# Assignments
# Non-linear Systems

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This course is evaluated by means of a set of homework assignments (the student is required to make an individual report), which involve the study of a number of dynamical systems via analytical derivations, numerical simulations and bifurcation calculations. In order to help you getting through these assignments, a number of guided sessions are organized.

Each session is focused around one exercise, and is meant to get you started. It is impossible to finish the exercise within the session; instead, you should make sure that you know how to proceed independently. Also, at the beginning of each session, we will take a little time to deal with questions regarding the previous exercise. You are therefore encouraged not to leave the assignments until the end of the semester. Note that we cannot (and do not intend to) give all the answers to the assignments, since this is the examination.

**Session 1:** Wednesday 28/02 week 9
- a) Introduction to software: `PPLANE`
- b) Stability of equilibrium points and bifurcations

**Session 2:** Wednesday 14/03 week 11
- a) Introduction to software: `COCO`
- b) Imperfect bifurcations

**Session 3:** Wednesday 21/03 week 12
Predator-prey model

**Session 4:** Wednesday 25/04 week 17
Aero-elastic galloping

**Session 5:** Wednesday 09/05 week 19
Lorenz attractor, Chua and Hindmarsh-Rose model

**Session 6:** Wednesday 16/05 week 20
Pattern formation

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# 1 Stability of equilibrium points and bifurcations

**Note:** There is no need to repeat the information provided during the guided session. Answer the questions as concise and accurate as possible. Use graphical and/or tabular presentation rather than full sentences. The *page limit* for this exercise is *4 printed pages*, including figures.

## 1.1 A simple population model

The population of the Charlotte metropolitan area in North Carolina, U.S., can be modeled as

$$\dot{N} = \alpha \frac{(K - N)}{K} N - \beta N,$$

where $N$ is the number of inhabitants, $K$ is the carrying capacity, $\alpha > 0$ is the per capita growth rate, and $\beta > 0$ is the per capita mortality rate.

1. Study the equilibrium points and their stability as a function of $\alpha$ and $\beta$. Summarize your conclusions by using a table. Which type of bifurcation occurs?

2. Data from 2016 estimates a growth rate $\alpha = 6.44\%$ and a mortality rate $\beta = 0.27\%$[1]. Assume the carrying capacity is $K = 4.2$ million. In 2010, the number of inhabitants was 1 687 440. Describe qualitatively what will happen with the population as $t \to \infty$, according to this model.

[1] https://www.opendatanetwork.com/entity/310M200US16740/Charlotte_Metro_Area_NC_SC/demographics.population.count?year=2016

### 1.2 Gene control model

Consider the coupled system describing the time evolution of non-negative real number numbers $x$ and $y$ given by:

$$\dot{x} = \frac{\alpha_1}{1 + y^r} - x, \tag{1}$$

$$\dot{y} = \frac{\alpha_2}{1 + x^r} - y, \tag{2}$$

where $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and $r \geq 0$. This model could describe protein abundances of genes. The first term on the right hand side of equations (1) and (2) represents a repression, the second term represents a degradation.

1. For a *repression rate* $r = 0$, find and classify the fixed points.

2. Consider the special case $\alpha_1 = \alpha_2 = 2$ and $r > 0$.

   (a) Find an obvious equilibrium point and verify numerically (e.g., using `PPLANE`) that it is the only one for $0 \leq r \leq 2$.

   (b) Study analytically the stability of this equilibrium.

   (c) At which value of the repression rate $r$ do you expect a bifurcation? Identify its type. Do you think this kind of bifurcation is likely based on the properties of the system?

   (d) Draw numerically all the qualitatively different phase portraits that occur as the parameter $r$ is varied. Sketch the bifurcation diagram.

## 2 Imperfect bifurcations

> **Note:** There is no need to repeat the information provided during the guided session. Answer the questions as concise and accurate as possible. Use graphical and/or tabular presentation rather than full sentences. The *page limit* for this exercise is *6 printed pages*, including figures.

Consider a simplified equilibrium equation with two parameters $r$ and $h$

$$-\frac{1}{3}u^3 + ru + h = 0,$$

in which $r \in [-1, 1]$.

Note that the terms *fold point*, *limit point*, *turning point* and *saddle node bifurcation* are all synonyms, and are used interchangeably in the literature (and in this assignment).

1. Use `COCO` to plot the bifurcation diagram as a function of $h$ for different values of $r$. For which values of $r$ can we observe turning points with respect to $h$? When do we encounter a non-generic turning point?
2. Use `COCO` to plot the bifurcation diagram as a function of $r$ for different values of $h$ (e.g. $h \in \{-0.1, 0, 0.1\}$). Identify the turning points. Show the connection with the disappearance of bifurcation points in 1-parameter problems under perturbation of the model. What is the meaning of the parameter $h$?
3. For the 2-parameter problem, determine the fold curve (i.e., the branch of turning points). Draw and discuss the projection of the fold curve in the $(u, r), (u, h)$ and $(r, h)$ plane. Compare the latter with your previous analysis. How does the fold curve evolve in the $(u, r, h)$-space?
4. The parameters of the continuation strategy in `COCO` can influence the computed results significantly. Determine the solution branches for $h \in \{-0.0025, -0.0005\}$, using $r$ as parameter. Experiment with the parameters (continuation step length `h_min` and `h_max`, maximum number of iterations `ItMX`, etc.) of the continuation process in `COCO`. What is the meaning of these parameters? When does the numerical procedure lead to wrong results?

## 3 Study of a predator-prey model

**Note:** There is no need to repeat the information provided during the guided session. Answer the questions as concise and accurate as possible. Use graphical and/or tabular presentation rather than full sentences. The *page limit* for this exercise is *6 pages text*, not including figures.

We consider a predator-prey model

$$\dot{x} = x(x-a)(1-x) - bxy,$$
$$\dot{y} = xy - cy - d,$$

in which $a, b, c$ and $d$ are physical parameters.

The values of the parameters $a$ and $b$ are fixed and given to you (see `parameters.pdf` on Toledo).

Give an ecological interpretation to the terms and parameters in the equations (not for the third-order term).

### 3.1 A qualitative study for $d = 0$

We consider the parameter $d = 0$, and confine ourselves to parameter values $c \in [0, 1.5]$.

1. First consider the case $y = 0$. Perform a numerical simulation for a number of initial values $x_0 \in [0, 1.5]$ and describe qualitatively the behaviour as a function of time.

2. For the two-dimensional problem, compute analytically the steady state solutions and determine for each steady state its stability and the topological structure of the phase diagram in the neighbourhood. For the equilibria in which $y = 0$, give the slow eigendirection of the nodes, and determine analytically the stable and the unstable manifolds for the saddles.

   **Hint:** Compute the Jacobian matrix, and evaluate the matrix for each fixed point. Use the trace and determinant of the Jacobian matrix or compute its the eigenvalues to find the topological structure around an equilibrium as a function of the parameter $c$.

**Example:** For the equilibrium $(c, (c-a)(1-c)/b)$, the Jacobian matrix is given by

$$J = \begin{bmatrix} c(1+a-2c) & -bc \\ (c-a)(1-c)/b & 0 \end{bmatrix}$$

The trace $\tau = c(1+a-2c)$ is zero if $c=0$ or $c=(1+a)/2$. The determinant $\Delta = c(c-a)(1-c)$ is zero if $c=0$, $c=a$ or $c=1$.

3. Confirm your analysis by drawing phase diagrams for some (relevant) values of $c$. Make sure your results are consistent with your analysis in item 2. How can you find the separatrices using a combination of analytical and numerical techniques? Determine the regions of attraction of the attractors.

4. What is the relation between (stable and unstable) manifolds of saddle points and separatrices? Indicate this on your figures.

5. Use the results of the previous questions to give a qualitative overview of the changes in the phase diagram when $c$ varies in $[0, 1.5]$. Give a qualitative picture of the evolution of the eigenvalues of the Jacobian in the complex plane for the steady state $(c, (1-c)(c-a)/b)$ as a function of the parameter $c$ and indicate which bifurcations occur. Make sure that your results are consistent with the bifurcation analysis in the next section.

6. Given the interval $[c_1, c_2]$ (see Toledo): between $c_1$ and $c_2$, an important global bifurcation phenomenon occurs. Monitor how the periodic solutions change as the parameter $c$ changes; also look at the period. Which bifurcation occurs? At which value of $c$? Draw the phase diagram for the critical value of $c$.

7. Compare the evolution of $x$ and $y$ as a function of time for

   - a limit cycle corresponding to an almost harmonic oscillation;
   - a limit cycle close to a heteroclinic cycle.

   Briefly discuss the qualitative difference between these two orbits.

### 3.2 Bifurcation analysis

For this part of the assignment, we confine ourselves to $c \in [0.1, 1.5]$.

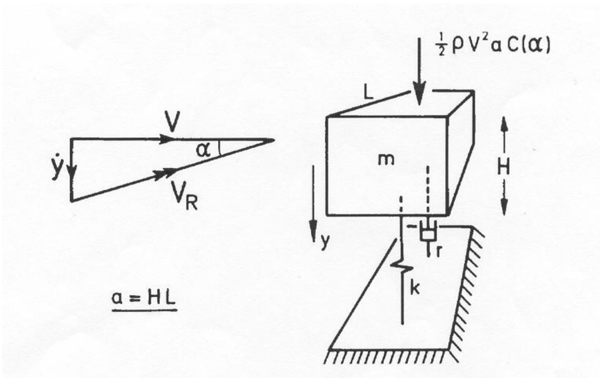
1. Draw the bifurcation diagram for $d = 0$. Consider the $(x, c)$ projection. Discuss the branches, and discuss how the stability changes at all relevant bifurcation points. Also compute the periodic solutions. Make sure that your bifurcation diagram is consistent with the phase diagrams and analysis of the first part of the assignment.

2. Draw the bifurcation diagrams for $d = 0.01$ and $d = -0.01$; Consider both $(x, c)$ and $(y, c)$ projections. Compare the results with the bifurcation diagram for $d = 0$, and indicate how the transcritical bifurcations disappear. It might be useful to extend your figures to include negative values of $x$, $y$ and $c$ to see all branches.

# 4 Aero-elastic galloping

**Note:** There is no need to repeat the information provided during the guided session. Answer the questions as concise and accurate as possible. Use graphical and/or tabular presentation rather than full sentences. The *page limit* for this exercise is *8 printed pages*, including figures.

A constant wind over a flexible, elastic structure can produce or sustain large amplitude oscillations, as is testified by the Tacoma bridge disaster. One of the mechanisms that can cause this phenomenon is aero-elastic galloping.




The square prism in the figure above represents an infinitesimal element of a bridge. The forces that act on the element are:

- *Inertia* : $m\ddot{y}$
- *Linear damping*: $r\dot{y}$
- *Elastic force*: $ky$
- *Driving force*: The wind speed $V_R$, relative to the prism, is composed of the prism's speed $\dot{y}$ and the constant absolute wind speed $V$. The resulting force depends of the angle $\alpha$ between the relative wind speed $V_R$ and the line perpendicular to the prisms direction of motion. For small $\alpha$, or equivalently large $V$, $\alpha$ can be approximated by $\alpha = \frac{\dot{y}}{V}$. The force along the *direction of the speed* $\dot{y}$ is then given by

  $$\frac{1}{2}\rho V^2 a C(\alpha).$$

  We approximate $C(\alpha)$ by the 7th order polynomial

  $$C(\alpha) = A_1\ \alpha - A_3\ \alpha^3 + A_5\ \alpha^5 - A_7\ \alpha^7,$$

where $\alpha$ is expressed in radials and $A_1 = 0.04695$, $A_3 = 8.932 \times 10^{-4}$, $A_5 = 1.015 \times 10^{-5}$ and $A_7 = 2.955 \times 10^{-8}$. Since the driving force is a function of the speed $\dot{y}$, it can be seen as a non-linear damping.

In the subsequent analysis, use $m = 1$, $\rho = 1$, $r = 1$, $k = 100$ and $a = 1$.

1. Write down the equations for the motion of the prism.
2. Make a linear approximation of the model. Analyze the stability of possible fixed points as a function of the parameter $V$. At which critical wind speed $V_C$ do we have a bifurcation? Identify the type of bifurcation of this linear model. Provide a tabular classification of the fixed point.
3. Make a model in MATLAB of the non-linear model (using the `ode45` solver) where you can adapt the parameter $\frac{V}{V_C}$ during simulation.
4. When looking at the structure of $C(\alpha)$, use your intuition to describe what will happen in the neighbourhood of the bifurcation if the non-linear terms are not neglected. Verify using your MATLAB–model.
5. When $V$ grows larger than $V_C$ another bifurcation happens. Lowering back $V$ shows the existence of a third bifurcation. Identify the type(s) of these bifurcations.

   **Hint:** plot $y$ as a function of $\dot{y}$.
6. Use `COCO` to obtain

   (a) a bifurcation diagram, plotting $\frac{A}{V_C}$ as a function of $\frac{V}{V_C}$, where $A$ is the maximal value of $y$ during a period, and

   (b) a plot of the periodic orbits in three dimensions (parameter $\frac{V}{V_C}$ and two phase space coordinates, e.g., $\frac{y}{V_C}$ and $\frac{\dot{y}}{V_C}$).

   Specify the parameters you used to compute the periodic orbits and perform the continuation.

In the following, make sure that the orbit discretization is not changed between subsequent continuation steps (i.e., set `NAdapt` equal to `0`).

7. The period of a limit cycle can be computed in `COCO` using

   ```
   period = coco_bd_col(po_data, 'po.period').
   ```

   Compute the period of the (first) limit cycle at $\frac{V}{V_C} = 1.5$. Experiment with the number of subintervals `NTST` and the number of collocation points `NCOL` in each subinterval, while keeping all other parameters fixed. What is the effect on the computed period? Make a tabular representation of the relative error in the computed value for different combinations of `NTST` and `NCOL`, e.g., `NTST` $\in \{12, 24, 36, 48\}$ and `NCOL` $\in \{4, 5, 6, 7\}$. What do you observe?

# 5 Lyapunov exponents, Hindmarsh-Rose model and Chua's circuit

**Note:** There is no need to repeat the information provided during the guided session. Answer the questions as concise and accurate as possible. Use graphical and/or tabular presentation rather than full sentences. The *page limit* for this exercise is *7 printed pages*, including figures.

## 5.1 Lyapunov exponents of the Lorenz Equations

The Matlab source code for this part of the assignment will be distributed during the session. You will get the files `rhs_lorenz.m` and `lyapunov.m`. The file `rhs_lorenz.m` contains the righthand side of the famous Lorenz equations, as well as the corresponding variational equations. The file `lyapunov.m` contains a function that computes and plots the Lyapunov exponents; it takes as input a function such as `rhs_lorenz.m` and a number of method parameters.

Take a careful look at this code and explain how/why it computes the Lyapunov coefficients. What do the variables `st` and `kkmax` mean? For the Lorenz equations, the Lyapunov exponents are approximately 0.9, 0 and -14.57. Verify this using the Matlab code. Experiment with the method parameters and discuss how they influence the accuracy of the obtained results.

## 5.2 The Hindmarsh-Rose model

The Hindmarsh-Rose model of neuronal activity models the spiking-bursting behavior of the membrane potential observed in experiments made with a single neuron. **The relevant variable is the membrane potential**, x(t), which is written in dimensionless units. There are two more variables, y(t) and z(t), which take into account the transport of ions across the membrane through the ion channels. The transport of sodium and potassium ions is made through fast ion channels and its rate is measured by y(t), which is called the spiking variable. The transport of other ions is made through slow channels, and is taken into account through z(t), which is called the bursting variable.

The mathematical form of the system is given by:[2]

$$\begin{aligned}\frac{dx}{dt} &= y - ax^3 + bx^2 + I - z\\ \frac{dy}{dt} &= c - dx^2 - y\\ \frac{dz}{dt} &= r(s(x - x_1) - z),\end{aligned}$$

where $a = 1$, $b = 3$, $c = 1$, $d = 5$ and **starting point** $x_1 = -\frac{8}{5}$. Simulate the system in:

1. Random burst structure: $r = 0.005$, $s = 4$, $I = 3.25$.
2. Burst generation: $r = 0.001$, $s = 4$, $I = 0.4$.

Discuss the system behavior and the sensitivity with respect to the initial conditions by changing the starting position of $y$. Illustrate your claims by making use of time plots and phase diagrams.

**Hint:** Use Matlab's **ode45** solver to simulate the time behaviour. See the Matlab help for details. Make sure that you simulate long enough to capture the full behavior of the system.

### 5.3 Chua's circuit

Consider the following non-linear (but piecewise linear) circuit.

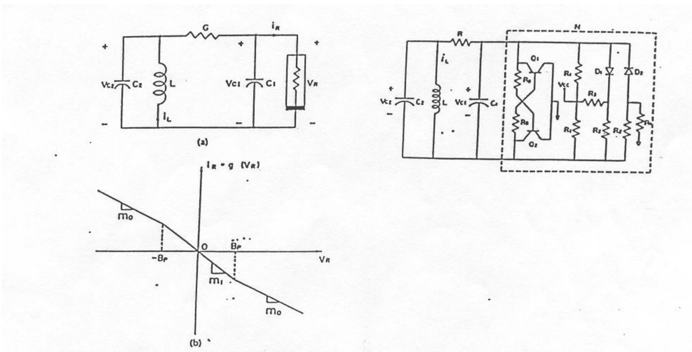


Fig. 1. A simple autonomous circuit with a chaotic attractor. (a) The circuitry. (b) v-i characteristic of the nonlinear resistor.

The dynamics of this circuit are given by

$$\begin{cases} C_1\frac{dv_{C_1}}{dt} &= G(v_{C_2} - v_{C_1}) - g(v_{C_1})\\ C_2\frac{dv_{C_2}}{dt} &= G(v_{C_1} - v_{C_2}) + i_t\\ L_{di_L}dt &= -v_{C_2}\end{cases}$$

[2] `https://en.wikipedia.org/wiki/Hindmarsh%E2%80%93Rose_model`

with $g(\cdot)$ the piecewise linear characteristic that is depicted in the figure. The parameter values are chosen to be

$$\frac{1}{C_1} = 11 \quad \frac{1}{C_2} = 2 \quad \frac{1}{L} = 7 \quad G = 0.7 \quad m_0 = -0.5 \quad m_1 = -0.8 \quad B_p = 1$$

The system can be written as

$$\begin{cases} \frac{dx}{d\tau} & = \alpha(y - h(x)) \\ \frac{dy}{d\tau} & = x - y + z \\ \frac{dz}{d\tau} & = -\beta y \end{cases} \qquad h(x) = \begin{cases} bx + a - b & x > 1 \\ ax & |x| \leq 1 \\ bx - a + b & x < -1 \end{cases}$$

via

$$\begin{array}{lll} x = \frac{v_{C_1}}{B_p} & y = \frac{v_{C_2}}{B_p} & z = \frac{i_L}{B_p G} \\ \tau = \frac{tG}{C_2} & a = \frac{m_1}{G} + 1 & b = \frac{m_0}{G} + 1 \\ \alpha = \frac{C_2}{C_1} & \beta = \frac{C_2}{LG^2} & \end{array}$$

1. Simulate the system behaviour. Show phase portraits in $x, y$ and $z$, as well as individual trajectories starting from two sets of nearby initial conditions. Illustrate your suspicion about the (chaotic) behaviour of the system.

2. Use the code from the previous exercise to compute the Lyapunov exponents.

# 6 Pattern formation

**Note:** There is no need to repeat the information provided during the guided session.
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**A. Theory**

Chapter 9 of the book [Misbah]: check that the necessary condition for the occurence of a Turing instability is given by (9.9) and that the critical wavevector $q_c$ (which is a scalar for 1D spatial problems) is given by (9.10).

**B. Brusselator model**

We consider a system of two interacting species, the famous *Brusselator* model, which exhibit Turing patterns. The equations for a one-dimensional spatial domain are given by

$$\begin{aligned} u_t &= D_u u_{xx} + f(u,v) = D_u u_{xx} + A - (B+1)u + u^2 v, \\ v_t &= D_v v_{xx} + g(u,v) = D_v v_{xx} + Bu - u^2 v, \end{aligned} \tag{3}$$

where $D_u$ and $D_v$ are the diffusion constants of the two species $u$ and $v$, and $A$ and $B$ are concentrations which are kept constant by coupling them to a reservoir. This is a *activator-inhibitor* model due to the nonlinear coupling term. The system has a spatially uniform steady state, in which $u_0 = A$, $v_0 = B/A$.
Perform a linear stability analysis of the uniform steady state:

1. Write the linearization of the Brusselator model around $(u_0, v_0) = (A, B/A)$, cf. section 9.3 of Chapter 9 of [Misbah]

2. Assume a solution of the form $(u_1, v_1) = (C_u, C_v)e^{iqx}e^{\omega t}$ and write the eigenvalue problem (9.2) for the Brusselator model (note that the eigenvector is now denoted as $(C_u, C_v)^T$ instead of $(A, B)^T$ because the symbols $A$ and $B$ are already used to denote concentrations in the Brusselator model.

3. Write the dispersion relation (9.4)-(9.5) specifically for the Brusselator model: show that

$$\begin{aligned} &S = B - 1 - A^2 - q^2(D_u + D_v), \\ &P = A^2 + q^2(A^2 D_u + (1-B)D_v) + q^4 D_u D_v. \end{aligned}$$

4. Find the error in formula (9.6). Rewrite the (correct) conditions (9.6) for the Brusselator model. Check whether conditions (9.12) or (9.13) can be satisfied for the Brusselator model.

5. Assume that the diffusion coefficients $D_u$ and $D_v$ and the concentration $A$ are fixed. For very small values of $B$ the spatially uniform solution is stable. We would like to increase $B$, and are interested in the minimal value for $B$ for which the Turing instability occurs. Compute $B_c$, the critical value of $B$, and the corresponding critical wavenumber $q_c$. Show that
$$q_c = \left(\frac{A^2}{D_u D_v}\right)^{1/4}.$$
and
$$B_c = (1 + A\eta)^2,$$
Here, $\eta = \sqrt{D_u/D_v}$.

6. When ignoring the diffusion term, one can easily check that the resulting system of two ordinary differential equation exhibits a Hopf bifurcation when $B = 1 + A^2$. Show that the Turing instability sets in before the Hopf bifurcation when
$$\eta < \frac{\sqrt{A^2+1} - 1}{A}.$$

### C. Numerical experiments

Use the Matlab-code `brusselator.m` (on Toledo) or the applet on `http://www.cmp.caltech.edu/~mcc/Patterns/Demo4_5.html` .
The Matlab-code and the applet perform time integration of the *two-dimensional* Brusselator model, defined on a unit square, i.e. Eq.(1) with $u_{xx}$ replaced by $u_{xx} + u_{yy}$ and $v_{xx}$ replaced by $v_{xx} + v_{yy}$. Periodic boundary conditions and a random initial condition are imposed. Note that in this model the diffusion coefficients are scaled by the actual size of the physical domain, i.e. $D_u = \overline{D_u}/L$ and $D_v = \overline{D_v}/L$ where $L$ is the length of each size of the physical square domain and $\overline{D_u}$ and $\overline{D_v}$ are the physical diffusion coefficients.

1. Start with the following values: $A$=4.5; $B = 7$; $D_u = 1$; $D_v = 8$. Use the analytical results to check whether the conditions for a Turing instability are satisfied. Describe what you observe.

2. Lower the value of $B$ so that the conditions for a Turing instability are not satisfied and the homogeneous steady state is stable.
Note that the $u$ (or $v$) values are greycoded so that black and white always correspond to the minimal and maximal values. For a correct interpretation of the values you must check the actual minimal and maximal values (Min and Max).

3. Reset $B = 7$ and increase $L$. Describe what you observe.

4. Reset the diffusion coefficients and vary $B$ in the interval (6.5,9). Describe what you observe.

**Reference**

[Misbah] C, Misbah, Complex dynamics and morphogenesis, Springer, 2017