A Theory Paper on Rank List Length

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In its current iteration, this paper advances three avenues of thought on the size of the set of equilibria in matching markets, seeking to expand on the work by Immorlica and others who argue persuasively that rank list length is intimately related to core size. In section II I develop an example where agents are content with short rank lists, preferences are highly correlated, and yet a large fraction of agents have more than one equilibrium allocation. This section is well developed. In section III, I develop a framework to partition markets in a way that I hope will shed light on why some markets have small equilibria sets, and others large equilibria sets. This section is a work in progress–there are a number op unproven conjectures remaining, and the algorithm may still need modification to generate the desired properties. Finally, I advance the conjecture that in a setting where players may defect from the allocation provided by the mechanism, imposing short rank lists cannot shrink the equilibrium set without giving some players an incentive to defect from the market allocation.

I. Introduction

For reference I summarize the preference framework Immorlica and others use to show that short rank lists cause a small equilibrium set. Immorlica Preferences I now will describe the rank list construction method used by (Immorlica and Mahdian 2005) (which i will henceforth refer to as Immorlica preferences).

1) A distribution of hospitals is fixed (Uniform distribution for (Immorlica and Mahdian 2005), arbitrary distribution for (Kojima and Pathak 2009))

2) Each doctor draws a hospitals from the distribution (iid) until k distinct hospitals have been chosen these k hospitals form the rank list, in order of their draw

3) Hospitals have fully random preferences over doctors

II. Analysis of small perturbations from perfect correlation

This section develops a proof technique to build a plausible example of a large market where preferences are highly correlated, rank lists can be short, and the market has multiple stable equilibria in the large market limit.

A. Example

The following is an example in which the two sides of the market are nearly perfectly correlated, rank lists can be short, but a constant non-zero fraction of the market has more than one stable matching.

- To construct this market, first consider countable men indexed $\{m_0, m_1, ...\}$ and women indexed $\{w_0, w_1, ...\}$
- Now, to begin, assign preferences such that, for each man m_i, woman w_j is preferred to woman $w_{j'}$ iff $j' > j$, and for each woman w_j, man m_i is preferred to woman $m_{i'}$ iff $i' > i$. In other words, all women agree that man m_0 is most preferred, and so on.
- Next, we perturb the preferences of men by the following process. All entries on each man's preference list are simultaneously exposed to a Poisson process, with an expected success rate of ϵ, for one unit of time. If the entry for woman w_i on the preference list of man m_j experiences a Poisson success, he switches her with the woman she is immediately preferred to on his preference list, unless she is already his first choice. (I.e. he promotes her by one position on his rank list).

  For a given matching ϕ we categorize perturbations to a man m_j's rank list as follows:

  - Upward: The Poisson process hits the entry for $\phi(m_j)$
  - Downward: The Poisson process hits the entry in m_j's list immediately dis-preferred to $\phi(w_j)$
  - Irrelevant: The Poisson process hits any other entry

  Now we repeat the perturbation process for the women, applying the same categorizations.

B. Framework of the math

Now I develop a framework to analyze the above example

DEFINITION 1: *Next Preferred Available Woman (NPAW): each man's NPAW is the man's most preferred woman who is neither currently his match, nor is matched to a man she prefers to him.*

DEFINITION 2: *Immediately preferred: m_i is immediately preferred to $m_{i'}$ by woman w_j if she prefers m_i to $m_{i'}$ and there is no other man $m_{i''}$ such that she prefers $m_{i''}$ to $m_{i'}$, but not m_i.*

LEMMA 1: *Consider the stable matching ϕ under complete preferences. This algorithm will terminate at ϕ^w, the woman optimal stable matching.*

1) *Each man points to the current partner of his NPAW*
2) *Identify any cycles*
3) *In each cycle, each man re-matches with his NPAW*
4) *repeat until no more cycles are identified*

PROOF:

1) First we show by contradiction that the matching after step 3 is stable (call this matching ϕ):
   a) $m \succ_w \phi(w)$
      - Suppose there exists a blocking pair $\{m, w\}$ to the new match ϕ'.
      - $m \succ_w \phi'(w)$ as m, w blocks ϕ'
      - $\phi'(w) \succsim_w \phi(w)$ by construction
      - thus $m \succ_w \phi(w)$
   b) $\phi(m) \succ_m w \succ_m \phi'(m)$
      - $\phi(m) \succ_m w$ by combing conclusion 1a with the fact that m, w does not block ϕ
      - $w \succ_m \phi'(m)$

      This is a contradiction, since $\phi'(m)$ is m's NPAW in matching ϕ

2) Now, we show by contradiction that the algorithm terminates on the woman-optimal stable matching
   - Suppose the algorithm stops at stable matching $\phi' \neq \phi^w$, and call M_c the set of agents for whom $\phi'(m) \neq \phi^w(m)$ or $\phi'(w) \neq \phi^w(w)$
   - Since the algorithm has terminated, there is some man $m \in M_c$ whose NPAW is not in M_c (otherwise there would be a cycle).
   - now consider $w = \phi^w(m)$: Since ϕ^w is woman optimal and man pessimal, $\phi'(m) \succ_m w$ and $m \succ_w \phi'(w)$
   - since w is in M_c, she is not m's NPAW, so he must have some other NPAW w^* for whom:
     - $m \succ_{w^*} \phi'(w^*)$
     - $\phi'(m) \succ_m w^* \succ_m w$
     - $\phi'(w^*) = \phi^w(w^*)$
   - Thus $\{m, w^*\}$ blocks ϕ^w, a contradiction

C. Putting it together

- First consider man proposing DA with $\epsilon = 0$. Each man m_i's NPAW is m_{i+1}, so there are no cycles, and there is exactly one stable matching
- Now consider if man m_i has had an upwards perturbation, and w_{i-1} has a downward perturbation. The relevant preference lists are given below, with the man-optimal stable matchings underlined:
  - m_i: $\{..., w_{i-2}, \underline{w_i}, w_{i-1}, ...\}$
  - w_{i-1}: $\{..., \underline{m_i}, m_{i-1}, m_{i+1}, ...\}$
  - m_{i-1}: $\{..., \underline{w_{i-1}}, w_i, ...\}$
  - w_i: $\{..., m_{i-1}, \underline{m_i}, ...\}$

  We see that m_i and m_{i-1} are each matched to their NPAW, inducing a cycle and thus multiple stable matches.
- Next consider if man m_i has had a downward perturbation and woman w_{i+1} experiences an upwards perturbation. The relevant preference lists are given below, with the man-optimal stable matchings underlined:
  - m_i: $\{..., w_{i-1}, \underline{w_{i+1}}, w_i, ...\}$
  - w_i: $\{..., m_i, \underline{m_{i+1}}, ...\}$
  - w_{i+1}: $\{..., m_{i-1}, m_{i+1}, \underline{m_i}, ...\}$
  - m_{i+1}: $\{..., \underline{w_i}, w_{i+1}, ...\}$

  We see that m_i and m_{i+1} are each matched to their NPAW, inducing a cycle and thus multiple stable matches.
- For small ϵ values, each man expects to be involved in one of these cycles with probability $2\epsilon^2$, regardless of the size of the market.
- Finally, since perturbations are rare, agents are nearly guaranteed to match within a very small range around the match they would have if $\epsilon = 0$, so a short rank list is equivalent to listing complete preferences.

III. An Algorithm to Partition the Market

This is a related approach: I conjecture that it will be conceptually valuable to partition the market into "segments" that are functionally separate . To do this,

Call the universe of unmatched participants U, the first choice of man m among the unmatched women $p_1(m)$, and likewise for women.

To start, initialize each m_i with the attribute $m_i.R = \emptyset$

initialize counter $i = 1$ The algorithm proceeds as follows:

1) Call the man with the lowest index number m^*, and call the set $C = \{p_1(m^*), p_1(p_1(m^*)), ...\}$ (Note that this will form a cycle, since the set of participants is finite)

2) Call ϕ_m the matching produced by man proposing deferred acceptance within set C and ϕ_w likewise for woman proposing deferred acceptance. Call the set of all matched participants ϕ

3) For each man $m \in C$, add to C each woman from the universe of agents that he prefers to $\phi_w(m)$, and likewise for each woman $w \in C$ and $\phi_m(w)$

4) If step 3 adds members to C, return to step 2. Otherwise proceed

5) Terminate if $\phi = \emptyset$, otherwise remove ϕ from U, rename $\{\phi_m, \phi_w, C\}$ as $\phi_i = \{\phi_{mi}, \phi_{wi}, C_i\}$, increment i and return to 1

LEMMA 2: *ϕ_{mi} is a re-arrangement of the matched agents in ϕ_{wi}*

PROOF:
This is an application of the rural hospitals theorem

PROPOSITION 1: *$\phi_m = \{\phi_{m1}, \phi_{m2}, ...\}$ is identical to the man proposing deferred acceptance matching Φ_m*

PROOF:
First we prove this for ϕ_{m1}

- S'pose not. There is some man m such that $\Phi_m(m) \neq \phi_{m1}(m)$
- $w = \Phi_m(m) \in C_1$ because of step 3
- Call $m' = \phi_{m1}(\phi_{m1}(m))$ and note that $\phi_{m1}(m') \neq \Phi_m(m')$
- There is thus a finite cycle of men $\{m, m', m'', ..., m^n\}$ such that $\phi_{m1}(m')$ rejected m in favor of m', $\phi_{m1}(m'')$ rejected m' in favor of m'', and so on until $\phi_{m1}(m)$ rejected m^n in favor of m. This is impossible

We now note that after removing ϕ_{m1} from U, the same logic holds, and the proof continues by induction.

PROPOSITION 2: *If there is a man $m \in \{C_i \setminus \phi_i\}$ then there is no woman $w \in \{C_i \setminus \phi_i\}$*

PROOF:
This is obvious as all participants rank all other participants

CONJECTURE 1: *A non-trivial partition of the market has more than one stable allocation if it is balanced* each man is inclrded because he was prefered to some woman's match in the mpda matching. thus he has a NPAW. If each man has an NPAW, then there is a cycle, and thus an alternats stable matching.

CONJECTURE 2: *A partition of the market has only one stable allocation if it is not balanced*

IV. A Conjecture on short rank lists

CONJECTURE 3: *In a setting where players may defect from the allocation provided by the mechanism, imposing short rank lists cannot shrink the equilibrium set without giving some players an incentive to defect from the market allocation.*

PROOF:

The basic idea here is that short rank lists shrink the Equilibrium set by forcing agents to leave unranked (ie declare unacceptable) matches that they would have in their less prefered allocations. If they have enough information to choose this cutoff optimally, then they wouldnt need to be forced to use a short rank list (they would do so anyway). If, however, they don't have very good information, then they will sometimes be unmatched, and be able to induce someone to defect from the allocation with them.

Mathematical Appendix