1 Stability of equilibrium points & bifurcations

1.1 A simple population model

The population model has in general two solutions (and hence two fixed points) for $\dot{N} = 0$, namely

$$N_1 = 0 \quad \text{and} \quad N_2 = K\frac{\alpha - \beta}{\alpha}.$$

The stability of these fixed points in function of α and β can be summarised as follows:

Parameter region	Fixed points
$\alpha < \beta$	$N_1 = 0$: stable $N_2 < 0$: unstable
$\alpha = \beta$	$N_1 = N_2 = 0$: half-stable (unstable for $N < 0$, stable for $N > 0$)
$\alpha > \beta$	$N_1 = 0$: unstable $N_2 > 0$: stable

The system thus undergoes a transcritical bifurcation at $\alpha = \beta$. Note that the fixed point $N_2 < 0$ is not meaningful in this model, as N represents a non-negative population count.

For the given parameter values, $\alpha > \beta$. Using the above results, we therefore find an unstable fixed point $N_1 = 0$, and a stable fixed point $N_2 = K(\alpha - \beta)/\alpha = 4\,023\,913$. As the population starts at $N > 0$, it will evolve towards N_2. The difference between $N(t)$ and $N(\infty) = N_2$ decays exponentially, as a Taylor approximation of $\dot{N}$ around N_2 can show.

1.2 Gene control model

For $r = 0$, the system equations become decoupled:

$$\dot{x} = \frac{\alpha_1}{2} - x$$
$$\dot{y} = \frac{\alpha_2}{2} - y.$$

We can therefore analyse them separately. It is clear that there is one fixed point, at $x^* = \alpha_1/2$ and $y^* = \alpha_2/2$. It is a globally stable attractor, as $\forall x < x^*, \dot{x} > 0$ and $\forall x > x^*, \dot{x} < 0$ (and analogously for $\dot{y}$). In other words, the fixed point is an attracting star.