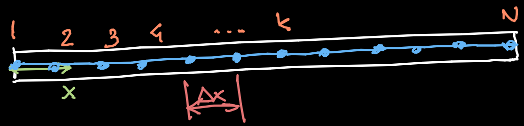
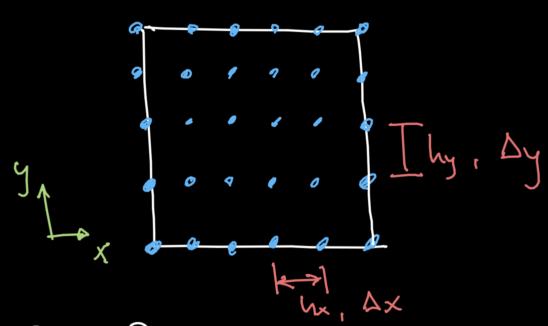
Método das diferenças finitas





Eq. de calor: $\rho c \dot{T} + \nabla \cdot (-k \nabla T) - s = 0$

CASO ESTACIONÁRIO s/ GERAÇÃO DE CALOR

$$\nabla \cdot (-k \nabla T) = 0$$

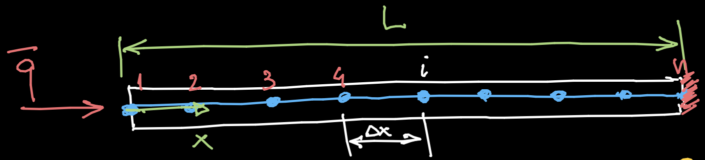
1D

$$-k \boxed{\frac{d^2 T}{d x^2}} = 0$$

2D

$$-k \left(\boxed{\frac{\partial^2 T}{\partial x^2}} + \boxed{\frac{\partial^2 T}{\partial y^2}} \right) = 0$$

Problema de transferência de calor unidimensional transiente



$$\rho c \boxed{\frac{\partial T}{\partial t}} - k \boxed{\frac{\partial^2 T}{\partial x^2}} = 0$$

C.I. $T(x, t=0) = T_0$

$$\text{C.C.} \begin{cases} q(x=0,t) = \bar{q} \\ q(x=L,t) = 0 \quad \text{(adiabático)} \end{cases}$$

Aproximação das derivadas:

t^k
$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i+1}^k - 2T_i^k + T_{i-1}^k}{\Delta x^2}$$

t^{k+1}

$t^k + \Delta t$
$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^{k+1} - 2T_i^{k+1} + T_{i-1}^{k+1}}{\Delta x^2}$$

$$\boxed{\frac{\partial T}{\partial t} = \frac{T_i^{k+1} - T_i^k}{\Delta t}}$$

$$\Delta t = t^{k+1} - t^k$$

Problema de transferencia de calor unidimensional transiente

$$\rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad \alpha = \frac{k}{\rho c}$$

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i+1}^{k} - 2T_i^{k} + T_{i-1}^{k}}{\Delta x^2} \qquad \frac{\partial T}{\partial t} = \frac{T_i^{k+1} - T_i^{k}}{\Delta t}$$

$$\rho c \left(\frac{T_i^{k+1} - T_i^{k}}{\Delta t} \right) - k \left(\frac{T_{i+1}^{k} - 2T_i^{k} + T_{i-1}^{k}}{\Delta x^2} \right) = 0$$

$$T_i^{k+1} - T_i^{k} = \frac{\alpha \Delta t}{\Delta x^2} \left(T_{i+1}^{k} - 2T_i^{k} + T_{i-1}^{k} \right)$$

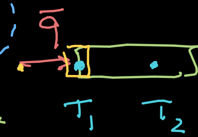
$$\boxed{T_i^{k+1} = \frac{\alpha \Delta t}{\Delta x^2} T_{i+1}^{k} + \left(1 - 2\alpha \frac{\Delta t}{\Delta x^2} \right) T_i^{k} + \frac{\alpha \Delta t}{\Delta x^2} T_{i-1}^{k}}$$

EXPLÍCITO

Problema de transferência de calor unidimensional transiente

$$T_i^{k+1} = \frac{\alpha \Delta t}{\Delta x^2} T_{i+1}^k + \left(1 - 2\alpha \frac{\Delta t}{\Delta x^2}\right) T_i^k + \frac{\alpha \Delta t}{\Delta x^2} T_{i-1}^k$$

T_1^{k+1} $\qquad T_2^k \qquad T_1^k \qquad \bar{q}\,\frac{2\Delta x}{k} + T_2^k$



Condiçoés de contorno

$$q(x=0,t) = \bar{q}$$
$$q(x=L,t) = 0$$

EXPLÍCITO

Lei de Fourier: $q = -k\nabla T \overset{1D}{=} -k\frac{\partial T}{\partial x}$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} \approx \frac{T(\Delta x,t) - T(-\Delta x,t)}{2\Delta x} = \frac{T_2^k - T_0^k}{2\Delta x}$$

Da condição de contorno: $-k\left(\frac{T_2^k - T_0^k}{2\Delta x}\right) = \bar{q}$

$$T_0^k = \bar{q}\,\frac{2\Delta x}{k} + T_2^k$$

Problema de transferência de calor unidimensional transiente

$$\rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0$$

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i+1}^{k+1} - 2T_i^{k+1} + T_{i-1}^{k+1}}{\Delta x^2} \qquad \frac{\partial T}{\partial t} = \frac{T_i^{k+1} - T_i^{k}}{\Delta t}$$

$$\rho c \left(\frac{T_i^{k+1} - T_i^{k}}{\Delta t} \right) - k \left(\frac{T_{i+1}^{k+1} - 2T_i^{k+1} + T_{i-1}^{k+1}}{\Delta x^2} \right) = 0$$

$$\boxed{-\alpha \frac{\Delta t}{\Delta x^2} T_{i+1}^{k+1} + \left(1 + 2\alpha \frac{\Delta t}{\Delta x^2} \right) T_i^{k+1} - \alpha \frac{\Delta t}{\Delta x^2} T_{i-1}^{k+1} = T_i^{k}}$$

$$\Downarrow$$

IMPLÍCITO

$$\underline{\underline{A}}\ \underline{T} = \underline{b}$$

Problema de transferência de calor unidimensional transiente

$$\boxed{-\alpha\frac{\Delta t}{\Delta x^2}T_{i+1}^{k+1}+\left(1+\frac{2\alpha\Delta t}{\Delta x^2}\right)T_i^{k+1}-\alpha\frac{\Delta t}{\Delta x^2}T_{i-1}^{k+1}=T_i^k}$$

Condições de contorno

$$q(x=0,t)=\bar{q}$$
$$q(x=L,t)=0$$

IMPLÍCITO

Lei de Fourier: $q=-k\nabla T=-k\dfrac{\partial T}{\partial x}$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0}\approx\frac{T(\Delta x,t)-T(-\Delta x,t)}{2\Delta x}=\frac{T_2^{k+1}-T_0^{k+1}}{2\Delta x}$$

Da condição de contorno: $-k\left(\dfrac{T_2^{k+1}-T_0^{k+1}}{2\Delta x}\right)=\bar{q}$

$$T_0^{k+1}=\bar{q}\frac{2\Delta x}{k}+T_2^{k+1}$$

Von Neumann Stability Analysis

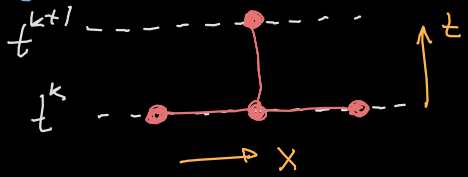
$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \qquad \alpha = \frac{k}{\rho c}$$

EXPLÍCITO

FTCS

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(u_{j+1}^k - 2u_j^k + u_{j-1}^k\right)$$

$$\frac{\varepsilon_j^{k+1} - \varepsilon_j^k}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(\varepsilon_{j+1}^k - 2\varepsilon_j^k + \varepsilon_{j-1}^k\right)$$



Fourier series :

$$\varepsilon(x,t) = \sum_{m=1}^{M} A_m e^{at} e^{i\beta_m x} = \sum_{m=1}^{M} \varepsilon_m$$

$$\varepsilon_j^{n+1} = \sum A_m e^{a(t+\Delta t)} e^{i\beta_m x} = \varepsilon_j^n e^{a\Delta t}$$

$$\varepsilon_{j+1}^n = \sum A_m e^{at} e^{i\beta_m (x+\Delta x)} = \sum \varepsilon_m \left(\sum A_m e^{at} e^{i\beta_m \Delta x}\right)$$

...

Von Neumann Stability Analysis

$$\frac{\varepsilon_j^{k+1} - \varepsilon_j^{k}}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(\varepsilon_{j+1}^{k} - 2\varepsilon_j^{k} + \varepsilon_{j-1}^{k}\right)$$

$$\varepsilon_j^{n+1} = \sum A_m e^{a(t+\Delta t)} e^{i\beta_m x} = \varepsilon_j^{n} e^{a\Delta t}$$

$$\varepsilon_{j+1}^{n} = \sum A_m e^{at} e^{i\beta_m (x+\Delta x)} = \sum \varepsilon_m \left(\sum A_m e^{at} e^{i\beta_m \Delta x}\right)$$

$$\dots$$

$$e^{a\Delta t} - 1 = \alpha \frac{\Delta t}{\Delta x^2}\left[e^{\beta_m \Delta x\, i} - 2 + e^{-\beta_m \Delta x\, i}\right]$$

Stability criteria $\quad G = \dfrac{\varepsilon_j^{n+1}}{\varepsilon_j^{n}} \qquad |G| \leq 1$

Von Neumann Stability Analysis

$$e^{a\Delta t} - 1 = \alpha \frac{\Delta t}{\Delta x^2}\left[e^{\beta m \Delta x\, i} - 2 + e^{-\beta m \Delta x\, i}\right]$$

Stability criteria $\quad G = \frac{\varepsilon_j^{n+1}}{\varepsilon_j^{n}} = e^{a\Delta t} \qquad |G| \leq 1$

$$G = e^{a\Delta t} = \frac{\alpha \Delta t}{\Delta x^2}\left[e^{\beta m \Delta x\, i} - 2 + e^{-\beta m \Delta x\, i}\right] + 1$$

$$= 2\alpha \frac{\Delta t}{\Delta x^2}\left[\cos \beta m \Delta x - 1\right] + 1$$

$$= 1 - 4\alpha \frac{\Delta t}{\Delta x^2} \sin^2\left(\beta m \frac{\Delta x}{2}\right)$$

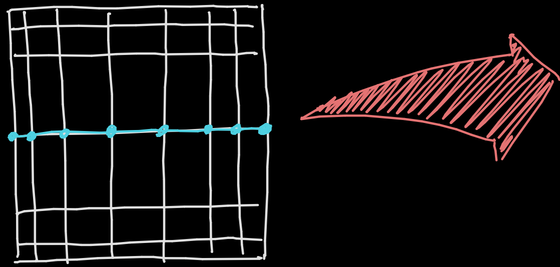
$$\boxed{\alpha \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}}$$

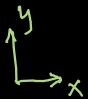
$$|G| \leq 1 \Rightarrow \quad 1 - 4\alpha \frac{\Delta t}{\Delta x^2} \sin^2\left(\beta m \frac{\Delta x}{2}\right) \leq 1$$

$$-1 + 4\alpha \frac{\Delta t}{\Delta x^2} \sin^2\left(\beta m \frac{\Delta x}{2}\right) \leq 1$$

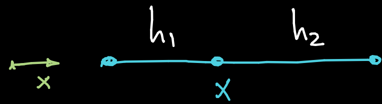
Malhas não-uniformes

CASO 2D





CASO 1D



$$u(x+h_2) = u(x) + h_2\, u'(x) + \frac{h_2^2}{2}\, u''(x) + \dots \qquad h_1^2$$

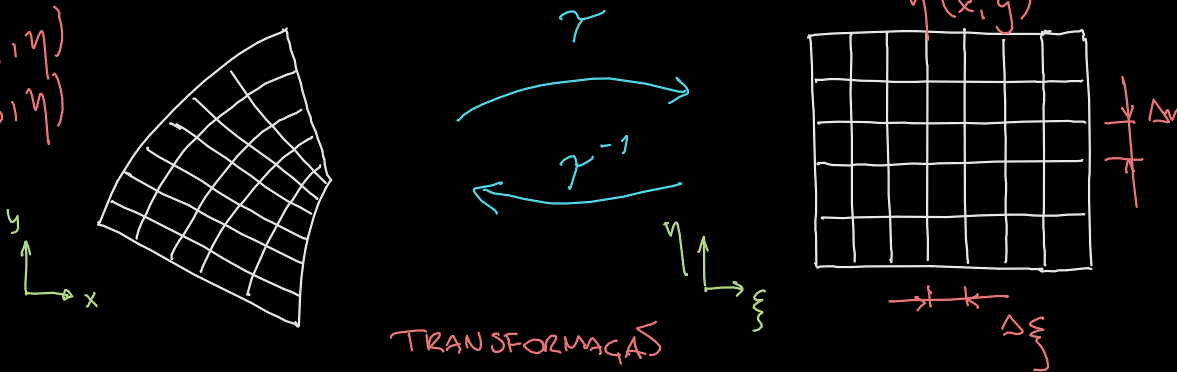
$$u(x-h_1) = u(x) - h_1\, u'(x) + \frac{h_1^2}{2}\, u''(x) + \dots \qquad h_2^2$$

$$\boxed{u'(x) = \frac{h_1^2\, u(x+h_2) + (h_1^2 - h_2^2)\, u(x) - h_2^2\, u(x-h_1)}{h_1^2 h_2 + h_1 h_2^2}}$$

Outros domínios

$x(\xi,\eta)$
$y(\xi,\eta)$

$\xi(x,y)$
$\eta(x,y)$



TRANSFORMAÇÃO CONFORME

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \qquad\qquad \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \; ; \; \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial x^2} = \boxed{\frac{\partial^2 u}{\partial \xi^2}}\boxed{\left(\frac{\partial \xi}{\partial x}\right)^2} + \boxed{\frac{\partial^2 u}{\partial \eta \partial \xi}}\boxed{\left[\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\right]} + \boxed{\frac{\partial^2 u}{\partial \eta^2}}\boxed{\left(\frac{\partial \eta}{\partial y}\right)^2}$$