3 Study of a predator-prey model

We study the system

$$\begin{aligned}\dot{x} &= x(x-a)(1-x) - bxy \\ \dot{y} &= xy - cy - d,\end{aligned}$$

with $a = 0.1$ and $b = 1.5$. The following could be an ecological interpretation of this system as a predator-prey model:

The $-d$ term represents a constant decline of species y; This would correspond to a linear decrease in y over time, with slope d, if this was the only term present. Maybe an environmental agency eliminates a fixed number d of y-type animals every time period, to keep the ecosystem balanced.

The $-cy$ term represents a proportional pressure on y, corresponding to an exponential decline of y over time with time constant $1/c$ (i.e. faster decline for larger c). There might be a fixed amount of resources available for the y species. Then, a larger number of y animals will result in a proportionally smaller amount of resources per animal.

The xy term represents a growth of y that is both proportional to the other species and to itself. For constant x, this would correspond to exponential growth of y with time constant $1/x$ (i.e. faster growth for more x). y could be a multiplying parasite, and x could be its host.

The $-bxy$ term represents a decline of the x species proportional to both itself and to the other species y. For constant y, this would correspond to an exponential decline of x with time constant $1/(by)$. The y parasite might be pathological for x. Both more parasites y and more hosts x yield a higher probability of transmitting the parasite between hosts.

Finally, the $x(1-x)$ factor of the first term describes logistic growth (i.e. exponential growth from the origin, which switches halfway to exponential decay up to a carrying capacity – which is 1 in this case). This is a common model for constrained species growth. The $(x-a)$ multiplier has the effect that the growth does not start until x reaches a: for $x < a$, the species will decline instead of grow. This could model the fact that more than a few individuals are necessary for succesful long-term reproduction.

3.1 A qualitative study for $d = 0$

Simulating the system for $y = 0$ confirms the predictions made above for the standalone behaviour of $x(t)$ (fig. 9): logistic growth above the threshold a, and decay to zero below this threshold.

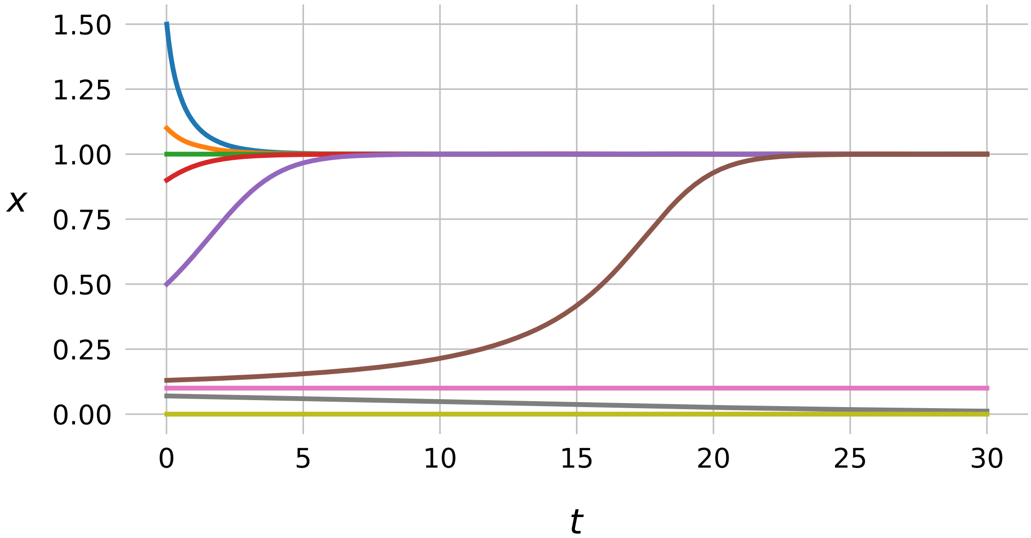

Figure 9: **Behaviour of x without y.** Simulated trajectories $x(t)$ for $y = 0$ and different initial values x_0 (from top to bottom: 1.5, 1.1, 1.0, 0.9, 0.5, 0.13, 0.1, 0.07 and 0). Note the stable fixed points at 0 and at the carrying capacity 1, and the unstable fixed point at $a = 0.1$.

Fixed point	Eigenvalues	Eigenvectors
$(0,\ 0)$	$-a$	$(1,\ 0)$
	$-c$	$(0,\ 1)$
$(a,\ 0)$	$a - a^2$	$(1,\ 0)$
	$a - c$	$(ab/(c - a^2),\ 1)$
$(1,\ 0)$	$a - 1$	$(1,\ 0)$
	$1 - c$	$(-b/(2 - c - a),\ 1)$

Table 1: **Fixed points on the x-axis.** (x, y)-coordinates of the fixed points, and eigenvalue-eigenvector pairs of the Jacobian.

The system has four fixed points for $d = 0$. Three of these lie on the x-axis. They are listed in table 1, together with the eigenvalues and corresponding eigenvectors of the Jacobian in these points. The fourth fixed points has coordinates $(c,\ (c-a)(1-c)/b)$, and the eigenstructure of its Jacobian is rather more.. complex. Its trace τ and determinant Δ have simpler analytical expressions however: $\tau = c(1 + a - 2c)$ and $\Delta = c(c - a)(1 - c)$. The following paragraphs describe the topological structure near these four fixed points.

$(0,\ 0)$ is an attractor node. In the common case that $c > a = 0.1$, the x-axis is the slow eigendirection. When $c < a$, the y-axis is the slow eigendirection.

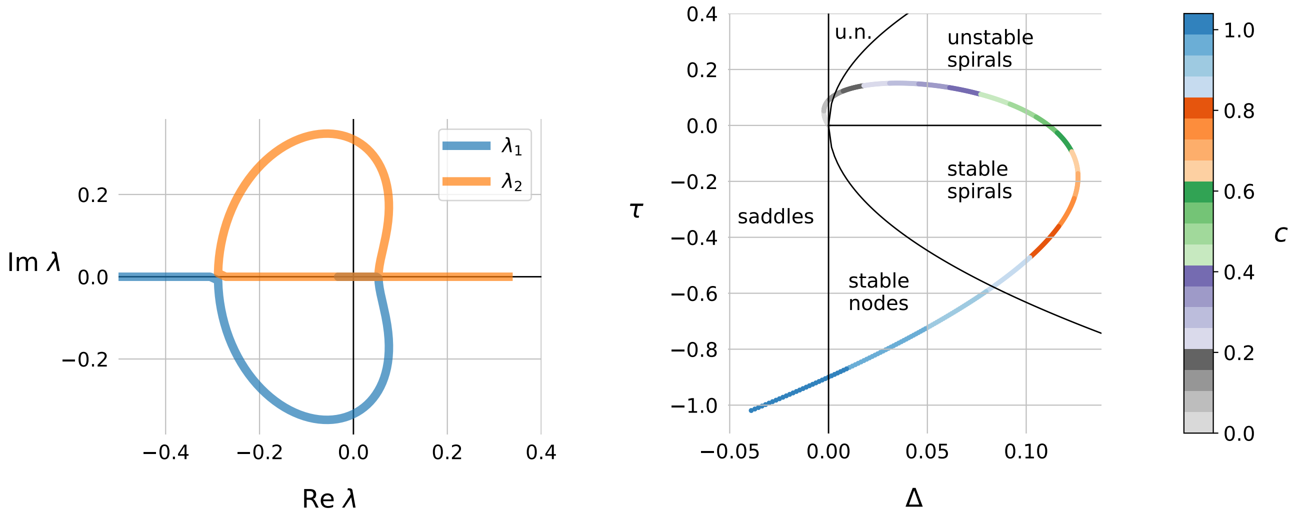

Figure 10: **Linear system analysis of the fixed point $(c,\ (a-c)(c-1)/b)$**. *Left*: eigenvalues of the Jacobian at the fixed point, for $a = 0.1$ and $c \in [0, 1.5]$ ($c = 1.5$ at the far right, and at the far left outside the figure). *Right*: trace τ and determinant Δ of the Jacobian at the fixed point, for $a = 0.1$ and $c \in [0, 1.04]$ (for larger c, the (τ, Δ) curve simply extends further down in the 3rd quadrant). u.n.: unstable nodes.

$(a,\ 0)$ is a saddle when $c > a$. Its stable manifold is then the x-axis, and its unstable manifold is locally spanned by $(ab/(c-a^2),\ 1)$, which is an upwards pointing vector, rotated slightly right. When $c < a$, both dimensions are unstable, and $(a,\ 0)$ is then a repellor node. For $c > a^2$, the slow eigendirection is $(ab/(c-a^2),\ 1)$. For $c < a^2$, the x-axis is the slow eigendirection.

$(1,\ 0)$ is a saddle when $c < 1$. With $a = 0.1$, the stable manifold is the x-axis, and the unstable manifold is locally spanned by $(-b/(2-c-a),\ 1)$, which is an upwards pointing vector, rotated slightly *left*. When $c > 1$, both dimensions become stable, and $(1,\ 0)$ is then an attracting node. Because $c < 1.5$ in this exercise, the slow eigendirection is $(-b/(2-c-a),\ 1)$.

The final fixed point, $(c,\ (c-a)(1-c)/b)$, has a wider range of behaviours (fig. 10). It is a saddle for $c > 1$, and for $0 < c < a$. It is an unstable node for $a < c < c_u$, an unstable spiral for $c_u < c < (a+1)/2$, a stable spiral for $(a+1)/2 < c < c_s$, and a stable node for $c_s < c < 1$. For $a = 0.1$, c_u and c_s can be numerically determined to be $c_u = 0.1259..$ and $c_s = 0.8783...$