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In[ ]:= ClearAll["Global`*"]
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Non-Commutative Kirillov Formula - SU(2)

We use non-commutative Kirillov Formula to calculate π_λ of SU(2) for highest weight $\lambda = 1, 2, 3$.

A matrix of SU(2) has representation

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \text{ and } |\alpha|^2 + |\beta|^2 = 1$$

The orthonormal basis of the irreducible unitary representations of SU (2) is given by the homogeneous polynomials of two complex variables (α, β), that is

$$\sqrt{\frac{(\lambda+1)!}{j!\,(\lambda-j)!}}\ \alpha^j \beta^{\lambda-j}\ :\ 0 \leqslant \mathrm{j} \leqslant \lambda$$

and each matrix coefficient of π_λ can be calculated by the analytic formula:

$$\pi_{j,k}{}^{(\lambda)} = \sqrt{\frac{(\lambda+1)!}{j!\,(\lambda-j)!}}\ \int_0^1 (\overline{\alpha}\ e^{i\,2\,\pi\,t} + \beta)^k\,(-\overline{\beta}\,e^{i\,2\,\pi\,t} + \alpha)^{\lambda-k}\ e^{-i 2\pi\,j\,t}\,dt$$

Examples of π_λ for $\lambda = 1, 2, 3$:

$$\pi_1 = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} \alpha^2 & \sqrt{2}\ \alpha\,\beta & \beta^2 \\ -\sqrt{2}\ \alpha\,\overline{\beta} & \alpha\,\overline{\alpha} - \beta\,\overline{\beta} & \sqrt{2}\ \beta\,\overline{\alpha} \\ \overline{\beta}^2 & -\sqrt{2}\ \overline{\alpha}\,\overline{\beta} & \overline{\alpha}^2 \end{pmatrix},$$

$$\pi_3 = \begin{pmatrix} \alpha^3 & \sqrt{3}\ \alpha^2\,\beta & \sqrt{3}\ \alpha\,\beta^2 & \beta^3 \\ -\sqrt{3}\ \alpha^2\,\overline{\beta} & \alpha^2\,\overline{\alpha} - 2\,\alpha\,\beta\,\overline{\beta} & 2\,\alpha\,\beta\,\overline{\alpha} - \beta^2\,\overline{\beta} & \sqrt{3}\ \beta^2\,\overline{\alpha} \\ \sqrt{3}\ \alpha\,\overline{\beta}^2 & -2\,\alpha\,\overline{\alpha}\,\overline{\beta} + \beta\,\overline{\beta}^2 & \alpha\,\overline{\alpha}^2 - 2\,\beta\,\overline{\alpha}\,\overline{\beta} & \sqrt{3}\ \beta\,\overline{\alpha}^2 \\ -\overline{\beta}^3 & \sqrt{3}\ \overline{\alpha}\,\overline{\beta}^2 & -\sqrt{3}\ \overline{\alpha}^2\,\overline{\beta} & \overline{\alpha}^3 \end{pmatrix}$$

(Notice that if we multiply each row of π_λ by $\sqrt{\lambda}$, then we recover the orthonormal basis same as above)

The Lie algebra of su(2) is spanned by the standard basis:

$$X_1 = \begin{pmatrix} i & \square \\ \square & -i \end{pmatrix},\ \ X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \ X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and the invariant differential operators induced by the basis are:

$$D_{X_1} = i\,\alpha\,\frac{\partial}{\partial\,\alpha} - i\,\beta\,\frac{\partial}{\partial\,\beta}\,,\quad D_{X_2} = \alpha\,\frac{\partial}{\partial\,\beta} - \beta\,\frac{\partial}{\partial\,\alpha}\,,\quad D_{X_3} = i\,\alpha\,\frac{\partial}{\partial\,\beta} + i\,\beta\,\frac{\partial}{\partial\,\alpha}$$

Hence, the infinitesimal representation $d\pi_\lambda$ with respect to these invariant differential operators are:

$$d\pi_1 = \begin{pmatrix} i h & x + i y \\ -x + i y & -i h \end{pmatrix}, \quad d\pi_2 = \begin{pmatrix} 2 i h & \sqrt{2}\,(x + i y) & 0 \\ -\sqrt{2}\,(x - i y) & 0 & \sqrt{2}\,(x + i y) \\ 0 & -\sqrt{2}\,(x - i y) & -2 i h \end{pmatrix}, \quad d\pi_3 =$$

$$\begin{pmatrix} 3 i h & \sqrt{3}\,(x + i y) & 0 & 0 \\ -\sqrt{3}\,(x - i y) & i h & 2\,(x + i y) & 0 \\ 0 & -2\,(x - i y) & -i h & \sqrt{3}\,(x + i y) \\ 0 & 0 & -\sqrt{3}\,(x - i y) & -3 i h \end{pmatrix}$$

with respect to $\gamma = (h, x, y) \in \mathbb{R}^3$.

Let $x \in \mathbb{R}^d$ and X the d – tuple of a basis of Lie algebra of G, then he Fourier transform of the non-commutative Kirillov formula is :

$$e^{d\pi_\lambda(x \cdot X)} = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \sum_{m=0}^{j} (-1)^{n-k-j-1} \frac{\phi_{n-k-j-1}(x \cdot X)}{(n-j+m-1)!} (x \cdot X)^k \, \psi(m-1)\, (v_{I_\lambda})^\vee$$

where v_{I_λ} is the Liouville measure of moment set I_λ, and ψ is a recursive formula

$$\psi(p) = \sum_{q=0}^{p} (-1)^{p-q} \frac{p!}{q!} \left(\gamma \frac{\partial}{\partial x}\right) \psi(p-1), \quad \text{and} \quad \psi(-1) = 1$$

The Fourier transform of $(v_{I_\lambda})^\vee$ is given by

$$(v_{I_\lambda})^\vee\ (h, x, y) = \left(\frac{\operatorname{Sin}\sqrt{h^2 + x^2 + y^2}}{\sqrt{h^2 + x^2 + y^2}}\right)^\lambda$$

The operator of polynomials are calculated as follows:

In[33]:=

```
dπ1 = {{ⅈ h, x + ⅈ y}, {-(x - ⅈ y), -ⅈ h}};
dπ2 = {{2 ⅈ h, √2 (x + ⅈ y), 0},
   {-√2 (x - ⅈ y), 0, √2 (x + ⅈ y)}, {0, -√2 (x - ⅈ y), -2 ⅈ h}};
dπ3 = {{3 ⅈ h, √3 (x + ⅈ y), 0, 0}, {-√3 (x - ⅈ y), ⅈ h, 2 (x + ⅈ y), 0},
   {0, -2 (x - ⅈ y), -ⅈ h, √3 (x + ⅈ y)}, {0, 0, -√3 (x - ⅈ y), - 3 ⅈ h}};
```

In[43]:=
```
ψ[p_] := Sum[Binomial[p, q] (-1)^(p-q) (p - q)! (dγ) * If[q == 0, 1, ψ[q - 1]], {q, 0, p}];
i = 2; n = 2;
π1 =
  (1 / (n - 1)!) * Sum[Sum[Sum[ (-1)^(n-k-j-1) * Binomial[j, m] * ((n - 1)! / (n - j + m - 1)!) *
        If[k == 0, IdentityMatrix[i], MatrixPower[dπ1, k]] *
        Tr[Minors[dπ1, n - j - k - 1]] * If[m == 0, 1, ψ[m - 1]], {m, 0, j}], {j, 0, n-k-1}], {k, 0, n-1}];
ExpandAll[π1] // MatrixForm
```

Out[46]//MatrixForm=

$$\begin{pmatrix} 1 + d\gamma + i h & x + i y \\ -x + i y & 1 + d\gamma - i h \end{pmatrix}$$

In[∘]:=
```
(* The highest weight matrix coefficient π_{1,1}^1 given that r^2 = h^2 + x^2 + y^2,
r ∂/∂r = h ∂/∂h + x ∂/∂x + y ∂/∂y *)
Sin[r]/r + r * D[Sin[r]/r, r] + i h Sin[r]/r // FullSimplify
```

Out[∘]= $\mathrm{Cos}[r] + \frac{i\, h\, \mathrm{Sin}[r]}{r}$

```
(* This is (α · exp) *)
```

In[47]:=
```
i = 3; n = 3;
π2 =
  (1 / (n - 1)!) * Sum[Sum[Sum[ (-1)^(n-k-j-1) * Binomial[j, m] * ((n - 1)! / (n - j + m - 1)!) *
        If[k == 0, IdentityMatrix[i], MatrixPower[dπ2, k]] *
        Tr[Minors[dπ2, n - j - k - 1]] * If[m == 0, 1, ψ[m - 1]], {m, 0, j}], {j, 0, n-k-1}], {k, 0, n-1}];
ExpandAll[π2] // MatrixForm
```

Out[49]//MatrixForm=

$$\begin{pmatrix} 1 + \frac{3 d\gamma}{2} + \frac{d\gamma^2}{2} + 2 i h + i d\gamma h + x^2 + y^2 & \sqrt{2} x + \frac{d\gamma x}{\sqrt{2}} + i \sqrt{2} h x + i \sqrt{2} y + \frac{i d\gamma y}{\sqrt{2}} - \sqrt{2} h y \\ -\sqrt{2} x - \frac{d\gamma x}{\sqrt{2}} - i \sqrt{2} h x + i \sqrt{2} y + \frac{i d\gamma y}{\sqrt{2}} - \sqrt{2} h y & 1 + \frac{3 d\gamma}{2} + \frac{d\gamma^2}{2} + 2 h^2 \\ x^2 - 2 i x y - y^2 & -\sqrt{2} x - \frac{d\gamma x}{\sqrt{2}} + i \sqrt{2} h x + i \sqrt{2} y + \frac{i d\gamma y}{\sqrt{2}} + \sqrt{2} h y \end{pmatrix}$$

```
(* The highest weight matrix coefficient π_{1,1}^2 given that r^2 = h^2 + x^2 + y^2,
r ∂/∂r = h ∂/∂h + x ∂/∂x + y ∂/∂y *)
(Sin[r]/r)^2 + 3/2 r * D[(Sin[r]/r)^2, r] + 1/2 r * D[r * D[(Sin[r]/r)^2, r], r] +
  2 i h * (Sin[r]/r)^2 + i h * r * D[(Sin[r]/r)^2, r] + (r^2 - h^2) (Sin[r]/r)^2 // FullSimplify
```

Out[∘]= $\frac{(r\, \mathrm{Cos}[r] + i\, h\, \mathrm{Sin}[r])^2}{r^2}$

```
(* This is (α · exp)^2 *)
```

In[50]:=

```
i = 4; n = 4;
π3 =
  (1 / (n - 1) !) * Sum[Sum[Sum[(-1)^(n-k-j-1) * Binomial[j, m] * ((n - 1) ! / (n - j + m - 1) !) *
       If[k == 0, IdentityMatrix[i], MatrixPower[ dπ3, k]] *
       Tr[Minors[ dπ3, n - j - k - 1]] * If[m == 0, 1, ψ[m - 1]], {m, 0, j}], {j, 0, n-k-1}], {k, 0, n-1}];
ExpandAll[π3] // MatrixForm
```

Out[52]//MatrixForm=

$$\begin{pmatrix} 1 + \frac{11\,d\gamma}{6} + d\gamma^2 + \frac{d\gamma^3}{6} + 3\,i\,h + \frac{5\,i\,d\gamma\,h}{2} + \frac{1}{2}\,i\,d\gamma^2\,h + \frac{h^2}{2} + \frac{d\gamma\,h^2}{6} + \frac{i\,h^3}{2} + \frac{7\,x^2}{2} + \frac{7\,d\gamma\,x^2}{6} \\ -\sqrt{3}\,x - \frac{5\,d\gamma\,x}{2\sqrt{3}} - \frac{d\gamma^2\,x}{2\sqrt{3}} - 2\,i\,\sqrt{3}\,h\,x - \frac{2\,i\,d\gamma\,h\,x}{\sqrt{3}} + \frac{1}{2}\sqrt{3}\,h^2\,x - \frac{\sqrt{3}\,x^3}{2} + i\,\sqrt{3}\,y + \frac{5\,i\,d\gamma\,y}{2\sqrt{3}} + \frac{i\,d\gamma^2\,y}{2\sqrt{3}} - 2\sqrt{3}\,h\,y - \\ \sqrt{3}\,x^2 + \frac{d\gamma\,x^2}{\sqrt{3}} + i\,\sqrt{3}\,h\,x^2 - 2\,i\,\sqrt{3}\,x\,y - \frac{2\,i\,d\gamma\,x\,y}{\sqrt{3}} + 2\sqrt{3}\,h\,x\,y - \sqrt{3} \\ -x^3 + 3\,i\,x^2\,y + 3\,x\,y^2 - i\,y^3 \end{pmatrix}$$

In[54]:= `ExpandAll[π3[[1, 1]]]`

Out[54]=

$$1 + \frac{11\,d\gamma}{6} + d\gamma^2 + \frac{d\gamma^3}{6} + 3\,i\,h + \frac{5\,i\,d\gamma\,h}{2} + \frac{1}{2}\,i\,d\gamma^2\,h + \frac{h^2}{2} + \frac{d\gamma\,h^2}{6} + \frac{i\,h^3}{2} + \frac{7\,x^2}{2} + \frac{7\,d\gamma\,x^2}{6} + \frac{3}{2}\,i\,h\,x^2 + \frac{7\,y^2}{2} + \frac{7\,d\gamma\,y^2}{6} + \frac{3}{2}\,i\,h\,y^2$$

In[55]:=

```
(* The highest weight matrix coefficient π_{1,1}^3 given that r^2 = h^2 + x^2 + y^2,
r ∂/∂r = h ∂/∂h + x ∂/∂x + y ∂/∂y *)
(Sin[r]/r)^3 + 11/6 * r * D[(Sin[r]/r)^3, r] + r * D[r * D[(Sin[r]/r)^3, r], r] +
  1/6 r * D[r * D[r * D[(Sin[r]/r)^3, r], r], r] + 3 * ⅈ h * (Sin[r]/r)^3 +
  5/2 * ⅈ h * r * D[(Sin[r]/r)^3, r] + 1/2 * ⅈ h * r * D[r * D[(Sin[r]/r)^3, r], r] +
  1/2 * h^2 * (Sin[r]/r)^3 + 1/6 h^2 * r * D[(Sin[r]/r)^3, r] + 1/2 * ⅈ h^3 * (Sin[r]/r)^3 +
  (r^2 - h^2) * 7/2 * (Sin[r]/r)^3 + (r^2 - h^2) * 7/6 * r * D[(Sin[r]/r)^3, r] +
  (r^2 - h^2) * 3/2 * ⅈ h * (Sin[r]/r)^3 // FullSimplify
```

Out[55]=

$$\frac{(r\,\mathrm{Cos}[r] + i\,h\,\mathrm{Sin}[r])^3}{r^3}$$

```
(* This is (α · exp)^3 *)
```