Ejercicio 89

$\mu : \mathcal{A} \to \mathbb{R}$ casi medida

$\mu^* : \mathcal{P}(X) \to \mathbb{R}$ medida exterior

$\mathcal{A}^* =$ medibles $\supseteq \Sigma(\mathcal{A})$

$\bar{\mu} = \mu^*|_{\mathcal{A}^*}$

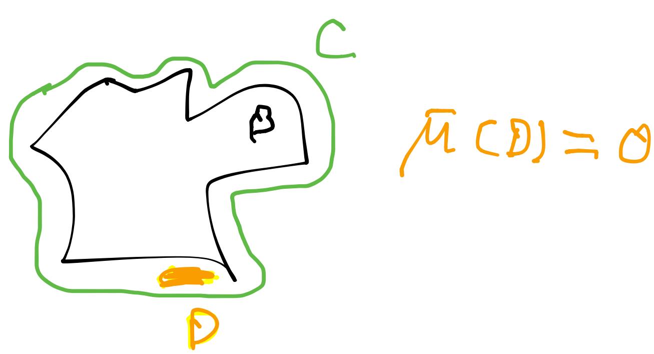
Ejer 86:

$B \subseteq X$, $\mu^*(B) < +\infty$ $\Rightarrow$ $\exists\, C \in \Sigma(\mathcal{A})$:

(i) $B \subseteq C$

(ii) $D \in \Sigma(\mathcal{A})$, $D \subseteq C \setminus B \Rightarrow \bar{\mu}(D) = 0$

(iii) $\bar{\mu}(C) = \mu^*(B)$



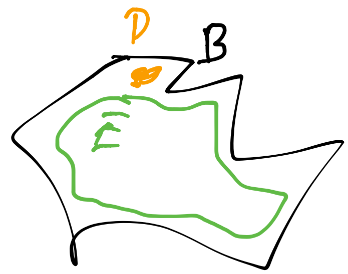
C es una cubierta medible

Núcleo medible: $E \in \Sigma(\mathcal{A})$:

(i) $E \subseteq B$

(ii) $D \subseteq B \setminus E$, $D \in \Sigma(\mathcal{A})$ $\Rightarrow \bar{\mu}(D) = 0$

(iii) $\underbrace{\mu_*(B) = \bar{\mu}(E)}_{\text{ejer 87}}$



$\bar{\mu}(D) = 0$

Ejer 89: $A \in \mathcal{A}^*$, $\bar{\mu}(A) < +\infty$
$B \subseteq A$.

Probar:

$B \in \mathcal{A}^* \iff \mu^*(A) = \mu^*(B) + \mu^*(A \setminus B)$

$\Rightarrow$]: Def. que $B \in \mathcal{A}^*$

$\Leftarrow$]: Ejer 87-ii:

$\mu^*(B) < +\infty$ entonces

$B \in \mathcal{A}^* \iff \mu^*(B) = \mu_*(B)$

Obs: $\mu^*(B) \leq \mu^*(A) < +\infty$

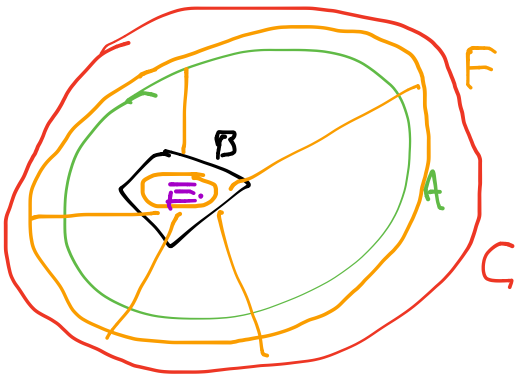
P.D. $\mu_*(B) = \mu^*(B)$

Ejer 86 $\Rightarrow$

$\exists F \in \mathcal{C}_i(A)$, $A \setminus B \subseteq F$, cubierta

$\exists\ C \in \Sigma'(A),\quad A \subseteq C$, cubierta

Tomamos $\quad E = F^c \cap C \in \Sigma'(A)$



Af 1: E núcleo medible de B

$E \in \Sigma'(A)$ ✓

$$\underline{E = F^c \cap C} \subseteq (A \setminus B)^c \cap C \quad ✓$$
$$= (A^c \cup B) \cap C$$
$$= B \cap C \subseteq B$$

Sea $D \in \Sigma'(A)$, $D \subseteq B \setminus E$

p.d. $\bar{\mu}(D) = 0$

$$B \setminus E = B \cap E^c = B \cap (F^c \cap C)^c$$
$$= B \cap (F \cup C^c)$$
$$= B \cap F$$
$$\subseteq B \cap F \cup A^c \cap F$$
$$= (B \cup A^c) \cap F$$
$$= F \setminus (A \setminus B)$$

Ya que F es cubierta de $A \setminus B$

$$D \subseteq B \setminus E \subseteq F \setminus (A \setminus B)$$

$$\Rightarrow \bar{\mu}(D) = 0$$

Fin Afi 1

Afi 2: $\mu^*(A \setminus B) = \mu^*(C \setminus E)$

Razón: $\mu^*(C \setminus E) = \mu^*(C \cap (F^c \cap C)^c)$

$$= \mu^*(C \cap F) \quad \text{— (1)}$$

$$\Sigma_1(A) \ni F \cap C^c \subseteq F \setminus (A \setminus B) = F \cap B \cup F \cap A^c$$

F cubierta $\Rightarrow \bar{\mu}(F \cap C^c) = 0$

$$\bar{\mu}(F) = \bar{\mu}(F \cap C) + \bar{\mu}(F \cap C^c)$$

$$= \bar{\mu}(F \cap C) \quad \text{— (2)}$$

(1) y (2) $\Rightarrow \mu^*(C \setminus E) = \mu^*(F) = \mu^*(A \setminus B)$

Fin Afi 2

Finalmente:

$$\begin{cases} \mu^*(A) = \mu^*(B) + \mu^*(A \setminus B) & \text{hipótesis} \\ \mu^*(C) = \mu^*(E) + \mu^*(C \setminus E) \end{cases}$$

$$E \subseteq B \subseteq A \subseteq C$$

$E \in \Sigma_1(A)$, $C \in \Sigma_1(A)$

$\bar{\mu}(E) = \mu_*(B)$ Afi 1 + ejer 87-i

$\mu^*(A \setminus B) = \mu^*(C \setminus E)$ Afi. 2

$\mu^*(A) = \mu^*(C)$ ejer 86

$$\Rightarrow \boxed{\mu^*(B) = \bar{\mu}(E) = \mu_*(B)}$$

Fin 89