69. $\int \log|f| \, d\mu \leq \log\left(\int |f| \, d\mu \right)$ (espacio de proba $\mu(X)=1$, $f \in \mathcal{L}_1$)

Hint: $\log(t) \leq t-1, \ \forall t \in (0,\infty)$.

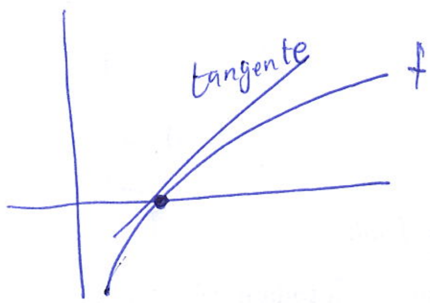
Razón: $f(t) := \log(t), \quad t \in [0,\infty)$

Recta tangente en $t=1$:

$$y = f(1) + f'(1)(t-1) = \log(1) + \frac{1}{1}(t-1) = t-1$$

log cóncava hacia abajo $\Rightarrow$

$$\log(t) \leq t-1 \quad \text{igualdad} \Leftrightarrow t=1$$



Ojo: Ya que $\log(0) = -\infty$ la desigualdad se extiende:

$$\log(t) \leq t-1, \quad t \in [0,\infty)$$

Si $\int |f| d\mu = 0 \Rightarrow |f| = 0$ c.d $\Rightarrow \log|f| = -\infty$ c.d. $\Rightarrow \int \log|f| = -\infty$

$$\log\left(\int |f| d\mu\right) = \log(0) = -\infty$$

Suponemos $\int |f| d\mu > 0$. Tomar $t = \frac{|f|}{\|f\|_1}$

$$\Rightarrow \log\frac{|f|}{\|f\|_1} \leq \frac{|f|}{\|f\|_1} - 1 \Rightarrow \int \log\frac{|f|}{\|f\|_1} d\mu \leq \frac{\int |f| d\mu}{\|f\|_1} - 1 = 1-1 = 0$$

$$\| \qquad \int (\log|f| - \log\|f\|_1) d\mu$$

$$\Rightarrow \boxed{\int \log|f| d\mu \leq \log\|f\|_1}$$

1) $\lim_{r\downarrow 0} \|f\|_r = \exp\left[\int \log|f|\,d\mu\right] = \lim_{n\to\infty} \left[1+\frac{\int\log|f|}{n}\right]^n = \lim_{r\to 0}\left[1+\frac{\int \log|f|\,r}{\cancel{}}\right]^{1/r}$

$\frac{1}{r}=n$

Notar: Como $\mu(X)=1$, $0<r<s\leq 1 \underset{\text{ejer 62}}{\Rightarrow} \|f\|_r \leq \|f\|_s \leq \|f\|_1$

$\therefore \lim_{r\downarrow 0}\|f\|_r = \inf_{r>0}\{\|f\|_r\}$ existe.

Hint: $\frac{t^r-1}{r} \underset{r\downarrow 0}{\downarrow} \log(t) \quad \forall t\in[0,\infty)$

$t=0$ por convención $0^r=0$, $-\frac{1}{r}\underset{r\to 0}{\to}-\infty$.
Podemos suponer $0<t<\infty$.

Razón. $g(r) := t^r = e^{r\log(t)}$, $g'(r) = t^r\log(t)$

$$\frac{t^r-1}{r} = \frac{g(r)-g(0)}{r-0} = g'(\xi) \qquad 0<\xi<t$$
$$= \xi^r \log(t)$$

$$\lim_{r\downarrow 0}\frac{t^r-1}{r} = \lim_{r\downarrow 0}\xi^r\log(t) = \log(t)$$

$$\frac{d}{dr}\left(\frac{t^r-1}{r}\right) = \frac{r(t^r\log(t)) - (t^r-1)}{r^2}$$
$$= \frac{t^r[r\log(t)-1]+1}{r^2}$$

para $r\to 0$
$r\log(t)-1<0$
a partir de cierto instante

$\therefore \frac{t^r-1}{r}$ es decreciente en r.

Como $\int|f|\,d\mu<\infty \Rightarrow \int\frac{|f|-1}{1}\,d\mu<\infty$. Por ejer (48):

$$\lim_{r\downarrow 0}\int\frac{|f|^r-1}{r}\,d\mu = \int\log|f|\,d\mu \; ;$$

or otro lado:

$$\int \log|f|\,d\mu = \frac{1}{r}\int \log|f|^r\,d\mu \underset{(i)}{\leq} \frac{1}{r}\log\Big[\underbrace{\int |f|^r\,d\mu}_{=\,t}\Big] \leq \frac{1}{r}(t-1)$$

$$\frac{1}{r}(t-1) \overset{\mu(X)=1}{=} \frac{1}{r}\left(\int |f|^r - 1\,d\mu\right) \xrightarrow{r \downarrow 0} \int \log|f|\,d\mu$$

$$\therefore \int \log|f|\,d\mu = \lim_{r\downarrow 0} \frac{1}{r}\log\Big[\int |f|^r\,d\mu\Big] = \lim_{r\downarrow 0} \log\|f\|_r$$

$$\Rightarrow \lim_{r\downarrow 0} \|f\|_r = \exp\Big[\int \log|f|\,d\mu\Big]$$