Assignment:
Bayesian Analysis of the Normal Regression Model

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Let us assume that we have collected n observations from the normal regression model with p explanatory variables

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip} + \varepsilon_i, \quad \text{with } \varepsilon_i \sim \mathcal{N}(0, \sigma^2), \tag{1}$$

where $x_{ij} \in \mathbb{R}$, $j = 1, \ldots, p$, $i = 1, \ldots, n$, are known and ε_i are (unobserved) independent and identically distributed random variables. Assume that the prior distribution of the parameters $(\boldsymbol{\beta}, \sigma^2)$ is the conjugate normal–inverse gamma model

$$(\boldsymbol{\beta}, \sigma^2) \sim \mathcal{N}_q\left(\boldsymbol{\mu}_\beta, \sigma^2 V\right) \mathcal{IG}(a, b)$$

with probability density function

$$\begin{aligned}\pi(\boldsymbol{\beta}, \sigma^2) &= \pi(\boldsymbol{\beta}|\sigma^2)\pi(\sigma^2)\\ &= \frac{b^a}{(2\pi)^{q/2}\Gamma(a)|V|^{1/2}(\sigma^2)^{a+1+q/2}}\\ &\qquad \times \exp\left\{-\frac{1}{2\sigma^2}(\boldsymbol{\beta} - \boldsymbol{\mu}_\beta)^t V^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu}_\beta) - \frac{b}{\sigma^2}\right\} \mathrm{I}_{\mathbb{R}^q \times (0,\infty)}(\boldsymbol{\beta}, \sigma^2),\end{aligned}$$

where $q = p + 1$, while the prior mean $\boldsymbol{\mu}_\beta$ is a $q \times 1$ vector, V denotes $q \times q$ positive definite symmetric matrix and $a,\ b > 0$, corresponding to the known hyper-parameters of the model. Recall that the posterior distribution is also normal-inverse gamma: $(\boldsymbol{\beta}, \sigma^2|\boldsymbol{y}) \sim \mathcal{N}_q\left(\widetilde{\boldsymbol{\beta}}, \sigma^2\widetilde{\Sigma}\right) \mathcal{IG}(\widetilde{a}, \widetilde{b})$.

1. Define a function in `R` that will summarize the basic characteristics of the posterior distribution $(\boldsymbol{\beta}, \sigma^2|\boldsymbol{y})$. The input of your function should be:

   (α′) a *vector* `y` with length n containing the observed values of the response variable $\boldsymbol{y}$

   (β′) an $n \times q$ *matrix* `x` containing the observed design matrix X

   (γ′) a *list* `prior_parameters` containing the hyper-parameters of the model, that is

   - `prior_parameters$a`: scalar $a > 0$.
   - `prior_parameters$b`: scalar $b > 0$.
   - `prior_parameters$mu`: vector $\boldsymbol{\mu}_\beta \in \mathbb{R}^q$. The default value should be equal to $\boldsymbol{\mu}_\beta = (0, \ldots, 0)^t$.

- `g`: positive scalar $g > 0$ that corresponds to the g hyper-parameter for the Zellner's g-prior. The prior parameter V will be set equal to $V = g(X^t X)^{-1}$ where X denotes the design matrix. The default value should be equal to $g = n$.

and should return the following objects:

- a *list* containing the parameters of the posterior distribution, named as `posterior_parameters`, consisting of the following entries:
  - `posterior_parameters$a`: parameter (scalar) $\widetilde{a} > 0$ of the posterior distribution.
  - `posterior_parameters$b`: parameter (scalar) $\widetilde{b} > 0$ of the posterior distribution.
  - `posterior_parameters$beta`: parameter (vector) $\widetilde{\boldsymbol{\beta}} \in \mathbb{R}^q$ of the posterior distribution.
  - `posterior_parameters$Sigma`: parameter ($q \times q$ positive definite symmetric matrix) $\widetilde{\Sigma}$ of the posterior distribution.
- the posterior mean of β_j, $j = 0, 1 \ldots, p$ and σ^2.
- the posterior variance of β_j, $j = 0, 1 \ldots, p$ and σ^2.
- the 95% and 99% equally tailed credible intervals of β_j, $j = 0, 1 \ldots, p$ and σ^2.
- the (natural) logarithm of the marginal likelihood $m(\boldsymbol{y})$.

2. For the accompanying dataset assume that $a = b = 0.01$, $\boldsymbol{\mu}_\beta = (0, \ldots, 0)^t$ and use the Zellner's g-prior with $g = n$.

   (α′) Fit the full model (containing a constant term and $p = 9$ explanatory variables) and report a summary of the posterior distribution.

   (β′) Calculate the posterior probabilities

   1. $\mathrm{P}(\beta_1 > 0 | \boldsymbol{y}, X)$
   2. $\mathrm{P}(\beta_3 > 0 | \boldsymbol{y}, X)$
   3. $\mathrm{P}(\sigma^2 > 3 | \boldsymbol{y}, X)$
   4. $\mathrm{P}(\beta_1 > 0 | \sigma^2 = 4, \boldsymbol{y}, X)$
   5. $\mathrm{P}(\beta_3 > 0 | \sigma^2 = 4, \boldsymbol{y}, X)$

   (γ′) Use the Bayes Factor in order to test the hypothesis

$$H_0\colon \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = \beta_9 = 0 \quad \text{vs} \quad H_1\colon \text{not } H_0.$$

Καλή επιτυχία!