AN ANALOGY OF APPROXIMATION METHODS

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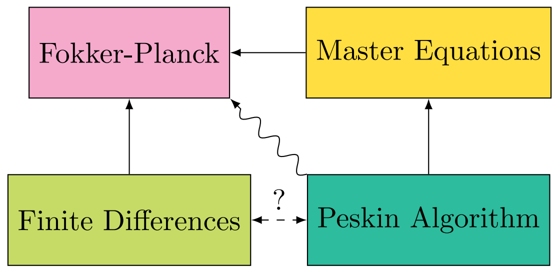

Figure 1. Approximation methods and their relationships to the Fokker-Planck equation.

The probability density $u(x,t)$ evolves according to the Fokker-Planck equation

$$\partial_t u(x,t) = \partial_x \left(u(x,t)\phi'(x) + \partial_x u(x,t)\right) \tag{1}$$

where $\phi(x)$ is the potential energy. In one spatial dimension, (1) is a simple convection-diffusion equation.

First we will write an explicit finite difference scheme for (1). We assume we work on the closed domain $[0,1]\times[0,t_F]$. Note that we must use a finite time interval, but t_F can be made as large as desired. We denote the spacings by h and s for space and time, respectively. Then the mesh points are

$$(x_j = jh, t_n = ns),\ j = 0,1,2\dots,J,\ n = 0,1,2,\dots,$$

where $h = 1/J$. We use the notation

$$U_j^n \approx u(x_j,t_n).$$

to mean the approximations of the solution at the mesh points.

The left side of (1) is approximated using a forward difference for the time derivative:

$$\frac{U_j^{n+1} - U_j^n}{s} \approx \frac{\partial u(x_j,t_n)}{\partial t}. \tag{2}$$

Next will we derive the approximation for the right side of (1). We first approximate $\phi'(x)$ as a constant at any space step j with the forward difference

$$\frac{\phi(x_{j+1}) - \phi(x_j)}{h} \approx \phi'(x_j). \tag{3}$$

Therefore, the RHS can be written as

$$\text{RHS} = u_x\phi' + u_{xx}.$$

Now using a forward difference and a centered second difference to approximate u_x and u_{xx} respectively, we have

$$\left(\frac{\phi_{j+1}-\phi_j}{h}\right)\left(\frac{U_{j+1}^n - U_j^n}{h}\right) + \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} \approx \partial_x\left(u(x_j,t_n)\phi'(x_j) + \partial_x u(x_j,t_n)\right). \tag{4}$$

Date: April 21, 2022.

Thus, the explicit scheme is

$$U_j^{n+1} = U_j^n + \frac{s}{h^2}\left(U_{j+1}^n(1+\Delta\phi_j) - U_j^n(2+\Delta\phi_j) + U_{j+1}^n\right) \tag{5}$$

where $\Delta\phi_j = \phi(x_{j+1}) - \phi(x_j)$.