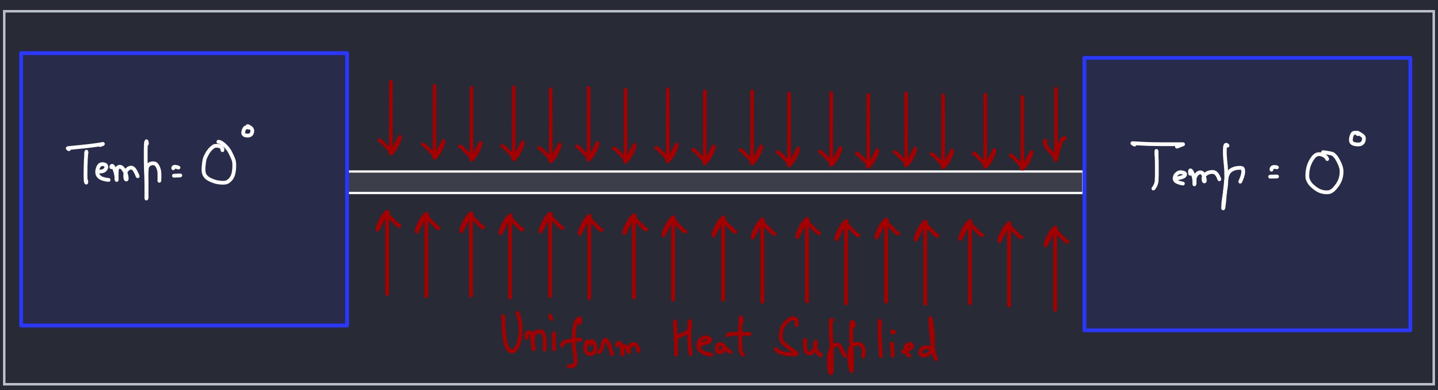
Lecture 2: Differential and Difference Equations

Introduction

Consider a scenario where a thin bar is put on an infinite block of ice at both ends and there a blowtorch heating it along it's length. The task is to find the temperature at several locations along the bar when it reaches steady state (i.e. after a very very long time of heating).



The differential equation (with boundary conditions) explaining this phenomenon is given as

$$-\frac{d^2u}{dx^2} = f(x) = 1; \qquad u(0) = 0, u(1) = 0 \tag{1}$$

where $u(x)$ is the temperature at location $x \in [0, 1]$ and the "ice at the ends" mean the temperature at $u(0) = u(1) = 0$ always! The heat from blowtorch is represented using the function $f(x)$ and it's set to a constant because it's uniform.

Let's look at this differential equation analytically. What we need is some function $u(x)$ that satisfies $-\frac{d^2u}{dx^2} = 1$. In other words, the second derivative of $u(x)$ should be -1. A function $u(x) = -\frac{1}{2}x^2$ satisfies this differential equation. So does another function $u(x) = -\frac{1}{2}x^2 + Cx + D$, where C and D can be any constants! This is where Boundary

Conditions come in and fix C and D values! This way we get an exact solution as $u(x) = -\frac{1}{2}x^2 + \frac{1}{2}x$

> Boundary Conditions enables a unique solution to the PDE! Hence they are very important in solving PDEs.

Difference Equations

In order to solve the differential equation numerically, we have to represent it as a difference equation where the derivatives are defined or approximated numerically.

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} = f(x_i) = 1 \tag{2}$$

The above mentioned approximation to the derivative is defined using Finite Differences.

Let's take a look at a few finite difference formulae approximating the 1st order derivative:

- **Forward Difference**

$$\Delta_F u = \frac{u(x+h) - u(x)}{h} \approx u' + \mathcal{O}(h) \tag{3}$$

- **Backward Difference**

$$\Delta_B u = \frac{u(x) - u(x-h)}{h} \approx u' + \mathcal{O}(h) \tag{4}$$

- **Centered Difference**

$$\Delta_C u = \frac{u(x+h) - u(x-h)}{2h} \approx u' + \mathcal{O}(h^2) \tag{5}$$

These finite difference formulae are derived using Taylor Series, which says that

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + \cdots \tag{6}$$

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2!}u''(x) - \frac{h^3}{3!}u'''(x) + \cdots \tag{7}$$

Rearranging Equation (6), we can get the Forward Difference approximation to the 1st order derivative:

$$\begin{aligned} \frac{u(x+h)-u(x)}{h} &= u'(x) + \frac{h}{2!}u''(x) + \frac{h^2}{3!}u'''(x) + \cdots \\ &= u'(x) + \mathcal{O}(h) \end{aligned} \tag{8}$$

Similarly using Equation (7), we can get the Backward Difference approximation:

$$\begin{aligned} \frac{u(x)-u(x-h)}{h} &= u'(x) - \frac{h}{2!}u''(x) + \frac{h^2}{3!}u'''(x) + \cdots \\ &= u'(x) + \mathcal{O}(h) \end{aligned} \tag{9}$$

Finally, subtracting (6) from (7), we get Centered Difference approximation:

$$\begin{aligned} \frac{u(x+h)-u(x-h)}{2h} &= u'(x) + \frac{2h^2}{3!}u'''(x) + \cdots \\ &= u'(x) + \mathcal{O}(h^2) \end{aligned} \tag{10}$$

> Notice how we get 1st and 2nd order accuracy depending on the terms left after moving them around.

Now we can get 2nd order derivatives in three ways:

- $\Delta_F \Delta_B$
- $\Delta_B \Delta_F$
- $\Delta_C \Delta_C$: This would be a weird choice because it will stretch too far on the stencil in a way that we get coefficients as $1 \quad 0 \quad -2 \quad 0 \quad 1$.

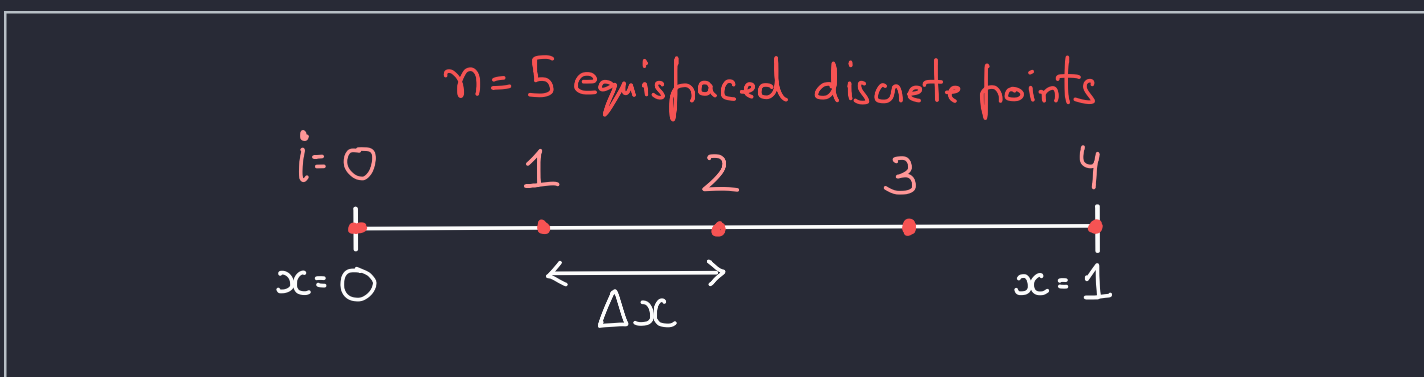
> This might sound unfamiliar, but I've discussed discretisation and stencil in the next section.

Apart from this, we can also get $\Delta_F\Delta_B$ or $\Delta_B\Delta_F$ by adding (6) and (7) together:

$$\begin{aligned}\frac{u(x+h)-2u(x)+u(x-h)}{h^2} &= u''(x)+\frac{2h^2}{4!}u''''(x)+\cdots \\ &= u''(x)+\mathcal{O}(h^2)\end{aligned} \tag{11}$$

Numerical Discretisation and Solution

The very first step to solving a differential equation numerically is discretisation of the domain. Consider the PDE in Equation (1), where the domain lies between 0 and 1. What discretisation does is instead of using a continuous domain, it uses a set of finite points on the domain to represent it (shown in diagram below). In this example, we will discretise this into 5 equispaced points with $\Delta x = 0.25$ as the distance between two consecutive points. Also, each point is indexed using the small letter $i = 0, 1, 2, 3, 4, 5.$



> Using 5 points to discretise any practical problem might be a bad idea, but for understanding it's better to keep n small.

The problem now is to find u_0, u_1, u_2, u_3 and u_4, as these 5 values will essentially represent the function describing u. This is where we turn our attention to Equation (2), and start setting different values to subscript i and form a set of equations.

- When $i = 1$

$$-u_2 + 2u_1 - u_0 = (0.25)^2 \tag{12}$$

- When $i = 2$

$$-u_3 + 2u_2 - u_1 = (0.25)^2 \tag{13}$$

- When $i = 3$

$$-u_4 + 2u_3 - u_2 = (0.25)^2 \tag{14}$$

-

We now have 3 equations and 3 unknowns (u_1, u_2, u_3). Remember that we already know the boundary conditions which states that $u_0 = u_4 = 0$.

The above defined set of equations can also be rewritten in terms of matrix vector multiplication,

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} (0.25)^2 \\ (0.25)^2 \\ (0.25)^2 \end{pmatrix} \tag{15}$$

If you remember from Lecture 1, the left side 3×3 matrix is K_3 and it's invertible! For now, let's just use some computer program to solve this $A\mathbf{u} = b$ and get the vector $\mathbf{u}$.

```
# Total internal points excluding boundaries
n = 3

# x values
x = range(0, 1, n+2)
# i indices
i = range(0, n-1, n+2);

# dx
Δx = x[2] - x[1];

# Function define matrix A
function coefficient(n::Int)
```

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    # A matrix
    A = zeros(n, n)
    for i=1:n
        # Diagonal Elements
        A[i,i] = 2
        if i ≠ 1
            # Lower Diagonal
            A[i, i-1] = -1
        end
        if i ≠ n
            # Upper diagonal
            A[i, i+1] = -1
        end
    end
    return A
end

# Coefficient matrix
A = coefficient(n)

# RHS
b = ones(n) * (Δx^2)

# Solve
u = A\b

println("Solution u: ", u)

# Appending boundary values to solution
pushfirst!(u, 0);
push!(u, 0);

# Plotting
scatter(x, u, label="u", xlabel="x", ylabel="u", title="Solution u(x)")
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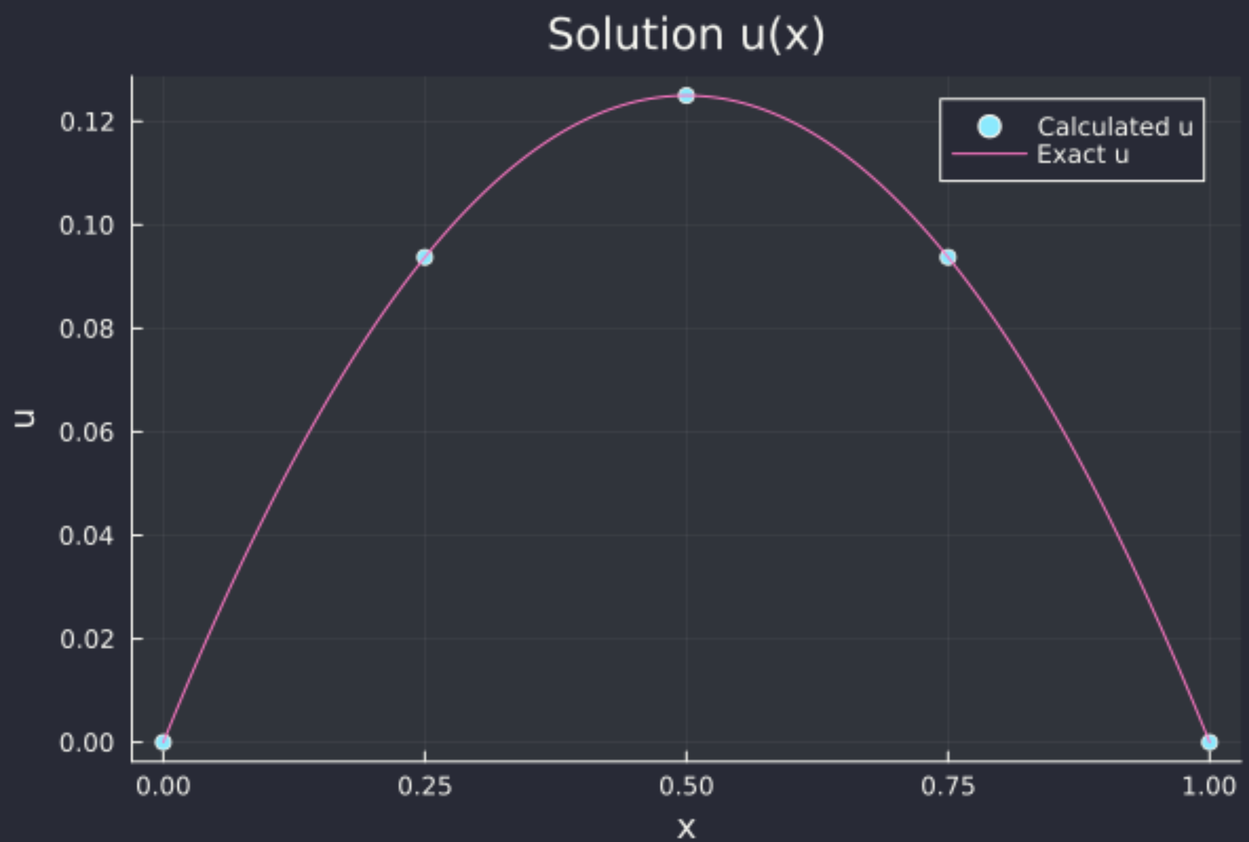
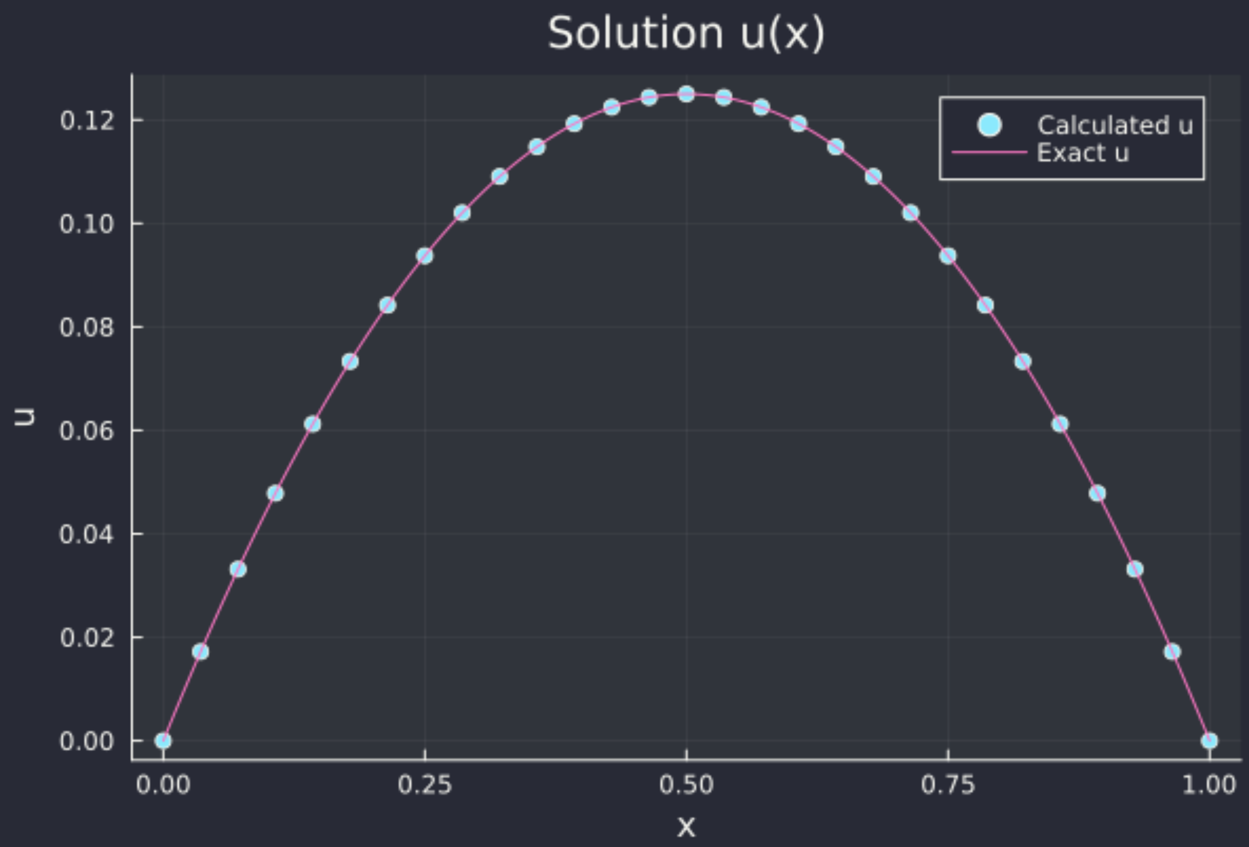


We can now increase the number of points n to get more accurate solution, for example if $n = 30$.



Changing the Boundary Conditions

Let's change something with the problem setup and remove the ice block at $x = 0$! What this means is that we now have a PDE with different boundary conditions

$$-\frac{d^2u}{dx^2} = f(x) = 1; \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, u(1) = 0 \tag{16}$$

> Remember that the problem is of steady state, hence the change in temperature at $x = 0$ is set to 0!

Again setting $n = 5$, the problem now is to find u_0, u_1, u_2, u_3 and u_4. In this case we know that, $u_4 = 0$ which leaves 4 unknown variables. Again using the approximations to convert differential equation to difference equation and also approximating the boundary condition at $x = 0$ using Equation (8) we get a set of equations as follows

- Boundary Condition Approximation

$$u_1 - u_0 = 0 \tag{17}$$

- When $i = 1$, we can use the fact that $u_1 = u_0$ and reduce the unknown variables to 3!

$$\begin{gathered} -u_2 + 2u_1 - u_0 = (0.25)^2 \\ \text{or} \\ -u_2 + u_1 = (0.25)^2 \end{gathered} \tag{18}$$

- When $i = 2$

$$-u_3 + 2u_2 - u_1 = (0.25)^2 \tag{19}$$

- When $i = 3$

$$-u_4 + 2u_3 - u_2 = (0.25)^2 \tag{20}$$

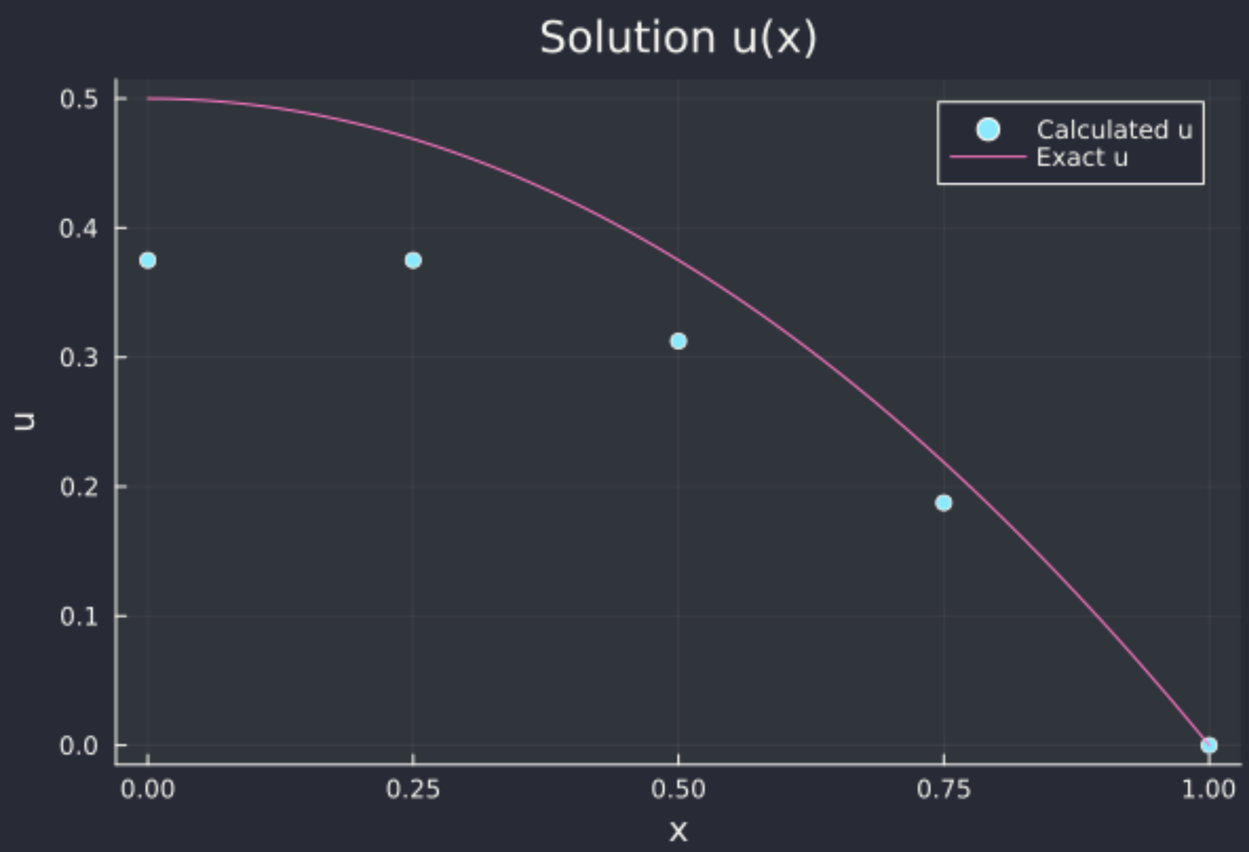
Now, we have 3 equations and 3 variables, which can again be represented as a matrix vector multiplication

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} (0.25)^2 \\ (0.25)^2 \\ (0.25)^2 \end{pmatrix} \tag{21}$$

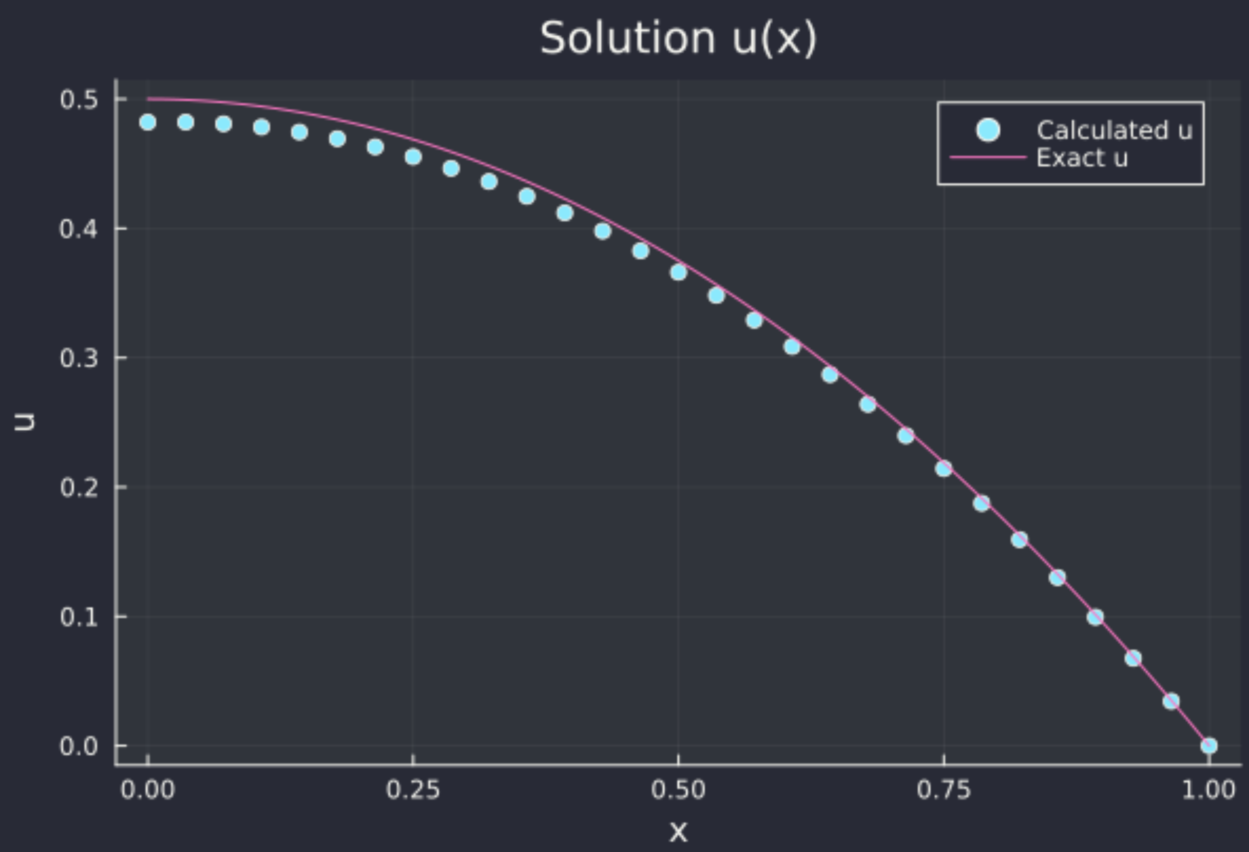
Again from Lecture 1, the left side 3×3 matrix is T_3 and it's invertible! Let's again use the inbuilt matrix solver to solve this $A\mathbf{u} = b$ and get the vector $\mathbf{u}$.

```
function coefficient_T(n::Int)
    # A matrix
    A = zeros(n, n)
    for i=1:n
        # Diagonal Elements
        if i == 1
            A[i,i] = 1
        else
            A[i,i] = 2
```

```julia
            end
            if i ≠ 1
                # Lower Diagonal
                A[i, i-1] = -1
            end
            if i ≠ n
                # Upper diagonal
                A[i, i+1] = -1
            end
        end
        return A
    end
```



Similarly, if we increase the number of points, we get a more accurate solution!



> An interesting observation is that for the previous case where the boundary conditions were fixed, we got accurate solution even when $n = 3$, but that is not the case here. It's just a matter of getting lucky where the equispaced points and the constant force term matched up such that it ended up as a perfect scenario. Rest be assured, if you change anything like variable forcing term, it'll give less accurate solution for smaller n values but that will still be $\mathcal{O}(h^2)$ accurate.
>
> However, in this case we approximated the left boundary using the backward difference formula which is $\mathcal{O}(h)$ accurate. Hence, we'll get the final solution which is also $\mathcal{O}(h)$ accurate (error propagates!).

For this simple problem, we can actually get a $\mathcal{O}(h^2)$ accuracy! Let's do that...

Now, we have an idea that the solution should be parabolic (this can be confirmed using the exact solution). What we can do is use this fact which will give that $u_{-1} = u_1$, and instead of using forward difference we can use centered difference

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \frac{u_1 - u_{-1}}{2\Delta x} = 0 \tag{22}$$

By using this, we again increase the unknown variables to 4, i.e. we need to find $x_0, x_1, x_2 \text{ and } x_3$.

- When $i = 0$, we can use the fact that $u_1 = u_{-1}$

$$\begin{gathered} -u_1 + 2u_0 - u_{-1} = (0.25)^2 \\ \text{or} \\ 2u_0 - 2u_1 = (0.25)^2 \end{gathered} \tag{23}$$

- When $i = 1$

$$-u_2 + 2u_1 - u_0 = (0.25)^2 \tag{24}$$

- When $i = 2$

$$-u_3 + 2u_2 - u_1 = (0.25)^2 \tag{25}$$

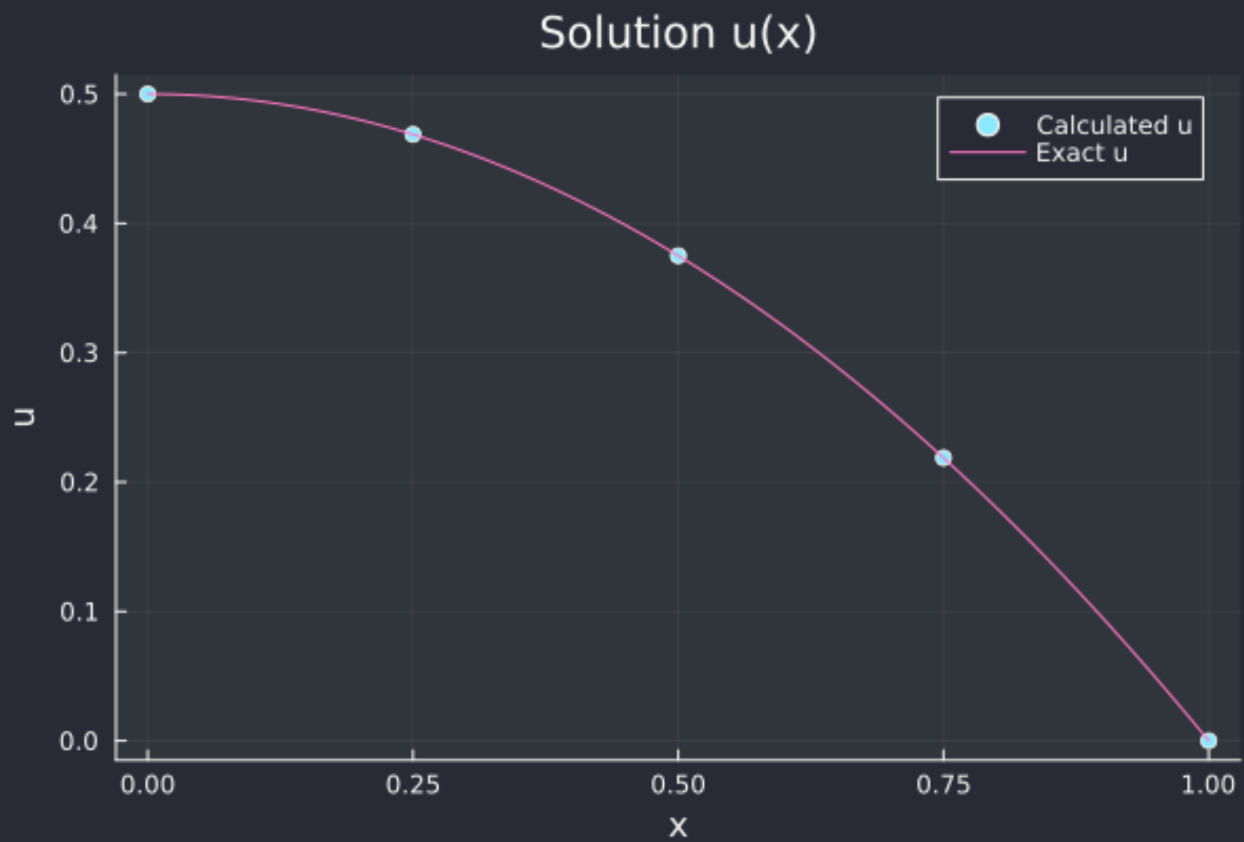
- When $i = 3$

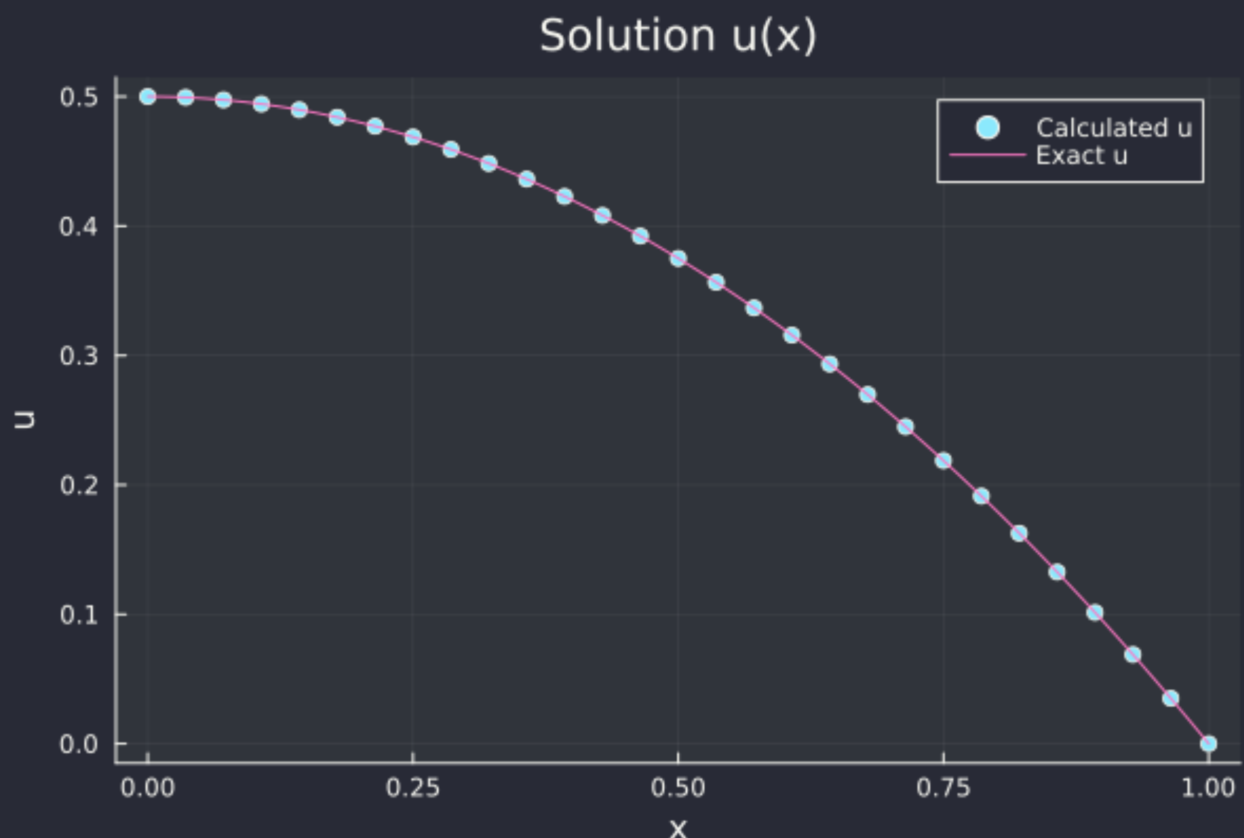
$$-u_4 + 2u_3 - u_2 = (0.25)^2 \tag{26}$$

Now, we have 4 equations and 4 variables, which can again be represented as a matrix vector multiplication

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{(0.25)^2}{2} \\ (0.25)^2 \\ (0.25)^2 \\ (0.25)^2 \end{pmatrix} \tag{21}$$

Solving this system of equation, we get the following solutions...





> This is $\mathcal{O}(h^2)$ accurate!