Complementary note on:
VFO stabilization of a unicycle-like robot with bounded curvature of motion

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Abstract

This note is complementing the analysis presented in conference paper [1]. It mainly contains detailed derivations of relations presented in the conference paper.

1 Introduction

Notational conventions from [1] are maintained in this note. References to equations from [1] will be enclosed in double parentheses. Unless stated otherwise, perfect motion execution by the VFO controller will be assumed throughout this document, that is $e_{ai}^{i+1}(t) \equiv 0$. The sign function used in the sequel is defined as follows

$$\operatorname{sign}(x) \triangleq \begin{cases} 1 & \text{for} \quad x \geq 0, & (1) \\ -1 & \text{for} \quad x < 0. & (2) \end{cases}$$

2 Integral curve of the closed-loop system

2.1 Derivation of differential equations

We will now begin the detailed derivation of integral curve ((17)). From the definition of u_2 in ((5)) one has

$$u_2 \triangleq \rho_i \, \sigma_i \cos\left(e_{ai}^{i+1}\right). \tag{3}$$

Moreover, in ((9)) convergence vector field $\bar{\boldsymbol{h}}_i^{i+1}$ was defined:

$$\bar{\boldsymbol{h}}_i^{i+1} \triangleq \begin{bmatrix} h_{xi}^{i+1} \\ h_{yi}^{i+1} \end{bmatrix} = -k_p \bar{\boldsymbol{q}}^{i+1} + \bar{\boldsymbol{v}}_i^{i+1} = -k_p \begin{bmatrix} x^{i+1} + \sigma_i \mu_i \, \| \, \bar{\boldsymbol{q}}^{i+1} \| \\ y^{i+1} \end{bmatrix}. \tag{4}$$

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Observe that, since $e_{ai}^{i+1} \equiv 0 \Rightarrow \theta^{i+1} \equiv \theta_{ai}^{i+1}$, the following relations hold:

$$\cos(\theta^{i+1}) = \cos\left(\theta_{ai}^{i+1}\right) \stackrel{((8))}{=} \cos\left(\text{Atan2c}\left(\sigma_i h_{yi}^{i+1}, \sigma_i h_{xi}^{i+1}\right)\right) = \sigma_i \frac{h_{xi}^{i+1}}{||\bar{\boldsymbol{h}}_i^{i+1}||}, \tag{5}$$

$$\sin(\theta^{i+1}) = \sin\left(\theta_{ai}^{i+1}\right) \stackrel{((8))}{=} \sin\left(\text{Atan2c}\left(\sigma_i h_{yi}^{i+1}, \sigma_i h_{xi}^{i+1}\right)\right) = \sigma_i \frac{h_{yi}^{i+1}}{||\bar{\boldsymbol{h}}_i^{i+1}||}, \tag{6}$$

$$\dot{\boldsymbol{q}}^{i+1} = u_2 \begin{bmatrix} \cos(\theta^{i+1}) \\ \sin(\theta^{i+1}) \end{bmatrix} \stackrel{(3)}{=} \rho_i \, \sigma_i \cos\left(e_{ai}^{i+1}\right) \begin{bmatrix} \cos(\theta^{i+1}) \\ \sin(\theta^{i+1}) \end{bmatrix} \stackrel{(5),(6)}{=} \frac{\rho_i}{||\bar{\boldsymbol{h}}_i^{i+1}||} \begin{bmatrix} h_{xi}^{i+1} \\ h_{yi}^{i+1} \end{bmatrix}. \tag{7}$$

Since for $e_{ai}^{i+1} \equiv 0$ robot's motion direction, and in consequence the considered integral curve, depends only on θ_{ai}^{i+1} and $\forall S > 0$ $\text{Atan2c}\,(Sy, Sx) = \text{Atan2c}\,(y, x)$, the integral curve does not depend on the choice of $\rho_i > 0$. That is why one can assume $\rho_i = ||\bar{\boldsymbol{h}}_i^{i+1}||$, and simplify (7) to $\dot{\boldsymbol{q}}^{i+1} = \bar{\boldsymbol{h}}_i^{i+1}$. As a consequence relations ((13)) and ((14)) hold:

$$\dot{y}^{i+1} \stackrel{(7)}{=} -k_p y^{i+1} \quad \Rightarrow \quad y^{i+1} = y_i^{i+1} \exp(-k_p \tau_i), \tag{8}$$

$$\dot{x}^{i+1} \stackrel{(7)}{=} -k_p \left(x^{i+1} + \sigma_i \mu_i ||\bar{\boldsymbol{q}}^{i+1}||\right). \tag{9}$$

2.2 Solution of differential equations - the integral curve

Now, (9) will be solved by considering two cases. It is known from (8), that

$$y_i^{i+1} = 0 \quad \Rightarrow \quad \forall t \geq 0 \quad y^{i+1} = 0 \tag{10}$$

for every motion segment. Thus, for $y_i^{i+1} = 0$ equation (9) degenerates to

$$\dot{x}^{i+1} = -k_p \left(1 + \sigma_i \mu_i \,\text{sign}\left(x^{i+1}\right)\right) x^{i+1}, \tag{11}$$

which implies convergence of $x^{i+1}(t)$ to 0, since $(1 + \sigma_i \mu_i \,\text{sign}\,(x^{i+1})) > 0$. Based on that, one concludes that for $y_i^{i+1} = 0$, closed-loop system follows the straight line $y = 0$.

When $y_i^{i+1} \neq 0$, we utilize a substitution $x^{i+1} = z_i y^{i+1}$ in (9) to obtain:

$$\dot{z}_i y^{i+1} + \dot{y}^{i+1} z_i = -k_p \left(z_i y^{i+1} + \sigma_i \mu_i \sqrt{z_i^2 (y^{i+1})^2 + (y^{i+1})^2} \right) \Rightarrow \tag{12}$$

$$\dot{z}_i y^{i+1} + \dot{y}^{i+1} z_i = -k_p y^{i+1} \left(z_i + \sigma_i \mu_i \text{sign}\left(y^{i+1}\right) \sqrt{(z_i)^2 + 1} \right), \tag{13}$$

which can be further simplified using (8):

$$\left(\dot{z}_i - k_p z_i\right) y^{i+1} = -k_p y^{i+1} \left(z_i + \sigma_i \mu_i \text{sign}\left(y^{i+1}\right) \sqrt{(z_i)^2 + 1} \right) \Rightarrow \tag{14}$$

$$\dot{z}_i = -k_p \sigma_i \mu \text{sign}\left(y^{i+1}\right) \sqrt{(z_i)^2 + 1} \Rightarrow \tag{15}$$

$$\frac{dz_i}{\sqrt{(z_i)^2 + 1}} = -k_p \sigma_i \mu_i \text{sign}\left(y^{i+1}\right) d\tau_i, \tag{16}$$

which after integrating both sides yields:

$$\text{arsinh}\left(z_i\right) = -k_p \sigma_i \mu_i \text{sign}\left(y^{i+1}\right) \tau_i + C_i, \tag{17}$$

where C_i is the integration constant, while $\tau_i \triangleq t - t_i$ denotes time from the beginning of i-th motion segment execution. It is evident that $\tau_i = 0 \Rightarrow C_i = \operatorname{arsinh}\left(\frac{x_i^{i+1}}{y_i^{i+1}}\right)$, $\operatorname{sign}\left(y^{i+1}\right) \overset{(8)}{=} \operatorname{sign}\left(y_i^{i+1}\right)$, and $z_i = x^{i+1}/y^{i+1}$ by definition, which is why (17) can be transformed to ((15)):

$$x^{i+1}(\tau_i) = \sinh\left[-k_p\beta_i\mu_i\tau_i + C_i\right] y^{i+1}(\tau_i), \tag{18}$$

$$C_i = \operatorname{arsinh}\left(\frac{x_i^{i+1}}{y_i^{i+1}}\right), \quad \beta_i \triangleq \sigma_i \operatorname{sign}\left(y_i^{i+1}\right). \tag{19}$$

2.3 Curve parametrization

There are infinitely many pairs $\left(x_i^{i+1}, y_i^{i+1}\right)$ corresponding to the same value of constant C_i (cf. (19)) and in consequence corresponding to the same integral curve (18). To achieve a more convenient (i.e. unique) curve parametrization and express curve (18) as a function of y^{i+1}, we will compute the value of y^{i+1}, for which Y^{i+1} axis is crossed by integral curve (18). Let us use (18) and solve the equation w.r.t. τ_i for $x^{i+1} = 0$ to compute the time at which Y^{i+1} axis will be crossed:

$$0 = \sinh\left[-k_p\sigma_i\mu_i \operatorname{sign}\left(y_i^{i+1}\right)\tau_i + \operatorname{arsinh}\left(\frac{x_i^{i+1}}{y_i^{i+1}}\right)\right] y^{i+1} \Rightarrow \tag{20}$$

$$k_p\mu_i\sigma_i \operatorname{sign}\left(y_i^{i+1}\right)\tau_i = \operatorname{arsinh}\left(\frac{x_i^{i+1}}{y_i^{i+1}}\right) \Rightarrow \tag{21}$$

$$\tau_i = \frac{\operatorname{arsinh}\left(x_i^{i+1}/y_i^{i+1}\right)}{k_p\,\mu_i\sigma_i \operatorname{sign}\left(y_i^{i+1}\right)}, \tag{22}$$

and eliminate time using (8) as follows

$$y^{i+1} = y_i^{i+1}\exp(-k_p\tau_i) \Rightarrow \tag{23}$$

$$\exp(-k_p\tau_i) = \frac{y^{i+1}}{y_i^{i+1}} \Rightarrow \tag{24}$$

$$\tau_i = \frac{-1}{k_p}\ln\left(\frac{y^{i+1}}{y_i^{i+1}}\right), \tag{25}$$

to obtain

$$\frac{-1}{k_p}\ln\left(\frac{y^{i+1}}{y_i^{i+1}}\right) \overset{(22),(25)}{=} \frac{\operatorname{arsinh}\left(x_i^{i+1}/y_i^{i+1}\right)}{k_p\,\mu_i\sigma_i \operatorname{sign}\left(y_i^{i+1}\right)} \Rightarrow \tag{26}$$

$$\ln\left(\frac{y^{i+1}}{y_i^{i+1}}\right) = \frac{\operatorname{arsinh}\left(x_i^{i+1}/y_i^{i+1}\right)}{-\mu_i\sigma_i \operatorname{sign}\left(y_i^{i+1}\right)} \Rightarrow \tag{27}$$

$$y^{i+1} = \exp\left(\frac{\operatorname{arsinh}\left(x_i^{i+1}/y_i^{i+1}\right)}{-\mu_i\sigma_i \operatorname{sign}\left(y_i^{i+1}\right)}\right) y_i^{i+1} \Rightarrow \tag{28}$$

$$\psi_i = \exp\left(\frac{-\operatorname{arsinh}\left(x_i^{i+1}/y_i^{i+1}\right)}{\mu_i\sigma_i \operatorname{sign}\left(y_i^{i+1}\right)}\right) y_i^{i+1} \Rightarrow \tag{29}$$

$$\psi_i \triangleq \exp\left(\frac{-C_i}{\mu_i\beta_i}\right) y_i^{i+1}, \tag{30}$$

where curve parameter $\psi_i \neq 0$ corresponds to $y^{i+1} \neq 0$ such that $x^{i+1} = 0$. It means that curve (18) crosses Y^{i+1} axis at point $(0, \psi_i)$. Such a point exists for every curve described by (18) and can be computed from (30), which is why one can assume $y_i^{i+1} := \psi_i$ and $x_i^{i+1} := 0$ in (18) without loss of generality and obtain:

$$x^{i+1} \overset{(18)}{=} \sinh\left[-k_p \sigma_i \mu_i \operatorname{sign}\left(y_i^{i+1}\right) \tau_i\right] y^{i+1} \overset{(25)}{\Rightarrow} \tag{31}$$

$$x^{i+1} = \sinh\left[\sigma_i \mu_i \operatorname{sign}\left(y_i^{i+1}\right) \ln\left(\frac{y^{i+1}}{y_i^{i+1}}\right)\right] y^{i+1} \overset{(8)}{\Rightarrow} \tag{32}$$

$$x^{i+1} = \sigma_i \sinh\left[\mu_i \ln\left(\frac{y^{i+1}}{y_i^{i+1}}\right)\right] \left|y^{i+1}\right| \overset{y_i^{i+1} := \psi_i}{\Rightarrow} \tag{33}$$

$$x^{i+1} = \sigma_i \sinh\left[\mu_i \ln\left(\frac{y^{i+1}}{\psi_i}\right)\right] \left|y^{i+1}\right|, \tag{34}$$

which can be transformed to ((17)) using the definition of sinh:

$$x^{i+1}\left(y^{i+1}\right) = \frac{\sigma_i |y^{i+1}|}{2} \left[\left(\frac{y^{i+1}}{\psi_i}\right)^{\mu_i} - \left(\frac{y^{i+1}}{\psi_i}\right)^{-\mu_i}\right], \tag{35}$$

where ψ_i is given by (30) and corresponds to a particular value of y^{i+1} when $x^{i+1} = 0$. Integral curve (35) describes robot's motion in perfect conditions, i.e. when $e_{ai}^{i+1} = 0$ and the auxiliary orientation θ_{ai}^{i+1} is tracked perfectly by robot's orientation θ^{i+1}. Robot will move along curve (35) regardless of chosen velocity profile $\rho_i > 0$.

Remark 1 *Negative time τ_i can be obtained during some calculations. It corresponds to valid positions on the integral curve, which will not be traversed by the robot in the particular motion segment. They could be traversed if the beginning of motion segment (i.e point $\left(x_i^{i+1}, y_i^{i+1}\right)$) was placed on the same integral curve, but further away from the $(i+1)$-st waypoint.*

3 Satisfaction of curvature constraint

Let us derive equation ((19)) describing curvature of convergence vector field $\bar{\boldsymbol{h}}_i^{i+1}$. From definition of control input $\boldsymbol{u}$ one has:

$$\kappa_i = \frac{u_1}{u_2} = \frac{\dot{\theta}_{ai}^{i+1}}{\sigma_i \rho_i} = \frac{\dot{h}_{yi}^{i+1} h_{xi}^{i+1} - \dot{h}_{xi}^{i+1} h_{yi}^{i+1}}{\sigma_i \rho_i ||\bar{\boldsymbol{h}}_i^{i+1}||^2}. \tag{36}$$

Moreover, assuming $e_{ai}^{i+1} \equiv 0$, one can write:

$$
\begin{aligned}
\dot{h}_{yi}^{i+1} &\overset{(4)}{=} -k_p \dot{y}^{i+1} \overset{(7)}{=} -k_p \rho_i \frac{h_{yi}^{i+1}}{||\bar{\boldsymbol{h}}_i^{i+1}||} \overset{(4)}{=} k_p^2 \rho_i \frac{y^{i+1}}{||\bar{\boldsymbol{h}}_i^{i+1}||}, && (37)\\
\dot{h}_{yi}^{i+1} h_{xi}^{i+1} &\overset{(37),(4)}{=} \frac{-k_p^3 \rho_i}{||\bar{\boldsymbol{h}}_i^{i+1}||} \left(x^{i+1} y^{i+1} + y^{i+1} \sigma_i \mu_i ||\bar{\boldsymbol{q}}^{i+1}|| \right), && (38)\\
\dot{h}_{xi}^{i+1} &\overset{(4)}{=} -k_p \left(\dot{x}^{i+1} + \sigma_i \mu_i \frac{\dot{x}^{i+1} x^{i+1} + \dot{y}^{i+1} y^{i+1}}{||\bar{\boldsymbol{q}}^{i+1}||} \right) \\
&\overset{(7)}{=} \frac{-k_p \rho_i}{||\bar{\boldsymbol{h}}_i^{i+1}||} \left(h_{xi}^{i+1} + \sigma_i \mu_i \frac{h_{xi}^{i+1} x^{i+1} + h_{yi}^{i+1} y^{i+1}}{||\bar{\boldsymbol{q}}^{i+1}||} \right) \\
&\overset{(4)}{=} \frac{k_p^2 \rho_i}{||\bar{\boldsymbol{h}}_i^{i+1}||} \left(\left(x^{i+1} + \sigma_i \mu_i ||\bar{\boldsymbol{q}}^{i+1}|| \right) + \sigma_i \mu_i \frac{(y^{i+1})^2 + x^{i+1} \left(x^{i+1} + \sigma_i \mu_i \parallel \bar{\boldsymbol{q}}^{i+1} \parallel \right)}{||\bar{\boldsymbol{q}}^{i+1}||} \right) \\
&= \frac{k_p^2 \rho_i}{||\bar{\boldsymbol{h}}_i^{i+1}||} \left(x^{i+1} + \sigma_i \mu_i ||\bar{\boldsymbol{q}}^{i+1}|| + \sigma_i \mu_i \frac{||\bar{\boldsymbol{q}}^{i+1}||^2 + \sigma_i \mu_i x^{i+1} ||\bar{\boldsymbol{q}}^{i+1}||}{||\bar{\boldsymbol{q}}^{i+1}||} \right) \\
&= \frac{k_p^2 \rho_i}{||\bar{\boldsymbol{h}}_i^{i+1}||} \left(x^{i+1} + 2 \sigma_i \mu_i ||\bar{\boldsymbol{q}}^{i+1}|| + \mu_i^2 x^{i+1} \right), && (39)\\
\dot{h}_{xi}^{i+1} h_{yi}^{i+1} &\overset{(39),(4)}{=} \frac{-k_p^3 \rho_i}{||\bar{\boldsymbol{h}}_i^{i+1}||} \left[2 \sigma_i \mu_i y^{i+1} ||\bar{\boldsymbol{q}}^{i+1}|| + x^{i+1} y^{i+1} \left(1 + \mu_i^2 \right) \right], && (40)
\end{aligned}
$$

hence:

$$
\begin{aligned}
\kappa_i &\overset{(36)}{=} \frac{\dot{h}_{yi}^{i+1} h_{xi}^{i+1} - \dot{h}_{xi}^{i+1} h_{yi}^{i+1}}{\sigma_i \rho_i ||\bar{\boldsymbol{h}}_i^{i+1}||^2} \\
&\overset{(38),(40)}{=} \frac{-k_p^3}{\sigma_i ||\bar{\boldsymbol{h}}_i^{i+1}||^3} \left[x^{i+1} y^{i+1} + y^{i+1} \sigma_i \mu_i ||\bar{\boldsymbol{q}}^{i+1}|| - 2 \sigma_i \mu_i y^{i+1} ||\bar{\boldsymbol{q}}^{i+1}|| - x^{i+1} y^{i+1} \left(1 + \mu_i^2 \right) \right] \\
&= \frac{-k_p^3}{\sigma_i ||\bar{\boldsymbol{h}}_i^{i+1}||^3} \left[-\mu_i^2 x^{i+1} y^{i+1} - y^{i+1} \sigma_i \mu_i ||\bar{\boldsymbol{q}}^{i+1}|| \right] \\
&= \frac{k_p^3}{\sigma_i ||\bar{\boldsymbol{h}}_i^{i+1}||^3} y^{i+1} \left[\mu_i^2 x^{i+1} + \sigma_i \mu_i ||\bar{\boldsymbol{q}}^{i+1}|| \right], && (41)
\end{aligned}
$$

but since from the definition of $\bar{\boldsymbol{h}}_i^{i+1}$ we have:

$$
\begin{aligned}
||\bar{\boldsymbol{h}}_i^{i+1}|| &= \sqrt{\left(h_{xi}^{i+1} \right)^2 + \left(h_{yi}^{i+1} \right)^2} \\
&\overset{(4)}{=} \sqrt{k_p^2 \left(x^{i+1} + \sigma_i \mu_i \parallel \bar{\boldsymbol{q}}^{i+1} \parallel \right)^2 + k_p^2 \left(y^{i+1} \right)^2} \\
&= \sqrt{k_p^2 \left[\left(x^{i+1} \right)^2 + 2 \sigma_i \mu_i x^{i+1} \parallel \bar{\boldsymbol{q}}^{i+1} \parallel + \mu_i^2 \parallel \bar{\boldsymbol{q}}^{i+1} \parallel^2 \right] + k_p^2 \left(y^{i+1} \right)^2} \\
&= \sqrt{k_p^2 \left[\left(x^{i+1} \right)^2 + \left(y^{i+1} \right)^2 + \mu_i^2 ||\bar{\boldsymbol{q}}^{i+1}||^2 + 2 \sigma_i \mu_i x^{i+1} ||\bar{\boldsymbol{q}}^{i+1}|| \right]} \\
&= \sqrt{k_p^2 \left[||\bar{\boldsymbol{q}}^{i+1}||^2 + \mu_i^2 ||\bar{\boldsymbol{q}}^{i+1}||^2 + 2 \sigma_i \mu_i x^{i+1} ||\bar{\boldsymbol{q}}^{i+1}|| \right]} \\
&= k_p ||\bar{\boldsymbol{q}}^{i+1}|| \sqrt{1 + \mu_i^2 + 2 \sigma_i \mu_i x^{i+1} / ||\bar{\boldsymbol{q}}^{i+1}||}, && (42)
\end{aligned}
$$

one can write:

$$\kappa_i \overset{(41),(42)}{=} \frac{y^{i+1}\left(\mu_i^2 x^{i+1} + \sigma_i \mu_i ||\bar{\boldsymbol{q}}^{i+1}||\right)}{\sigma_i ||\bar{\boldsymbol{q}}^{i+1}||^3 \sqrt{1+\mu_i^2 + 2\sigma_i\mu_i x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}||}^3} \Rightarrow \tag{43}$$

$$\kappa_i\left(x^{i+1}, y^{i+1}\right) = \frac{\mu_i y^{i+1}}{||\bar{\boldsymbol{q}}^{i+1}||^2} m_i\left(\bar{\boldsymbol{q}}^{i+1}\right), \tag{44}$$

$$m_i\left(\bar{\boldsymbol{q}}^{i+1}\right) = \frac{\sigma_i + \mu_i\left(x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}||\right)}{\sigma_i\sqrt{1+\mu_i^2+2\sigma_i\mu_i\left(x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}||\right)}^3} \tag{45}$$

as in ((19)).

In [1] we define $\hat{m}_i \triangleq \lim_{\bar{\boldsymbol{q}}^{i+1}\to\mathbf{0}} m_i\left(\bar{\boldsymbol{q}}^{i+1}\right) = const$ used in analysis of limit ((22)). This claim is verified by examining definition of $m_i\left(\bar{\boldsymbol{q}}^{i+1}\right)$ in (45) and taking into account that:

- $\sigma_i \neq 0$,
- $\forall(x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}||) \quad 1+\mu_i^2+2\sigma_i\mu_i\left(x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}||\right) \neq 0$,
- $\lim_{\bar{\boldsymbol{q}}^{i+1}\to\mathbf{0}} x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}|| = \operatorname{sign}\left(x_i^{i+1}\right)$ along integral curve (35).

It remains to show that $\lim_{\bar{\boldsymbol{q}}^{i+1}\to\mathbf{0}} x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}|| = 1$ along integral curve (35). Let us recall that $\operatorname{sign}\left(x^{i+1}\right) = \operatorname{sign}\left(x_i^{i+1}\right)$ (cf. (35)) and compute

$$\begin{aligned}
x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}|| &= \operatorname{sign}\left(x^{i+1}\right)\sqrt{\left(x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}||\right)^2} \\
&= \operatorname{sign}\left(x_i^{i+1}\right)\sqrt{\left(x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}||\right)^2} \\
&= \operatorname{sign}\left(x_i^{i+1}\right)\sqrt{\left(\frac{x^{i+1}}{\sqrt{\left(x^{i+1}\right)^2+\left(y^{i+1}\right)^2}}\right)^2} \\
&\overset{(35)}{=} \operatorname{sign}\left(x_i^{i+1}\right)\sqrt{\left(\frac{0.5\,\sigma_i|y^{i+1}|\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{\mu_i}-\left(\frac{y^{i+1}}{\psi_i}\right)^{-\mu_i}\right]}{\sqrt{0.25\,|y^{i+1}|^2\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{\mu_i}-\left(\frac{y^{i+1}}{\psi_i}\right)^{-\mu_i}\right]^2+\left(y^{i+1}\right)^2}}\right)^2} \\
&= \operatorname{sign}\left(x_i^{i+1}\right)\sqrt{\left(\frac{\sigma_i\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{\mu_i}-\left(\frac{y^{i+1}}{\psi_i}\right)^{-\mu_i}\right]}{\sqrt{\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{\mu_i}-\left(\frac{y^{i+1}}{\psi_i}\right)^{-\mu_i}\right]^2+1}}\right)^2} \\
&= \operatorname{sign}\left(x_i^{i+1}\right)\sqrt{\frac{\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{\mu_i}-\left(\frac{y^{i+1}}{\psi_i}\right)^{-\mu_i}\right]^2}{D\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{\mu_i}-\left(\frac{y^{i+1}}{\psi_i}\right)^{-\mu_i}\right]^2+D}},
\end{aligned} \tag{46}$$

$$D \triangleq \operatorname{sign}\left(\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{\mu_i}-\left(\frac{y^{i+1}}{\psi_i}\right)^{-\mu_i}\right]^2+1\right), \tag{47}$$

which can be simplified to

$$x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}|| \overset{(46)}{=} \operatorname{sign}\left(x_i^{i+1}\right)\sqrt{\frac{\left(\frac{y^{i+1}}{\psi_i}\right)^{-2\mu_i}\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{2\mu_i}-1\right]^2}{\left(\frac{y^{i+1}}{\psi_i}\right)^{-2\mu_i}\left\{D\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{2\mu_i}-1\right]^2+D\left(\frac{y^{i+1}}{\psi_i}\right)^{2\mu_i}\right\}}}$$

$$= \operatorname{sign}\left(x_i^{i+1}\right)\sqrt{\frac{\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{2\mu_i}-1\right]^2}{D\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{2\mu_i}-1\right]^2+D\left(\frac{y^{i+1}}{\psi_i}\right)^{2\mu_i}}}, \tag{48}$$

hence

$$\lim_{\bar{\boldsymbol{q}}^{i+1}\to\mathbf{0}} x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}|| \overset{(48)}{=} \lim_{\bar{\boldsymbol{q}}^{i+1}\to\mathbf{0}} \operatorname{sign}\left(x_i^{i+1}\right)\sqrt{\frac{\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{2\mu_i}-1\right]^2}{D\left[\left(\frac{y^{i+1}}{\psi_i}\right)^{2\mu_i}-1\right]^2+D\left(\frac{y^{i+1}}{\psi_i}\right)^{2\mu_i}}}$$

$$= \operatorname{sign}\left(x_i^{i+1}\right)\sqrt{\frac{1}{D}} \overset{(47)}{=} \operatorname{sign}\left(x^{i+1}\right). \tag{49}$$

3.1 Curvature scaling

To verify that for a constant $S > 0$ one can write $\kappa_i\left(Sx^{i+1}, Sy^{i+1}\right) = \kappa_i\left(x^{i+1}, y^{i+1}\right)/S$ we observe that

$$x^{i+1}/||\bar{\boldsymbol{q}}^{i+1}|| = Sx^{i+1}/||S\bar{\boldsymbol{q}}^{i+1}||. \tag{50}$$

This relation implies that $m_i\left(\bar{\boldsymbol{q}}^{i+1}\right) = m_i\left(S\bar{\boldsymbol{q}}^{i+1}\right)$ (cf. (45)). It is now evident that

$$\kappa_i\left(Sx^{i+1}, Sy^{i+1}\right) = \frac{\mu_i S y^{i+1}}{S^2||\bar{\boldsymbol{q}}^{i+1}||^2} m_i\left(\bar{\boldsymbol{q}}^{i+1}\right) = \kappa_i\left(x^{i+1}, y^{i+1}\right)/S. \tag{51}$$

To show that κ_i along integral curve (35) is inversely proportional to ψ_i one must show that both $x^{i+1}\left(y^{i+1}\right)$ and y^{i+1} (cf. (35)) scale linearly with ψ_i. Let us recall that ψ_i corresponds to a particular value of y^{i+1} when $x^{i+1} = 0$, hence by definition y^{i+1} scales linearly with ψ_i. If one considers that y^{i+1}/ψ_i is constant under scaling, because y^{i+1} scales linearly with ψ_i, it can be seen from (35) that $x^{i+1}\left(Sy^{i+1}\right) = Sx^{i+1}\left(y^{i+1}\right)$. After considering the above claims, one concludes that κ_i along integral curve (35) is indeed inversely proportional to ψ_i, which means that we must compute only particular curvature supremum $\hat{\kappa}_i\left(0,1\right)$ and use

$$\hat{\kappa}_i\left(x_i^{i+1}, y_i^{i+1}\right) = \hat{\kappa}_i\left(0, \psi_i\right) = \hat{\kappa}_i\left(0,1\right)/\psi_i, \tag{52}$$

for other cases. Note that such a solution is general, since ψ_i can be computed for any integral curve using (30). $\hat{\kappa}_i\left(0,1\right)$ represents maximal curvature of motion along a particular integral curve described by (35) with $\psi_i = 1$. One might interpret this particular curve as an archetype of all other integral curves described by (35). All other integral curves can be obtained by scaling of this curve. The scaling corresponds to different values of parameter ψ_i. We will show how to compute $\hat{\kappa}_i\,(0,1)$ in the next subsection.

3.2 Maximum of curvature

To find $\hat{\kappa}_i\,(0,1) = \kappa_i\,(x^{i+1}(\hat{y}^{i+1}), \hat{y}^{i+1})$ (cf. ((20))), one must only compute $\hat{y}^{i+1}$ corresponding to the point at which maximum of curvature is reached by curve (35). The value of $x^{i+1}(\hat{y}^{i+1})$ is given by (35).

To simplify the notation, we will assume $\mathcal{Y} \equiv |y^{i+1}|$ in the sequel. We begin the derivation of the exact value of $\hat{y}^{i+1}$ by substituting (35) into (44) and taking $x_i^{i+1} = 0 \,\wedge\, \psi_i = 1$ (because $\hat{\kappa}_i\,(0,1)$ must be computed) to obtain:

$$\tilde{\kappa}\,(\mathcal{Y}) \overset{(35)}{=} \frac{4\,\mu_i\,\left(\mu_i\,\mathcal{Y}^{2\,\mu_i} - \mu_i + \mathcal{Y}^{2\,\mu_i} + 1\right)}{\mathcal{Y}^{\mu_i+1}\left(\frac{\left(\mu_i + \mu_i\,\mathcal{Y}^{2\,\mu_i} + \mathcal{Y}^{2\,\mu_i} - 1\right)^2}{\mathcal{Y}^{2\,\mu_i}} + 4\right)^{\frac{3}{2}}}, \tag{53}$$

where $\tilde{\kappa}\,(\mathcal{Y})$ denotes curvature along the considered particular integral curve, which can be differentiated to write

$$\frac{d}{dy^{i+1}}\tilde{\kappa} = -\frac{N}{M} = 0, \tag{54}$$

where:

$$\begin{aligned} N = {}& 4\mu_i\,(2\,\mu_i\,\mathcal{Y}^{4\,\mu_i} - 2\,\mu_i\,\mathcal{Y}^{2\,\mu_i} - 2\,\mu_i + 2\,\mu_i\,\mathcal{Y}^{6\,\mu_i} - 3\,\mu_i + 3\,\mathcal{Y}^{2\,\mu_i} + \\ & 3\,\mathcal{Y}^{4\,\mu_i} + \mathcal{Y}^{6\,\mu_i} + 2\,\mu_i^3 - \mu_i^4 + 2\,\mu_i^3\,\mathcal{Y}^{2\,\mu_i} - 3\,\mu_i^4\,\mathcal{Y}^{2\,\mu_i} - 2\,\mu_i^3\,\mathcal{Y}^{4\,\mu_i} - 3\,\mu_i^4\,\mathcal{Y}^{4\,\mu_i} - \\ & 2\,\mu_i^3\,\mathcal{Y}^{6\,\mu_i} - \mu_i^4\,\mathcal{Y}^{6\,\mu_i} + 9\,\mu_i^2 - 9\,\mu_i^3 + 3\,\mu_i^4 - 3\,\mu_i\,\mathcal{Y}^{2\,\mu_i} + 3\,\mu_i\,\mathcal{Y}^{4\,\mu_i} + 3\,\mu_i\,\mathcal{Y}^{6\,\mu_i} + \\ & 3\,\mu_i^2\,\mathcal{Y}^{2\,\mu_i} + 3\,\mu_i^3\,\mathcal{Y}^{2\,\mu_i} + 3\,\mu_i^2\,\mathcal{Y}^{4\,\mu_i} - 3\,\mu_i^4\,\mathcal{Y}^{2\,\mu_i} - 3\,\mu_i^3\,\mathcal{Y}^{4\,\mu_i} + \\ & 9\,\mu_i^2\,\mathcal{Y}^{6\,\mu_i} - 3\,\mu_i^4\,\mathcal{Y}^{4\,\mu_i} + 9\,\mu_i^3\,\mathcal{Y}^{6\,\mu_i} + 3\,\mu_i^4\,\mathcal{Y}^{6\,\mu_i} + 1), \end{aligned} \tag{55}$$

$$M = \mathcal{Y}^{3\,\mu_i+2}\big(2\,\mu_i\,\mathcal{Y}^{2\,\mu_i} - 2\,\mu_i\mathcal{Y}^{-2\,\mu_i} + \mathcal{Y}^{-2\,\mu_i} + \mathcal{Y}^{2\,\mu_i} + 2\,\mu_i^2 + \mu_i^2\mathcal{Y}^{-2\,\mu_i} + \mu_i^2\,\mathcal{Y}^{2\,\mu_i} + 2\big)^{\frac{5}{2}}. \tag{56}$$

Since $\mathcal{Y} \neq 0 \Rightarrow M \neq 0$, we can solve $N = 0$ instead. Solution of this equation is ((21)). It can be obtained by utilizing a substitution $\mathcal{Y}^{2\mu_i} = v$ in the equation $N = 0$ and solving the resulting cubic polynomial

$$\begin{aligned} & 4\mu_i\,[v^3\left(2\,\mu_i^4 + 7\,\mu_i^3 + 9\,\mu_i^2 + 5\,\mu_i + 1\right) + v^2\left(-6\,\mu_i^4 - 5\,\mu_i^3 + 3\,\mu_i^2 + 5\,\mu_i + 3\right) + \\ & v\left(-6\,\mu_i^4 + 5\,\mu_i^3 + 3\,\mu_i^2 - 5\,\mu_i + 3\right) + 2\,\mu_i^4 - 7\,\mu_i^3 + 9\,\mu_i^2 - 5\,\mu_i + 1] = 0 \end{aligned} \tag{57}$$

w.r.t. $\mathcal{Y}$ to obtain $\hat{y}^{i+1}$. Polynomial (57) has one real root $\hat{y}^{i+1}$ and two conjugate complex roots, hence $\hat{\kappa}_i$ is maximum of curvature in i-th motion segment.

It remains to show that $\mathcal{Y} \neq 0 \Rightarrow M \neq 0$. Let us factorize M as follows

$$\begin{aligned} M &\overset{(56)}{=} \mathcal{Y}^{3\,\mu_i+2}\big\{\mathcal{Y}^{-2\mu_i}\left(\mathcal{Y}^{2\mu_i} + 1\right)\left[\mathcal{Y}^{2\mu_i}\left(\mu_i^2 + 2\mu_i + 1\right) + \left(\mu_i^2 - 2\mu_i + 1\right)\right]\big\}^{\frac{5}{2}} \\ &= \mathcal{Y}^{3\,\mu_i+2}\big\{\mathcal{Y}^{-2\mu_i}\left(\mathcal{Y}^{2\mu_i} + 1\right)\left[\mathcal{Y}^{2\mu_i}\,(\mu_i + 1)^2 + (\mu_i - 1)^2\right]\big\}^{\frac{5}{2}} \\ &= \mathcal{Y}^{2-2\mu_i}\big\{\left(\mathcal{Y}^{2\mu_i} + 1\right)\left[\mathcal{Y}^{2\mu_i}\,(\mu_i + 1)^2 + (\mu_i - 1)^2\right]\big\}^{\frac{5}{2}}. \end{aligned} \tag{58}$$

It follows from (58) that $M = 0$ iff

$$\mathcal{Y} = 0 \quad \vee \quad \mathcal{Y} = (-1)^{\frac{1}{2\mu_i}} \quad \vee \quad \mathcal{Y} = \left(-\frac{(\mu_i - 1)^2}{(\mu_i + 1)^2} \right)^{\frac{1}{2\mu_i}}, \tag{59}$$

which means that M will have value of 0 only when $\mathcal{Y}$ will be either 0 or complex. We know that such a situation cannot occur by definition.

Remark 2 *Using the reasoning from Subsection 3.1 one can express curvature along any integral curve* (35) *as*

$$\kappa_i(\mathcal{Y}) = \frac{\tilde{\kappa}(\mathcal{Y})}{\psi_i}. \tag{60}$$

4 Relations used in multi-segment stabilization planning

Let us begin by deriving ((27)). From definition of θ_{ai}^{i+1} we have:

$$\begin{aligned} \theta_{ai}^{i+1} &= \text{Atan2c}\left(\sigma_i h_{yi}^{i+1}, \sigma_i h_{xi}^{i+1}\right) \\ &\overset{(4)}{=} \text{Atan2c}\left(-k_p \sigma_i y^{i+1}, -k_p \sigma_i \left(x^{i+1} + \sigma_i \mu_i ||\bar{\boldsymbol{q}}^{i+1}||\right)\right), \end{aligned} \tag{61}$$

which after taking into account that $y^{i+1} = a_i^{i+1} x^{i+1}$ takes the form:

$$\begin{aligned} \theta_{ai}^{i+1} &= \text{Atan2c}\left(-k_p \sigma_i \text{sign}\left(x_i^{i+1}\right) |x^{i+1}| a_i^{i+1}, -k_p |x^{i+1}| \left(\sigma_i \text{sign}\left(x_i^{i+1}\right) + \mu_i \sqrt{1 + \left(a_i^{i+1}\right)^2}\right)\right) \\ &= \text{Atan2c}\left(-\alpha_i a_i^{i+1}, -\alpha_i - \mu_i \sqrt{1 + \left(a_i^{i+1}\right)^2}\right), \end{aligned} \tag{62}$$

$$\alpha_i \triangleq \sigma_i \text{sign}\left(x_i^{i+1}\right). \tag{63}$$

To derive ((38)) one can use (62) as follows:

$$\tan(\theta_{ai}^{i+1}) \triangleq T_i = \frac{\text{sign}\left(x_i^{i+1}\right) a_i^{i+1}}{\text{sign}\left(x_i^{i+1}\right) + \mu_i \sigma_i \sqrt{1 + \left(a_i^{i+1}\right)^2}} \Rightarrow \tag{64}$$

$$T_i \mu_i \sqrt{1 + \left(a_i^{i+1}\right)^2} = \text{sign}\left(x_i^{i+1}\right)\left(a_i^{i+1} - T_i\right) \Rightarrow \tag{65}$$

$$(T_i)^2 (\mu_i)^2 \left(1 + \left(a_i^{i+1}\right)^2\right) = \left(a_i^{i+1} - T_i\right)^2 \Rightarrow \tag{66}$$

$$(T_i)^2 (\mu_i)^2 \left(1 + \left(a_i^{i+1}\right)^2\right) = (a_i^{i+1})^2 - 2T_i a_i^{i+1} + (T_i)^2, \tag{67}$$

which is a quadratic polynomial w.r.t a_i^{i+1}

$$(a_i^{i+1})^2 \left((T_i)^2 (\mu_i)^2 - 1\right) + a_i^{i+1}(2T_i) + (T_i)^2 \left((\mu_i)^2 - 1\right) = 0 \tag{68}$$

with discriminant:

$$\begin{aligned}
\Delta &= 4(T_i)^2 - 4\left((T_i)^2(\mu_i)^2 - 1\right)(T_i)^2\left((\mu_i)^2 - 1\right) \\
&= 4(T_i)^2\left(1 + \left((\mu_i)^2 - 1\right)\left(1 - (T_i)^2(\mu_i)^2\right)\right) \\
&= 4(T_i)^2\left(1 + (\mu_i)^2 - (\mu_i)^4(T_i)^2 - 1 + (T_i)^2(\mu_i)^2\right) \\
&= 4(T_i)^2(\mu_i)^2\left(1 + (T_i)^2(1 - (\mu_i)^2)\right) > 0, \qquad (69) \\
\sqrt{\Delta} &= 2\mu_i\sqrt{(T_i)^4\left(1 - (\mu_i)^2\right) + (T_i)^2}, \qquad (70)
\end{aligned}$$

and two solutions:

$$a_i^{i+1} = \frac{\pm\mu_i\sqrt{(T_i)^4\left(1 - (\mu_i)^2\right) + (T_i)^2} - T_i}{(\mu_i)^2(T_i)^2 - 1}, \qquad (71)$$

as in ((28))).

Before deriving the conservative constraint ((31)), let us recall that $|\psi_i| \geq \hat{\psi}_i$ must be satisfied for integral curve which will be followed by the robot to satisfy the curvature constraint ((2)). It is also known from the definition of ψ_i that for any constant value of y_i^{i+1}, x_i^{i+1} grows linearly w.r.t. ψ_i.

Since the considered waypoint lies on a straight line $y^{i+1} = a_i^{i+1}x^{i+1}$ with known a_i^{i+1}, one can define a constraint on x_i^{i+1}

$$|x_i^{i+1}| \geq \hat{\Psi}_i > 0. \qquad (72)$$

where $\hat{\Psi}_i$ is a conservative estimate which we will now compute. Let us begin by computing x^{i+1} coordinate of the point where boundary integral curve (one with $\psi_i = \hat{\psi}_i$) intersects with straight line $y^{i+1} = a_i x^{i+1}$. It is given by ((30)) and can be computed using (35):

$$\begin{aligned}
x^{i+1}\left(y^{i+1}\right) &\overset{(35)}{=} \frac{\sigma_i|y^{i+1}|}{2}\left(\left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{\mu_i} - \left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{-\mu_i}\right) \Rightarrow \qquad (73) \\
\frac{x^{i+1}}{y^{i+1}} &= \frac{\operatorname{sign}\left(y^{i+1}\right)\sigma_i}{2}\left(\left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{\mu_i} - \left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{-\mu_i}\right) \Rightarrow \qquad (74) \\
\frac{2\sigma_i\operatorname{sign}\left(y^{i+1}\right)}{a_i} &= \left(\left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{\mu_i} - \left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{-\mu_i}\right) \Rightarrow \qquad (75) \\
\left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{\mu_i}\frac{2\sigma_i\operatorname{sign}\left(y^{i+1}\right)}{a_i^{i+1}} &= \left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{2\mu_i} - 1 \Rightarrow \qquad (76) \\
0 &= \left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{2\mu_i} - \left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{\mu_i}\frac{2\sigma_i\operatorname{sign}\left(y^{i+1}\right)}{a_i^{i+1}} - 1, \qquad (77)
\end{aligned}$$

and solving (77) for $\left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{\mu_i}$:

$$\Delta = \left(2/a_i^{i+1}\right)^2 + 4 > 0, \quad \sqrt{\Delta} = 2\sqrt{1/(a_i^{i+1})^2 + 1}, \tag{78}$$

$$\left(\frac{y^{i+1}}{\hat{\psi}_i}\right)^{\mu_i} = \frac{\sigma_i \mathrm{sign}\left(y^{i+1}\right)}{a_i^{i+1}} \pm \sqrt{1/(a_i^{i+1})^2 + 1} \Rightarrow \tag{79}$$

$$x^{i+1} = \frac{\hat{\psi}_i}{a_i^{i+1}} \left(\frac{\sigma_i \mathrm{sign}\left(y_i^{i+1}\right)}{a_i^{i+1}} \pm \sqrt{\left(1/a_i^{i+1}\right)^2 + 1}\right)^{1/\mu_i}. \tag{80}$$

Expression (80) has two values, only one of which corresponds to the actual intersection point with the considered boundary integral curve. To simplify the analysis, we estimate conservatively to obtain ((31)) based on ((30)) and ((20)):

$$\left|x^{i+1}\right| \geq \hat{\Psi}_i = \frac{\hat{\psi}_i}{|a_i^{i+1}|} \left(\frac{1}{|a_i^{i+1}|} + \sqrt{\left(1/a_i^{i+1}\right)^2 + 1}\right)^{1/\mu_i}. \tag{81}$$

References

[1] T. Gawron and M. M. Michałek. VFO stabilization of a unicycle robot with bounded curvature of motion. Submitted to RoMoCo 2015, January 2015.