Reaction-diffusion spatial modeling of COVID-19 in Chicago

Trent Gerew*

Department of Applied Mathematics, Illinois Institute of Technology, Chicago, Illinois

November 6, 2021

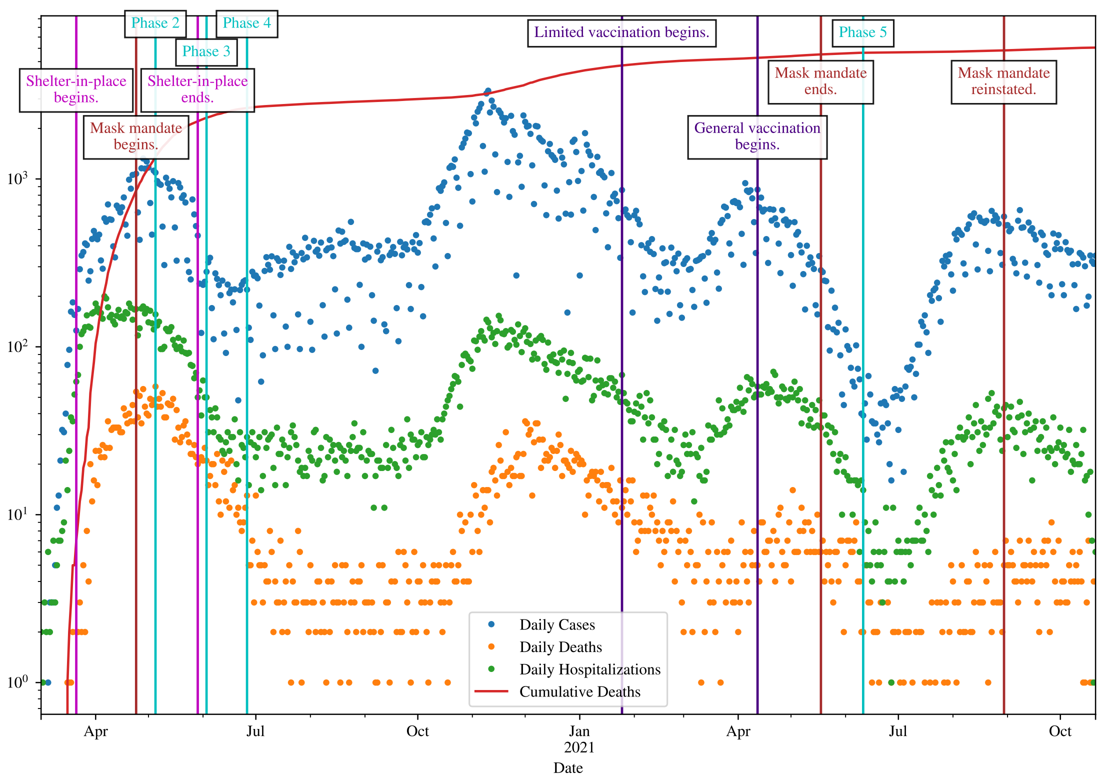


Figure 1: Timeline of the progression of COVID-19 in Chicago with key public policy events marked. The COVID-19 data was obtained from the City of Chicago Data Portal [3]. The dates of the policy events were gathered from the Illinois.gov press releases [9], [7], [8], [10], [6], the Chicago Tribune [2], and NBC Chicago [1]. Note the logarithmic scale.

1 Model Setup

We begin by explaining the ODE model. This is obtained from the full PDE model in Equations (1)-(8) by simply removing the diffusion terms.

*tgerew@hawk.iit.edu

We extend the standard SIR model to incorporate key aspects of the COVID-19 virus.

Given the time scale of the pandemic, we model a weak effect of the change in the susceptible population from births or other mortality factors with the term $-\mu S$. We estimate $\mu = 1.8997 \times 10^{-5}$ [per day] using the average United States death rate in urban regions from 2019 [4].

Table 1: Time sequence of events and simulation times.

Initial simulation time	Imposed lockdown	Effective lockdown	Last fitting day
March 18, 2020	March 21, 2020	April 1, 2020	June 24, 2020
$t_i = 1$	$t_q = 4$	$t_q = 15$	$t_f = 99$

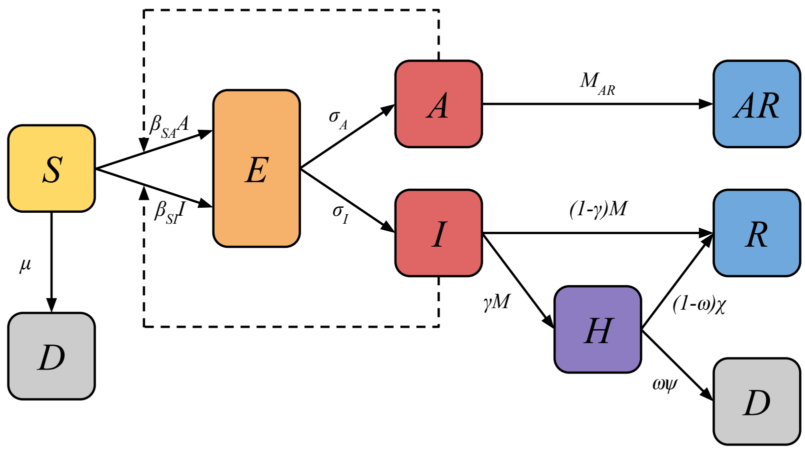


Figure 2: Schematic diagram of the model. The dashed lines indicate the interaction of the infected populations with the susceptible populations that leads to infection.

$$
\begin{aligned}
S_t &= \mathfrak{D}_S \Delta S - \beta_{SA} SA - \beta_{SI} SI - \mu S, && (1)\\
E_t &= \mathfrak{D}_E \Delta E + \beta_{SA} SA + \beta_{SI} SI - (\sigma_A + \sigma_I) E, && (2)\\
AR_t &= M_{AR} A, && (3)\\
A_t &= \mathfrak{D}_A \Delta A + \sigma_A E - M_{AR} A, && (4)\\
I_t &= \sigma_I E - MI, && (5)\\
H_t &= \gamma MI - (1-\omega)\chi H - \omega\psi H, && (6)\\
R_t &= (1-\gamma) MI + (1-\omega)\chi H, && (7)\\
D_t &= \omega\psi H. && (8)
\end{aligned}
$$

To create the PDE model, we first must find suitable parameters using the ODE model as in previous studies. We utilize the MATLAB nonlinear optimization algorithm `fminsearch` for this purpose. The optimal parameters are determined by minimizing the Euclidean distance $\mathcal{N}$ between the time series generated by the model (subscript "num") and the corresponding observed data time series (subscript "obs")

$$\mathcal{N} = \sum_i \left(|\ln C_{\text{num}}(t_i) - \ln C_{\text{obs}}(t_i)| + |\ln D_{\text{num}}(t_i) - \ln D_{\text{obs}}(t_i)| \right) \qquad (9)$$

where the index i identifies a point in the time series. The parameters are chosen to reproduce the time series of the total number of cases $C(t) = I(t) + H(t) + R(t) + D(t)$, and the total number of deceased $D(t)$.

Table 2: Population values for Chicago. Initial populations are determined from March 13, 2020.

		Population
Total population	N	2,695,598
Initial infected	I_0	162
Initial hospitalized	H_0	38
Initial deceased	D_0	3

To account for changes in virus transmission due to the shelter-in-place order, we impose a time dependence on the transmission rates β as in Equations (10) and (11).

$$\beta_{SI}(t) = \beta_{SI}\left(\eta_{SI} + (1-\eta_{SI})\frac{1-\tanh[2(t-t_q)]}{2}\right) \tag{10}$$

$$\beta_{SA}(t) = \beta_{SA}\left(\eta_{SA} + (1-\eta_{SA})\frac{1-\tanh[2(t-t_q)]}{2}\right) \tag{11}$$

Table 3: ODE parameters: optimal (best-fitting), median and interquartile range, and variation range used in the optimization algorithm. Initial parameter guesses were uniformly sampled within these ranges.

		Median (IQR)	Initial value
Transmission rate, $S \to I$ [per day][a]	β_{SI}		$c \in \mathcal{U}[0,1]$
Transmission rate, $S \to A$ [per day][a]	β_{SA}		$c \in \mathcal{U}[0,1]$
Lockdown effect, $S \to I$	η_{SI}		$c \in \mathcal{U}[0,1]$
Lockdown effect, $S \to A$	η_{SA}		$c \in \mathcal{U}[0,1]$
Incubation period, $E \to I$ [days]	$1/\sigma_I$		$1/k, k \in \mathcal{U}[2,7]$
Latent period, $E \to A$ [days]	$1/\sigma_A$		$1/k, k \in \mathcal{U}[2,7]$
Infectivity period [days]	$1/M$		$1/k, k \in \mathcal{U}[5,12]$
Recovery period, $A \to AR$ [days]	$1/M_{AR}$		$1/k, k \in \mathcal{U}[5,12]$
Recovery period, $H \to R$ [days]	$1/\chi$		$1/k, k \in \mathcal{U}[5,20]$
Period to deceased, $H \to D$ [days]	$1/\psi$		$1/k, k \in \mathcal{U}[5,20]$
Conversion fraction ($I \xrightarrow{\gamma} H,\ I \xrightarrow{1-\gamma} R$)	γ		$c \in \mathcal{U}[0.25, 0.75]$
Conversion fraction ($H \xrightarrow{\omega} D,\ H \xrightarrow{1-\omega} R$)	ω		$c \in \mathcal{U}[0.1, 0.5]$
Initial population fraction, exposed	E_0/I_0		$c \in \mathcal{U}[1,5]$
Initial population fraction, asymptomatic	A_0/I_0		$c \in \mathcal{U}[1,5]$

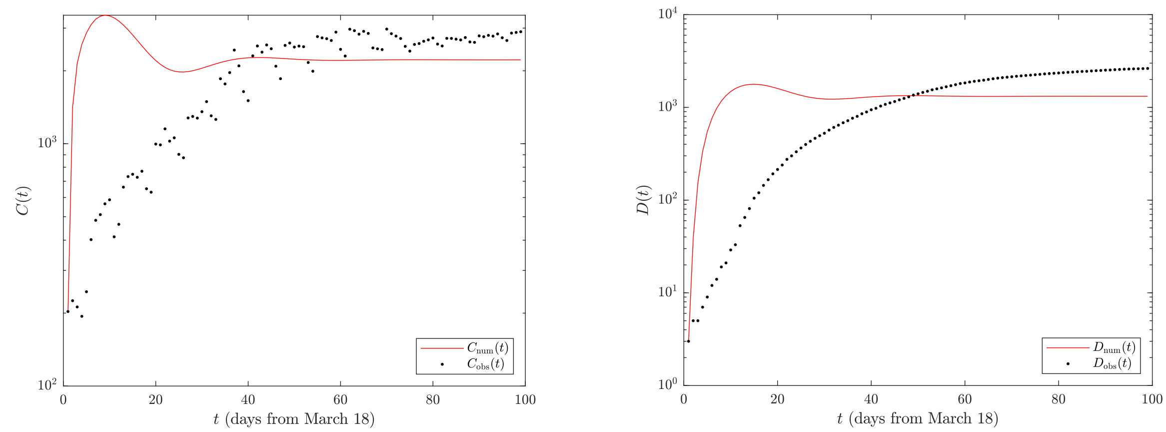
[a] Note: The transmission rates β must be divided by N when used in the ODE model.

2 ODE Dynamics

We want to understand the trajectories of the dynamics of the ODE system (1)-(8) under different initial conditions. To do this we first find the equilibrium points by solving

$$S_t = E_t = A_t = I_t = H_t = R_t = D_t = 0$$

simultaneously for $\mathbf{x} = (S, E, A, AR, I, H, R, D)^\intercal$. The solutions of this system are of the form $\mathbf{x}^* = (0, 0, 0, AR, 0, 0, R, D)^\intercal$. This implies there are infinitely many non-isolated equilibrium points. We determine the stability of these equilibrium points by analyzing the linearized system near the



(a) Confirmed cases $C(t) = I(t) + R(t) + H(t) + D(t)$. (b) Number of deaths $D(t)$.

Figure 3: ODE model with fitting to official data from March 18, 2020 ($t_i = 1$) to June 24, 2020 ($t_f = 99$). Here we show the case when β changes exactly on the imposed lockdown on March 21, 2020 ($t_q = 4$).

points. The Jacobian of the system is

$$\mathsf{J} = \begin{pmatrix} -A\beta_{SA} - I\beta_{SI} - \mu & 0 & -S\beta_{SA} & 0 & -S\beta_{SI} & 0 & 0 & 0 \\ A\beta_{SA} + I\beta_{SI} & -\sigma_A - \sigma_I & S\beta_{SA} & 0 & S\beta_{SI} & 0 & 0 & 0 \\ 0 & \sigma_A & -M_{AR} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{AR} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_I & 0 & 0 & -M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M\gamma & -\chi(1-\omega) - \psi\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & M(1-\gamma) & \chi(1-\omega) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi\omega & 0 & 0 \end{pmatrix}. \tag{12}$$

Now evaluating J at the equilibrium point $\mathbf{x}^*$ and calculating the eigenvalues, we have

$$\lambda = \{0, 0, 0, -M, -M_{AR}, -\mu, -\sigma_A - \sigma_I, -\chi + \chi\omega - \psi\omega\}. \tag{13}$$

Note that the first three eigenvalues are 0, which implies the equilibrium points are non-isolated. This agrees with our earlier observation.

The equilibrium points are stable when $\lambda_i < 0$ for $4 \leq i \leq 8$. Since all the system parameters are positive, this implies $\lambda_i < 0$ for $4 \leq i \leq 7$. Thus the stability depends on the sign of λ_8. There are two cases when $\lambda_8 = -\chi + \chi\omega - \psi\omega < 0$ is true:

1. $0 < \omega \leq 1$ implies $\lambda_8 < 0$, and
2. $\omega > 1$ and $\chi < \frac{\psi\omega}{\omega - 1}$ implies $\lambda_8 < 0$.

That is, whenever we have either of these conditions the equilibrium points are stable. If $\lambda_8 > 0$, the equilibrium points are unstable.

We can further analyze the evolution of the pandemic by calculating the basic reproduction number R_0. We use the next generation matrix approach of the system (1)-(8) without the spatial terms, as in [5] and [11]. In particular, we rewrite the model in the form $\mathbf{x}_t = \mathbf{F} - \mathbf{V}$. The components F_i represents the rate of appearance of new infections in compartment i. The vector $\mathbf{V} = \mathbf{V}^- - \mathbf{V}^+$, where V_i^+ represents the rate of transfer of individuals into compartment i by all other means, and V_i^- represents the rate of transfer of individuals out of compartment i. Reordering the compartments so $\mathbf{x_t'} = (E_t, A_t, I_t, H_t, S_t, AR_t, R_t, D_t)^\intercal$, we have

$$\mathbf{F} = \begin{pmatrix} \beta_{SA}SA + \beta_{SI}SI \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} (\sigma_A + \sigma_I)E \\ -\sigma_A E + M_{AR}A \\ -\sigma_I E + MI \\ -\gamma MI + (1-\omega)\chi H + \omega\psi H \\ \beta_{SA}SA + \beta_{SI}SI + \mu S \\ M_{AR}A \\ -(1-\gamma)MI - (1-\omega)\chi H \\ -\omega\psi H \end{pmatrix}.$$

We focus on just the infectious/infected compartments, E, A, I, H, and find the Jacobians of $\mathbf{F}$ and $\mathbf{V}$ with respect to these populations in the order in which they appear. Evaluating at the the disease-free equilibrium ($S = S^*,\ E = AR = A = I = H = R = D = 0$) yields

$$\mathsf{F} = \begin{pmatrix} 0 & \beta_{SA}S^* & \beta_{SI}S^* & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{V} = \begin{pmatrix} \sigma_A + \sigma_I & 0 & 0 & 0 \\ -\sigma_A & M_{AR} & 0 & 0 \\ -\sigma_I & 0 & M & 0 \\ 0 & 0 & -\gamma M & \chi(1-\omega) + \psi\omega \end{pmatrix}.$$

The matrix FV^{-1} is the next-generation matrix. Then $R_0 = \rho(\mathsf{FV}^{-1})$, which is

$$R_0 = \frac{\beta_{SA}S^*\sigma_A}{m_{AR}(\sigma_A + \sigma_I)} + \frac{\beta_{SI}S^*\sigma_I}{M(\sigma_A + \sigma_I)}. \tag{14}$$

References

[1] Read the full 'restore Illinois' plan aimed at reopening the state during coronavirus. *NBC Chicago*, May 2020. `https://www.nbcchicago.com/news/coronavirus/read-the-full-restore-illinois-plan-aimed-at-reopening-the-state-during-coronavirus/2267039/`.

[2] J. Byrne, D. Petrella, and A. Lukach. Mayor Lori Lightfoot says Chicago will move to phase 3 of her reopening plan on June 3 but warns: 'COVID-19 is still very much part of our present'. *Chicago Tribune*, May 2020. `https://www.chicagotribune.com/coronavirus/ct-coronavirus-chicago-lightfoot-reopening-20200528-cefwiuidwnfd7a57m25uavq6me-story.html`.

[3] City of Chicago. Daily chicago covid-19 cases, deaths, and hospitalizations, 2021. Data retrieved from Chicago Data Portal, `https://data.cityofchicago.org/Health-Human-Services/Daily-Chicago-COVID-19-Cases-Deaths-and-Hospitaliz/kxzd-kd6a`.

[4] S. C. Curtin and M. R. Spencer. Trends in death rates in urban and rural areas: United States, 1999-2019. *National Center for Health Statistics*, NCHS Data Brief, no 417, September 2021. `https://www.cdc.gov/nchs/products/databriefs/db417.htm`.

[5] O. Diekmann, J. Heesterbeek, and J. Metz. On the definition and the computation of the basic reproduction ratio r_0 in models for infectious diseases in heterogeneous populations. *Journal of Mathematical Biology*, 28:365–382, 1990.

[6] Illinois.gov. Gov. Pritzker releases guidelines to safely reopen additional businesses and industries as state advances to next phase of restore Illinois. Press Release, June 2020. `https://www.illinois.gov/news/press-release.21714.html`.

[7] Illinois.gov. Gov. Pritzker aligns Illinois mask guidance with CDC for fully vaccinated people. Press Release, May 2021. https://www.illinois.gov/news/press-release.23322.html.

[8] Illinois.gov. Gov. Pritzker announces metrics-based pathway for Illinois to fully reopen; expands vaccine eligibility to all residents 16+ on April 12. Press Release, March 2021. https://www.illinois.gov/news/press-release.22961.html.

[9] Illinois.gov. Gov. Pritzker issues guidelines for Illinois reopening on June 11. Press Release, June 2021. https://www.illinois.gov/news/press-release.23399.html8.

[10] Illinois.gov. United Center vaccination appointments open Thursday for Illinois seniors. Press Release, March 2021. https://www.illinois.gov/news/press-release.22868.html.

[11] P. van den Driessche and J. Watmough. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences*, 180(1):29–48, 2002.