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Load Balancing Unstructured Meshes for Massively Parallel Transport Sweeps

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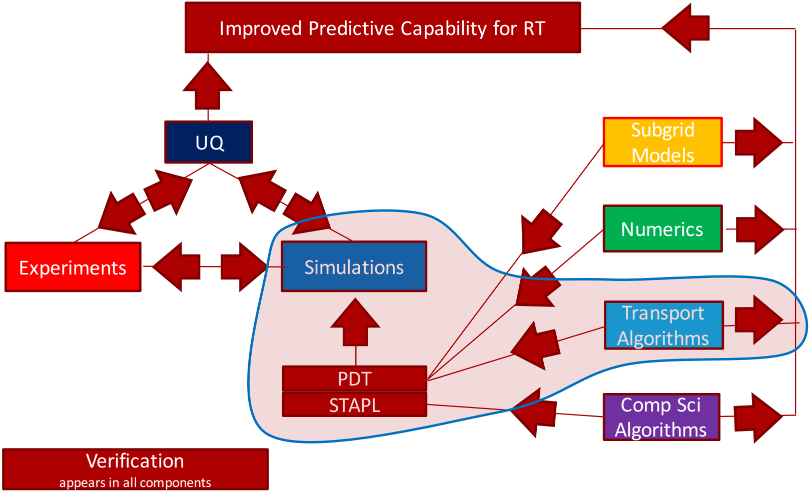
Project Components and Integration

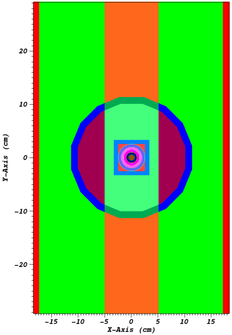


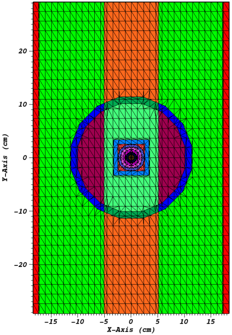
Motivation

- When running any massively parallel code, load balancing is a priority in order to achieve the best possible parallel efficiency.
- A load balanced problem has an equal number of degrees of freedom per processor.
- Load balancing a logically Cartesian mesh is “not difficult”, as the user specifies the number of cells being used.
- In an unstructured mesh, the user cannot always specify the number of cells they want per processor, and obtaining a load balanced problem is more difficult.
- The goal is to implement a load balancing algorithm for unstructured meshes in PDT.

The Triangle Mesh Generator

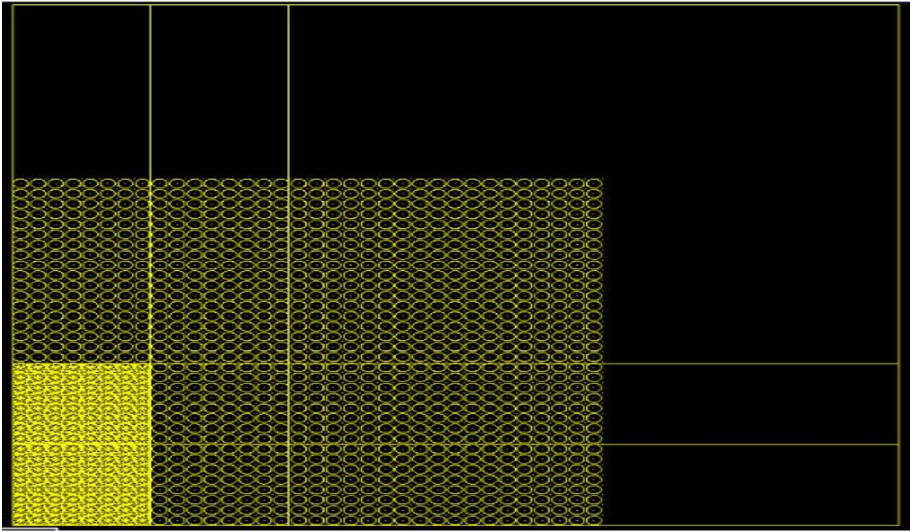
- Unstructured meshes in PDT are generated in 2D using the Triangle Mesh Generator.
- These can be extruded to create 3D meshes.





- The user inputs coordinates for cut lines in the X and Y directions.
- The cut lines will determine the number of "subsets" the problem is partitioned into.
- Optimizing the location of these cut lines is the basis of the load balancing algorithm.
- A "subset" is an orthogonal unit that is formed by intersecting cut lines.

The Subset



Goal and Definitions

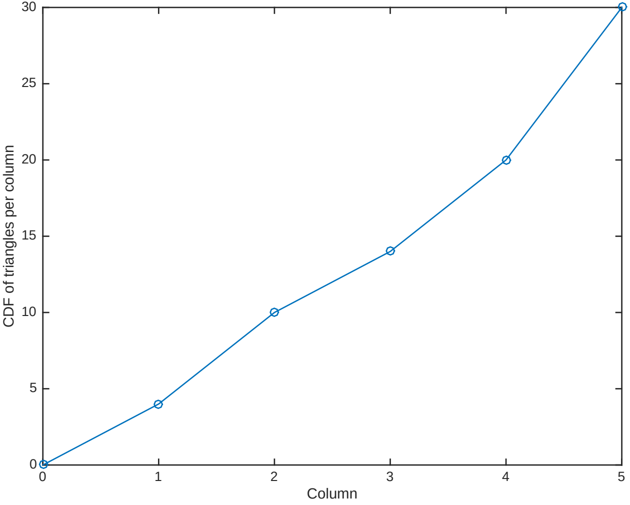
- **Goal:** Obtain an equal number of cells per processor, which for our purposes means an equal number of cells per subset.
- Achieved by optimizing the location of X_i and Y_j, the location of the cut lines.
- **Define**:
  - N_{ij}: The number of cells in subset i, j
  - $f = \frac{\max_{ij}(N_{ij})}{\frac{N_{tot}}{I \cdot J}}$
  - $f_I = \max_i [\sum_j N_{ij}] / \frac{N_{tot}}{I}$
  - $f_J = \max_j [\sum_i N_{ij}] / \frac{N_{tot}}{J}$

Load Balancing Algorithm

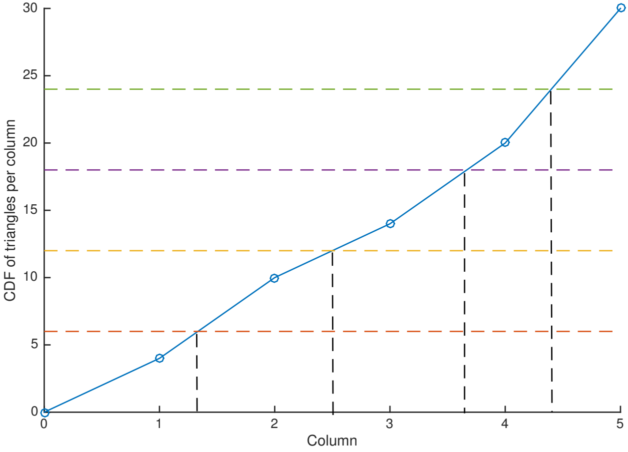

```
//I,J subsets specified by user
//Check if all subsets meet the tolerance
while (f > tol_subset)
{
  //Mesh all subsets
  if (f_I > tol_column)
  {
    Redistribute(X);
  }
  if (f_J > tol_row)
  {
    Redistribute(Y);
  }

}
```

Redistribution Function

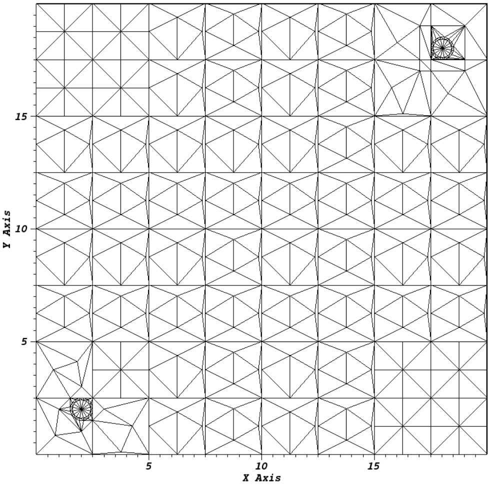


Redistribution Function



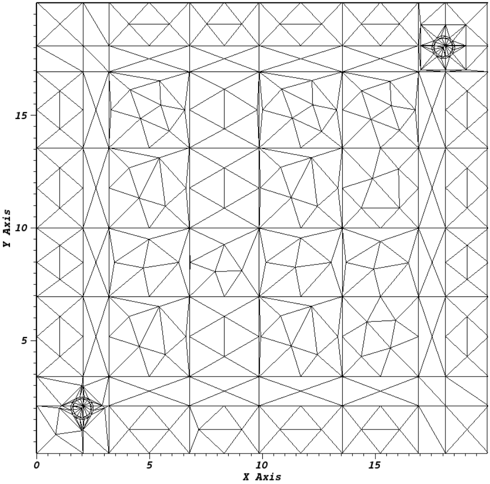
Iter 0

$f = 7.20583$



Iter 1

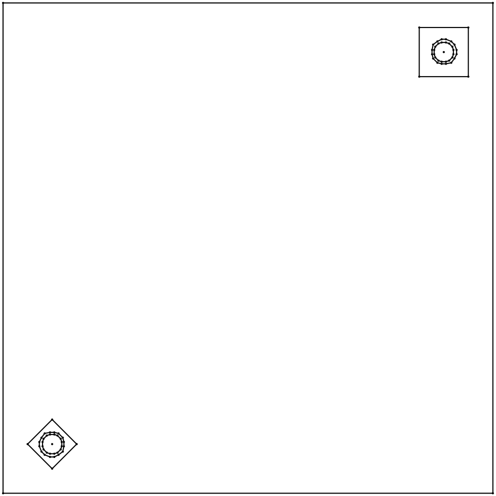
$f = 3.61695$



Load Balancing Results

- Three test cases were used to study the behavior of the load balancing algorithm.
- For each test case, 162 inputs were constructed by varying:
  - The number of subsets
  - The spatial resolution of the mesh (maximum triangle area).

Test Case 1



1: The metric behavior of the first test case run with **no load balancing** iterations.

Area	N=4	N=9	N=16	N=25	N=36	N=49	N=64	N=81	N=100
Coarse	1.95	4.12	6.76	9.60	12.44	14.21	16.44	8.60	6.77
1.8	1.46	2.32	4.11	4.64	7.84	8.61	**24.77**	6.14	4.58
1.6	1.42	2.21	4.20	4.64	6.86	8.52	24.71	5.94	4.58
1.4	1.32	2.05	2.98	4.64	6.23	8.58	19.98	5.90	4.51
1.2	1.30	1.95	3.02	4.93	4.51	7.25	19.97	4.30	4.51
1	1.35	1.75	2.90	4.93	4.52	6.02	20.01	4.62	4.51
0.8	1.26	1.65	2.95	3.31	4.45	4.40	19.74	4.58	2.92
0.6	1.14	1.45	2.05	3.01	3.55	4.22	14.28	2.87	3.10
0.4	1.09	1.35	1.79	2.02	2.74	3.33	14.09	2.80	2.06
0.2	1.05	1.14	1.34	1.55	1.65	2.05	8.78	1.82	1.45
0.1	1.02	1.04	1.11	1.17	1.29	1.36	4.43	1.41	1.24
0.08	1.01	1.03	1.09	1.19	1.21	1.29	3.39	1.32	1.18
0.06	1.01	1.03	1.04	1.10	1.09	1.20	2.93	1.28	1.06
0.05	1.02	1.02	1.06	1.09	1.08	1.11	2.61	1.22	1.09
0.04	1.00	1.01	1.00	1.06	1.07	1.07	2.20	1.17	1.11
0.03	1.00	1.02	1.02	1.05	1.07	1.05	1.93	1.13	1.03
0.02	1.00	1.01	1.01	1.03	1.02	1.03	1.57	1.08	1.05
0.01	**1.00**	1.01	1.01	1.01	1.04	1.02	1.28	1.04	1.01

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2: The metric behavior of the first test case after **10 load balancing iterations**.

Area	N=4	N=9	N=16	N=25	N=36	N=49	N=64	N=81	N=100
Coarse	1.95	1.60	3.37	2.10	2.28	2.68	2.53	2.81	3.05
1.8	1.46	1.94	2.81	2.59	2.98	2.89	2.97	4.50	4.33
1.6	1.42	1.95	2.43	2.42	3.00	3.05	2.71	4.11	4.09
1.4	1.32	1.87	2.65	3.13	2.45	3.03	4.14	4.39	4.15
1.2	1.30	1.77	2.46	2.66	2.59	3.18	4.02	4.28	**5.05**
1	1.35	1.64	2.26	2.33	2.35	3.01	3.93	3.67	4.34
0.8	1.26	1.51	2.02	2.79	2.02	2.61	3.27	3.37	3.63
0.6	1.14	1.45	1.79	2.41	2.81	2.09	2.90	2.87	3.63
0.4	1.09	1.35	1.45	1.87	2.40	1.84	1.96	2.35	2.26
0.2	1.05	1.14	1.34	1.55	1.65	2.05	1.40	1.79	1.71
0.1	1.02	1.04	1.11	1.17	1.29	1.36	1.32	1.41	1.22
0.08	1.01	1.03	1.09	1.19	1.21	1.29	1.20	1.32	1.38
0.06	1.01	1.03	1.04	1.10	1.09	1.20	1.15	1.28	1.07
0.05	1.02	1.02	1.06	1.09	1.08	1.11	1.14	1.22	1.18
0.04	1.00	1.01	1.00	1.06	1.07	1.07	1.16	1.17	1.17
0.03	1.00	1.02	1.02	1.05	1.07	1.05	1.93	1.13	1.04
0.02	1.00	1.01	1.01	1.03	1.02	1.03	1.57	1.08	1.09
0.01	**1.00**	1.01	1.01	1.01	1.04	1.02	1.28	1.04	1.02

3: The ratio of the metric with no iteration and 10 iterations. The closer the z-value to zero, the better the improvement.

Area	N=4	N=9	N=16	N=25	N=36	N=49	N=64	N=81	N=100
Coarse	1.00	0.39	0.50	0.22	0.18	0.19	0.15	0.33	0.45
1.8	1.00	0.83	0.68	0.56	0.38	0.34	0.12	0.73	0.95
1.6	1.00	0.88	0.58	0.52	0.44	0.36	**0.11**	0.69	0.89
1.4	1.00	0.91	0.89	0.67	0.39	0.35	0.21	0.74	0.92
1.2	1.00	0.90	0.81	0.54	0.58	0.44	0.20	1.00	1.12
1	1.00	0.93	0.78	0.47	0.52	0.50	0.20	0.79	0.96
0.8	1.00	0.92	0.68	0.84	0.45	0.59	0.17	0.74	**1.24**
0.6	1.00	1.00	0.87	0.80	0.79	0.50	0.20	1.00	1.17
0.4	1.00	1.00	0.81	0.93	0.88	0.55	0.14	0.84	1.10
0.2	1.00	1.00	1.00	1.00	1.00	1.00	0.16	0.99	1.19
0.1	1.00	1.00	1.00	1.00	1.00	1.00	0.30	1.00	0.98
0.08	1.00	1.00	1.00	1.00	1.00	1.00	0.35	1.00	1.17
0.06	1.00	1.00	1.00	1.00	1.00	1.00	0.39	1.00	1.00
0.05	1.00	1.00	1.00	1.00	1.00	1.00	0.44	1.00	1.08
0.04	1.00	1.00	1.00	1.00	1.00	1.00	0.52	1.00	1.05
0.03	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.01
0.02	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.04
0.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.01

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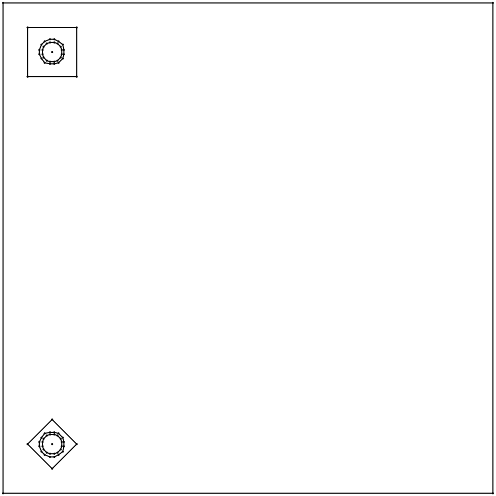

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4: The metric behavior of the second test case after **no load balancing** iterations.

Area	N=4	N=9	N=16	N=25	N=36	N=49	N=64	N=81	N=100
Coarse	1.95	4.12	6.76	9.60	12.44	14.21	16.44	8.60	6.77
1.80	1.45	2.31	4.10	4.91	7.90	8.61	**22.67**	6.37	6.19
1.60	1.42	2.24	4.19	4.91	6.94	8.50	20.91	6.29	6.19
1.40	1.31	2.12	2.97	4.41	6.22	8.58	19.84	6.25	5.99
1.20	1.30	1.96	3.02	4.65	4.53	7.09	19.83	4.30	6.23
1.00	1.34	1.78	2.90	4.35	4.49	5.88	19.85	4.62	4.98
0.80	1.26	1.64	2.95	3.09	4.47	4.45	17.42	4.58	4.18
0.60	1.14	1.42	2.05	2.72	3.50	4.09	12.90	2.80	4.18
0.40	1.09	1.34	1.79	2.08	2.73	3.34	11.39	2.83	2.68
0.20	1.06	1.15	1.34	1.56	1.72	2.03	7.02	1.85	1.72
0.10	1.02	1.04	1.15	1.22	1.29	1.37	4.12	1.36	1.37
0.08	1.01	1.04	1.08	1.15	1.20	1.30	3.47	1.33	1.26
0.06	1.01	1.03	1.04	1.10	1.08	1.20	2.79	1.26	1.19
0.05	1.02	1.03	1.05	1.07	1.06	1.12	2.57	1.23	1.16
0.04	1.00	1.03	1.01	1.06	1.08	1.07	2.22	1.18	1.11
0.03	1.01	1.02	1.01	1.04	1.07	1.05	1.86	1.11	1.08
0.02	1.01	1.02	1.01	1.04	1.04	1.03	1.57	1.09	1.07
0.01	**1.00**	1.01	1.02	1.02	1.02	1.02	1.29	1.04	1.02

Test Case 2

5: The metric behavior of the second test case after **10 load balancing iterations**.

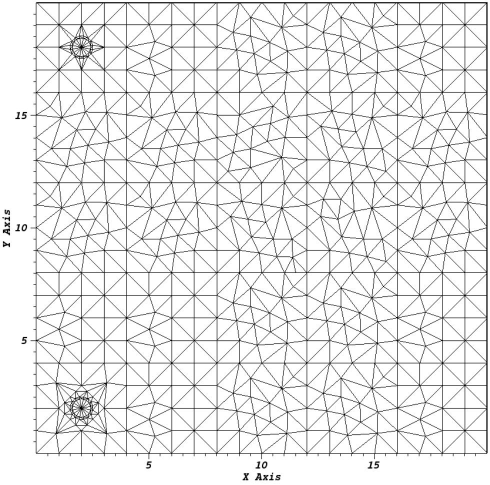
Area	N=4	N=9	N=16	N=25	N=36	N=49	N=64	N=81	N=100
Coarse	1.85	1.36	1.76	1.48	1.74	1.60	1.79	1.82	1.92
1.8	1.15	1.33	1.65	2.08	2.58	2.41	2.69	3.83	**3.99**
1.6	1.12	1.34	1.65	2.35	2.67	2.47	2.96	2.59	2.97
1.4	1.12	1.37	1.79	1.86	1.83	2.71	2.82	2.58	3.74
1.2	1.15	1.50	1.54	1.56	1.71	2.13	2.81	2.79	2.87
1	1.15	1.45	1.73	1.74	1.74	2.39	2.48	2.81	3.07
0.8	1.14	1.40	1.47	1.44	1.58	2.26	2.38	2.60	3.39
0.6	1.05	1.31	1.49	1.85	1.57	1.81	1.81	2.42	2.36
0.4	1.09	1.19	1.37	1.77	1.71	1.87	1.57	1.72	2.26
0.2	1.06	1.15	1.18	1.35	1.63	1.67	1.73	1.52	1.72
0.1	1.02	1.04	1.15	1.22	1.29	1.34	1.25	1.26	1.37
0.08	1.01	1.04	1.08	1.15	1.20	1.30	1.22	1.21	1.26
0.06	1.01	1.03	1.04	1.10	1.08	1.20	1.18	1.26	1.19
0.05	1.02	1.03	1.05	1.07	1.06	1.12	1.15	1.23	1.16
0.04	1.00	1.03	1.01	1.06	1.08	1.07	1.13	1.18	1.11
0.03	1.01	1.02	1.01	1.04	1.07	1.05	1.32	1.11	1.08
0.02	1.01	1.02	1.01	1.04	1.04	1.03	1.15	1.09	1.07
0.01	**1.00**	1.01	1.02	1.02	1.02	1.02	1.29	1.04	1.02

6: The ratio of the metric with no iteration and 10 iterations. The closer the z-value to zero, the better the improvement.

Area	N=4	N=9	N=16	N=25	N=36	N=49	N=64	N=81	N=100
Coarse	0.95	0.33	0.26	0.15	0.14	0.11	**0.11**	0.21	0.28
1.8	0.79	0.57	0.40	0.42	0.33	0.28	0.12	0.60	0.65
1.6	0.79	0.60	0.39	0.48	0.38	0.29	0.14	0.41	0.48
1.4	0.85	0.64	0.60	0.42	0.29	0.32	0.14	0.41	0.62
1.2	0.89	0.77	0.51	0.34	0.38	0.30	0.14	0.65	0.46
1	0.85	0.81	0.60	0.40	0.39	0.41	0.12	0.61	0.62
0.8	0.91	0.85	0.50	0.47	0.35	0.51	0.14	0.57	0.81
0.6	0.92	0.92	0.73	0.68	0.45	0.44	0.14	0.86	0.57
0.4	1.00	0.89	0.76	0.85	0.63	0.56	0.14	0.61	0.84
0.2	1.00	1.00	0.89	0.86	0.95	0.82	0.25	0.82	1.00
0.1	1.00	1.00	1.00	1.00	1.00	0.98	0.30	0.92	1.00
0.08	1.00	1.00	1.00	1.00	1.00	1.00	0.35	0.91	1.00
0.06	1.00	1.00	1.00	1.00	1.00	1.00	0.42	1.00	1.00
0.05	1.00	1.00	1.00	1.00	1.00	1.00	0.45	1.00	1.00
0.04	1.00	1.00	1.00	1.00	1.00	1.00	0.51	1.00	1.00
0.03	1.00	1.00	1.00	1.00	1.00	1.00	0.71	1.00	1.00
0.02	1.00	1.00	1.00	1.00	1.00	1.00	0.74	1.00	1.00
0.01	**1.00**	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Iter 0

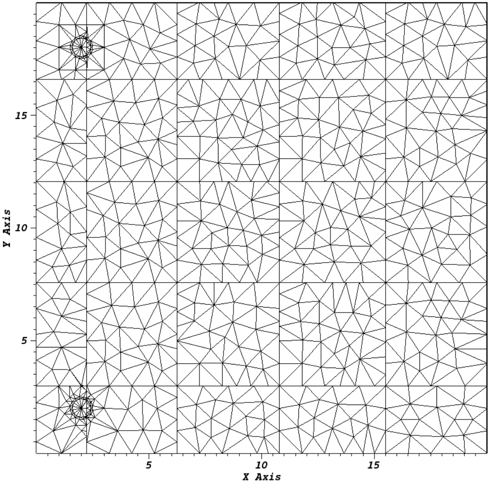
$f = 2.72$



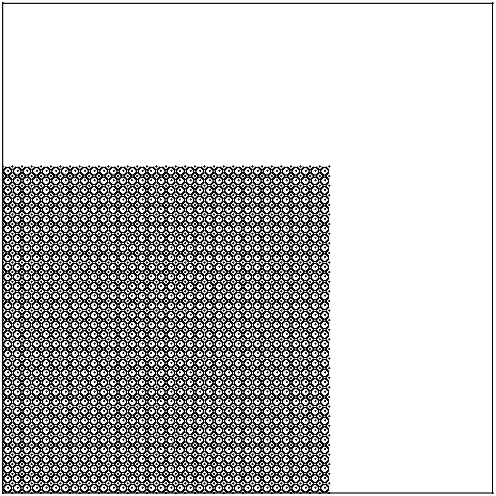
A Closer Look at Test Case 2

Iter 10

f =1.85



Test Case 3



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7: The metric behavior of the third test case after **no load balancing iterations**.

Area	N=4	N=9	N=16	N=25	N=36	N=49	N=64	N=81	N=100
Coarse	2.24	2.24	2.28	2.27	2.24	2.29	2.32	2.26	2.29
1.8	2.13	2.13	2.16	2.42	2.13	2.43	2.23	2.17	**2.65**
1.6	2.11	2.12	2.15	2.40	2.11	2.42	2.22	2.16	2.63
1.4	2.09	2.10	2.13	2.38	2.10	2.39	2.20	2.12	2.61
1.2	2.07	2.07	2.11	2.35	2.08	2.37	2.18	2.11	2.59
1	2.04	2.04	2.07	2.32	2.04	2.33	2.15	2.08	2.54
0.8	1.99	1.99	2.02	2.27	1.99	2.28	2.10	2.03	2.50
0.6	1.91	1.92	1.95	2.18	1.92	2.20	2.03	1.96	2.41
0.4	1.78	1.79	1.82	2.04	1.79	2.06	1.90	1.83	2.27
0.2	1.47	1.48	1.51	1.70	1.49	1.72	1.59	1.52	1.91
0.1	1.09	1.10	1.12	1.28	1.11	1.29	1.21	1.16	1.45
0.08	1.03	1.02	1.03	1.13	1.02	1.15	1.07	1.03	1.31
0.06	1.03	1.04	1.04	1.15	1.04	1.18	1.09	1.08	1.28
0.05	1.02	1.02	1.03	1.11	1.03	1.13	1.09	1.06	1.20
0.04	1.06	1.06	1.06	1.12	1.08	1.12	1.09	1.10	1.20
0.03	1.08	1.08	1.09	1.12	1.10	1.11	1.10	1.11	1.15
0.02	**1.02**	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.06
0.01	1.03	1.03	1.03	1.04	1.03	1.04	1.04	1.03	1.05

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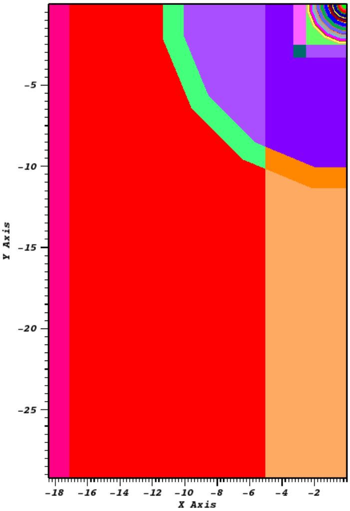
8: The metric behavior of the third test case after **10 load balancing iterations**.

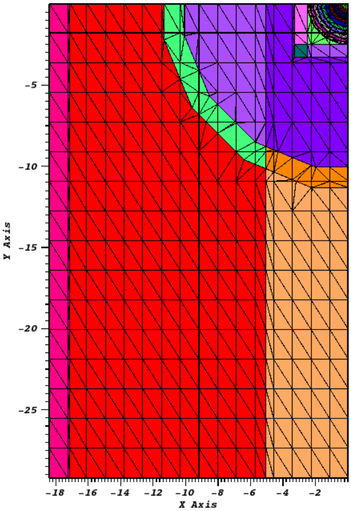
Area	N=4	N=9	N=16	N=25	N=36	N=49	N=64	N=81	N=100
Coarse	**1.00**	1.01	1.04	1.05	1.01	1.06	1.06	1.06	1.08
1.8	1.02	1.03	1.15	1.21	1.20	1.23	1.36	1.42	1.54
1.6	1.03	1.04	1.08	1.20	1.18	1.23	1.54	1.69	1.58
1.4	1.02	1.06	1.09	1.25	1.32	1.39	1.37	1.52	1.62
1.2	1.03	1.06	1.24	1.24	1.30	1.32	1.48	1.56	1.84
1	1.02	1.05	1.15	1.25	1.31	1.35	1.49	1.80	2.15
0.8	1.04	1.06	1.10	1.23	1.27	1.53	1.79	1.84	1.95
0.6	1.03	1.11	1.13	1.38	1.51	1.61	1.79	1.96	2.17
0.4	1.04	1.19	1.26	1.39	1.66	1.47	1.90	1.83	**2.27**
0.2	1.06	1.17	1.16	1.33	1.49	1.62	1.59	1.52	1.78
0.1	1.09	1.10	1.12	1.14	1.11	1.19	1.21	1.16	1.19
0.08	1.03	1.02	1.03	1.13	1.02	1.15	1.07	1.03	1.14
0.06	1.03	1.04	1.04	1.15	1.04	1.18	1.09	1.08	1.28
0.05	1.02	1.02	1.03	1.11	1.03	1.13	1.09	1.06	1.20
0.04	1.06	1.06	1.06	1.12	1.08	1.12	1.09	1.10	1.20
0.03	1.08	1.08	1.09	1.12	1.10	1.11	1.10	1.11	1.15
0.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.06
0.01	1.03	1.03	1.03	1.04	1.03	1.04	1.04	1.03	1.05

9: The ratio of the metric with no iteration and 10 iterations. The closer the z-value to zero, the better the improvement.

Area	N=4	N=9	N=16	N=25	N=36	N=49	N=64	N=81	N=100
Coarse	**0.45**	0.45	0.46	0.46	0.45	0.46	0.45	0.47	0.47
1.8	0.48	0.48	0.53	0.50	0.56	0.51	0.61	0.65	0.58
1.6	0.49	0.49	0.50	0.50	0.56	0.51	0.69	0.78	0.60
1.4	0.49	0.50	0.51	0.52	0.63	0.58	0.62	0.72	0.62
1.2	0.50	0.51	0.59	0.53	0.62	0.56	0.68	0.74	0.71
1	0.50	0.51	0.56	0.54	0.64	0.58	0.69	0.86	0.85
0.8	0.52	0.53	0.54	0.54	0.64	0.67	0.85	0.90	0.78
0.6	0.54	0.58	0.58	0.63	0.79	0.73	0.88	1.00	0.90
0.4	0.59	0.66	0.70	0.68	0.93	0.71	1.00	1.00	1.00
0.2	0.72	0.79	0.77	0.78	1.00	0.94	1.00	1.00	0.93
0.1	1.00	1.00	1.00	0.89	1.00	0.92	1.00	1.00	0.83
0.08	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.87
0.06	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.05	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.04	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.03	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.02	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.01	**1.00**	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

IM1-B





$f = 42.1526$

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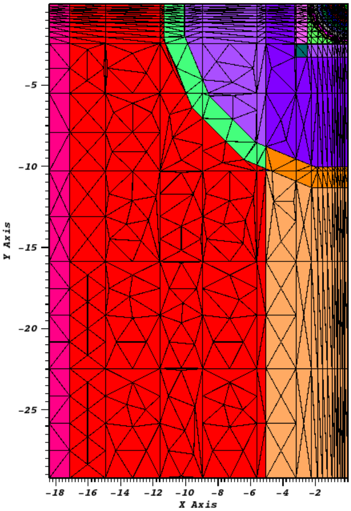
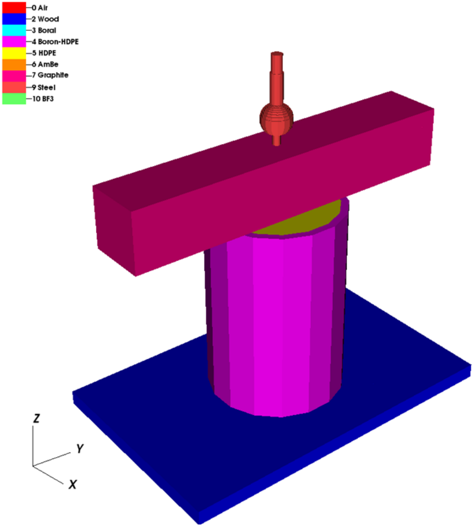
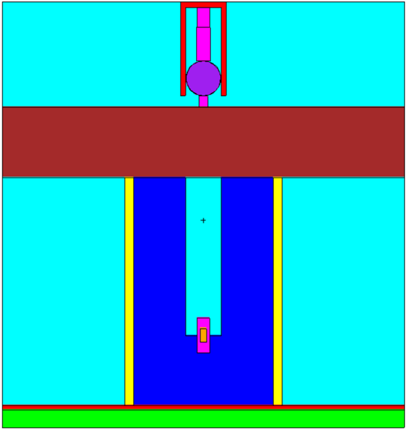
$f = 2.99$

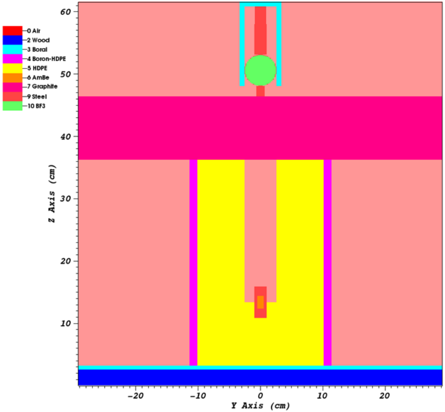


Extruded Mesh Capability

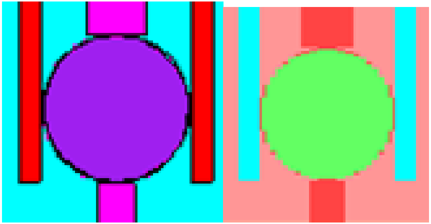


Extruded Mesh Capability





Extruded Mesh Capability



MCNP vs. PDT

10: The results of MCNP and PDT Compared for IM1-B

Code and Setup	Abs. Rate w/Air (s^{-1})	Abs. Rate w/Graphite s^{-1}
MCNP	70.1	12.66
PDT Unstructured	68.93	12.75

Conclusions

- The effectiveness of the load balancing algorithm depends on the spatial distribution of fine geometric features, the maximum triangle area used, and the number of subsets the domain is decomposed into.
- Good improvement is seen for all test cases, particularly Test Cases 1,2, and IM1-B.
- Improvements to the algorithm must be made, as the user will often need to decide on the number of subsets based on how many processors are wanted.

Future Work

- Three paths for improving the load balancing algorithm have been outlined.
  - Adaptively splitting the subsets that have large cell counts into smaller subsets, and redistributing subsets amongst processors.
  - Load balancing by column only (which our algorithm does effectively), then in each column individually load balancing rows.
  - Taking advantage of nested parallelism to assign more parallel processes at subsets that require more work to be done.

Initial Setup

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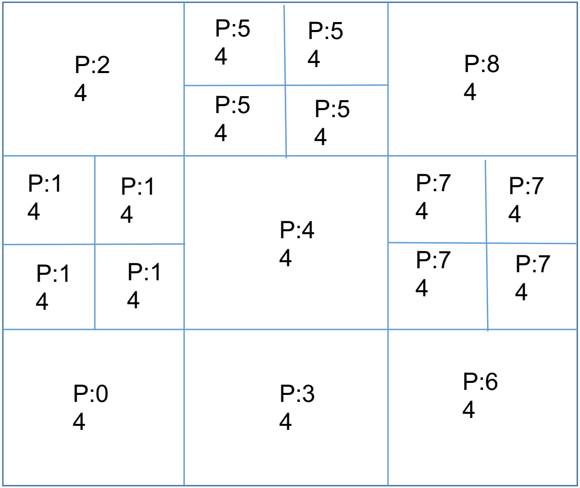
P:2 4	P:5 16	P:8 4
P:1 16	P:4 4	P:7 16
P:0 4	P:3 4	P:6 4

P: Processor_ID
Number of Cells

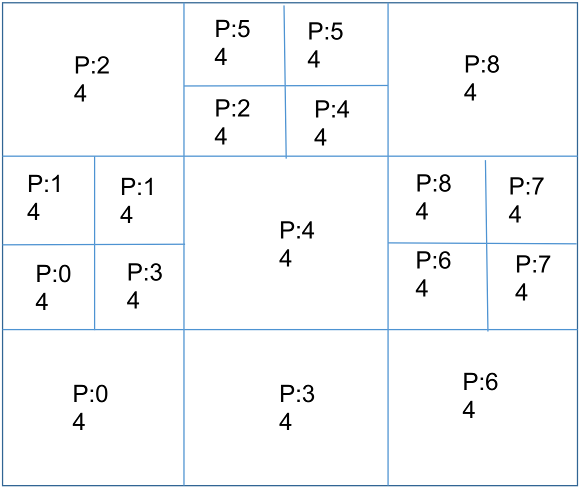
$f = 2$

Adaptively Refine Subsets



Subset Redistribution

$f = 1$



Load Balancing by Dimension

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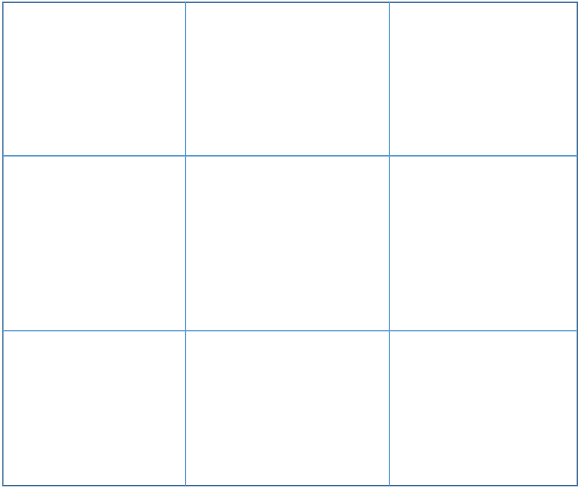
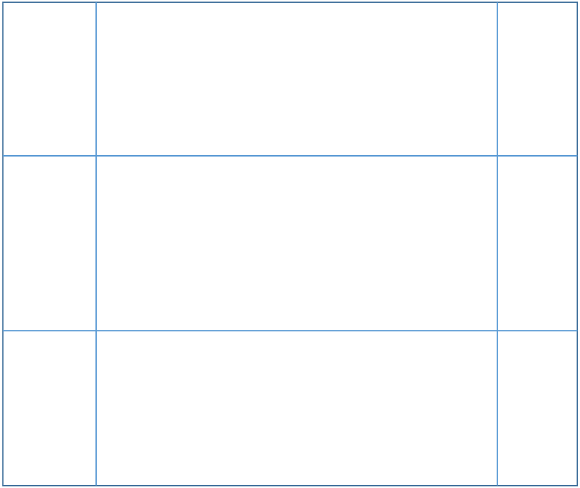
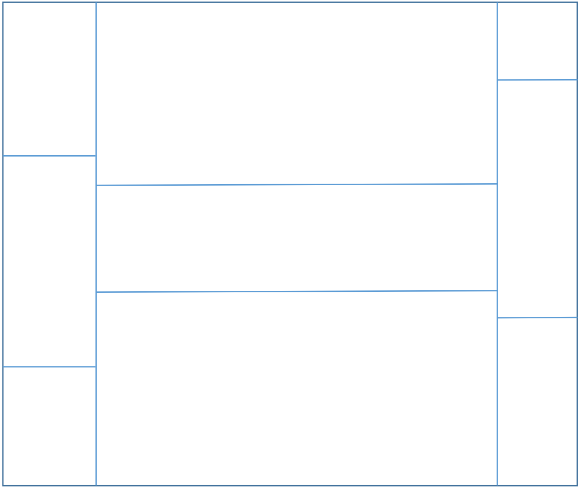


Load Balance Columns



Load Balancing by Dimension



Nested Parallelism

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P:2 4	P:5 16	P:8 4
P:1 16	P:4 4	P:7 16
P:0 4	P:3 4	P:6 4

P: Processor_ID
Number of Cells

f = 2

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- Drs. Ragusa, Morel, Adams, and Popov
- Michael Adams, Daryl Hawkins, and Dr. Timmie Smith
- Dr. Andrew Till
- The CERT team and fellow grad students
- PSAAP-II

Backup Slides

- Two benchmark problems were set up to verify that the scalar flux was being computed correctly on unstructured meshes in PDT.
- Both problems utilized a 1 cm$\times$1 cm square domain, with opposing reflecting boundaries on the y boundaries, an incident isotropic angular flux on the left boundary, and a vacuum boundary on the right.

The error presented when comparing numerical to analytical solutions is defined as follows:

$$\epsilon = \frac{\|\text{Analytical} - \text{Numerical}\|_{l2}}{\|\text{Analytical}\|_{l2}},$$

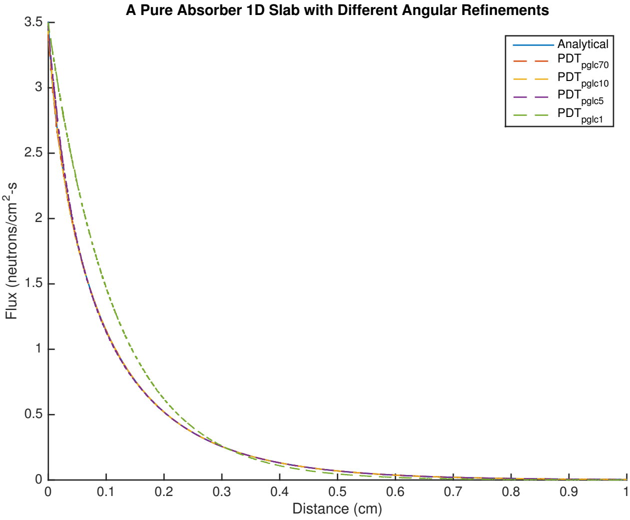
Pure Absorber

The analytical scalar flux solution of the 1D Pure Absorber is:

$$\begin{aligned}\phi(x) &= \int_0^1 \psi(x, \mu > 0) d\mu \\ &= \int_0^1 \psi_{inc} \exp(-\frac{\Sigma_a}{\mu} x) d\mu = \psi_{inc} E_2(\Sigma_a x),\end{aligned}$$

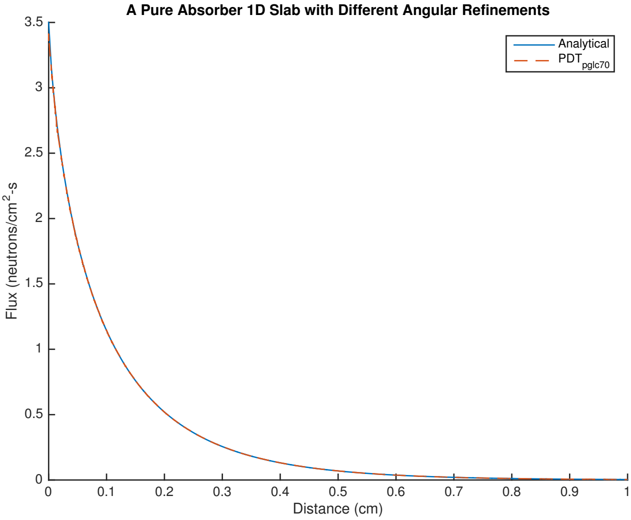
The pure absorber was run with $\psi_{inc} = 3.5 \frac{\text{n}}{\text{cm}^2\text{-s-ster}}$ and $\Sigma_a = 5\ \text{cm}^{-1}$.

PDT Results vs. Analytical for the Pure Absorber



Analysis with 70 Positive Polar Angles

$\epsilon = 0.012$



Pure Scatterer

The transport solution for a pure scatterer reaches the diffusion limit, and the solution is:

$$\phi(x) = \frac{4j_{inc}}{1+4D}(-x + x_{\max} + 2D).$$

This problem was run with $\Sigma_t = 100\ \mathrm{cm}^{-1}$ and $j_{inc} = \frac{7}{4}\frac{\mathrm{n}}{\mathrm{cm}^2\text{-s}}$.

ϵ=4.25E-04