Exercise 1

a) The shower generation after a propagated distance X is $n = X/X_{1/2}$, where $X_{1/2} = X_0 \ln(2)$ is the characteristic length after which the energy is halved and X_0 is the characteristic interaction length for pair production and Bremsstrahlung. In the lecture is has been shown that when the shower maximum is reached $n = n_{\max} = \ln(E_0/E_\mathrm{c})/\ln(2)$ so that

$$X_{\max} = n_{\max} X_{1/2} = \frac{\ln(E_0/E_\mathrm{c})}{\ln(2)} X_0 \ln(2) = \ln\left(\frac{E_0}{E_\mathrm{c}}\right) X_0 \tag{1}$$

b) With an approximate atomic number of $Z \approx 7.2$ for air, the critical energy becomes

$$E_c = \frac{710}{Z + 0.92} \mathrm{MeV} \approx 89\mathrm{MeV} \tag{2}$$

so that (assuming as done in Ex. 4 of Sheet 4 that $X_0 = 36$ g/cm^2) $X_{\max}(E_0 = 1\mathrm{TeV}) = 335.4$ g/cm^2 and $X_{\max}(E_0 = 30\mathrm{GeV}) = 209.1$ g/cm^2.

c) In the Heitler model

$$R_\mathrm{el} = \frac{dX_{\max}}{d\ln(E_0)} = X_0 \frac{d(\ln(E_0) - \ln(E_c))}{d\ln(E_0)} = X_0 \tag{3}$$

so that for and increase in the energy E_0 of order $e \approx 2.7$, the maximum length $X_{\max}$ increases of a factor X_0.

d) According to the definitions on the exercise sheet

$$X^\mathrm{A}_{\max} = X^\mathrm{p}_{\max} - X_0 \ln(\mathrm{A}) \tag{4}$$

with

$$X^\mathrm{p}_{\max} = \lambda_\mathrm{I} \ln(2) + X_0 \ln\left(\frac{E_0}{6N_\pi E^\gamma_\mathrm{c}}\right) \tag{5}$$

where λ_I is the interaction length

$$\lambda_\mathrm{I} = \left(90 - 9\ln\left(\frac{E_0}{\mathrm{EeV}}\right)\right) \frac{\mathrm{g}}{\mathrm{cm}^2} \tag{6}$$

and

$$N_\pi = \left(\frac{E_0}{\mathrm{PeV}}\right)^{1/5} \tag{7}$$

Putting it all together we obtain that

$$
\begin{aligned}
X_{\max}^{\mathrm{A}} &= X_{\max}^{\mathrm{p}} - X_0 \ln(\mathrm{A}) = \\
&= \lambda_{\mathrm{I}} \ln(2) + X_0 \ln\left(\frac{E_0}{6 N_\pi E_{\mathrm{c}}^{\gamma}}\right) - X_0 \ln(\mathrm{A}) = \\
&= \ln(2)\left(90 - 9\ln\left(\frac{E_0}{\mathrm{EeV}}\right)\right)\frac{\mathrm{g}}{\mathrm{cm}^2} + X_0 \ln\left(\frac{E_0}{6E_{\mathrm{c}}^{\gamma}}\frac{\mathrm{PeV}^{1/5}}{E_0^{1/5}}\right) + \\
&\quad - X_0 \ln(\mathrm{A}) = \\
&\approx \left[62 - 6\ln\left(\frac{E_0}{\mathrm{EeV}}\right) - 36\ln(6) + 36\ln\left(\frac{E_0}{E_{\mathrm{c}}^{\gamma}}\right) + \right. \\
&\quad \left. +36\ln\left(\frac{\mathrm{PeV}^{1/5}}{E_0^{1/5}}\right) - 36\ln(\mathrm{A})\right]\frac{\mathrm{g}}{\mathrm{cm}^2} = \\
&\approx \left\{-2 - 36\ln(\mathrm{A}) + 6\left[-\ln\left(\frac{E_0}{\mathrm{EeV}}\right) + 6\ln\left(\frac{E_0}{E_{\mathrm{c}}^{\gamma}}\right) + \right.\right. \\
&\quad \left.\left. + \frac{6}{5}\ln\left(\frac{\mathrm{PeV}}{E_0}\right)\right]\right\}\frac{\mathrm{g}}{\mathrm{cm}^2} = \\
&\approx \left[-2 - 36\ln(\mathrm{A}) + 6\ln\left(\frac{10^{18}\mathrm{eV}}{E_0}\frac{E_0^6}{(E_{\mathrm{c}}^{\gamma})^6}\frac{(10^{15}\mathrm{eV})^{6/5}}{E_0^{6/5}}\right)\right]\frac{\mathrm{g}}{\mathrm{cm}^2} = \\
&\approx \left[-2 - 36\ln(\mathrm{A}) + 6\ln\left(\frac{E_0^4 10^{36}\mathrm{eV}^2}{(E_{\mathrm{c}}^{\gamma})^6}\right)\right]\frac{\mathrm{g}}{\mathrm{cm}^2} = \\
&\approx \left[-2 - 36\ln(\mathrm{A}) + 12\ln\left(\frac{E_0^2\mathrm{EeV}}{(E_{\mathrm{c}}^{\gamma})^3}\right)\right]\frac{\mathrm{g}}{\mathrm{cm}^2}
\end{aligned}
$$

so that

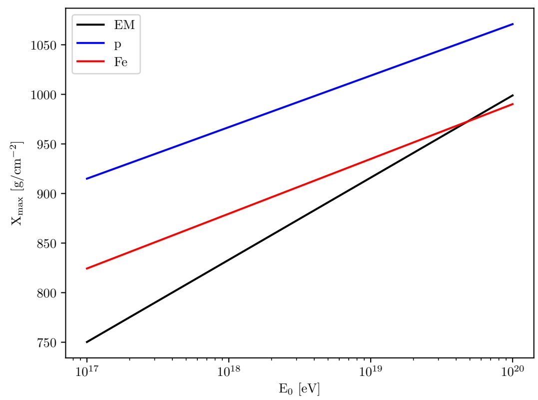
$$
c^{\mathrm{A}} = [-2 - 36\ln(\mathrm{A})]\frac{\mathrm{g}}{\mathrm{cm}^2} \tag{8}
$$

and

$$
R_{\mathrm{el}} = 12?? \tag{9}
$$

which is independent on A, as to prove.

e) By plotting $X_{\max}$ for photons-, protons- and iron-induced air showers we obtain the following plot:



f)

Exercise 2

a) By solving the integral

$$N_{\mathrm{e}}(x) = 2\pi \int r\rho_{\mathrm{e}}(r,x)dr \tag{10}$$

with

$$\rho_{\mathrm{e}}(r,x) = \frac{N_{\mathrm{e}}(x)C_1(s)}{r_{\mathrm{s}}^2}\left(\frac{r}{r_{\mathrm{s}}}\right)^{s-2}\left(\frac{r_{\mathrm{s}}+r}{r_{\mathrm{s}}}\right)^{s-4.5} \tag{11}$$

one obtains that

$$C_1(s) = \frac{1}{2\pi\int \frac{r}{r_{\mathrm{s}}^2}\left(\frac{r}{r_{\mathrm{s}}}\right)^{s-2}\left(\frac{r_{\mathrm{s}}+r}{r_{\mathrm{s}}}\right)^{s-4.5}} \tag{12}$$

For $s = 1.25$ we get that $C_1 = 0.447$.

b) According to the lecture notes, the lateral distribution of the particle density for hadronic showers is given by

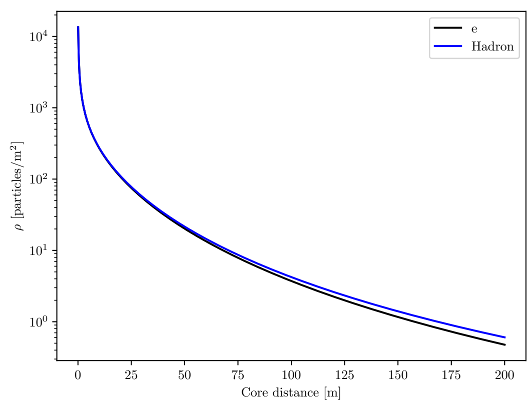
$$\rho(r,x) = \frac{N(x)C_1(s)}{r_{\mathrm{s}}^2}\left(\frac{r}{r_{\mathrm{s}}}\right)^{s-2}\left(\frac{r_{\mathrm{s}}+r}{r_{\mathrm{s}}}\right)^{s-4.5}\left(1+C_2\left(\frac{r}{r_{\mathrm{s}}}\right)^{\delta}\right) \tag{13}$$

Using $\delta = 1$ and $C_2 = 0.088$ analogously to exercise a) we obtain

$$C_1(s) = \frac{1}{2\pi \int \frac{r}{r_s^2} \left(\frac{r}{r_\mathrm{s}}\right)^{s-2} \left(\frac{r_\mathrm{s}+r}{r_\mathrm{s}}\right)^{s-4.5} \left(1 + C_2 \frac{r}{r_\mathrm{s}}\right)} \tag{14}$$

so that, for $s = 1.25$, $C_1 = 0.403$.

c) The following plot displays the lateral distribution for EM (black line) and hadronic (blue line) cascades for $N = 10^6$. To define r_s we used the definition (5.173) of the lecture notes to obtain $r_s \approx 8.5$ g cm^{-2} and than we divided it by the air density at sea level ($\rho_0 = 1.3$ g cm^{-3}) to obtain a Moliere radius of 6,5 cm.

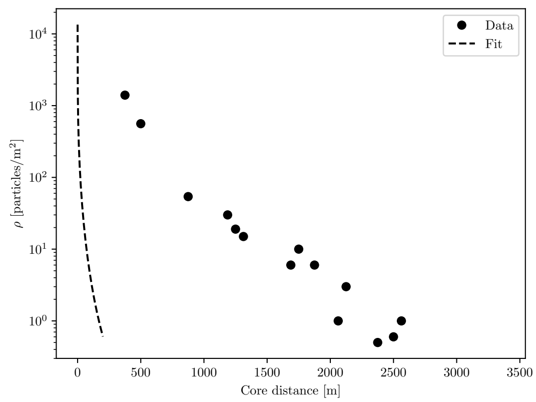


It is possible to observe a deviation for large distances from the core, where the density of EM cascades decreases faster than the density of hadronic cascades. This behavior could be explained with the fact that "hadronic decay particles typically carry larger transversal momenta than EM particles" so that "the transversal extension of hadronic cascades is larger" (Lecture notes p. 109) and with it also the density for very large distances from the shower core.

d) As possible to observe in Fig. , a density of 1 particle per m^2 is reached at a distance from the shower core of ≈ 165 m.

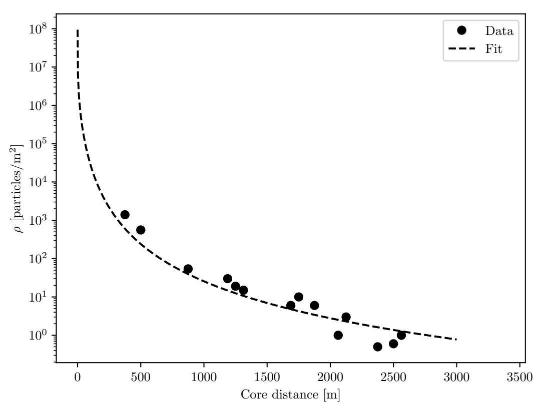
Exercise 3

a) By using the data from the previous exercise, we obtain the following plot.



From the figure it is clear that the number of particle must be higher than the 1×10^6 assumed before.

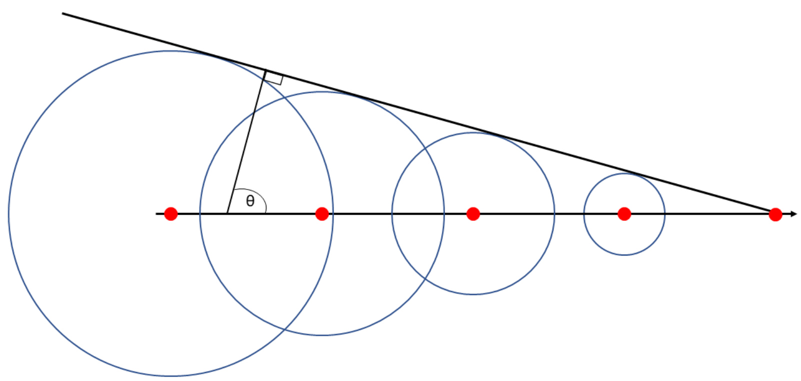
b) By increasing the number of particle to 7×10^9 we obtain the following (more realistic) result.



c) Assuming that approximately between 0.1% and 1% of the particles reaching ground are muons, i.e. between 7×10^6 and 7×10^7, then according to Fig. 5.128 of the lecture notes (assuming an energy threshold of 1 GeV) the primary energy of the event is approximately between 1×10^{18} and 1×10^{19} eV.

Exercise 4

a) According to Huygens' principle each point of a wavefront is source of a elementary wave. In the graphic below it is shown how a particle moves from left to right and emits light at each position (shown are just a few examples). The resulting wavefront is thus the upper line, tangent to all elementary waves.



Here one can see that

$$cos(\theta) = \frac{distance\ of\ light}{distance\ of\ particle} = \frac{c't}{\beta ct} = \frac{1}{\beta n}.$$

In the last step it is used that the speed of light in medium is given by $c' = c/n$, where n is the refractive index of the medium.

b) For relativistic particles - $\beta \approx 1$ - the opening angle for water is 41.30° and for 1.38°.

c) To induce Cherenkov light the particles' speed needs to be higher than the speed of light in the medium. For water that is $c' = c/n < \beta c \to 1/n = 0.751 < \beta$. From relativistic kinematics and by inserting $\beta > 0.751$ one gets in natural units ($c = 1$)

$$E_{kin} = K = E - E_0 = (\gamma - 1)m_0 > 0.514 \cdot m_0.$$

Now inserting the restmasses of the particles one gets $K_\mu > 54.414\ MeV$, $K_p > 483.210\ MeV$, $K_n > 483.876\ MeV$, $K_e > 0.263\ MeV$.

d)

$$
\begin{aligned}
\frac{d^2N}{dxd\lambda} &= \frac{2\pi\alpha}{\lambda^2}\left(1-\frac{1}{\beta^2n^2}\right) \\
\frac{dN}{dx} &= \int_{300\ nm}^{550\ nm}\frac{2\pi\alpha}{\lambda^2}\left(1-\frac{1}{\beta^2n^2}\right)d\lambda \\
&= \frac{2\pi\alpha}{660\ nm}\left(1-\frac{1}{\beta^2n^2}\right) \\
&= 3.026\cdot 10^{-5}\ nm^{-1} && \text{for water} \\
&= 4.029\cdot 10^{-8}\ nm^{-1} && \text{for air} \\
\rightarrow N &= 3.026\cdot 10^{4} && \text{for water} \\
N &= 4.029\cdot 10^{1}\approx 40 && \text{for air}
\end{aligned}
$$

Integrating the $1/\lambda^2$ term from 300 nm to 550 nm gives the factor $1/(660\ nm)$. Again it is assumed $\beta \approx 1$. $\alpha = 1/137$ is used in this calculation.

e) The energy loss is given by the number of particles multiplied by their energy $E = hc/\lambda$. On gets

$$
\begin{aligned}
\frac{d^2E_{loss}}{dxd\lambda} &= \frac{2\pi\alpha hc}{\lambda^3}\left(1-\frac{1}{\beta^2n^2}\right) \\
\frac{dE_{loss}}{dx} &= \int_{300\ nm}^{550\ nm}\frac{2\pi\alpha hc}{\lambda^3}\left(1-\frac{1}{\beta^2n^2}\right)d\lambda \\
\frac{dE_{loss}}{dx} &= \frac{34\pi\alpha hc}{4356000\ nm^2}\left(1-\frac{1}{\beta^2n^2}\right) \\
&= 9.665\cdot 10^{1}\frac{eV}{m} && \text{for water} \\
&= 1.287\cdot 10^{-1}\frac{eV}{m} && \text{for air}
\end{aligned}
$$

Again $\beta \approx 1$ and $\alpha = 1/137$ is used.

f) The source given on the sheet shows that an IceTop tank contains a vertical 90 cm of ice. A VEM thus produces $3.026\cdot 10^4 \cdot 0.9 = 2.723\cdot 10^4$ photons. The tanks at Pierre Auger observatory have a depth of 120 cm (Wikipedia) resulting in $3.631\cdot 10^4$ photons.

g) The light emitted by fluorescence is isotropic and thus the fraction in direction of the Cherenkov cone is given by $\theta/180°$. The total number of photons can be estimated as 4.8 m^{-1} (script). The photons produced from fluorescence in direction of the Cherenkov cone is then given by $d/m \cdot 4.8 \cdot \theta/180° = 184$.

Using $\theta = 1.38$ from part b). The number of photons from Cherenkov light is $d/m \cdot 40.29 = 2.015 \cdot 10^5$. So 0.091 % of the photons arriving at the ground are produced by fluorescence.
Here the results from the previous tasks are used. That means in particular the result in exercise d) of approximately 40 Cherenkov photons per meter in air.

Exercise 5