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On the Design of the Kinematic Structure of Seven-Axes Redundant Manipulators for Maximum Conditioning

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Abstract

The kinematic design of redundant seven-axes manipulators is addressed here, the focus being on the optimization of the kinematic conditioning of the manipulators of interest. It is shown that isotropic seven-axes manipulators are possible, and structural considerations pertaining to the design of such manipulators are discussed.

1 Introduction

The kinematic structure of a serial manipulator is understood as the set of parameters that do not change from configuration to configuration. In this paper, we aim at the optimization of a performance index of the manipulator from the kinematic and static viewpoints, based on the *condition number* of the associated Jacobian matrix. Moreover, we assume at the outset that the manipulator under design has seven axes and, hence, is of the redundant type.

We begin by discussing the merits of a performance measure based on the condition number of the manipulator Jacobian and compare it with another well-known performance index, namely, the manipulability index [1]. Moreover, the problem of dimensional inhomogeneity of the Jacobian entries, that prevents the definition of a physically meaningful condition number, is readily overcome by defining a *characteristic length* of the manipulator. Based on the characteristic length, a normalization of the Jacobian can be performed, thus rendering it dimensionally homogeneous. Furthermore, we conclude that the manipulability index fails to give a reliable measure of the kinetostatic performance of the manipulator because it is independent of the operation point of the end effector. On the contrary, the condition number does depend on the operation point and hence, the location of the point is crucial in obtaining a good kinematic performance.

Two-link manipulators capable of attaining configurations with a minimum condition number of unity were called "isotropic" by Salisbury and Craig [2]. An extension of isotropy to other manipulators is given in [3]. We show here that the kinematic design of a seven-axis redundant manipulator with an isotropic architecture is possible. In fact, we give two examples of isotropic architectures and a third example of a nearly-isotropic architecture, all of which were determined numerically.

2 Measures of Dexterity

Various attempts have been made for devising means to assess the kinematic performance of robotic manipulators. Moreover, the said kinematic performance assesses the behavior of the manipulator *locally*, and hence, the Jacobian matrix $\mathbf{J}$ of the manipulator becomes a centerpiece in this tudy. We recall that the Jacobian matrix of a serial-type manipulator is defined as the transformation mapping the joint rates into the Cartesian velocities, defined at an operation point of the end effector. For an n-axes manipulator, this transformation is written as:

$$\mathbf{J}\dot{\boldsymbol{\theta}} = \mathbf{t} \tag{1}$$

where $\dot{\boldsymbol{\theta}}$ is the n-dimensional vector of joint rates and $\mathbf{t}$ is the 6-dimensional *twist* vector at the operation point of the end effector, defined as

$$\mathbf{t} \equiv \begin{bmatrix} \boldsymbol{\omega} \\ \dot{\mathbf{p}} \end{bmatrix} \tag{2}$$

where $\boldsymbol{\omega}$ and $\dot{\mathbf{p}}$ denote the angular velocity of the end-effector and the velocity of the operation point P of the end-effector, respectively. On the other hand, the transpose of the Jacobian matrix can be interpreted as

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a linear transformation that maps the 6-dimensional wrench $\mathbf{w}$ acting at the operation point into the n-dimensional torque vector, i.e,

$$\mathbf{J}^T\mathbf{w} = \boldsymbol{\tau} \tag{3}$$

where the wrench $\mathbf{w}$ is defined as the 6-dimensional vector containing the moment $\mathbf{n}$ acting on the end effector and the force $\mathbf{f}$ applied at point P, namely,

$$\mathbf{w} \equiv \begin{bmatrix} \mathbf{n} \\ \mathbf{f} \end{bmatrix} \tag{4}$$

The two well-known control schemes for robotic operations, namely, rate control and force feedback control, are based on the foregoing transformations. This, in turn, has motivated the Jacobian matrix to become a central issue in the quantitative assessment of the robot performance. Since both equations (1) and (3) entail the inversion of $\mathbf{J}$, one needs to know when $\mathbf{J}$ is singular or rank-deficient. As well, one needs to monitor the roundoff-error amplification upon inverting either $\mathbf{J}$ or its transpose, which is crucial in enhancing the accuracy of the operation.

As one of the first attempts to define the kinematic performance of a manipulator, Paul and Stevenson [4] used the determinant of the Jacobian matrix in order to assess the kinematic performance of spherical wrists. However, the magnitude of the determinant of a matrix is not adequate for assessing the invertibility of that matrix [5]. Yoshikawa [1] proposed the concept of ***manipulability***, denoted here by μ, as a quantitative measure for the local dexterity of serial manipulators, which he defined as

$$\mu = \sqrt{\det(\mathbf{J}\mathbf{J}^{\mathbf{T}})} \tag{5}$$

The manipulability index μ can be applied to both redundant and non-redundant manipulators. Klein and Blaho [6] proposed the minimum singular value of $\mathbf{J}$ as a measure of ***distance*** to a singular configuration of the manipulator.

In this paper, we adopt the 2-norm condition number of the Jacobian matrix, or, equivalently, the conditioning index (CI), as defined in [3], as the main criterion in manipulator design. The CI is defined as a percentage of the reciprocal of the minimum value that the condition number of the Jacobian matrix can attain. The CI thus measures the invertibility of the Jacobian matrix, i.e., the roundoff-error amplifications associated with the local inverse kinematics of both redundant and non-redundant manipulators. The condition number of $\mathbf{J}$ was first used in [2], as a measure of the accuracy with which forces can be exerted at the operation point. Furthermore, an isotropic configuration for a two-link manipulator was defined as a configuration at which the condition number of the Jacobian attains its minimum value of unity. It is recalled that the 2-norm condition number of a matrix can be calculated as the ratio of the greatest to the smallest singular values of that matrix [7]. The condition number of a matrix thus indicates how far the said matrix is from singularity.

Next we show that μ, as defined in eq.(5), is insensitive to changes in operation point, which disables it as a reliable measure of kinematic or static performance. This result was first outlined by Li [8], who made an authoritative investigation on different measures of dexterity and pointed out that Yoshikawa's manipulability index is independent of the operation point—in fact, he called this property ***translation-invariance***—, a property with which neither the minimum singular value nor the condition number, in general, are endowed. Below we will expand on the foregoing result and highlight features left out by Li. To this end, let A and B be two different operation points of the end effector, and $\mathbf{J}_A$ and $\mathbf{J}_B$ be the corresponding Jacobian matrices, i.e.,

$$\mathbf{J}_A\dot{\boldsymbol{\theta}} = \mathbf{t}_A \tag{6a}$$

$$\mathbf{J}_B\dot{\boldsymbol{\theta}} = \mathbf{t}_B \tag{6b}$$

where $\mathbf{t}_A$ and $\mathbf{t}_B$ are the 6-dimensional twist vectors for A and B, respectively. If we express the twist for point B in terms of that for point A and denote with $\mathbf{a}$ and $\mathbf{b}$ the position vectors of these points, then we can write

$$\mathbf{t}_B \equiv \begin{bmatrix} \omega \\ \dot{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \omega \\ \dot{\mathbf{b}} + \omega \times (\mathbf{b} - \mathbf{a}) \end{bmatrix} \tag{7}$$

Furthermore, we make use of the skew-symmetric matrices $\mathbf{A}$ and $\mathbf{B}$, associated with vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, to represent the cross product appearing in eq.(7), which allows us to derive the relation between both twists, namely,

$$\mathbf{t}_B = \begin{bmatrix} \omega \\ \dot{\mathbf{a}} + (\mathbf{A} - \mathbf{B})\omega \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{A} - \mathbf{B} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \omega \\ \dot{\mathbf{a}} \end{bmatrix} \tag{8}$$

a relation that we term the ***twist-transfer formula***. The twists for A and B are therefore related as

$$\mathbf{t}_B = \mathbf{U}_{AB}\mathbf{t}_A, \tag{9}$$

with $\mathbf{U}_{AB}$ defined as

$$\mathbf{U}_{AB} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{A}-\mathbf{B} & \mathbf{1} \end{bmatrix} \tag{10}$$

If we substitute for $\mathbf{t}_B$ in eq.(6b) and compare the result with eq.(6a), then we can express the Jacobian transformation upon a change in the operation point from A to B as

$$\mathbf{J}_B = \mathbf{U}_{AB}\mathbf{J}_A \tag{11}$$

which can be referred to as the *Jacobian transfer formula.* Now it is apparent from eq.(10) that $\mathbf{U}_{AB}$ is a matrix of the *unimodular group*, i.e., its determinant is unity, and hence,

$$\begin{aligned} \det(\mathbf{J}_B\mathbf{J}_B^T) &= \det(\mathbf{U}_{AB}\mathbf{J}_A\mathbf{J}_A^T\mathbf{U}_{AB}^T) \\ &= \det(\mathbf{J}_A\mathbf{J}_A^T) \end{aligned} \tag{12}$$

This proves that the manipulability index remains unchanged under changes in the operation point.

The importance of the changes made on the condition number upon changing the operation point will become apparent by employing an example from everyday experience. Consider performing a rather common task such as writing by hand, the pen being regarded as an integral part of the end effector—the hand—to which it is rigidly attached for writing purposes. The tip of the pen is then the operation point, that can be changed if the pen is held from, say, its other end. The handwriting that results when holding the pen from that end will be very different from that which results when holding the pen in the usual way, which indicates remarkable differences in manipulability for the two situations.

Moreover, the evaluation of the condition number of the manipulator Jacobian is elusive because of the absence of a Euclidean norm of the said matrix, which in turn arises due to the dimensional inhomogeneity of the Jacobian. This will be overcome by introducing the *characteristic length* of the manipulator, based on which a normalization of the Jacobian can be performed.

3 Isotropic Manipulators

First we recall the concept of isotropy for rectangular matrices. Isotropic rectangular matrices are those with a minimum condition number of unity [7]. The condition number of a matrix can be defined in various ways. By adopting a definition of the condition number based on the singular values of a matrix, we can apply it to both square and rectangular matrices. For instance, if $\mathbf{A}$ is an $m \times n$ matrix, with $m < n$, then the singular values of $\mathbf{A}$ are the square roots of the eigenvalues of the $m \times m$ symmetric positive-semidefinite matrix $\mathbf{A}\mathbf{A}^T$. Next, the condition number of $\mathbf{A}$ is defined as the square root of the ratio of the largest to the smallest eigenvalues of $\mathbf{A}\mathbf{A}^T$. Hence, a matrix is isotropic if all its singular values are identical and nonzero. This is equivalent to saying that, if $\mathbf{A}$ is isotropic, then a real number σ exists such that

$$\mathbf{A}\mathbf{A}^T = \sigma^2\mathbf{1} \tag{13}$$

If σ is defined as non-negative, then it is the common singular value of $\mathbf{A}$. Isotropic matrices bear interesting properties that explain why they do not produce roundoff-error amplifications at all, upon inverting them. Indeed, the inverse of an isotropic square matrix is a multiple of its transpose, while the generalized inverse of a rectangular isotropic matrix is a multiple of its transpose as well. Moreover, the 6×7 Jacobian matrix of a 7-axes manipulator takes on the form [9]

$$\mathbf{J} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_7 \\ \mathbf{e}_1 \times \mathbf{r}_1 & \mathbf{e}_2 \times \mathbf{r}_2 & \cdots & \mathbf{e}_7 \times \mathbf{r}_7 \end{bmatrix} \tag{14}$$

It is recalled that $\mathbf{e}_i$ is the unit vector parallel to the axis of the ith revolute and $\mathbf{r}_i$ is the vector directed from any point on the same axis to the operation point of the end-effector. In other words, the ith column of $\mathbf{J}$ represents the *Plücker coordinates* of the ith axis of the manipulator [9]. Thus, the first three of those coordinates are dimensionless, while the last three, representing the moment of the ith axis with respect to the operation point, have units of length. This dimensional inhomogeneity gives rise to inconsistencies when evaluating the condition number under discussion. We circumvent this problem by performing a normalization of the entries of the Jacobian matrix, which can be done by dividing the last three rows of $\mathbf{J}$ by a *characteristic length* L, thereby providing a dimensionally homogeneous $\mathbf{J}$. Moreover, L is chosen so as to minimize the condition number of the dimensionally homogeneous $\mathbf{J}$. If $\mathbf{J}$ is made isotropic then from eq.(13) and (14) it it can be shown that

$$L^2 = \frac{\sum_1^7 \|\mathbf{e}_k \times \mathbf{r}_k\|^2}{7} \tag{15}$$

Next we define an objective function z in order to minimize the squared Frobenius norm of the matrix

M, as given below:

$$\mathbf{M} \equiv \mathbf{J}\mathbf{J}^T - \sigma^2 \mathbf{1} \tag{16}$$

and hence,

$$z \equiv \sqrt{\mathrm{Tr}(\mathbf{M}\mathbf{M}^T)} \rightarrow \min_{\mathbf{x}} \tag{17}$$

where $\mathbf{x}$ is the set of design parameters defined below.

4 Isotropic Design

It is required to determine the set of Hartenberg-Denavit parameters [10], together with the characteristic length of the manipulator that render the Jacobian matrix isotropic. Moreover, for functional reasons, designs that resemble the human arm will be attempted. It is first noted that the 6×6 symmetric matrix $\mathbf{J}\mathbf{J}^T$ has 21 different entries. As a first attempt, we choose all the relevant and non-zero HD parameters together with the characteristic length as our design variables. On the other hand, from the total number of 28 HD parameters to be determined, the first offset distance b_1 and the last twist angle α_7 are taken here as zero. Moreover, noting that the first joint variable, which produces a rigid-body motion of the overall manipulator, does not affect the condition number, we can write $\theta_1 = 0$. By eliminating the three aforementioned variables, we have 25 HD parameters to determine plus L and σ. The latter can be readily found to be $\sqrt{7/3}$ from algebraic arguments. Thus, we are left with 26 unknowns to be determined, which are then grouped in the 26-dimensional design vector $\mathbf{x}$. Furthermore, we denote by $\mathbf{f}$ the 21-dimensional vector function containing the above-mentioned different entries of the symmetric matrix $\mathbf{M}$ of eq.(16), which allows us to reduce the design problem at hand to a least-squares problem, i.e.,

$$\min_{\mathbf{x}} ||\mathbf{f}(\mathbf{x})||^2 \tag{18}$$

The *MATLAB* function *fmins* was used for solving the foregoing least-squares problem. The results obtained for this design are given in Table 1.

These HD parameters produce a configuration whose Jacobian matrix is isotropic with its singular values being identical and equal to $\sqrt{7/3}$. For this isotropic design, eq.(15) produces the same value for L as the one obtained with *fmins* (Table 1).

Link i	a_i	b_i	α_i(deg)	θ_i(deg)
1	0.1154	0	104.6285	180.0000
2	1.5704	-0.0483	-86.3539	40.1118
3	0.1756	1.0226	60.6524	30.5779
4	1.0499	-0.7054	108.6141	-105.7290
5	0.9094	-0.0104	-110.1435	-69.0636
6	0.0053	-0.0614	-107.3289	146.9810
7	0.4810	0.8844	0	33.5665
Characteristic Length = 0.7502				

Table 1: HD parameters for the fully isotropic configuration. Design No. 1

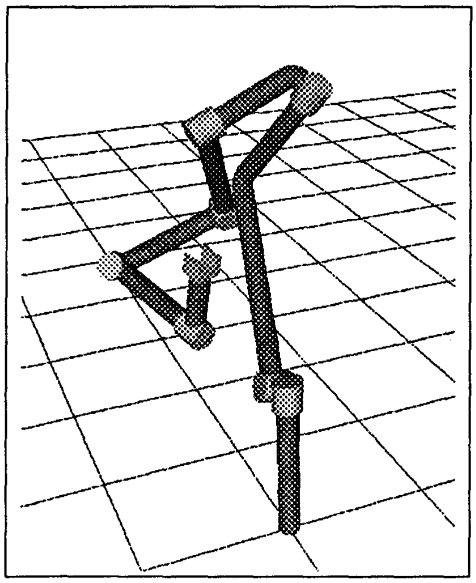

Figure 1: Fully isotropic seven-axes manipulator

Furthermore, Fig. 1 depicts a 3-dimensional rendering of this manipulator in its isotropic configuration. From a structural viewpoint, it would be advantageous for design considerations, if we could concentrate as much mass of the arm as possible over the base of the manipulator. This would enhance the dynamic performance and the structural rigidity of the manipulator. Hence, we attempted an alternative design by preassigning values to five of the components of $\mathbf{x}$. This in turn enabled us to formulate a set of 21 nonlinear equations in 21 unknowns. Therefore, we set a_1, a_3, a_5, b_2 and b_4 equal to zero. Elimination of these five parameters from the design vector $\mathbf{x}$ leaves us with a system of 21 nonlinear equations in 21 unknowns, which can be solved numerically. The

Link i	a_i	b_i	α_i(deg)	θ_i(deg)
1	0	0	-62.7126	0
2	0.0239	0	-11.0926	35.0924
3	0	0.1760	106.6820	62.7137
4	2.2620	0	72.8709	117.7082
5	0	-1.8796	55.8331	-24.6355
6	0.0738	3.2468	62.8430	-2.3164
7	1.2060	-1.4819	0	225.4504
Characteristic Length = 1.0444				

Table 2: HD parameters for the fully isotropic configuration. Design No. 2

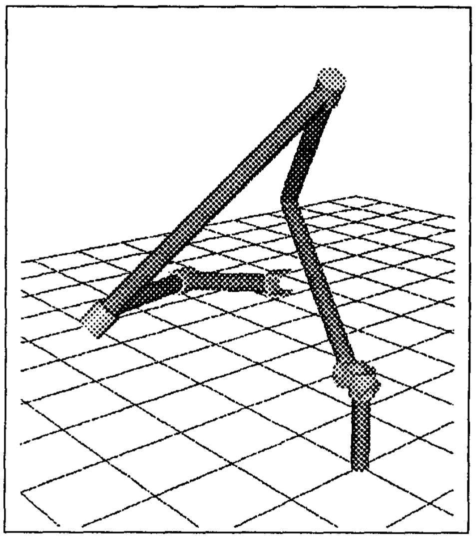

Figure 2: Fully isotropic seven-axes manipulator

subroutine *fsolve* of *MATLAB*, which is based on Newton's method, was employed for solving the said system of nonlinear equations. The results obtained for this design are given in Table 2, while the robot in its isotropic pose is shown in Fig. 2. The corresponding Jacobian matrix for this configuration can be shown to have a condition number of unity. Simillar to the previous example, eq.(15) results in the same value for L as the one obtained with *fsolve*.

From Fig. 2 it can be observed that, by having the first four joints concentrated very close to the base of the manipulator, the weight of the corresponding links and actuators will be concentrated closer to the base. However, the existence of large offset distances such as b_5 is still unacceptable with regard to both mechanical rigidity and desired anthropomorphism. In order to remedy this, a further design is attempted

Link i	a_i	b_i	α_i(deg)	θ_i(deg)
1	0	0	-77.8800	0
2	0	0	-80.1659	259.9013
3	1.5045	0	55.4774	3.6829
4	0	1.7420	95.4346	-108.8578
5	0	0	-93.0426	-87.9244
6	2.0629	0	188.9605	-101.4668
7	1.3420	0.0089	0	-145.6447
Characteristic Length = 1.0002				

Table 3: HD parameters for the Near-isotropic Anthropomorphic manipulator. Design No. 3

here. By preassigning zero values to eight elements of the unknowns, namely $a_1, a_2, a_4, a_5, b_2, b_3, b_5$ and b_6, a nonlinear optimization problem can be formulated, which consists of 21 equations to be satisfied by 18 variables. The aforementioned subroutine, *fmins*, was used to find the least-square approximation of this nonlinear overdetermined system. The solution obtained is given in Table 3, while the corresponding Jacobian matrix has a condition number of 1.3845.

Figure 3 shows a 3-dimensional rendering of this last design. It can be seen that this solution, as compared to the previous alternatives, has the closest resemblance to the human arm. Obviously, this improvement has been traded off by a small increase in the magnitude of the condition number. Thus, one cannot call this manipulator isotropic. However, its CI is 72.4% and can be accepted as suitably conditioned. We call this manipulator *nearly-isotropic.* For this example, since $\mathbf{J}$ is not isotropic we do not expect to obtain the same value for L by using eq.(15) as what was obtained numerically. In fact eq.(15) gives a value of 1.0039 instead of 1.0002. This very small change is an indication of the insensitivity of the well conditioned problem at hand. Next we further illustrate this robustness insensitivity with respect to perturbations in the design parametrs. If we perturb the link lengths and offsets of the three examples (Tables 1,2 and 3) by simply rounding them up to two significant figures and determine the new condition number of the corresponding Jacobian matrix, we will obtain values of 1.0929, 1.3878 and 1.7344 for examples 1,2 and 3, respectively. These results indicate that, in the presence of dimensional uncertainties conditioning index of the manipulators thus designed will remain close to 100%.

5 Conclusions

The emphasis of this paper was mainly on the application of kinetostatic performance indices in the

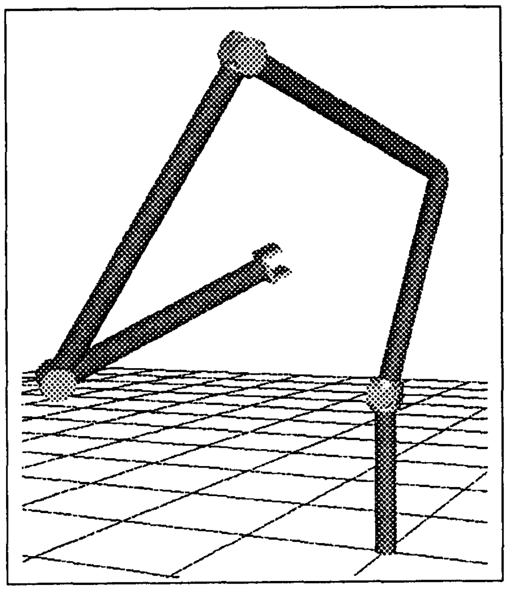

Figure 3: Near-isotropic anthropomorphic manipulator

design of seven-axes, revolute-coupled manipulators. Three different optimum solutions were presented: first, a nonlinear minimization problem was solved, which, in effect, rendered the Jacobian matrix fully isotropic. Next, it was argued that, despite the fully isotropic feature of the first solution, some of its structural features could be improved. This led to preassigning some of the parameters defining the structure of the manipulator, which lead to a system of 21 nonlinear equations in 21 unknowns. As our second solution, the Hartenberg-Denavit parameters of an isotropic manipulator were then obtained by solving the said system of nonlinear equations. Further constraints to make the manipulator structure anthropomorphic led to a third solution. This last solution gave rise to a manipulator which is not fully isotropic, but is very close to isotropy, and exhibits a greater resemblance to the structure of the human arm.

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