TG INVESTMENTS & RESEARCH

Mastering Interest Rate Derivatives

Introduction to Building of Interest Rate Curves

Tim Glauner

May 25, 2025

Copyright and Trademark

Outline

Outline

Purpose of Building Curves

- **"Everything starts with interest rate curves"** - Interest rate curves serve as the foundation for various financial analyses and applications
- Valuing any financial instrument depends on interest rates, which can be extracted or implied from these curves

Concept of Building Curves - 1

- Objective: Develop a rational approach to estimate fair interest rates
- Considerations:
  - Interest rates vary with the term of the instrument and include a risk premium
  - Arbitrage opportunities should be avoided
- Approach to build curve:
  - Utilize widely used instruments to estimate rates
  - Find rates to price all instruments to market
- Representation of built curve
  - Curves are stored as a term structure of zero rates

Concept of Building Curves - 2

- The curves we estimate represent the interest rates at which investors actually trade, rather than the expected rates
- The forward rate differs from the expected spot rate due to the inclusion of a risk premium

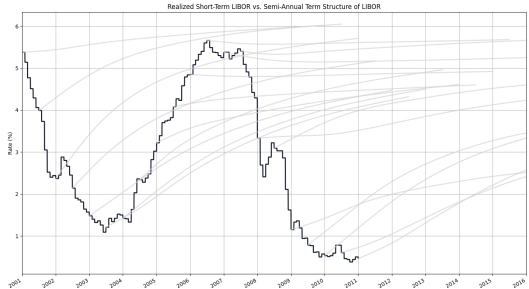


Figure: Term Structure of Rates Compared to the Realized Short-term Spot Rate

Outline

Simplified: Use Discount Bond - 1

- Assume we have 5 bonds maturing over the next 5 years
- They are zero coupon bonds with known prices
- For illustrative purposes, as these bonds are not liquid in reality

Maturity	# Days	Price
1 year	365	98.81
23 months	700	97.35
3 years	1,095	95.03
49 months	1,490	92.54
5 years	1,825	90.48

Table: 5 Bonds

Simplified: Use Discount Bond - 2

- To determine today's price, we need to discount the par value of the bond from its maturity value; otherwise, an arbitrage opportunity would arise
- We can calculate each zero rate by utilizing a closed-form formula based on this concept
- Let d_i represent the number of days between the maturity date of bond i and today
- The price P_i of bond i is given by $P_i = DF_{d_i} * \$100$
- The discount factor DF_{d_i} is calculated as $DF_{d_i} = e^{-z_{d_i} * d_i / 365}$
- Now, solving for z_{d_i} for each bond's maturity allows us to obtain the corresponding zero rates. As an exercise, you can derive the explicit formula for z_{d_i}

Simplified: Use Discount Bond - 3

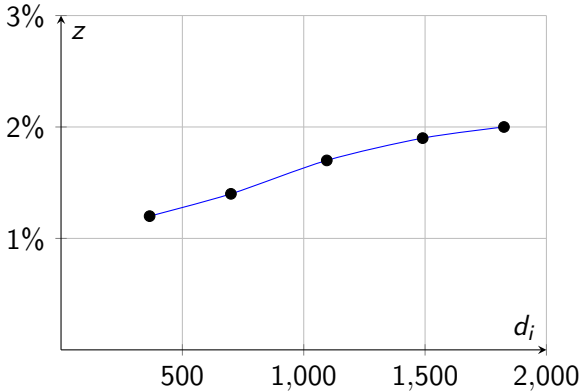
- After deriving zero rates, we have the following bond data:

Maturity	# Days	Price	Zero Rate
1 year	365	98.81	1.2%
23 months	700	97.35	1.4%
3 years	1,095	95.03	1.7%
49 months	1,490	92.54	1.9%
5 years	1,825	90.48	2%

Table: Bond Information including Zero Rates

Simplified: Use Discount Bond - 4

- Let's graph our first zero curve based on our simplified assumptions



Outline

Outline

Introduction to the Structure of the Curve and its Segments

- USD/LIBOR illustration to build curve [1]
- Deriving a zero curve from liquid instruments
- Conceptually good points, although slightly different instruments were used in practice

Market Data as of 29-Jan-2019		
Type	**Date**	**Rate**
MM	2D	4.875%
MM	1M	4.875%
MM	3M	4.9%
FUT	MAR-19	95.06
FUT	JUN-19	95.12
...	...	
SWAP	2Y	5.125%
SWAP	3Y	5.178%
...	...	

[1] LIBOR ceased to exist mid-2023 but still very good for illustration

Outline

Derive the First Zero Rate for 2D Money Market Point

Market Data as of 29-Jan-2019			
Type	**Date**	**Rate**	**Zero**
MM	2D	4.875%	
MM	1M	4.875%	
MM	3M	4.9%	
FUT	MAR-19	95.06	
FUT	JUN-19	95.12	
...	...		
SWAP	2Y	5.125%	
SWAP	3Y	5.178%	
...	...		

Flows for 2D rate of 4.875%		
Date	**Amount**	**Required NPV**
29-Jan-19	\$100	\$100
31-Jan-19	-\$100.0271	-\$100

- We need to define the discount factor (DF) to make the sum of NPV values \$0
- Recall, we have $DF = e^{(-z \cdot \frac{\#\text{ days}}{365})}$
- So, $z = \ln(\frac{100.0271}{100}) \cdot \frac{365}{2}$
- and, $z = 4.94204\%$

Derive the next Zero Rate for 1M Money Market Point

Market Data as of 29-Jan-2019			
Type	**Date**	**Rate**	**Zero**
MM	2D	4.875%	**4.94204%**
MM	1M	4.875%	
MM	3M	4.9%	
FUT	MAR-19	95.06	
FUT	JUN-19	95.12	
...	...		
SWAP	2Y	5.125%	
SWAP	3Y	5.178%	
...	...		

Flows for 1M rate of 4.875%		
Date	**Amount**	**Required NPV**
31-Jan-19	\$100	\$99.9729
28-Feb-19 [a]	-\$100.3792	-\$99.9729

- We know the zero rate of 31-Jan-19
- Find next z so, $\sum_{i=1}^{n} NPV(Flow_i) = \$0$
- Flows are at spot date and 28-Feb-19
- We receive \$100 spot and pay back $\$100 \cdot 4.875\% \cdot \frac{28}{360} = \$100.3792$
- So, $z = \ln(\frac{100.3792}{99.9729}) \cdot \frac{365}{30}$
- and therefore, $z = 4.93394\%$

[a] Rolled date to good business day with Modified Following rolling convention

Derive the next Zero Rate for 3M Money Market Point

Market Data as of 29-Jan-2019			
Type	**Date**	**Rate**	**Zero**
MM	2D	4.875%	4.94204%
MM	1M	4.875%	**4.93394%**
MM	3M	4.9%	
FUT	MAR-19	95.06	
FUT	JUN-19	95.12	
...	...		
SWAP	2Y	5.125%	
SWAP	3Y	5.178%	
...	...		

Flows for 3M rate of 4.9%		
Date	**Amount**	**Required NPV**
31-Jan-19	\$100	\$99.9729
30-Apr-19	-\$101.2114	-\$99.9729

- We know zero rates up to 28-Feb-19
- Find z for 30-Apr-19 so that $\sum_{i=1}^{n} NPV(Flow_i) = \$0$
- So, $z = \ln(\frac{101.2114}{99.9729}) \cdot \frac{365}{91}$
- and therefore, $z = 4.93829\%$

Outline

Interpolation of Zero Rates

- The zero curve consists of distinct data points
- Interpolation is required for values not present on the curve
- For the first futures contract, we require z on the starting date
- There are many interpolation methods
- Smoothing forward rates is common practice

Linear interpolation of Zero Rate

Given, z_1 at time t_1 and z_2 at time t_2 interpolate z at time t:

$$z = z_1 + (z_2 - z_1) \cdot \frac{t - t_1}{t_2 - t_1} = \frac{z_2 \cdot (t - t_1) + z_1 \cdot (t_2 - t)}{t_2 - t_1}$$

Outline

Derive the next Zero Rate for first Futures contract - 1

- Eurodollar future represents a 3 month forward rate
- Prices are available for up to 3-4 years into the future
- It's common to use futures up to 1-2 years
- The price of 95.06 implies rate of 4.94%

Futures Price to Rate Conversion

$$Rate = \frac{100 - Futures\ Price}{100}$$

- The length of the future can be interpreted as 3M or contract to contract
- The start and end dates are determined by exchanges

Derive the next Zero Rate for first Futures contract - 2

Market Data as of 29-Jan-2019			
Type	**Date**	**Rate**	**Zero**
MM	2D	4.875%	4.94204%
MM	1M	4.875%	4.93394%
MM	3M	4.9%	**4.93829%**
FUT	MAR-19	95.06	
FUT	JUN-19	95.12	
...	...		
SWAP	2Y	5.125%	
SWAP	3Y	5.178%	
...	...		

Flows for MAR-19 contract with price 95.06		
Date	**Amount**	**Required NPV**
20-Mar-19	\$100	\$99.3263
19-Jun-19	-\$101.2487	-\$99.3263

- Imply fwd rate: $\frac{100-95.06}{100} = 4.94\%$
- Interpolate z for 20-Mar-19 to get the zero rate on the start date of the future
- Calculate t, t_1, t_2 based on days to today
- $z = z_1 + (z_2 - z_1) \cdot \frac{20}{59} = 4.93537\%$
- We can now present value the start flow of the future
- $NPV = \$100 \cdot e^{-4.93537\% \cdot \frac{50}{365}} = \$99.3263$

Derive the next Zero Rate for first Futures contract - 3

Market Data as of 29-Jan-2019			
Type	**Date**	**Rate**	**Zero**
MM	2D	4.875%	4.94204%
MM	1M	4.875%	4.93394%
MM	3M	4.9%	**4.93806%**
FUT	MAR-19	95.06	
FUT	JUN-19	95.12	
...	...		
SWAP	2Y	5.125%	
SWAP	3Y	5.178%	
...	...		

Flows for MAR-19 contract with price 95.06		
Date	**Amount**	**Required NPV**
20-Mar-19	\$100	\$99.3263
19-Jun-19	-\$101.2487	-\$99.3263

- Calculate 19-Jun-19 flow using standard formula
- Basis for interest calculation is A/360 by convention
- $\$100 \cdot (1 + 4.94\% \cdot \frac{91}{360}) = 100.2487$
- Find next z so, $\sum_{i=1}^{n} NPV(Flow_i) = \$0$
- So, $z = \ln(\frac{100.2487}{99.3263}) \cdot \frac{365}{141}$
- and therefore, $z = 4.96262\%$

Recap and Graph Zero Rates

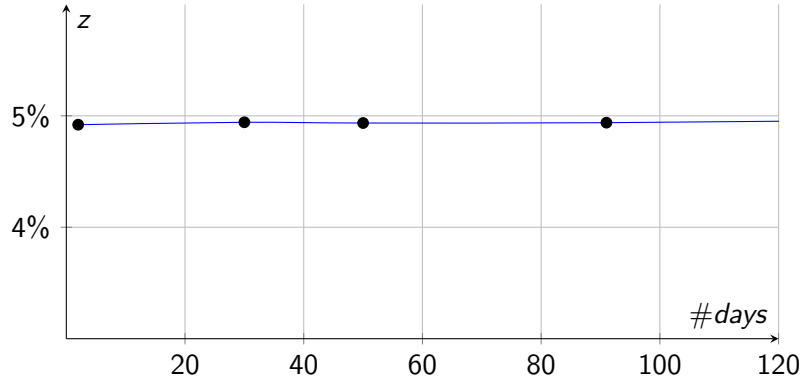


Figure: Zero Rates and Number of Days

Next Point is JUN-19 Futures Contract - 1

Market Data as of 29-Jan-2019			
Type	**Date**	**Rate**	**Zero**
MM	2D	4.875%	4.94204%
MM	1M	4.875%	4.93394%
MM	3M	4.9%	4.93829%
FUT	MAR-19	95.06	**4.96262%**
FUT	JUN-19	95.12	
...	...		
SWAP	2Y	5.125%	
SWAP	3Y	5.178%	
...	...		

Flows for JUN-19 contract with price 95.12		
Date	**Amount**	**Required NPV**
19-Jun-19	\$100	\$98.1012
18-Sep-19	-\$101.2336	-\$98.1012

- Imply fwd rate: $\frac{100-95.12}{100} = 4.88\%$
- Flow for 18-Sep-19 is \$101.2336
- z on start date is known from previous Steps
- So, we can imply $z = 4.94493\%$ for the end date of the futures contract

Adding Remaining Futures

Market Data as of 29-Jan-2019			
Type	**Date**	**Rate**	**Zero**
MM	2D	4.875%	4.94204%
MM	1M	4.875%	4.93394%
MM	3M	4.9%	4.93394%
FUT	MAR-19	95.06	4.96262%
FUT	JUN-19	95.12	4.94493%
FUT	SEP-19	95.13	4.93438%
FUT	DEC-19	94.76	5.00989%
FUT	MAR-20	95.01	5.01309%
FUT	JUN-20	94.94	5.02602%
SWAP	2Y	5.125%	
SWAP	3Y	5.178%	
...	...		

- We use six contracts
- In practice, futures often overlap with the first Swap
- You can then either specify which has priority or merge the zero rates
- Convexity adjustment is required but we won't cover it in this lecture

Outline

Next Point is the First Swap Point - 2Y - 1

- We assume the projection index equals the discount index for simplicity
- The floating side can then be ignored when building the curve
- Reason is that with notional exchanges, the value will always be \$0
- Post 2008, LIBOR flows were discounted using the Fed Fund rate. With SOFR, the market is returning to equal indices
- LIBOR swaps pay semi-annually, resulting in 4 cash flows for a 2-year period
- As we are ignoring the floating side we need to add the notional exchanges to the fixed leg
- Find the z of the last cash flows, such that $\sum_{i=1}^{6} NPV(Flow_i) = \$0$
- The known $\{z_i\}$ values can be obtained from the curve

Next Point is the First Swap Point - 2Y - 2

Market Data as of 29-Jan-2019			
Type	**Date**	**Rate**	**Zero**
MM	2D	4.875%	4.94204%
MM	1M	4.875%	4.93394%
MM	3M	4.9%	4.93394%
FUT	MAR-19	95.06	4.96224%
FUT	JUN-19	95.12	4.94431%
...	...	...	...
FUT	MAR-20	95.01	5.01261%
FUT	JUN-20	94.94	5.02535%
SWAP	2Y	5.125%	
SWAP	3Y	5.178%	
...	...		

Flows - 2Y Swap Pay 5.125%		
Date	**Amount**	**NPV**
31-Jan-19	$100	$99.9729
31-Jul-19	-$2.5625	-$2.4996
31-Jan-20	-$2.5625	-$2.4375
31-Jul-20	-$2.5625	-$2.3761
29-Jan-21	-$2.5482	-$2.3025
29-Jan-21	-$100	-$90.3570

- Fixed leg of 2Y swap has 4 cashflows, $\{flow_i\}$
- To find z we require notional exchanges at start and end
- Requirement for z: $\sum_{i=1}^{6} NPV(flow_i) = \$0$
- The known values, $\{z_i\}$, are extracted from the curve
- So, $z = \ln\left(\frac{100+flow_4}{\sum_{i=1}^{3} NPV(flow_i)+NPV(100,Spot)}\right) \cdot \frac{365}{731}$
- and therefore, $z = 5.06314\%$

Next Point is the 3Y Swap Point

Market Data as of 29-Jan-2019			
Type	**Date**	**Rate**	**Zero**
MM	2D	4.875%	4.94204%
MM	1M	4.875%	4.93394%
MM	3M	4.9%	4.93394%
FUT	MAR-19	95.06	4.96224%
FUT	JUN-19	95.12	4.94431%
...	...	...	...
FUT	MAR-20	95.01	5.01261%
FUT	JUN-20	94.94	5.02535%
SWAP	2Y	5.125%	**5.06314%**
SWAP	3Y	5.178%	
...	...		

Flows - 3Y Swap Pay 5.178%		
Date	**Amount**	**NPV**
31-Jan-19	$100	$99.9729
31-Jul-19	-$2.5890	-$2.5254
31-Jan-20	-$2.5890	-$2.4627
31-Jul-20	-$2.5890	-$2.4007
29-Jan-21	-$2.5746	-$2.3263
30-Jul-21	-$2.6033	-$2.2922
31-Jan-22	-$2.5890	-$2.2199
31-Jan-22	-$100	-$85.7453

- Requirement for z: $\sum_{i=1}^{8} NPV(flow_i) = \$0$
- Known $\{z_i\}$ are taken from the curve
- So, $z = \ln\left(\frac{100+flow_6}{\sum_{i=1}^{5} NPV(flow_i)+NPV(100,Spot)}\right) \cdot \frac{365}{1098}$
- and therefore, $z = 5.11228\%$

Complete Cashflow Schedule Information for a 3Y Swap

INVESTMENTS & RESEARCH

Semi-Annual Fixed Leg Cashflows										
Begin	**End**	**Period**	**Rate**	**Notional**	**Flow**	**Date**	**Days**	**Zero**	**DF**	**PV(Flow)**
1/31/2019	7/31/2019	180	5.178	1,000,000.00	-25,890.00	7/31/2019	183	4.95445	0.97547	-25,254.81
7/31/2019	1/31/2020	180	5.178	1,000,000.00	-25,890.00	1/31/2020	367	4.97089	0.95125	-24,627.79
1/31/2020	7/31/2020	180	5.178	1,000,000.00	-25,890.00	7/31/2020	549	5.01934	0.92728	-24,007.36
7/31/2020	1/29/2021	179	5.178	1,000,000.00	-25,746.17	1/29/2021	731	5.06314	0.90357	-23,263.47
1/29/2021	7/30/2021	181	5.178	1,000,000.00	-26,033.83	7/30/2021	913	5.08751	0.88051	-22,922.97
7/30/2021	1/31/2022	180	5.178	1,000,000.00	-25,890.00	1/31/2022	1098	5.11228	0.85745	-22,199.47

Quarterly LIBOR Floating Leg Cashflows											
Begin	**End**	**Reset**	**Period**	**Rate**	**Notional**	**Flow**	**Date**	**Days**	**Zero**	**DF**	**PV(Flow)**
1/31/2019	4/30/2019	1/29/2019	89	4.9	1,000,000.00	12,113.89	4/30/2019	91	4.93829	0.98776	11,965.66
4/30/2019	7/31/2019	4/26/2019	92	4.93319	1,000,000.00	12,607.05	7/31/2019	183	4.95445	0.97547	12,297.75
7/31/2019	10/31/2019	7/29/2019	92	4.87391	1,000,000.00	12,455.54	10/31/2019	275	4.93994	0.96347	12,000.48
10/31/2019	1/31/2020	10/29/2019	92	5.02604	1,000,000.00	12,844.31	1/31/2020	367	4.97089	0.95125	12,218.12
1/31/2020	4/30/2020	1/29/2020	90	5.13841	1,000,000.00	12,846.03	4/30/2020	457	5.0114	0.93918	12,064.77
4/30/2020	7/31/2020	4/28/2020	92	5.02143	1,000,000.00	12,832.53	7/31/2020	549	5.01934	0.92728	11,899.39
7/31/2020	10/30/2020	7/29/2020	91	5.11358	1,000,000.00	12,925.99	10/30/2020	640	5.03812	0.91545	11,833.09
10/30/2020	1/29/2021	10/28/2020	91	5.20123	1,000,000.00	13,147.57	1/29/2021	731	5.06314	0.90357	11,879.75
1/29/2021	4/30/2021	1/27/2021	91	5.13538	1,000,000.00	12,981.10	4/30/2021	822	5.07532	0.89199	11,579.03
4/30/2021	7/30/2021	4/28/2021	91	5.15973	1,000,000.00	13,042.65	7/30/2021	913	5.08751	0.88051	11,484.14
7/30/2021	10/29/2021	7/28/2021	91	5.18408	1,000,000.00	13,104.20	10/29/2021	1004	5.09969	0.86912	11,389.09
10/29/2021	1/31/2022	10/27/2021	94	5.20996	1,000,000.00	13,603.78	1/31/2022	1098	5.11228	0.85745	11,664.60

Adding Remaining Swaps

Market Data as of 29-Jan-2019				
Type	**Date**	**Rate**	**Zero**	**Date**
MM	2D	4.875%	4.94204%	31-Jan-2019
MM	1M	4.875%	4.93394%	28-Feb-2019
MM	3M	4.9%	4.93394%	30-Apr-2019
FUT	MAR-19	95.06	4.96262%	19-Jun-2019
FUT	JUN-19	95.12	4.94493%	18-Sep-2019
FUT	SEP-19	95.13	4.93438%	18-Dec-2019
FUT	DEC-19	94.76	5.00989%	18-Mar-2020
FUT	MAR-20	95.01	5.01309%	17-Jun-2020
FUT	JUN-20	94.94	5.02602%	16-Sep-2020
SWAP	2Y	5.125%	5.06314%	29-Jan-2021
SWAP	3Y	5.178%	**5.11228%**	31-Jan-2022
SWAP	5Y	5.394%	**5.42569%**	31-Jan-2024
SWAP	7Y	5.548%	**5.57245%**	29-Jan-2026
SWAP	10Y	5.709%	**5.72934%**	31-Jan-2029
SWAP	15Y	5.863%	**5.90024%**	31-Jan-2034
SWAP	20Y	5.932%	**5.97185%**	31-Jan-2039
SWAP	30Y	5.96%	**5.98892%**	29-Jan-2049

- We employ the same methodology as before to add remaining swaps
- For large gaps, synthetic swaps are constructed using the cubic spline method
- We may also consider utilizing the 40Y or 50Y points
- It is important to consider extrapolation assumptions

Zero Rates Graph

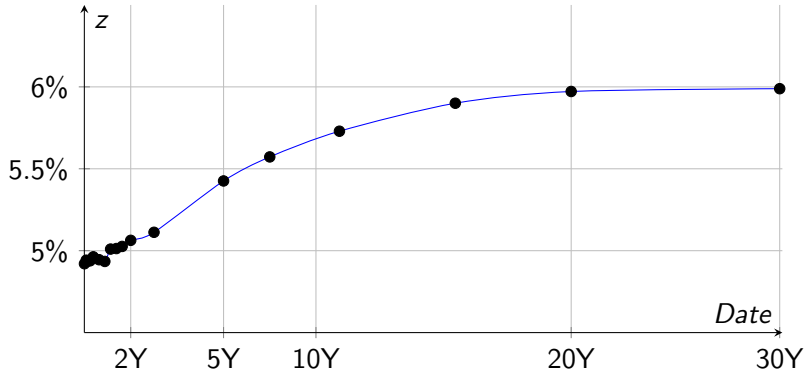


Figure: Zero Rate for Different Maturities

Summary

1. Overview, Purpose and Concept of Building Curves
2. Simplified Example of Building Curves using Discount Bonds
3. Building Curves using Money Market Rates, Futures and Swaps
   - Introduction to the Structure of the Curve and its Segments
   - Money Markets
   - Zero Rate Interpolation
   - Interest Rate Futures
   - Swaps

Thank You!

INVESTMENTS & RESEARCH

Questions?