```
ClearAll["Global`*"]
(*
Root system of A₂:
   Φ = {α₁, α₂, α₁+α₂, -α₁, -α₂, -(α₁+α₂)}
*)
```

In[1]:=
```
(* The infinitesimal representation of fundamental weight λ₁ of SU(3) *)
dX1 = {{0, 1, 0}, {0, 0, 0}, {0, 0, 0}};
dY1 = {{0, 0, 0}, {1, 0, 0}, {0, 0, 0}};
dX2 = {{0, 0, 0}, {0, 0, 1}, {0, 0, 0}};
dY2 = {{0, 0, 0}, {0, 0, 0}, {0, 1, 0}};
dX3 = {{0, 0, 1}, {0, 0, 0}, {0, 0, 0}};
dY3 = {{0, 0, 0}, {0, 0, 0}, {1, 0, 0}};
dH1 = {{1, 0, 0}, {0, -1, 0}, {0, 0, 0}};
dH2 = {{0, 0, 0}, {0, 1, 0}, {0, 0, -1}};

T = (ⅈ h₁ dH1 + ⅈ h₂ dH2) + x₁ (dX1 - dY1) + ⅈ y₁ (dX1 + dY1) +
   x₂ (dX2 - dY2) + ⅈ y₂ (dX2 + dY2) + x₃ (dX3 - dY3) + ⅈ y₃ (dX3 + dY3);
T // MatrixForm
```

Out[10]//MatrixForm=

$$\begin{pmatrix} i h_1 & x_1 + i y_1 & x_3 + i y_3 \\ -x_1 + i y_1 & -i h_1 + i h_2 & x_2 + i y_2 \\ -x_3 + i y_3 & -x_2 + i y_2 & -i h_2 \end{pmatrix}$$

In[90]:=
```
T2 = (ⅈ h₁ dH1 + ⅈ h₂ dH2); (* restrict to Cartan subalgebra *)
T2 // MatrixForm
```

Out[91]//MatrixForm=

$$\begin{pmatrix} i h_1 & 0 & 0 \\ 0 & -i h_1 + i h_2 & 0 \\ 0 & 0 & -i h_2 \end{pmatrix}$$

In[97]:=
```
(* Non-comm Kirillov formula for fundamental weight λ₁ of SU(3) *)
ψ[p_] := Sum[Binomial[p, q] (-1)^(p-q) (p - q) ! (dγ) * If[q == 0, 1, ψ[q - 1]], {q, 0, p}];
i = 3; n = 3;
π_λ₁ = Sum[Sum[Sum[(-1)^(n-k-j-1) * Binomial[j, m] *
      ((n - 1) ! / (n - j + m - 1) !) * If[k == 0, IdentityMatrix[i], MatrixPower[T, k]] *
      Tr[Minors[T, n - j - k - 1]] * If[m == 0, 1, ψ[m - 1]], {m, 0, j}], {j, 0, n-k-1}], {k, 0, n-1}];
ExpandAll[π_λ₁] // MatrixForm
```

Out[100]//MatrixForm=

$$\begin{pmatrix} 2 + 3\, d\gamma + d\gamma^2 + 2\, i\, h_1 + i\, d\gamma\, h_1 - h_1 h_2 + h_2^2 + x_2^2 + y_2^2 & 2\, x_1 + d\gamma \\ -2\, x_1 - d\gamma\, x_1 - i\, h_2\, x_1 - x_2\, x_3 + 2\, i\, y_1 + i\, d\gamma\, y_1 - h_2\, y_1 - i\, x_3\, y_2 + i\, x_2\, y_3 - y_2\, y_3 & 2 \\ x_1 x_2 - 2\, x_3 - d\gamma\, x_3 - i\, h_1\, x_3 + i\, h_2\, x_3 - i\, x_2\, y_1 - i\, x_1\, y_2 - y_1 y_2 + 2\, i\, y_3 + i\, d\gamma\, y_3 - h_1 y_3 + h_2 y_3 & -2\, x_2 - d \end{pmatrix}$$

In[56]:=
```
ExpandAll[π_λ₁[[1, 1]]]
```

Out[56]= $2 + 3\, d\gamma + d\gamma^2 + 2\, i\, h_1 + i\, d\gamma\, h_1 - h_1 h_2 + h_2^2 + x_2^2 + y_2^2$

```
(* The Fourier transform of Iλ₁,  (νIλ₁)ᵛ :

      Find a basis {H1,H2} such that
      α1(H1)  =  2    α1(H2)  = -1
      α2(H1)  = -1    α2(H2)  =  2

      Hence, with respect to dot product
       H1 = (1,1),  H2 = (-1,0)
       α1 = (1,1),  α2 = (-2,1)
 *)
a1 = {1, 1};  a2 = {-2, 1} ; δ = a1 + a2;
```

$$\lambda 1 = \frac{1}{3} * (2 * a1 + 1 * a2);\ \lambda 2 = \frac{1}{3} * (1 * a1 + 2 * a2);$$

$$\nu_1[x\_, y\_] := \frac{e^{i\,(\lambda 1.\{x,y\})}}{-(a1.\{x, y\}) * ((a1 + a2).\{x, y\})} + \frac{e^{i\,((\lambda 1 - a1).\{x,y\})}}{(a1.\{x, y\}) * (a2.\{x, y\})} + \frac{e^{i\,((\lambda 1 - a1 - a2).\{x,y\})}}{-(a2.\{x, y\}) * ((a1 + a2).\{x, y\})};$$

```
ν1[
 h1 -
  h2,
 h1]
```

Out[87]= $$\frac{e^{i\,(-h1+h2)}}{(h1 - 2\,(h1 - h2))\,(2\,h1 - h2)} + \frac{e^{-i\,h2}}{(-h1 + 2\,(h1 - h2))\,(h1 + h2)} + \frac{e^{i\,h1}}{(-2\,h1 + h2)\,(h1 + h2)}$$

```
(* Character formula for fundamental weight λ1 of SU(3) *)

Tr[πλ1] // Simplify
```

Out[101]= $6 + 9\,d\gamma + 3\,d\gamma^2 + h_1^2 - h_1\,h_2 + h_2^2 + x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2$

In[96]:= `Tr[πλ1] // Simplify  (* restrict to Cartan subalgebra *)`

Out[96]= $6 + 9\,d\gamma + 3\,d\gamma^2 + h_1^2 - h_1\,h_2 + h_2^2$

In[119]:= $$f1[h1\_, h2\_] := \frac{e^{i\,(-h1+h2)}}{(h1 - 2\,(h1 - h2))\,(2\,h1 - h2)} + \frac{e^{-i\,h2}}{(-h1 + 2\,(h1 - h2))\,(h1 + h2)} + \frac{e^{i\,h1}}{(-2\,h1 + h2)\,(h1 + h2)};$$

```
f2[h1_, h2_] := h1 * D[f1[h1, h2], h1] + h2 * D[f1[h1, h2], h2];  (* dγ *)
f3[h1_, h2_] := h1 * D[f2[h1, h2], h1] + h2 * D[f2[h1, h2], h2];  (* dγ^2 *)

6 * f1[h1, h2] + 9 * f2[h1, h2] + 3 * f3[h1, h2] + (h1^2 - h1 h2 + h2^2) * f1[h1, h2] //
 FullSimplify  (* Calculation *)
```

Out[122]= $e^{i\,h1} + e^{-i\,(h1-h2)} + e^{-i\,h2}$

In[125]:=
```
(* Comparing with Weyl character formula  *)
(e^(i (λ1+δ).{h1-h2,h1}) - e^(i (λ1-a1+a2).{h1-h2,h1}) + e^(i (λ1-a1-2 a2).{h1-h2,h1}) -
    e^(i (λ1-2 δ).{h1-h2,h1}) + e^(i (λ1-2 a1).{h1-h2,h1}) - e^(i (λ1+a1).{h1-h2,h1})) /
  (e^(i (δ).{h1-h2,h1}) - e^(i (a2).{h1-h2,h1}) + e^(i (-a2).{h1-h2,h1}) - e^(i (-δ).{h1-h2,h1}) +
    e^(i (-a1).{h1-h2,h1}) - e^(i (a1).{h1-h2,h1})) // FullSimplify
```

Out[125]= $e^{i\,h1} + e^{-i\,(h1-h2)} + e^{-i\,h2}$