Digital Image Processing

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Contents

Overview

SPATIAL DOMAIN

- Intensity transformation → Operates on single pixels
  - → Contrast manipulation
  - → Image Thresholding
- Spatial filtering → Operates on neighborhoods
  - → Image Sharpening

Spatial domain processes can be denoted by

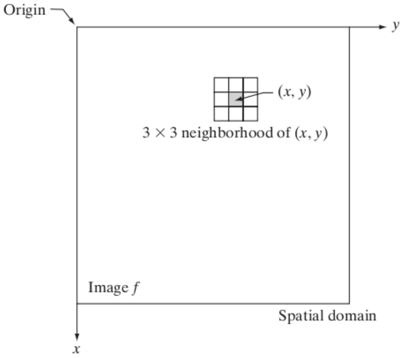
$$g(x, y) = T\left[f(x, y)\right] \tag{1}$$

- $f(x, y)$ - Input image.
- $g(x, y)$ - Output image.
- $T[\cdot]$ - Operator on f defined over a neighborhood of point (x, y).

Spatial domain processes can be denoted by

$$g(x, y) = T\,[f(x, y)] \tag{2}$$

- Spatial filtering using a 3×3 mask.

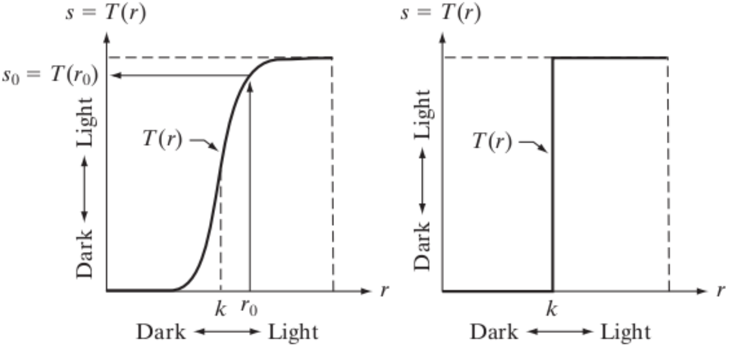


- Smallest neighborhood size 1×1.
- In this case, we have an *intensity transformation* function

$$s = T(r). \tag{3}$$

  where s and r denote the intensity of g and f at point (x, y), respectively.

Examples of contrast stretching and thresholding.



Basic intensity transformations:

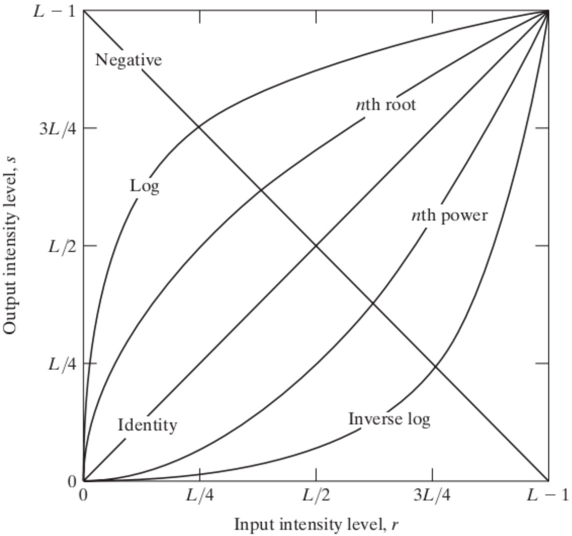
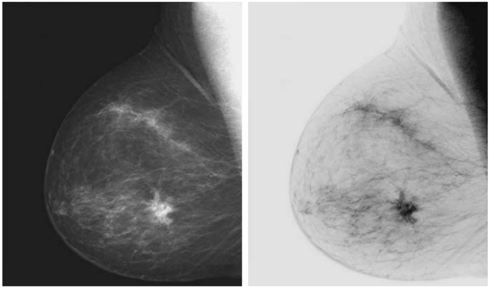


Image negative

$$s = L - 1 - r \tag{4}$$

particularly suited for enhancing white or gray detail embedded in dark regions.



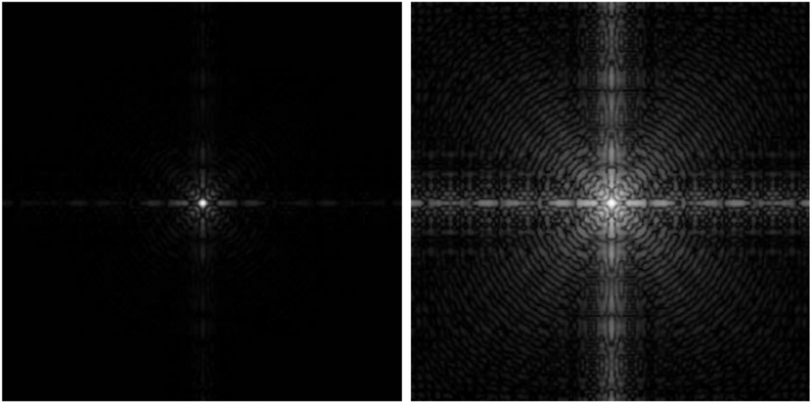
Log transformations

$$s = c\log(1 + r), \tag{5}$$

- Where c is a constant and it is assumed that $r \geq 0$.
- The log transform maps a narrow range of low intensity values in the input into a wider range of output levels.
- Used to expand the dark pixels in an image while compressing the higher-level values.
- The opposite is true of the inverse log transformation.
- Compresses the dynamic range of images with large variations in pixel values.

Classical application of the log transform; Fourier Spectra:

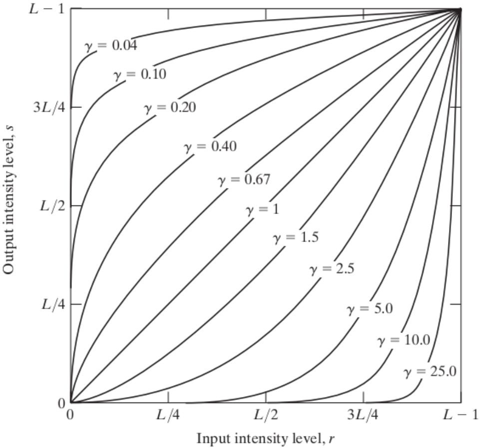
- It has values typically within range $\in [0, 10^6]$ or higher.
- Such wide range cannot be faithfully represented without the aid of a transform:



Basic form:

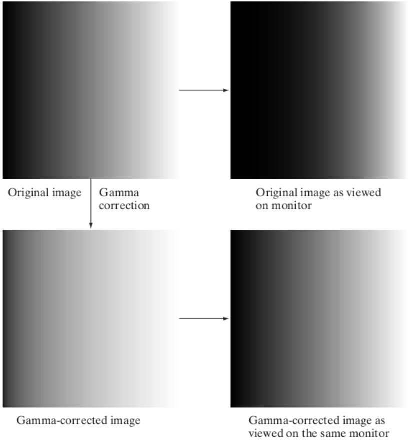
$$s = cr^{\gamma}, \tag{6}$$

where c and γ are positive constants.



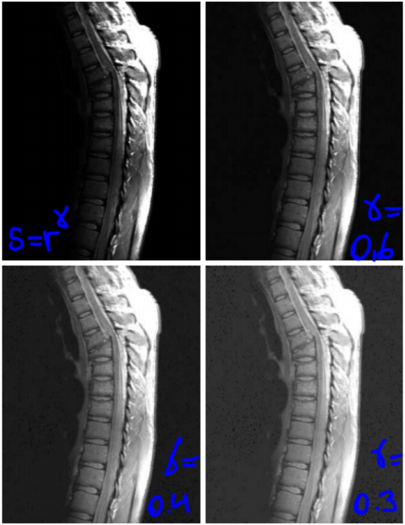
Gamma correction.

- Cathode Ray Tube (CRT) have intensity-to-voltage response that is a power function (exponents $\in [1.8, 2.5]$.
- Example of gamma-correction using $s = r^{1/2.5} = r^{0.4}$:



Gamma correction is important:

- Accurately display an image in a computer screen.
- An image shown in a website is seen by millions of different screens.
- Varying gamma affects color.


s=r^γ
γ=
0.6
γ=
0.4
γ=
0.3


γ=3
4
5

- Piece-wise linear functions can be arbitrarily complex.
- Require, however, considerably more user input.

Low-contrast images can result from;

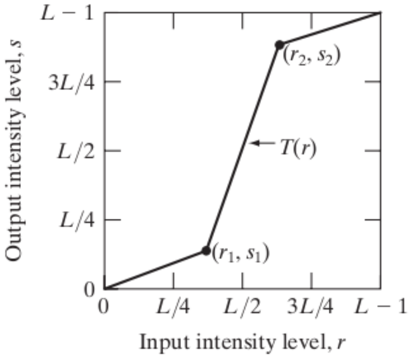
- poor illumination;
- lack of dynamic range in the imaging sensor, or even a;
- wrong setting of a lens aperture during image acquisition.

Contrast stretching

Process that expands the range of intensity levels in an image so that it spans the full intensity range of the recording medium or display device.

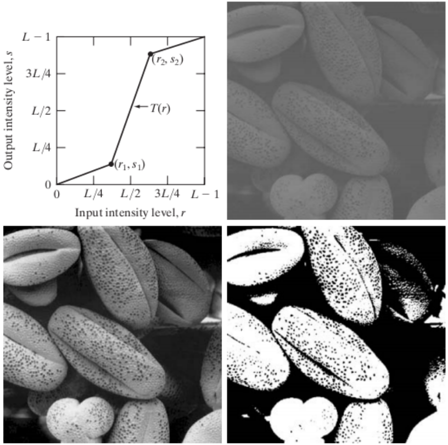
Picewise-linear transformation.

- $r_1 = s_1$ and $r_2 = s_2 \rightarrow$ linear transformation, i. e., no change.
- $r_1 = r_2$, $s_1 = 0$ and $s_2 = L - 1 \rightarrow$ *thresholding*.
- In general, $r_1 \leq r_2$ and $s_1 \leq s_2$, preserving the order of intensity levels and preventing the creation of intensity artifacts.



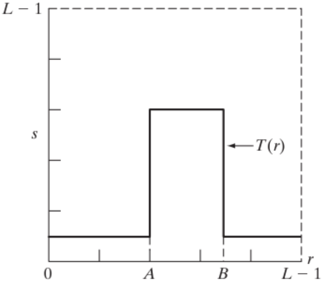
Scanning electron microscope image of pollen $\times 700$.

1. $(r_1, s_1) = (r_{min}, 0)$ and $(r_2, s_2) = (r_{max}, L-1)$.
2. $(r_1, s_1) = (m, 0)$ and $(r_2, s_2) = (m, L-1)$.



Highlight specific ranges of intensity;

1. binarize range of interest, or;
2. brightens (or darkens) desired range leaving the rest unchanged.

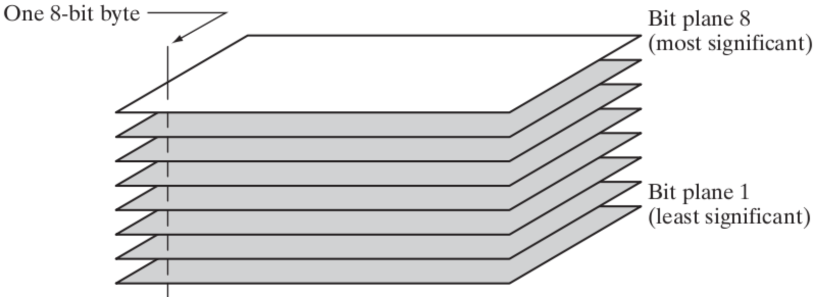


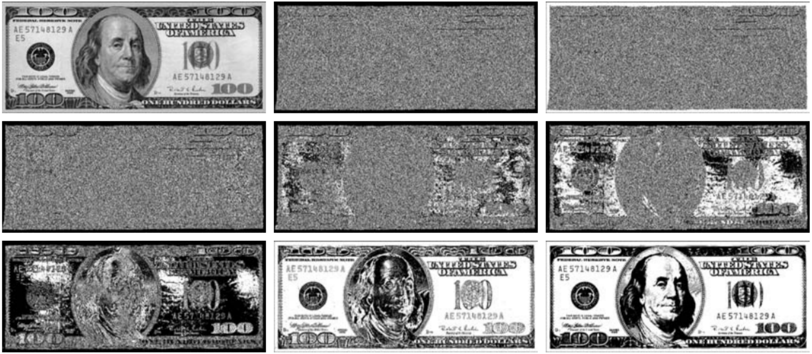
Aortic angiogram near the kidney area.

150 < I < 240 = 255

64 < I < 144 = 0

Highlight the contribution made to total image appearance by specific bits.




FEDERAL RESERVE NOTE
THE UNITED STATES OF AMERICA
AE 57148129 A
E5
100
ONE HUNDRED DOLLARS

An example application of bit slicing: Image compression:

a b c

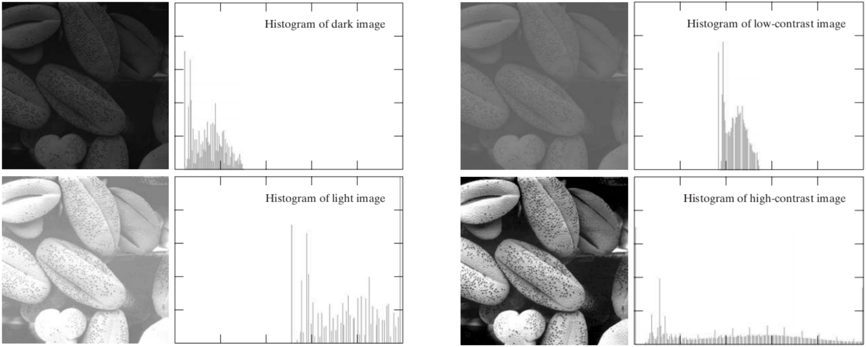
FIGURE 3.15 Images reconstructed using (a) bit planes 8 and 7; (b) bit planes 8, 7, and 6; and (c) bit planes 8, 7, 6, and 5. Compare (c) with Fig. 3.14(a).

- Notice that the 4 most significant bits are sufficient to reasonably "reconstruct" the original image (a reduction of 50% in bit memory storage);

Histogram processing

- The histogram of a digital image with intensity levels in the range $[0, L-1]$ is a discrete function $h(r_k) = n_k$ where r_k is the kth intensity value and n_k is the number of pixels in the image with intensity r_k.
- The histogram is commonly normalized by $1/n$, with $n = MN$.
- The probability of occurrence of intensity value r_k is $p(r_k) = n_k/n$, for $k = 0, \ldots, L-1$.

Examples:


Histogram of dark image
Histogram of low-contrast image
Histogram of light image
Histogram of high-contrast image

Histogram is an important tool for image processing. It is used in

- Image enhancement.
- Statistical processing.
- Compression.
- Segmentation.

Furthermore:

- It is simple to compute.
- Cheap hardware implementation.
- Automatic.

Histogram equalization

- Consider (momentarily) continuous intensity values.
- $r \in [0, L-1]$ denotes the intensities of an image to be processed.
- Consider intensity mappings of the form

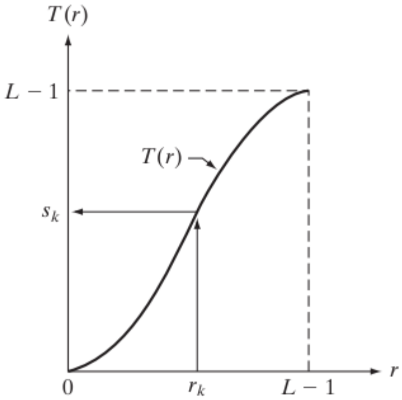
$$s = T(r) \quad 0 \leq r \leq L-1. \tag{7}$$

Assume:

1. $T(r)$ is monotonically increasing in the interval $0 \leq r \leq L-1$, and;
2. $0 \leq T(r) \leq L-1$ for $0 \leq r \leq L-1$.

Considering the inverse

$$r = T^{-1}(s) \tag{8}$$



- Intensity levels in an image may be viewed as random variables in the interval $[0, L-1]$.
- Let $p_r(r)$ and $p_s(s)$ denote the Probability Density Functions (PDFs) of r and s.
- From probability theory, if $p_r(r)$ and $T(r)$ are known, and $T(r)$ is continuous and differentiable over the range of values of interest, then the PDF of the transformed variable s can be obtained using

$$p_s(s) = p_r(r) \left| \frac{dr}{ds} \right| . \tag{9}$$

Consider the transformation function

$$s = T(r) = (L-1)\int_0^r p_r(w)dw, \tag{10}$$

which is the Cumulative Distribution Function (CDF) of random variable r, which satisfies previous conditions.
From previous equation, one can derive

$$\frac{ds}{dr} = (L-1)\frac{d}{dr}\left[\int_0^r p_r(w)dw\right] = (L-1)p_r(r). \tag{11}$$

Substituting this result in $p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$ yields...

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$$

$$p_r(r)\left|\frac{1}{(L-1)p_r(r)}\right|$$

$$\boxed{p_s(s) = \frac{1}{L-1}}, \quad 0 \leq s \leq L-1. \tag{12}$$

which is a *uniform* probability density function.

In other words, using the transformation

$$\boxed{s = T(r) = (L-1)\int_0^r p_r(w)dw},$$

yields a random variable s characterized by a uniform PDF.
Note that:

- $T(r)$ depends on $p_r(r)$, but;
- the resulting $p_s(s)$ is *always* uniform, independently of $p_r(r)$.

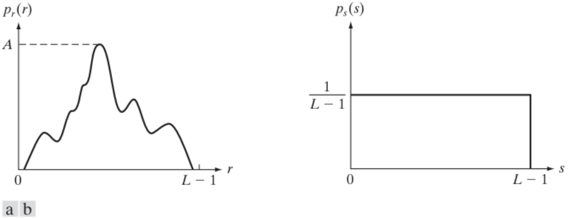


FIGURE 3.18 (a) An arbitrary PDF. (b) Result of applying the transformation in Eq. (3.3-4) to all intensity levels, r. The resulting intensities, s, have a uniform PDF, independently of the form of the PDF of the r's.

$$p_r(r_k) = \frac{n_k}{n}, \quad k = 0, 1, \ldots, L-1.$$

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j).$$

$$\boxed{s_k = \frac{(L-1)}{n}\sum_{j=0}^{k} n_j}, \quad k = 0, 1, \ldots, L-1. \tag{13}$$

where

$$n = MN$$

Each input intensity r_k is mapped into a level s_k in the output image, using Eq. (13).

Example: Hypothetical $64 \times 64 = 4096$ image with $L = 8$ intensity levels.

r_k	n_k	n_k/n	s_k	s_k
0	790	0.19	1.35	1
1	1023	0.25	3.10	3
2	850	0.21	4.55	5
3	656	0.16	5.67	6
4	329	0.08	6.23	6
5	245	0.06	6.65	7
6	122	0.03	6.86	7
7	81	0.02	7.00	7

Illustration of histogram equalization:

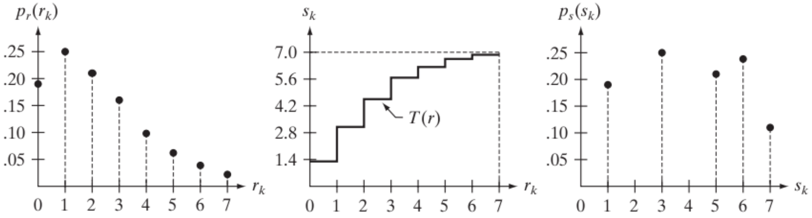


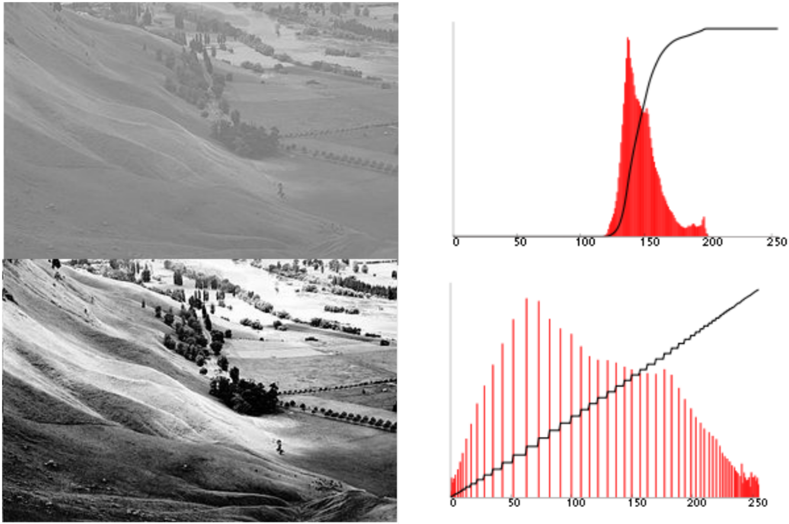
Figure: Illustration of histogram equalization of a 3-bit (8 intensity levels) image. (left) Original histogram. (middle) Transformation function. (right) Equalized histogram.

Step-by-step histogram equalization:

1. Compute image's probabilities $p_r(r_k) = n_k/n$.
2. Compute cumulative probability distribution $s_k = [(L-1)/n] \sum_{j=0}^{k} n_j$.
3. Map pixels with values r_k to intensity values (s_k).

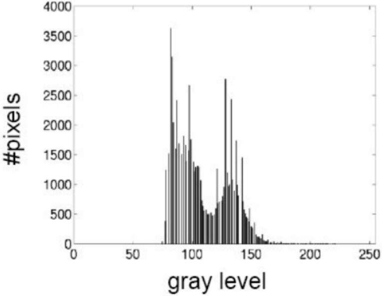
Notice

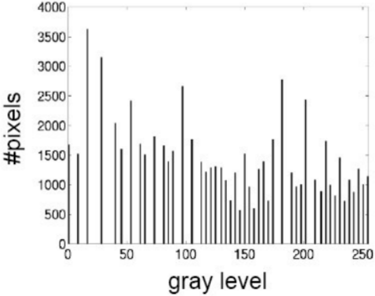
Histogram equalization has the tendency to wider the range of the intensity values, enhancing the contrast.



0
50
100
150
200
250
0
50
100
150
200
250

Figure: Original (left) and histogram equalized image (right).

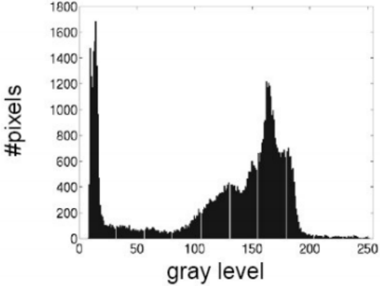

#pixels
4000
3500
3000
2500
2000
1500
1000
500
0
0
50
100
150
200
250
gray level


#pixels
4000
3500
3000
2500
2000
1500
1000
500
0
0
50
100
150
200
250
gray level

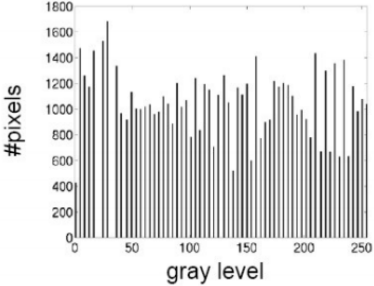
Original image
Cameraman

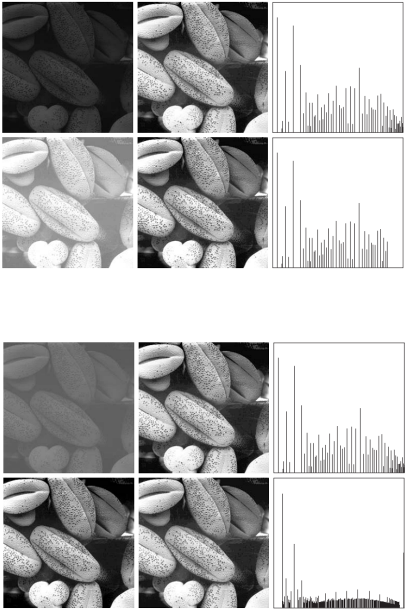
Cameraman
after histogram equalization

Original image *Cameraman*



. . . after histogram equalization





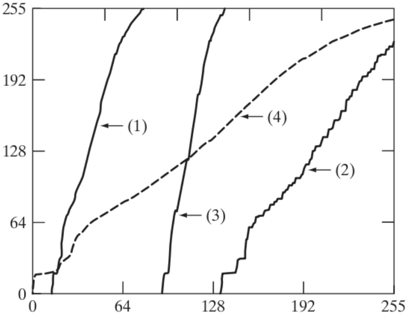


Figure: Transformations (1) through (4) were obtained from the histograms of the images (top-bottom) in the left column of using $s_k = (L-1/MN)\sum_{j=0}^{k} n_j,\ j = 0, 1, \ldots, L-1.$

Histogram equalization:

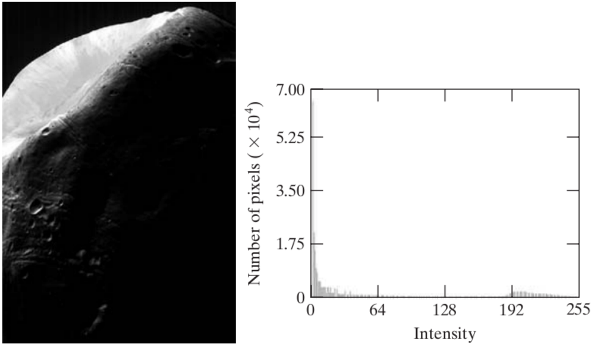
- Automatically determines a transformation function that seeks to produce an output image that has a uniform histogram.
- If automatic enhancement is desired, this is a good approach because;
  - the results from this technique are predictable, and;
  - the method is simple to implement.
- However:

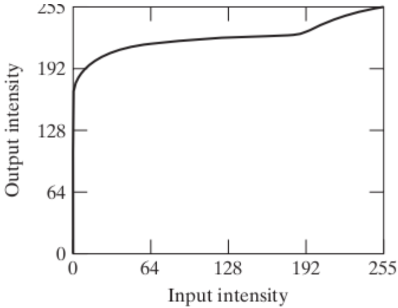
Histogram matching

Sometimes it is useful to be able to specify the shape of the histogram that we wish the processed image to have.

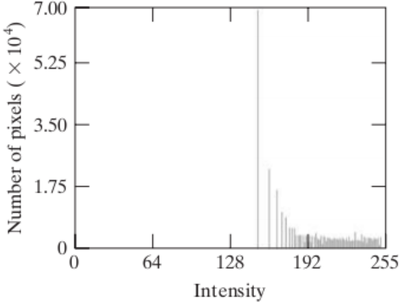
a b

FIGURE 3.23
(a) Image of the Mars moon Phobos taken by NASA's *Mars Global Surveyor.* (b) Histogram. (Original image courtesy of NASA.)

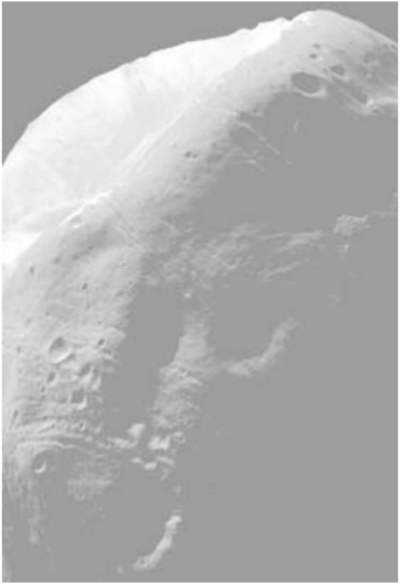


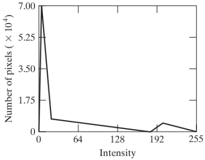


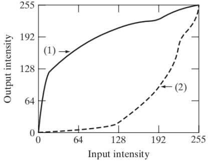
Output intensity
Input intensity
0
64
128
192
255

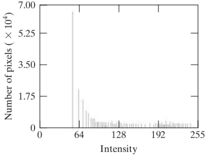


Number of pixels (× 10^4)
Intensity
7.00
5.25
3.50
1.75
0
64
128
192
255




7.00
5.25
3.50
1.75
0
Number of pixels (× 10^4)
0
64
128
192
255
Intensity


255
192
128
64
0
Output intensity
(1)
(2)
0
64
128
192
255
Input intensity

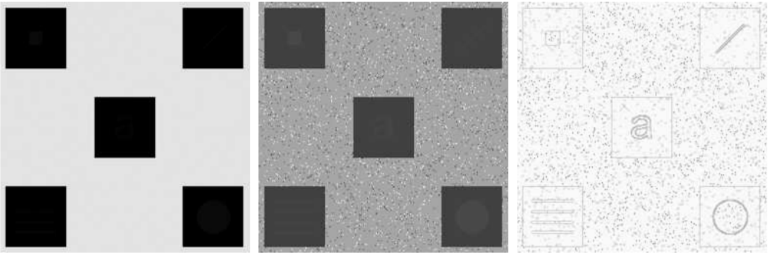

7.00
5.25
3.50
1.75
0
Number of pixels (× 10^4)
0
64
128
192
255
Intensity

Histogram matching is:

- A trial-and-error process.
- Sometimes, the user might know what an “average” histogram should look like.
- There are no general rules to specifying histograms, varying between different cases.

Local histogram processing

- Histogram equalization and matching are **global** techniques.
- How to enhance details locally? Consider small windows centered on each pixel.
- The global techniques presented can be applied locally.



a b c

FIGURE 3.26 (a) Original image. (b) Result of global histogram equalization. (c) Result of local histogram equalization applied to (a), using a neighborhood of size 3×3.

Statistics obtained from an image can be used for image enhancement:

- r denotes a discrete random variable representing intensities within range $[0, L-1]$.
- $p(r_i)$ denotes the normalized histogram corresponding to intensity r_i.
- The mean
$$m = \sum_{i=0}^{L-1} r_i p(r_i). \tag{14}$$
- The nth moment
$$\mu_n = \sum_{i=0}^{L-1} (r_i - m)^n \, p(r_i). \tag{15}$$

The second moment is particularly important:

$$\mu_2 = \sigma^2 = \sum_{i=0}^{L-1} (r_i - m)^2 \, p(r_i). \tag{16}$$

- The mean m is a measure of average intensity.
- The variance σ^2 is a measure of contrast.

Notation when computing mean and moment on sub-images:

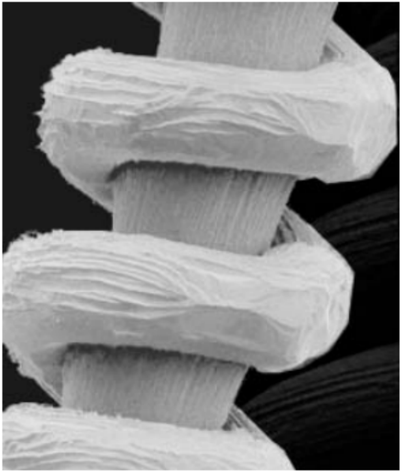
$$m_{S_{xy}} = \sum_{(s,t)\in S_{xy}} r_{s,t} p(r_{s,t}). \tag{17}$$

$$\mu^n_{S_{xy}} = \sum_{(s,t)\in S_{xy}} \left(r_{s,t} - m_{S_{xy}}\right)^n p\left(r_{s,t}\right) \tag{18}$$

As before, the local mean is a measure of **average intensity** in neighborhood and the local variance $\sigma^2 = \mu^2$ is a measure of **intensity contrast** in that neighborhood.

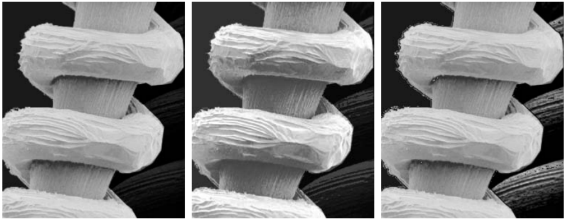
Consider the SEM image of a tungsten filament.
We want to enhance areas which

- Have low intensity average *wrt* the whole image.
- Have low contrast.
- Have not a constant contrast.



$$g(x,y) = \begin{cases} E \cdot f(x,y) & \text{if } m_{S_{xy}} \leq k_0 m_G \text{ and } k_1\sigma_G \leq \sigma_{S_{xy}} \leq k_2\sigma_G \\ f(x,y) & \text{otherwise} \end{cases}. \tag{19}$$

Where E increases (or decreases) the gray levels satisfying (19).

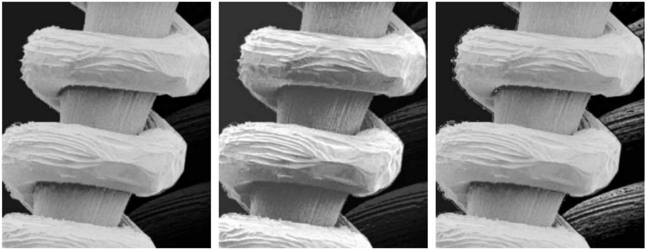


a b c

FIGURE 3.27 (a) SEM image of a tungsten filament magnified approximately 130×. (b) Result of global histogram equalization. (c) Image enhanced using local histogram statistics. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)

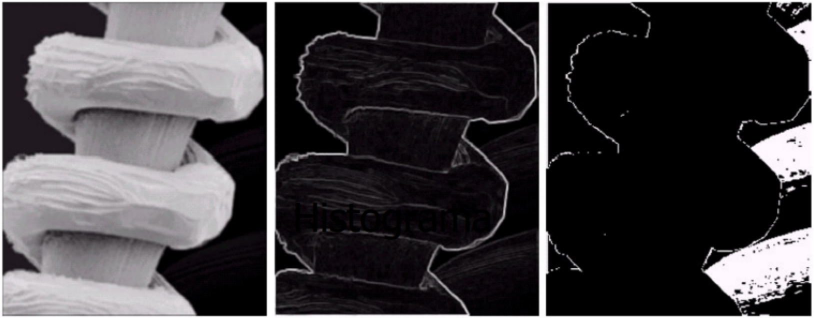
Values (require experimentation):

- $E = 4.0$. S_{xy} is a 3×3 region.
- $k_0 = 0.4$, $k_1 = 0.02$, $k_2 = 0.4$.



a b c

FIGURE 3.27 (a) SEM image of a tungsten filament magnified approximately 130×. (b) Result of global histogram equalization. (c) Image enhanced using local histogram statistics. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)



a b c

FIGURE 3.25 (a) Image formed from all local means obtained from Fig. 3.24 using Eq. (3.3-21). (b) Image formed from all local standard deviations obtained from Fig. 3.24 using Eq. (3.3-22). (c) Image formed from all multiplication constants used to produce the enhanced image shown in Fig. 3.26.

Spatial filtering

- Spatial filtering is often used for image enhancement.
- Term *filtering* was borrowed from the frequency domain.
- Spatial filtering can be used for non-linear filtering, which cannot be done in the frequency domain.

A spatial filter consists in

1. A *neighborhood*.
2. A *predefined operation*.

The spatial filter can be;

- linear, or;
- nonlinear.

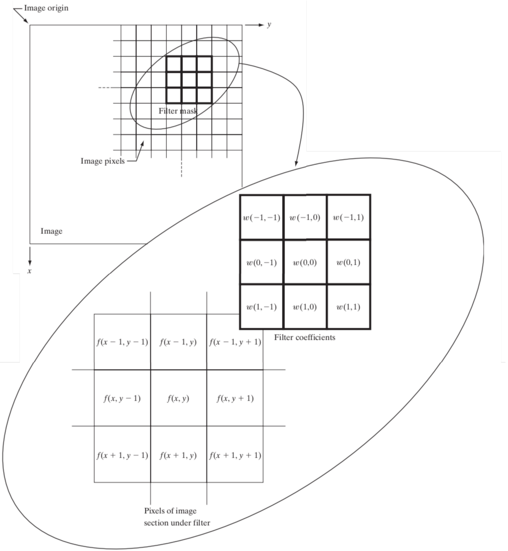


FIGURE 3.28 The mechanics of linear spatial filtering using a 3×3 filter mask. The form chosen to denote the coordinates of the filter mask coefficients simplifies writing expressions for linear filtering.

$$g(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x+s, y+t), \quad \forall\, x, y. \tag{20}$$

Where:

- $a = (m-1)/2$ and $b = (n-1)/2$ form the window dimension $m \times n$ (both odd numbers).
- Also called *convolution*.
- Note: the result of the operation does not change the original image.

The basic procedure consists in summing the products between the mask coefficients and the gray levels at the local image region.

$$R = w_1 z_1 + w_2 z_2 + \ldots + w_9 z_9.$$

In general;

$$R = w_1 z_1 + w_2 z_2 + \ldots + w_{mn} z_{mn}$$

$$R = \sum_{k=1}^{mn} w_k z_k$$

$$R = \mathbf{w}^T \mathbf{z} \qquad (21)$$

w_1	w_2	w_3
w_4	w_5	w_6
w_7	w_8	w_9

Smoothing filters

- Smoothing filters are used for *blurring* and *noise reduction*.
  - Blurring is used in pre-processing stages to remove small details for object extraction.
  - Noise reduction can be obtained also with non-linear filtering.
- Low-pass filtering.
  - Neighborhood average.
  - Results in a blurred image.

Smoothing filter mask examples:

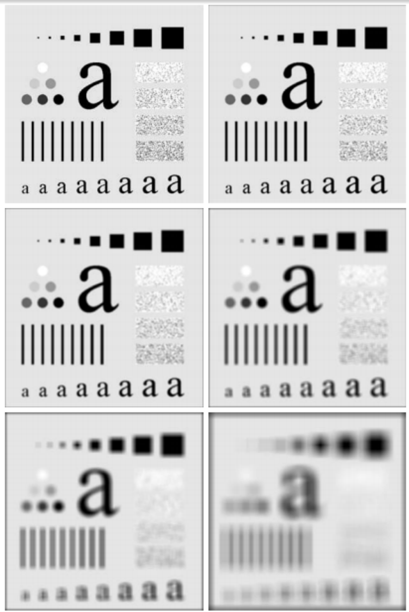
$$\frac{1}{9} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \frac{1}{16} \times \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

a b

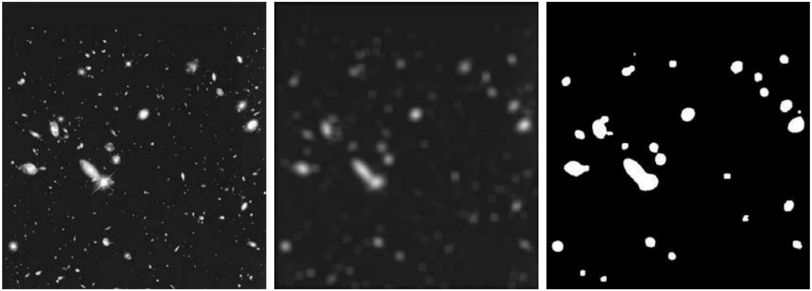
FIGURE 3.32 Two 3×3 smoothing (averaging) filter masks. The constant multiplier in front of each mask is equal to 1 divided by the sum of the values of its coefficients, as is required to compute an average.

Example: average filter (mask sizes $= 3, 5, 9, 15, 35$).
Notice:

- Blurring.
- Blending.
- Noise reduction.
- Object elimination.
- Black border.



Gross representation of objects. Note:



a b c

FIGURE 3.34 (a) Image of size 528 × 485 pixels from the Hubble Space Telescope. (b) Image filtered with a 15 × 15 averaging mask. (c) Result of thresholding (b). (Original image courtesy of NASA.)

Non-linear filters

Median filter:

- Useful for elimination of *salt-and-pepper* (or *impulse*) noise.
- The median ξ of a set of values is such that half the values in the set are less than or equal to ξ and half are greater than or equal to ξ.
- I. e., its main function is to force points with distinct intensity levels to be more like their neighbors.

5	5	6
3	4	5
3	4	7

Original

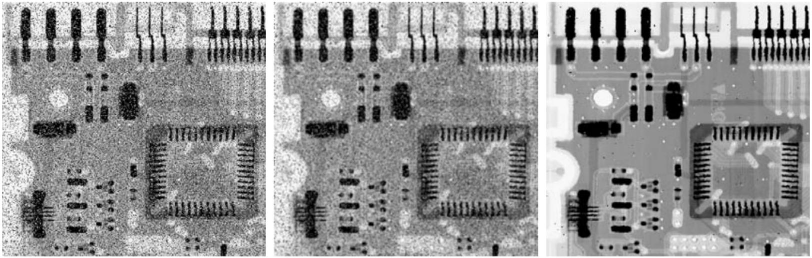
→ (3,3,4,4,**5**,5,5,6,7) →

Sorting

5	5	6
3	5	5
3	4	7

Filtered

Salt and pepper noise removal using the median filter:



a b c

FIGURE 3.35 (a) X-ray image of circuit board corrupted by salt-and-pepper noise. (b) Noise reduction with a 3×3 averaging mask. (c) Noise reduction with a 3×3 median filter. (Original image courtesy of Mr. Joseph E. Pascente, Lixi, Inc.)

Sharpening spatial filters

Sharpening filters aim at:

- Enhancing fine details.
- Recovering information compromised by blurring.

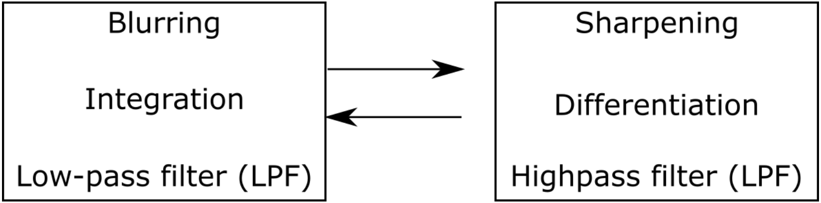
Applications:

- Electronic printing.
- Medical imaging.
- Industrial inspection.

In the spatial domain:

- Blurring (averaging) is analogous to integration.
- Sharpening is analogous do differentiation.

Image differentiation enhances edges and other discontinuities (e.g., noise) and deemphasizes areas with slowly varying intensities.



A first derivative:

- Must be zero in areas of constant intensity.
- Must be nonzero at the onset of an intensity step or ramp.
- Must be nonzero along ramps.

Basic definition of a first-order derivative of a one-dimensional function:

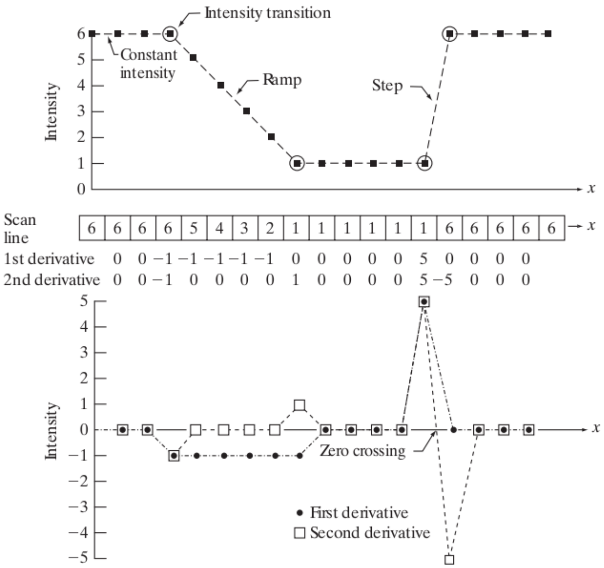
$$\frac{\partial f}{\partial x} = f(x+1) - f(x). \tag{22}$$

A second derivative:

- Must be zero in constant areas.
- Must be nonzero at the onset and end of an intensity step or ramp.
- Must be zero along ramps of constant slope.

Definition of second order derivative

$$\frac{\partial^2 f}{\partial x^2} = f(x+1) + f(x-1) - 2f(x). \tag{23}$$



a
b
c

FIGURE 3.36 Illustration of the first and second derivatives of a 1-D digital function representing a section of a horizontal intensity profile from an image. In (a) and (c) data points are joined by dashed lines as a visualization aid.

a b
c

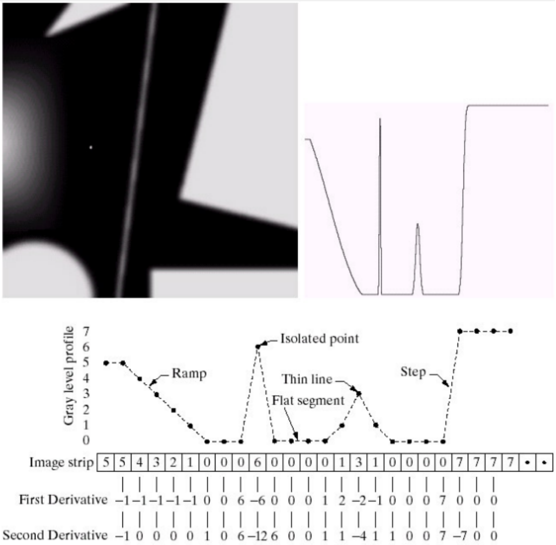


FIGURE 3.38 (a) A simple image. (b) 1-D horizontal gray-level profile along the center of the image and including the isolated noise point. (c) Simplified profile (the points are joined by dashed lines to simplify interpretation).

Notes on derivatives:

- 1st derivative:
  - Produces thicker borders.
  - Higher response to big differences (e.g., step).
  - Often used for border detection.
- 2nd derivative
  - More sensitive to finer details.
  - Better for image enhancement, since its more sensitive.
  - Produces double response to step.

Simplest *isotropic*[1] derivative operator: The Laplacian:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}. \tag{24}$$

Since:

$$\frac{\partial^2 f}{\partial x^2} = f(x+1) + f(x-1) - 2f(x)$$

and

$$\frac{\partial^2 f}{\partial y^2} = f(y+1) + f(y-1) - 2f(y).$$

We have

$$\boxed{\nabla^2 f = f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)}. \tag{25}$$

[1] Independent of orientation

$$\boxed{\nabla^2 f = f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)}$$

0	1	0
1	−4	1
0	1	0

1	1	1
1	−8	1
1	1	1

0	−1	0
−1	4	−1
0	−1	0

−1	−1	−1
−1	8	−1
−1	−1	−1

a b
c d

FIGURE 3.37
(a) Filter mask used to implement Eq. (3.6-6). (b) Mask used to implement an extension of this equation that includes the diagonal terms. (c) and (d) Two other implementations of the Laplacian found frequently in practice.

Because the Laplacian is a derivative operator:

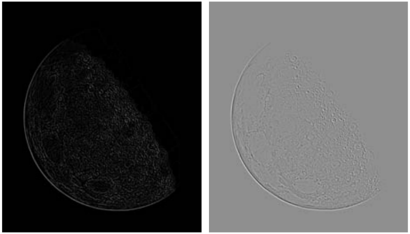
- It highlights intensity discontinuities in an image.
- De-emphasizes regions with slowly varying intensity levels.

The basic way to use the Laplacian for image sharpening is:

$$\boxed{g(x, y) = f(x, y) + c\nabla^2 f(x, y)} \tag{26}$$

where:

- $f(x, y)$ and $g(x, y)$ are the input and sharpened images.
- c is -1 or $+1$ if the center of the filter is negative or positive, respectively.



a
b c
d e

FIGURE 3.38
(a) Blurred image of the North Pole of the moon. (b) Laplacian without scaling. (c) Laplacian with scaling. (d) Image sharpened using the mask in Fig. 3.37(a). (e) Result of using the mask in Fig. 3.37(b). (Original image courtesy of NASA.)

High-boost[2] filtering

1. Blur the original image.
2. Subtract the blurred image from the original (the resulting difference is called the *mask*).
3. Add the mask to the original.

[2] "Alto reforço"

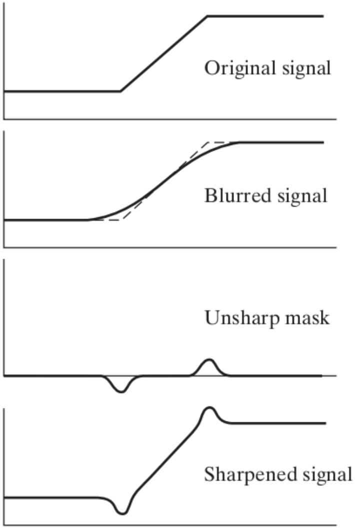
More formally:

$$g_{mask} = f(x,y) - \underbrace{\bar{f}(x,y)}_{\text{Blurred image}}$$

$$g(x,y) = f(x,y) + k \cdot g_{mask}(x,y). \tag{27}$$

Where:

- $k \geq 0$.
- $k = 1$, we have unsharp masking.
- When $k > 1$ we have *highboost filtering*.



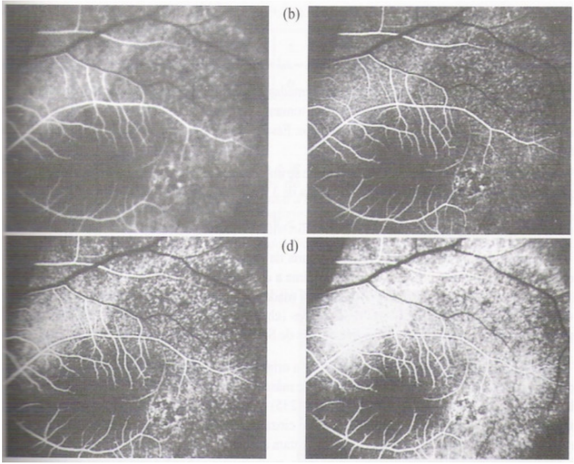
a
b
c
d

FIGURE 3.39 1-D illustration of the mechanics of unsharp masking. (a) Original signal. (b) Blurred signal with original shown dashed for reference. (c) Unsharp mask. (d) Sharpened signal, obtained by adding (c) to (a).

FIGURE 3.40
(a) Original image.
(b) Result of blurring with a Gaussian filter.
(c) Unsharp mask. (d) Result of using unsharp masking.
(e) Result of using highboost filtering.

Note: In fig 3.40 (d) and (e), k=1 and 4.5.

Examples with $k = 1.1$, 1.15 and 1.2.



The gradient of $f(x, y)$ is defined as the two dimensional column vector:

$$\nabla f \equiv \begin{pmatrix} g_x \\ g_y \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{pmatrix} \tag{28}$$

This vector points in the direction of the greatest rate of change of f at (x, y).
The *magnitude* of vector ∇f is

$$M(x, y) = mag(\nabla f) = \sqrt{g_x^2 + g_y^2}. \tag{29}$$

is the value at (x, y) of the rate of change in the direction of the gradient vector.

Note that $M(x, y)$ is an image of the same size as the original. In some implementations, it is more suitable computationally to approximate the squares and square root operations by absolute values:

$$M(x, y) \approx |g_x| + |g_y|. \tag{30}$$

Now we define discrete approximations to the preceding equations and formulate the masks.

- The smallest filter masks are of size 3×3.

z_1	z_2	z_3
z_4	z_5	z_6
z_7	z_8	z_9

Approximations to g_x and g_y using a 3×3 neighborhood centered on z_5 are as follows (The *Sobel* operators):

$$g_x = \frac{\partial f}{\partial x} = (z_7 + 2z_8 + z_9) - (z_1 + 2z_2 + z_3) \tag{31}$$

and

$$g_y = \frac{\partial f}{\partial y} = (z_3 + 2z_6 + z_9) - (z_1 + 2z_4 + z_7) \tag{32}$$

z_1	z_2	z_3
z_4	z_5	z_6
z_7	z_8	z_9

The *Sobel* operators:

−1	−2	−1		−1	0	1
0	0	0		−2	0	2
1	2	1		−1	0	1

Notes:

- The weight of 2 in the center coefficient is to achieve some smoothing by giving more importance to the center point.
- The magnitude of the gradient is obtained with:

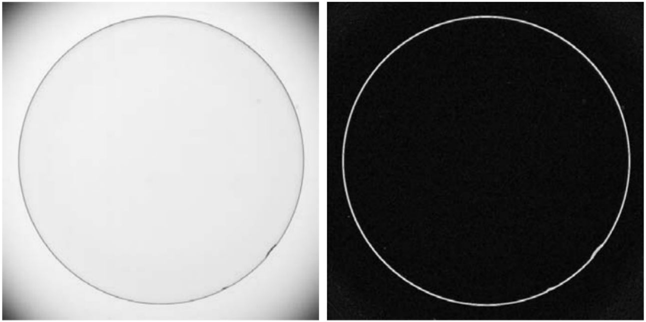
$$M(x, y) \approx |g_x| + |g_y|. \tag{33}$$

Note

The coefficients in all masks sum to 0 (null response in an area of constant intensity).

Application: Industrial inspection.

- Aid humans.
- Automatic detection.



a b

FIGURE 3.42 (a) Optical image of contact lens (note defects on the boundary at 4 and 5 o'clock). (b) Sobel gradient. (Original image courtesy of Pete Sites, Perceptics Corporation.)

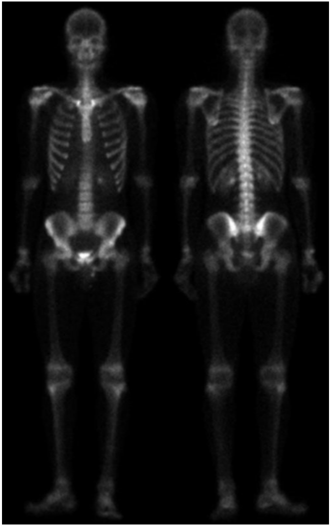
Example: Sharpening an image of nuclear whole body bone scan (detect bone infection and tumors).

Observe:

- Narrow dynamic range.
- High noise.

Strategy:

1. Laplacian to enhance detail.
2. Gradient to enhance borders.



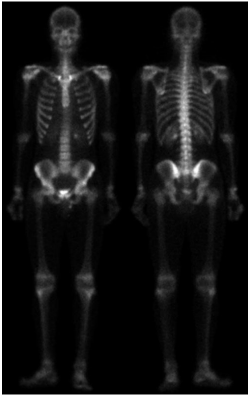


Figure: $f(x, y)$

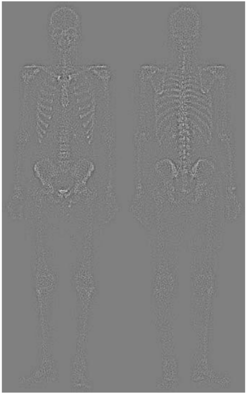


Figure: $\nabla^2 f(x, y)$ (scaled).

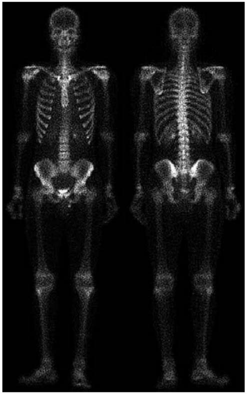


Figure: $g(x, y) = f(x, y) - c\nabla^2 f(x, y)$

The Laplacian:

- Second-order derivative operator capable of enhancing fine details.
- Noisy.

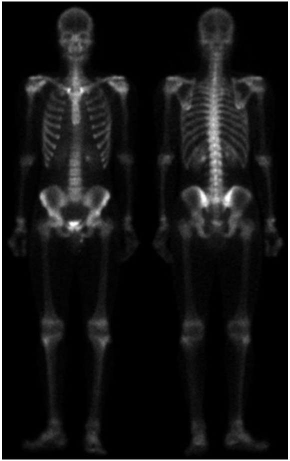
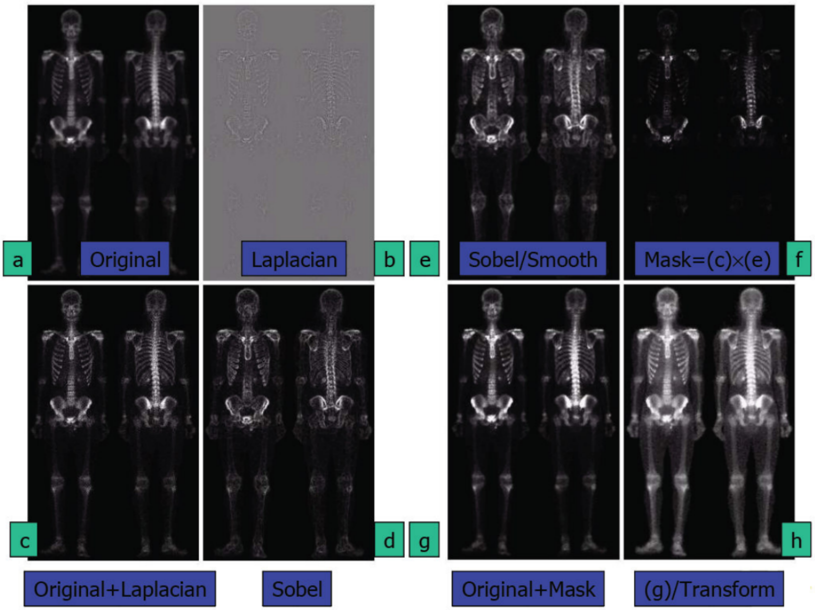


Figure: $f(x, y)$

Figure: $M(x, y) = |g_x| + |g_y|$.

The gradient:

- Stronger average response in areas of significant image transition (ramps and steps).
- Lower response to noise and fine detail.



a
Original
Laplacian
b
e
Sobel/Smooth
Mask=(c)×(e)
f
c
d
g
h
Original+Laplacian
Sobel
Original+Mask
(g)/Transform

Conclusions:

- Processes can be linked to achieve results better than single techniques.
- Strategy depends on the application.

Applications

- Radiology.
- Physicians are unlikely to derive diagnosis from enhanced images, but they serve as clues for further investigations.

Areas where the enhanced images are used:

- Printing industry.
- Image-based product inspection.
- Forensics.
- Microscopy.
- Surveillance.

Application of fuzzy sets to:

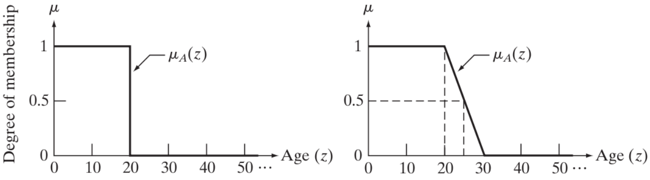
- Intensity transformation.
- Spatial filtering.

Definitions:

- A *set* is a collection of elements.
- *Set theory* conveys tools dealing with operations **on** and **among** sets.
- *Crispy* set membership: an element **is** or **isn't** inside a set.

Example:

- $Z =$ set of all people.
- $A =$ subset of young people.
- Suppose *threshold* $= 20$ years.
- Someone 20 years (and 1 second) old?!



a b

FIGURE 3.44 Membership functions used to generate (a) a crisp set, and (b) a fuzzy set.

A person now can be:

- Young.
- Relatively young.
- 50% young.
- Not so young.
- Not young, etc.

Fuzzy

Infinite valued membership functions;

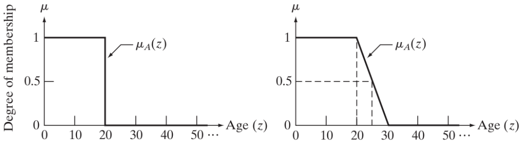
- are the foundation of *fuzzy logic*, and;
- the sets generating them may be viewed as *fuzzy sets*.

Define:

- Z = set of elements.
- z = generic element of Z; $Z = \{z\}$.

Definition

A *fuzzy set* in Z is characterized by a membership function $\mu_A(z)$, that associates with each element of Z a real number in the interval $[0, 1]$.



a b

FIGURE 3.44 Membership functions used to generate (a) a crisp set, and (b) a fuzzy set.

- The value of $\mu_A(z)$ represents the grade of membership of z in A.

An element z is;

- A full member of A if $\mu_A(z) = 1$.
- Not a member of A if $\mu_A(z) = 0$.
- A partial member of A if $0 < \mu_A(z) < 1$.

- Crispy sets are a particular case of Fuzzy sets.

Basic definitions:

- **Empty set**: A fuzzy set is *empty* if and only if its membership function is identically zero in Z.
- **Equality**: Two fuzzy sets A and B are equal, written $A = B$, if and only if $\mu_A(z) = \mu_B(z)$ for all $z \in Z$.
- **Complement**: The *complement* (NOT) of a fuzzy set A, denoted by $\bar{A}$, or NOT(A), is defined as the set whose membership function is

$$\mu_{\bar{A}}(z) = 1 - \mu_A(z), \quad \forall z \in Z.$$

Basic definitions:

- **Subset**: A fuzzy set A is a subset of a fuzzy set B if and only if

$$\mu_A(z) \leq \mu_B(z), \quad \forall z \in Z.$$

- **Union**: The *union* (OR) of two fuzzy sets A and B, denoted $A \cup B$, or A OR B, is a fuzzy set U with membership function

$$\mu_U(z) = \max\left[\mu_A(z), \mu_B(z)\right], \quad \forall z \in Z.$$

Basic definitions:

- **Intersection**: The intersection (AND) of two fuzzy sets A and B, denoted, $A \cap B$ or A AND B, is a fuzzy set I with membership function

$$\mu_U(z) = \min\left[\mu_A(z), \mu_B(z)\right], \quad \forall z \in Z.$$

a b
c d

FIGURE 3.45
(a) Membership functions of two sets, A and B. (b) Membership function of the complement of A. (c) and (d) Membership functions of the union and intersection of the two sets.

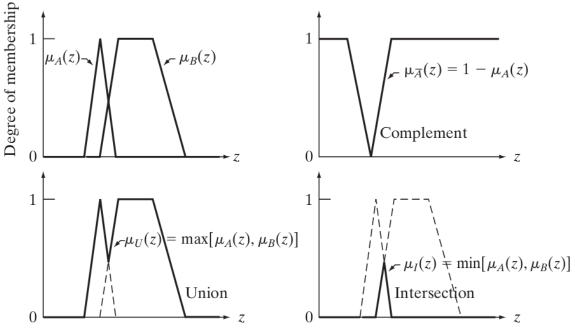
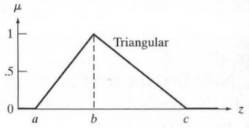
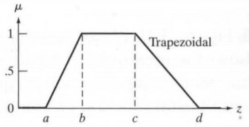
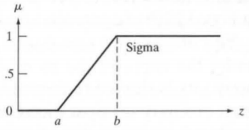
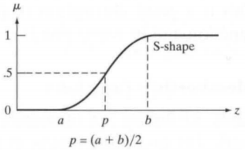
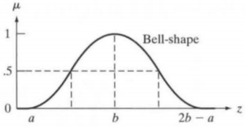
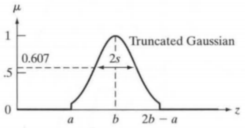
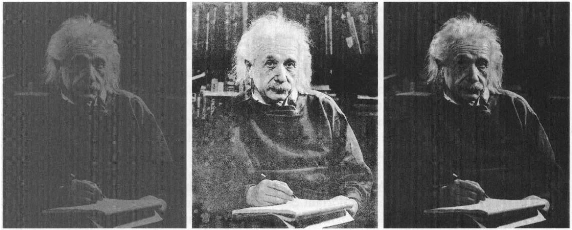


TABLE 3.6 Some commonly-used membership functions and corresponding plots.

Name	Equation	Plot
Triangular	$\mu(z) = \begin{cases} 0 & z < a \\ (z-a)/(b-a) & a \le z < b \\ 1-(z-b)/(c-b) & b \le z < c \\ 0 & c \le z \end{cases}$	
Trapezoidal	$\mu(z) = \begin{cases} 0 & z < a \\ (z-a)/(b-a) & a \le z < b \\ 1 & b \le z < c \\ 1-(z-b)/(c-b) & c \le z < d \\ 0 & d \le z \end{cases}$	
Sigma	$\mu(z) = \begin{cases} 0 & z < a \\ (z-a)/(b-a) & a \le z < b \\ 1 & b \le z \end{cases}$	

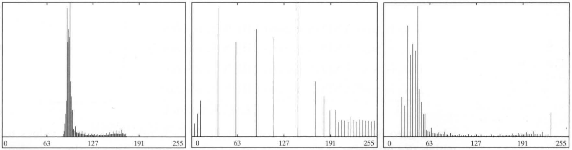
<table>
<tr><td>S-shape[†]</td><td>$$S(z,a,b)=\begin{cases}0 & z<a\\ 2\left[\dfrac{z-a}{b-a}\right]^2 & a\le z<p\\ 1-2\left[\dfrac{z-b}{b-a}\right]^2 & p\le z<b\\ 1 & b\le z\end{cases}$$</td><td></td></tr>
<tr><td>Bell-shape</td><td>$$\mu(z)=\begin{cases}S(z,a,b) & z<b\\ S(2b-z,a,b) & b\le z\end{cases}$$</td><td></td></tr>
<tr><td>Truncated Gaussian</td><td>$$\mu(z)=\begin{cases}e^{-\frac{(z-b)^2}{s^2}} & \lvert z-b\rvert\le(b-a)\\ 0 & \text{otherwise}\end{cases}$$</td><td></td></tr>
</table>

[†]Typically, only the independent variable, z, is used as an argument when writing $\mu(z)$ in order to simplify notation. We made an exception in the S-shape curve in order to use its form in writing the equation of the Bell-shape curve.



a b c

FIGURE 3.28 (a) Low-contrast image. (b) Result of histogram equalization. (c) Result of fuzzy, rule-based, contrast enhancement.



a b c

FIGURE 3.29 Histograms of the images in Fig. 3.28(a), (b), and (c), respectively.