1 Stability of equilibrium points & bifurcations

1.1 Simple population model

The population model has in general two solutions (and hence two fixed points) for $\dot{N} = 0$, namely

$$N_1 = 0 \quad \text{and} \quad N_2 = K\frac{\alpha - \beta}{\alpha}.$$

The stability of these fixed points in function of α and β can be summarised as follows:

Parameter region	Fixed points
$\alpha < \beta$	$N_1 = 0$: stable $N_2 < 0$: unstable
$\alpha = \beta$	$N_1 = N_2 = 0$: half-stable (unstable for $N < 0$, stable for $N > 0$)
$\alpha > \beta$	$N_1 = 0$: unstable $N_2 > 0$: stable

The system thus undergoes a transcritical bifurcation at $\alpha = \beta$. Note that the fixed point $N_2 < 0$ is not meaningful in this model, as N represents a non-negative population count.

For the given parameter values, $\alpha > \beta$. Using the above results, we therefore find an unstable fixed point $N_1 = 0$, and a stable fixed point $N_2 = K(\alpha - \beta)/\alpha = 4\,023\,913$. As the population starts at $N > 0$, it will evolve towards N_2. The difference between $N(t)$ and $N(\infty) = N_2$ decays exponentially, as a Taylor approximation of $\dot{N}$ around N_2 can show.

1.2 Gene control model

For $r = 0$, the system equations become decoupled:

$$\dot{x} = \frac{\alpha_1}{2} - x$$
$$\dot{y} = \frac{\alpha_2}{2} - y.$$

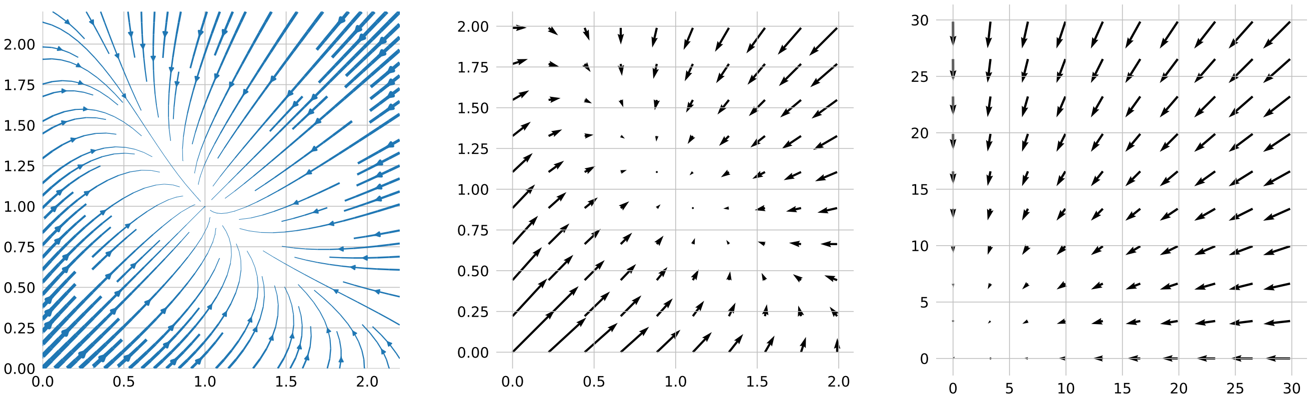


Figure 1: **There is only one fixed point for $0 \leq r \leq 2$.** Phase space plots of the gene control model, for $r = 1$ and $\alpha_1 = \alpha_2 = 2$. *Left*: some (partial) trajectories in phase space. Thicker lines represent a higher local speed. Note the attractor at $(1, 1)$. *Middle*: local velocities, evaluated on a grid. *Right*: same as middle, but for a larger region in phase space.

We can therefore analyse them separately. It is clear that there is one fixed point, at $x^* = \alpha_1/2$ and $y^* = \alpha_2/2$. It is a globally stable attractor, as $\forall x < x^*$, $\dot{x} > 0$ and $\forall x > x^*$, $\dot{x} < 0$ (and analogously for $\dot{y}$ and y^*). The fixed point is thus an attracting star.

For $r \geq 0$ and $\alpha_1 = \alpha_2 = 2$, the equilibrium equations become

$$x(1 + y^r) = 2$$
$$y(1 + x^r) = 2.$$

It is easily verified that $(1, 1)$ is a solution and hence a fixed point. We have already shown that it is the only fixed point for $r = 0$. Plotting the gradient in phase space for different $r \in (0, 2)$ strongly suggests that it is also the only fixed point for nonzero $r < 2$ (at least for $x \geq 0$ and $y \geq 0$). See e.g. fig. 1 for $r = 1$. As a final piece of evidence, different trajectories simulated back in time all either tend towards (∞, ∞) or cross into forbidden $x < 0,\ y < 0$ territory.

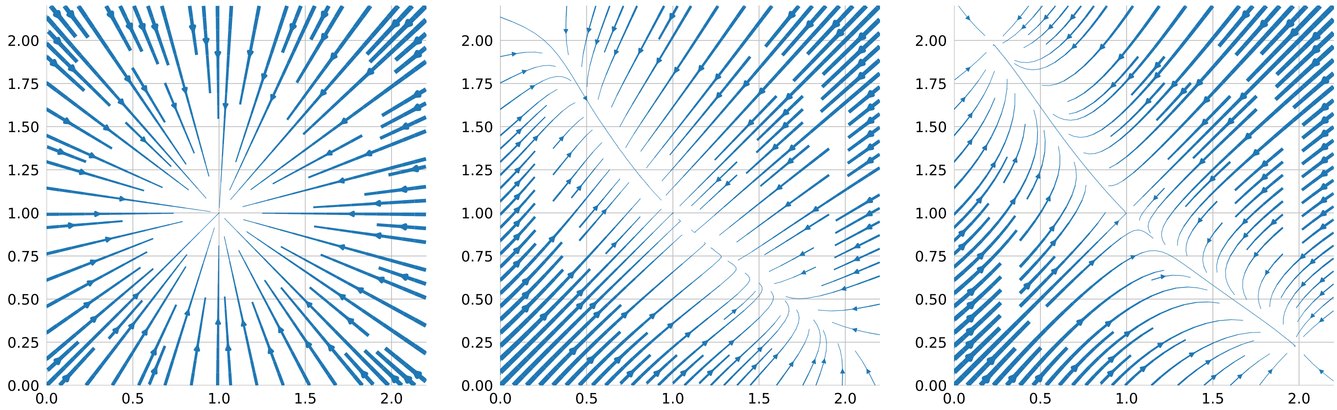
To analyse the stability of the $(1, 1)$ fixed point, we approximate $(\dot{x}, \dot{y})$ as a linear system around this point. The Jacobian of $(\dot{x}, \dot{y})$ evaluated in $(1, 1)$ is:

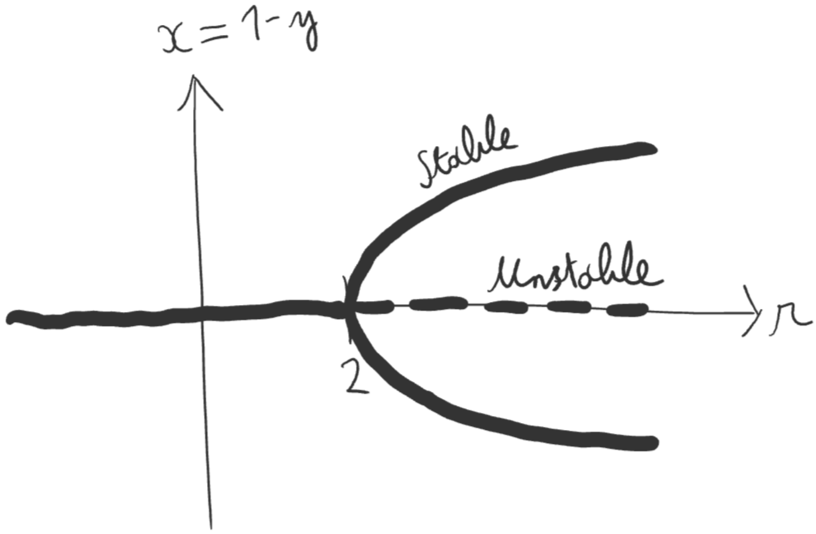
$$\begin{pmatrix} -1 & -\frac{r}{2} \\ -\frac{r}{2} & 1 \end{pmatrix}.$$

It has two distinct eigenvalue-eigenvector pairs:

$$\lambda_1 = -\frac{r}{2} - 1, \quad \mathbf{v}_1 = (1,\ 1)$$
$$\lambda_2 = \frac{r}{2} - 1, \quad \mathbf{v}_2 = (-1,\ 1).$$

For $0 \leq r < 2$, $\lambda_1 \in (-2, -1]$ and $\lambda_2 \in [-1, 0)$. Both eigenvalues are negative, and $(1, 1)$ is therefore a stable node. Because $\lambda_1 < \lambda_2$, $\mathbf{v}_1 = (1,\ 1)$ is the fast eigendirection





and $\mathbf{v}_2 = (-1,\ 1)$ is the slow eigendirection, as can be seen in fig. 1. (For $r = 0$, both eigenvalues are equal; i.e. $(1, 1)$ is then a star node).

Based on phase space plots for different values of r, it seems that a supercritical pitchfork bifurcation occurs at $r = 2$. This is a plausible bifurcation given the system that is modelled: when the mutual repression rate r is high, a slight abundance of one gene (say x) over the other will create a positive feedback loop: x represses y more than y represses x, resulting in more x and less y. This again results in even more x and even less y, etcetera. Vice versa for an initial slight abundance of y. There is thus a precarious balance when $x = y$ (the unstable fixed point at $(1, 1)$ for $r > 2$). On the other hand, when the mutual repression rate r is low, both genes can be expressed in equal amounts. (For large r, this system might be called a winner-takes-all model, or a competitive model).

reffff