JOURNAL OF FUNCTIONAL ANALYSIS 66, 323–346 (1986)

Addition of Certain Non-commuting Random Variables*

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Communicated by A. Connes

Received September 1, 1985

The non-commuting random variables in the title can be illustrated by the following example.

Let G be the (non-commutative) free group on two generators and let u_j $(j = 1, 2)$ be the unitaries in $l^2(G)$ corresponding to left translation by the generators and consider $\xi \in l^2(G)$ the function $\xi(g) = \delta_{g,e}$. Further, let X_j be operators of the form $X_j = \varphi_j(u_j)$. The operators X_j may be viewed as "random variables" with moments $\langle X_j^k \xi, \xi \rangle$, or equivalently with distributions given by the analytic functionals μ_j, where $\mu_j(f) = \langle f(X_j)\, \xi, \xi \rangle$. With these conventions, the distribution of $X_1 + X_2$ depends only on the distributions of X_1 and X_2 and the aim of the present paper is to explicate this relationship. This may be viewed as a non-commutative analogue of the addition of independent random variables. Indeed if G in the above example is replaced by the abelian free group on two generators, then we have precisely the usual situation of independent random variables for which addition means convolution of the distributions.

In [11], where we began the study of this kind of non-commutative independence of random variables, we constructed a certain functor from real Hilbert spaces and contractions to operator algebras with specified trace states and unital completely positive maps. This functor may be regarded as the analogue in our non-commutative framework of the functor which associates to a real Hilbert space the L^∞-algebra with respect to the gaussian measure on it with the trace state corresponding to this measure. Related to this we also obtained the corresponding central limit theorem, the limit distribution being a semiellipse law.

In the commutative situation, addition of random variables corresponds to convolution of distributions, which in turn corresponds to addition of the logarithms of their Fourier transforms (near the origin). In the present

* The present paper has been circulated as INCREST preprint No. 64/1984.

0022-1236/86 \$3.00