83 - (i)

$$N \subseteq [0, \infty) \,, \quad \lambda^*(N) = 0$$

$$N^2 := \{ x^2 \mid x \in N \}$$

$$\sqrt{N} := \{ \sqrt{x} \mid x \in N \}$$

Probar $\lambda^*(N^2) = \lambda^*(\sqrt{N}) = 0$.

<u>Dem</u>

$$N = \bigcup_{n=1}^{\infty} \left[\tfrac{1}{n}, \infty\right) \cap N = \bigcup_{n=1}^{\infty} N_n$$

$$\Rightarrow \forall n, \quad \lambda^*(N_n) = 0$$

Además:

$$\sqrt{N} = \bigcup_{n=1}^{\infty} \sqrt{N_n}$$

Es suf. probar $\lambda^*(\sqrt{N_n}) = 0$

$$\lambda^*(N_n) = 0 = \inf \left\{ \sum_{i=1}^{\infty} \lambda(A_i) \;\middle|\; \begin{array}{l} (A_i) \subseteq \mathcal{A} \\ N_n \subseteq \bigcup_{i=1}^{\infty} A_i \end{array} \right\}$$

$$\mathcal{A} = \{ \text{uniones finitas disjuntas de } (a, b] \}$$

Dada $\varepsilon > 0$ encontrar

$$A_i = (a_i, b_i]$$

[illegible]

$$N_n \cap [1,\infty) = N_n \subseteq \bigcup_{i=1} [a_i, b_i]$$

$$\sum_{i=1}^{\infty} b_i - a_i < \varepsilon$$

$$N_n \subseteq \bigcup_{i=1}^{\infty} [a_i, b_i]$$

$$\Rightarrow \sqrt{N_n} \subseteq \bigcup_{i=1}^{\infty} (\sqrt{a_i}, \sqrt{b_i}]$$

$$(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) = b - a$$

$$\sqrt{b} - \sqrt{a} = \frac{b-a}{\sqrt{b} + \sqrt{a}}$$

$$\sqrt{b_i} - \sqrt{a_i} = \frac{b_i - a_i}{\sqrt{b_i} + \sqrt{a_i}} \leq 2\sqrt{n}\,(b_i - a_i)$$

Obs: se puede suponer $a_i, b_i \geq \frac{1}{n}$

$$\Rightarrow \frac{1}{\sqrt{a_i} + \sqrt{b_i}} < 2\sqrt{n}$$

$\Rightarrow$

$$\sum_{i=1}^{\infty} \sqrt{b_i} - \sqrt{a_i} \le 2\sqrt{n}\left(\sum_{i=1}^{\infty} b_i - a_i\right)$$
$$< 2\sqrt{n}\,\varepsilon$$

$\therefore$ $\{(\sqrt{a_i}, \sqrt{b_i}]\}_{i=1}^{\infty}$ cubierta medible de $\sqrt{N_n}$, $(\sqrt{a_i}, \sqrt{b_i}] \in \mathcal{A}$ $\forall i$

y $\sum_{i=1}^{\infty} \sqrt{b_i} - \sqrt{a_i} < 2\sqrt{n}\,\varepsilon$

Como $\varepsilon > 0$ es arbitraria

$$\Longrightarrow \quad \lambda^*(\sqrt{N_n}) = 0$$

Para N^2 es similar

$$N \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i], \quad \lambda^*(N) = 0, \quad \sum_{i=1}^{\infty} b_i - a_i < \varepsilon$$

$$\Downarrow$$

$$N^2 \subseteq \bigcup_{i=1}^{\infty} (a_i^2, b_i^2]$$

Necesito $\sum_{i=1}^{\infty} b_i^2 - a_i^2 < \varepsilon$

$$(b_i - a_i)\underbrace{(b_i + a_i)} = b_i^2 - a_i^2$$

$$N = \bigcup_{n=1} N \cap [0, n] = \bigcup_{n=1} N_n$$

Para N_n, s.g. $(a_i, b_i] \subseteq [0, n]$

$$\therefore \underline{b_i^2 - a_i^2} \leq 2n \, \underline{(b_i - a_i)}$$

Fin