot{x} < 0$  (and analogously for  $\dot{y}$  and  $y^*$ ). The fixed point is thus an attracting star.

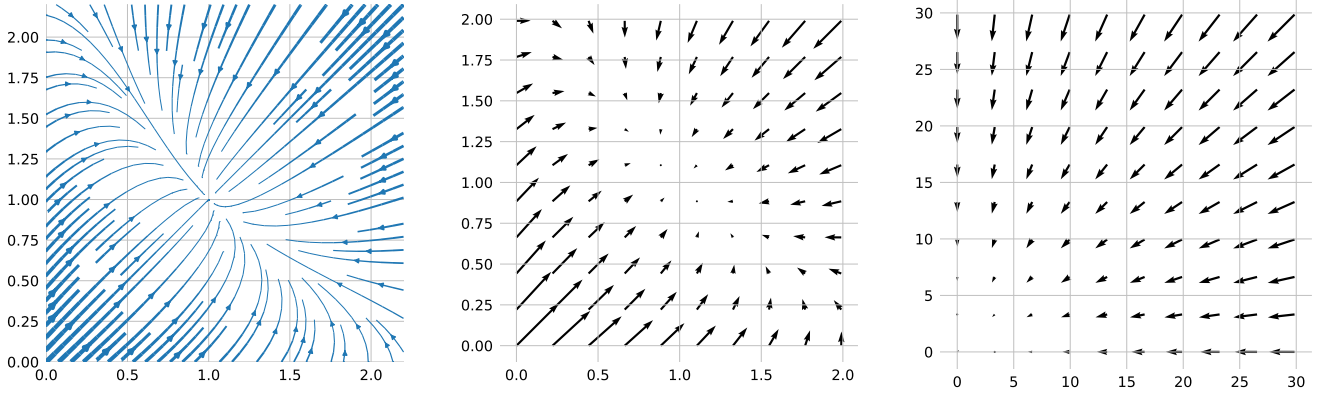


Figure 1: **There is only one fixed point for  $0 \leq r \leq 2$ .** Phase space plots of the gene control model, for  $r = 1$  and  $\alpha_1 = \alpha_2 = 2$ . *Left*: some (partial) trajectories in phase space. Thicker lines represent a higher local speed. Note the attractor at  $(1, 1)$ . *Middle*: local velocities, evaluated on a grid. *Right*: same as middle, but for a larger region in phase space.

For  $r \geq 0$  and  $\alpha_1 = \alpha_2 = 2$ , the equilibrium equations become

$$\begin{aligned} x(1 + y^r) &= 2 \\ y(1 + x^r) &= 2. \end{aligned}$$

It is easily verified that  $(1, 1)$  is a solution and hence a fixed point. We have already shown that it is the only fixed point for  $r = 0$ . Plotting the gradient in phase space for different  $r \in (0, 2)$  strongly suggests that it is also the only fixed point for nonzero  $r < 2$  (at least for  $x \geq 0$  and  $y \geq 0$ ). See e.g. [fig. 1](#) for  $r = 1$ . As a final piece of evidence, different trajectories simulated back in time all either tend towards  $(\infty, \infty)$  or cross into forbidden  $x < 0, y < 0$  territory.

To analyse the stability of the  $(1, 1)$  fixed point, we approximate  $(\dot{x}, \dot{y})$  as a linear system around this point. The Jacobian of  $(\dot{x}, \dot{y})$  evaluated in  $(1, 1)$  is:

$$\begin{pmatrix} -1 & -\frac{r}{2} \\ -\frac{r}{2} & 1 \end{pmatrix}.$$

It has two distinct eigenvalue-eigenvector pairs:

$$\begin{aligned} \lambda_1 &= -\frac{r}{2} - 1, & \mathbf{v}_1 &= (1, 1) \\ \lambda_2 &= \frac{r}{2} - 1, & \mathbf{v}_2 &= (-1, 1). \end{aligned}$$

For  $0 \leq r < 2$ ,  $\lambda_1 \in (-2, -1]$  and  $\lambda_2 \in [-1, 0)$ . Both eigenvalues are negative, and  $(1, 1)$  is therefore a stable node. Because  $\lambda_1 < \lambda_2$ ,  $\mathbf{v}_1 = (1, 1)$  is the fast eigendirection and  $\mathbf{v}_2 = (-1, 1)$  is the slow eigendirection, as can be seen in [fig. 1](#). (For  $r = 0$ , both eigenvalues are equal; i.e.  $(1, 1)$  is then a star node).

Based on phase space plots for different values of  $r$ , it seems that a supercritical pitchfork bifurcation occurs at  $r = 2$ . As the problem is symmetrical for  $\alpha_1 = \alpha_2$ , this is expected (Strogatz 1994 [1]).

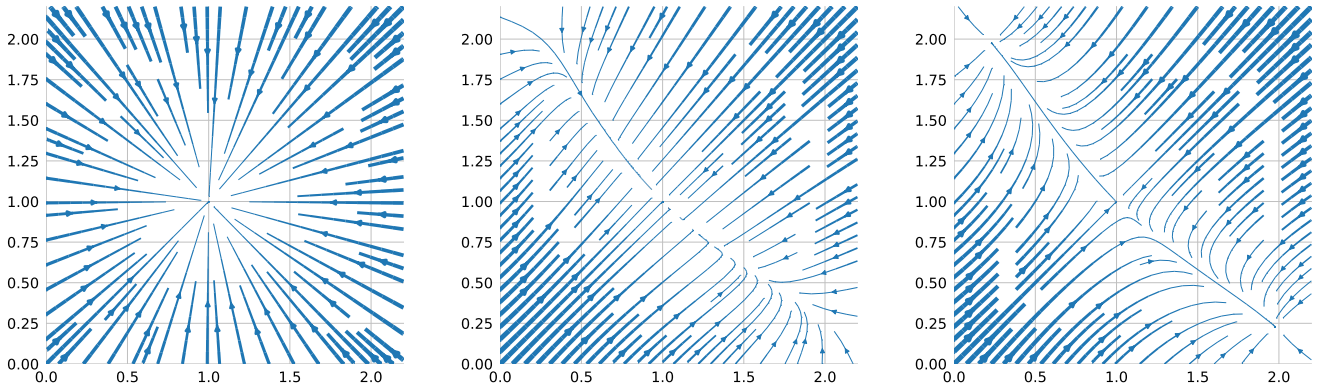


Figure 2: **Topologies of phase space.** *Left:*  $r = 0$ . The single fixed point is an attracting star node. (For the intermediate case between  $r = 0$  and  $r = 2$ , see [fig. 1](#)). *Middle:*  $r = 2$ . There is still only one fixed point. Because we are exactly at the bifurcation point, the approach to the fixed point occurs very slowly (i.e. no longer exponentially fast). This is known as critical slowing down. *Right:*  $r = 3$ . There are three fixed points: a saddle point at  $(1, 1)$ , and two attracting nodes (mirror symmetric around  $y = x$ ).

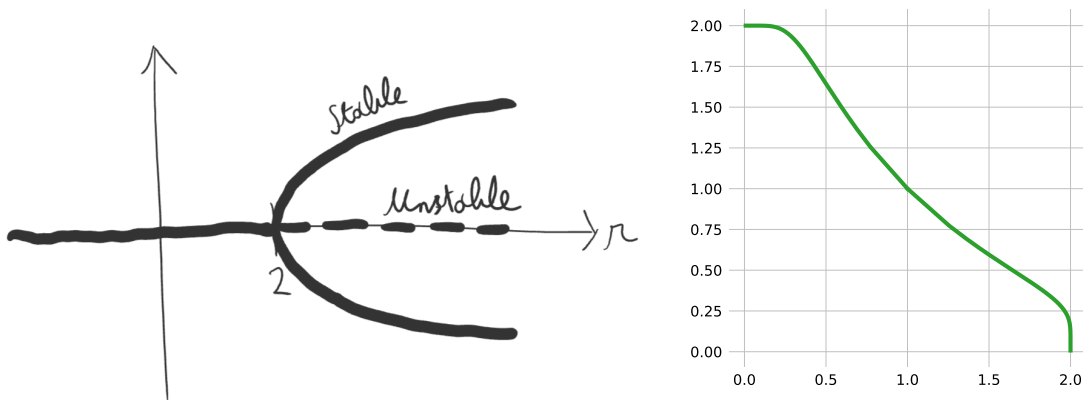


Figure 3: **The gene control model undergoes a supercritical pitchfork bifurcation.** *Left:* sketch of the bifurcation diagram. The vertical axis denotes distance in the bifurcation subspace from the  $(1, 1)$  point. *Right:* the one-dimensional bifurcation subspace embedded in two-dimensional phase space.

[Figure 2](#) shows the different types of phase portrait that occur as  $r \geq 0$  is varied. [Figure 3](#) sketches the bifurcation diagram.

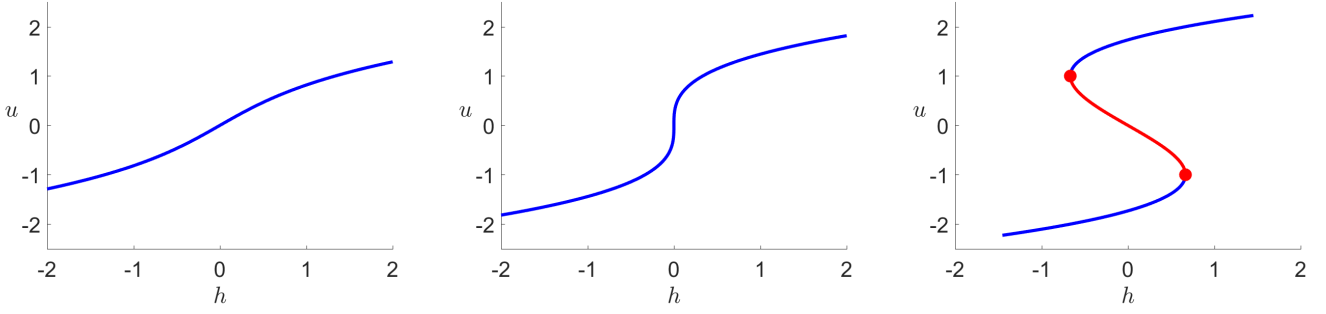


Figure 4:  $(h, u)$ -bifurcation diagrams. Blue lines denote stable equilibria, red lines denote unstable equilibria, and red circles indicate saddle-node bifurcations. Left:  $r = -1$ . (Phase space topologies for other  $r < 0$  are equivalent). Center:  $r = 0$ . Right:  $r = 1$ . (Phase space topologies for other  $r > 0$  are equivalent).

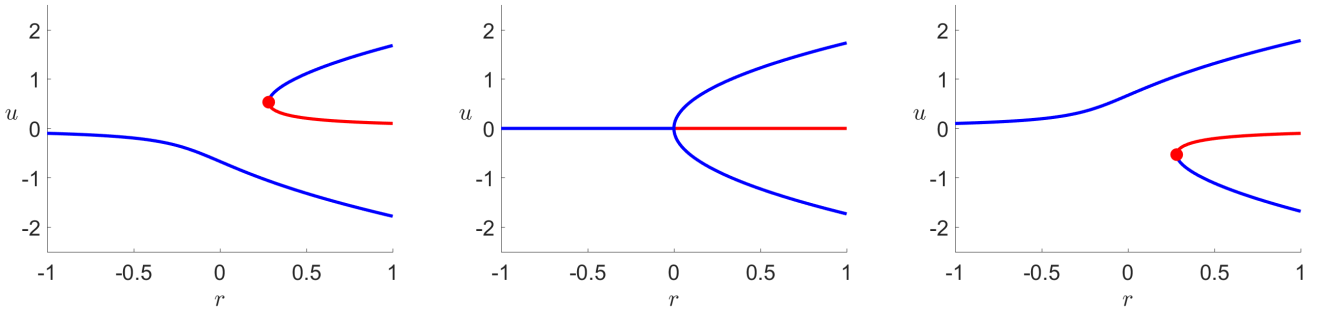


Figure 5:  $(r, u)$ -bifurcation diagrams. Colors as in [fig. 4](#). Left:  $h = -0.1$ . Center:  $h = 0$ . Right:  $h = 0.1$ .

## 2 Imperfect bifurcations

When  $h$  is varied, we observe saddle-node bifurcations for  $r > 0$  (see [fig. 4](#)). For  $r = 0$ , these two points coalesce into a single degenerate bifurcation point at  $h = 0$ , where  $u$  experiences critical slowing down as it approaches the origin. This is a codimension-2 bifurcation.

When  $r$  is varied, only one saddle-node bifurcation can be observed for each fixed  $h$  ([fig. 5](#)). The saddle-node appears at positive  $u$  for negative  $h$ , and vice versa. For  $h = 0$ , the system undergoes a supercritical pitchfork bifurcation (at  $r = 0, u = 0$ ).  $h$  is an ‘imperfection’ parameter: without it, the system is symmetric (i.e. identical for  $u = -u$ ).

[Figure 6](#) shows the continuation curve of the saddle-node bifurcations. We note that there are indeed no saddle-node bifurcations for  $r < 0$ , and that there are two possible saddle-node bifurcations for  $r > 0$ , as already seen in [figs. 4](#) and [5](#). The symmetries observed in those figures are also apparent here. The codimension-2 bifurcation is visible

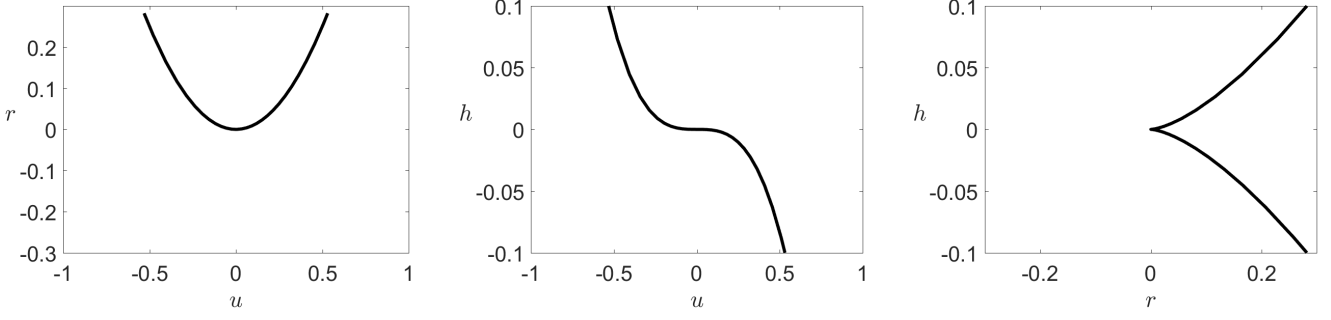


Figure 6: **Fold curve.** Continuation of the saddle-node bifurcations (the red circles in [figs. 4](#) and [5](#)), projected on the  $(u, r)$ ,  $(u, h)$ , and  $(r, h)$  planes.

in the  $(r, h)$  plane as the cusp point at  $(0, 0)$  (where the two branches meet tangentially). In  $(u, r, h)$ -space, the continuation curve has a spiral-like shape. It starts out in the  $(+, +, -)$ -octant, climbing towards the  $(0, 0, 0)$  cusp point, where it is locally flat along the  $h$ -axis. It then continues its climb in the  $(-, -, +)$ -octant.

We now discuss some parameters of the numeric continuation algorithm used [\[2\]](#). `cont.ItMX` is the maximum number of continuation steps taken along a branch. Reducing this parameter (in conjunction with shorter step sizes) results in shorter found branch segments.

For each continuation step, a solution (i.e. a fixed point) is searched for some parameter step size  $\Delta h$ . The maximum number of iterations of this root-finding phase can be specified with `corr.ItMX`. If no root is found within this number of iterations, the parameter step size  $\Delta h$  is reduced, and the root-finding phase repeats. The maximum and minimum step sizes  $\Delta h$  can be set using `h_max` and `h_min`, respectively.

Choosing step sizes `h_max` that are too large can result in missed branches ([fig. 7](#)). The continuation algorithm assumes a relatively smooth branch shape. When  $h$  becomes very small however, the derivative along the branch at  $(0, 0)$  becomes very steep, and this assumption is violated. Only with small step sizes can the branch still be followed. Choosing *minimum* step sizes `h_min` that are too large can also miss branches, or even report non-existent saddle-node bifurcations ([fig. 8](#)).

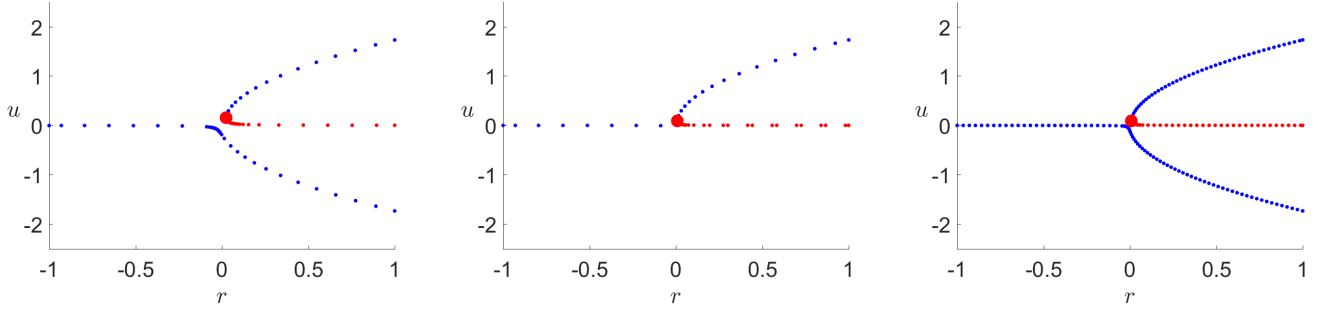


Figure 7: **Continuation is a discrete process.** Bifurcation diagrams for  $h = -0.0025$  (left) and  $h = -0.0005$  (center and right). Each dot is the solution of one continuation step. Colors as in [fig. 4](#). The (maximum) step size  $h_{\max} = 0.2$  is identical in the left and center figures; yet in the center figure, the lower stable branch is not found. When the step size is decreased to  $h_{\max} = 0.05$  however, the lower branch is found (right). (Other parameters were left to their default COCO settings: `cont.ItMX` = 100, `corr.ItMX` = 10,  $h_{\max} = 0.01$ , etc).

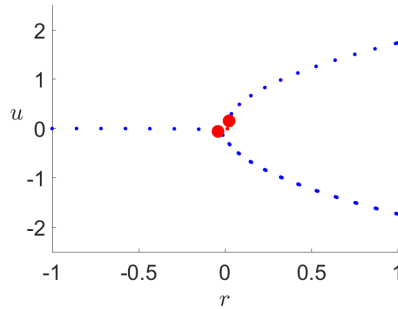


Figure 8: **Incorrect second saddle-node bifurcation.** Bifurcation diagram for  $h = -0.0025$  and  $h_{\min} = h_{\max} = 0.2$ . (Other parameter settings as in [fig. 7](#)). Note the missed unstable branch, and the two saddle-node bifurcations (where there should only be one).



### 3 Study of a predator-prey model

We study the system

$$\begin{aligned}\dot{x} &= x(x - a)(1 - x) - bxy \\ \dot{y} &= xy - cy - d,\end{aligned}$$

with  $a = 0.1$  and  $b = 1.5$ . The following could be an ecological interpretation of this system as a predator-prey model:

The  $-d$  term represents a constant decline of species  $y$ ; This would correspond to a linear decrease in  $y$  over time, with slope  $d$ , if this was the only term present. Maybe an environmental agency eliminates a fixed number  $d$  of  $y$ -type animals every time period, to keep the ecosystem balanced.

The  $-cy$  term represents a proportional pressure on  $y$ , corresponding to an exponential decline of  $y$  over time with time constant  $1/c$  (i.e. faster decline for larger  $c$ ). There might be a fixed amount of resources available for the  $y$  species. Then, a larger number of  $y$  animals will result in a proportionally smaller amount of resources per animal.

The  $xy$  term represents a growth of  $y$  that is both proportional to the other species and to itself. For constant  $x$ , this would correspond to exponential growth of  $y$  with time constant  $1/x$  (i.e. faster growth for more  $x$ ).  $y$  could be a multiplying parasite, and  $x$  could be its host.

The  $-bxy$  term represents a decline of the  $x$  species proportional to both itself and to the other species  $y$ . For constant  $y$ , this would correspond to an exponential decline of  $x$  with time constant  $1/(by)$ . The  $y$  parasite might be pathological for  $x$ . Both more parasites  $y$  and more hosts  $x$  yield a higher probability of transmitting the parasite between hosts.

Finally, the  $x(1 - x)$  factor of the first term describes logistic growth (i.e. exponential growth from the origin, which switches halfway to exponential decay up to a carrying capacity – which is 1 in this case). This is a common model for constrained species growth. The  $(x - a)$  multiplier has the effect that the growth does not start until  $x$  reaches  $a$ : for  $x < a$ , the species will decline instead of grow. This could model the fact that more than a few individuals are necessary for succesful long-term reproduction.

In summary, the parameters  $(a, b, c, d)$  could be given the following names:

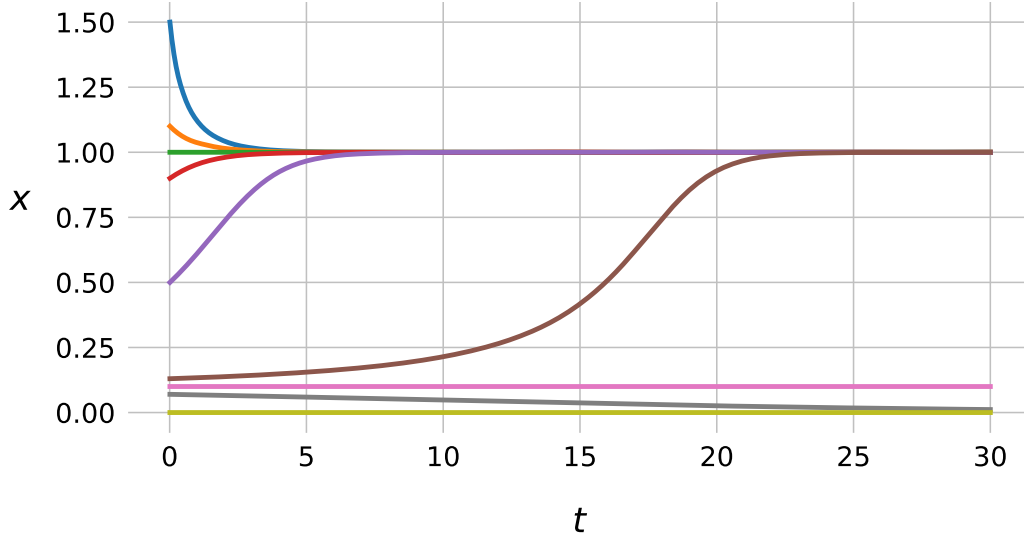


Figure 9: **Prey without predators.** Simulated trajectories  $x(t)$  for  $y = 0$  and different initial values  $x_0$  (from top to bottom: 1.5, 1.1, 1.0, 0.9, 0.5, 0.13, 0.1, 0.07 and 0). Note the stable fixed points at 0 and at the carrying capacity 1, and the unstable fixed point at  $a = 0.1$ .

- a* Minimum prey population
- b* Predator lethality
- c* Predator decay rate
- d* External predator elimination rate

### 3.1 A qualitative study for $d = 0$

Simulating the system for  $y = 0$  confirms the predictions made above for the standalone behaviour of  $x(t)$  (fig. 9): logistic growth above the threshold  $a$ , and decay to zero below this threshold.

The system has four fixed points for  $d = 0$ . Three of these lie on the  $x$ -axis. They are listed in table 1, together with the eigenvalues and corresponding eigenvectors of the Jacobian in these points. The fourth fixed point has coordinates  $(c, (c - a)(1 - c)/b)$ , and the eigenstructure of its Jacobian is rather more.. complex. Its trace  $\tau$  and determinant  $\Delta$  have simpler analytical expressions however:  $\tau = c(1 + a - 2c)$  and  $\Delta = c(c - a)(1 - c)$ . The following paragraphs describe the topological structure near these four fixed points.

$(0, 0)$  is an attractor node for  $c > 0$ . In the common case that  $c > a = 0.1$ , the  $x$ -axis is the slow eigendirection. When  $c < a$ , the  $y$ -axis is the slow eigendirection.

$(a, 0)$  is a saddle when  $c > a$ . Its stable manifold is then the  $x$ -axis, and its unstable manifold is locally spanned by  $(ab/(c - a^2), 1)$ , which is an upwards pointing vector,

Fixed point	Eigenvalues	Eigenvectors
(0, 0)	$-a$	(1, 0)
	$-c$	(0, 1)
(a, 0)	$a - a^2$	(1, 0)
	$a - c$	$(ab/(c - a^2), 1)$
(1, 0)	$a - 1$	(1, 0)
	$1 - c$	$(-b/(2 - c - a), 1)$

Table 1: **Fixed points on the  $x$ -axis.**  $(x, y)$ -coordinates of the fixed points, and eigenvalue-eigenvector pairs of the Jacobian.

rotated slightly right. When  $c < a$ , both dimensions are unstable, and  $(a, 0)$  is then a repeller node. For  $c > a^2$ , the slow eigendirection is  $(ab/(c - a^2), 1)$ . For  $c < a^2$ , the  $x$ -axis is the slow eigendirection.

$(1, 0)$  is a saddle when  $c < 1$ . With  $a = 0.1$ , the stable manifold is the  $x$ -axis, and the unstable manifold is locally spanned by  $(-b/(2 - c - a), 1)$ , which is an upwards pointing vector, rotated slightly *left*. When  $c > 1$ , both dimensions become stable, and  $(1, 0)$  is then an attracting node. Because  $c < 1.5$  in this exercise, the slow eigendirection is  $(-b/(2 - c - a), 1)$ .

The final fixed point,  $(c, (c - a)(1 - c)/b)$ , has a wider range of behaviours (fig. 10). It is a saddle for  $c > 1$ , and for  $0 < c < a$ . It is an unstable node for  $a < c < c_u$ , an unstable spiral for  $c_u < c < (a + 1)/2$ , a stable spiral for  $(a + 1)/2 < c < c_s$ , and a stable node for  $c_s < c < 1$ .  $c_u$  and  $c_s$  are the solutions to  $4\Delta = \tau^2$  (Strogatz 1994 [1]). For  $a = 0.1$ ,  $c_u$  and  $c_s$  are numerically determined to be  $c_u = 0.1259..$  and  $c_s = 0.8783..$

The separatrices between basins of attraction are the stable manifolds of saddles. The stable manifold of a saddle at  $(x^*, y^*)$  can be found numerically by simulating a trajectory back in time. The initial point of this trajectory is  $(x^* + \Delta x, y^* + \Delta y)$ , where  $\Delta x$  and  $\Delta y$  are small perturbations on the line spanned by the eigenvector of the Jacobian corresponding to this stable manifold (i.e. the eigenvector with a negative eigenvalue).

This technique is used in fig. 11 to draw the separatrices between basins of attraction. Note that the separatrices themselves are also one-dimensional ‘manifolds of attraction’, namely for the saddles.

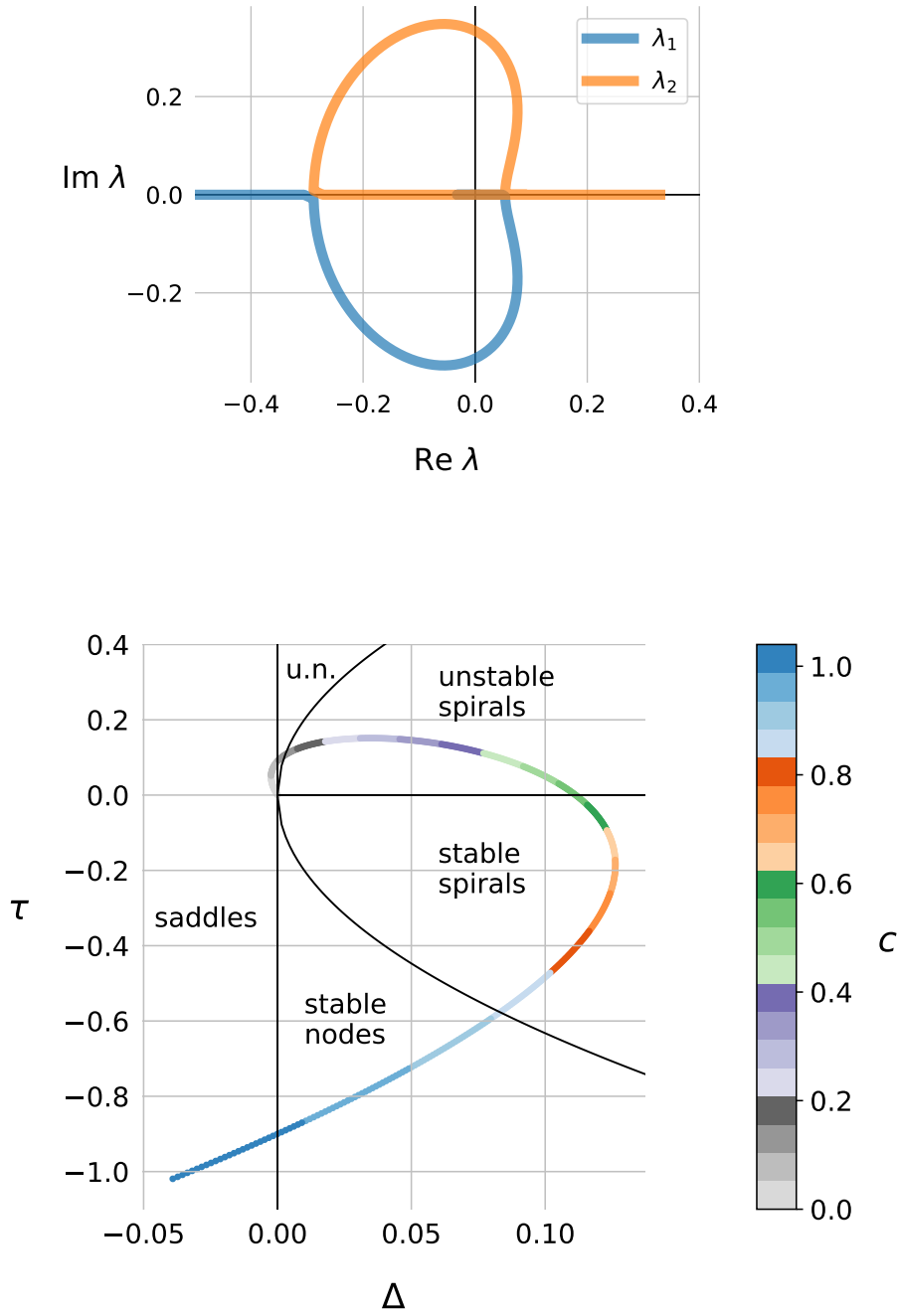


Figure 10: **Linear system analysis of the fixed point  $(c, (a - c)(c - 1)/b)$ .** *Top:* eigenvalues of the Jacobian at the fixed point, for  $a = 0.1$  and  $c \in [0, 1.5]$  ( $c = 0$  in the origin, and  $c = 1.5$  at the far right for  $\lambda_2$  and at the far left, outside the figure, for  $\lambda_1$ ). *Bottom:* trace  $\tau$  and determinant  $\Delta$  of the Jacobian at the fixed point, for  $a = 0.1$  and  $c \in [0, 1.04]$  (for larger  $c$ , the  $(\tau, \Delta)$  curve simply extends further down in the 3rd quadrant). u.n.: unstable nodes.

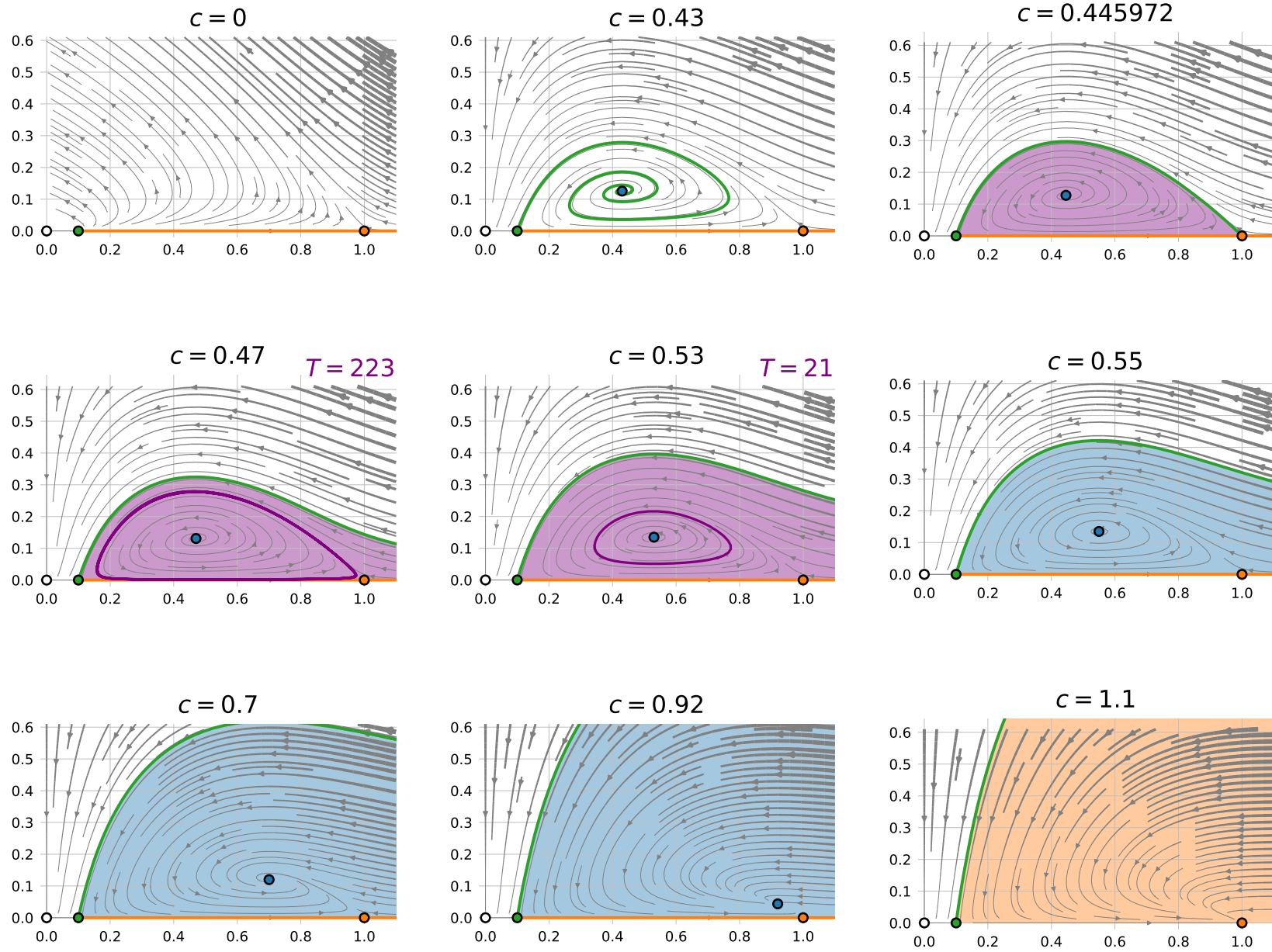


Figure 11 is consistent with the analysis made above: the origin is indeed always an attractor node for  $c > 0$ ; the role of  $(a, 0)$  indeed switches from repeller node to saddle (top row), just as  $(1, 0)$  switches from saddle to attractor node (bottom row); and finally, the fixed point  $(c, (c - a)(1 - c)/b)$  takes on the subsequent roles predicted in fig. 10: from unstable spiral (top and middle row), to center (middle row, rightmost column), to stable spiral, attractor node, and saddle (bottom row).

This sequence of behaviours of the last fixed point can also be observed by following the eigenvalues  $\lambda_i$  of its Jacobian in the complex plane (fig. 10). For  $c > 1$ , the eigenvalues start out on opposite ends of the real axis, and the fixed point is indeed a saddle. When  $c$  then decreases, the eigenvalues move closer together, until the right eigenvalue (say  $\lambda_2$ ) crosses the origin and enters stable territory, at  $c = 1$ . The fixed point is then a stable node, until  $c = c_s = 0.8783...$ . There, the eigenvalues meet on the negative real axis, only to immediately part again. The eigenvalues leave the real axis in opposite directions, tracing symmetric paths that bend towards the  $y$ -axis. The fixed point is then a stable spiral, up until the eigenvalues cross the  $y$ -axis. This happens at  $c = 0.55$ , and the fixed point is then locally surrounded by closed orbits. When  $c$  decreases further, the eigenvalues continue their heart-shaped paths in the right half plane, while the fixed point is an unstable spiral. The eigenvalues meet again on the positive real axis, when  $c = c_u = 0.1259...$ . The fixed point is then a repeller node, while the eigenvalues move in opposite directions on the real axis, towards their initial locations (left for  $\lambda_1$ , right for  $\lambda_2$ ). At  $c = a = 0.1$ ,  $\lambda_1$  crosses into the left half plane, making the fixed point a saddle point. This continues until  $c = 0$ , when both eigenvalues arrive at the origin, and the fixed point disappears.

Between  $0.446... < c < 0.55$ , the phase portrait contains a limit cycle (the purple orbit in fig. 11). The parameter value where the limit cycle first appears is determined numerically to be  $c_h = 0.44597197...$ . For increasing  $c$ , the limit cycle tightens around the central fixed point, until it coalesces with it at  $c = (1 + a)/2 = 0.55$ . The fixed point is then an attracting spiral with a large basin of attraction. Going back in the other direction, for decreasing  $c$ , both the orbit period and the limit cycle become increasingly large. In the limit for  $c = c_h$ , the limit cycle coincides with the stable manifolds of the two neighbouring saddle points  $((a, 0)$  and  $(1, 0))$ , and the orbit period becomes infinite. At this bifurcation point  $c = c_h$ , the two saddle points on the  $x$ -axis are connected in a heteroclinic orbit. Analogously to the ‘homoclinic bifurcation’ from Strogatz 1994 [1], this bifurcation could be called a ‘heteroclinic bifurcation’.

These differences in limit cycle behaviour can be observed in fig. 12. A near-harmonic oscillation yields regular, sine-like time series. Their amplitude is roughly the same as their initial amplitude (closed orbit-like behaviour). On the other hand, a near-heteroclinic cycle yields more exotic time series, with a longer period. The amplitude also grows quickly to the limit cycle amplitude, irrespective of the initial condition. Finally, each oscillation period contains a long interval where both time series are minimal and near constant. This happens when the trajectory comes close to the slow, stable manifold on the  $x$ -axis

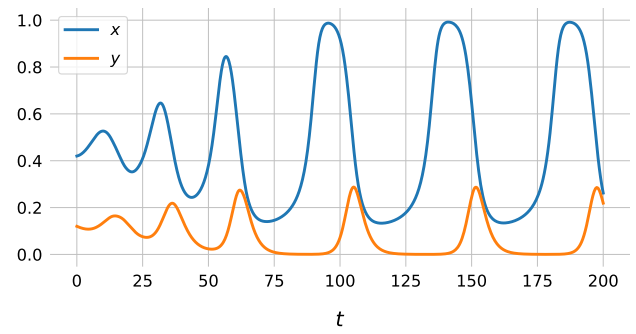
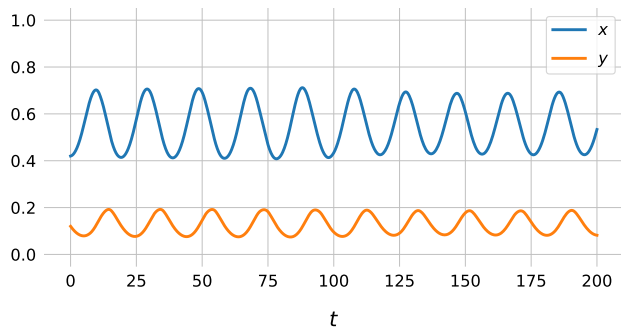


Figure 12: **Limit cycle simulations.** Simulated trajectories  $x(t)$  and  $y(t)$  for a near-harmonic oscillation (left,  $c = 0.55$ ) and a near-heteroclinic orbit (right,  $c = 0.46$ ). In both simulations, the initial phase point was  $(0.42, 0.12)$ .

## 4 Aero-elastic galloping



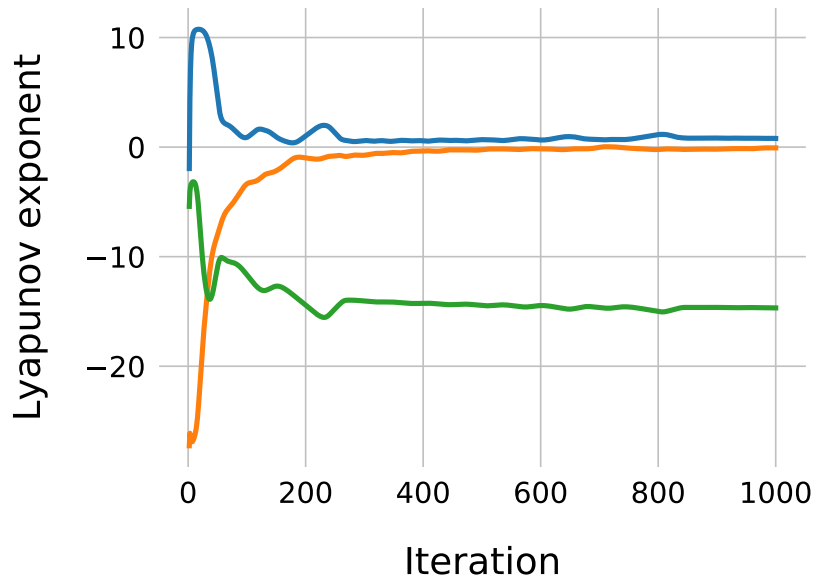


Figure 13: **Estimating Lyapunov exponents of the Lorenz system.** Initial phase point  $(1, 1, 1)$ . Final exponent estimates are  $0.79$ ,  $-0.07$ , and  $-14.7$ .

## 5 Chaos

### 5.1 Lyapunov exponents of the Lorenz equations

The algorithm integrates both an initial phase point and the spatial gradient of the phase point along the trajectory over fixed time steps  $st$  (via the simplest first order Euler method). Next, an orthogonal basis is sought for the new spatial gradient, via the Gram-Schmidt algorithm. The cumulative sum of logarithms of the Gram-Schmidt scaling factors divided by the total time elapsed is then the estimate of the Lyapunov exponent. This is integration-orthogonalisation loop is repeated for a given number of iterations  $kk_{max}$ .

When the time step is too large (e.g.  $st = 0.1$  here), the exponents diverge. When it is too small on the other hand ( $st = 0.001$  here), the convergence is extremely slow. Even for a balanced time step (like  $st = 0.01$  here), enough iterations need to be taken yield a decent result (e.g.  $kk_{max} > 400$  here; see [fig. 13](#)). The initial phase point also influences the results. A different phase point as in [fig. 13](#)  $((6, 6, 6))$  yields better estimates for example  $(0.89, -0.04, \text{ and } -14)$ .

Interestingly, using a more advanced ODE solver (like a higher order Runge-Kutta method) or orthogonalisation algorithm (like a singular value decomposition via the QR-algorithm) yields markably worse results.

## 5.2 Hindmarsh-Rose neuron model

The system behaves chaotically in random burst mode (fig. 14, top): even the tiniest perturbation to the initial phase point renders  $x(t)$  eventually unpredictable. The burst generation mode is not chaotic: even relatively large perturbations do not qualitatively change the phase space trajectories (fig. 14, bottom).

## 5.3 Chua's circuit

A simulation of the Chua circuit displays chaotic behaviour (fig. 15). Estimating the Lyapunov exponents according to the method of section 5.1 confirms this: the largest exponent  $\approx 0.16 > 0$  (fig. 16).

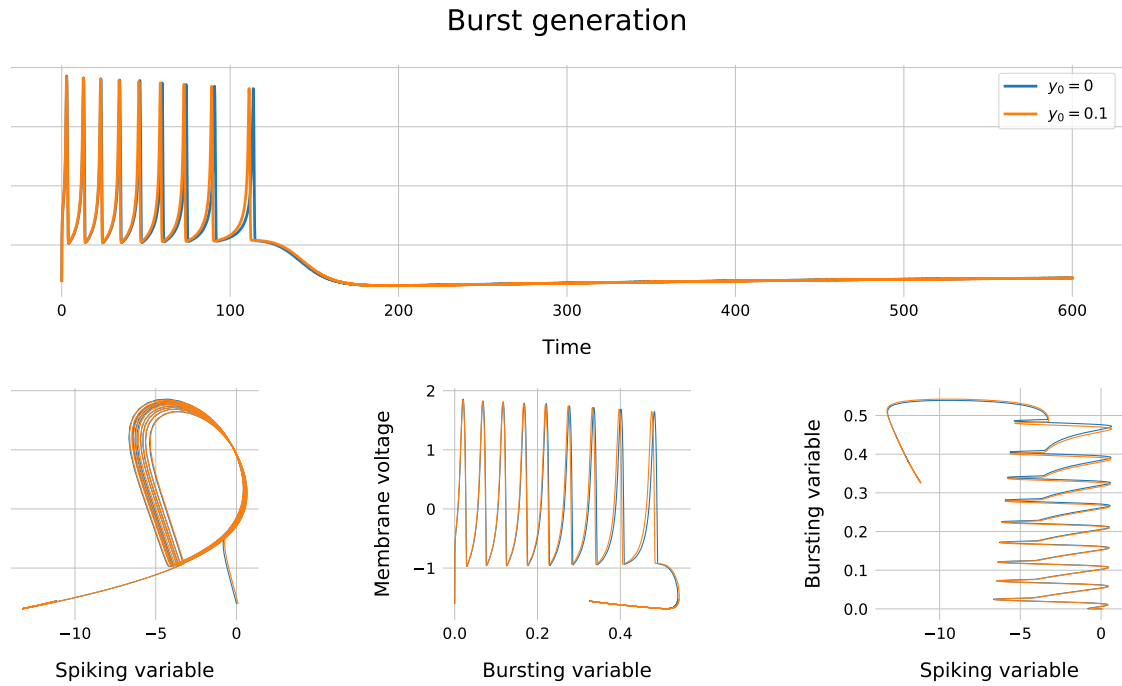
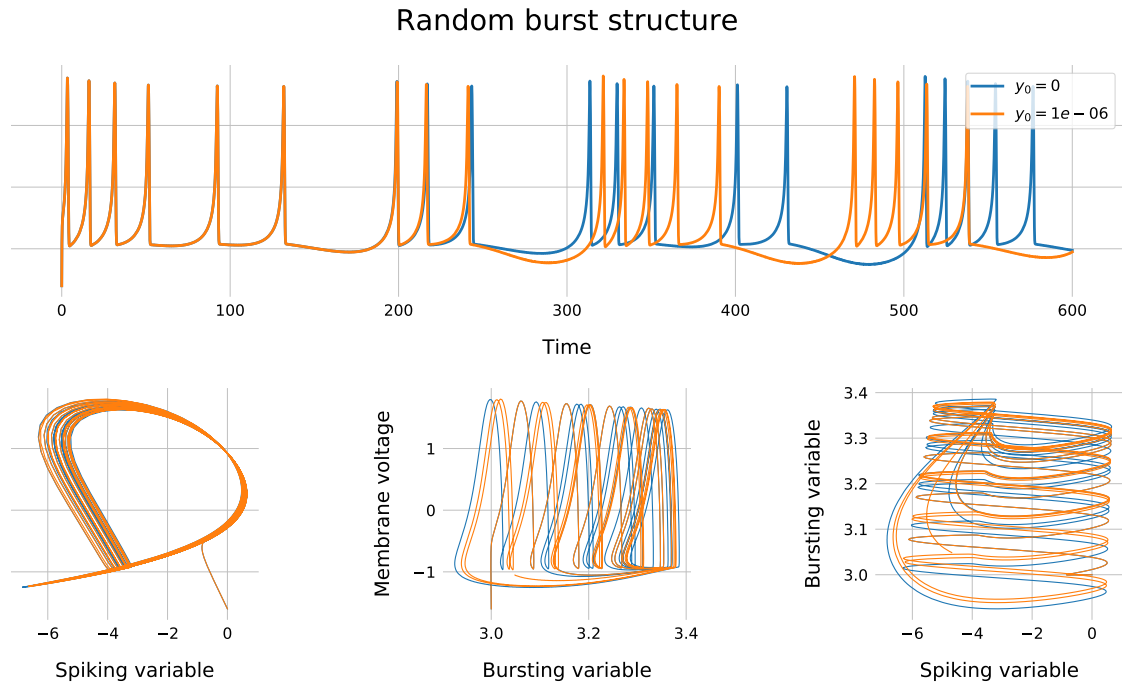


Figure 14: **Hindmarsh-Rose neuron model**. See text for interpretation. (In random burst mode, the initial slow current was  $z(0) = 3$ , while in burst generation mode, it was  $z(0) = 0$ . Other parameters as instructed).

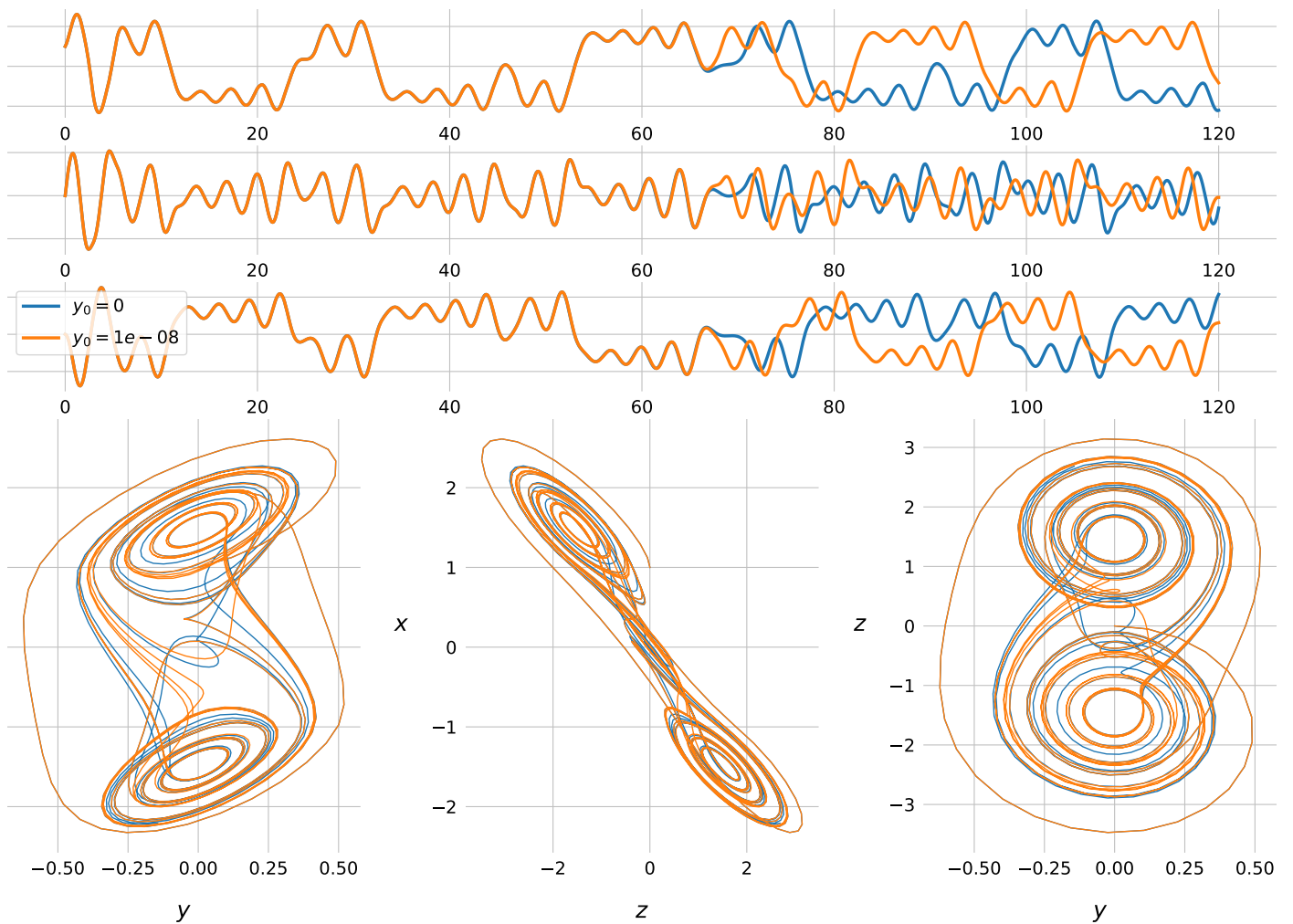


Figure 15: **Chua circuit model.** Time series (top) and phase space projections (bottom). Note the chaotic behaviour and the double-scroll dynamics. (Initial condition  $(1, y_0, 0)$ , with  $y_0$  as indicated).

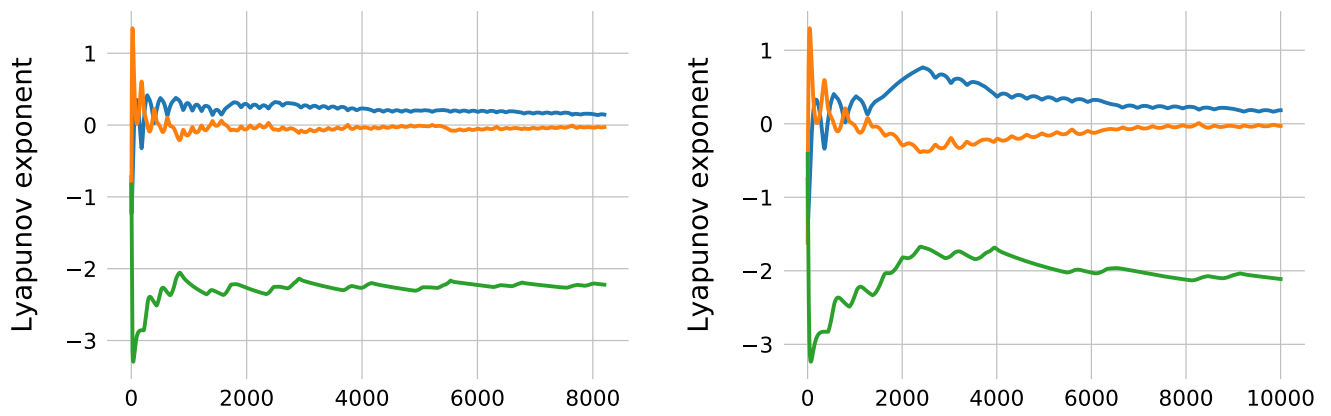


Figure 16: **Estimating Lyapunov exponents of the Chua circuit.** *Left:*  $st = 0.01$ , final largest exponent 0.146. *Right:*  $st = 0.005$ , final largest exponent 0.184.

## References

- [1] Steven H. Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*. Perseus Books, 1994.
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URL: <https://sourceforge.net/projects/cocotools/>.