Reaction-diffusion spatial modeling of COVID-19 in Chicago

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Outline

Background

The study of infectious diseases began with Bernoulli in 1760.
The study of a critical level of transmissibility for a disease is fundamental in epidemiology.
This is useful for

- predicting counterintuitive effects of treatments, and
- considering containment measures

The main flavors of models fit under deterministic/stochastic and discrete/continuous.
Here we consider a deterministic and continuous compartment model (Kermack-McKendrick model).

Assumptions

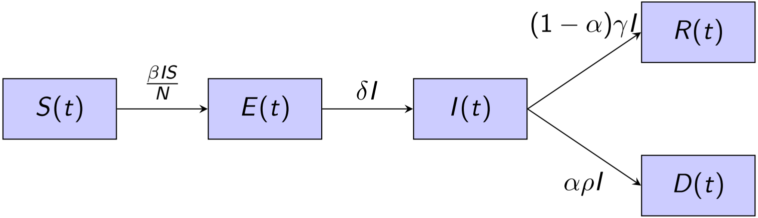
- Birth rate balances the mortality rate.
- No immunity in early stages of epidemic.
- Removal provides complete immunity against reinfection.

Outline

1. COVID-19 in Chicago and the Impact of Demographics

2. Accounting for Mobility

Chicago Model



$$
\begin{aligned}
\dot{S} &= -\frac{\beta IS}{N} \\
\dot{E} &= \frac{\beta IS}{N} - \delta E \\
\dot{I} &= \delta E - (1-\alpha)\gamma I - \alpha\rho I \\
\dot{R} &= (1-\alpha)\gamma I \\
\dot{D} &= \alpha\rho I
\end{aligned}
$$

- N is the total population of the group
- β is the transmission rate
- α is the infection fatality rate
- δ, γ, and ρ are the transition rates

Definition (Basic Reproduction Number)

$$R_0 = \frac{\beta}{\gamma}$$

Basic Reproduction Number

The website Rt Live tracked R_0 since the beginning of the COVID-19 pandemic using data from The COVID Tracking Project.

Model ([?])

Search for the most likely curve that produced the new cases per day that is observed:

- Assume a seed number of people and a R_0 curve,
- Distribute these cases into the future using a known delay distribution,
- Scale and add noise.

Based the calculation for Illinois, we assume a logistic form for R_0 over time:

$$R_0(t) = \frac{R_{\text{start}} - R_{\text{end}}}{1 + \exp(-k(x_0 - t))} + R_{\text{end}}$$

Demographics and Death Rates

The City of Chicago Data Portal provides COVID data in demographic categories for age, race, and sex. We begin by fitting α for each category independently, where

- δ, γ, and ρ are constants determined medical literature, and
- R_0 is determined from the Rt Live model.

Final model trained on the entire data set and predicted 10 days out (from July 16).

The average death rate of Chicago is then calculated by a weighted mean:

$$\alpha = \frac{1}{\sum N_i} \sum \alpha_i N_i$$

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2. Accounting for Mobility

Local Model

$$S(t) \xrightarrow{\lambda S} I(t) \xrightarrow{\gamma I} R(t)$$

$$\dot{S} = -\lambda S$$
$$\dot{I} = \lambda S - \gamma I$$
$$\dot{R} = \gamma I$$

The force of infection describes the infection rate:

$$\lambda = \frac{\beta_I I}{N}$$

- $N = S + I + R$ is the total population
- β_I is the transmission rate
- γ is the removal rate

Basic Reproduction Number

Consider the disease-free equilibrium: $S = N$, $I = R = 0$.

Then the Jacobian matrix of the infection subsystem is

$$\mathrm{J}_0 = \left[\frac{\partial \dot{I}}{\partial I}\right] = \left[-\gamma + \frac{\beta_I SI}{N^2} + \frac{\beta_I S}{N}\right]_{S=N, I=R=0} = -\gamma + \beta_I$$

J_0 can be decomposed into transmission ($\mathrm{T} = \beta_I$) and transition ($\Sigma = -\gamma$) parts such that $\mathrm{J}_0 = \mathrm{T} + \Sigma$.

The next-generation matrix is

$$\mathrm{K_L} = -\mathrm{T}\Sigma^{-1} = \frac{\beta_I}{\gamma}$$

and the basic reproduction number is

$$\mathcal{R}_0 = \rho(\mathrm{K_L}) = \frac{\beta_I}{\gamma}$$

Dynamics

Let $k = \frac{\beta_I}{N}$, then $\dot{S} = -kSI$ and $\dot{I} = kSI - \gamma I$
There are infintely non-isolated fixed points of the form $(a, 0)$.
Linearizing, we get

$$\mathrm{A}_{(a,0)} = \begin{bmatrix} 0 & -ka \\ 0 & ka - \gamma \end{bmatrix}$$

so for $a < \frac{\gamma}{k}$ the nodes are stable, and they are unstable for $a > \frac{\gamma}{k}$.
Taking $\frac{\dot{I}}{\dot{S}} = -1 + \frac{\gamma}{kS}$ and solving this differential equation gives $I = -S + \frac{\gamma}{k}\ln(S) + C$. Thus

$$E(S, I) = S + I - \frac{\gamma}{k}\ln(S)$$

is a conserved quantity for the system. So there cannot be any attracting fixed points ([?]).

Spatial Model

Now suppose we have a set of n connected communities. Assume γ is community-independent, while λ_i is the force of infection for community i.

$$\begin{aligned} \dot{S}_i &= -\lambda_i S_i \\ \dot{I}_i &= \lambda_i S_i - \gamma I_i \\ \dot{R}_i &= \gamma I_i \end{aligned}$$

Definition ([?])

$$\lambda_i = \sum_{j=1}^{n} C_{ij}^{S} \frac{\sum_{k=1}^{n}(\beta_I C_{kj}^{I} I_k)}{\sum_{k=1}^{n}(C_{kj}^{S} S_k + C_{kj}^{I} I_k + C_{kj}^{R} R_k)}$$

C_{ij}^{X} (with $X \in \{S, I, R\}$) is the probability that individuals who belong to compartment X and are from community i contact individuals in community j.

Mobile Contact Probability

If reliable mobility data exists, we can explicitly define the contact probabilities C_{ij}^X.

Definition ([?])

$$C_{ij}^X = \begin{cases} (1-p_i) + (1-r_X)p_i + r_X p_i q_{ij} & \text{if } i = j, \\ r_X p_i q_{ij} & \text{otherwise} \end{cases}$$

- p_i is the fraction of mobile people in i
- q_{ij} is the fraction of mobile people beween i and $j = 1, \ldots, n$
- $r_X (0 \leq r_X \leq 1)$ is the fraction of contacts occuring when people of X travel

Mobility data can be often be found in the form of Origin-Destination matrices.

- The US DoT and MIT provide such data on a national scale.
- Within Chicago, we can access tranportation datasets provided by the Chicago Data Portal.

Mobility Models

In data-scare contexts, models are needed to predict mobility patterns.
In general, the probability for a person in location i to travel from i to j is given by

$$\pi_{ij} = \frac{F_{ij}}{T_i}$$

where F_{ij} is the number of trips from i to j, and T_i is the total number of trips leaving i, per day.

Gravity Model

$$F_{ij} = k\frac{P_i^n P_j^m}{f_\gamma(d_{ij})} \qquad \pi_{ij} = \frac{\frac{P_j^m}{f_\gamma(d_{ij})}}{\sum_{i\neq j} \frac{P_j^m}{f_\gamma(d_{ij})}}$$

Radiation Model

$$\pi_{ij} = \frac{P_i P_j}{(P_i + S_{ij})(P_i + P_j + S_{ij})}$$

Impedance Model ([?])

$$F_{ij} = \alpha\frac{P_i + P_j}{d_{ij}} \qquad \pi_{ij} = \frac{\frac{P_i+P_j}{d_{ij}}}{\sum_{i\neq j}\left(\frac{P_i+P_j}{d_{ij}}\right)}$$

Mobile Contact Probability

Using one of the definitions of π_{ij} we can define the contact probabilities.

Definition

$$C_{ij}^{X} = \begin{cases} (1 - \sum_{j=1}^{n} \pi_{ij}) + (1 - r_X)(\sum_{j=1}^{n} \pi_{ij}) + r_X \pi_{ij} & \text{if } i = j, \\ r_X \pi_{ij} & \text{otherwise} \end{cases}$$

- $r_X (0 \leq r_X \leq 1)$ is the fraction of contacts occuring when people of X travel