COMP 251

Algorithms & Data Structures (Winter 2022)

Graphs – Minimum Spanning Trees

School of Computer Science
McGill University

Slides of (Comp321 ,2021), Langer (2014), Kleinberg & Tardos, 2005 & Cormen et al., 2009, Jaehyun Park' slides CS 97SI, Top-coder tutorials, T-414-AFLV Course, Programming Challenges books, slides from D. Plaisted (UNC) and Comp251-Fall McGill.

Announcements

Outline

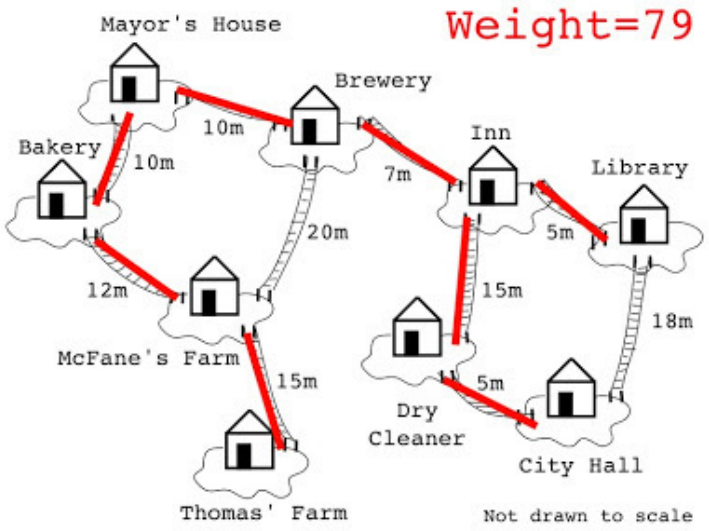
- Graphs.
  - Introduction.
  - Topological Sort. / Strong Connected Components
  - Network Flow 1.
    - Introduction
    - Ford-Fulkerson
  - Network Flow 2.
    - Min-cuts
  - Shortest Path.
  - Minimum Spanning Trees.
  - Bipartite Graphs.

Minimum Spanning Tree- Problem

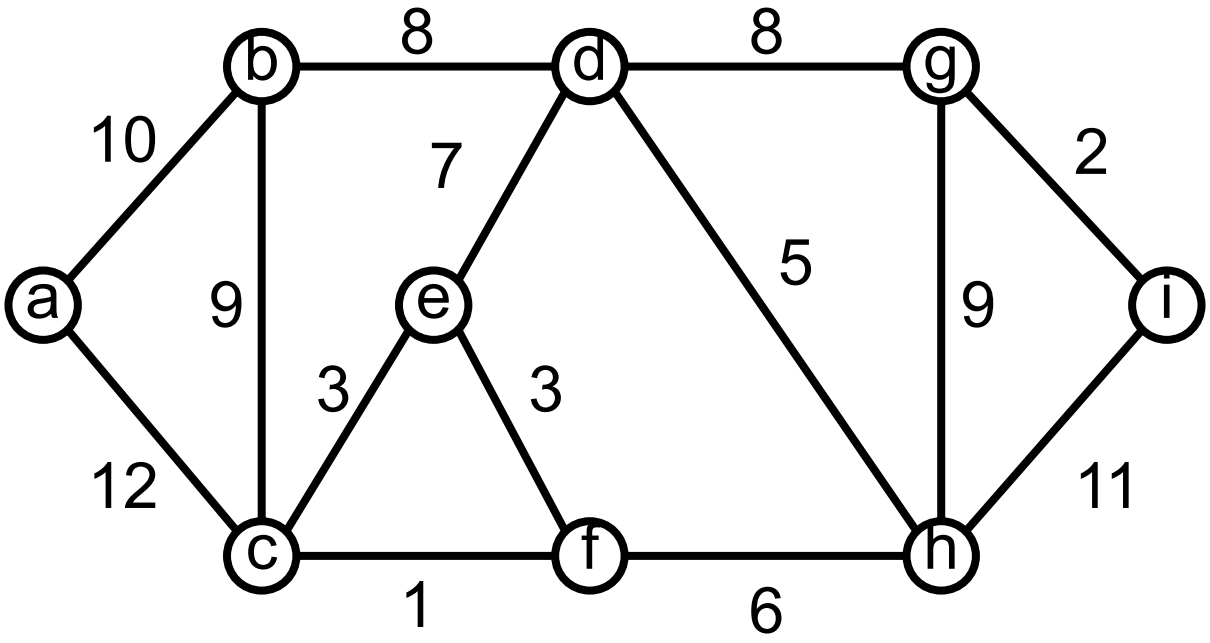
- A town has a set of houses and a set of roads.
- A road connects 2 and only 2 houses.
- A road connecting houses *u* and v has a repair cost w(*u*, v).

Goal: Repair enough (and no more) roads such that:

1. everyone stays connected: can reach every house from all other houses, and
2. total repair cost is minimum.

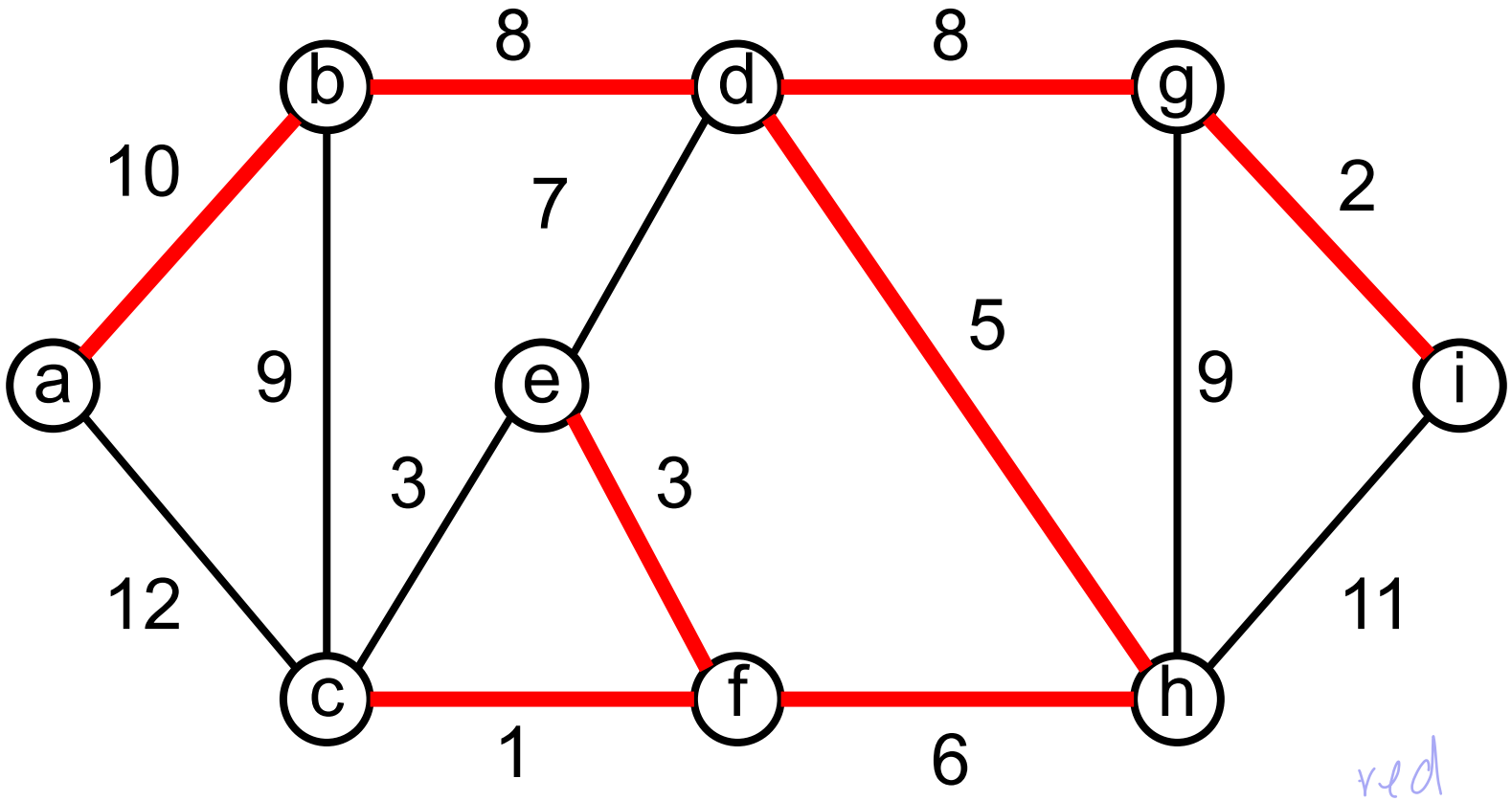


Minimum Spanning Tree – As a graph problem



- Undirected graph $G = (V, E)$.
- ***Weight*** w(u, v) on each edge (u, v) $\in E$.
- Find $T \subseteq E$ such that:
  1. T connects all vertices (T is a ***spanning tree***),
  2. $w(T) = \sum_{(u,v)\in T} w(u,v)$ is minimized.

Minimum Spanning Tree – As a graph problem



- It has $|V| - 1$ edges.
- It has no cycles.
- It might not be unique.

red edges are safe

MST– Generic Algorithm

- Initially, A has no edges.
- Add edges to A and maintain the **loop invariant**: *"A is a subset of some MST"*.

```
A ← ∅;
while A is not a spanning tree do
        find a edge (u, v) that is safe for A;
        A ← A ∪ {(u, v)}
return A
```

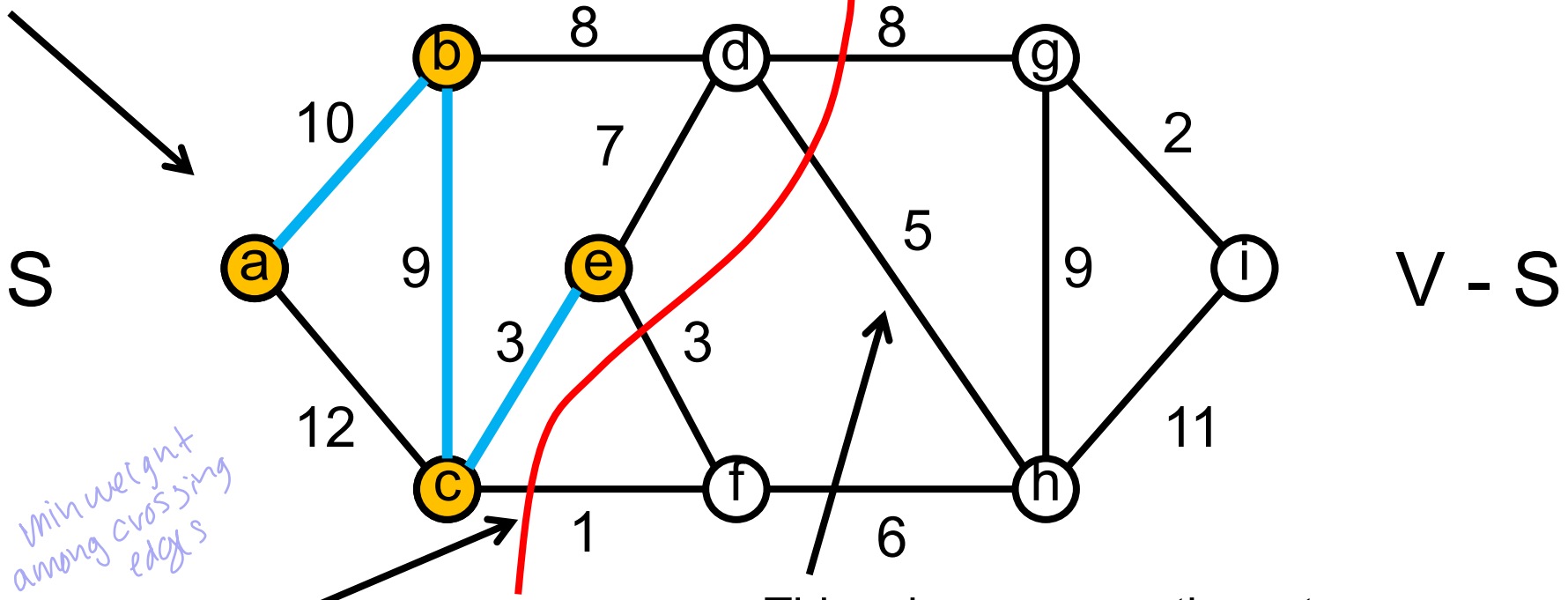
add appropriate edges one at a time

- **Initialization:** The empty set trivially satisfies the loop invariant.
- **Maintenance:** We add only safe edges, *A* remains a subset of some MST.
- **Termination:** All edges added to *A* are in an MST, so when we stop, *A* is a spanning tree that is also an MST.

MST– Definitions

A cut **respects** *A* if and only if no edge in *A* crosses the cut.

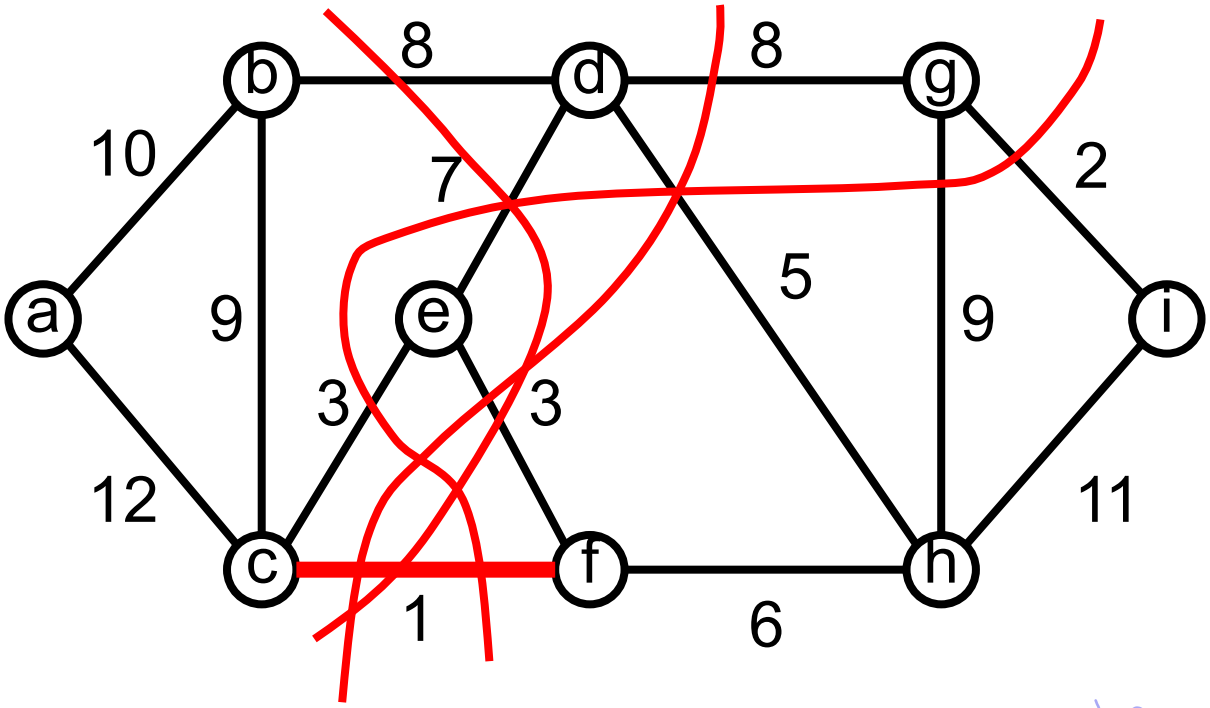
cut partitions vertices into disjoint sets, *S* and *V* – *S*.



A **light** edge crossing cut (may not be unique)

This edge **crosses** the cut. (one endpoint is in *S* and the other is in *V* – *S*.)

MST– What is a safe edge?



Intuitively: Is (c,f) safe when A=∅? - depends on cuts

- Let S be any set of vertices including c but not f.
- There has to be one edge (at least) that connects *S* with *V* − *S*.
- Why not choosing the one with the minimum weight?

MST– Safe edge

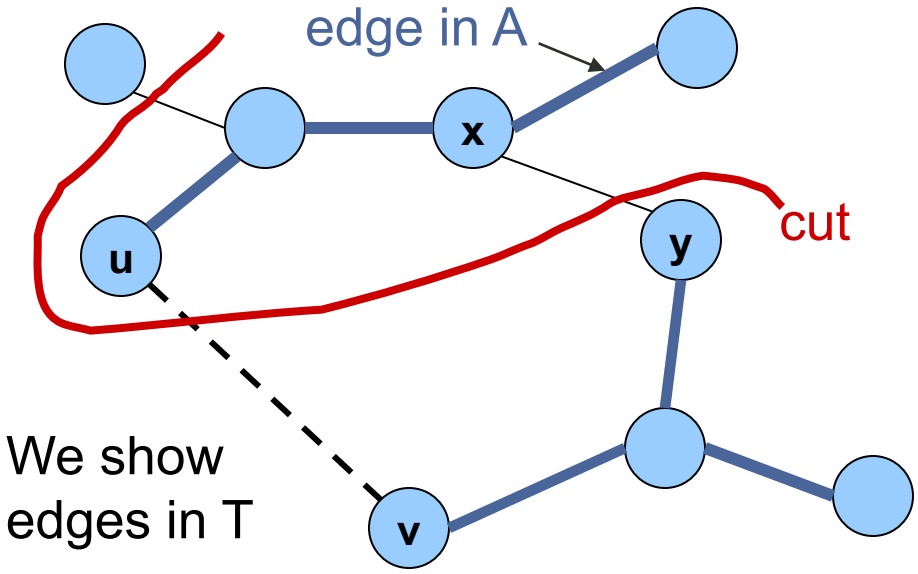
Theorem 1: Let (S, V-S) be any cut that respects A, and let (u, v) be a light edge crossing (S, V-S). Then, (u, v) is safe for A.

Proof:

Let T be a MST that includes A.

Case 1: (u, v) in T. We're done.

Case 2: (u, v) not in T. We have the following:



(x, y) crosses the cut.
Let $T' = T - \{(x, y)\} \cup \{(u, v)\}$.
Because (u, v) is light for cut,
$w(u, v) \leq w(x, y)$. Thus,
$w(T') = w(T) - w(x, y) + w(u, v) \leq w(T)$.
Hence, T´ is also a MST.
So, **(u, v) is safe for A**.

MST– Safe edge

In general, A will consist of several connected components.

Corollary: If (u, v) is a light edge connecting one CC in $G_A=(V, A)$ to another CC in $G_A=(V, A)$, then (u, v) is safe for A.

A ← ∅;

while A is not a spanning tree **do**

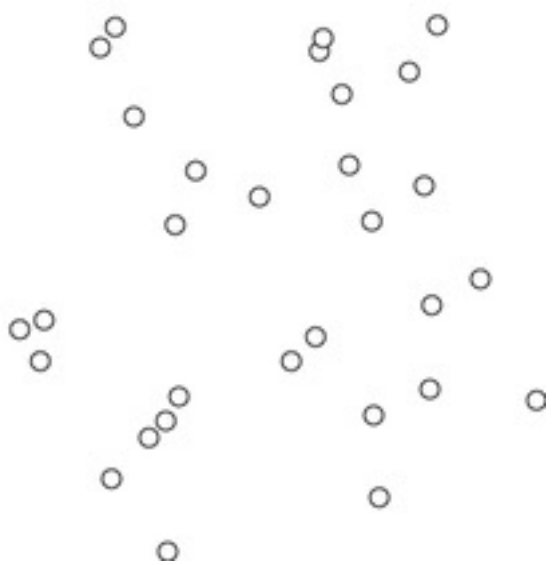
find a edge (u, v) that is safe for A;

A ← A ∪ {(u, v)}

return A

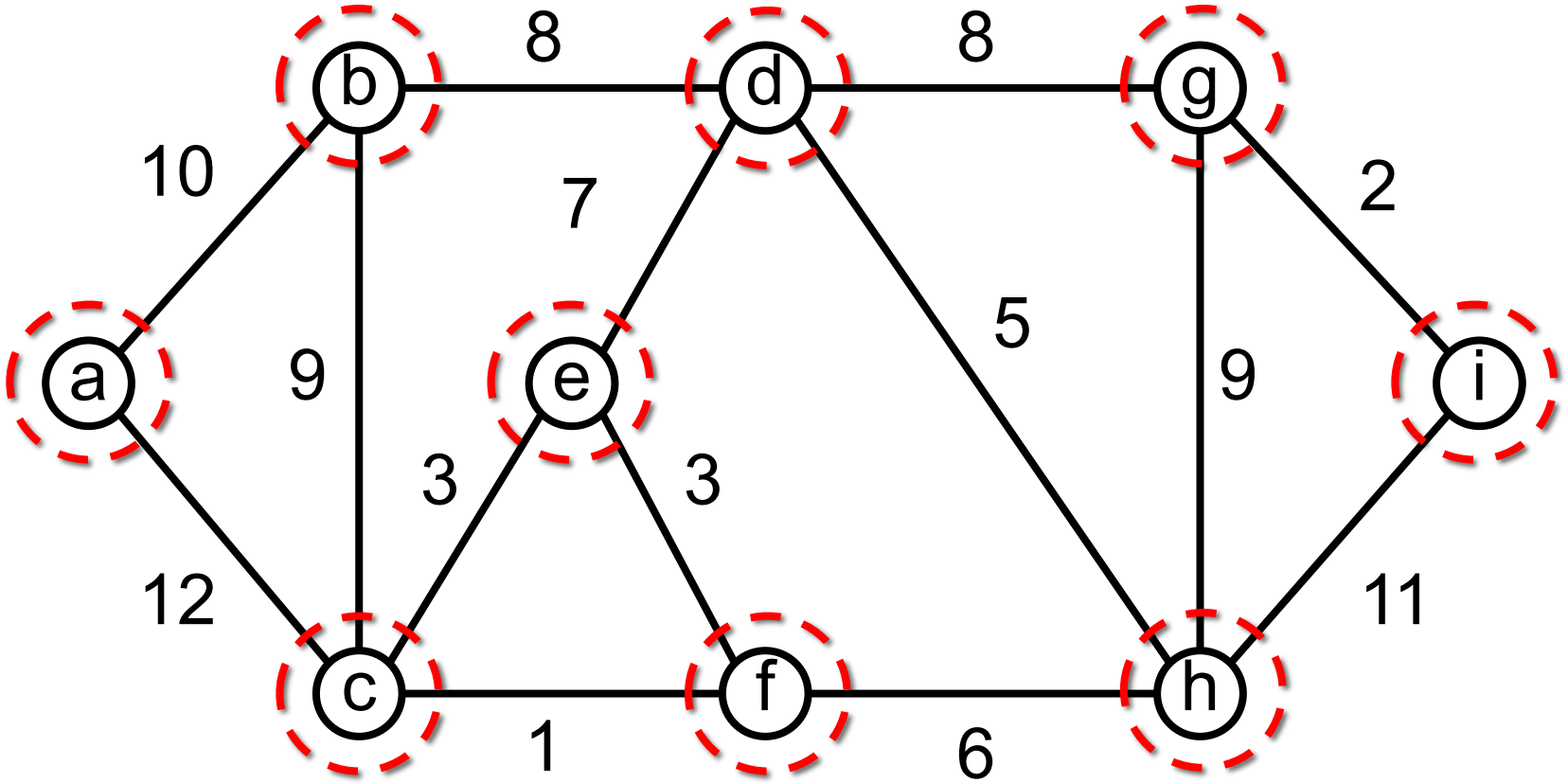
MST– Kruskal's Algorithm

1. Starts with each vertex in its own component.
2. Repeatedly merges two components into one by choosing a light edge that connects them (i.e., a light edge crossing the cut between them).
3. Scans the set of edges in monotonically increasing order by weight.
4. Uses a **disjoint-set data structure** to determine whether an edge connects vertices in different components.

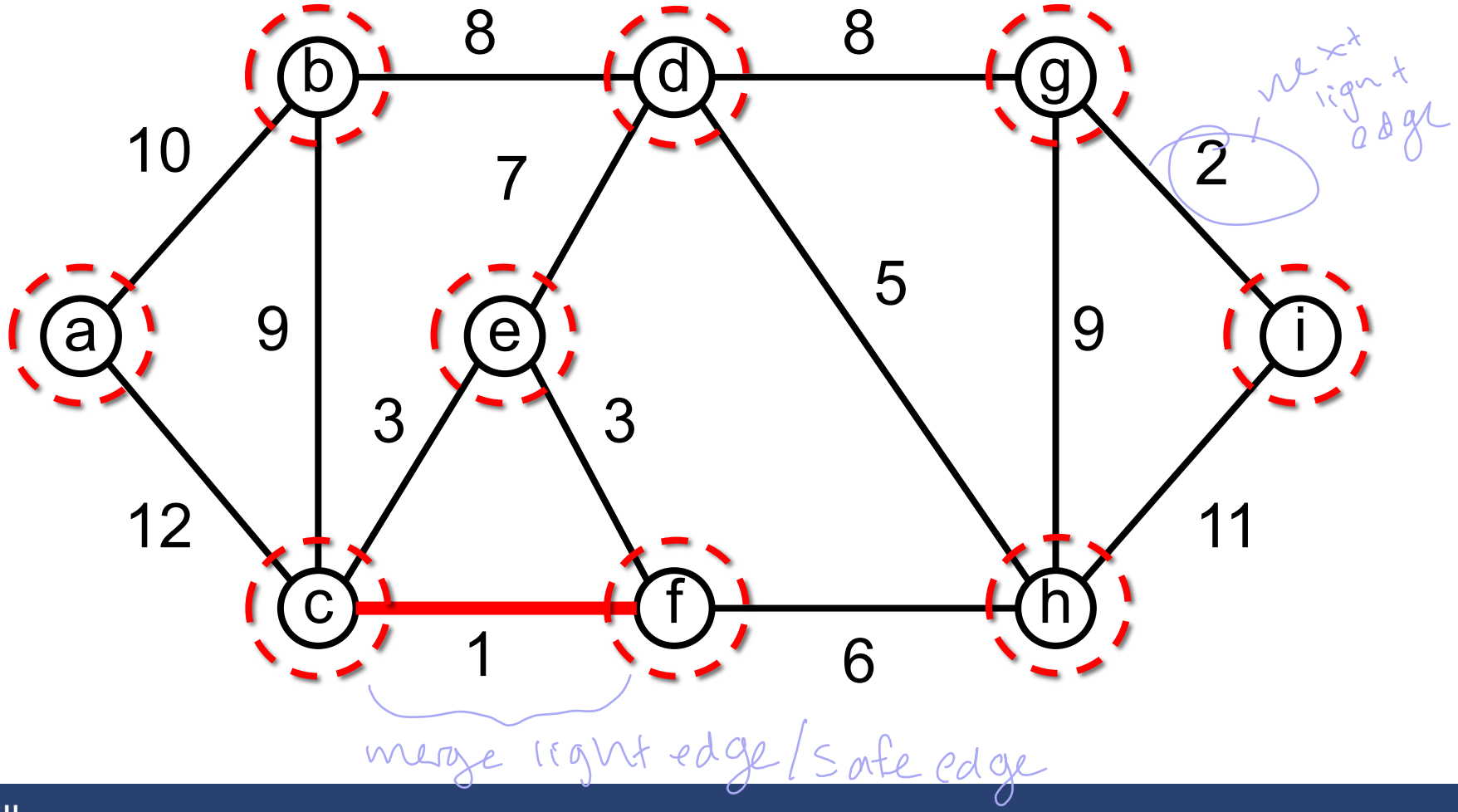


Kruskal's Algorithm - Example

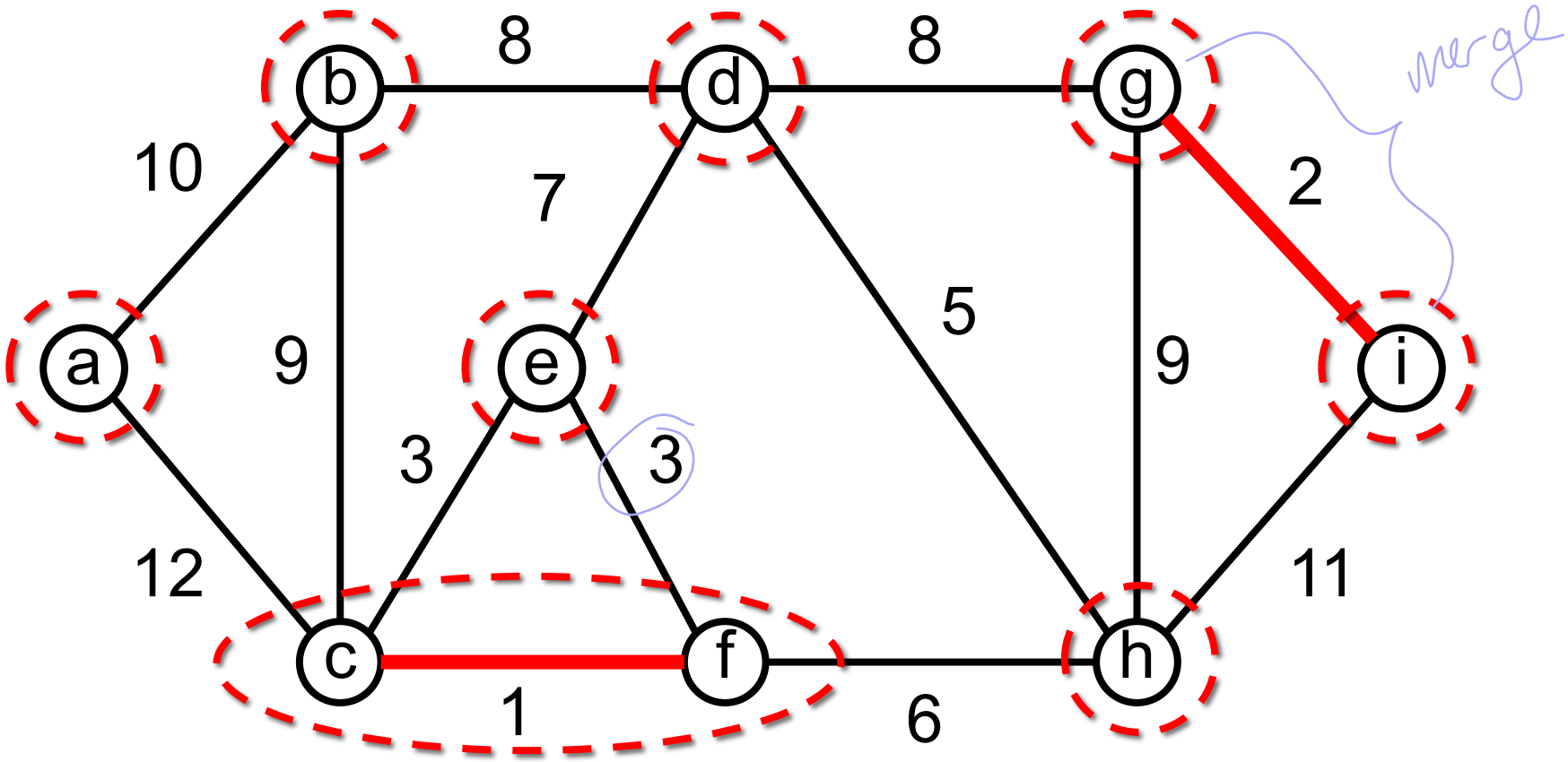
each vertex an individual set



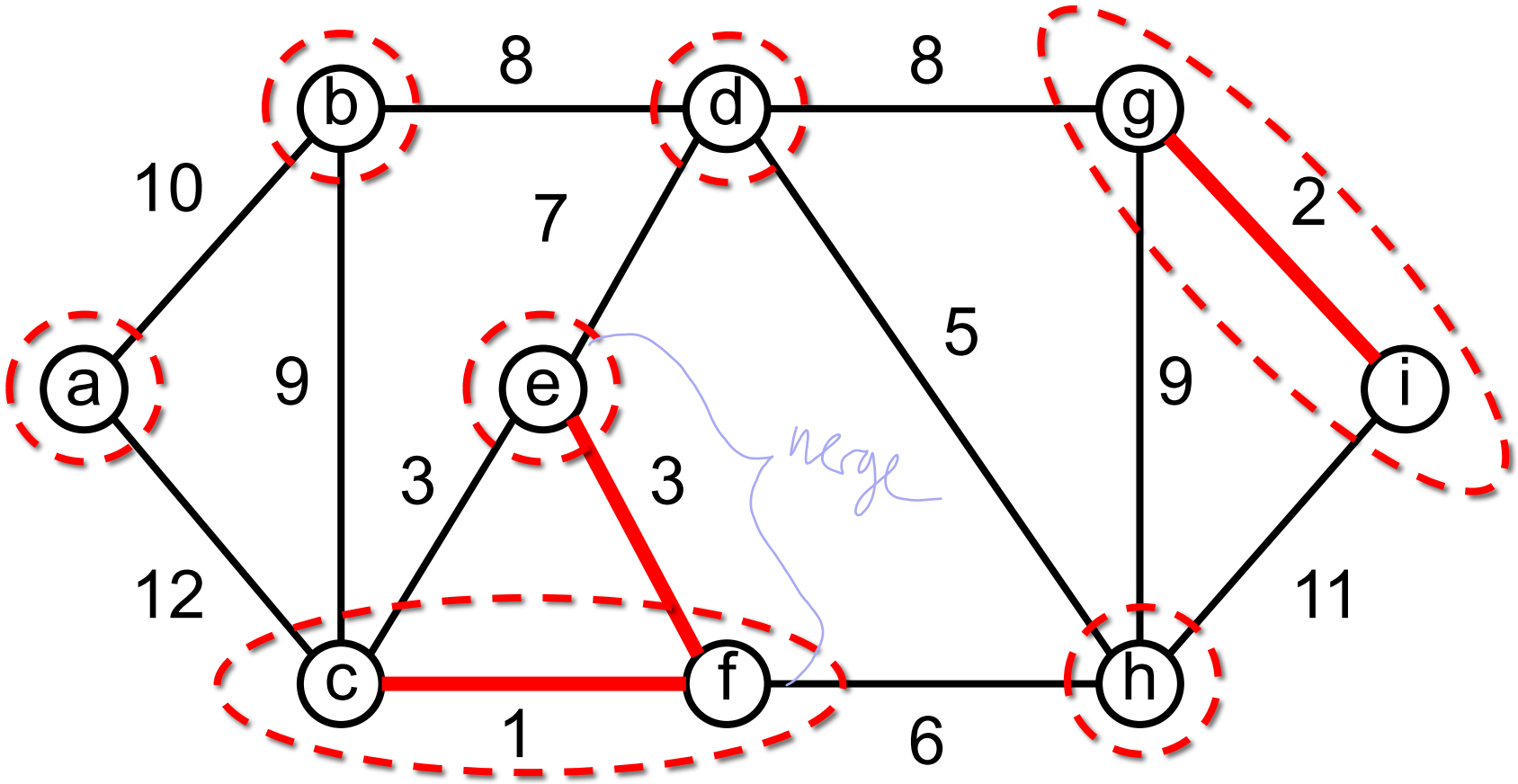
Kruskal's Algorithm - Example



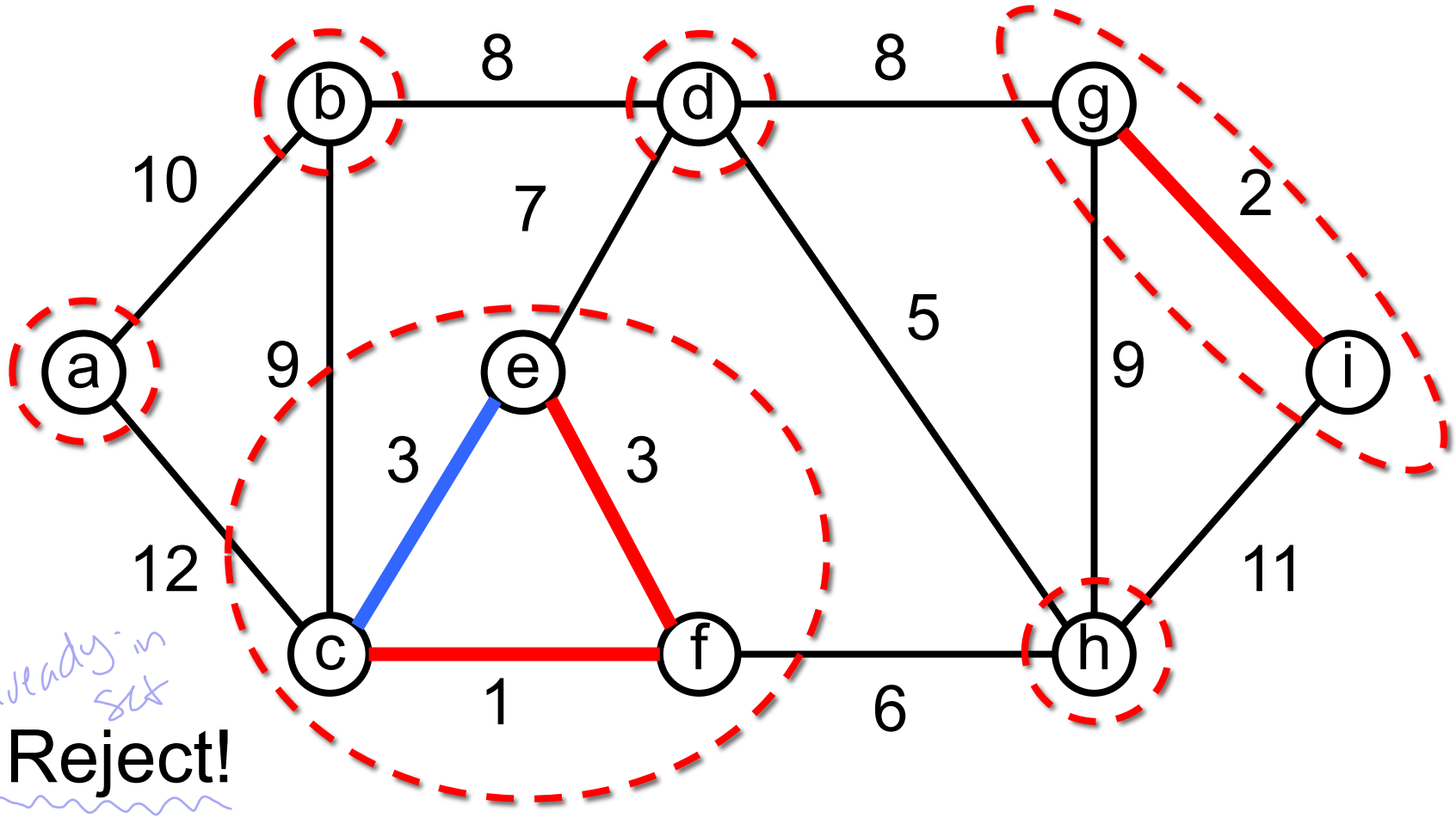
Kruskal's Algorithm - Example



Kruskal's Algorithm - Example



Kruskal's Algorithm - Example



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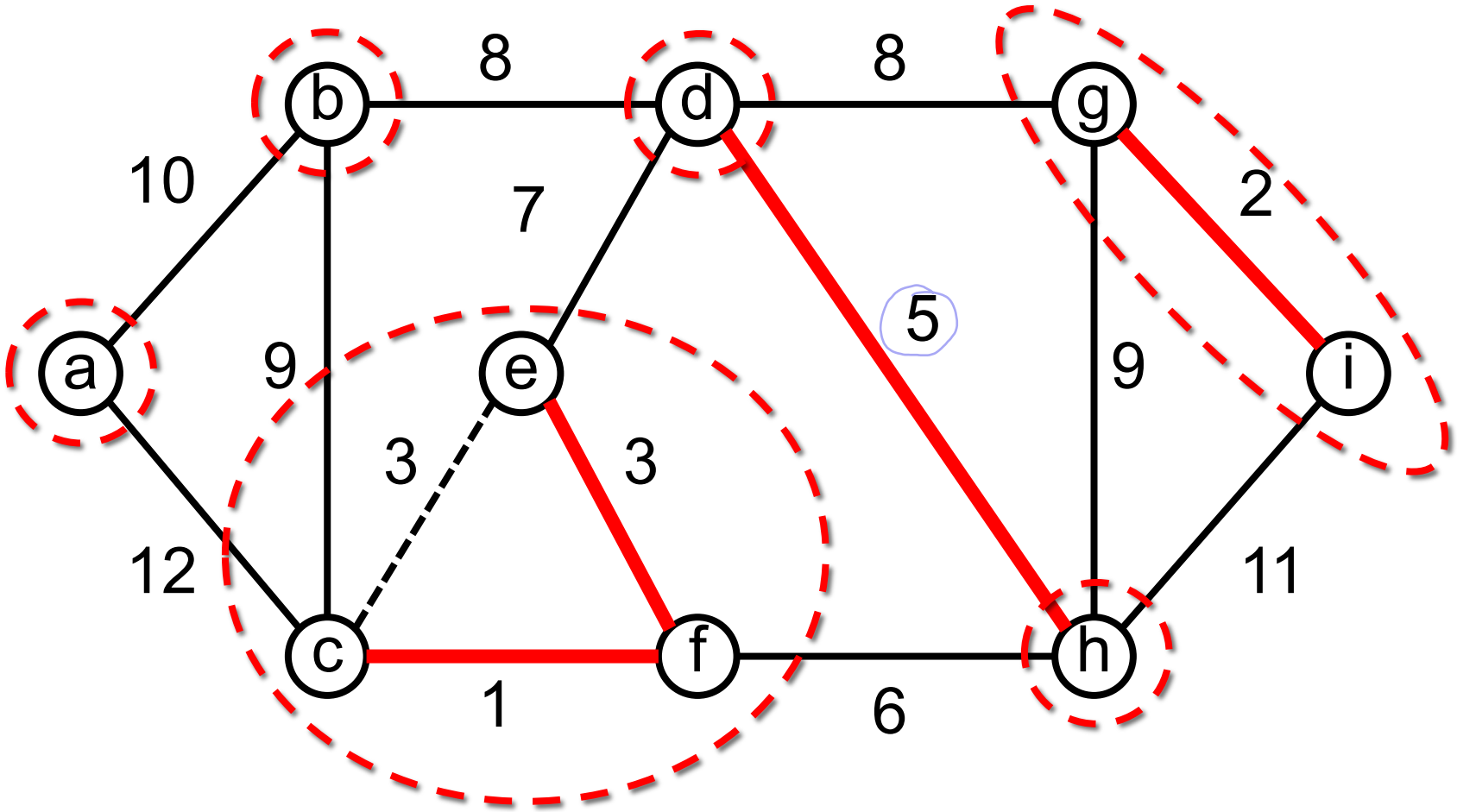
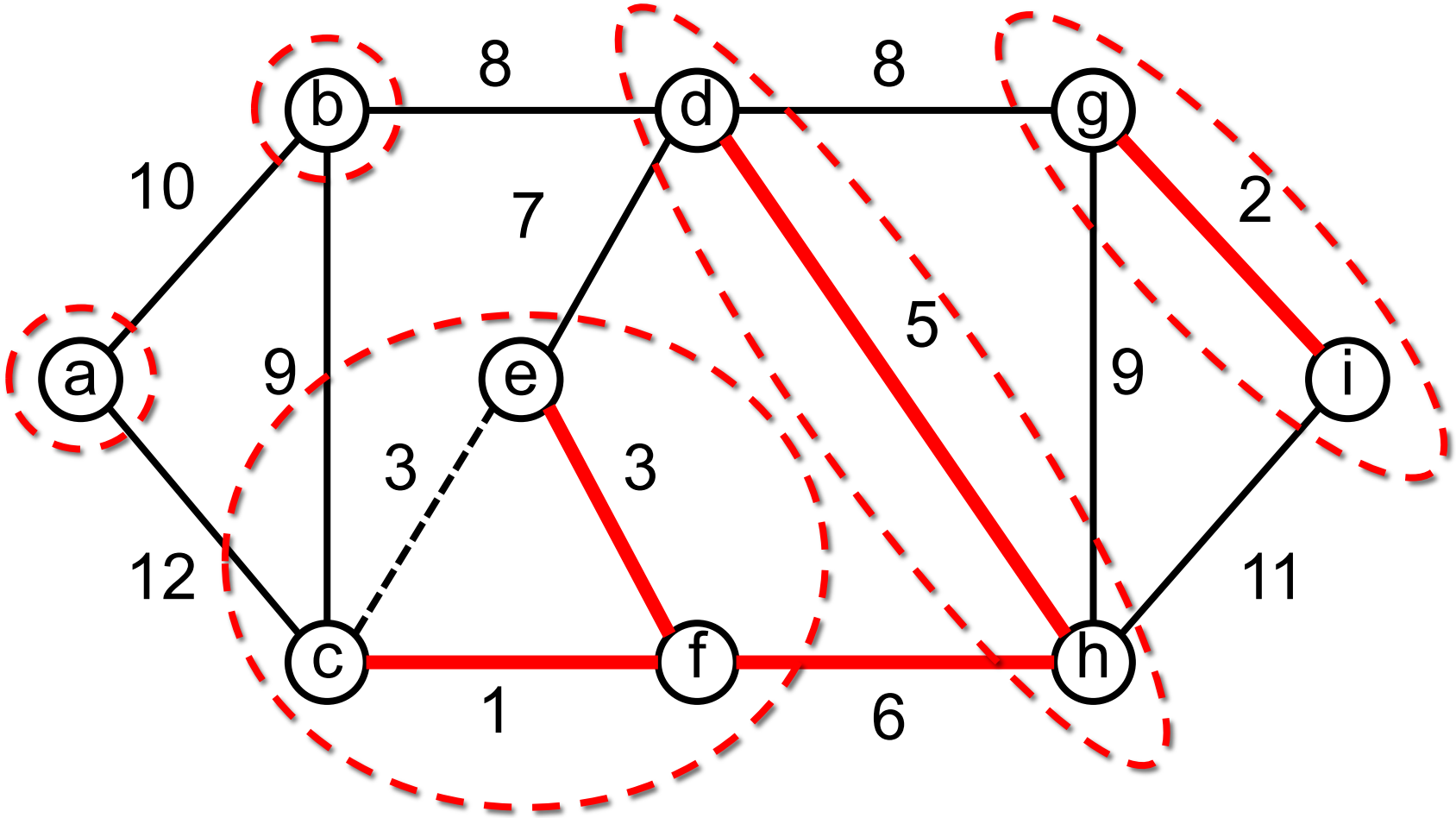
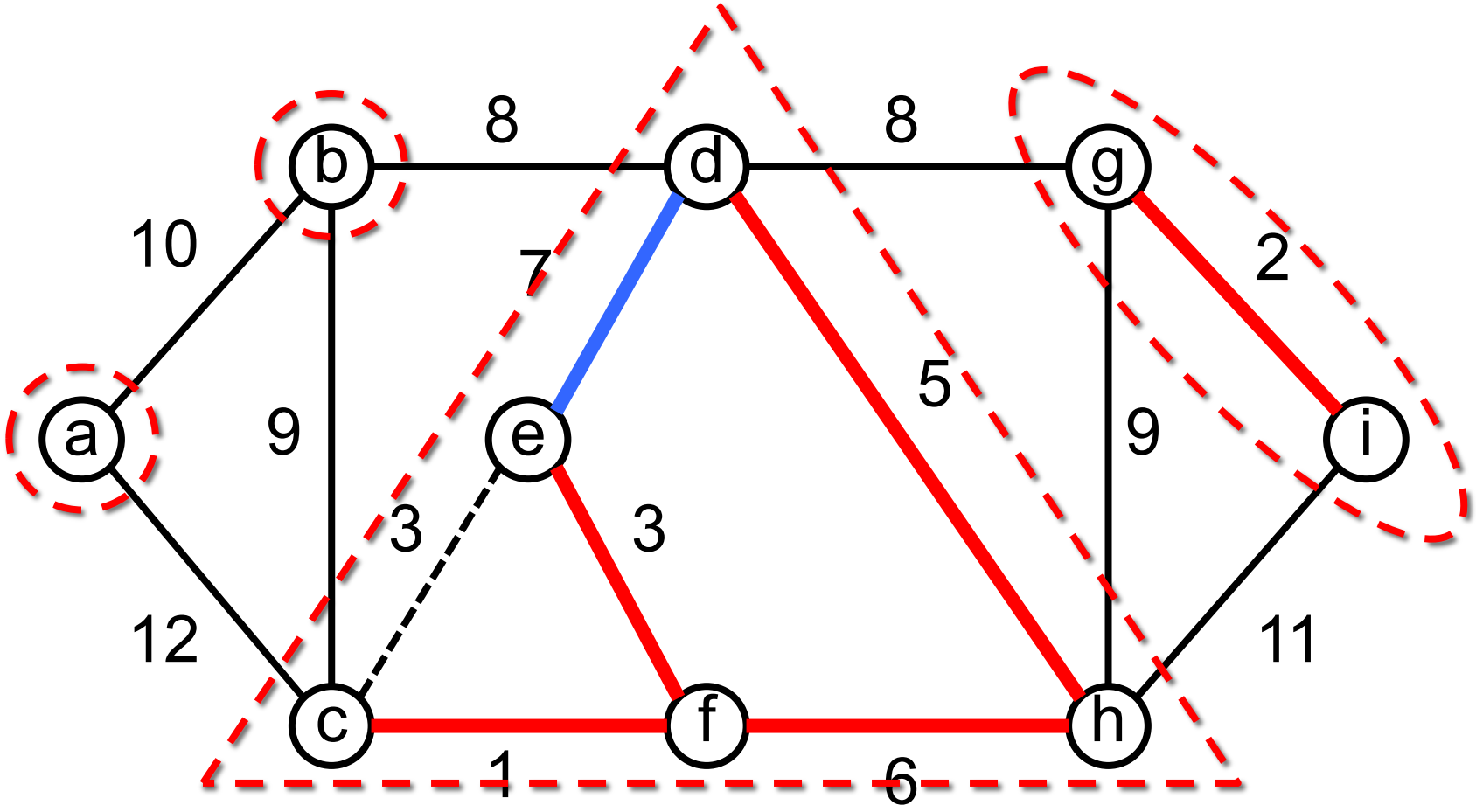


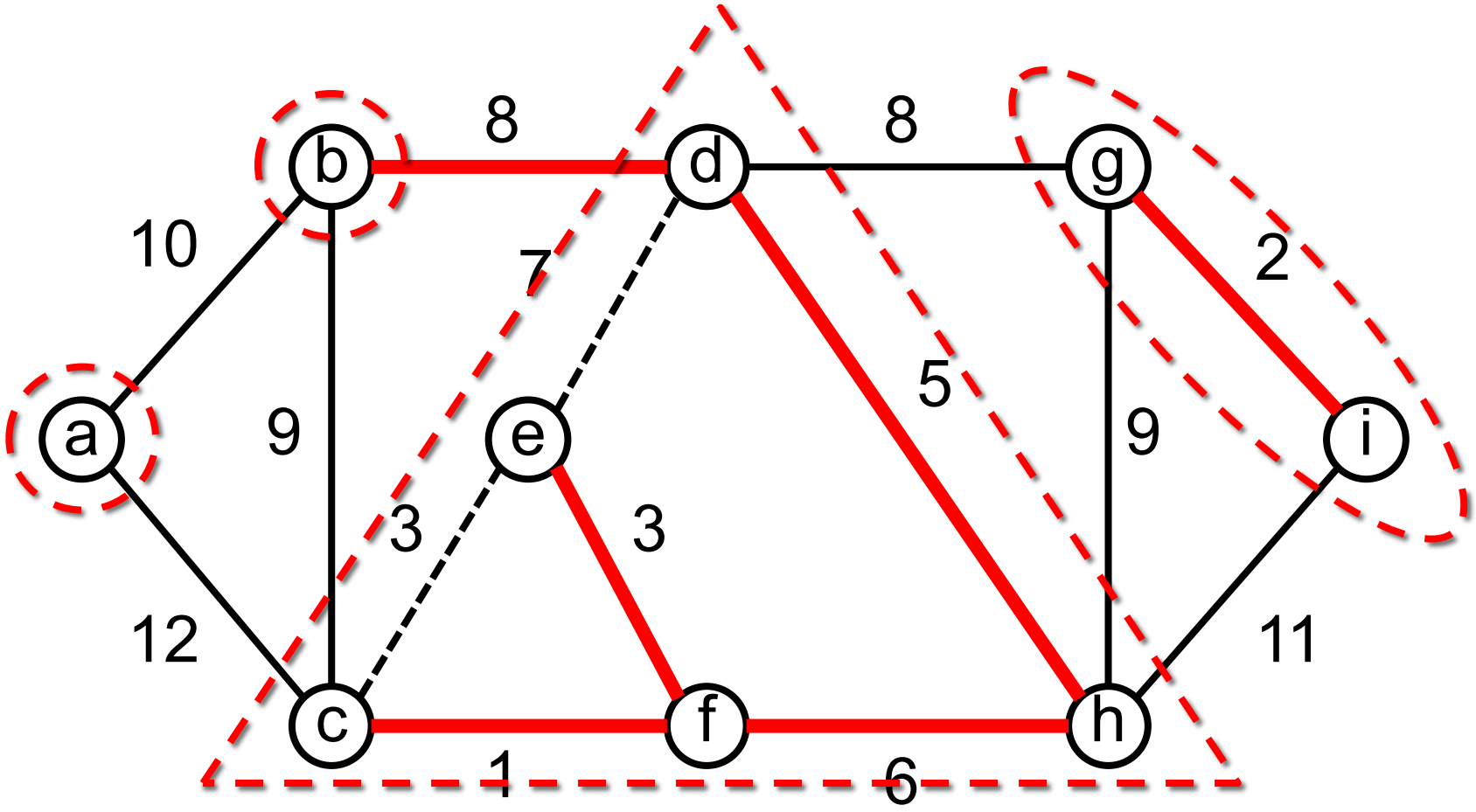
Kruskal's Algorithm - Example



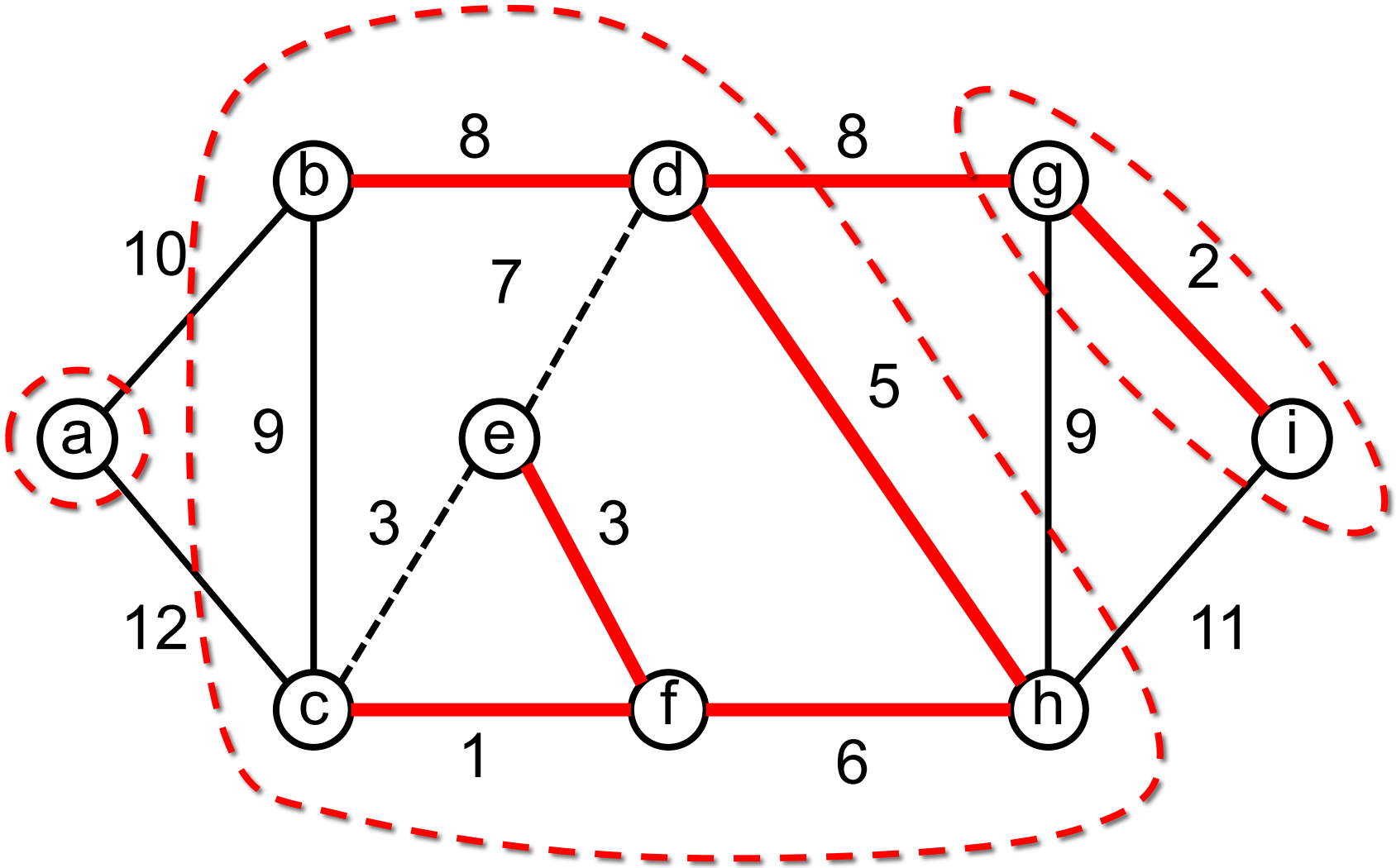
Kruskal's Algorithm - Example



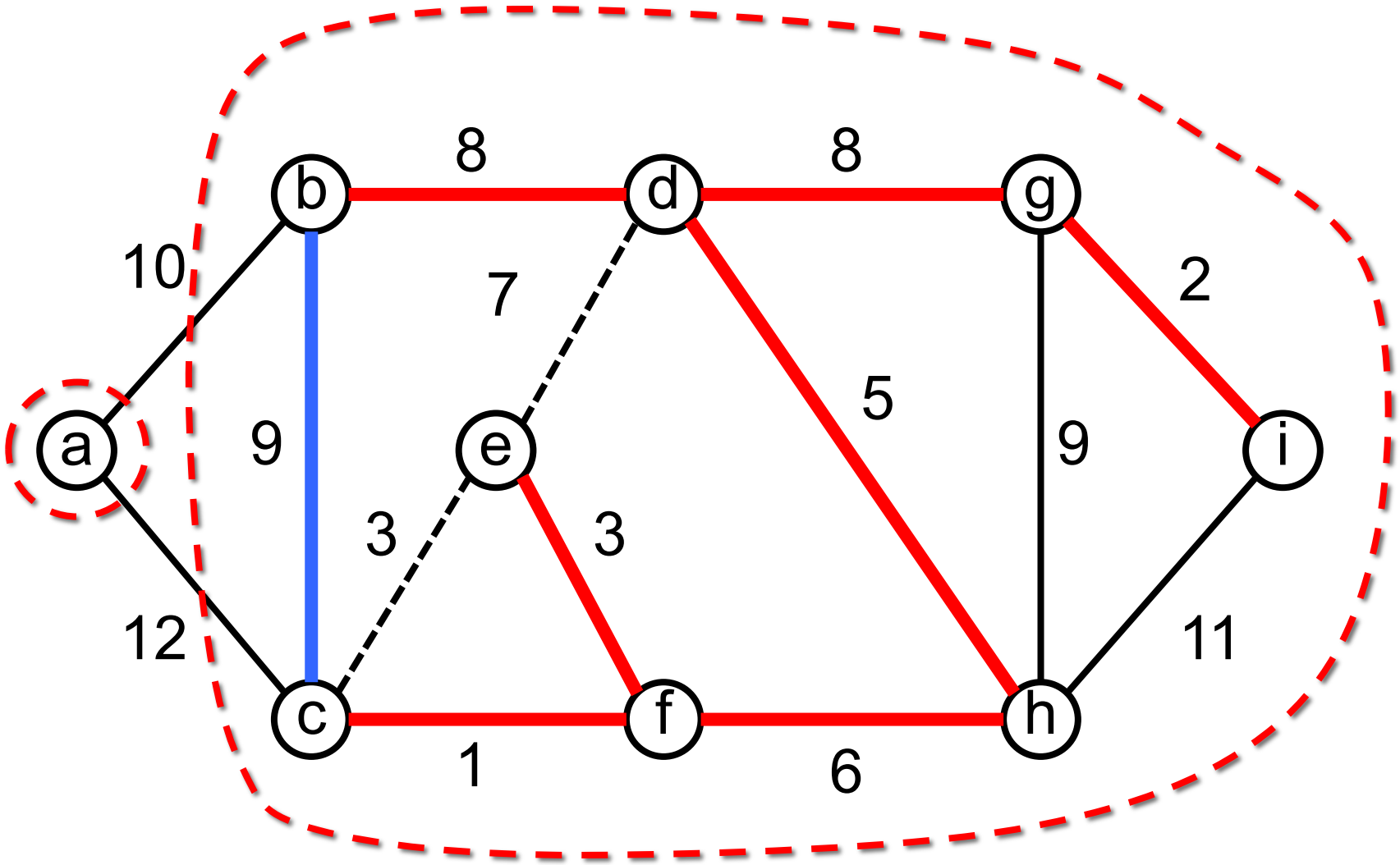
Kruskal's Algorithm - Example



Kruskal's Algorithm - Example



Kruskal's Algorithm - Example



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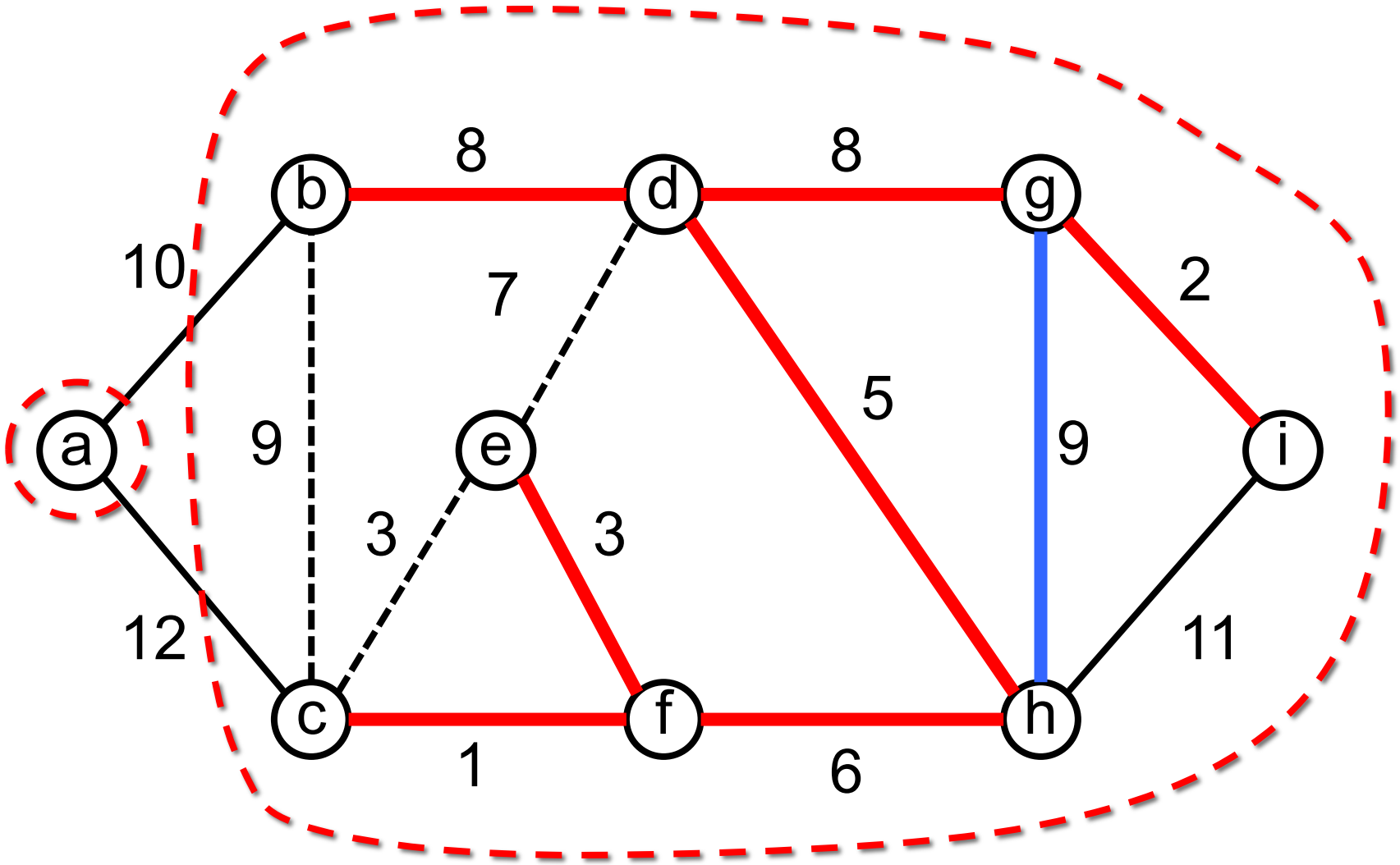

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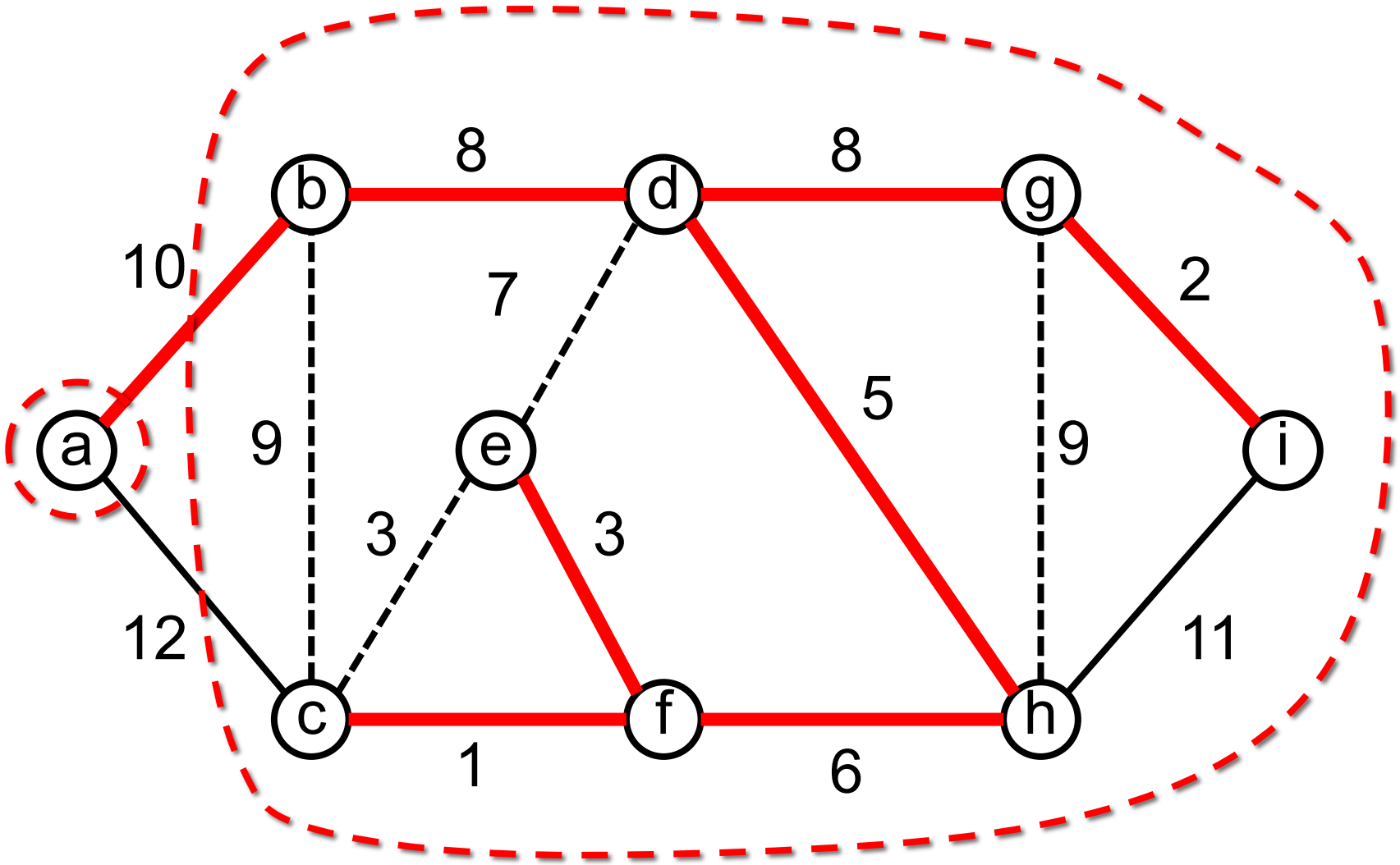
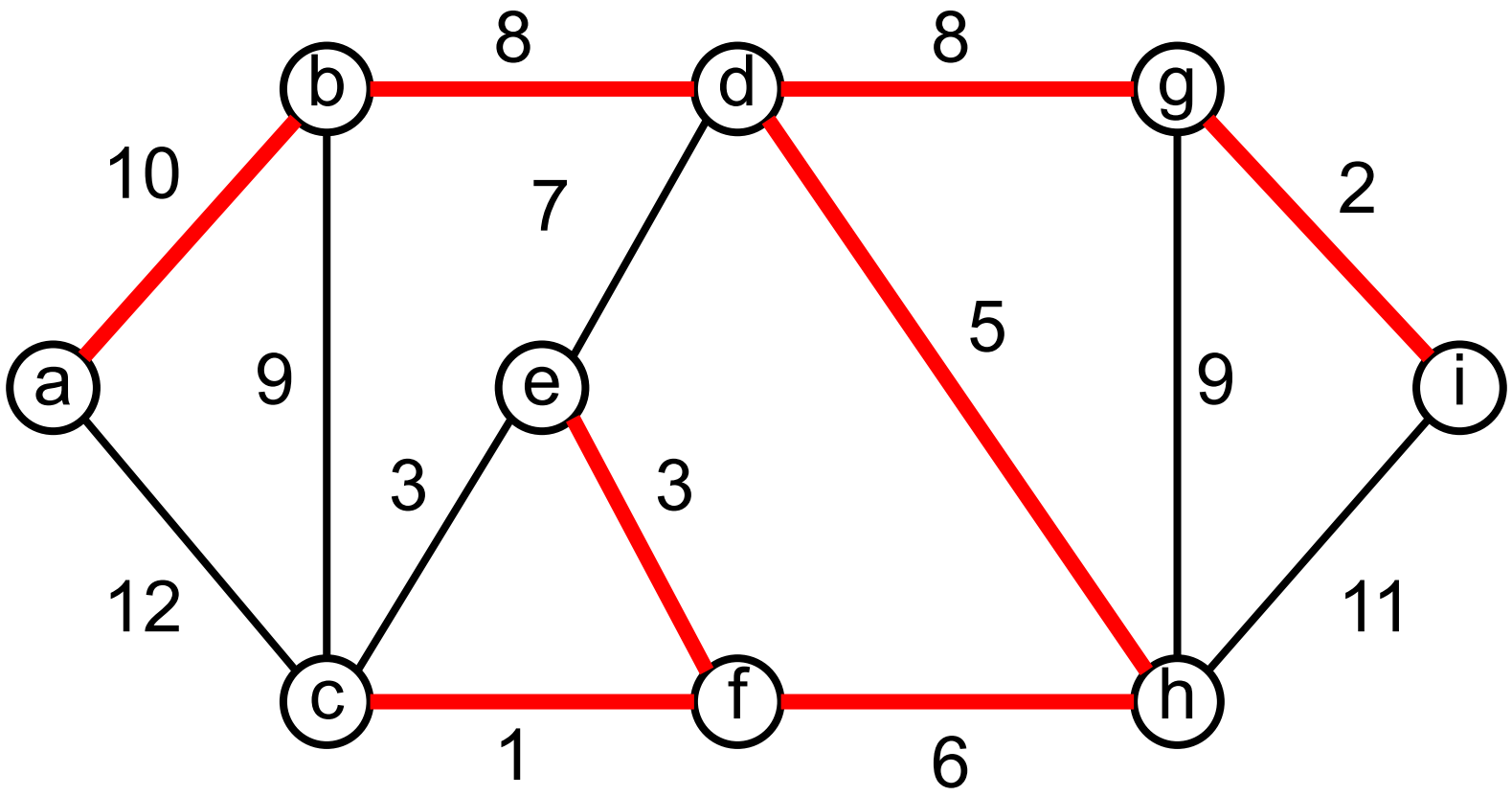


Kruskal's Algorithm - Example



Kruskal's Algorithm - Complexity

```
MST-KRUSKAL(G, w)
1  A = ∅
2  for each vertex v ∈ G.V
3      MAKE-SET(v)
4  sort the edges of G.E into nondecreasing order by weight w
5  for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
6      if FIND-SET(u) ≠ FIND-SET(v)
7          A = A ∪ {(u, v)}
8          UNION(u, v)
9  return A
```

each node an individual set

1	O(1)
2-3	O(v) MAKE-SETs
4	O(E lg E)
5-8	O(E) FIND-SETs and UNIONs

Kruskal's Algorithm - Complexity

- Initialize A: $O(1)$
- First **for** loop: $|V|$ MAKE-SETs
- Sort E: $O(E \lg E)$
- Second **for** loop: $O(E)$ FIND-SETs and UNIONs

Assuming union by rank and path compression, m find/union operations on a set with n objects is $O(m \cdot \alpha(n))$:

$$\Longrightarrow O\big(E \cdot \alpha(V)\big) + O(E \cdot \log(E))$$

Moreover, $\alpha(V) = O(\log V) = O(\log E)\,; (|E| \geq |V| - 1)$

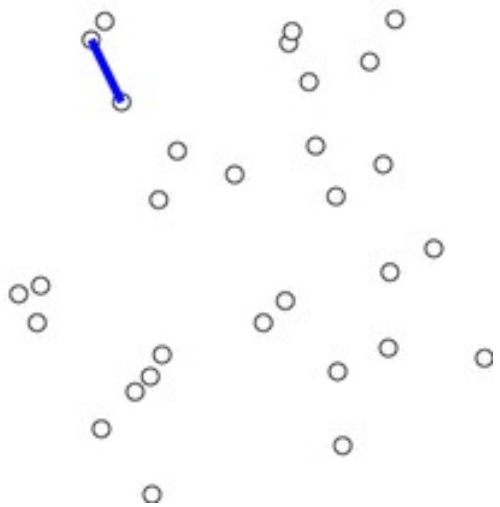
$$\Longrightarrow O(E \cdot \log(E))$$

Since, $|E| \leq |V|^2 \Longrightarrow \log|E| = O(2 \log V) = O(\log V)$

$$\Longrightarrow O(E \cdot \log V)$$

MST– Prim's Algorithm

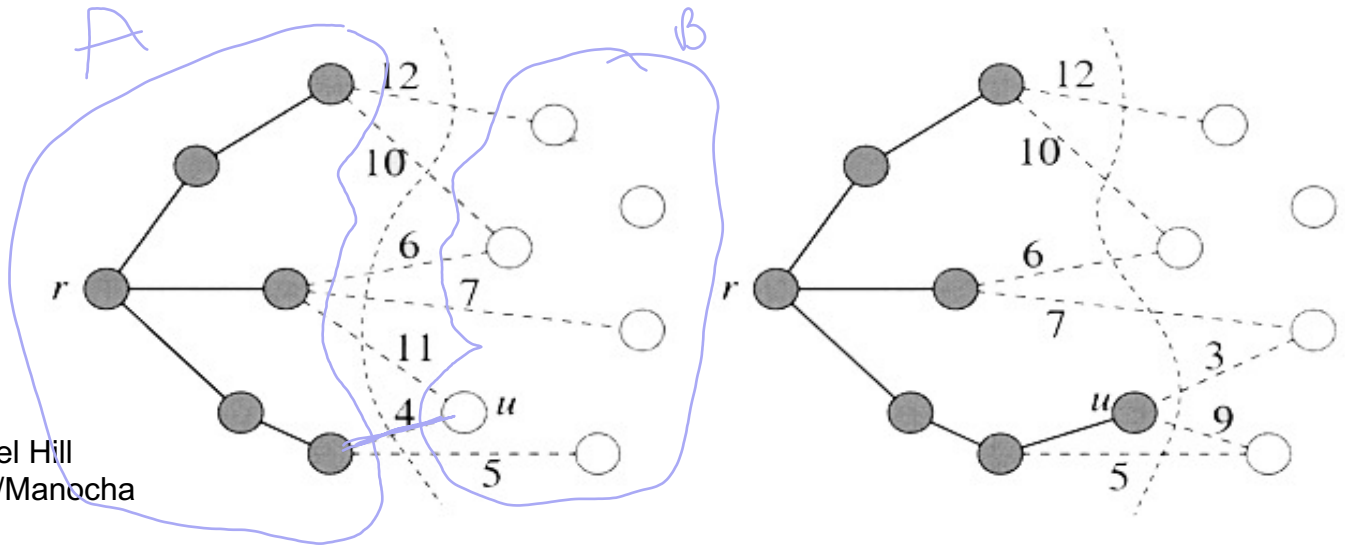
1. Builds **one tree**, so A is always a tree.
2. Starts from an arbitrary "root" r .
3. At each step, **adds a light edge** crossing cut $(V_A, V - V_A)$ to A.
   - Where V_A = vertices that A is incident on.



Differs from Kruskal's in that we grow a single supernode (tree) A instead of growing multiple ones (forest) at the same time

MST– Prim's Algorithm - Intuition

- Consider the set of vertices S currently part of the tree, and its complement (V-S). We have a cut of the graph and the current set of tree edges A is respected by this cut.
- Which edge should we add next? *Light edge!*



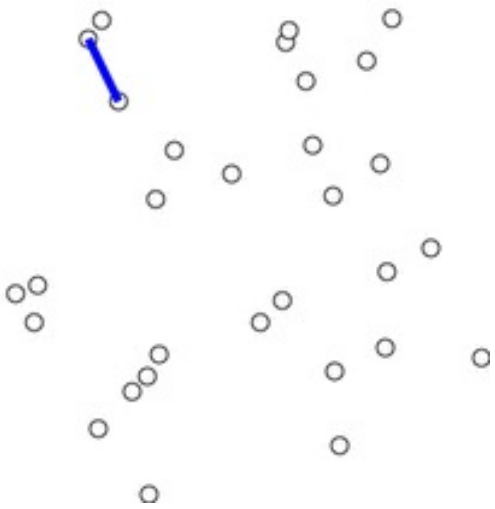
UNC Chapel Hill
Lin/Foskey/Manocha

Prim's Algorithm – Finding a light edge

1. Uses a **priority queue *Q*** to find a light edge quickly.
2. Each object in *Q* is a vertex in $V - V_A$.
3. Key of *v* has minimum weight of any edge (*u, v*), where $u \in V_A$.
4. Then the vertex returned by Extract-Min is *v* such that there exists $u \in V_A$ and (*u, v*) is light edge crossing (V_A, $V - V_A$).
5. Key of *v* is ∞ if *v* is not adjacent to any vertex in V_A.

Prim's Algorithm – Basics

- It works by adding leaves one at a time to the current tree.
  - Start with the root vertex r (it can be any vertex). At any time, the subset of edges A forms a single tree. S = vertices of A.
  - At each step, a light edge connecting a vertex in S to a vertex in V- S is added to the tree.
  - The tree grows until it spans all the vertices in V.
- Implementation Issues:
  - How to update the cut efficiently?
  - How to determine the light edge quickly?



Prim's Algorithm – Implementation – Priority Queue

- Priority queue implemented using heap can support the following operations in $O(lg\ n)$ time:
  - Insert (Q, u, key): Insert u with the key value key in Q
  - u = Extract_Min(Q): Extract the item with minimum key value in Q
  - Decrease_Key(Q, u, new_key): Decrease the value of u's key value to new_key
- All the vertices that are *not* in S reside in a priority queue Q based on a key field. When the algorithm terminates, Q is empty.

Prim's Algorithm - faster but harder to implement

```
Q := V[G];
for each u ∈ Q do
    key[u] := ∞
    π[u] := Nil;

Decrease-Key(Q,r,0);
while Q ≠ ∅ do
    u := Extract-Min(Q);
    for each v ∈ Adj[u] do
        if v ∈ Q ∧ w(u, v) < key[v] :
            π[v] := u;
            Decrease-Key(Q,v,w(u,v));
```

add vertexes to the Q

initialize

make root 0 to min

set to weight

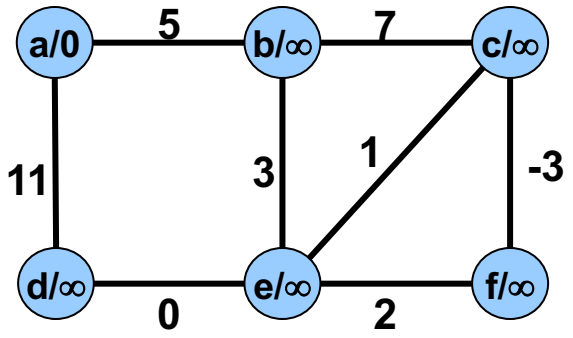
Complexity:

Using binary heaps: O(E lg V).
- Building initial queue: O(V).
- Initialization: O(V).
- V Extract-Min: O(V lgV).
- E Decrease-Key: O(E lg V).

Using Fibonacci heaps:
O(E + V lg V).

Notes: (i) r is the root.

Prim's Algorithm - Example

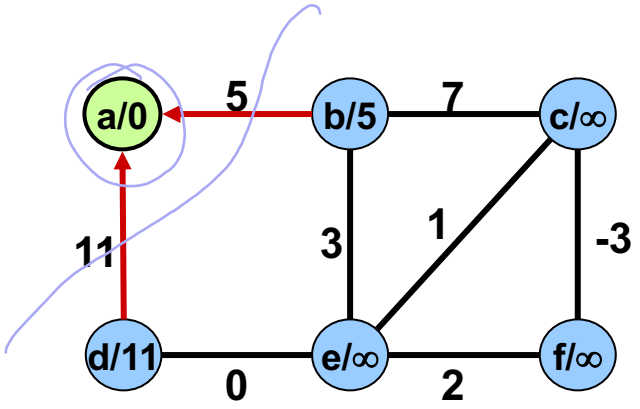


Not in tree

Q = a b c d e f

0 ∞ ∞ ∞ ∞ ∞

Prim's Algorithm - Example

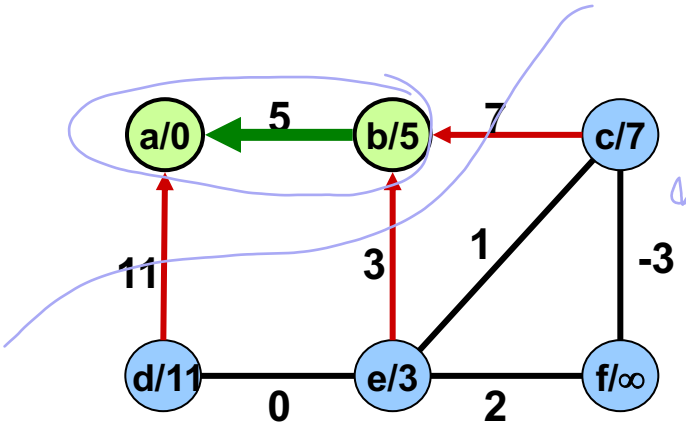


Q = b d c e f

5 11 ∞ ∞ ∞

min

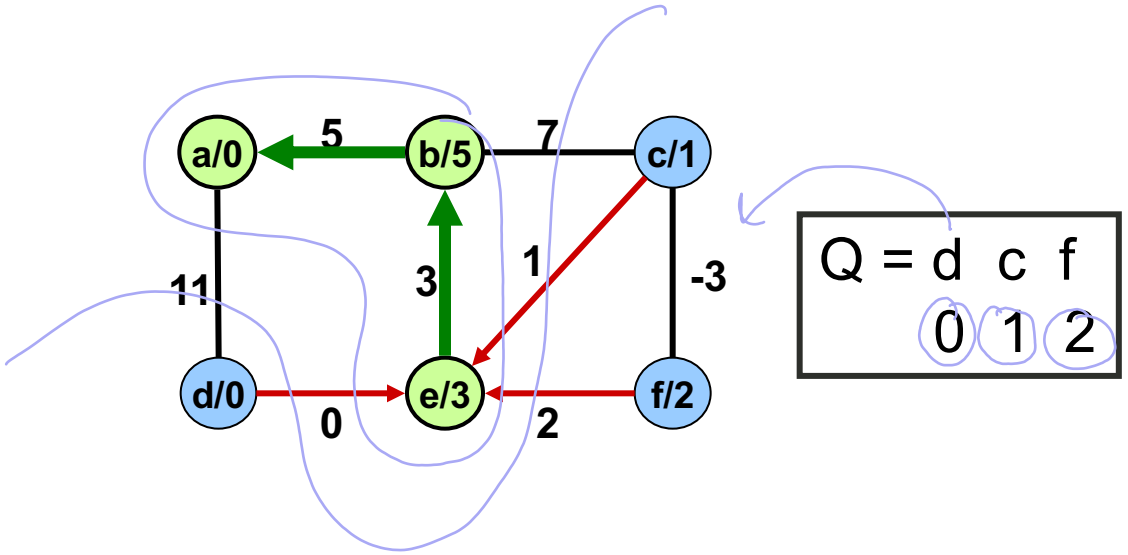
Prim's Algorithm - Example



Q = e c d f
3 7 11 ∞

extract min weight edge because it is safe

Prim's Algorithm - Example



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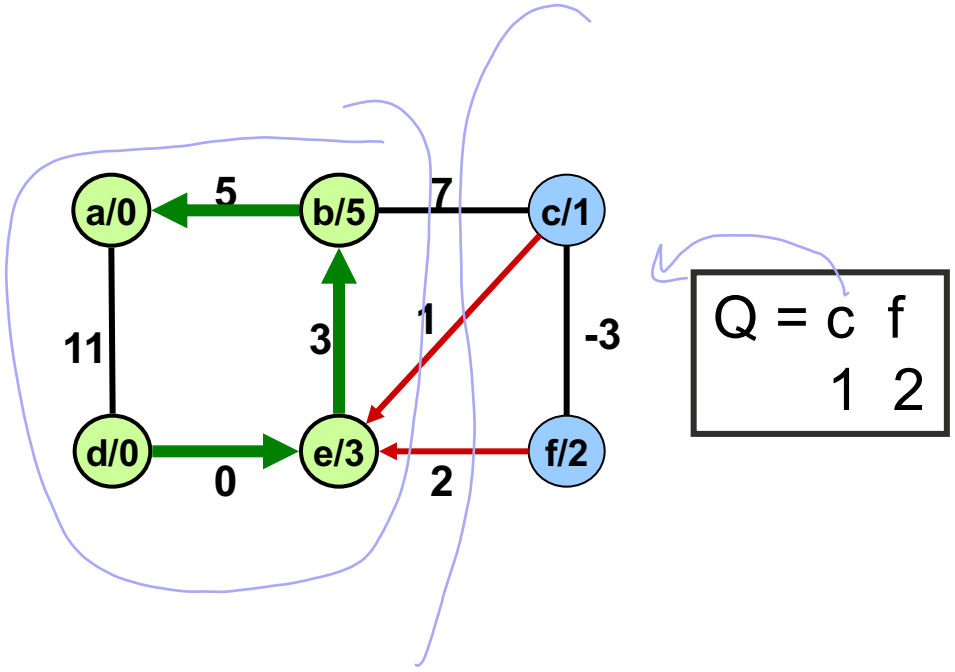

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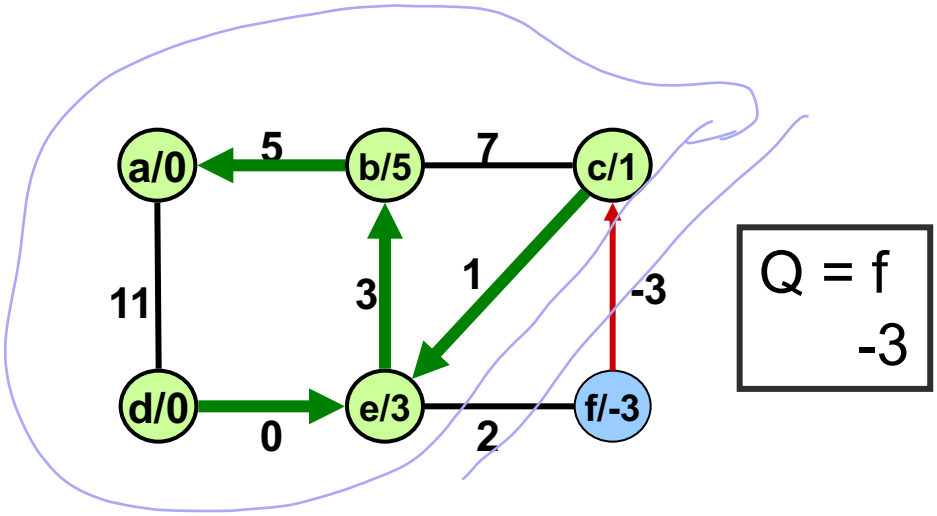

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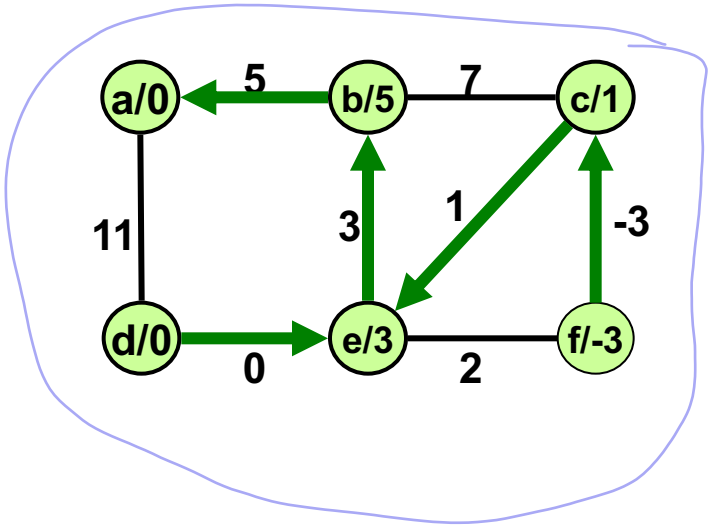
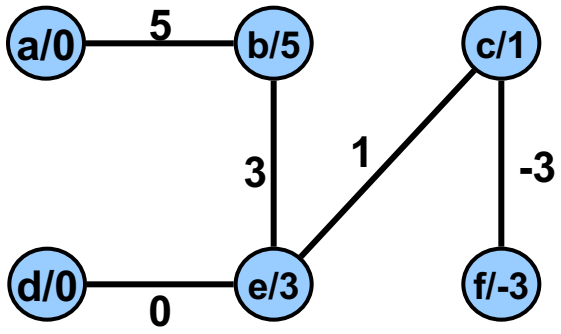


Q = ∅

Prim's Algorithm - Example



Prim's Algorithm - Correctness

same proof

- Again, show that every edge added is a safe edge for A
- Assume (u, v) is next edge to be added to A.
- Consider the cut $(A, V\text{-}A)$.
  - This cut respects A
  - and (u, v) is the light edge across the cut
- Thus, by the Theorem 1, (u,v) is safe.

based in selecting the light / safe edge

Prim's Algorithm – Time complexity

- Initialization: $O(V)$
- Extract the light edge from the queue: $\mathrm{O}(\mathrm{V}\log V)$
- Relax the neighbour edges: $\mathrm{O}(E\log V)$

$\Longrightarrow O(V\log V + E\log V) = O(E\log V)$ // same as Kruskall

Note: Using Fibonacci heaps, we can obtain $O(E + V\log V)$.

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  - Network Flow 1.
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  - Network Flow 2.
    - Min-cuts
  - Shortest Path.
  - Minimum Spanning Trees.
  - Bipartite Graphs.


time for questions