Exercise 1

a) The shower generation after a propagated distance X is $n = X/X_{1/2}$, where $X_{1/2} = X_0 \ln(2)$ is the characteristic length after which the energy is halved and X_0 is the characteristic interaction length for pair production and Bremsstrahlung. In the lecture is has been shown that when the shower maximum is reached $n = n_{\max} = \ln(E_0/E_c)/\ln(2)$ so that

$$X_{\max} = n_{\max} X_{1/2} = \frac{\ln(E_0/E_c)}{\ln(2)} X_0 \ln(2) = \ln\left(\frac{E_0}{E_c}\right) X_0 \tag{1}$$

b) With an approximate atomic number of $Z \approx 7.2$ for air, the critical energy becomes

$$E_c = \frac{710}{Z + 0.92}\text{MeV} \approx 89\text{MeV} \tag{2}$$

so that (assuming as done in Ex. 4 of Sheet 4 that $X_0 = 36$ g/cm^2) $X_{\max}(E_0 = 1\text{TeV}) = 335.4$ g/cm^2 and $X_{\max}(E_0 = 30\text{GeV}) = 209.1$ g/cm^2.

c) In the Heitler model

$$R_{\text{el}} = \frac{dX_{\max}}{d\ln(E_0)} = X_0 \frac{d(\ln(E_0) - \ln(E_c))}{d\ln(E_0)} = X_0 \tag{3}$$

so that for and increase in the energy E_0 of order $e \approx 2.7$, the maximum length $X_{\max}$ increases of a factor X_0.

d) According to the definitions on the exercise sheet

$$X_{\max}^{\text{A}} = X_{\max}^{\text{p}} - X_0 \ln(\text{A}) \tag{4}$$

with

$$X_{\max}^{\text{p}} = \lambda_{\text{I}} \ln(2) + X_0 \ln\left(\frac{E_0}{6N_\pi E_c^\gamma}\right) \tag{5}$$

where λ_{I} is the interaction length

$$\lambda_{\text{I}} = \left(90 - 9\ln\left(\frac{E_0}{\text{EeV}}\right)\right) \frac{\text{g}}{\text{cm}^2} \tag{6}$$

and

$$N_\pi = \left(\frac{E_0}{\text{PeV}}\right)^{1/5} \tag{7}$$

Putting it all together we obtain that

$$
\begin{aligned}
X_{\max}^{\mathrm{A}} &= X_{\max}^{\mathrm{p}} - X_0 \ln(\mathrm{A}) = \\
&= \lambda_{\mathrm{I}} \ln(2) + X_0 \ln\left(\frac{E_0}{6 N_\pi E_{\mathrm{c}}^{\gamma}}\right) - X_0 \ln(\mathrm{A}) = \\
&= \ln(2) \left(90 - 9 \ln\left(\frac{E_0}{\mathrm{EeV}}\right)\right) \frac{\mathrm{g}}{\mathrm{cm}^2} + X_0 \ln\left(\frac{E_0}{6 E_{\mathrm{c}}^{\gamma}} \frac{\mathrm{PeV}^{1/5}}{E_0^{1/5}}\right) + \\
&\quad - X_0 \ln(\mathrm{A}) = \\
&\approx \left[62 - 6 \ln\left(\frac{E_0}{\mathrm{EeV}}\right) - 36 \ln(6) + 36 \ln\left(\frac{E_0}{E_{\mathrm{c}}^{\gamma}}\right) + \right. \\
&\quad \left. + 36 \ln\left(\frac{\mathrm{PeV}^{1/5}}{E_0^{1/5}}\right) - 36 \ln(\mathrm{A})\right] \frac{\mathrm{g}}{\mathrm{cm}^2} = \\
&\approx \left\{-2 - 36 \ln(\mathrm{A}) + 6 \left[- \ln\left(\frac{E_0}{\mathrm{EeV}}\right) + 6 \ln\left(\frac{E_0}{E_{\mathrm{c}}^{\gamma}}\right) + \right.\right. \\
&\quad \left.\left. + \frac{6}{5} \ln\left(\frac{\mathrm{PeV}}{E_0}\right)\right]\right\} \frac{\mathrm{g}}{\mathrm{cm}^2} = \\
&\approx \left[-2 - 36 \ln(\mathrm{A}) + 6 \ln\left(\frac{10^{18}\mathrm{eV}}{E_0} \frac{E_0^6}{(E_{\mathrm{c}}^{\gamma})^6} \frac{(10^{15}\mathrm{eV})^{6/5}}{E_0^{6/5}}\right)\right] \frac{\mathrm{g}}{\mathrm{cm}^2} = \\
&\approx \left[-2 - 36 \ln(\mathrm{A}) + 6 \ln\left(\frac{E_0^4 10^{36} \mathrm{eV}^2}{(E_{\mathrm{c}}^{\gamma})^6}\right)\right] \frac{\mathrm{g}}{\mathrm{cm}^2} = \\
&\approx \left[-2 - 36 \ln(\mathrm{A}) + 12 \ln\left(\frac{E_0^2 \mathrm{EeV}}{(E_{\mathrm{c}}^{\gamma})^3}\right)\right] \frac{\mathrm{g}}{\mathrm{cm}^2}
\end{aligned}
$$

so that

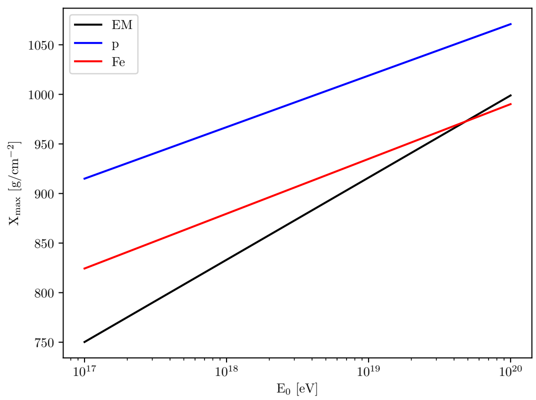
$$c^{\mathrm{A}} = [-2 - 36 \ln(\mathrm{A})] \frac{\mathrm{g}}{\mathrm{cm}^2} \tag{8}$$

and

$$R_{\mathrm{el}} = 12?? \tag{9}$$

which is independent on A, as to prove.

e) By plotting $X_{\max}$ for photons-, protons- and iron-induced air showers we obtain the following plot:



f)

Exercise 2

a) By solving the integral

$$N_{\mathrm{e}}(x) = 2\pi \int r \rho_{\mathrm{e}}(r, x) dr \tag{10}$$

with

$$\rho_{\mathrm{e}}(r, x) = \frac{N_{\mathrm{e}}(x) C_1(s)}{r_{\mathrm{s}}^2} \left(\frac{r}{r_{\mathrm{s}}}\right)^{s-2} \left(\frac{r_{\mathrm{s}} + r}{r_{\mathrm{s}}}\right)^{s-4.5} \tag{11}$$

one obtains that

$$C_1(s) = \frac{\Gamma(4.5 - s)}{2\pi \Gamma(s) \Gamma(4.5 - 2s)} \tag{12}$$

so that then $C_1(s = 1.25) = 0.45$.

b) According to the lecture notes, the lateral distribution of the particle density for hadronic showers is given by

$$\rho(r, x) = \frac{N(x) C_1(s)}{r_{\mathrm{s}}^2} \left(\frac{r}{r_{\mathrm{s}}}\right)^{s-2} \left(\frac{r_{\mathrm{s}} + r}{r_{\mathrm{s}}}\right)^{s-4.5} \left(1 + C_2 \left(\frac{r}{r_{\mathrm{s}}}\right)^{\delta}\right) \tag{13}$$

Using $\delta = 1$ it becomes

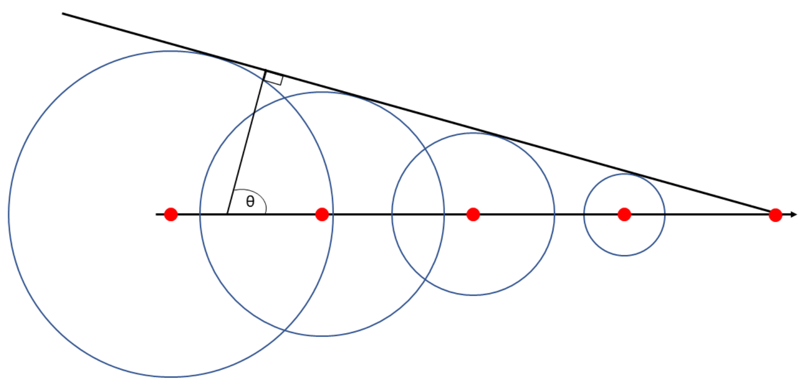
$$\rho(r,x) = \frac{N(x)C_1(s)}{r_\mathrm{s}^2}\left(\frac{r}{r_\mathrm{s}}\right)^{s-2}\left(\frac{r_\mathrm{s}+r}{r_\mathrm{s}}\right)^{s-4.5} + \\ + \frac{N(x)C_1(s)C_2}{r_\mathrm{s}^2}\left(\frac{r}{r_\mathrm{s}}\right)^{s-1}\left(\frac{r_\mathrm{s}+r}{r_\mathrm{s}}\right)^{s-4.5} \tag{14}$$

and, analogously to exercise a), we obtain

$$C_1(s) = \frac{\Gamma(4.5-s)}{2\pi\Gamma(s)\Gamma(4.5-2s)} + C_2?? \tag{15}$$

Exercise 4

a) According to Huygens' principle each point of a wavefront is source of a elementary wave. In the graphic below it is shown how a particle moves from left to right and emits light at each position (shown are just a few examples). The resulting wavefront is thus the upper line, tangent to all elementary waves.



Here one can see that

$$cos(\theta) = \frac{distance\ of\ light}{distance\ of\ particle} = \frac{c't}{\beta ct} = \frac{1}{\beta n}.$$

In the last step it is used that the speed of light in medium is given by $c' = c/n$, where n is the refractive index of the medium.

b) For relativistic particles - $\beta \approx 1$ - the opening angle for water is 41.30° and for 1.38°0.

c) To induce Cherenkov light the particles' speed needs to be higher than the speed of light in the medium. For water that is $c' = c/n < \beta c \rightarrow 1/n < \beta$. From relativistic kinematics one gets in natural units ($c = 1$)

$$E_{kin} = E - E_0 = (\gamma - 1)m_0$$