Status Update for PhD Progress

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Outline for Today

1. Goals
   - Oral in 6 Months?
   - Seek MD specific feedback
   - 3 *Good* Papers

2. Research Agenda
   - Animating Conjecture
   - Pieces of this
     - Characterize early contracting
     - Characterize when matching games have 'large' cores
     - Characterize the strategic interactions within the matching game
     - Formally combine these ideas

3. Technical Summary of 'Large Core' Work

Oral in 6 Months?

- Discuss timing for oral exam

Seek MD specific feedback

- Fujito Kojima seems like a good place to start once i have something short and polished
- Should I seek out a market design specialist to have on my committee?

3 *Good* Papers

- I want to finish up reasonably soon, but definitely don't want to skimp on paper quality

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Early contracting will result from a matching process where there is lots of scope for strategy

Characterize early contracting

- This is my “Unraveling” line of work
- In this line of work I have a model where there is a self-fulfilling process of early contracting that is welfare destroying (the state where all participate in the organized market is pareto preferred to equilibrium)

Characterize when matching games have ‘large’ cores

This is my most current work, which I will reprise in detail below

Characterize the strategic interactions within the matching game

I believe this to be an intractable problem in the general case but I think I can solve it in my simpler model of preferences that we have been looking at more recently

Formally combine these ideas

- This is ambitious–I'd like to say something in general about matching markets where the choice to participate is a strategic option.
- I believe this is also the biggest gap in market design. MD mechanisms are embedded in a larger dame that the designers do not control, and thats not well enough covered in the literature yet

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Technical Summary of ‘Large Core’ Work

the following is a technical summary of my work on the size of the set of stable matches

Model Definitions

- M men and W women: $U = W \cup M$.
- Men have preferences $\succ_m$ over $W \cup \{m\}$ and women likewise
- A matching ϕ is a mapping from U onto itself such that
  - for every m, $|\phi(m)| = 1$, $\phi(m) \in W \cup \{m\}$ and for every w, $|\phi(w)| = 1$, $\phi(w) \in M \cup \{w\}$
  - $\phi(m) = w$ iff $\phi(w) = m$
- ϕ is blocked by a pair $\{m, w\}$ if $w \succ_m \phi(m)$ and $m \succ_w \phi(w)$
- It is individually rational if $\forall a \in U, \phi(a) \succeq a$
- ϕ is stable if it is individually rational and not blocked

Deferred Acceptance

The deferred acceptance mechanism is a mainstay of this paper. In the man-proposing version of this mechanism, the algorithm iteritavely selects a man m who proposes to his most preferred woman who has not yet rejected him, unless he has been rejected by all women who he prefers to being unmatched. If she prefers him to her current tenative assignment (and prefers him to being unmatched), she holds his proposal and rejects her current assignment (if one exists). Otherwise, she rejects m. This process repeats until all men are either tentatively matched or have run out of proposals to make, and arrives at a stable matching that is weakly preferred by all men to any other stable matching. We call this the man-optimal stable match, because it is weakly preferred by all men to any other stable match. A parallel result exists for the woman-proposing algorithm.

Market Participants

- n men and n women
- indexed $\{m_1, m_2, ..., m_n\}$ and $\{w_1, w_2, ..., w_n\}$
- Preference formation
  - m_1 first ranks all women according to their index number, so w_1 is his favorite, w_n his least favorite, and so on.
  - With probability ϵ he transposes the rankings of his first and second favorite woman, then with probability ϵ he transposes the rankings of his second and third favorite women, and so on through his list.
  - each man and woman also forms their preferences the same way

An Algorithm to Partition the Market

Setup:

- $U_0 = M \cup W$, define $p_1(m)$ is man m's first choice among U_0, and likewise for women.
- m, the man in U_0 with the lowest index number, and select $w = p_1(m)$, $m' = p_1(w)$, and so on, until a cycle is found. Call the members of that cycle C^*.

An Algorithm to Partition the Market

Process:

1. define the correspondence R from U_0 to $\mathcal{P}(U_0)$ such that $\forall a \in U_0, R(a) = \emptyset$
2. for each $a \in C^*$, alter R by replacing $R(a)$ with $R(a) \cup \{p_1(m)\}$
3. for each $a \in U_0$, and $b \in U_0$, if $b \in R(a)$ replace $R(b) = \{a\} \cup R(b)$
4. for each $a \in U_0$ such that $R(a) \neq \emptyset$, call c their least preferred member of $R(a)$. Now, for each $d \in U_0 | d \succ_a c$, replace $R(a) = R(a) \cup d$
5. if R has been altered in step 3 or 4, return to step 3, otherwise proceed
6. call C_0 the set of all agents for whom $R(a) \neq \emptyset$
7. Perform man-proposing deferred acceptance on C_0, and call the result ϕ_{m0}

To find the next partition, call $U_1 = U_0 \setminus \{\phi_{m0}\}$, and repeat the entire process to find C_1 and ϕ_{m1}

An Algorithm to Partition the Market

Iterate:
To find the next partition, call $U_1 = U_0 \setminus \{\phi_{m0}\}$, and repeat the entire process to find C_1 and ϕ_{m1}

An Algorithm to Partition the Market

Lemma

$\{\phi_{mi}\} = \{\phi_{wi}\}$

Proof.

This is an application of the rural hospitals theorem, which says that the set of unmatched agents is the same in all stable matchings (Roth 1986). $\square$

An Algorithm to Partition the Market

Proposition

$\phi_m = \{\phi_{m0}, \phi_{m1}, ...\}$ is identical to the man proposing deferred acceptance matching Φ_m

An Algorithm to Partition the Market

Proof.

First we prove this for ϕ_{m0}, proceeding by contradiction. Call
$w = \phi_{m0}(m) \neq \Phi_m(m)$
Call $m' = \Phi_m(w)$
Either $m' \succ_w m$ or $m \succ_w m'$. We will handle each case separately.

An Algorithm to Partition the Market

Proof.

First suppose $m' \succ_w m$

- Since w is always matched to her favorite man of all who ever propose to her, m' must never have proposed to w. Since men propose in descending order of prefernce until all acceptable options are exhausted, m' must be matched to $w' = \phi_{m0}(m')$ where $w' \succ_{m'} w$
- call $m'' = \Phi_m(w')$
- as the pair $\{m', w'\}$ are not a blocking pair for Φ_m, $m'' \succ_{w'} m'$
- we can continue as above to form the infinate sequence $\{m, w, m', w', m'', ...\}$ contained within the finite set C_0. Since this is an infinite sequence on a finite set, we have a cycle such that each man prefers the woman he is matched with in ϕ_{m0}, and each women prefers her match in Φ_m

An Algorithm to Partition the Market

Proof.

First suppose $m' \succ_w m$ (Continuing:)

- this cycle represents a pareto improvement over Φ_m for men, and thus it cannot be a stable matching on U. So there must be a man in $U \setminus C_0$ such that he forms a blocking pair with one of the women in this cycle.
- the man should have been added to C_0 at step 4, thus a contradiction

An Algorithm to Partition the Market

Proof.

Now suppose $m \succ_w m'$

- call $w^a = \Phi_m(m)$
- as $\{m, w\}$ do not block ϕ_{m0}, $w^a \succ_m w$
- call $m^a = \phi_{m0}(w^a)$
- now as $\{m, w^a\}$ do not block Φ_m, $m^a \succ_{w^a} m$
- now call $w^b = \Phi_m(m^a)$ and continue as above to form an infinite sequence on the finite set C_0
- we now have a cycle where Φ_m is a pareto improvement over ϕ_{m0} for men

An Algorithm to Partition the Market

Proof.

Now suppose $m \succ_w m'$ (Continuing:)

- Since this is a stable match in U but not in C_0, there must be a man in $C_0 \setminus U$ such that he forms a blocking pair with one of the women in this cycle.
- $C_0 \subseteq U$, so $C_0 \setminus U = \emptyset$, and so we have a contradiction

An Algorithm to Partition the Market

Proof.

Thus ϕ_{m0} is a subset of Φ_m
We now note that after removing ϕ_{m0} from U, the same logic holds, and the proof continues by induction. $\square$

An Algorithm to Partition the Market

Proposition

If there is a man $m \in \{C_i \setminus \phi_i\}$ *then there is no woman* $w \in \{C_i \setminus \phi_i\}$

Proof.

As all participants rank all other participants, if there were an unmatched man and woman, they would wish to match with each other. $\square$

An Algorithm to Partition the Market

Proposition

ϕ_{mi} depends only on the preferences held by the agents in $\{C_0 \cup C_1 \cup ... \cup C_i\}$

Proof.

Given proposition 1, it is sufficient to show that there is no set of preferences that could be held by $U \setminus \{C_0 \cup C_1 \cup ... \cup C_i\}$ that would change the makeup of C_i.
Since no member of C^* holds any member of $U \setminus \{C_0 \cup C_1 \cup ... \cup C_i\}$ as their first choice, no other preference profile held by members of $U \setminus \{C_0 \cup C_1 \cup ... \cup C_i\}$ would alter the composition of C^*.
Since steps 3 and 4 of the algorithm in subsection **??** only depend on the preferences of individuals for whom $R(a) \neq \emptyset$, no preferences held by any member of the set $U \setminus \{C_0 \cup C_1 \cup ... \cup C_i\}$ influences the workings of the algorithm.
Thus, the preferences of $U \setminus \{C_0 \cup C_1 \cup ... \cup C_i\}$ are irrelevant to ϕ_{mi} $\square$

Analyzing the model

We begin by partitioning the market as above.
With probability $p = \epsilon^2(1-\epsilon)^2$, w_1 and m_2 invert their first and second choices, yielding the following preference lists with the man-optimal stable matching underlined, and the woman optimal matching denoted with a bar.

- m_1: $\{\underline{w_1}, \overline{w_2}, ...\}$
- m_2: $\{\underline{w_2}, \overline{w_1}, ...\}$
- w_1: $\{\overline{m_2}, \underline{m_1}, ...\}$
- w_2: $\{..., \overline{m_1}, \underline{m_2}, ...\}$

$\{m_1, m_2, w_1, w_2\}$ form a set C, and within that set the match $\phi_m = \{m_{i-1} : w_{i-1}\}, \{m_i : w_i\}$ and the match $\phi_w = \{m_{i-1} : w_i\}, \{m_i : w_{i-1}\}$ are both stable assignments.

Analyzing the model

Next consider a similar but different set of preferences (a scenario that also occurs with probability $p = \epsilon^2(1-\epsilon)^2$). The man-optimal stable matching is underlined, and the woman optimal matching is denoted with a bar.

- m_1: $\{..., \underline{w_2}, \overline{w_1}, ...\}$
- m_2: $\{..., \underline{w_1}, \overline{w_2}, ...\}$
- w_1: $\{..., \overline{m_1}, \underline{m_2}, ...\}$
- w_2: $\{..., \overline{m_2}, \underline{m_1}, ...\}$

We can see that m_1 and m_2 expect to be involved in one of these cycles with probability $p = 2\epsilon^2(1-\epsilon)^2$.

Analyzing the model

Next we must show that this possibility is very likely to apply to each man, and so each agent is involved in such a cycle with a probability of approximately $p \approx 4\epsilon^2(1-\epsilon)^2$.
Finally, I need to bound the chances of a “far away” match to show that nobody would mind submitting a short rank list.

Bibliography

Roth, Alvin E. 1986. "On the allocation of residents to rural hospitals: a general property of two-sided matching markets." *Econometrica: Journal of the Econometric Society*, 425—-427.