COMP 251

Algorithms & Data Structures (Winter 2021)

Extras – Amortized Analysis

School of Computer Science
McGill University

Slides of Langer (2014) & Cormen et al., 2009 & Comp251-Fall McGill

Announcements

Outline

- Extras.
  - Amortized Analysis.
  - Randomized algorithms.
  - Probabilistic Analysis.
  - Review Final Exam.

Amortized Analysis – Introduction 1

- Make good use of class time (minimum requirements).
  - I assume you are working for 40 hours a week and you are taking five courses: 8 work hours per week per course * 13 weeks = 104 hours total, which breaks down to:
    - 39 hours of scheduled lecture time (3 hours per week)
    - 39 hours of review/exercises, including studying for midterm exams (3 hours per week)
    - 26 hours for 3 assignments ('amortized' 2 hours per week)

When I say that you spend an average of 2 hours per week on assignments, I mean that this is the amount of work you do per week, averaged or "amortized" over the semester.

- There is nothing random going on here.

Amortized Analysis – Introduction 2

When you buy a house for 500K, you do not (usually) pay the whole amount up front and then live in it for free for n = 25 years. Rather, you pay say 100K and the bank pays 400K, and you make regular (equal) mortgage payments to the bank for n = 25 years. The amount is determined by an amortization table.

- There is nothing random going on here.

Wikipedia:

Amortization is the process of accounting for an amount over a period.

Wikipedia: In computer science, **amortized analysis** is a method of analyzing the execution cost of algorithms over a sequence of operations.

Amortized Analysis

- Analyze a sequence of operations on a data structure.
- We will talk about average performance of each operation in the worst case (i.e. not averaging over a distribution of inputs. No probability!)
- **Goal:** Show that although some individual operations may be expensive, on average the cost per operation is small.
- 3 methods:
  1. aggregate analysis
  2. accounting method
  3. potential method (See textbook for more details)

Amortized Analysis – Aggregate Analysis

- **Goal:** Show that although some individual operations may be expensive, on average the cost per operation is small.

- 3 methods:
  1. aggregate analysis.
     - A sequence of n operations takes worst-case time T(n) in total. In the worst case, the average cost, or amortized cost, per operation is therefore T(n)/n.
     - Note that this amortized cost applies to each operation, even when there are several types of operations in the sequence.
     - Although some operations might be expensive, many others might be cheap.
     - 26 hours during the term (13 weeks) . You spend an average of 2 hours per week on assignments. T(n) = 26, T(n)/n = 2.

Aggregate Analysis – Example 1

Stack operations

- PUSH(S, x): $O(1)$ each ⇒ $O(n)$ for any sequence of n operations.
- POP(S): $O(1)$ each ⇒ $O(n)$ for any sequence of n operations.
- MULTIPOP(S,k):

```
while S≠∅ and k>0 do
    POP(S)        O(1)
    k←k−1         O(1)
```

how many times?
↳ min(S, k)

Running time of MULTIPOP?

Aggregate Analysis – Example 1

Running time of MULTIPOP:

- Let each PUSH/POP cost 1.
- # of iterations of **while** loop is min(s, k), where s = # of objects on stack. Therefore, total cost = min(s, k). → $O(n)$

Sequence of n PUSH, POP, MULTIPOP operations:

- Worst-case cost of MULTIPOP is $O(n)$.
- Have n operations.
- Therefore, worst-case cost of sequence is $O(n^2)$. not a tight upper limit

But:

- Each object can be popped only once per time that it is pushed.
- Have ≤ n PUSHes ⇒ ≤ n POPs, including those in MULTIPOP.
- Therefore, total cost = $O(n)$.
- Average over the n operations ⇒ $O(1)$ per operation on average.

Aggregate Analysis – Example 2

- k-bit binary counter $A[0 \,..\, k-1]$ of bits, where $A[0]$ is the least significant bit and $A[k-1]$ is the most significant bit.
- Counts upward from 0.
- Value of counter is: $\sum_{i=0}^{k-1} A[i] \cdot 2^i$ decimal rep of counter
- Initially, counter value is 0, so $A[0 \,..\, k-1] = 0$.
- To increment, add 1 (mod 2^k):

```
Increment(A,k):
i←0
while i<k and A[i]=1 do
    A[i]←0
    i←i+1
if i < k then
    A[i] ← 1
```

increment of counter

Aggregate Analysis – Example 2

Let
k=3

Counter Value	A 2 1 0	cost
0	0 0 0	0
1	0 0 1	1
2	0 1 0	3
3	0 1 1	4
4	1 0 0	7
5	1 0 1	8
6	1 1 0	10
7	1 1 1	11
0	0 0 0	14

We underline the bits we will flip at the next `increment`

3+1 flipped
4+3 flipped
8+1 flipped

Cost of INCREMENT = Θ(# of bits flipped)

Analysis: Each call could flip k bits, so n INCREMENTs takes $O(nk)$ time.

Aggregate Analysis – Example 2

Bit	Flips how often	Time in n INCREMENTs
0	Every time	n
1	½ of the time	floor(*n*/2)
2	¼ of the time	floor(*n*/4)
…	…	
i	$1/2^i$ of the time	floor($n/2^i$)
…	…	
i≥k	Never	0

Thus, total # flips = $\sum_{i=0}^{k-1} \lfloor n/2^i \rfloor < n \cdot \sum_{i=0}^{\infty} 1/2^i = n\left(\frac{1}{1-1/2}\right) = 2 \cdot n$

Therefore, *n* INCREMENTs costs *O*(*n*). – tighter upper bound

Average cost per operation = *O*(1).

Amortized Analysis – Accounting Method

- **Goal:** Show that although some individual operations may be expensive, on average the cost per operation is small.

- 3 methods:
  2. Accounting Method.
     - Assign different charges to different operations.
       - Some are charged more than actual cost.
       - Some are charged less.
     - ***Amortized cost*** = amount we charge.
       - When amortized cost is higher than the actual cost, store the difference *on specific objects* in the data structure as ***credit*** ***(we need to guarantee that the credit never goes negative!!!!!)***. overcharge / saving
       - Use credit later to pay for operations whose actual cost is higher than the amortized cost.
     - 26 hours during the term (13 weeks) . You worked on the assignments only on the week of the due date. Use the credit (sleep - energy) stored in the other weeks to pay (work ~8hours) the week of the due date

Amortized Analysis – Accounting Method

- **Goal:** Show that although some individual operations may be expensive, on average the cost per operation is small.

- 3 methods:
  2. Accounting Method.
     - Differs from aggregate analysis:
       - In the aggregate analysis, all operations have same cost.
       - In accounting method, different operations can have different costs.

result of both methods need to be the same

Amortized Analysis – Accounting Method

Let c_i = cost of actual i[th] operation.

$\hat{c}_i$ = amortized cost of i[th] operation.

upper limit of actual cost

Then require $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$ for all sequences of n operations.

Total credit stored $= \sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0$

credit is always positive

Accounting Method – Example 1

Operation	Actual cost	Amortized cost
PUSH	1	2
POP	1	0
MULTIPOP	min(k,s)	0

assigned cost

save for pop/multipop

Intuition: When pushing an object, pay $2.

- $1 pays for the PUSH.
- $1 is prepayment for it being popped by either POP or MULTIPOP.
- Since each object has $1, which is credit, the credit can never go negative.
- Total amortized cost (= $O(n)$) is an upper bound on total actual cost.

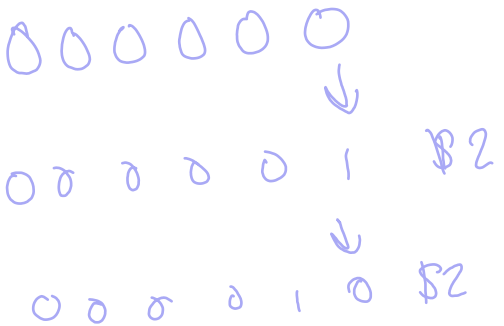
same result as aggregate method

Accounting Method – Example 2

Charge $2 to set a bit to 1.

- $1 pays for setting a bit to 1.
- $1 is prepayment for flipping it back to 0.
- Have $1 of credit for every 1 in the counter.
- Therefore, credit ≥ 0.

$1 to set to 1
$1 to prepay setting to 0



Accounting Method – Example 2

Amortized cost of INCREMENT:

- Cost of resetting bits to 0 is paid by credit.
- At most 1 bit is set to 1.
- Therefore, amortized cost ≤ $2.
- For n operations, amortized cost = $O(n)$.

```
Increment(A,k):
i←0
while i<k and A[i]=1 do
    A[i]←0
    i←i+1
if i < k then
    A[i] ← 1
```

Amortized Analysis – Example 3 – Dynamic Tables

Scenario

- Have a table (maybe a hash table).
- Don't know in advance how many objects will be stored in it.
- When it fills, must reallocate with a larger size, copying all objects into the new, larger table.
- When it gets sufficiently small, *might* want to reallocate with a smaller size.

Goals

1. $O(1)$ amortized time per operation.
2. Unused space always ≤ constant fraction of allocated space.

Load factor α = (# items stored) / (allocated size)

Never allow $\alpha > 1$; Keep $\alpha >$ a constant fraction ⇒ Goal 2.

Dynamic Tables - Expansion

Consider only insertion.

- When the table becomes full, double its size and reinsert all existing items.
- Guarantees that $\alpha \geq ½$.
- Each time we insert an item into the table, it is an ***elementary insertion***.

```
TABLE-INSERT(T,x)
if size[T ]=0
      then allocate table[T] with 1 slot
           size[T]←1
if num[T]=size[T] then
      allocate new-table with 2 · size[T] slots
      insert all items in table[T] into new-table
      free table[T]
      table[T]←new-table
      size[T]←2·size[T]
insert x into table[T]
num[T]←num[T] + 1
```

(Initially, $num[T]=size[T]= 0$)

Dynamic Tables – Aggregate Analysis

- Cost of 1 per elementary insertion.
- Count only elementary insertions (other costs = constant).

c_i = actual cost of i^{th} operation

- If not full, $c_i = 1$.
- If full, have $i-1$ items in the table at the start of the i^{th} operation. Have to copy all $i-1$ existing items, then insert i^{th} item $\Rightarrow c_i = i$.

Naïve: n operations $\Rightarrow c_i = O(n) \Rightarrow O(n^2)$ time for n operations

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is power of } 2 \\ 1 & \text{Otherwise} \end{cases}$$

→ when you need to double size of table

loose upper limit

$$\text{Total cost} = \sum_{i=1}^{n} c_i \le n + \sum_{j=0}^{\lfloor \log n \rfloor} 2^j = n + \frac{2^{\lfloor \log n \rfloor + 1} - 1}{2-1} < n + 2n = 3n$$

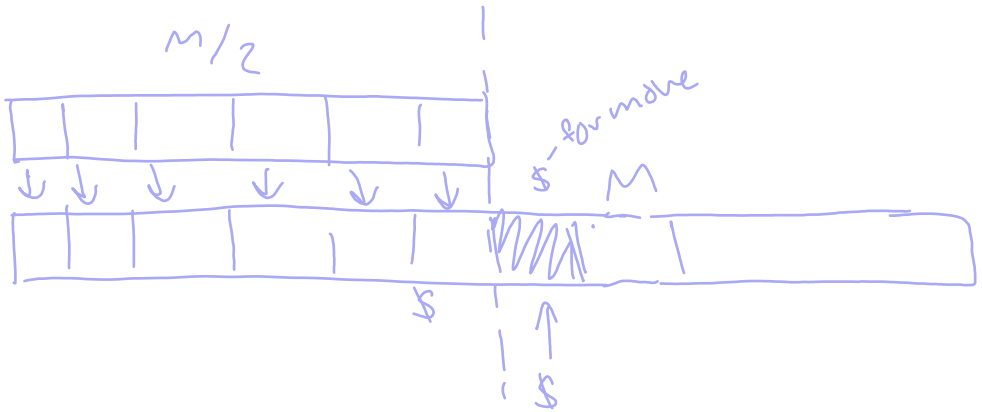
Amortized cost per operation = 3.

Dynamic Tables – Accounting method

Charge $3 per insertion of *x*.

- $1 pays for *x*'s insertion.
- $1 pays for *x* to be moved in the future.
- $1 pays for some other item to be moved.

on the next table growth



once full, all from the original table have a new a prepaid dollar

Dynamic Tables – Accounting method

Charge \$3 per insertion of *x*.

- \$1 pays for *x*'s insertion.
- \$1 pays for *x* to be moved in the future.
- \$1 pays for some other item to be moved.

Prove the credit never goes negative:

- *size*=*m* before and *size*=2*m* after expansion.
- Assume that the expansion used up all the credit, thus that there is no credit available after the expansion.
- We will expand again after another *m* insertions.
- Each insertion will put \$1 on one of the *m* items that were in the table just after expansion and will put \$1 on the item inserted.
- Have \$2*m* of credit by next expansion, when there are 2*m* items to move. Just enough to pay for the expansion…

Outline

- Extras.
  - Amortized Analysis.
  - Randomized algorithms.
  - Probabilistic Analysis.
  - Review Final Exam.


time for questions