Exercise 1

a)

$$
\begin{aligned}
E_{kin} &= \gamma m_0 c^2 - m_0 c^2 = m_0 c^2(\gamma - 1) \\
\frac{E_{kin}}{m_0 c^2} &= \frac{1}{\sqrt{1-\beta^2}} - 1 \\
\beta &= \sqrt{1 - \left(\frac{E_{kin}}{m_0 c^2} + 1\right)^{-2}}
\end{aligned}
$$

Now rewriting the energy density as

$$\rho_E = \rho_{E_0} + \rho_{E_{kin}}$$

where ρ_{E_0} is the energy density occurring from the rest mass and $\rho_{E_{kin}}$ is the kinetic energy density. Analogous to the calculation in the script one gets:

$$
\begin{aligned}
\rho_E = \rho_{E_0} &+ \int_{E_{min}}^{\infty} dE_{kin}\, \frac{dn_{CR}}{dE_{kin}} E_{kin} \\
= \rho_{E_0} &+ \int_{E_{min}}^{\infty} dE_{kin}\, \frac{4\pi}{\beta c} \frac{d\Phi}{dE_{kin}} E_{kin} \\
= \rho_{E_0} &+ \frac{4\pi\Phi_0}{c} \int_{E_{min}}^{\infty} dE_{kin}\, \frac{E_{kin}^{-\gamma+1}}{\sqrt{1 - \left(\frac{E_{kin}}{m_0 c^2} + 1\right)^{-2}}}
\end{aligned}
$$

In the last step the power-law for the kinetic energy flux is inserted and the expression for β from before.

b) From the graph one can read the points $P_1 = (10|2 \cdot 10^{-3})$ and $P_2 = (10^2|5 \cdot 10^{-6})$. This results in $\Phi_0 = 0.8\ (cm^2\ s\ sr\ GeV)^{-1}$ and $\gamma = 2.602$.

c) Inserting into the integral values from b) gives a value

$$\int_{1\ GeV}^{\infty} dE_{kin}\, \frac{E_{kin}^{-\gamma+1}}{\sqrt{1 - \left(\frac{E_{kin}}{m_0 c^2} + 1\right)^{-2}}} = 0.9182$$

Resulting in $\rho_{E_{kin}} = 3.0790 \cdot 10^{-10}\ \frac{GeV}{cm^3}$.

d) C is approximately 0.1. For constant flux one finds the same integral just with $\gamma = 0$. This again results in $\rho_{E_kin} = 3.6552\ \frac{GeV}{cm^3}$.

Exercise 4

a) The energy gain per time is given by

$$\begin{aligned}\frac{dE}{dt} =& Collisionrate \cdot \delta E \\ =& \frac{\xi E}{t_{cycle}} = \frac{E}{\alpha\ t_{esc}} \\ => E(t) =& E_0 \cdot \exp\left(\frac{\xi}{t_{cycle}} \cdot t\right) = E_0 \cdot \exp\left(\frac{t}{\alpha\ t_{esc}}\right)\end{aligned}$$

Here ξ is the relative energy gain per acceleration. t_{cycle} is the time between to cycles, α is the spectral index and t_{esc} is the escape time.

b) The acceleration time is given by

$$t_{acceleration} = \frac{t_{cycle}}{\xi} \cdot ln\left(\frac{E}{E_0}\right)$$

$\xi = \frac{4}{3}\beta_V^2 = 3.7088 \cdot 10^{-8}$, t_{cycle} is limited by d to $\approx 3\ a$. So one gets

$$t_{acceleration} = 1.21 \cdot 10^9\ a$$

c)

$$\frac{E(t = 6 \cdot 10^6\ a)}{E_0} = \exp\left(\frac{\xi}{t_{cycle}} \cdot 6 \cdot 10^6\ a\right) = 1.077$$

d) The new energy gain is given by $\xi = 3.7088 \cdot 10^{-6}$. The new cycle time $t_{cycle} \approx 3.2616 \cdot 10^6\ a$. One gets for the acceleration from 1 GeV to $10^{19.5}\ eV$ a time of $2.1261 \cdot 10^{13}\ a$.
For the acceleration from $10^{15}\ eV$ one gets a time of $0.1122 \cdot 10^{12}\ a$.