Mastering Interest Rate Derivatives

Interest Rate Fundamentals

Tim Glauner

December 28, 2024

Copyright and Trademark

Outline

Outline

Derive Spot Rates based on Bond Prices - 1

- Bond prices are expressions of interest rates
- Zero-coupon bonds pay only par at maturity

Example

- Assume we have six zero-coupon bonds with different maturities
- Each bond has a face value of $100 and matures at the end of each year for the next six years
- The transaction price for each of these bonds is listed below

Maturity	Price
1 year	94.88
2 years	90.75
3 years	87.25

Maturity	Price
4 years	84.05
5 years	80.78
6 years	77.63

Derive Spot Rates based on Bond Prices - 2

Example

continued

- Define term and compounding convention for spot rate
- We assume an annual rate and annual compounding
- Using standard discounting we can determine the spot rates, $\{r_i\}$
- $P = \frac{\$100}{(1+r)^n}$
- where P is the bond price, n the number of years

Maturity	Price	Rate (%)
1 year	94.88	5.40
2 years	90.75	4.97
3 years	87.25	4.65

Maturity	Price	Rate (%)
4 years	84.05	4.44
5 years	80.78	4.36
6 years	77.63	4.31

Outline

1. Term Structure of Interest Rates and Bonds
2. Compounding
3. Basis and Cashflow Calculation
4. Spot, Zero and Short Rates
5. Forward Rates

Compounding

- Compounding is the process where the interest earned is reinvested to generate additional earnings
- Compounding usually happens multiple times and create exponential growth in interest earned
- Various compounding conventions exist
  - Annual, semi-annual, quarterly, monthly, weekly
  - Daily is now the most common convention due to LIBOR cessation
  - LIBOR has been discontinued and replaced by SOFR
  - SOFR is an overnight rate, hence daily compounding is natural
  - Continuous compounding is also common in quantitative models
- Conversions from one rate to another can be explored as an exercise

Compounding Intuition - 1

INVESTMENTS & RESEARCH

- Let's create some intuition around compounding
- Assume an annual rate and quarterly compounding
- P_i is the principal after i compounding periods
- r is the annual rate, I is the interest
- Let's illustrate how much principal we have after one year

Example

After 3 months, which is the end of the first compounding period, we have:

$$\begin{aligned} P_1 &= P_0 + I \\ &= P_0 + P_0 \cdot \frac{r}{4} \\ &= P_0 \cdot \left(1 + \frac{r}{4}\right) \end{aligned}$$

Compounding Intuition - 2

Example

After 6 months, which is the end of the second compounding period, we calculate the value of the principal using the compound interest formula:

- P_0: Initial principal
- P_1: Value at end of first compounding period
- P_2: Value at end of second compounding period

$$
\begin{aligned}
P_2 &= P_1 + I \\
&= P_1 + P_1 \cdot \frac{r}{4} \\
&= P_1 \cdot \left(1 + \frac{r}{4}\right) \\
&= P_0 \cdot \left(1 + \frac{r}{4}\right) \cdot \left(1 + \frac{r}{4}\right) \\
&= P_0 \cdot \left(1 + \frac{r}{4}\right)^2
\end{aligned}
$$

Compounding Intuition - 3

Example

After 9 months, which corresponds to the end of the third compounding period, the final amount P_3 is calculated as follows:

- P_0: Initial principal
- P_2: Value at end of second compounding period
- P_3: Value at end of third compounding period

$$
\begin{aligned}
P_3 &= P_2 + I \\
&= P_2 + P_2 \cdot \frac{r}{4} \\
&= P_2 \cdot \left(1 + \frac{r}{4}\right) \\
&= P_0 \cdot \left(1 + \frac{r}{4}\right)^2 \cdot \left(1 + \frac{r}{4}\right) \\
&= P_0 \cdot \left(1 + \frac{r}{4}\right)^3
\end{aligned}
$$

Compounding Intuition - 4

Example

Finally, after 1 year we get

P_0: Initial principal

P_3: Value at end of third compounding period

P_4: Value at end of fourth and last compounding period

$$
\begin{aligned}
P_4 &= P_3 + I \\
&= P_3 + P_3 \cdot \frac{r}{4} \\
&= P_3 \cdot (1 + \frac{r}{4}) \\
&= P_0 \cdot (1 + \frac{r}{4})^3 \cdot (1 + \frac{r}{4}) \\
&= P_0 \cdot (1 + \frac{r}{4})^4
\end{aligned}
$$

Compounding Intuition - 5

Corollary

We can now generalize and state that for n compounding periods

$$P_n = P_0 \cdot \left(1 + \frac{r}{n}\right)^n$$

r: *Annual rate*

n: *Number of coumpounding periods*

P_0: *Initial principal*

P_n: *Value at end of n^{th} compounding period*

Outline

Basis Example

- We start with an example to illustrate the significance of the basis

Example

Imagine we have a financial instrument with an interest rate period that starts on 5th January 2023 and ends on 5th July 2023
For a Money Market trade with a 1% coupon, we calculate the cashflow as follows:

$$\mathit{Cashflow} = \$1,000,000 \cdot 1\% \cdot \frac{181}{360} = \$5,027.78$$

For a Treasury bond trade with a 1% coupon, we calculate the cashflow as follows:

$$\mathit{Cashflow} = \$1,000,000 \cdot 1\% \cdot \frac{180}{360} = \$5,000$$

The basis differs, defining how to calculate the length of the interest period

Basis and Cashflow Calculation

- Day count or basis defines the year fraction between two dates
- Measure is necessary to define calculation of cash flows

We calculate cash flows for an interest rate period as

$$\text{Cashflow} = \text{Notional} \cdot \text{Rate} \cdot \text{Year Fraction}$$
$$\text{Year Fraction} = \frac{\text{Basis Numerator}}{\text{Basis Denominator}}$$

where numerator and denominator are defined by the basis

Numerator, Denominator and Sample Basis

- Measure for basis in numerator calculates days between the start and end date
  - Actual number of days
  - Assumed 30 days per month
  - Business days only
- Measure for basis in denominator calculates days in a year
  - 360 days
  - 365 days
  - 365 days with leap year adjustments
  - 252 days (business days)
- Many bases are used in the market, including
  - Act/360 - used in SOFR swap and money market calculations
  - 30/360 - used in swap fixed leg and corporate bond calculations
  - Act/Act - used in treasury bond calculations
  - BUS/252 - common in Brazil

Outline

Spot and Zero Rates

TG INVESTMENTS & RESEARCH

- A **spot rate** is the interest rate from today for a set period
- **Zero rates** (denoted as z) are continuously compounded spot rates with a basis of A/365
- Hence, the discount factor using zero rates is

$$DF(T_0, T_1) = e^{-z \cdot \frac{ActualDays(T_0, T_1)}{365}} = e^{-z \cdot dt}$$

DF: Discount factor

z: Zero rate

Short Rates

- An **instantanous short rate** is a spot rate for an infinitesimally short period of time
- Provides a foundational building block in continuous-time stochastic interest rate models for quants and traders
- Requires understanding of stochastic calculus
- The Heath-Jarrow-Morton (HJM) model uses stochastic instantaneous short rate
- Instantaneous short rates are usually continuously compounded
- Such rates are not observed but are used as theoretical constructs
- Continous nature allows derivation of complex models

Outline

Forward rates - 1

- Forward rates are set today but apply to specified future periods
- Essential for swaps, alongside discount factors

Example

- Let's assume today's date is 3-Jan-2023
- Today's 3-month rate 6 months forward applies from 3-Jul-2023 to 3-Oct-2023
- We define it as $f(\text{03-Jan-2023}, \text{03-Jul-2023}, \text{3-Oct-2023})$
- More generally, we can denote it as $f(T_0, T_1, T_2)$

Forward rates - 2

- Forward rates span specific start and end dates
- Determined in markets
- Market product for these is the Forward Rate Agreement (FRA)
- Consider them as the slope or first derivative of spot rates

Summary

1. Term Structure of Interest Rates and Bonds
2. Compounding
3. Basis and Cashflow Calculation
4. Spot, Zero and Short Rates
5. Forward Rates

Thank You!

Questions?