Abstraction in Decision-Makers with Limited Information Processing Capabilities

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Abstraction

- Neglecting **irrelevant** information with respect to a particular **purpose**
  - Example: chair for the purpose of sitting down
  - separation of structure from noise
  - treating different things as equal

- Crucial if information processing capabilities are limited
- Traditionally, the formation of abstractions is associated with high computational cost (e.g. clustering)
- We argue that abstractions arise as a consequence of limited computational capacity
- … which could be induced by time- or memory-constraints, limited sample size or high signal-to-noise ratio

Multi-Task Decision-Making scenario

- Here, we consider the following multi-task problem - the qualitative findings carry over to more general inference or decision-making scenarios

- Each task is formalized via a (relative) utility function U
- Agent emits an action x conditioned on an observation y: $p(x|y)$
- Assumptions:
  - Action-space is shared among all tasks
  - Observations are **fully** informative about the current task
  - Discrete, i.i.d. setting

- Multi-task abstractions – different tasks will be treated equally

Thermodynamic Model for DM

- Decision-making with information processing limits
- Behavior/policy is modeled by a probability distribution
- Trade-off between gain in expected utility and transformation cost (from prior behavior p_0 to optimized policy $p(x|y)$)

$$\underset{p(x|y)}{\operatorname{argmax}} \mathbf{E}_{p(x|y)}[U_y(x)] - \frac{1}{\beta} D_{\mathrm{KL}}(p(x|y)||p_0(x))$$

- Variational problem has very similar mathematical form as a *free-energy difference* minimization
- Closed-form solution:

$$p(x|y) = \frac{1}{Z} p_0(x) e^{\beta U_y(x)}$$

Temperature as rationality-parameter

$$\underset{p(x|y)}{\operatorname{argmax}}\, \mathbf{E}_{p(x|y)}[U_y(x)] - \frac{1}{\beta} D_{\mathrm{KL}}(p(x|y)||p_0(x))$$

$$p(x|y) = \frac{1}{Z} p_0(x) e^{\beta U_y(x)}$$

- Limits:
  - Fully rational actor: $\beta \rightarrow \infty$
  - Fully bounded actor: $\beta \rightarrow 0$

- Normative framework for changing from **prior** belief/behavior to **posterior** belief/behavior with information processing cost
  - Bayes rule can be recovered as a special case

Rate-Distortion for Decision-Making

- Extend free energy model by taking the average over observations and optimizing over the prior as well:

$$\underset{p_0(x)}{\operatorname{argmax}} \sum_y p(y) \left[\underset{p(x|y)}{\operatorname{argmax}} \mathbf{E}_{p(x|y)}[U_y(x)] - \frac{1}{\beta} D_{\mathrm{KL}}(p(x|y)||p_0(x)) \right]$$

- … which can be rewritten:

$$\underset{p(x|y)}{\operatorname{argmax}} \underbrace{\sum_{x,y} p(x,y)U(x,y)}_{\mathbf{E}_{p(x,y)}[U]} - \frac{1}{\beta} \underbrace{\sum_y p(y) D_{\mathrm{KL}}(p(x|y)||p(x))}_{I(x;y)}$$

- Same mathematical form as rate-distortion problem

Rate-Distortion for Decision-Making

$$\underset{p(x|y)}{\operatorname{argmax}} \underbrace{\sum_{x,y} p(x,y)U(x,y)}_{\mathbf{E}_{p(x,y)}[U]} - \frac{1}{\beta} \underbrace{\sum_{y} p(y) D_{\mathrm{KL}}(p(x|y)||p(x))}_{I(x;y)}$$

- Trade off between **high expected utility** and **low mutual information** between actions and observations (tasks)
- Optimal prior implies minimal average transformation cost for adapting to a particular task
  - If memory is limited, $p(x)$ would be the optimal information to store

- Rate-distortion – a framework for lossy compression
  - Lossy compresion requires neglecting “the right” information
  - Separation of structure and noise

Temperature as rationality-parameter

- Well known (self-consistent) solution:

$$p(x|y) = \frac{1}{Z} p(x) e^{\beta U_y(x)}$$
$$p(x) = \sum_y p(y) p(x|y)$$

- Temperature governs trade-off between high expected utility and low task-specificity.
  - $\beta \to \infty$: optimal adaptation to each task
  - $\beta \to 0$: full abstraction – same policy for all tasks
- Agent with limited information processing capabilities (finite β) can not adapt optimally to each task
  - Emergence of robustness/abstraction
  - Temperature governs the trade-off and thus the granularity of the abstraction

Experiment I

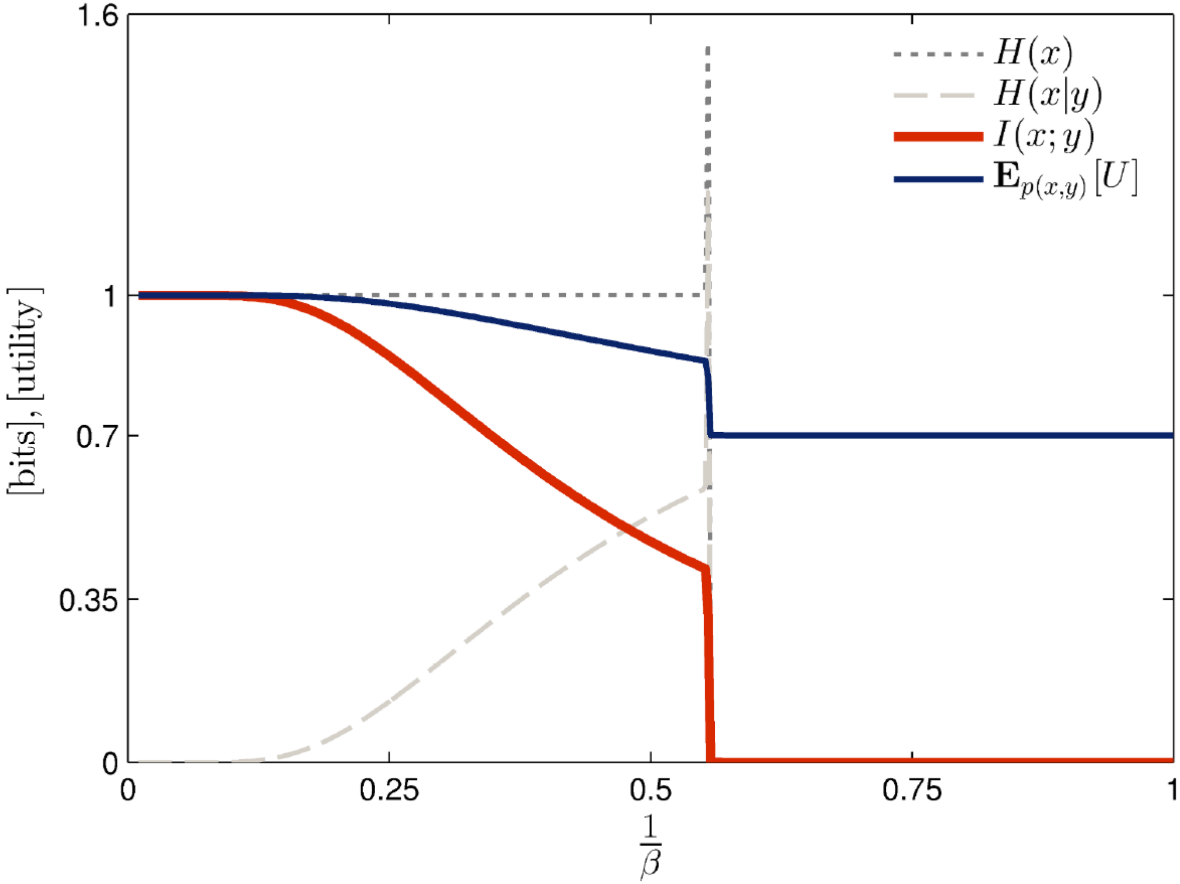
- Four actions, two tasks
- Two temperature values
  - High inverse temperature: rational actor
  - Low inverse temperature: bounded actor

			$\beta = 100$			$\beta = 1$		
x	$U(x, y_1)$	$U(x, y_2)$	$p(x)$	$p(x\|y_1)$	$p(x\|y_2)$	$p(x)$	$p(x\|y_1)$	$p(x\|y_2)$
$[0, 0]$	0	0	0	0	0	0	0	0
$[0, 1]$	0	1	0.5	0	1	0	0	0
$[0.7, 0]$	0.7	0.7	0	0	0	1	1	1
$[1, 1]$	1	0	0.5	1	0	0	0	0

- Fully rational actor: selects maximum utility action in each task
- Fully bounded actor: always chooses suboptimal action
  - → mutual information between action and observation is zero

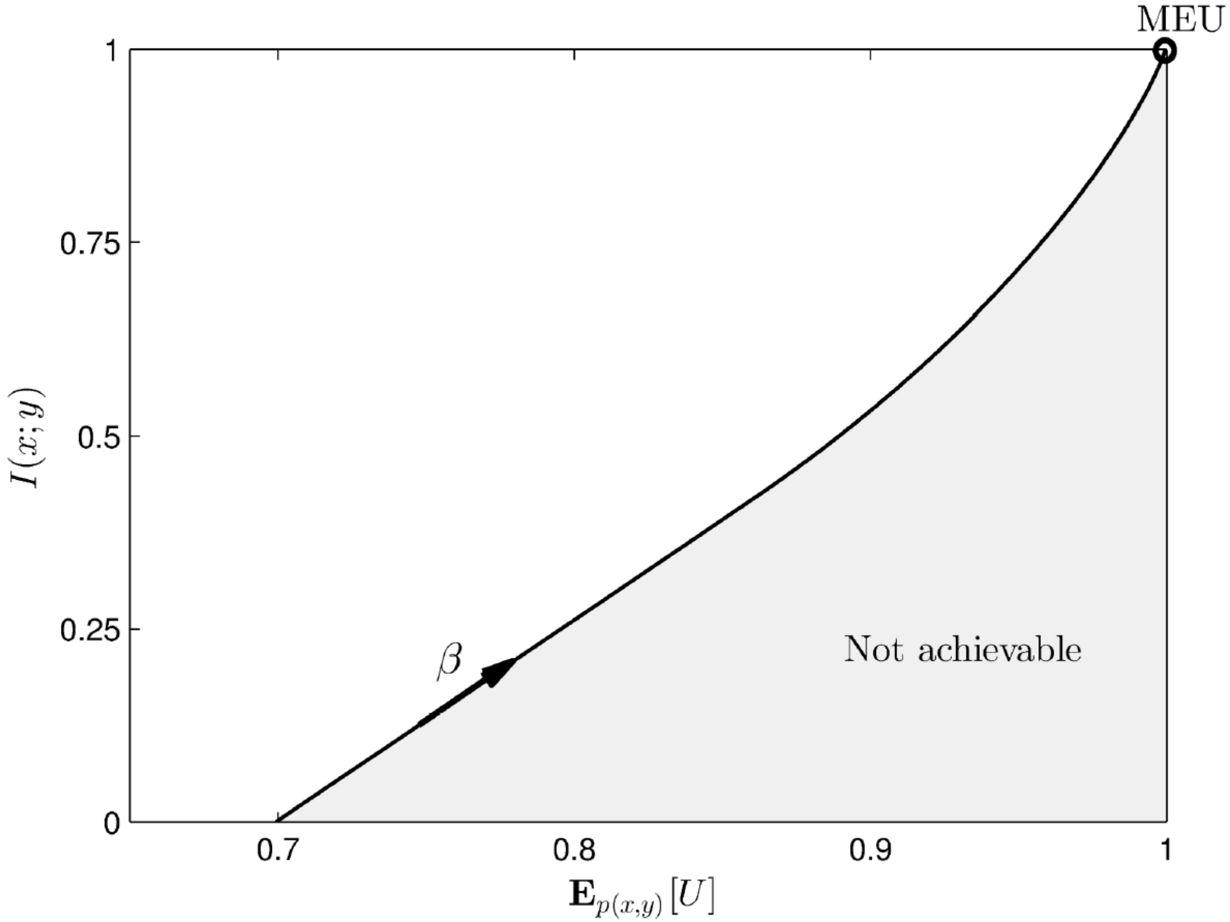
Experiment I

- Initial mixing of maximum utility actions (increase in stochasticity)
- Gradual decrease of MI up to a "phase-transition"



Experiment I

- Bounded-optimal actors lie on the rate-utility curve
- The shaded region is theoretically impossible to achieve



Experiment II

- Actions are binary colorings of 3×3 grids
- 512 possible actions/patterns

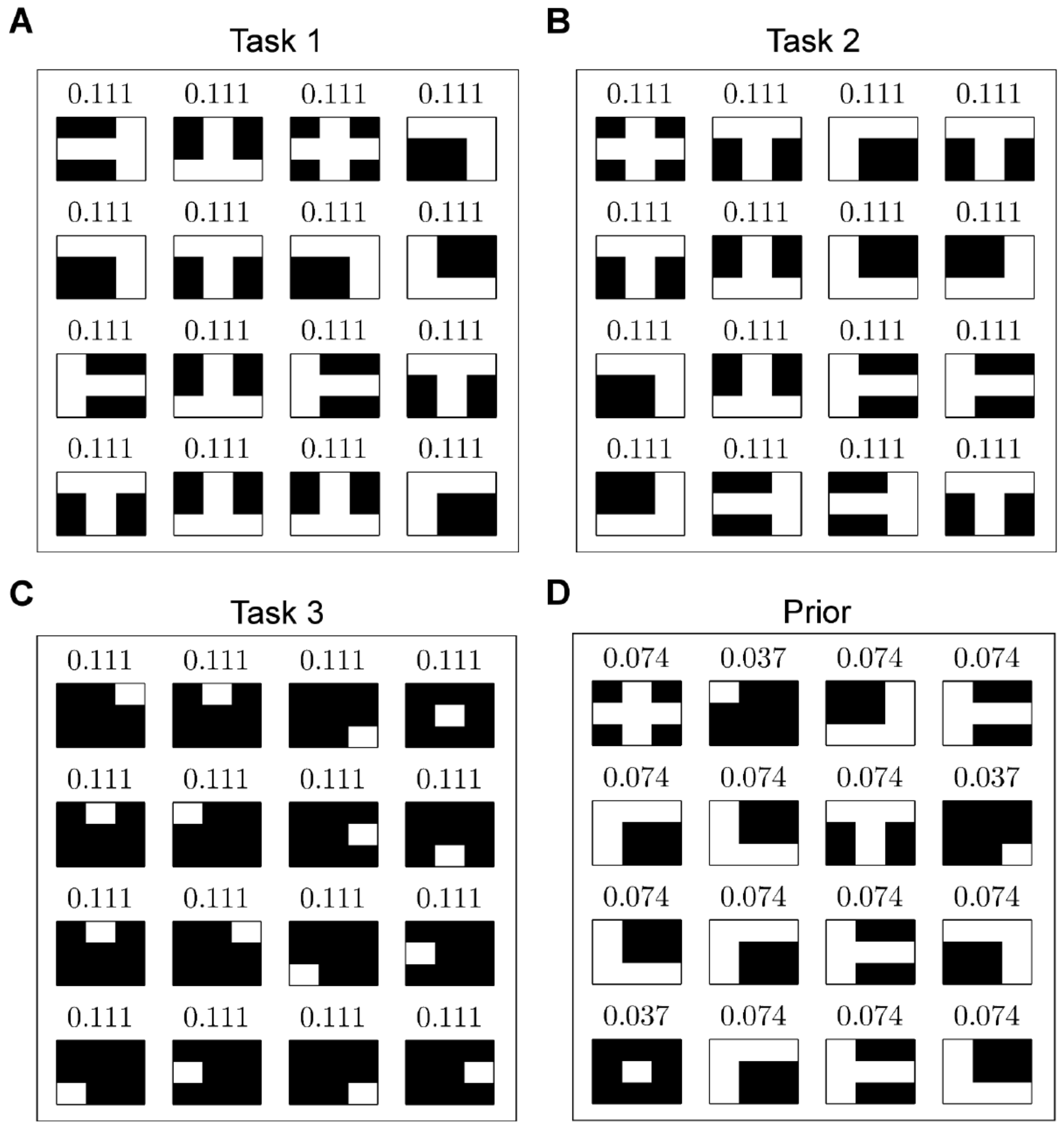
- Three environments
  1) One row and one column has to be all-white, each colored pixel yields +1 utility
  2) Any pattern with exactly four colored pixels yields +4 utility
  3) Any pattern with an even number of colored pixels yields +1 utility for each colored pixel

- Effective cardinality of the action-distribution is affected by boundedness of the actor

Experiment II

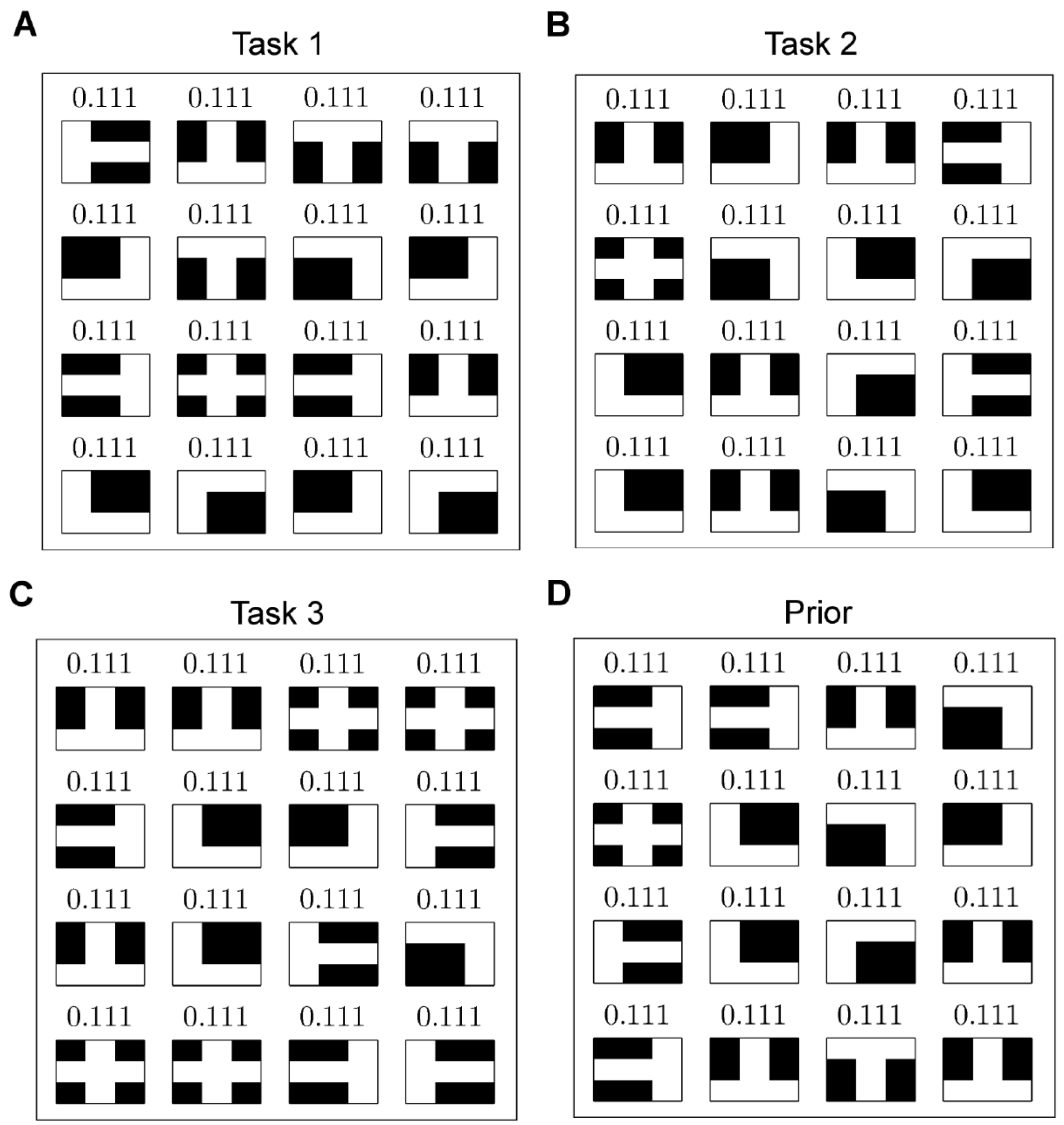
$\beta = 10$

- Samples from conditionals and prior



Experiment II
$\beta = 0.1$

- Samples from conditionals and prior



Discussion

- Formalism also carries over to more general decision-making or inference cases
- Similar work
  - VAN DIJK, S. G. & POLANI, D. (2013). Informational Constraints-Driven Organization in Goal-Directed Behavior. *Advances in Complex Systems*.
  - STILL, S & CRUTCHFIELD, J. P. (2008). Structure or Noise? *arXiv:0708.0654v2 [physics.data-an]*
  - Information Bottleneck Method, Relevant Information
  - Rational Inattention
- Certain regularizers might be elegant ways of implementing rate-distortion

- Continuous cases
  - No closed-form solutions for self-consistent equations
- Sampling representation
  - Representing prior and conditionals with samples
- Modelling **several layers** of abstraction
  - Unclear how to break down utility functions across layers