Partial Fractions

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of the preceding example, it consists of finding constants A and B such that

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}. \qquad (1)$$

We call the fractions $A/(x+1)$ and $B/(x-3)$ **partial fractions** because their denominators are only part of the original denominator x^2-2x-3. We call A and B **undetermined coefficients** until suitable values for them have been found.

To find A and B, we first clear Equation (1) of fractions and regroup in powers of x, obtaining

$$5x-3 = A(x-3) + B(x+1) = (A+B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. Example 3 of this section illustrates such a case.
- We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known. A quadratic polynomial (or factor) is **irreducible** if it cannot be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.

Method of Partial Fractions When $f(x)/g(x)$ Is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2+px+q)} + \frac{B_2x + C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx + C_n}{(x^2+px+q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}$$

Solution Note that each of the factors $(x - 1)$, $(x + 1)$, and $(x + 3)$ is raised only to the first power. Therefore, the partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

To find the values of the undetermined coefficients A, B, and C, we clear fractions and get

$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x, obtaining

$$\begin{aligned} &\text{Coefficient of } x^2: && A + B + C = 1 \\ &\text{Coefficient of } x^1: && 4A + 2B = 4 \\ &\text{Coefficient of } x^0: && 3A - 3B - C = 1 \end{aligned}$$

There are several ways of solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables or the use of a calculator or computer. The solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\frac{6x + 7}{(x + 2)^2}$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2} \qquad \text{Two terms because } (x + 2) \text{ is squared}$$

$$6x + 7 = A(x + 2) + B \qquad \text{Multiply both sides by } (x + 2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2}$$

Solution The denominator has an irreducible quadratic factor $x^2 + 1$ as well as a repeated linear factor $(x - 1)^2$, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \tag{2}$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 \\ &\quad + (A - 2B + C)x + (B - C + D). \end{aligned}$$

Equating coefficients of like terms gives

$$\begin{aligned} &\text{Coefficients of } x^3: & 0 &= A + C \\ &\text{Coefficients of } x^2: & 0 &= -2A + B - C + D \\ &\text{Coefficients of } x^1: & -2 &= A - 2B + C \\ &\text{Coefficients of } x^0: & 4 &= B - C + D \end{aligned}$$

We solve these equations simultaneously to find the values of A, B, C, and D:

$$-4 = -2A, \qquad A = 2 \qquad \text{Subtract fourth equation from second.}$$

$$C = -A = -2 \qquad \text{From the first equation}$$

$$B = (A + C + 2)/2 = 1 \qquad \text{From the third equation and } C = -A$$

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

$$\frac{1}{x(x^2+1)^2}:$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned} 1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A. \end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

Solving this system gives $A=1$, $B=-1$, $C=0$, $D=-1$, and $E=0$. Thus, Find A, B, and C in the partial fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Solution If we multiply both sides of Equation (3) by $(x-1)$ to get

$$\frac{x^2+1}{(x-2)(x-3)} = A + \frac{B(x-1)}{x-2} + \frac{C(x-1)}{x-3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2+1}{(1-2)(1-3)} = A + 0 + 0,$$

$$A = 1.$$

In exactly the same way, we can multiply both sides by $(x-2)$ and then substitute in $x = 2$. This gives

$$\frac{(2)^2+1}{(2-1)(2-3)} = B.$$

So $B = -5$. Finally, we multiply both sides by $(x-3)$ and then substitute in $x = 3$, which yields

$$\frac{(3)^2+1}{(3-1)(3-2)} = C,$$

and $C = 5$. ∎

Find A, B, and C in the expression

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

by assigning numerical values to x.

Solution Clear fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$\begin{aligned}
x = 1: &\quad (1)^2 + 1 = A(-1)(-2) + B(0) + C(0) \\
&\quad 2 = 2A \\
&\quad A = 1 \\
x = 2: &\quad (2)^2 + 1 = A(0) + B(1)(-1) + C(0) \\
&\quad 5 = -B \\
&\quad B = -5 \\
x = 3: &\quad (3)^2 + 1 = A(0) + B(0) + C(2)(1) \\
&\quad 10 = 2C \\
&\quad C = 5.
\end{aligned}$$

Conclusion:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}:$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r}
2x \phantom{{}- 4x^2 - x - 3} \\
x^2 - 2x - 3 \overline{\smash{)}\, 2x^3 - 4x^2 - x - 3} \\
\underline{2x^3 - 4x^2 - 6x} - 3 \\
5x - 3
\end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

Functions of Single variable

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a *unique* value $f(x)$ in Y to each x in D.

The symbol f represents the function, the letter x is the **independent variable** representing the input value to f, and y is the **dependent variable** or output value of f at x.

FIGURE 1.1 A diagram showing a function as a kind of machine.

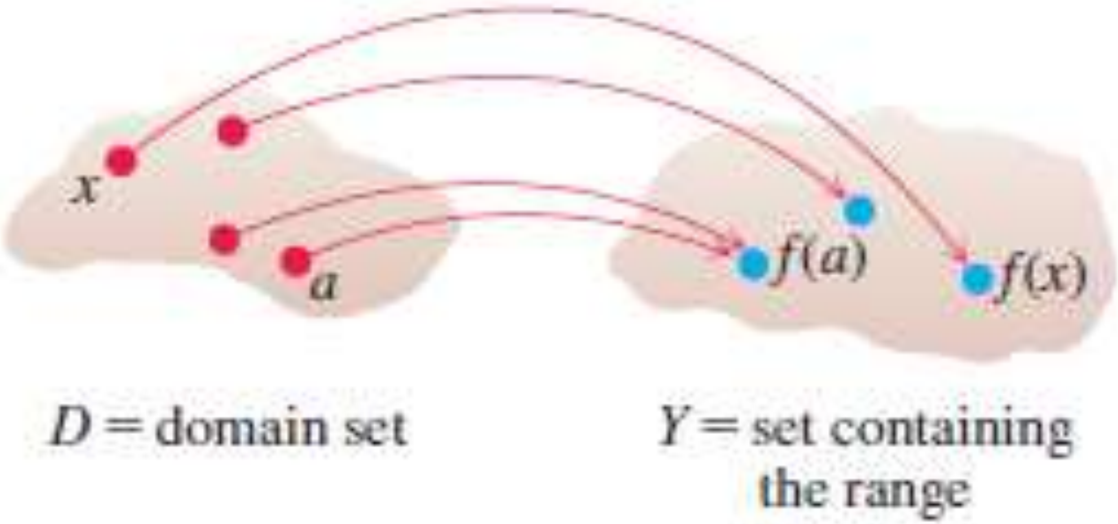
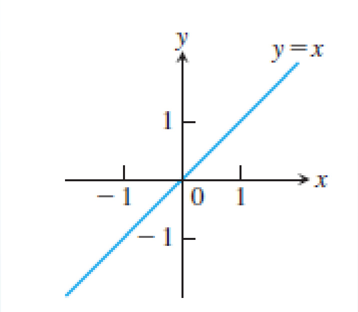
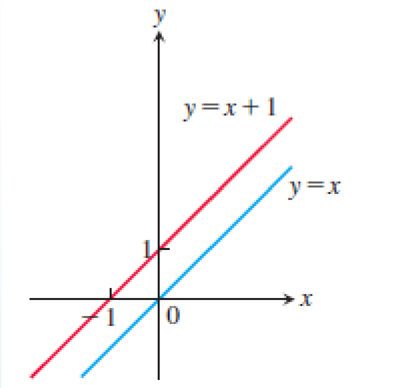


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

Graphs of basic curves





(b)

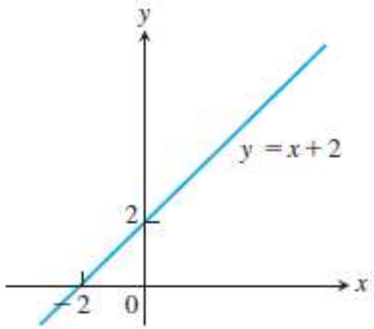
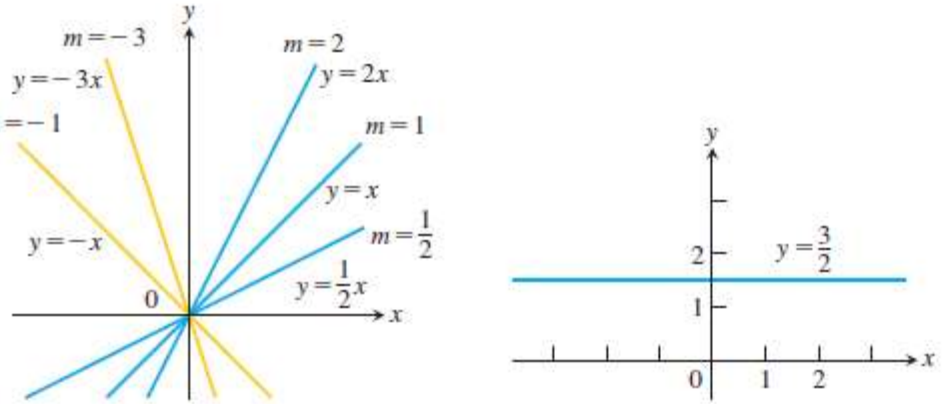


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

Figure 1.14a shows an array of lines $f(x) = mx$.



(a) (b)

FIGURE 1.14 (a) Lines through the origin with slope m. (b) A constant function with slope $m = 0$.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

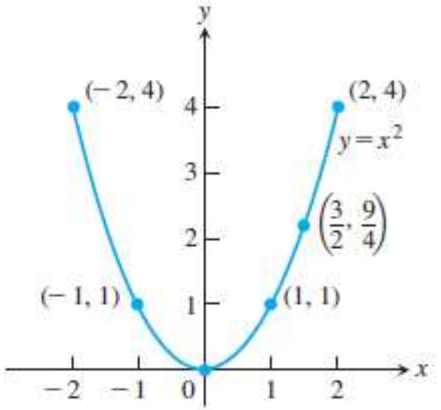
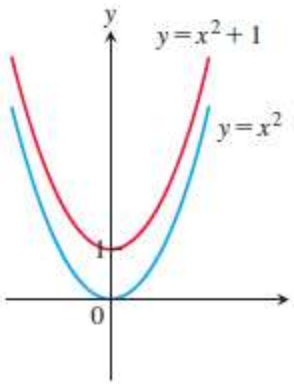


FIGURE 1.5 Graph of the function in Example 2.



(a)

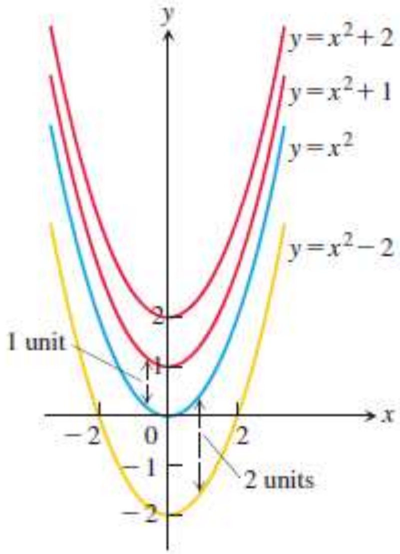


FIGURE 1.29 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Examples 3a and b).

Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$ — Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ — Shifts the graph of f *left* h units if $h > 0$
Shifts it *right* $|h|$ units if $h < 0$