Econometrics - Revisiting and developing (in progress)

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1 Introduction

The linear regression analysis is based on the book I used, Econometric Methods by Jack Johnston, and the class notes from Econometrics 1 (and possibly 2) that I took at FEA-USP.

Several excerpts have been rewritten/adapted based on the main book (and class notes) with the aim of developing my writing skills in R and LaTeX.

Most of the statistical calculations were carried out in Rstudio with the same objective. Excel was also used to clean the database and integrate with Rstudio.

I will try to use some techniques and concepts presented in Quantitative Economics with R; A Data Science Approachlink. Its PDF can be easily found by searching online.

The statlect website, with its clear demonstrations and examples, will also be used

The main objective here will not be to present a formal approach to econometrics. Instead, I will aim to provide a concise overview of the theory necessary to understand the topic, and proceed with practical examples using R, Excel, and other tools.

2 The base of Linear Rgression

To understand Linear Regression, we need to grasp some statistical concepts. Let's start with relationships between two variables.

2.1 Bivariate Frequency Distribution

Note: The columns refer to chest circumference (inches), and the rows refer to height (inches)

	33-35	36-38	39-41	42-44	45-over	Row totals
64-65	39.00	331.00	326.00	26.00	0.00	722.00
66-67	40.00	591.00	1010.00	170.00	4.00	1815.00
68-69	19.00	312.00	1144.00	488.00	18.00	1981.00
70-71	5.00	100.00	479.00	290.00	23.00	897.00
72-73	0.00	17.00	120.00	153.00	27.00	317.00
Column totals	103.00	1351.00	3079.00	1127.00	72.00	5732.00

Note: The second table displays conditional means

	1	2	3	4	5
mean of height given chest-inches	66.31	66.84	67.89	69.16	70.53
mean of chest given height-inches	38.41	39.19	40.26	40.76	41.80

The second table shows a lot of numbers. But how were they calculated? Let's demonstrate that: The number 66.84 in the first row, second column of the second table was derived by performing the following calculation: $(64.5 \cdot 331 + 66.5 \cdot 591 + 68.5 \cdot 312 + 70.5 \cdot 100 + 72.5 \cdot 17)/1351 = 66.84$.

In a similary way, the number 38.41 in the second row, first column of the second table: $(39 \cdot 34 + 331 \cdot 37 + 326 \cdot 40 + 26 \cdot 43)/722 = 38.41$

2.2 The correlation coefficient

The correlation coefficient measures the direction and closeness of the linear association between two variables. Let´s denote the observations by (X_i, Y_i) with $i = 1, 2, 3...n$. The data can be expressed in deviation form as: $(x_i = X_i - \overline{X}), (y_i = Y_i - \overline{Y})$, where $\overline{X}, \overline{Y}$ denote sample means of X and Y. Def: $Cov(X, Y) = \sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})/n = \sum_{i=1}^{n} x_i y_i/n$. One problem with the sample covariance is that it is sensitive to the unit. Suppose X is measured in dollars, and so is Y. The covariance gives $dollars^2$ measure. If X and Y change to centes, it gives a coeficient that is $1 * 10^4$ the old.

The covariance of the standardized deviations is the correlation coefficient, **r** namely, measures linear association, and is dimensionless. **r** is limited,$-1 \leq \mathbf{r} \leq 1$

$\mathbf{r} = \sum_{i=1}^{n}(x_i/s_x)(y_i/s_y)/n$, where $s_x = \sqrt{(\sum_{i=1}^{n}(x_i)^2/n)}$

2.3 Practical examples using R

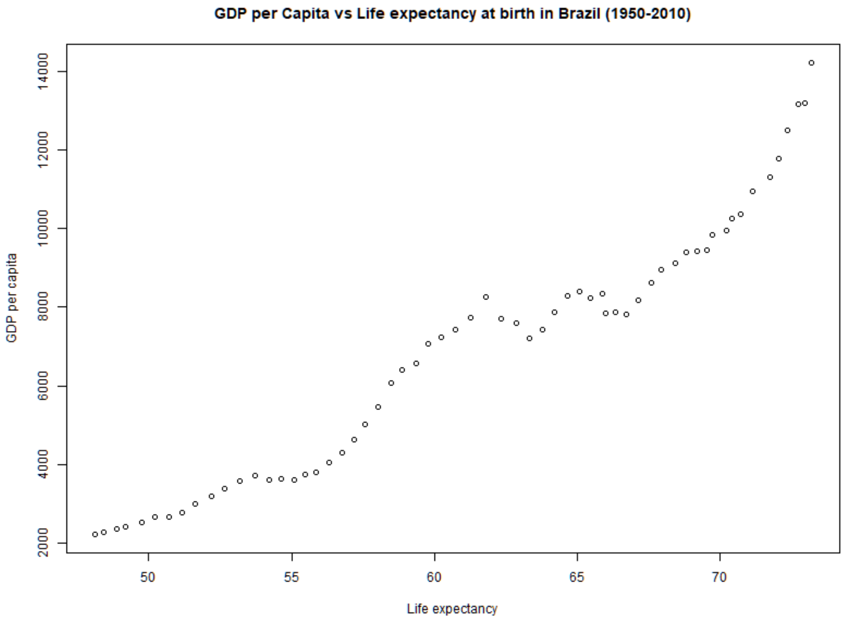
Here, we can calculate the correlation coefficient between life expectancy and GDP per capita in Brazil. We have 61 observations from 1950 to 2010. This means that for each year, we have a pair (life expectancy, GDP), and for this, we apply the formula above. The data is available here life expectancy vs GDP per-capita

	Year	Life expectancy	GDP per Capita
1	1950.00	48.12	2236.00
2	1951.00	48.43	2279.00
3	1952.00	48.87	2377.00
4	1953.00	49.20	2418.00
5	1954.00	49.75	2531.00
6	1955.00	50.22	2675.00
7	1956.00	50.71	2672.00
8	1957.00	51.17	2793.00
9	1958.00	51.64	3005.00
10	1959.00	52.19	3201.00
11	1960.00	52.66	3398.00
12	1961.00	53.18	3585.00
13	1962.00	53.71	3711.00
14	1963.00	54.21	3623.00
15	1964.00	54.65	3637.00
16	1965.00	55.08	3617.00
17	1966.00	55.47	3747.00
18	1967.00	55.87	3795.00
19	1968.00	56.31	4050.00
20	1969.00	56.75	4313.00
21	1970.00	57.17	4635.00
22	1971.00	57.59	5024.00
23	1972.00	58.03	5480.00
24	1973.00	58.47	6086.00
25	1974.00	58.88	6416.00
26	1975.00	59.35	6582.00
27	1976.00	59.79	7079.00
28	1977.00	60.24	7248.00
29	1978.00	60.72	7425.00
30	1979.00	61.25	7736.00
31	1980.00	61.78	8249.00
32	1981.00	62.33	7709.00
33	1982.00	62.86	7587.00
34	1983.00	63.33	7203.00
35	1984.00	63.77	7438.00
36	1985.00	64.20	7862.00
37	1986.00	64.64	8281.00
38	1987.00	65.08	8402.00
39	1988.00	65.45	8230.00
40	1989.00	65.87	8333.00
41	1990.00	65.98	7842.00
42	1991.00	66.31	7888.05

43	1992.00	66.71	7812.79
44	1993.00	67.11	8166.24
45	1994.00	67.57	8615.69
46	1995.00	67.92	8951.69
47	1996.00	68.41	9124.52
48	1997.00	68.81	9409.95
49	1998.00	69.19	9419.11
50	1999.00	69.52	9441.76
51	2000.00	69.74	9834.42
52	2001.00	70.19	9953.31
53	2002.00	70.41	10245.07
54	2003.00	70.72	10354.60
55	2004.00	71.13	10949.66
56	2005.00	71.75	11305.77
57	2006.00	72.04	11766.60
58	2007.00	72.37	12500.01
59	2008.00	72.72	13164.01
60	2009.00	72.95	13180.89
61	2010.00	73.18	14215.61

The correlation between GDP per capita and life expectancy at birth in Brazil (1950-2010) is 0.972979.

The graph below shows every point (Life expectancy, GDP per capita). It is reasonable to say that we can observe a positive correlation between the two variables. As one variable increases, the other also tends to increase. In fact, this relationship helps us understand the significance of the exact value of the correlation coefficient.



2.4 Probability Models for two variables

2.4.1 Discrete Bi-variate Probability Distribution

First, let show a table of a discrete bi-variate probability distribution.

TABLE : A bivariate probability distribution

	X_1	$\dots$	X_i	Marginal probability
Y_1	p_{11}	$\dots$	p_{i1}	$p_{.1}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
Y_j	p_{1j}	$\dots$	p_{ij}	$P.j$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
Y_p	p_{1p}	$\dots$	p_{ip}	$P.p$
Marginal probability	$p_{1.}$	$\dots$	$p_{i.}$	1

The covariance is:

$$\sigma_{X,Y} = cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)] = \sum_i \sum_j p_{ij}(X_i - \mu_x)(Y_j - \mu_y)$$

The population correlation coefficient is defined as:

$$corr(X,Y) = \rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

2.4.2 Conditional Probabilities

Consider the X_i column in the the table above. Each cell probability may be divided by the column total, $p_i.$, to give a conditional probability for Y given X_i. Thus,

$$\frac{p_{ij}}{p_{i.}} = probability\, that\, Y = Y_i\, given\, that\, X = X_i = Prob(Y_j|X_i)$$

The mean of this distribution is the conditional expectation of Y, given X_i, that is:

$$\mu_{y|x_i} = E(Y|X_i) = \sum_j (\frac{p_{ij}}{p_i.}) Y_j$$

Similarly, the variance of this distribution is a conditional variance, or

$$\sigma^2_{y|x_i} = var(Y|X_i) = \sum_j (\frac{p_{ij}}{p_i})(Y_j - \mu_{y|x_i})^2$$

The conditional means and variances are both functions of X, so there is a set of "i" conditional means and variances.

Columns refer to Income (X) and rows refer to Vacation Expenditure (Y)

	20000	30000	40000
1000	0.28	0.03	0.00
2000	0.08	0.15	0.03
3000	0.04	0.06	0.06
4000	0.00	0.06	0.15
5000	0.00	0.00	0.03
6000	0.00	0.00	0.03
Marginal Probability	0.40	0.30	0.30
Mean(Y|X)	1.40	2.50	3.90
Var(Y|X)	0.44	0.85	1.09

From the table above, come the conditional probabilities:

Table: Conditional Probabilities. Columns refer to Income (X) and rows refer to Vacation Expenditure (Y).

	1000	2000	3000	4000	5000	6000
20000	0.70	0.20	0.10	0.00	0.00	0.00
30000	0.10	0.50	0.20	0.20	0.00	0.00
40000	0.00	0.10	0.20	0.50	0.10	0.10

Using the table of conditional probabilities, it's possible to calculate the last two lines of the first table: the conditional means and variances.

2.5 End of first part and considerations

Certainly, there are additional statistical concepts necessary to fully understand linear regression, which is the core of Econometrics 1. If needed for any demonstration, I will incorporate these concepts into the text. The initial part of this document is derived from the book mentioned in the introduction. For now, I will likely focus on linear regression using the class notes.

In addition to these, understanding the following concepts is also necessary: the density function of probability; knowledge of well-known distributions such as Normal, Bernoulli, and Uniform; the expected value of a continuous variable; the conditional expectation of continuous variables; and matrices, including fundamental operations like sum, multiplication, and diagonalization.

3 Linear Regression Model

The main objective here will not be to present a formal approach to econometrics. Instead, I will aim to provide a concise overview of the theory necessary to understand the topic, and proceed with practical examples using R, Excel, and other tools.

3.1 The k-variable model

For this part, I am going to use class notes and statlect by Marcos Taboga.
$y_i = B_1 x_{i1} + B_2 x_{i2} ... + B_k x_{ik} + \varepsilon_i$,
$x_{1i} = 1 \; \forall i$
Using matrix notation:

$$B_{K\times 1} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{bmatrix}, \; x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{ik} \end{bmatrix} \rightarrow y_i = {x_i}' B + \varepsilon_i, \; i = 1, 2...n$$

In this form, y_i represents a scalar output variable, called dependent variable or regressand.

${x_i}'$ is a kx1 vector of input variables, called independent variables or regressors.

N is the sample size.

B is a kx1 vector of constants, called regression coefficients.

ε_i is an unobservable error term.This includes the sources of variability in y_i that are not accounted for in the input vector x_i,such as measurement errors and input variables that are not observed by the statistician.

3.1.1 Example:

Suppose there is a sample of countries for which GDP, life expectancy, and crime rates are observed. We aim to establish a linear regression model to predict GDP based on life expectancy and crime rates. $g_i = B_1 + B_2 l_i + B_3 c_i + \varepsilon_i$

g_i, l_i, c_i denote GDP, life expectancy and crime rates.
B_1, B_2, B_3 are regression coefficients.
ε_i is an error term.
The equation can be written in a vector notation as:

$$y_i = {x_i}' B + \varepsilon_i$$

by defining

$$y_i = g_i$$

$$x_i = \begin{bmatrix} 1 \\ l_i \\ c_i \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

3.1.2 Matrix Notation

Denote y by the Nx1 vector of outputs

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

X by the NxK matrix of inputs

$$X = \begin{bmatrix} {x_1}' \\ {x_2}' \\ \vdots \\ {x_N}' \end{bmatrix}$$

and ε, the Nx1 error terms by:

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{bmatrix}$$

we have, then:

$$y_{Nx1} = X_{NxK} B_{Kx1} + \varepsilon_{Nx1}$$

3.1.3 Design Matrix

Definition: A design matrix contains information/data about multiple characteristics of many objects of interest. Each row corresponds to an individual and each column to a characteristic.

In the context of Linear Regression, it is often represented by the X.

Example: Consider the linear regression $y_i = x_i'B + \varepsilon_i$, where y_i is the dependent variable, x_i' is a Kx1 vector containing the K explanatory variables, B is a Kx1 vector of regression coefficients, ε_i is the error term and there are N observations

All the observations can be collected in the design matrix:

$$X = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_N' \end{bmatrix} = \begin{bmatrix} 1 & x_{12} & \cdots & x_{1k} \\ 1 & x_{22} & \cdots & x_{2k} \\ \vdots & & & \vdots \\ 1 & x_{N2} & \cdots & x_{Nk} \end{bmatrix}$$

where it sets the first column, $x_{i1} = 1$, because in the matrix form $Y = XB + \varepsilon$, this 1 in each row for each individual forms the intercept of the regression line.

3.1.4 Initiating a practical example using R

Not considering the context of biased estimator and similar concepts, we can begin the visualization of the regression I mentioned earlier: Suppose there is a sample of countries for which GDP, life expectancy, and crime rates are observed. We aim to establish a linear regression model to predict GDP based on life expectancy and crime rates. $g_i = B_1 + B_2 l_i + B_3 c_i + \varepsilon_i$, which is equivalent to $y_i = x_i'B + \varepsilon_i$ and $Y = XB + \varepsilon$