In[1]:= (* Convolution of the projection of coadjoint orbits of SU(2)

$$\mu_{\lambda_1}*\ldots*\mu_{\lambda_n}= \int_{t^+} \sum_W \mathrm{sgn}(w)\; e_{w.\lambda_1} \;*\; \mu^p{}_{\lambda_2}*\ldots*\mu^p{}_{\lambda_n}(\lambda'')\; \mu_{\lambda''}\;\; d\lambda''$$

and

$$\mu^p{}_{\lambda_1}*\ldots*\mu^p{}_{\lambda_n}= \int_{t^+} \sum_W \mathrm{sgn}(w)\; e_{w.\lambda_1} \;*\; \mu^p{}_{\lambda_2}*\ldots*\mu^p{}_{\lambda_n}(\lambda'')\; \mu^p{}_{\lambda''}\;\; d\lambda''$$

*)

```
Clear["Global`*"]
```

In[12]:=
```
(*1. Specify the continuous Kostant function P *)
scale = 100;
f[x_] := Piecewise[{{1, 0 ≤ x < ∞}}, 0];
```

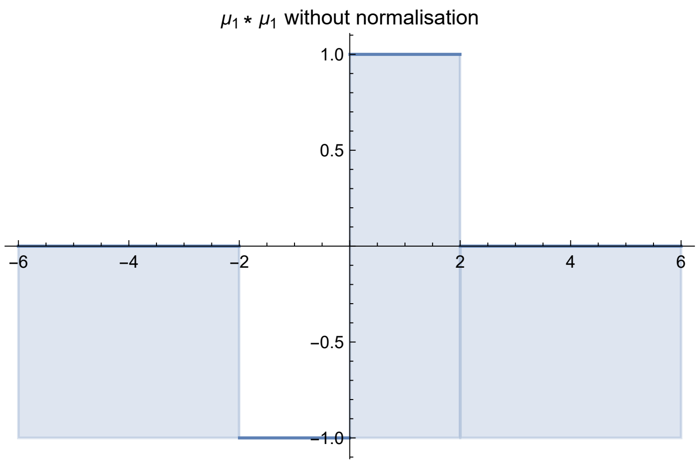
In[14]:=
```
(*2. write down convolution formula *)
λ1 = 1; λ2 = 1;
```

$$\phi[x\_] := \left(\sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^{i+j} * f\left[\;(-1)^{i+1}\,\lambda 1 \;-\; (-1)^{j+1}\,\lambda 2 + x\right]\right); \quad (*\ \nu(1,1,\lambda'')\ *)$$

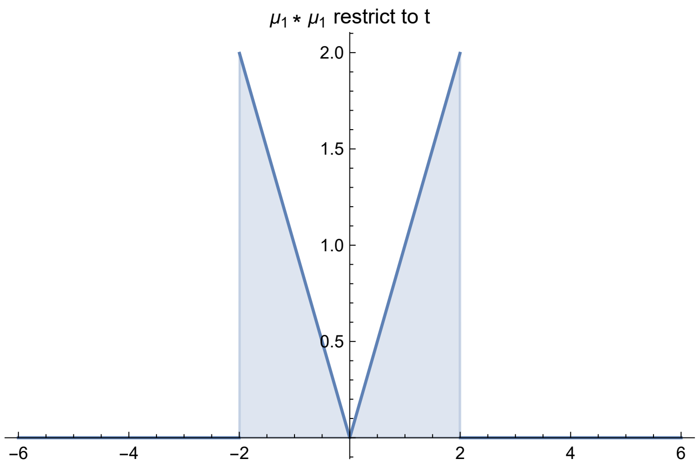
$$\phi 1[x\_] := \left(\sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^{i+j} * f\left[\;(-1)^{i+1}\,\lambda 1 \;-\; (-1)^{j+1}\,\lambda 2 + x\right]\right) * x;$$

```
Plot[ϕ[x], {x, -6, 6}, Filling → Bottom, PlotLabel → "μ1 * μ1 without normalisation"]
Plot[ϕ1[x], {x, -6, 6}, Filling → Bottom, PlotLabel → "μ1 * μ1 restrict to t"]
```

Out[17]=



Out[18]=



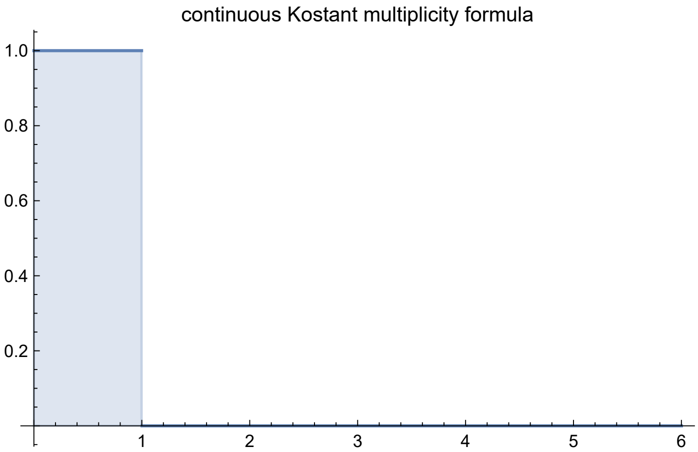
In[19]:=
```
ϕ1[x] // Simplify
```

Out[19]=

$$\begin{cases} -x & -2 \le x < 0 \\ x & 0 \le x < 2 \\ 0 & \text{True} \end{cases}$$

In[20]:=
```
(* Test out Kostant Multiplicity Formula *)
ff[x_, n_] := -(f[x - n] - f[x + n + 1]);
Plot[ff[x, 1], {x, 0, 6}, Filling → Bottom,
 PlotLabel → "continuous Kostant multiplicity formula"]
```

Out[21]=

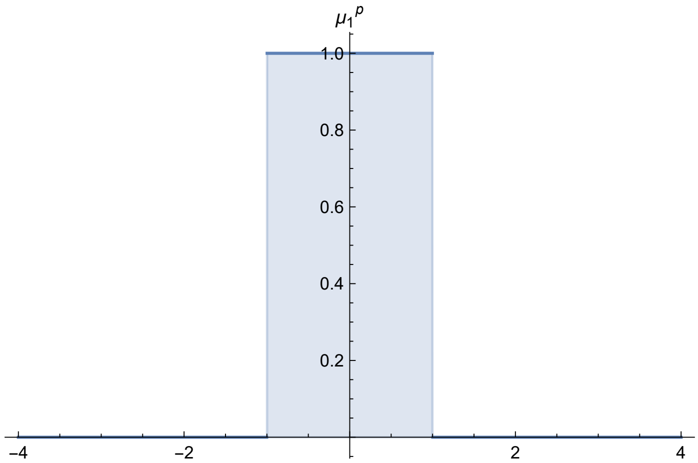


In[22]:= (*Projection of μ_1 : $\mu_1{}^{p}$ *)

$$\psi[x\_] := \sum_{i=1}^{2} (-1)^{i+1} * f\left[(-1)^{i+1} * 1 - x\right];$$

Plot[ψ[x], {x, -4, 4}, Filling → Bottom, PlotLabel → "$\mu_1{}^{p}$"]

Out[23]=

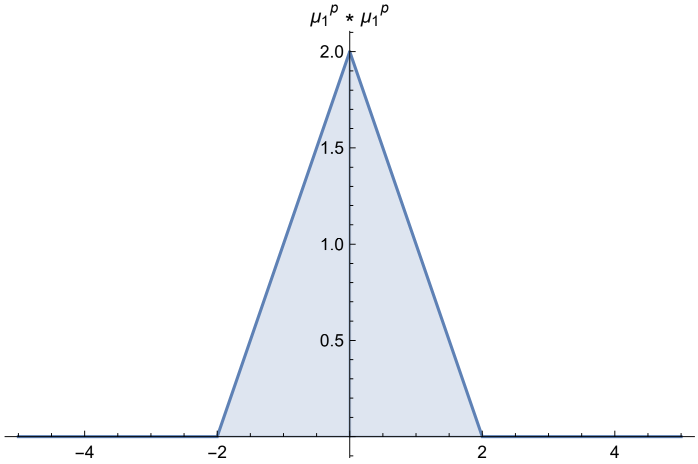


In[24]:=
```
(* Convolution of Projection of Measures μ1^p * μ1^p  *)
ff[x_] := Piecewise[{{1, 0 < x < ∞}}];
```

$$\phi 2[x\_, \lambda\_] := \sum_{i=1}^{2} (-1)^{i+1} * ff[(-1)^{i+1} \lambda - x];$$

```
T[x_] = Integrate[ϕ[y] * ϕ2[x, y], {y, 0, ∞}];
Plot[T[x], {x, -5, 5}, Filling → Bottom, PlotLabel → "μ1^p * μ1^p"]
```

Out[27]=



In[30]:= `T[x]`

Out[30]=

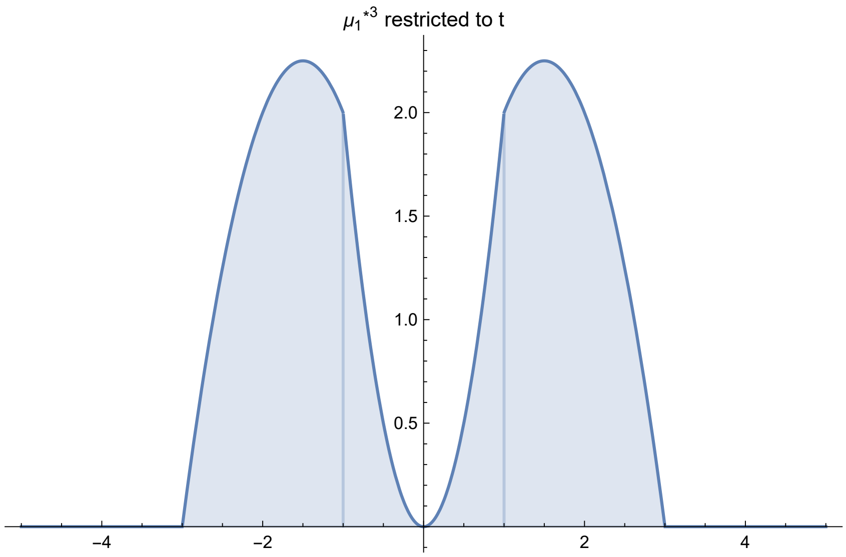
$$\begin{cases} 2 - x & 0 \leq x < 2 \\ 2 + x & -2 < x < 0 \\ 0 & \text{True} \end{cases}$$

In[31]:=
```
(* Convolution of of Measures μ1^*3 *)
```

$$\phi 3[x\_] := \sum_{i=1}^{2} (-1)^{i+1} * T[(-1)^{i+1} - x] * x;$$

```
Plot[ϕ3[x], {x, -5, 5}, Filling → Bottom, PlotLabel → "μ1^*3 restricted to t"]
```

Out[32]=



In[33]:=
```
ϕ3[x] // Simplify
```

Out[33]=

$$\begin{cases} -((-3+x)\,x) & 1 \le x < 3 \\ 2\,x^2 & -1 < x < 1 \\ -x\,(3+x) & -3 < x \le -1 \\ 0 & \text{True} \end{cases}$$

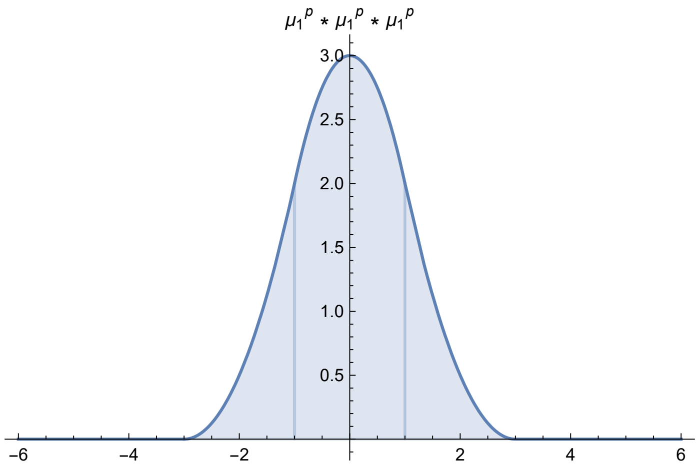
In[34]:=

$$T2[x\_] := \sum_{i=1}^{2} (-1)^{i+1} * T\left[(-1)^{i+1} - x\right];$$

```
P[x_] = Integrate[T2[y] * ϕ2[x, y], {y, 0, ∞}];
```

$\text{Plot}\left[P[x],\ \{x, -6, 6\},\ \text{Filling} \to \text{Bottom},\ \text{PlotLabel} \to \text{"}\mu_1{}^{p} * \mu_1{}^{p} * \mu_1{}^{p}\text{"}\right]$

Out[36]=



In[37]:=
```
P[x]
```

Out[37]=

$$\begin{cases} 2 & x == -1 \,||\, x == 1 \\ 3 - x^2 & -1 < x < 1 \\ \frac{1}{2} \times \left(9 - 6\,x + x^2\right) & 1 < x < 3 \\ \frac{1}{2} \times \left(9 + 6\,x + x^2\right) & -3 < x < -1 \\ 0 & \text{True} \end{cases}$$

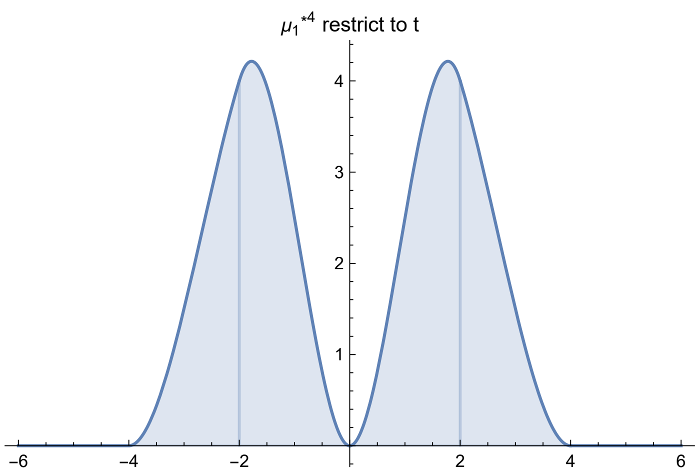
In[38]:=

$(*\ \text{Convolution of of Measures}\ \mu_1{}^{*4}\ *)$

$$\phi4[x\_] := \sum_{i=1}^{2} (-1)^{i+1} * P\left[(-1)^{i+1} - x\right] * x;$$

$\text{Plot}\left[\phi4[x],\ \{x, -6, 6\},\ \text{Filling} \to \text{Bottom},\ \text{PlotLabel} \to \text{"}\mu_1{}^{*4}\text{ restrict to t"}\right]$

Out[39]=



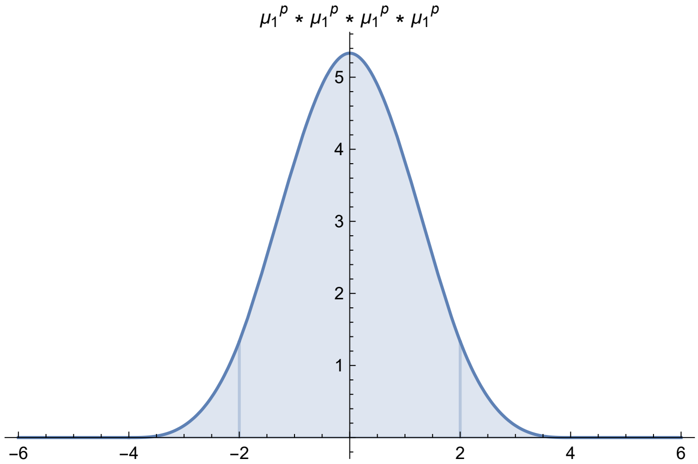
In[40]:= `ϕ4[x] // Simplify`

Out[40]=
$$\begin{cases} -\frac{1}{2}\,x\,(4+x)^2 & -4<x\le -2 \\ \frac{1}{2}\,(-4+x)^2\,x & 2\le x<4 \\ \frac{1}{2}\times(8-3\,x)\,x^2 & 0<x<2 \\ \frac{1}{2}\,x^2\,(8+3\,x) & -2<x<0 \\ 0 & \text{True} \end{cases}$$

In[41]:= `(* Convolution of Projection of Measures μ1^p * μ1^p * (μ1 * μ1)^p *)`

$$\text{T3}[x\_] := \sum_{i=1}^{2} (-1)^{i+1} * \text{P}\left[(-1)^{i+1} - x\right];$$

```
P2[x_] = Integrate[T3[y] * ϕ2[x, y], {y, 0, ∞}];
Plot[P2[x], {x, -6, 6}, Filling → Bottom, PlotLabel → "μ1^p * μ1^p * μ1^p * μ1^p"]
```

Out[43]=


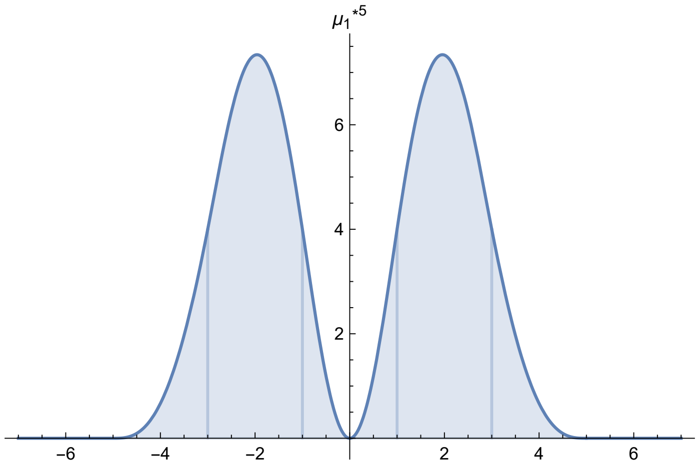
In[44]:= `P2[x]`

Out[44]=
$$\begin{cases} \frac{4}{3} & x == -2 \,||\, x == 2 \\ \frac{1}{6}\times\left(32-12\,x^2-3\,x^3\right) & -2<x<0 \\ \frac{1}{6}\times\left(64-48\,x+12\,x^2-x^3\right) & 2<x<4 \\ \frac{1}{6}\times\left(64+48\,x+12\,x^2+x^3\right) & -4<x<-2 \\ \frac{1}{6}\times\left(32-12\,x^2+3\,x^3\right) & 0\le x<2 \\ 0 & \text{True} \end{cases}$$

In[47]:=
```
(* Convolution of of Measures μ1*5 *)
φ5[x_] := Sum[(-1)^(i+1) * P2[(-1)^(i+1) - x] * x, {i, 1, 2}];
Plot[φ5[x], {x, -7, 7}, Filling → Bottom, PlotLabel → "μ1*5"]
```

$$\phi5[x\_] := \sum_{i=1}^{2} (-1)^{i+1} * P2\left[(-1)^{i+1} - x\right] * x;$$

Out[48]=



In[49]:=
```
φ5[x] // Simplify
```

Out[49]=

$$\begin{cases} -\frac{1}{6} x (5 + x)^3 & -5 < x \leq -3 \\ -\frac{1}{6} (-5 + x)^3 x & 3 \leq x < 5 \\ -x^2 \left(-5 + x^2\right) & -1 \leq x \leq 1 \\ \frac{1}{3} x \left(-5 + 30 x - 15 x^2 + 2 x^3\right) & 1 < x < 3 \\ \frac{1}{3} x \left(5 + 30 x + 15 x^2 + 2 x^3\right) & -3 < x < -1 \\ 0 & \text{True} \end{cases}$$