Reaction-diffusion spatial modeling of COVID-19 in Chicago

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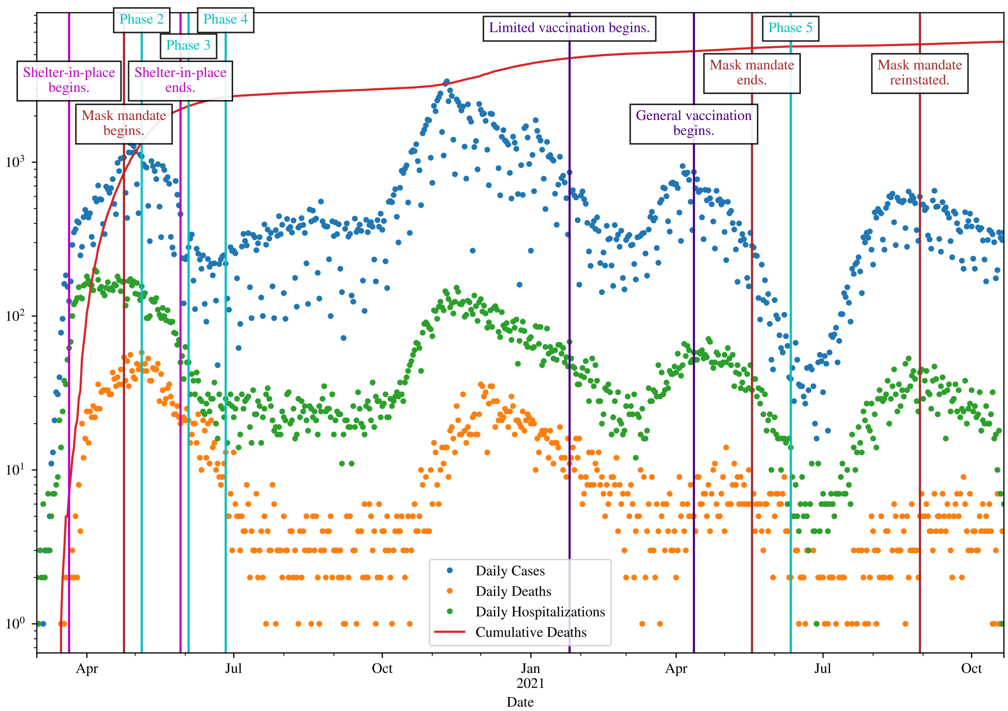

Figure 1: Timeline of the progression of COVID-19 in Chicago with key public policy events marked. The COVID-19 data was obtained from the City of Chicago Data Portal [3]. The dates of the policy events were gathered from the Illinois.gov press releases [8], [6], [7], [9], [5], the Chicago Tribune [2], and NBC Chicago [1]. Note the logarithmic scale.

1 Model Setup

The formulation presented here extends the standard SIR model to incorporate some key aspects of the COVID-19 virus. A schematic of the model is given in Figure 2.

We begin with a population of *susceptibles* S, which may become *exposed* E due to the appearance of the virus in the population. This represents the feature of COVID-19 to remain latent within the host for some time before the host becomes infectious. After the latent period, exposed

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individuals may become *asymptomatically infectious* A or *symptomatically infectious* I. We assume both A and I can interact with S to draw new members into E. A fraction of hosts in I may require hospitalization, generating the *hospitalized* H population. A fraction of the H population is successfully treated, moving to the *recovered* R population. The remaining population in H do not recover, leading the the population of *deceased* D. Meanwhile, asymptomatic hosts are assumed to all recover, moving to the *asymptomatic recovered* AR population.

Importantly, the transfer between population groups is defined by the Law of Mass Action. We consider I/N and A/N to be the probability of selecting an infective or asymptomatic individual, respectively. Then we define the force of infection on S as $\beta_I I/N$ and $\beta_A A/N$. Thus, the rate of new infections are described by $\beta_I IS/N$ and $\beta_A AS/N$.

Table 1: Time sequence of events and simulation times.

Initial simulation time	Imposed lockdown	Effective lockdown	Last fitting day
March 18, 2020	March 21, 2020	April 1, 2020	June 24, 2020
$t_i = 1$	$t_q = 4$	$t_q = 15$	$t_f = 99$

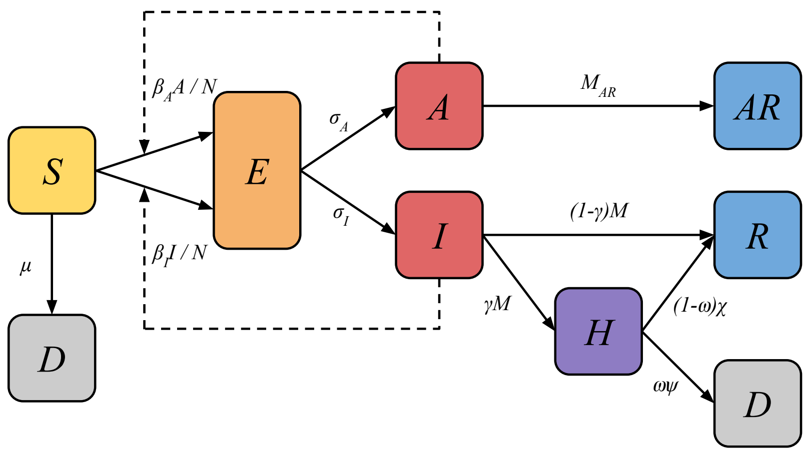


Figure 2: Schematic diagram of the model. The dashed lines indicate the interaction of the infected populations with the susceptible populations that leads to infection.

The full model, including spatial terms, can be described by the following system of equations.

$$
\begin{aligned}
S_t &= \mathfrak{D}_S \Delta S - \frac{\beta_A A}{N} S - \frac{\beta_I I}{N} S - \mu S, \\
E_t &= \mathfrak{D}_E \Delta E + \frac{\beta_A A}{N} S + \frac{\beta_I I}{N} S - (\sigma_A + \sigma_I) E, \\
A_t &= \mathfrak{D}_A \Delta A + \sigma_A E - M_{AR} A, \\
AR_t &= M_{AR} A, \\
I_t &= \sigma_I E - MI, \\
H_t &= \gamma MI - (1-\omega)\chi H - \omega\psi H, \\
R_t &= (1-\gamma) MI + (1-\omega)\chi H, \\
D_t &= \omega\psi H.
\end{aligned}
$$

The $-\mu S$ term models the net change in the population due to non-COVID factors.

2 ODE Model

We begin by reducing the model. We introduce the fractions

$$
\begin{aligned}
s &= \frac{S}{N}, & e &= \frac{E}{N}, & a &= \frac{A}{N}, & ar &= \frac{AR}{N}, \\
i &= \frac{I}{N}, & h &= \frac{H}{N}, & r &= \frac{R}{N}, & d &= \frac{D}{N}.
\end{aligned}
$$

Next, we ignore non-COVID deaths. That is, set $\mu = 0$. This gives the conservation law

$$
s + e + ar + a + i + h + r + d = 1. \tag{1}
$$

At this point it is worth noting the dimensions of the components of the system. Since $[X] =$ population where $X \in \{N, S, E, AR, A, I, H, R, D\}$, then $[x] = 1$ where $x \in \{s, e, ar, a, i, h, r, d\}$. Then,

$$
[\beta_A] = [\beta_I] = [\sigma_A] = [\sigma_I] = [M_{AR}] = [M] = [\chi] = [\psi] = \frac{1}{T}
$$

and $[\omega] = [\gamma] = 1$. We introduce the non-dimensional time variable $\tau = tM$. Then by the chain rule, $x_t = Mx_\tau$. Next, we define the following non-dimensional parameters

$$
\begin{aligned}
\alpha_A &= \frac{\beta_A}{M}, & \alpha_I &= \frac{\beta_I}{M}, & \lambda_A &= \frac{\sigma_A}{M}, & \lambda_I &= \frac{\sigma_I}{M}, \\
\eta &= \frac{M_{AR}}{M}, & \theta &= \frac{\chi}{M}, & \kappa &= \frac{\psi}{M}.
\end{aligned}
$$

Using these non-dimensional parameters and the conservation law in Equation 1, we can replace the ODE system with

$$
\begin{aligned}
s_\tau &= -\alpha_A a s - \alpha_I i s, && (2) \\
e_\tau &= \alpha_A a s + \alpha_I i s - (\lambda_A + \lambda_I) e, && (3) \\
a_\tau &= \lambda_A e - \xi a, && (4) \\
ar_\tau &= \xi a, && (5) \\
i_\tau &= \lambda_I e - i, && (6) \\
h_\tau &= \gamma i - (1-\omega)\theta h - \omega \kappa h, && (7) \\
r_\tau &= (1-\gamma) i + (1-\omega)\theta h. && (8)
\end{aligned}
$$

Then by the conservation law, we have

$$
d = n - (s + e + a + ar + i + r) \tag{9}
$$

so we need not solve the differential equation for d.

To create the PDE model, we first must find suitable parameters using the ODE model as in previous studies. We utilize the MATLAB nonlinear optimization algorithm `fminsearch` for this purpose. The optimal parameters are determined by minimizing the Euclidean distance $\mathcal{N}$ between the time series generated by the model (subscript "num") and the corresponding observed data time series (subscript "obs")

$$
\mathcal{N} = \sum_i \left(|\ln C_{\text{num}}(t_i) - \ln C_{\text{obs}}(t_i)| + |\ln D_{\text{num}}(t_i) - \ln D_{\text{obs}}(t_i)| \right) \tag{10}
$$

where the index i identifies a point in the time series. The parameters are chosen to reproduce the time series of the total number of cases $C(t) = I(t) + H(t) + R(t) + D(t)$, and the total number of deceased $D(t)$.

Table 2: Population values for Chicago. Initial populations are determined from March 13, 2020.

		Population
Total population	N	2,695,598
Initial infected	I_0	121
Initial hospitalized	H_0	38
Initial deceased	D_0	3

We seed the ODE model using data from March 13, 2020 as collected on the City of Chicago Data Portal [3]. The data collected is in Cases, Hospitalized, and Deceased. In order to get the initial infected population I_0, we take $I_0 = \text{Cases} - H_0 - D_0$. The relevant data is shown in Table 2.

To account for changes in virus transmission due to the shelter-in-place order, we impose a time dependence on the transmission rates β as in Equations (11) and (12).

$$\beta_I(t) = \beta_I \left(\eta_I + (1-\eta_I)\frac{1-\tanh[2(t-t_q)]}{2} \right) \tag{11}$$

$$\beta_A(t) = \beta_A \left(\eta_A + (1-\eta_A)\frac{1-\tanh[2(t-t_q)]}{2} \right) \tag{12}$$

Table 3: ODE parameters: optimal (best-fitting), median and interquartile range, and variation range used in the optimization algorithm. Initial parameter guesses were uniformly sampled within these ranges.

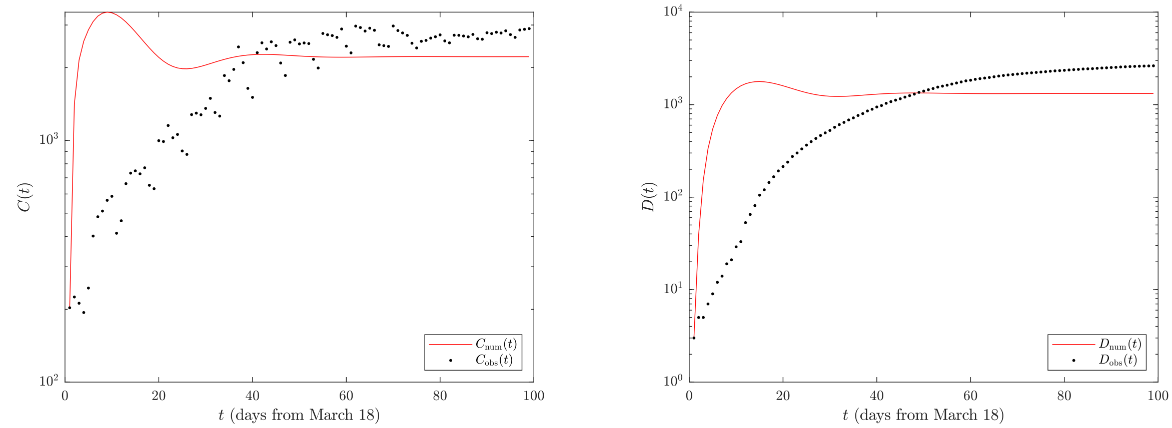
		Median (IQR)	Initial value
Transmission rate, $S \to I$ [per day]	β_I		$c \in \mathcal{U}[0,1]$
Transmission rate, $S \to A$ [per day]	β_A		$c \in \mathcal{U}[0,1]$
Lockdown effect, $S \to I$	η_I		$c \in \mathcal{U}[0,1]$
Lockdown effect, $S \to A$	η_A		$c \in \mathcal{U}[0,1]$
Incubation period, $E \to I$ [days]	$1/\sigma_I$		$1/k, k \in \mathcal{U}[2,7]$
Latent period, $E \to A$ [days]	$1/\sigma_A$		$1/k, k \in \mathcal{U}[2,7]$
Infectivity period [days]	$1/M$		$1/k, k \in \mathcal{U}[5,12]$
Recovery period, $A \to AR$ [days]	$1/M_{AR}$		$1/k, k \in \mathcal{U}[5,12]$
Recovery period, $H \to R$ [days]	$1/\chi$		$1/k, k \in \mathcal{U}[5,20]$
Period to deceased, $H \to D$ [days]	$1/\psi$		$1/k, k \in \mathcal{U}[5,20]$
Conversion fraction ($I \xrightarrow{\gamma} H,\ I \xrightarrow{1-\gamma} R$)	γ		$c \in \mathcal{U}[0.25, 0.75]$
Conversion fraction ($H \xrightarrow{\omega} D,\ H \xrightarrow{1-\omega} R$)	ω		$c \in \mathcal{U}[0.1, 0.5]$
Initial population fraction, exposed	E_0/I_0		$c \in \mathcal{U}[1,5]$
Initial population fraction, asymptomatic	A_0/I_0		$c \in \mathcal{U}[1,5]$

3 PDE Model

4 ODE Dynamics

We want to understand the trajectories of the dynamics of the ODE system (2)-(9) under different initial conditions. To do this we first find the equilibrium points by solving

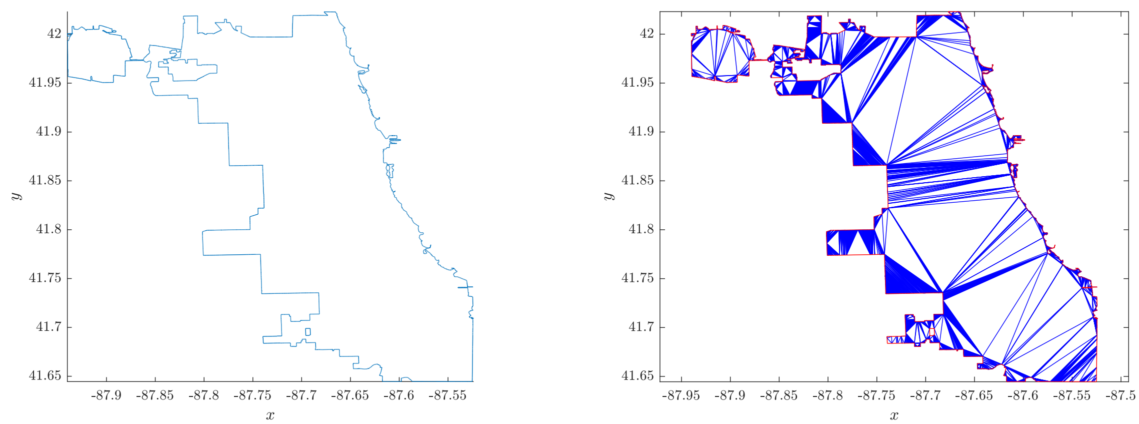
$$s_\tau = e_\tau = ar_\tau = a_\tau = i_\tau = h_\tau = r_\tau = 0$$



(a) Confirmed cases $C(t) = I(t) + R(t) + H(t) + D(t)$.

(b) Number of deaths $D(t)$.

Figure 3: ODE model with fitting to official data from March 18, 2020 ($t_i = 1$) to June 24, 2020 ($t_f = 99$). Here we show the case when β changes exactly on the imposed lockdown on March 21, 2020 ($t_q = 4$).



(a) Computational domain for the PDE model.

(b) Preliminary mesh of the domain.

Figure 4: Computational domain and initial meshing for the PDE model.

simultaneously for $\mathbf{x} = (s, e, a, ar, i, h, r)^\intercal$. The solutions of this system are of the form $\mathbf{x}^* = (0, 0, 0, AR, 0, 0, r)^\intercal$. This implies there are infinitely many non-isolated equilibrium points. We determine the stability of these equilibrium points by analyzing the linearized system near the points. The Jacobian of the system is

$$\mathsf{J} = \begin{pmatrix} -a\alpha_a - i\alpha_i & 0 & -s\alpha_a & 0 & -s\alpha_i & 0 & 0 \\ a\alpha_a + i\alpha_i & -\lambda_a - \lambda_i & s\alpha_a & 0 & s\alpha_i & 0 & 0 \\ 0 & \lambda_a & -\xi & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi & 0 & 0 & 0 & 0 \\ 0 & \lambda_i & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & -\theta(1-\omega) - \kappa\omega & 0 \\ 0 & 0 & 0 & 0 & 1-\gamma & \theta(1-\omega) & 0 \end{pmatrix}. \tag{13}$$

Now evaluating J at the equilibrium point $\mathbf{x}^*$ and calculating the eigenvalues, we have

$$\lambda = \{-1, 0, 0, 0, -\lambda_a - \lambda_i, -\xi, -\theta + \theta\omega - \kappa\omega\}. \tag{14}$$

Note that three of the eigenvalues are 0, which implies the equilibrium points are non-isolated. This agrees with our earlier observation.

The equilibrium points are stable when $\lambda_i < 0$ for $5 \leq i \leq 7$. Since all the system parameters are positive, this implies $\lambda_i < 0$ for $i = 5$ and $i = 6$. Thus the stability depends on the sign of λ_7. There are two cases when $\lambda_7 = -\theta + \theta\omega - \kappa\omega < 0$ is true:

1. $0 < \omega \leq 1$ implies $\lambda_8 < 0$, and
2. $\omega > 1$ and $\theta < \frac{\kappa\omega}{\omega - 1}$ implies $\lambda_8 < 0$.

That is, whenever we have either of these conditions the equilibrium points are stable. If $\lambda_8 > 0$, the equilibrium points are unstable.

We can further analyze the evolution of the pandemic by calculating the basic reproduction number R_0. We use the next generation matrix approach of the system (2)-(9) without the spatial terms, as in [4] and [10]. In particular, we rewrite the model in the form $\mathbf{x}_t = \mathbf{F} - \mathbf{V}$. The components F_i represents the rate of appearance of new infections in compartment i. The vector $\mathbf{V} = \mathbf{V}^- - \mathbf{V}^+$, where V_i^+ represents the rate of transfer of individuals into compartment i by all other means, and V_i^- represents the rate of transfer of individuals out of compartment i. Reordering the compartments so $\mathbf{x_t'} = (E_t, A_t, I_t, H_t, S_t, AR_t, R_t, D_t)^\intercal$, we have

$$\mathbf{F} = \begin{pmatrix} \beta_{SA}SA + \beta_{SI}SI \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} (\sigma_A + \sigma_I)E \\ -\sigma_A E + M_{AR}A \\ -\sigma_I E + MI \\ -\gamma MI + (1-\omega)\chi H + \omega\psi H \\ \beta_{SA}SA + \beta_{SI}SI + \mu S \\ M_{AR}A \\ -(1-\gamma)MI - (1-\omega)\chi H \\ -\omega\psi H \end{pmatrix}.$$

We focus on just the infectious/infected compartments, E, A, I, H, and find the Jacobians of $\mathbf{F}$ and $\mathbf{V}$ with respect to these populations in the order in which they appear. Evaluating at the the disease-free equilibrium ($S = S^*,\ E = AR = A = I = H = R = D = 0$) yields

$$\mathsf{F} = \begin{pmatrix} 0 & \beta_{SA}S^* & \beta_{SI}S^* & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{V} = \begin{pmatrix} \sigma_A + \sigma_I & 0 & 0 & 0 \\ -\sigma_A & M_{AR} & 0 & 0 \\ -\sigma_I & 0 & M & 0 \\ 0 & 0 & -\gamma M & \chi(1-\omega) + \psi\omega \end{pmatrix}.$$

The matrix $\mathsf{F}\mathsf{V}^{-1}$ is the next-generation matrix. Then $R_0 = \rho(\mathsf{F}\mathsf{V}^{-1})$, which is

$$R_0 = \frac{\beta_{SA}S^*\sigma_A}{m_{AR}(\sigma_A + \sigma_I)} + \frac{\beta_{SI}S^*\sigma_I}{M(\sigma_A + \sigma_I)}. \tag{15}$$

References

[1] Read the full ‘restore Illinois’ plan aimed at reopening the state during coronavirus. *NBC Chicago*, May 2020. `https://www.nbcchicago.com/news/coronavirus/read-the-full-restore-illinois-plan-aimed-at-reopening-the-state-during-coronavirus/2267039/`.

[2] J. Byrne, D. Petrella, and A. Lukach. Mayor Lori Lightfoot says Chicago will move to phase 3 of her reopening plan on June 3 but warns: ‘COVID-19 is still very much part of our present’. *Chicago Tribune*, May 2020. `https://www.chicagotribune.com/coronavirus/ct-coronavirus-chicago-lightfoot-reopening-20200528-cefwiuidwnfd7a57m25uavq6me-story.html`.

[3] City of Chicago. Daily chicago covid-19 cases, deaths, and hospitalizations, 2021. Data retrieved from Chicago Data Portal, https://data.cityofchicago.org/Health-Human-Services/Daily-Chicago-COVID-19-Cases-Deaths-and-Hospitaliz/kxzd-kd6a.

[4] O. Diekmann, J. Heesterbeek, and J. Metz. On the definition and the computation of the basic reproduction ratio r_0 in models for infectious diseases in heterogeneous populations. *Journal of Mathematical Biology*, 28:365–382, 1990.

[5] Illinois.gov. Gov. Pritzker releases guidelines to safely reopen additional businesses and industries as state advances to next phase of restore Illinois. Press Release, June 2020. https://www.illinois.gov/news/press-release.21714.html.

[6] Illinois.gov. Gov. Pritzker aligns Illinois mask guidance with CDC for fully vaccinated people. Press Release, May 2021. https://www.illinois.gov/news/press-release.23322.html.

[7] Illinois.gov. Gov. Pritzker announces metrics-based pathway for Illinois to fully reopen; expands vaccine eligibility to all residents 16+ on April 12. Press Release, March 2021. https://www.illinois.gov/news/press-release.22961.html.

[8] Illinois.gov. Gov. Pritzker issues guidelines for Illinois reopening on June 11. Press Release, June 2021. https://www.illinois.gov/news/press-release.23399.html8.

[9] Illinois.gov. United Center vaccination appointments open Thursday for Illinois seniors. Press Release, March 2021. https://www.illinois.gov/news/press-release.22868.html.

[10] P. van den Driessche and J. Watmough. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences*, 180(1):29–48, 2002.