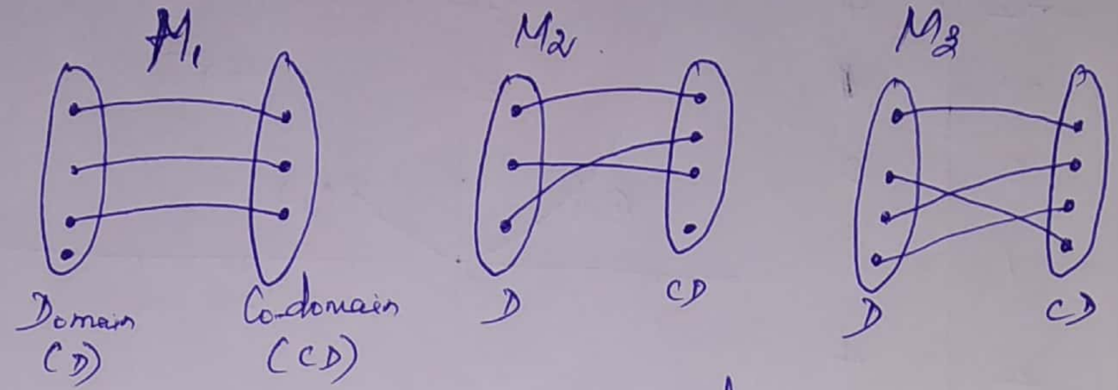
Functions & Graphs:

Function is a well defined mapping.



M_2 & M_3 both are functions.

Sketch the graphs of the following functions.

(i). $y = f(x) = x$, (ii) $y = x^2$ (or) $f(x) = x^2$

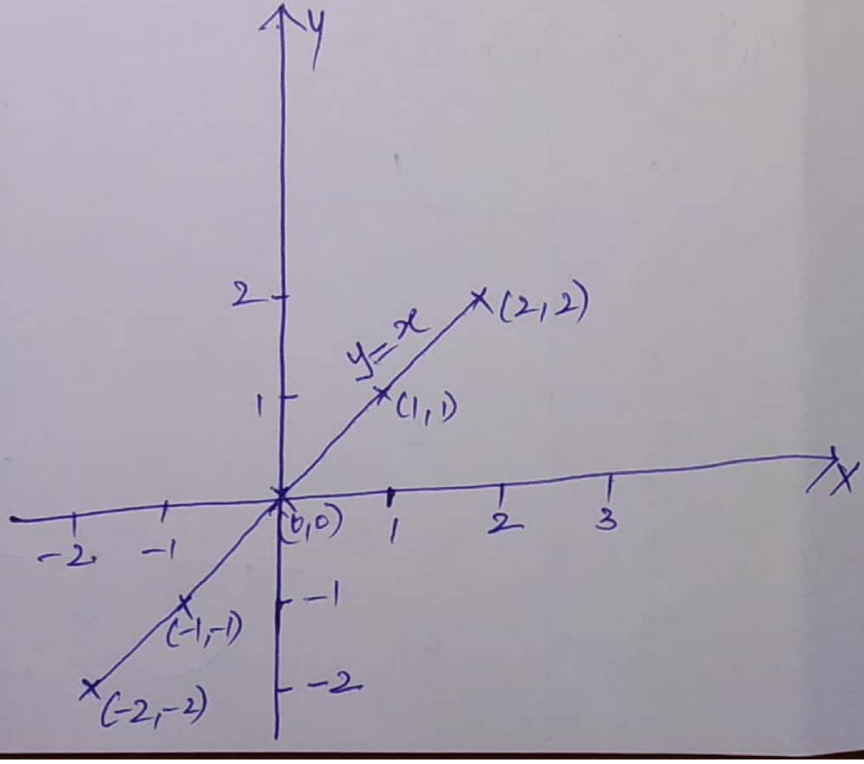
(iii) $y = \sin x$, (iv). $y = \cos x$ (v) $f(x) = |x|$

(vi) $y = |x| + 2$, (vii). $y = |x| - 2$ (viii). $y = |x+2|$

(ix) $y = |x-2|$

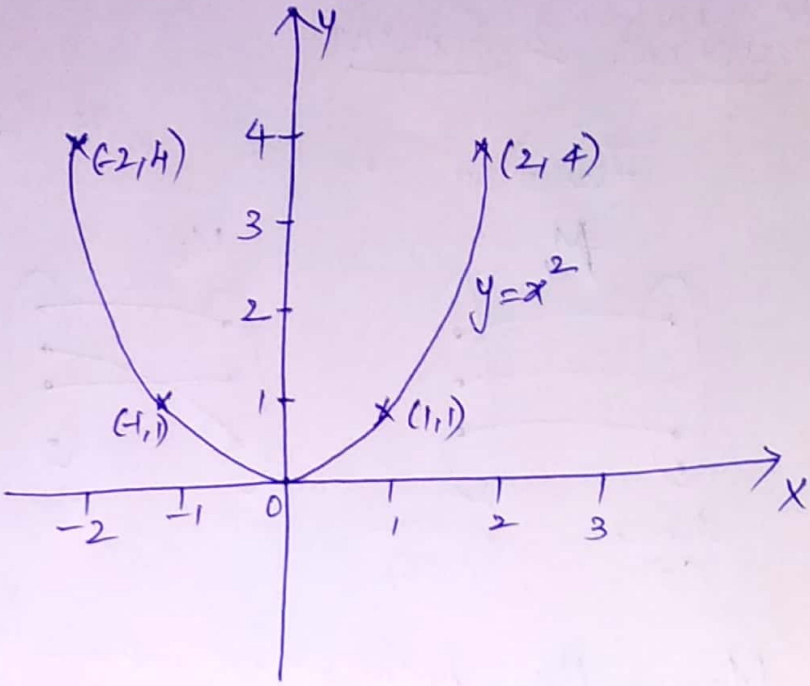
(i) $y = x$

x	y
-2	-2
-1	-1
0	0
1	1
2	2



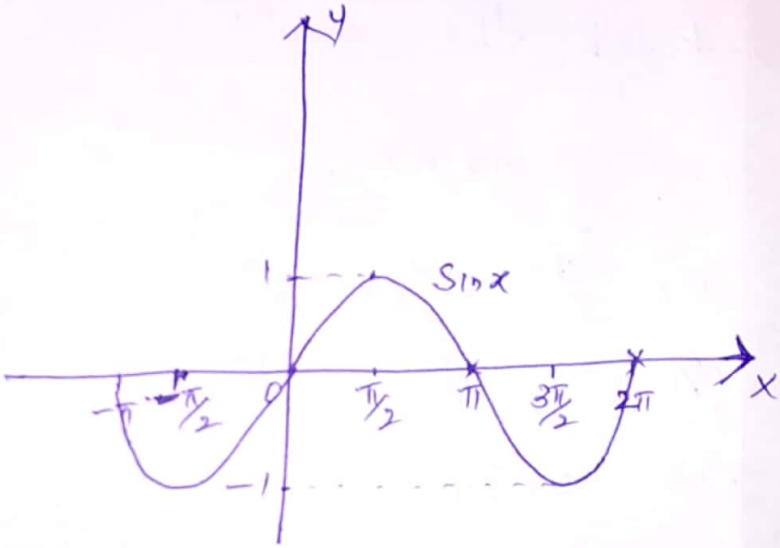
(ii) $y = x^2$

x	y
-2	4
-1	1
0	0
1	1
2	4



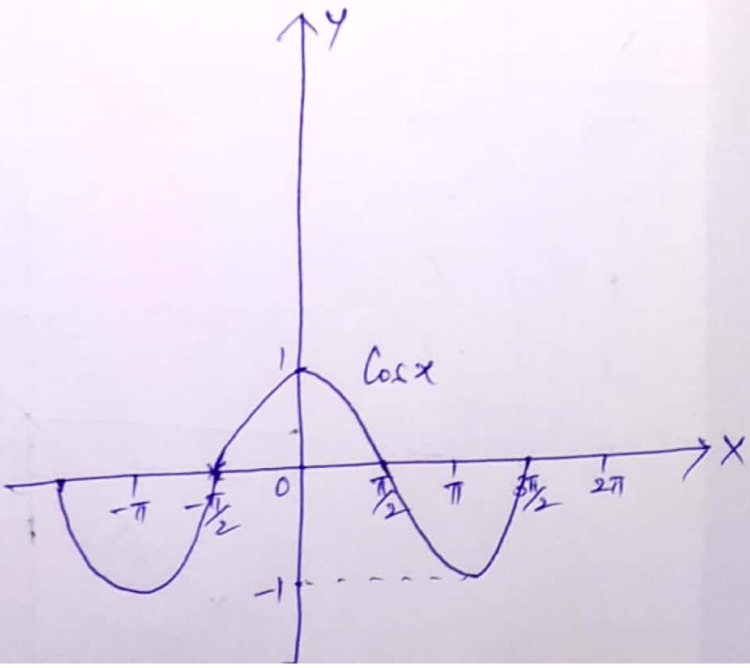
(iii). $y = \sin x$

x	y
0	0
$\pi/2$	1
π	0
$3\pi/2$	-1
2π	0



(iv) $y = \cos x$

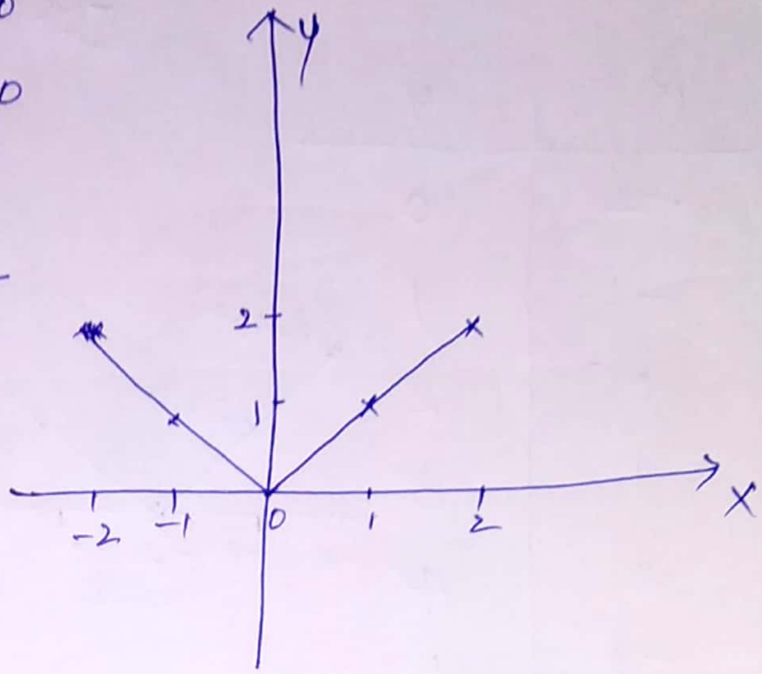
x	y
$-\pi/2$	0
0	1
$\pi/2$	0
π	-1
$3\pi/2$	0



(v). $y = f(x) = |x|$

$$y = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

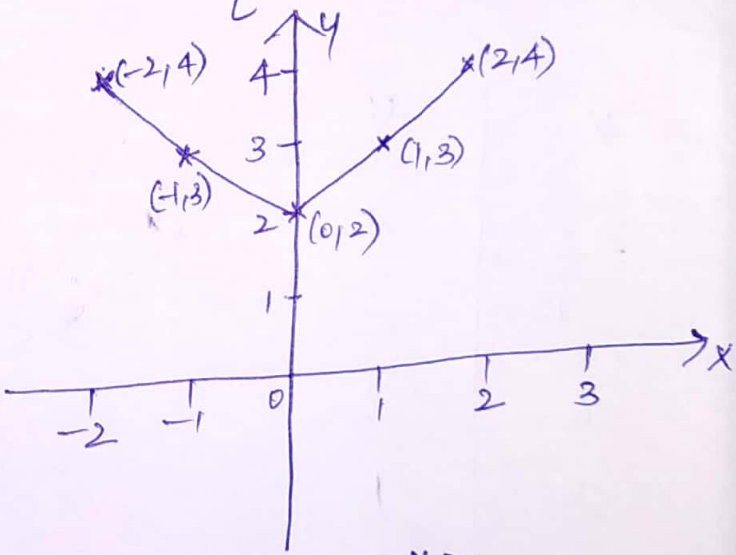
x	y
-2	$-(-2) = 2$
-1	$-(-1) = 1$
0	0
1	1
2	2



(vi) $y = |x| + 2$

$$y = 2 + \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

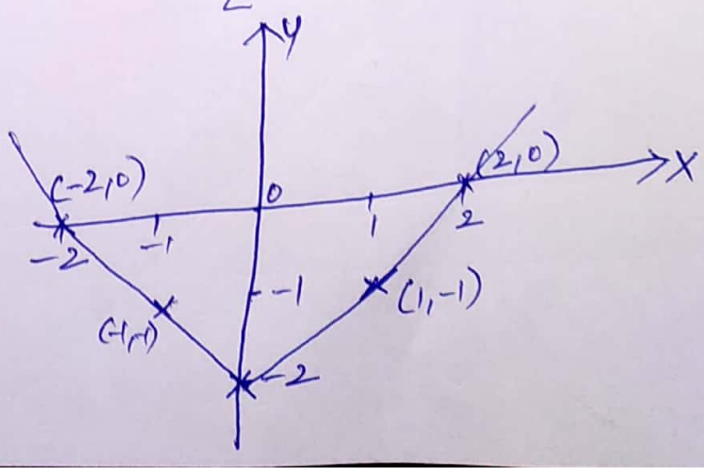
x	y
-2	4
-1	3
0	2
1	3
2	4



(vii). $y = |x| - 2$

$$y = -2 + \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

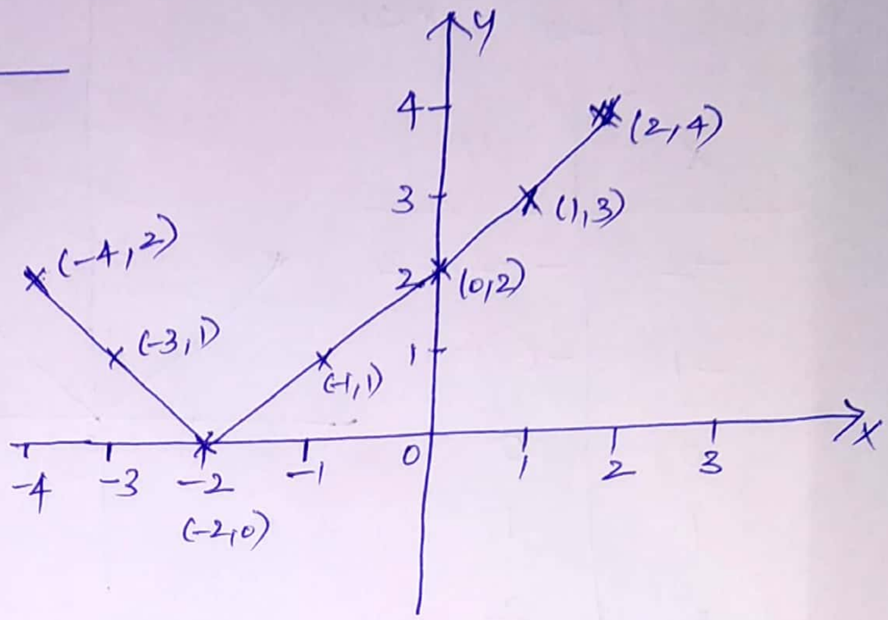
x	y
-2	0
-1	-1
0	-2
1	-1
2	0



(viii). $y = |x+2|$

$$y = \begin{cases} x+2, & x \geq -2 \\ -(x+2), & x < -2 \end{cases}$$

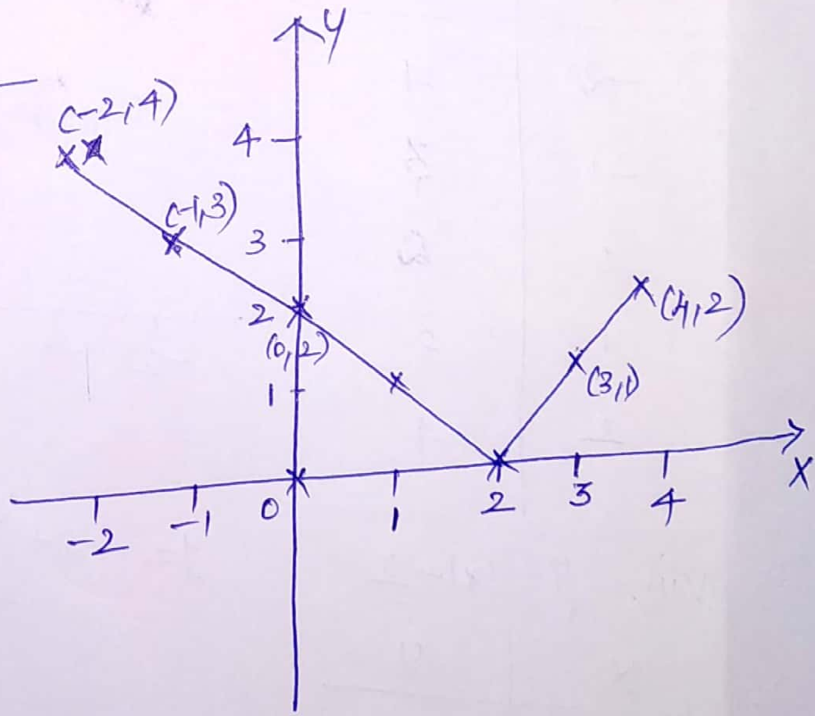
x	y
-2	0
-1	1
0	2
1	3
2	4
-3	1
-4	2



(ix): $y = |x-2|$

$$y = \begin{cases} x-2, & x \geq 2 \\ -(x-2), & x < 2 \end{cases}$$

x	y
-2	4
-1	3
0	2
1	1
2	0
3	1
4	2



Note:

Even and odd functions:

1. If a function $y=f(x)$ is symmetrical about the y-axis (ie) mathematically, $\boxed{f(-x)=f(x)}$ then the function is even.

Ex: x^2, $\cos x$, x^4, constant function, $|x|$ etc.

2. If a function $y=f(x)$ is symmetrical about the origin (ie) mathematically $\boxed{f(-x)=-f(x)}$ then the function is odd.

Ex: x^3, $\sin x$, x^5 etc.

3. If $f(-x)\neq f(x)$ and $f(-x)\neq -f(x)$ then the function $f(x)$ is neither even nor odd.

Ex: e^x, e^{-x}

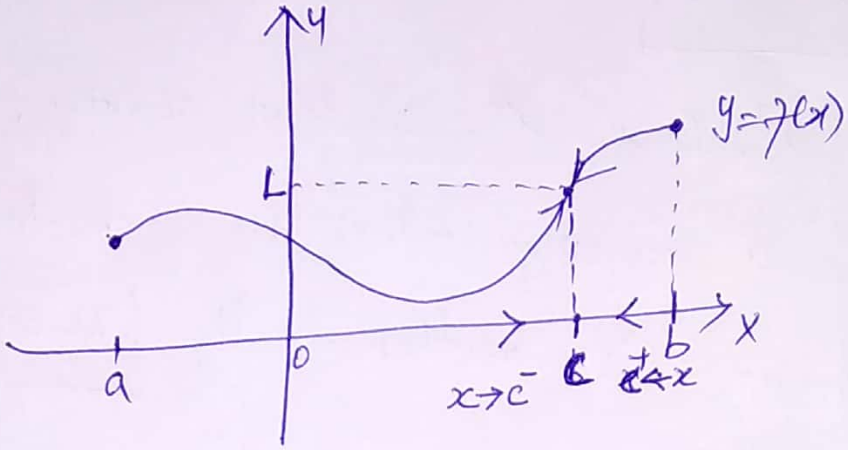
Problems: Sketch the graph of the following functions.

1. $y=e^x$, 2. $y=e^{-x}$ 3. $y=\frac{e^x+e^{-x}}{2}$

4. $y=\frac{e^x-e^{-x}}{2}$

Limit of a function $y = f(x)$:

Let $y = f(x)$ be any function defined over the interval $[a, b]$. and the graph is shown below. c be a point is (a, b).



Does $\underset{x \to c}{Lt} f(x)$ exist?

$$f(c^-) = \underset{x \to c^-}{Lt} f(x) \to L$$

$$f(c^+) = \underset{x \to c^+}{Lt} f(x) \to L$$

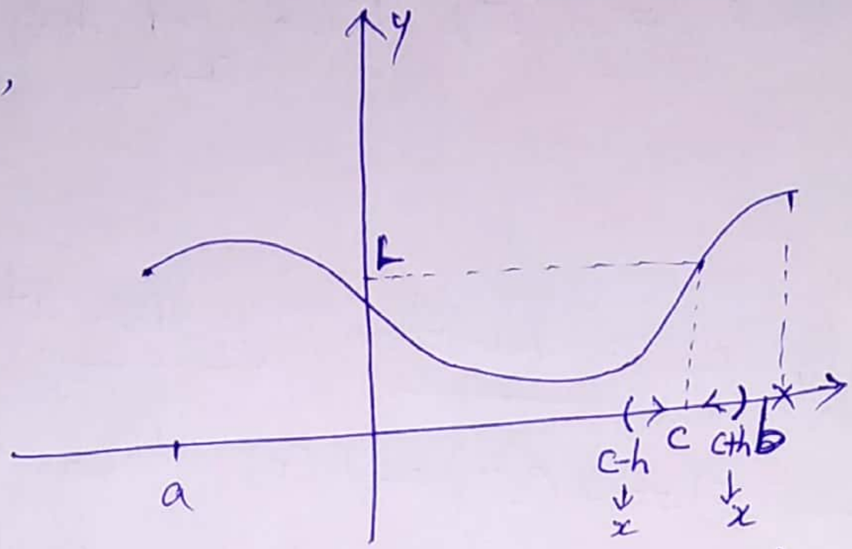
Left hand limit
Right hand limit

If $f(c^-) = f(c^+) = L$ (Both are same)

then $\underset{x \to c}{Lt} f(x)$ exists and the value is equal to L. (ie) $\underset{x \to c}{Lt} f(x) = L$.

<u>Note</u>: When we discuss about the existence of the limit at the point 'c', the function value exactly at $x = c$ (ie) $f(c)$ is not necessary.

Here,



$$f(c^-) = \underset{x \to c^-}{Lt} f(x) = \underset{c-h \to c}{Lt} f(c-h) = \underset{h \to 0}{Lt} f(c-h)$$

$$f(c^+) = \underset{x \to c^+}{Lt} f(x) = \underset{c+h \to c}{Lt} f(c+h) = \underset{h \to 0}{Lt} f(c-h)$$

(ie), $f(c^-) = \underset{h \to 0}{Lt} f(c-h)$ —— (1)

$f(c^+) = \underset{h \to 0}{Lt} f(c+h)$ —— (2)

If (1) & (2) are equal then $\underset{x \to c}{Lt} f(x)$ exists, and is equal 'l' if $f(c^-) = f(c^+) = l$.

Suppose the left hand limit of $f(x)$ at $x = c$ is not equal to the right hand limit of $f(x)$ at $x = c$ (ie)

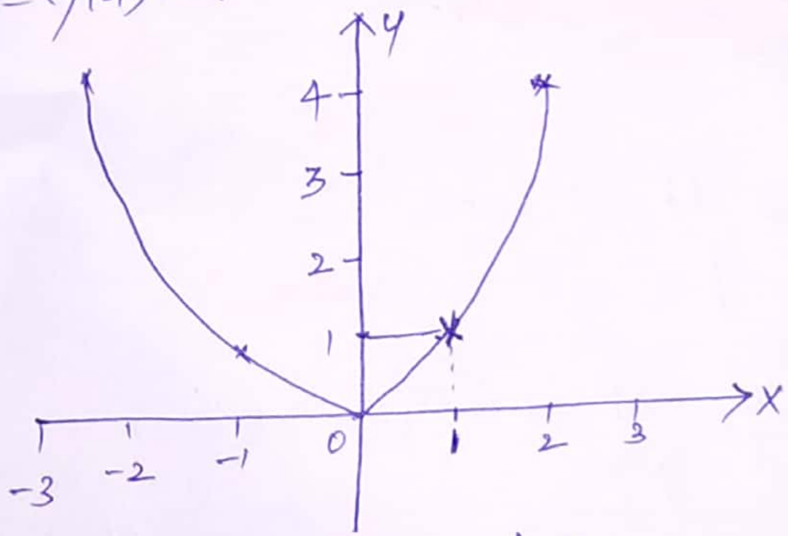
If $f(c^-) \neq f(c^+)$ then $\underset{x \to c}{Lt} f(x)$ does not exist.

~~Ex~~

Continuity of a function $y = f(x)$ at $x = c$:-

If $f(c^-) = f(c^+) = f(c)$ then the function is continuous at the point $x = c$.

1. Ex:- Let us consider the graph of a function $y = f(x) = x^2$ in $[-2, 2]$.



Does $\underset{x \to 1}{Lt} f(x)$ exist?

Using equation (1) and (2),

$f(1^-) = \underset{h \to 0}{Lt} f(1-h) = \underset{h \to 0}{Lt} (1-h)^2 = 1$

$f(1^+) = \underset{h \to 0}{Lt} f(1+h) = \underset{h \to 0}{Lt} (1+h)^2 = 1.$

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$\therefore \underset{x\to 1}{Lt}\, f(x)$ exist and is equal to 1.

(ie) $\underset{x\to 1}{Lt}\, f(x) = 1$ [Obvious that $f(1^-) = f(1^+)$].

<u>Note</u>:

If $f(x) = x^2$, then $f(1) = 1^2 = 1$.

$\therefore f(1^-) = f(1^+) = f(1) = 1$.

$\Rightarrow \underset{x\to c^-}{Lt}\, f(x) = \underset{x\to c^+}{Lt}\, f(x) = f(c)$ at $c = 1$.

$\Rightarrow$ The function is ~~continuous~~ continuous at the point $x = 1$.

(ie) $\boxed{\underset{x\to 1}{Lt}\, f(x) = 1 = f(1)}$.

2. Similarly, if $f(x) = |x|$, does $\underset{x\to 0}{Lt}\, f(x)$ exists?

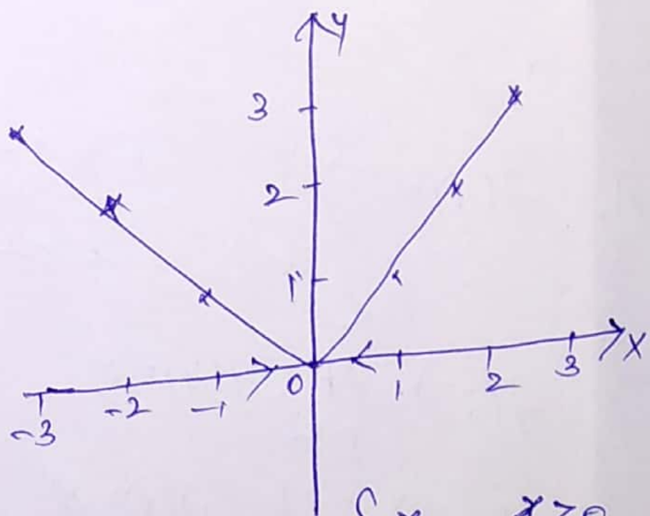
$f(0^-) = \underset{h\to 0}{Lt}\, f(0-h)$

$= \underset{h\to 0}{Lt}\, -(0-h)$

$= \underset{h\to 0}{Lt}\, h = 0$

$f(0^+) = \underset{h\to 0}{Lt}\, f(0+h)$

$= \underset{h\to 0}{Lt}\, 0+h = 0.$



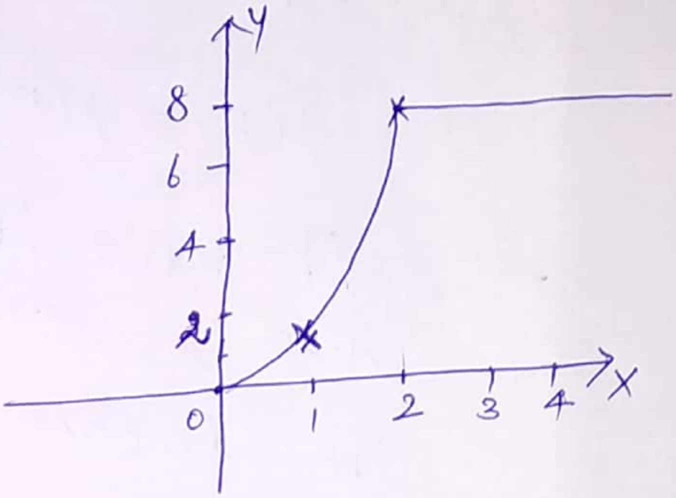
$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$f(0) = 0.$

$\therefore f(0^-) = f(0^+) \Rightarrow \lim\limits_{x \to 0} f(x)$ exists. and is equal to 0. $\therefore f(0^-) = f(0^+) = f(0) = 0$

$\Rightarrow f(x)$ is continuous at the point $x = 0$.

3. $f(x) = \begin{cases} x^3, & 0 \le x < 2 \\ 8, & x \ge 2 \end{cases}$



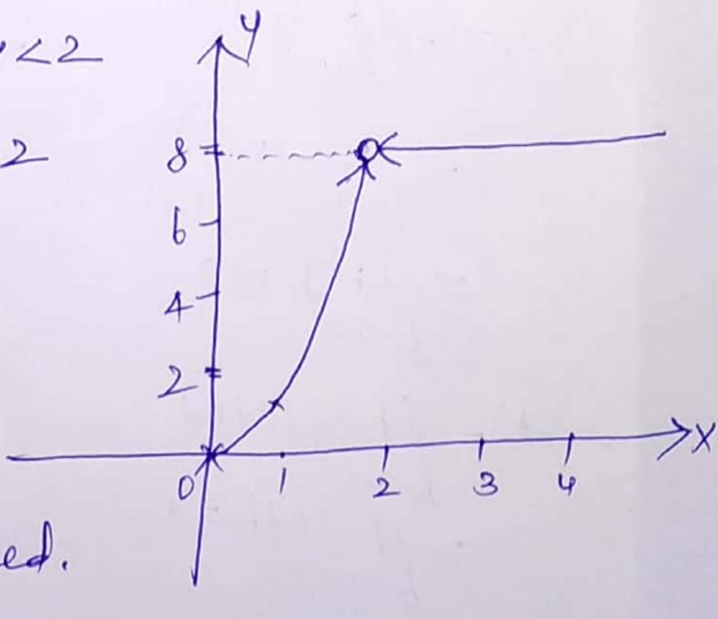
$f(2^-) = \lim\limits_{x \to 2^-} f(x) = 8$

$f(2^+) = \lim\limits_{x \to 2^+} f(x) = 8$

$f(2) = 8.$

$\Rightarrow \lim\limits_{x \to 2} f(x) = f(2) = 8.$ $\therefore f(x)$ is continuous at $x = 2$.

4. $f(x) = \begin{cases} x^3 & 0 \le x < 2 \\ 8 & x > 2 \end{cases}$



$f(2^-) = \lim\limits_{x \to 2^-} f(x) = 8.$

$f(2^+) = \lim\limits_{x \to 2^+} f(x) = 8$

But $f(2)$ not defined.

Here, $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = 8$.

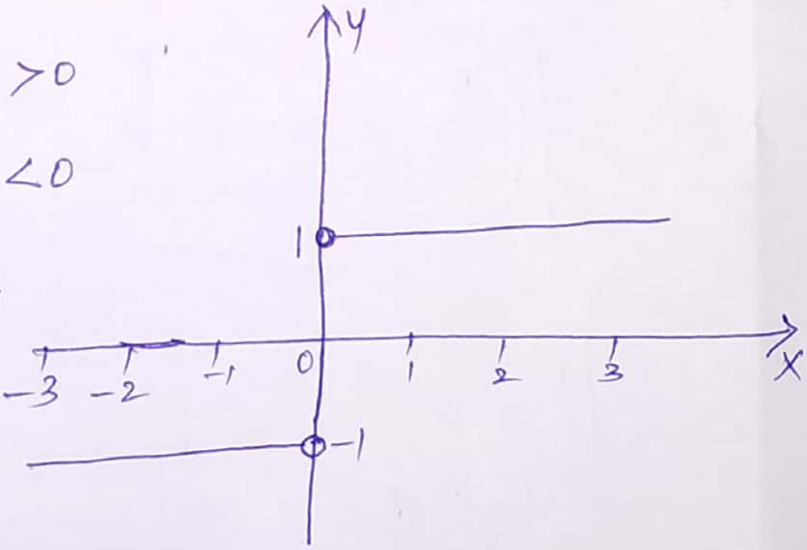
(ie) $f(2^-) = f(2^+)$.

$\Rightarrow$ $\lim\limits_{x\to 2} f(x)$ exist and is equal to 8.

But $\lim\limits_{x\to 2} f(x) \neq f(2)$. $\Rightarrow$ $f(x)$ is not continuous at $x=2$.

<u>Hint</u>:- Limit exist at $x=2$ but not continuous at $x=2$. $\therefore$ The discontinuity must be point discontinuity (or) Removable discontinuity.

5. $f(x) = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$

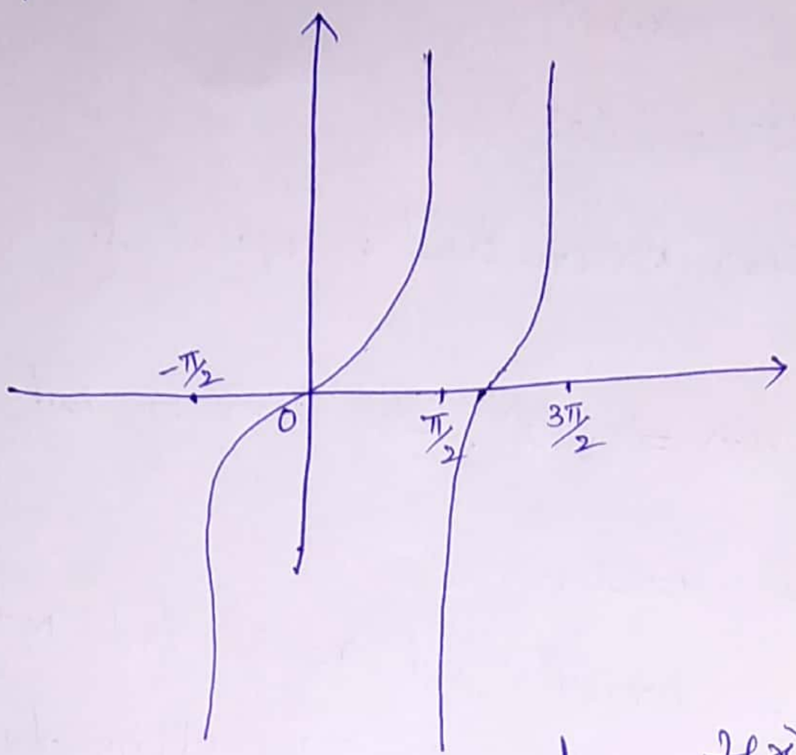


$f(0^-) = \lim\limits_{h\to 0^-} f(x) = -1$

$f(0^+) = \lim\limits_{h\to 0^+} f(x) = 1$

$\therefore$ $f(0^-) \neq f(0^+)$. $\Rightarrow$ $\lim\limits_{x\to 0} f(x)$ does not exists. (2 units gap). $\Rightarrow$ Jump discontinuity.

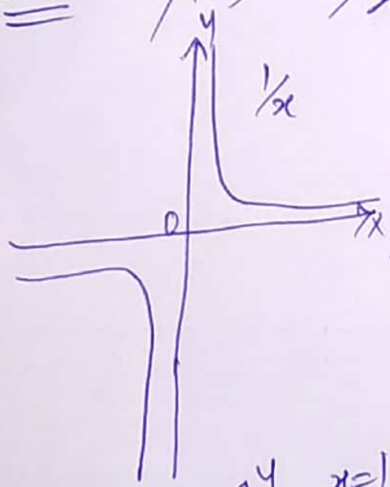
6. $f(x) = \tan x$

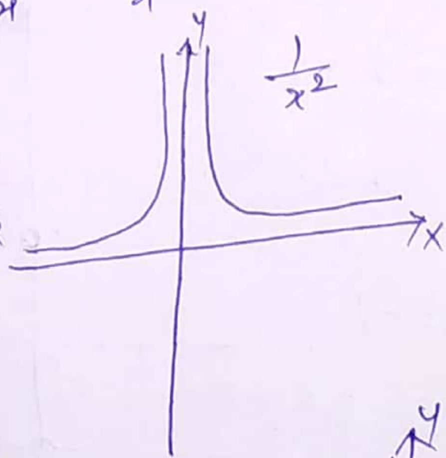


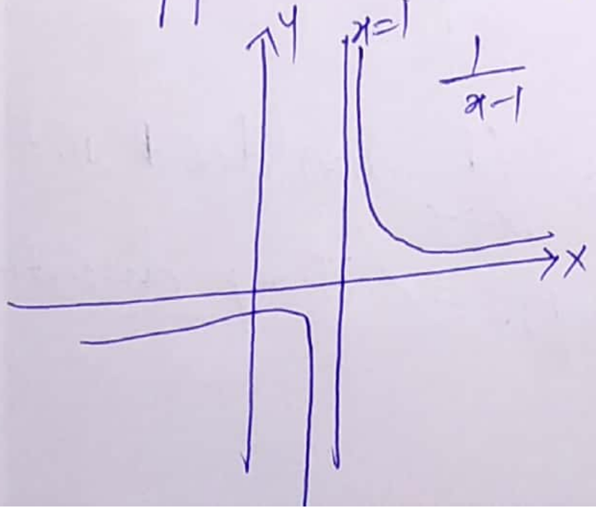
At $x = \pi/2$ the function $f(x)$ is infinite.

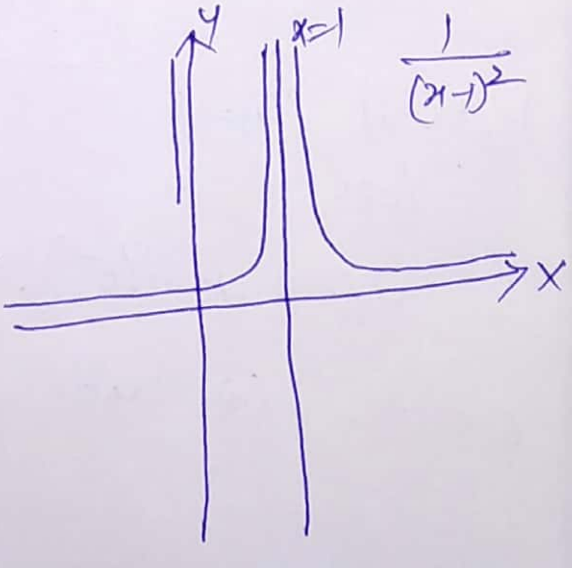
$\Rightarrow$ Infinite discontinuous functions.

Ex: $f(x) = 1/x, \frac{1}{x^2}, \frac{1}{x-1}, \frac{1}{(x-1)^2}$.



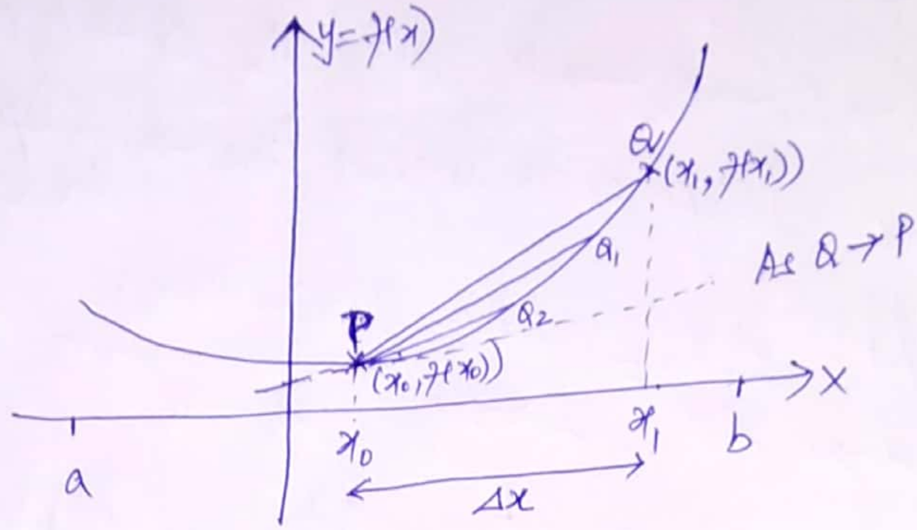






Differentiability of a function $f(x)$:

Let $y = f(x)$ be any function defined over the interval $[a, b]$. $[x_0, x_1] \subseteq [a, b]$.



~~Equation~~ To find the derivative of $f(x)$ at the point $x = x_0$. (At P):

Consider the equation of line joining the points $P(x_0, f(x_0))$ & $Q(x_1, f(x_1))$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

(ie) $$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$ $\left[y - y_1 = m(x - x_1)\right]$ Eqn of a chord.

Here, slope of a secant line is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Here, $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ is called

difference quotient.

As $Q \to Q_1 \to Q_2 \to P$ (ie) As $Q \to P$ (or)

$\Delta x \to 0$, [Slope of a secant line $\to$ slope of a tangent line].

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} =$$

$$= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \simeq \left(\frac{df}{dx}\right)_{x = x_0} \text{ (or) } \frac{dy}{dx}.$$

(ie) $$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$= \left(\frac{df}{dx}\right)_{x = x_0}$$

$\therefore$ If $y = f(x)$ then

$$\frac{dy}{dx} \text{ (or) } \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Instantaneous rate of change of a function $y = f(x)$ at the point x.