CS 6190: Probabilistic Machine Learning Spring 2022

Homework 3

Handed out: 15 Mar, 2022
Due: 11:59pm, 29 Mar, 2022

- You are welcome to talk to other members of the class about the homework. I am more concerned that you understand the underlying concepts. However, you should write down your own solution. Please keep the class collaboration policy in mind.
- Feel free discuss the homework with the instructor or the TAs.
- Your written solutions should be brief and clear. You need to show your work, not just the final answer, but you do *not* need to write it in gory detail. Your assignment should be **no more than 10 pages**. Every extra page will cost a point.
- Handwritten solutions will not be accepted.
- The homework is due by **midnight of the due date**. Please submit the homework on Canvas.

Analytical problems [100 points + 40 bonus]

1. [13 points] The joint distribution over three binary variables are given in Table 1. Show by direct evaluation that this distribution has the property that a and b are marginally dependent, so that $p(a,b) \neq p(a)p(b)$, but that they become independent conditioned on c, i.e., $p(a,b|c) = p(a|c)p(b|c)$.
2. [12 points] Using the d-separation algorithm/criterion (Bayes ball algorithm) to show that the conditional distribution for a node x in a directed graph, conditioned on all of the nodes in its Markov blanket, is independent of the remaining variables in the graph.
3. [15 points] See the graphical model in Figure 1. Recall what we have discussed in the class. Show that $a \perp\!\!\!\perp b|\emptyset$. Suppose we have observed the variable d. Show that in general $a \not\!\perp\!\!\!\perp b|d$.

a	b	c	p(a,b,c)
0	0	0	0.192
0	0	1	0.144
0	1	0	0.048
0	1	1	0.216
1	0	0	0.192
1	0	1	0.064
1	1	0	0.048
1	1	1	0.096

Table 1: Joint distribution of a, b, c.

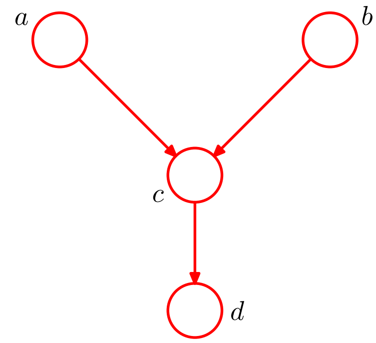


Figure 1: Graphical model.

4. [10 points] Convert the directed graphical model in Figure 1 into an undirected graphical model. Draw the structure and write down the definition of the potential functions.

5. [15 points] Write down every step of the sum-product algorithm for the graphical model shown in Figure 2. Note that you need to first choose a root node, and write down how to compute each message. Once all your messages are ready, please explain how to compute the marginal distribution $p(x_4, x_5)$.

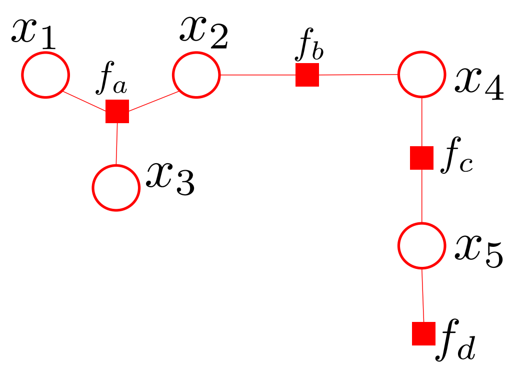


Figure 2: Factor graph.

6. [10 points] Now if x_2 in Figure 2 is observed, explain how to conduct the sum-product algorithm, and compute the posterior distribution $p(x_4, x_5|x_2)$.

7. [10 points] Suppose all the random variables in Figure 2 are discrete, and no one has been observed. Now we want to find the configuration of the $x_1, \ldots, x_5$ to maximize the joint probability. Write done every step of the max-sum algorithm to calculate the maximum joint probability and to find the corresponding configurations of each random variable.

8. [**Bonus**][20 points] Show the message passing protocol we discussed in the class is always valid on the tree-structured graphical models— whenever we compute a message (from a factor to a variable or a variable to a factor), the dependent messages are always available. Hint: use induction.

9. [15 points] Use d-separation algorithm (Bayes ball) to determine if $a \perp\!\!\!\perp d|e$ in the graphical model shown in Figure 3, and if $a \perp\!\!\!\perp d|b$ in the graphical model shown in Figure 4.

10. [**Bonus**][20 points] We have listed two examples in the class to show that in terms of the expressiveness (i.e., conditional independence) of the directed and undirected graphical models , there is not a guarantee that who is better than who.

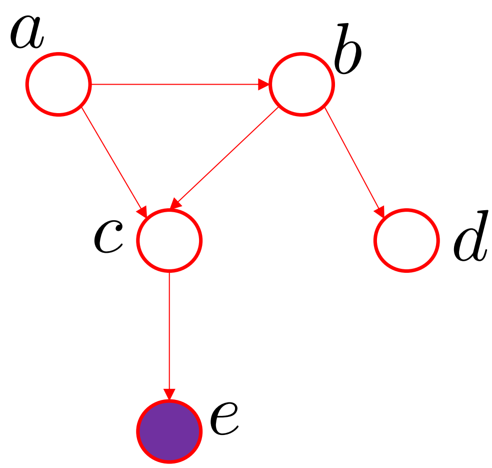


Figure 3: Model 1.

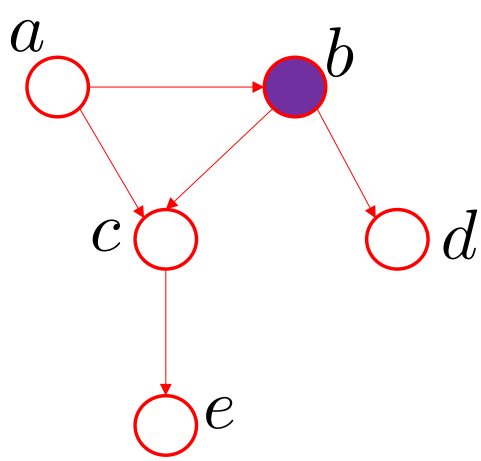
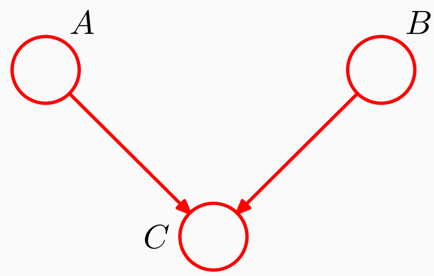


Figure 4: Model 2.

(a) [10 points] Now show that for the directed graphical model in Figure 5, we cannot find an equivalent undirected graphical model to express the same set of conditional independence.

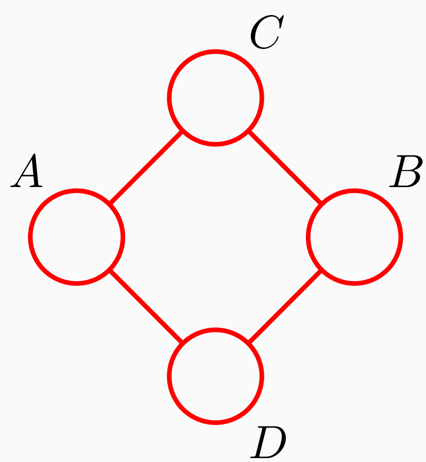
(b) [10 points] Show that for the undirected graphical model in Figure 6, we cannot find an equivalent directed graphical model to express the same set of conditional independence.



$$A \perp\!\!\!\perp B \mid \emptyset$$

$$A \not\!\perp\!\!\!\perp B \mid C$$

Figure 5: Directed.



$$A \not\!\perp\!\!\!\perp B \mid \emptyset$$

$$A \perp\!\!\!\perp B \mid C \cup D$$

$$C \perp\!\!\!\perp D \mid A \cup B$$

Figure 6: Undirected.