Lecture 1: Four Special Matrices

Introduction

Consider a matrix, $K_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$.

What are the special properties of this matrix?

- **Symmetric**: $K = K^T$
- **Sparse (Tridiagonal)**: The point on sparsity is especially true when matrix K is something like 100×100
- **Toeplitz**: Constant diagonals, i.e. we have 2 and -1 across the diagonals
- **Invertible**: Identifying that a matrix is invertible, is crucial to solving a system of equations. The standard way to find out if a matrix is invertible, is to row reduce it and represent it in upper triangular form. Let's see how to do that.

Row Reducing a Matrix

Using the same K_4 matrix discussed above,

- Picking the first column, where the pivot is k_{11} element. We make all the elements below the pivot, zero by performing row reduction operation $R_2 \to R_2 - (k_{11}/k_{21})R_1$.

$$K_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{1}$$

- Next, move onto the 2nd column and again ensure that all elements below the pivot k_{22} are zero by performing the row reduction operation $R_3 \rightarrow R_3 - (k_{22}/k_{32})R_2$.

$$K_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{2}$$

- Do the same thing for the 3rd column, $R_4 \rightarrow R_4 - (k_{33}/k_{43})R_3$.

$$K_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix} \tag{3}$$

- For the final column, there's nothing below the pivot value.

We now have K_4 reduced to the upper triangular form and the simple test of invertibility is to check if all pivots are non-zero. If they're non-zero, then the matrix is invertible.

> Don't touch determinants when dealing with invertibility! However, any numerical methods library will evaluate determinants this way by first converting the matrix into an upper triangular form and then taking the product of the diagonal elements. In this case, the determinant is 5!

Circulant Matrix

Consider another matrix similar to K_4, such that the value -1 wraps around the row, Let's call this matrix a circulant matrix and denote this as, $C_4 = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$.

Matrix C_4 shares many properties with K_4 such as:

- **Symmetric**
- **Sparse (not Tridiagonal though)**
- **Toeplitz**

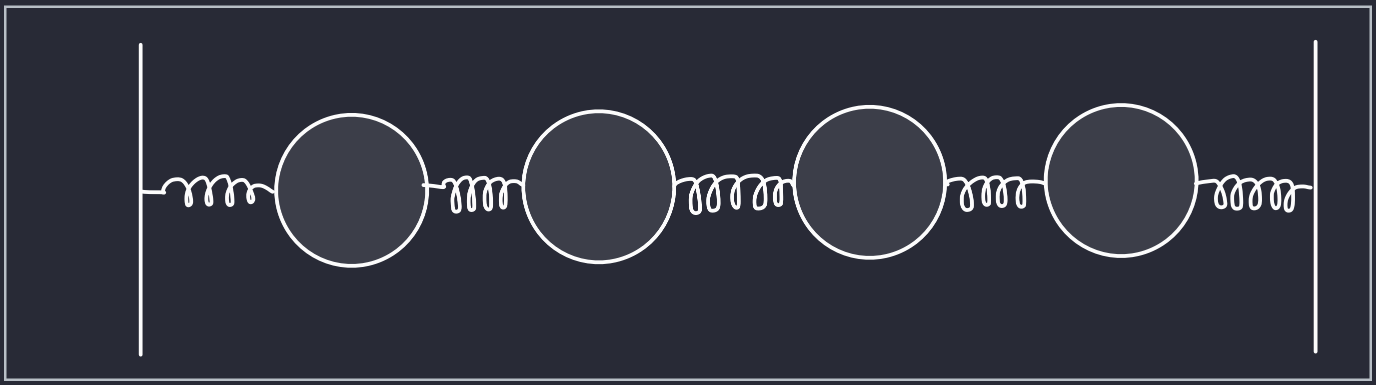
However, this matrix is not invertible! We can do the row reduction stuff to verify this but there is another way to see this. If we multiply C_4 with ones vector as shown in equation (4), we get all zeros as the resulting vector and by definition a matrix is invertible if and only if it has zero vector in its null space!

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

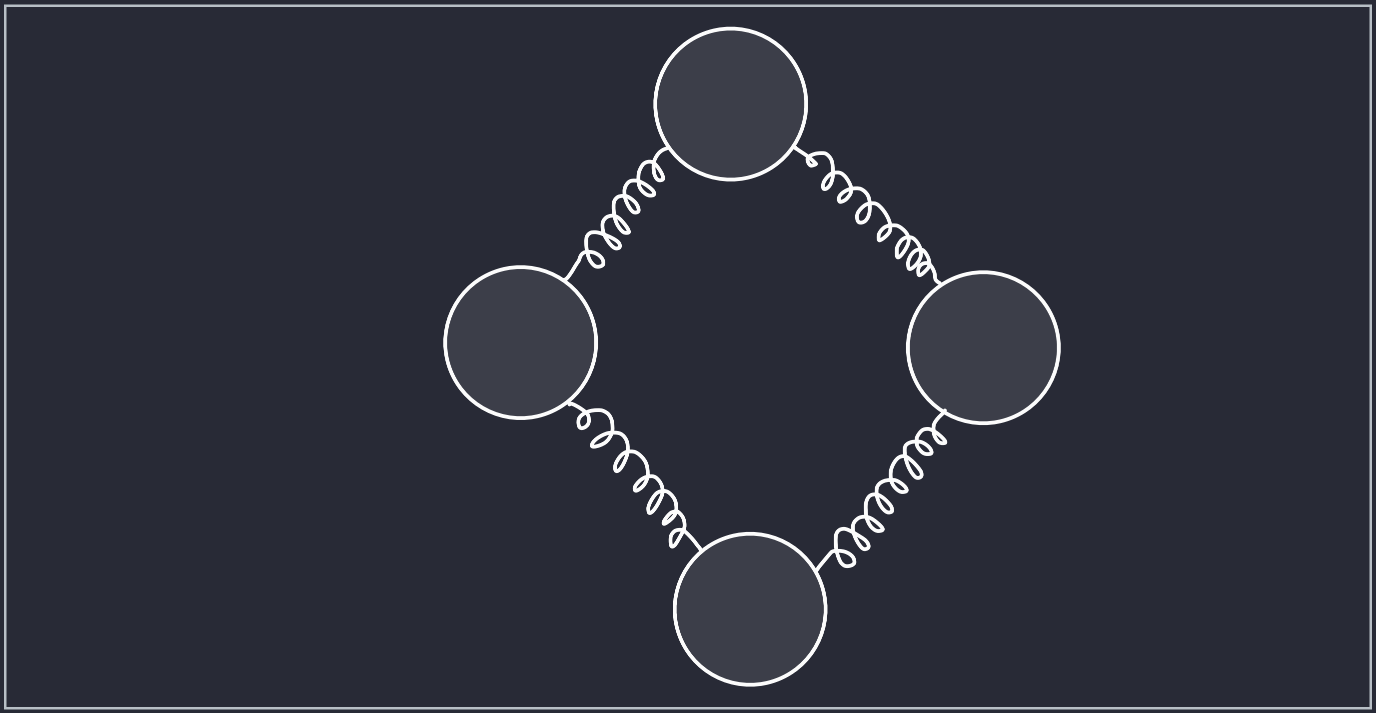
We can write the above statement mathematically as well. Consider $Cu = 0$ and if C^{-1} exists, then $C^{-1}Cu = C^{-1}0$ and $u = 0$. However, we have $u = 1$ hence it's not invertible!

Practical Origin of Matrices

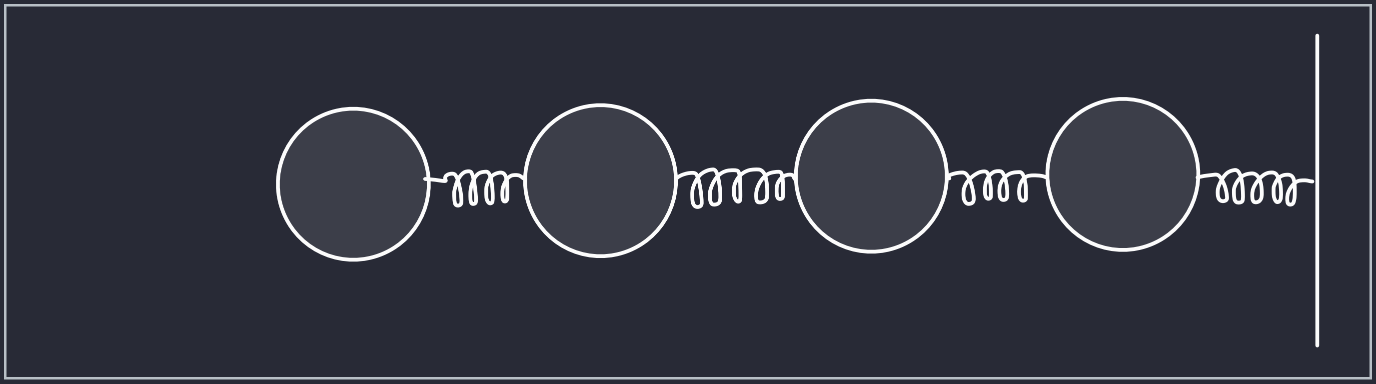
- Consider a system of 4 masses connected by springs (as shown below), and the matrix generated when trying to solve this system is K_4.



- Now, if we remove the fixed ends and just tie the masses together, we get matrix C_4. This is where the term circulant comes from.

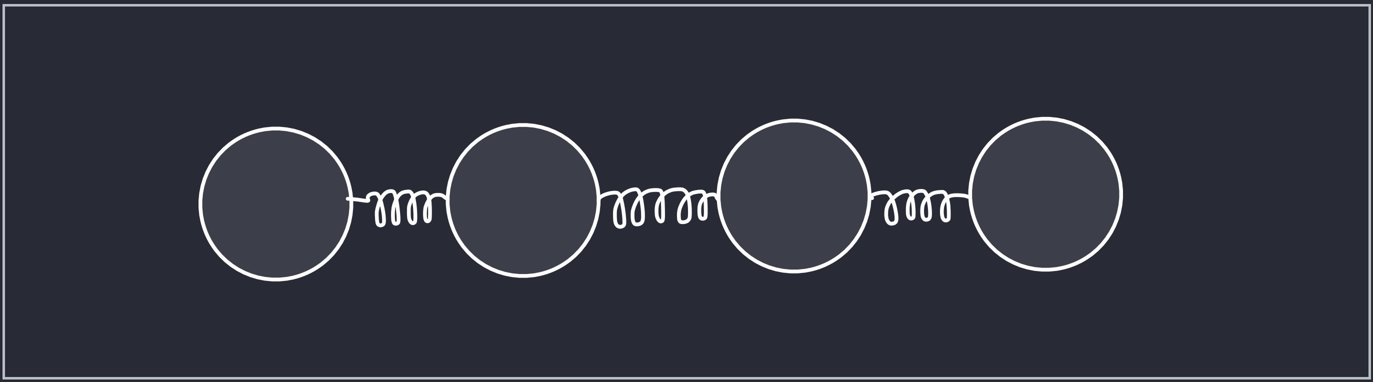


- Just as an example, take the spring mass system which gives K_4 and free one end of support.



In doing so, we get a new matrix $T_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$.

- For the sake of completeness, let's free both ends.



Now, we get another new matrix $B_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$.

All these four special matrices are **Symmetric** and **Sparse**. When it comes to invertibility, K and T are invertible, but C and B aren't. A physics intuition is that the systems with some sort of support (fixed end in this case) results in invertible matrices while the unsupported systems give non-invertible matrices.

> Also, T and B are not Toeplitz!

Another thing is, K and T are positive definite (pivots are all positive in upper triangular form) while C and B are positive semi-definite (pivots are zeros or positive in upper triangular form).

Recitation 1: Linear Algebra Review

Linear Algebra is just about **Vectors**, **Matrices** and **Subspaces**.

Vectors

We take linear combinations of vectors (add, subtract, multiply by scalars, etc), like

$$x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = x_1\begin{pmatrix}1\\-1\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\-1\end{pmatrix} + x_3\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{5}$$

Matrices

It's just a bunch of vectors put into columns of a matrix, i.e. $A = \begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\end{pmatrix}$, then

$$A\mathbf{x} = \begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1\\x_2 - x_1\\x_3 - x_2\end{pmatrix} \tag{6}$$

and this matrix A is known as the difference matrix. The thing to remember is that when a matrix multiplies a vector, the result is a combination of columns.

This is all good, but usually in practice we are given the right hand side and asked to find $\mathbf{x}$. This involves the inverse of A and in this case, intuitively speaking, the inverse would be a sum matrix! i.e. $A^{-1} = \begin{pmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\end{pmatrix}$, then

$$A^{-1}\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \end{pmatrix} \tag{7}$$

Subspaces

For an invertible matrix, the column vectors are also the basis vectors of a subspace. A subspace could be the origin point, a line, a plane inside a 3D space or the whole 3D space itself.