Representations of Compact Lie Groups

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Preface

Harmonic analysis is an extension of the classical Fourier analysis derived by replacing the real line $\mathbb{R}$ by an arbitrary locally compact group.

The theory of group representation lies at the heart of harmonic analysis. In particular, the irreducible representations of a group are used as a basis replacing the role of exponential function in classical Fourier analysis. [SAK05]

Informally, the representations of a finite or compact group G can be viewed as matrices, and the irreducible representations are the the minimal buidling blocks of these matrices.

In this project, we will look at how to work out the irreducible representations of compact Lie groups.

In Chapter 1, we will briefly review the knowledge of group theory used in this project, introduce the representations of finite groups and work out all the irreducible representations of the symmetric group S_3.

In Chapter 2, we will develop some topological aspects of compact Lie groups. In fact, this extends the theory of finite groups, as every finite group is a zero-dimensional compact Lie group where we define a discrete topology on it. Then, we will introduce topological properties of some matrix Lie groups, for example, orthogonal group $O(n)$ and unitary group $U(n)$ for $n \geq 1$. Finally, we will briefly talk about the structure of compact Lie groups.

In Chapter 3, we will introduce the unitary representations of compact groups on a Hilbert space $\mathcal{H}$, and some important theorems of irreducible representations for compact groups. Once we have these irreducible representations, we need to find a way to pass these representations to $L^2(G)$ to perform Fourier analysis. To do so, we introduce the notion of matrix coefficients. Also, we will introduce the Plancherel theorem for a non-abelian groups. Finally, we will work out all the irreducible representations of the special unitary group $SU(2)$.

In Chapter 4, we will briefly look at the concept of Lie algebra and representations of Lie algebra. If G is simply connected, there is a natural one-to-one correspondence between the representations of G and the representations of its Lie algebra. [Hal00]. We will briefly introduce the concept of tangent vector space, Lie bracket and adjoint representation.

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Contents

Chapter 1

Representations of Finite Groups

In this chapter, we will briefly review some aspects of group theory, including group axioms, subgroup, coset, normal subgroup and quotient group. Then, we will introduce the representations of finite groups, subrepresentations and irreducible representations. Also, we will look at concrete examples showing how to calculate the irreducible subrepresentations of the permutation representations of the symmetric group S_3. Finally, we will briefly talk about induced representations, which can be derived by extending the representations of a subgroup to the whole group.

The contents in this chapter are developed from these references:

[Ear14], [Bel17], [Ser71]

1.1 Group Axioms

<u>**Definition 1.1**</u> A ***binary operation*** $*$ on a set S is a map $*: S \times S \to S$, which is $x * y$ is the image of (x, y) under operation $*$ for any $x, y \in S$.

So a binary operation inputs two elements of S in a given order and returns a single output $x * y$ which has to be in the set S.

<u>**Example 1.2**</u> The following are examples of binary operations:

(i) $+, -, \times$ on $\mathbb{R}$, and $\div$ is not binary operation on $\mathbb{R}$, e.g. $1 \div 0$ is not defined.

(ii) $\wedge$, cross product on $\mathbb{R}^3$.

(iii) min and max on $\mathbb{N}$, e.g. $\min(1, 2) = 1$, $\max(1, 2) = 2$

(iv) Matrix multiplication on the set of $n \times n$ invertible matrices.

<u>**Definition 1.3**</u> A group **(G,*)** is a set G with a binary operation $*$ on G such that

$$*: G \times G \to G, \qquad (x, y) \mapsto x * y$$

and it has following properties:

(i) $*$ is associative, i.e. $x * (y * z) = (x * y) * z$ for any $x, y, z \in G$

(ii) there is an identity element $e \in G$ which satisfy $e * x = x * e$ for all $x \in G$.

(iii) for every $x \in G$, there exists an inverse x^{-1} such that $x * x^{-1} = x^{-1} * x = e$

When we identify $(G, *)$ as a group, we need to check if the above properties are satisfied, and importantly we need to check $x * y \in G$ for all $x, y \in G$. This is called *closure*.

<u>**Example 1.4**</u> The group $(\mathbb{Z}, +)$, contains of all integers $\mathbb{Z}$ under the operation $+$, and it has identity element 0, i.e

$$\mathbb{Z} = \{......, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...\}$$

(i) $+$ is associative, i.e. $x + (y + z) = (x + y) + z$ for any $x, y, z \in \mathbb{Z}$

(ii) an identity element $0 \in \mathbb{Z}$ which satisfy $0 + x = x + 0$ for all $x \in \mathbb{Z}$.

(iii) for every $x \in \mathbb{Z}$, there exists an inverse $-x$ such that $x + (-x) = -x + x = 0$.

(iv) $x + y \in \mathbb{Z}$ for all $x, y \in \mathbb{Z}$. So, $(Z, +)$ is closed.

Definition 1.6 A group $(G, *)$ is called ***abelian*** (after Norwegian mathematician Niels Abel [1802-1829]), if the group operation is commutative, that is

$$x * y = y * x, \quad \forall\ x, y \in G$$

Example 1.7 The set $\mathbb{R}$ of real numbers forms an abelian group under operation $+$, with $e = 0$, and $x^{-1} := -x$. Similarly, the n-dimensional Euclidean space $\mathbb{R}^n$ forms an abelian group under the operation of vector addition.

Example 1.8 The set of non-zero real numbers $\mathbb{R} \setminus \{0\}$ forms an abelian group under the operation $\times$, with $e = 1$, and $x^{-1} := 1/x$.

Example 1.9 The set of complex numbers of absolute value 1, $\{z \in \mathbb{C} \mid |z| = 1\}$ (i.e., of the form $e^{i\theta}$), forms an abelian group under operation $\times$, with $e = 1$ and $x^{-1} := 1/x$. This group is the so called **circle group** (unit circle), denoted $\mathbb{T}$. This group is extremely important in all aspect of mathematics.

We now introduce some groups which are **non-abelian**.

Example 1.10 The **General Linear group**, is the set of $n \times n$ invertible matrices over any fields F, forms a group under matrix multiplication, with identity element to be identity matrix I_n, is denoted by $GL(n, \mathbb{F})$.

Since matrix multiplication is not commutative, thus this group is non-abelian.

Example 1.11 The **Symmetric Group of degree n**, is the set of the permutations of a set $X = \{1, 2, ..., n\}$ under 'multiplication', it is denoted by S_n.

e.g. The symmetric group S_3 permutes the set $X = \{1, 2, 3\}$.

Let $S_3 = \{(), (1, 2, 3), (1, 3, 2), (1, 2), (1, 3), (2, 3)\}$, where $()$ is the identity element.

The 'multiplication' of Symmetric group is defined by: e.g. $(2, 3) \cdot (1, 2, 3) = (1, 3)$

$$\begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{matrix} \cdot \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix} = \begin{matrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{matrix}$$

The 'multiplication' is from right to left, which is $(1-2-2-3) = 3$, $(2-3-3-2) = 2$ and $(3-1-1-1) = 1$.

Also, S_3 is generated by $(1, 2, 3)$ and $(2, 3)$, since

$$\begin{aligned} (2, 3) \cdot (1, 2, 3) &= (1, 3), & (1, 2, 3) \cdot (2, 3) &= (1, 2) \\ (1, 2, 3) \cdot (1, 2, 3) &= (1, 3, 2), & (2, 3) \cdot (2, 3) &= () \end{aligned}$$

1.2 Subgroup, Coset, Normal Subgroup and Quotient Group

Definition 1.7 Let G be a group. A subset $H \subseteq G$ is said to be a **subgroup** if the group operation $*$ of G restricted to H forms a group, that is:

(i) $e \in H$.

(ii) if h_1, $h_2 \in H$, then $h_1 * h_2 \in H$.

(iii) if $h \in H$, then $h^{-1} \in H$.

(iv) e and G are called trivial subgroups of G

Example 1.12 The group $(\mathbb{Z}, +)$, i.e.integers under addition, is a subgroup of $(\mathbb{R}, +)$.

Example 1.13 The group $SL(n, \mathbb{F})$ is a group of $n \times n$ matrices over field $\mathbb{F}$ with determinant 1. $SL(n, \mathbb{F})$ is a subgroup of $GL(n, \mathbb{F})$.

Definition 1.8 Let H be a subgroup of G, then the **left cosets of** H are the sets:

$$gH = \{gh : h \in H\}$$

and the **right cosets of** H are the sets:

$$Hg = \{hg : h \in H\}$$

We write **G/H** for the set of right cosets of H in G, and the cardinality of G/H is called the **index** of H in G.

Obviously, $gH = Hg$ if G is an abelian group, otherwise, in general, $gH \neq Hg$.

Example 1.14 Let $G = S_3$, and $H = \{(), (1,2)\}$, then the left and right cosets of H are:

$$\begin{aligned} ()H &= \{(), (1,2)\}; & H() &= \{(), (1,2)\} \\ (1,3)H = (1,3,2)H &= \{(1,3), (1,3,2)\}; & H(1,3) = (1,2,3)H &= \{(1,3), (1,2,3)\} \\ (2,3)H = (1,2,3)H &= \{(2,3), (1,2,3)\}; & H(2,3) = (1,3,2)H &= \{(2,3), (1,3,2)\} \end{aligned}$$

Note that it's possible that $g_1H = g_2H$ for $g_1 \neq g_2$.

Example 1.15 Let $G = \mathbb{C}^\times$ and $H = \mathbb{T}$, then for $w \in \mathbb{C}^\times$, the set of cosets of $\mathbb{T}$ is:

$$w\mathbb{T} = \{z \in \mathbb{C}^\times : |z| = |w|\}$$

Example 1.16 Let $G = \mathbb{Z}$ and $H = n\mathbb{Z}$, then the cosets of $n\mathbb{Z}$ is $r + n\mathbb{Z}, r \in \mathbb{Z}$. There are n cosets, these are:

$$0 + n\mathbb{Z},\ 1 + n\mathbb{Z},\ ..., (n-1) + n\mathbb{Z} = -1 + n\mathbb{Z}$$

In fact, we can naturally identify $\mathbb{Z}_n$ with $\mathbb{Z}/n\mathbb{Z}$.

Definition 1.9 Let G and H be groups, a **homomorphism** $\phi : G \to H$ is a map such that:

$$\phi(g_1 *_G g_2) = \phi(g_1) *_H \phi(g_2)$$

for all $g_1, g_2 \in G$. Hence, an **isomorphism** between G and H is simply a *bijective* homomorphism.

Example 1.17 The exponential map $\phi : \mathbb{R} \to \mathbb{T}$ is homomorphism since

$$e^{i(x+y)} = e^{ix} \cdot e^{iy}$$

for any $x, y \in \mathbb{R}$.

Proposition 1.1 Let $\phi : G \to H$ be a homomorphism, then

(i) $\phi(e_G) = e_H$

(ii) $\phi(g^{-1}) = \phi(g)^{-1}$

(iii) $\phi(g^n) = \phi(g)^n$

for $g \in G$ and $n \in \mathbb{Z}$

Proof:

(i) We have $\phi(e_G) = \phi(e_G * e_G) = \phi(e_G)\phi(e_G)$, then if we apply $\phi(e_G)^{-1}$ to both sides, (i) follows.

(ii) We have $\phi(g)\phi(g^{-1}) = \phi(g * g^{-1}) = \phi(e_G) = e_H$.

(iii) For $n > 0$, this is obvious from Definition 1.9.
For $n = -k < 0$,

$$\phi(g^n) = \phi((g^{-1})^k) = \phi((g^{-1}))^k = \phi((g)^{-1})^k = \phi(g)^n$$

Definition 1.10 Let $\phi : G \to H$ be a homomorphism, then

(i) the **kernel** of ϕ, denoted by $ker(\phi)$, is

$$ker(\phi) = \{g \in G : \phi(g) = e_H\} \subseteq G$$

(ii) the **image** of ϕ, denoted by $im(\phi)$, is

$$im(\phi) = \{\phi(g) : g \in G\} \subseteq H$$

Definition 1.11 Let G be a group and H be a subgroup of G, then H is a **normal subgroup** of G if

$$gH = Hg$$

for all $g \in G$. Equivalently if

$$g^{-1}Hg \in H$$

for all $g \in G$ and all $h \in H$. If H is a normal subgroup of G, we write $H \triangleleft G$.

Example 1.18 The group $(\mathbb{Z}, +)$, i.e.integers under addition, is a normal subgroup of $(\mathbb{R}, +)$. In fact, any subgroup of abelian group is a normal subgroup.

Proposition 1.2 Let G be a group and H be a subgroup of G, then $\mathbf{ker}(\phi) \triangleleft \mathbf{G}$.

Proof: Let $k_1, k_2 \in ker(\phi)$ and $g \in G$, then

$$\phi(e_G) = e_H, \quad \phi(k_1k_2) = \phi(k_1)\phi(k_2) = e_He_H = e_H, \quad \phi(k_1^{-1}) = \phi(k)^{-1} = e_H^{-1} = e_H$$

shows $ker(\phi)$ is a subgroup of G. Also,

$$\phi(g^{-1}k_1g) = \phi(g^{-1})\phi(k_1)\phi(g) = \phi(g^{-1})e_H\phi(g) = \phi(g^{-1})phi(g) = e_H$$

shows $ker(\phi)$ is a normal subgroup of G.

Example 1.19 The map $\phi : \mathbb{R} \to \mathbb{T}$, where

$$\phi(x) = e^{ix}$$

has $ker(\phi) = 2\pi\mathbb{Z}$ and $im(\phi) = \mathbb{T}$.

Example 1.20 The map $det : GL(n, \mathbb{R}) \to \mathbb{R}^\times$ has $ker(\phi) = SL(n, \mathbb{R})$, $im(\phi) = \mathbb{R}^\times$.

Definition 1.12 If $H \triangleleft G$, then G/H is called the **quotient group**. The binary operation * on G/H given by

$$(g_1H) * (g_2H) = (g_1g_2)H$$

is well-defined if and only if $H \triangleleft G$.

Proposition 1.3 Let G be a group and H a subset of G. Then H is a normal subgroup of G if and only if it is the kernel of some homomorphism from G.

Proof: It's true by Proposition 1.2. Conversely, if $H \triangleleft G$, then

$$\phi : G \to G/H, \qquad \phi(g) = gH$$

is a homomorphism with $ker(\phi) = H$. Since,

$$\phi(g_1g_2) = (g_1g_2)H = (g_1H)(g_2H) = \phi(g_1)\phi(g_2)$$

Example 1.21 Let $G = S_n$ and $H = A_n$ (Alternating group, i.e. even permutation). Then,

$$S_n/A_n = \{A_n, S_n \backslash A_n\} = \{evens, odds\}$$

For example, in S_3, $A_3 = \{(), (1,2,3), (1,3,2)\}$ and

$$()A_3 = A_3, \quad (1,2)A_3 = (1,3)A_3 = (2,3)A_3 = S_3\backslash A_3 \quad (1,2,3)A_3 = (1,3,2)A_3 = A_3$$

So, $S_3/A_3 = \{A_3, S_3\backslash A_3\}$

Example 1.22 Let $G = \mathbb{R}^+$ and $H = 2\pi\mathbb{Z}$, then

$$R/2\pi\mathbb{Z} \cong [0, 2\pi)$$

In fact, the map $\phi : R/2\pi\mathbb{Z} \to \mathbb{T}$ given by $x \mapsto e^{ix}$ for $x \in [0, 2\pi)$ is a homomorphism, where $ker(\phi) = 2\pi\mathbb{Z}$. Aslo, this map is bijective, so it's an isomorphism,
i.e $R/2\pi\mathbb{Z} \cong \mathbb{T}$.

1.3 Finite Group Representations

Definition 1.13 Let G be a finite group, the **representation** of G is a *homomorphism*

$$\rho : G \to GL(n, \mathbb{C})$$

such that $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$. It has following properties:

(i) $\rho(g)$ is acting on a n-dimensional vector space V, over $\mathbb{C}$.

(ii) $\rho(e) = I_n$ and $\rho(g^{-1}) = \rho(g)^{-1}$.

(iii) If $\rho(g) = I_n$ for all $g \in G$, then $\rho(g)$ is called the *trivial representation.*

Example 1.23 Let $G = C_2 = \{e, g\}$ such that $eg = g$ and $g^2 = e$. Then we can define ρ, representation of C_2 acting on $\mathbb{C}^2$. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{C}^2$ and defined by:

$$\mathbb{C}^2 = \text{span}\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

The matrices $\rho(e)$ and $\rho(g)$ are:

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

such that

$$\rho(e)e_1 = e_1, \quad \rho(e)e_2 = e_2, \quad \rho(g)e_1 = e_2, \quad \rho(g)e_2 = e_1,$$
$$\rho(g^2)e_1 = \rho(g)\rho(g)e_1 = e_1, \quad \rho(g^2)e_2 = \rho(g)\rho(g)e_2 = e_2.$$

Definition 1.15 A representation $\rho(g)$ of G on V is **decomposable** if it is isomorphic to the direct sum of strictly smaller representations, e.g.

$$(V, \rho) \cong (V_1 \oplus V_2, \rho_1 \oplus \rho_2)$$

if there exists $P \in GL(V)$ such that $(\rho_1 \oplus \rho_2) = P\rho P^{-1}$, and $V_1 \perp V_2$.

Definition 1.16 Let $\rho(g)$ be a representation of G on V. A ***subrepresentation*** of $\rho(g)$ is

$$\rho(g)_W : W \to W$$

for all $g \in G$, where W is a vector subspace of V.

We say $\rho(g)$ *preserves* W.

Definition 1.17 A representation $\rho(g)$ of G on V is ***reducible*** if there is a non-zero proper subrepresentation, otherwise, V is ***irreducible***.

Remark: Every **1-dimensional** representation is irreducible.

Every finite-dimensional representation $\rho(g)$ of a finite group G on V is isomorphic to a direct sum of irreducible representations

$$V \cong V_1 \oplus V_2 \oplus ...V_n \quad \text{and} \quad \rho \cong \rho_1 \oplus \rho_2 \oplus ...\rho_n$$

where V_i is orthogonal to each other, and ρ_i is an **irreducible representation** of G on V.

Example 1.24 Let $G = C_2$ and the representations of C_2 is acting on $\mathbb{C}^2$, we want to find all the irreducible representations.

Suppose W is a 1-dimensional vector subspace of $\mathbb{C}^2$, where $W = \text{span}\{a_1e_1 + a_2e_2\}$, such that $\rho(g) : W \to W$. Since W is 1-dimensional, so we have $\rho(g)W = \lambda W$, where λ is the eigenvalue of $\rho(g)$. Then we can write

$$\rho(e)(a_1e_1 + a_2e_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

and

$$\rho(g)(a_1e_1 + a_2e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

From these actions, we can see that $\lambda = 1$ and $a_1 = a_2$.

Hence, $W = \text{span}\{e_1 + e_2\}$ and the restriction of $\rho(g)$ to W, denoted by $\rho_W(g)$, is written by $\rho_W(g) = 1$.

So, we can say $\rho(g)$ is a *trivial* representation on W.

Now, we want to find the orthogonal complement of W, denoted by W'.

Let $W' = \text{span}\{a_1'e_1 + a_2'e_2\}$. Since $W \perp W'$, so the dot product

$< e_1 + e_2\,,\, a_1'e_1 + a_2'e_2 >= 0$, where

$$< e_1 + e_2\,,\, a_1'e_1 + a_2'e_2 >= a_1' < e_1\,,\, e_1 > + a_2' < e_2\,,\, e_2 >= a_1' + a_2' = 0$$

Thus we have $W' = \text{span}\{a_1'e_1 + a_2'e_2 : \sum_i a_i' = 0\}$

Then, we can write down W and W'

$$W = \text{span}\left\{a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \qquad W' = \text{span}\left\{b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

such that

$$\rho(g)a = a, \qquad \rho(g)b = -b$$

and

$$\rho_W(g) = 1, \qquad \rho(g)_{W'}(g) = -1$$

Both $\rho_W(g)$ and $\rho(g)_{W'}$ are 1-dimensional, so they are irreducible representations. We can write $\rho(g) \cong \rho_W(g) \oplus \rho_{W'}(g)$, and in the matrix form, it is written by

$$\rho(g) \cong \begin{pmatrix} \rho_W(g) & 0 \\ 0 & \rho_{W'}(g) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1.4 Irreducible Representations of S_3

The symmetric groups S_n plays an important role in group theory, so we want to look at their representations. The general case can be described using **Young tableaux**.

Here I will specifically work out the irreducible representations of the symmetric group S_3.

We introduce three natural representations of S_3, these are

(i) Trivial Representation: 1-dimensional

$$\rho(g) = 1 \quad \text{for all } g \in G$$

(ii) Sign Representation: 1-dimensional

$$\rho(g) = \begin{cases} 1 & \text{even permutations} \quad e.g.(1,2,3) = (1,2)\cdot(1,3) \\ -1 & \text{odd permutations} \quad e.g.(1,2) \end{cases}$$

(iii) Permutation Representation: 3-dimensional

Every 1-dimensional representation is irreducible, so the trivial and sign representations are irreducible. In fact, we will show that the 3-dimensional permutation representation can be decomposed as a sum of the trivial representation and an irreducible representation which is 2-dimensional.

The permutation representation of S_3 acts on $\mathbb{C}^3$ as follows. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{C}^3$,

$$\rho(g)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_{g(1)} \\ a_{g(2)} \\ a_{g(3)} \end{pmatrix}$$

Since S_3 is generated by $(1,2,3)$ and $(2,3)$ (Example 1.11), we can describe this representation by working out $\rho((1,2,3))$ and $\rho((2,3))$. Notice

$$\rho((1,2,3)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \rho((2,3)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Suppose that W is the 1-dimensional vector subspace of $\mathbb{C}^2$, given by $W = \text{span}\{a_1e_1 + a_2e_2 + a_3e_3\}$, such that $\rho(g) : W \to W$. Since W is 1-dimensional, so we have $\rho(g)W = \lambda W$, where λ is common for all $\rho(g)$. Then we write

$$\rho((1,2,3))\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_3 \\ a_1 \end{pmatrix} = \lambda\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\rho((2,3))\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_3 \\ a_2 \end{pmatrix} = \lambda\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

From the second action, we find that $\lambda = 1$ and $a_2 = a_3$. From the first action, we find $a_1 = a_2 = a_3$.

Hence, $W = \text{span}\{e_1 + e_2 + e_3\}$ and the restriction of $\rho(g)$ to W, denoted by $\rho_W(g)$, satisfies $\rho_W(g) = 1$. So $\rho(g)$ is the trivial representation on W.

The orthogonal complement of W, denoted by W', is defined by

$$W' = \text{span}\{a_1' e_1 + a_2' e_2 + a_3' e_3 : \sum_i a_i' = 0\}$$

Notice that $W \perp W'$.

Now, we can write down W and W'

$$W = \text{span}\left\{ a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad W' = \text{span}\left\{ b_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, b_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

and we have

$$\rho((1,2,3))a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot a$$

and

$$\rho((2,3))a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot a$$

So W is invariant under $\rho(g)$, for all $g \in G$, as we found above.

On the other hand,

$$\rho((1,2,3))b_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = b_2 - b_1$$

$$\rho((1,2,3))b_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = -b_1$$

$$\rho((2,3))b_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = -b_1$$

$$\rho((2,3))b_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = b_2 - b_1$$

Hence, W' is invariant under any $\rho(g)$, putting b_1, b_2 in a vector form, we have

$$\rho((1,2,3)) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_2 - b_1 \\ -b_1 \end{pmatrix}, \qquad \rho((2,3)) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -b_1 \\ b2 - b_1 \end{pmatrix}$$

Thus, the matrices $\rho(g)$ acts on $W^{'}$ are the 2×2 matrices

$$\rho_{W'}((1,2,3)) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \qquad \rho_{W'}((2,3)) = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$$

Since S_3 is generated by $(1,2,3)$ and $(2,3)$, so we can multiply $\rho_{W'}((1,2,3))$ and $\rho_{W'}((2,3))$ to get rest of $\rho_{W'}(g)$ s. Here we omit the calculations.

Can we reduce the dimension of $\rho_{W'}$ further? The answer is no. Because the matrices $\rho_{W'}(g)$ don't have common eigenvectors, so there is no smaller, invariant 1-dimensional vector subspace of $W^{'}$.

Hence, we can write the permutation representation $\rho(g)$ of S_3 as $\rho(g) \cong \rho_W(g) \oplus \rho_{W'}(g)$, i.e,

$$\begin{pmatrix} \rho_W(g) & 0 \\ 0 & \rho_{W'}(g) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

where $\rho_{W'}(g)$ is a **2-dimensional irreducible representation**.

It turns out that this is the only other irreducible representation of S_3. To see this, we use of theorem of Frobenius, stated as follows. We omit the proof. See [Ser71] p18.

Theorem 1.1 (Frobenius) If G is a finite group, and its irreducible representations are of degree $d_1, d_2, ..., d_n$, then

$$d_1^2 + d_2^2 + \;.., + d_n^2 = |G|$$

Example 1.25 For S_3, $|S_3| = 6$ and the sum of the square of degree of irreducible representations is

$$1^2 + 1^2 + 2^2 = 6$$

1.5 Induced Representations of Finite Groups

An **induced representation** is a way of passing from a representation of a subgroup H to a representation of the whole group G.

Definition 1.17 Let G be a finite group, H is an subgroup of G. Given a representation of H, ρ, on vector space W, let $n = [G : H]$ be the index of H in G, and let $g_1, ..., g_n$ be a full set of representatives in G of the left cosets in G/H. The induced representation, denoted by $\rho \uparrow_H^G$ acts on the following space

$$V = \bigoplus_{i=1}^{n} g_i W$$

by

$$\rho \uparrow_H^G (g) \sum_{i=1}^{n} g_i w_i = \sum_{i=1}^{n} g_{j(i)} \rho(h_i) w_i$$

where $w_i \in W$, and $j(i) \in 1, 2, ..., n$ are such that $gg_i = g_{j(i)} h_i$.

Definition 1.18 More formally, let H be a subgroup of finite group G. Let V be an H-module, we define a G-module over the group algebra $\mathbb{C}[G]$

$$\mathbb{C}[G] = \text{span}\{g_i : \ g_i \in G\}$$

and

$$\mathbb{C}[G] \otimes_H V = \text{span}\{g \otimes v_j : \ v_j \in V\}$$

such that $g \cdot h \otimes v_j = g \otimes (h \cdot v_j)$.

Thus, our induced representation is $\rho \uparrow_H^G = \mathbb{C}[G] \otimes_H V$.

Theorem 1.2 **(Frobenius Reciprocity Theorem)**
Let G be a finite group and H a subgroup of G. Let ρ be a representation of H and σ be a representation of G. Then the number of times that $\rho \uparrow_H^G$ (induction) contains σ equals to the number of times that the restriction $\sigma|_H$ contains ρ.
We omit the proof. See [Ser71] p28.

Example 1.26 Let $G = S_3$ and $H = A_3$, the subgroup of even permutations. Note that $A_3 \cong \{1, x, x^2\}$, where $x = (1, 2, 3)$. A_3 is a cyclic group of 3 elements.

There are three irreducible representations (characters) of A_3, denoted by $\chi(g)$. These are:

(i) Trivial character, $\chi_0(h) = 1$ for all $h \in A_3$.

(ii) $\chi(1) = 1$, $\chi(x) = e^{i(2\pi/3)}$, $\chi(x^2) = e^{i(4\pi/3)}$

(iii) $\overline{\chi(1)} = 1$, $\overline{\chi(x)} = e^{i(4\pi/3)}$, $\overline{\chi(x^2)} = e^{i(2\pi/3)}$

These are all the irreducible representations of the cyclic group of A_3.

Let σ be a representation of S_3, then as we found above, there are three irreducible representations,

(i) σ_0 is the trivial representation.

(ii) σ_{sgn} is the sign representation.

(iii) σ_2 is the 2-dimensional irreducible representation.

and we also discussed the 3-dimensional permutation representation σ_3.

We can see that the restriction of the representations to A_3 are:

(i) $\sigma_0|_{A_3} = \sigma_0$.

(ii) $\sigma_{sgn}|_{A_3} = \sigma_0$.

(iii) $\sigma_2|_{A_3} = \chi \oplus \overline{\chi}$.

Therefore, by Frobrnius Reciprocity Theorem, the induced representations are:

(i) $\chi \uparrow_{A_3}^{S_3}$ is equivalent to σ_2 therefore contains it once.

(ii) $\overline{\chi} \uparrow_{A_3}^{S_3}$ is also equivalent to σ_2 therefore contains it once.

(iii) $\chi_0 \uparrow_{A_3}^{S_3}$ is the permutation representation σ_3 which as we saw before, contains σ_0 once and σ_2 once.

Chapter 2

Compact Lie Groups

A **compact lie group** is a topological group G whose topology is compact, and both the group binary operation and the function mapping group elements to their corresponding inverses are continuous. A group is a Lie group if it is also a smooth manifold, and the group operation is not just continuous, but also smooth.

In this chapter, we approach compact Lie group by studying smooth manifolds and topological groups. Then, we will show that every finite group is a zero-dimensional compact Lie group. In this project, we focus on studying the topological properties of some fundamental matrix Lie groups, such as orthogonal group $O(n)$ and unitary group $U(n)$. So, we will briefly talk about some topological aspects of these matrix Lie groups, including compactness, path-connectedness, and simply connectedness. Finally, we will briefly look at the structure of compact Lie groups, which provides a way of constructing a compact Lie group from a simply connected compact Lie group.

The contents in this chapter are developed from these references:

[Sch13], [BD85], [Fol94], [Hal00], [Pri77], [Doo77]

2.1 Topological Manifold

Informally, a topological manifold is a topological space that locally looks like an Euclidean space $\mathbb{R}^n$, where n is fixed. For example,

(i) S^1 circle locally looks like $\mathbb{R}^1$.

(ii) $\mathbb{T}^2$ torus locally looks like $\mathbb{R}^2$.

(iii) S^2 sphere locally looks like $\mathbb{R}^2$.

Definition 2.1 A paracompact Hausdorff topological space (M, τ_M) is called a n-dimensional (topological) *manifold* if for every $x \in M$, there exists an open neighbourhood of x, $U(x) \in \tau_M$, and a homeomorhism $f : U \to f(U) \subseteq \mathbb{R}^n$. Here $\mathbb{R}^n$ is equipped with standard topology, denoted by τ_{std}. We can also define *complex* topological manifold, by letting f be a homemorhpism onto $\mathbb{C}^n$.

Example 2.1 $\mathbb{R}^n$ is an n-dimensional manifold for any $n \geq 1$. S^1 is 1-dimensional manifold and S^2 and T^2 are 2-dimensional manifold.

Definition 2.2 Let (M, τ_M) be a n-dimensional topological space, and a subset $Y \subseteq X$, then $(Y, \tau_X|_Y)$ is called a *submanifold* if it is a manifold on its own.

Example 2.2 $(\mathbb{R}^n, \tau_{std})$ is a manifold. $S^1 \subset R^2$, and $(S^1, \tau_{std}|_{S^1})$ is a submanifold of $(\mathbb{R}^n, \tau_{std})$.

Definition 2.3 Let (M, τ_M) be (N, τ_N) be topological manifold, then $(M \times N, \tau_{M\times N})$ is a topological manifold of dimension $\dim((M, \tau_M)) + \dim((N, \tau_N))$, it is called *product manifold*.

Example 2.3

(i) The torus $\mathbb{T}^2 = S^1 \times S^1$ is 2-dimensional product manifold.

(ii) $\mathbb{T}^n = S^1 \times S^1 ... S^1$ (n times) is n-dimensional product manifold.

(iii) The cylinder $C = S^1 \times \mathbb{R}$ is 2-dimensional product manifold.

(iv) The Möbius strip is not a product manifold, but locally it looks like a product manifold.

Definition 2.4 Let (M, τ_M) be a topological manifold, then a pair (U, ϕ) where $U \in \tau_M$ and $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$, is called a *chart* of the manifold.

Since $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} ... \times \mathbb{R}$, and for each $\mathbb{R}$, we define the component functions of ϕ to be:

$$\phi^i : U \to \mathbb{R}, \qquad x \mapsto proj_i(\phi(x))$$

are called the *co-ordinates* of the point $x \in U$ with respect to the chart (U, ϕ).

There could exist overlapping charts covering same point, or many charts could have non-empty overlap.

Definition 2.5 An *atlas* for a manifold (M, τ_M) is a collection $\mathcal{A} = \{(U_i, \phi_i) \mid i \in I\}$ such that

$$\bigcup_{i \in I} U_i = X$$

Definition 2.6 Two charts (U, ϕ) and (V, ψ) are said to be $\mathcal{C}^0-$*compatible* if either $U \cap V = \varnothing$, or the map:

$$\psi \circ \phi^{-1} : \phi(U \cap V) \longrightarrow \psi(U \cap V)$$

is continuous.

$\psi \circ \phi^{-1}$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

$$\begin{tikzcd}
 & U \cap V \subseteq X \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
\phi(U \cap V) \subseteq \mathbb{R}^n \arrow[rr, "\psi\circ\phi^{-1}"] & & \psi(U \cap V) \subseteq \mathbb{R}^n
\end{tikzcd}$$

Since ϕ and ψ are homeomorphism, so their inverses exist and are continuous. The map $\psi \circ \phi^{-1}$ and its inverse $\phi \circ \psi^{-1}$ are called *coordinate change map* or *chart transition map.*

Definition 2.7 A $\mathcal{C}^0 -$ *atlas* of a manifold is an atlas of pairwise $\mathcal{C}^0$-compatible charts. Any atlas is also a $\mathcal{C}^0 -$ *atlas.*

Definition 2.8 A $\mathcal{C}^0$-atlas $\mathcal{A}$ is said to be a *maximal atlas* if $(U, \phi) \in \mathcal{A}$, then all charts (V, ψ) which are $\mathcal{C}^0$-compatible with (U, ϕ), which are also belong to $\mathcal{A}$.

Example 2.4 Consider a topological manifold $(\mathbb{R}, \tau_{std})$ and an atlas of this manifold is $\mathcal{A} = \{(\mathbb{R}, \mathrm{Id}_\mathbb{R}), ((-\infty, 0), \mathrm{Id}_\mathbb{R})\}$, where $\mathrm{Id}_\mathbb{R}$ is the identity mapping from $\mathbb{R}$ to $\mathbb{R}$, $x \mapsto x$. Notice that $((0, \infty), \mathrm{Id}_\mathbb{R})$ is also $\mathcal{C}^0$-compatible with $(\mathbb{R}, \mathrm{Id}_\mathbb{R})$ and $((-\infty, 0), \mathrm{Id}_\mathbb{R})$, but it is not in $\mathcal{A}$, so $\mathcal{A}$ is not a maximal atlas.

2.2 Differential Manifold

Definition 2.9 An atlas of a topological manifold is called $\mathcal{C}^*$-atlas if any two charts $(U, \phi), (V, \psi) \in \mathcal{A}$ are $\mathcal{C}^*$-compatible.

Definition 2.10 $\mathcal{C}^*$ denotes continuously differentiable maps of order $*$, and it has several different orders:

(i) $\mathcal{C}^0$: the transition maps are continuous (not differentiable at all)

(ii) $\mathcal{C}^k$: the transition maps are k-times continuously differentiable

(iii) $\mathcal{C}^\infty$: the transition maps are *smooth* (continuously differentiable infinite many times); it is equivalent to say the atlas is a $\mathcal{C}^k$-atlas for all $k \geq 0$.

(iv) $\mathcal{C}^\omega$: the transition maps are smooth, along with being *analytic*, i.e the Taylor expansion is absolutely convergent and equals the function on some open ball.

(v) *complex*: the dimension of M is even, M is a *complex manifold* if the transition maps are continuous and satisfy Cauchy-Riemann equations; its complex dimension is half of dim(M).

A space with a $\mathcal{C}^0$-atlas may not admit any $\mathcal{C}^1$-atlas. However, if the manifold contains a $\mathcal{C}^1$-atlas, then $\mathcal{C}^\infty$-atlas can be obtained by only keeping the charts which are $\mathcal{C}^\infty$-compatible and removing rest of charts.

Definition 2.11 A $\mathcal{C}^k$-manifold is a triple $(M, \tau_M, \mathcal{A})$, where (M, τ_M) is a topological manifold and $\mathcal{A}$ is a maximal $\mathcal{C}^k$-atlas.

Note that a given topological manifold can carry different incompatible atlases.

Definition 2.12 Two $\mathcal{C}^*$-atlases, $\mathcal{A}$ and $\mathcal{B}$ are compatible if their union $\mathcal{A} \cup \mathcal{B}$ is again a $\mathcal{C}^*$-atlas, otherwise they are incompatible.

Example 2.5 Let $(M, \tau_M) = (\mathbb{R}, \tau_{std})$. Suppose there are two atlases $\mathcal{A} = \{(\mathbb{R}, Id_{\mathbb{R}})\}$ and $\mathcal{B} = \{(\mathbb{R}, \phi)\}$, where $\phi : x \to \sqrt[3]{x}$. Since both atlases contain only one chart, their transition maps are compatible, i.e. $Id_{\mathbb{R}} \circ Id_{\mathbb{R}}^{-1}$ and $\phi \circ \phi^{-1}$ are smooth. Thus, $\mathcal{A}$ and $\mathcal{B}$ are both $\mathcal{C}^\infty$-atlas.

Now, consider $\mathcal{A} \cup \mathcal{B} = \{(\mathbb{R}, Id_{\mathbb{R}}), (\mathbb{R}, \phi)\}$. The transition map $Id_{\mathbb{R}} \circ \phi^{-1}$ is the map $x \mapsto x^3$, so it is a smooth map. On the other hand, the transition map $\phi \circ Id_{\mathbb{R}}^{-1}$ is just ϕ, which is not smooth, because its first derivative is not defined at 0. Therefore, $\mathcal{A}$ and $\mathcal{B}$ are not even $\mathcal{C}^1$-compatible.

This example shows that we can equip the real line with different incompatible $\mathcal{C}^\infty$-atlas, and this implies arbitrary choice of atlases could lead to incompatibility.

Definition 2.13 Suppose $(M, \tau_M, \mathcal{A}_M)$ and $(N, \tau_N, \mathcal{A}_N)$ are two $\mathcal{C}^k$-manifolds,$k \geq 1$, and there is a map $\phi : M \to N$. Then, ϕ is said to be $\mathcal{C}^k$**-differentiable** at $x \in U$, with $\phi(x) \in V$, if for some charts $(U, f) \in \mathcal{A}_M$ and $(V, g) \in \mathcal{A}_N$, the map $g \circ \phi \circ f^{-1} : \mathbb{R}^{dimM} \to \mathbb{R}^{dimN}$, is $\mathcal{C}^k$-differentiable at $f(x) \in f(U) \subseteq \mathbb{R}^{dimM}$.

$$\begin{tikzcd}
U \subseteq M \arrow[r, "\phi"] \arrow[d, "f"] & V \subseteq N \arrow[d, "g"] \\
f(U) \subseteq \mathbb{R}^{dimM} \arrow[r, "g\circ\phi\circ f^{-1}"] & g(V) \subseteq \mathbb{R}^{dimN}
\end{tikzcd}$$

Note that we need to check whether the differentiability of ϕ depends on the what charts have been chosen. What if we have chosen different charts? So, we need to show the differentiability still holds at different point(chart) of interest.

Now, we need to prove the 'lifting' notion of differentiability from the chart representation of ϕ to the manifold level is well-defined. In other words, suppose for some other charts $(\widetilde{U}, \widetilde{f}) \in \mathcal{A}_M$ and $(\widetilde{V}, \widetilde{g}) \in \mathcal{A}_N$, we need to show if $g \circ \phi \circ f^{-1}$ is differentiable, then $\widetilde{g} \circ \phi \circ \widetilde{f}^{-1}$ is differentiable.

$$\begin{tikzcd}
\widetilde{f}(U \cap \widetilde{U}) \subseteq \mathbb{R}^{\dim M} \arrow[r, "\widetilde{g}\circ\phi\circ\widetilde{f}^{-1}"] & \widetilde{g}(V \cap \widetilde{V}) \subseteq \mathbb{R}^{\dim N} \\
U \cap \widetilde{U} \subseteq M \arrow[r, "\phi"] \arrow[u, "\widetilde{f}"] \arrow[d, "f"] & V \cap \widetilde{V} \subseteq N \arrow[u, "\widetilde{g}"] \arrow[d, "g"] \\
f(U \cap \widetilde{U}) \subseteq \mathbb{R}^{\dim M} \arrow[r, "g\circ\phi\circ f^{-1}"] \arrow[uu, bend left, "\widetilde{f}\circ f^{-1}"] & g(V \cap \widetilde{V}) \subseteq \mathbb{R}^{\dim N} \arrow[uu, bend right, "\widetilde{g}\circ g^{-1}"']
\end{tikzcd}$$

From the diagram above, consider the map $\widetilde{f} \circ f^{-1}$, since $(U, f), (\widetilde{U}, \widetilde{f}) \in \mathcal{A}_M$, where $\mathcal{A}_M$ is C^k-atlas, then $\widetilde{f} \circ f^{-1}$ is C^k-differentiable map between subsets of $\mathbb{R}^{dimM}$, and similar analogy for $\widetilde{g} \circ g^{-1}$. Now we can represent $\widetilde{g} \circ \phi \circ \widetilde{f}^{-1}$ by:

$$\widetilde{g} \circ \phi \circ \widetilde{f}^{-1} = (\widetilde{g} \circ g^{-1}) \circ (g \circ \phi \circ f^{-1}) \circ (\widetilde{f} \circ f^{-1})^{-1}$$

This composition map is again $\mathcal{C}^k$**-differentiable**.

This illustration demonstrates the significance of restricting the topological manifolds to be $\mathcal{C}^k$-atlases. These atlases only contain charts with transition maps being $\mathcal{C}^k$-differentiable. By this way, the differentiable maps between manifolds can be well-defined.

Example 2.7 Consider the smooth manifolds $(\mathbb{R}^d, \tau_{std}, \mathcal{A}_d)$ and $(\mathbb{R}^{d'}, \tau_{std}, \mathcal{A}_{d'})$, and $\mathcal{A}_d$ and $\mathbb{R}^{d'}$ are containing the charts $(\mathbb{R}^d, Id_{\mathbb{R}^d})$ and $(\mathbb{R}^{d'}, Id_{\mathbb{R}^{d'}})$ respectively. Now, let $\phi : \mathbb{R}^d \to \mathbb{R}^{d'}$, then the diagram defining differentiability of ϕ is:

$$
\begin{tikzcd}
\mathbb{R}^d \arrow[r, "\phi"] \arrow[d, "Id_{\mathbb{R}^d}"] & \mathbb{R}^{d'} \arrow[d, "Id_{\mathbb{R}^{d'}}"] \\
\mathbb{R}^d \arrow[r, "Id_{\mathbb{R}^{d'}} \circ \phi \circ (Id_{\mathbb{R}^d})^{-1}"] & \mathbb{R}^{d'}
\end{tikzcd}
$$

By definition, the map ϕ is a smooth map between manifolds if any only if the map $Id_{\mathbb{R}^{d'}} \circ \phi \circ (Id_{\mathbb{R}^d})^{-1} = \phi$ is smooth.

Definition 2.14 Let $(M, \tau_M, \mathcal{A}_M)$ and $(N, \tau_N, \mathcal{A}_N)$ be smooth manifolds, and let $\phi : M \to N$ be a bijective map between smooth manifolds. If both ϕ and ϕ^{-1} are smooth, then ϕ is called **diffeomorphism**. Diffeomorphism is the structure preserving map between smooth manifolds.

Definition 2.15 Two smooth manifolds $(M, \tau_M, \mathcal{A}_M)$ and $(N, \tau_N, \mathcal{A}_N)$ are said to be *diffeomorphic* if there exists a diffeomorphism $\phi : M \to N$ between them. It is written as $M \cong_{diff} N$.

2.3 Topological Groups

Definition 2.16 A **topological group** is a group G equipped with a topology with the group operation $*$ being continuous, that is,

$$G \times G \to G, \quad (x, y) \mapsto xy$$

and

$$G \to G, \quad x \mapsto x^{-1}$$

are continuous for $x, y \in G$ (Here the group operation notation $*$ is omitted).

If G is a topological group, then the identity element of G is denoted as e. If $U \subset G$ and $x \in G$, we can define translations and inversion as:

$$xU := \{xy \mid y \in U\}, \quad Ux := \{yx \mid y \in U\}, \quad U^{-1} := \{y^{-1} \mid y \in U\}$$

and if $V \in G$, then we can define:

$$UV := \{xy \mid x \in U, y \in V\}$$

Remark:

The domain of the group operation $G \times G$ is equipped with product topology. Recall that, for any open set $U \subset G$, the map f is continuous if $f^{-1}(U)$ is open in the domain of f.

Thus the group operation is continuous if for any open subset $U \subset G$ with $xy \in U$, there exists and open $V \times W \subset G \times G$ with $(x, y) \in V \times W$, and $VW := \{pg \mid p \in V,\, q \in W\} \subseteq U$.

Similarly, the inverse is continuous if for any open set $U \in G$ with $x^{-1} \in U$, there exists an open subset $V \subset G$ with $x \in V$, such that $V^{-1} := \{p^{-1} \mid p \in V\} \subseteq U$.

Proposition 2.1 Let $U \subset G$, then U is called *symmetric* if $U = U^{-1}$. In addition, $U \cap V = \varnothing$ if and only if $e \notin U^{-1}V$.

Proof:

If $U \cap V \neq \varnothing$, then there exists an element g, $g \in U$ and $g \in V$ such that $g^{-1}g = e \in U^{-1}V$.

Now, we shall look at some basic properties of topological groups.

Proposition 2.2 Let G be a topological group, then

(i) The topology of G is invariant under group translations and inversion; that is, if $U \subset G$ is open, then xU, Ux and U^{-1} are open for any $x \in G$ (group translation and inversion of any open set after are still open in the topology of G). Also, if U is open, then VU and UV are open for any $V \in G$.

(ii) For every neighbourhood U of e, there is a symmetric neighbourhood V of e such that $VV \subset U$.

(iii) If H is a subgroup of G, so is $\bar{H}$ (Complement of H)

(iv) Every open subgroup of G is closed.

(v) If U and V are compact sets in G, so is UV.

Proof:

(i) The separate continuity of the map $(x, y) \mapsto xy$, i.e. fix either x or y. Also the continuity of map $x \mapsto x^{-1}$.

VU and UV are open since $VU = \bigcup_{x \in V} xU$ and $UV = \bigcup_{x \in V} Ux$ are open.

(ii) The continuity of map $(x, y) \mapsto xy$ at identity means for every neighbourhood U of identity, there are neighbourhoods V_1 and V_2 of identity such that $V_1V_2 \subset U$, and we can set V to be $V_1 \cap V_1^{-1} \cap V_2 \cap V_2^{-1}$.

(iii) If $x, y \in \bar{H}$ and there are sequences $\{x_n\}$ and $\{y_n\}$ converging to x, y. Then $x_ny_n \to xy$ and $x_n^{-1} \to x^{-1}$, which means xy and x^{-1} are in $\bar{H}$.

(iv) If H is open, then all its cosets xH for all $x \in G$ are open. The complement of H, $G\backslash H$ is just the union of all the cosets of H except H itself. So $G\backslash H$ is open and H is closed.

(v) For $x \in U$ and $y \in V$, UV is the image of the compact set $U \times V$ under continuous map $(x, y) \mapsto xy$, which is compact.

2.4 Compact Lie Groups

<u>**Definition 2.17**</u> A **Lie group** is a *differentiable* manifold G which is also a group such that the group operation $*$

$$G \times G \to G, \quad (x, y) \mapsto xy$$

and

$$G \to G, \quad x \mapsto x^{-1}$$

are differentiable maps.

A **homomorphism** of Lie groups is a differentiable group homomorphism between Lie groups.

Remark:

Differentiable means infinitely many times differentiable (Definition 2.10), denoted by $\mathcal{C}^\infty$.

<u>**Definition 2.18**</u> A Lie Group is a **compact Lie group** if its underlying manifold (topology) is compact.

<u>**Proposition 2.3**</u> Every **finite group** is a zero-dimensional **compact Lie group**.

Proof:

If $|G| < \infty$, then we can equip G with discrete topology, where the open sets are all subsets of G, i.e. 2^G.

Then, we need to show that both group multiplication and group inversion are smooth maps, which is $2^G \times 2^G \mapsto 2^G$ and $2^G \mapsto 2^G$. Let $U \subset 2^G$, since preimage(U) is just a tuple of open sets in 2^G, so the group multiplication is a smooth map. Similar argument hold for the group inversion map.

Also, we need to show that every finite group is a manifold, by Definition 2.1. Since we have a metric on 2^G i.e, $d(x, y) = 1$ if $x = y$, otherwise $d(x, y) = 0$. So 2^G is Hausdorff. The discrete topology is compact since every open cover has finite subcover ($|G| < \infty$), and compactness implies paracompact. In addtion, given a open set $U \subset 2^G$, we can define a homeomorphism

$$f : U \to \mathbb{R}^0$$

which is just the identity map (smooth) $f(g) = g$ and $\mathbb{R}^0$ is just a collection of discrete points. Certainly f and f^{-1} is bijective and continuous. So f is a homeomorphsim. Finally, we can form a maximal $\mathcal{C}^\infty$ atlas $\mathcal{A}$ (Definition 2.8)

$$\mathcal{A} = \{(U_i, f_i) \,|\, f_i : U_i \to \mathbb{R}^0 \text{ for } U_i \subset 2^G\}$$

So, we have proved that every finite group is a zero-dimensional compact Lie group.

Now, we want to look at some important Lie groups, especially the 'matrix' Lie groups.

Proposition 2.4 $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are **Lie groups**.

Proof:

Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. The set $\mathrm{Aut}(V)$ is the linear automorphisms of V, and it is also an open subset of the open subset of $\mathrm{End}(V)$ of linear maps $V \to V$.

Since $\mathrm{Aut}(V) = \{A \in$ End(V): $\det(A) \neq 0\}$ and determinant $\det : \mathrm{Aut}(V) \to \mathbb{R}^1$ is a continuous map. So $\mathrm{Aut}(V)$ has the structure of a differentiable manifold.

If we introduce V to be $\mathbb{R}^n$ and $\mathbb{C}^n$, then the group operation of $\mathrm{Aut}(V)$ is matrix multiplication and it is differentiable. Thus, $\mathrm{Aut}(V)$ has a structure of Lie group and we can write

$$GL(n, \mathbb{R}) = Aut_{\mathbb{R}}(\mathbb{R}^n), \qquad GL(n, \mathbb{C}) = Aut_{\mathbb{C}}(\mathbb{C}^n)$$

Definition 2.19 The **orthogonal group** is written as

$$O(n) = \{A \in GL(n, \mathbb{R}) : A^T A = I_n\}$$

such that the column vectors of A are orthonormal, that is

$$\sum_{i=1}^{n} A_{ij} A_{ik} = \delta_{jk}$$

It has following properties:

(i) $A^T = A^{-1}$.

(ii) A preserves inner product, i.e. $< x, y >= (Ax, Ay)$ for all vectors $x, y \in \mathbb{R}^n$.

(iii) $\det(A) = \pm 1$, since $\det(A^T) = \det(A)$ and $\det(A^T A) = (\det(A))^2 = \det(I_n) = 1$.

So, $O(n)$ can be split into two parts, by the values of determinant 1 or -1.

Definition 2.20 The **special orthogonal group** is written as

$$SO(n) = \{A \in O(n) : \det(A) = 1\}$$

Example 2.8 The special orthogonal group $\mathbf{SO(n)}$, is also called n-dimensional rotation group, is the group of all rotations about the origin of $\mathbb{R}^n$, with group operation matrix multiplication.

Definition 2.21 The **unitary group** is written as

$$U(n) = \{A \in GL(n, \mathbb{C}) : A^* A = I_n\}$$

such that the column vectors of A are orthonormal, that is

$$\sum_{i=1}^{n} \overline{A_{ij}} A_{ik} = \delta_{jk}$$

It has following properties:

(i) $A^* = A^{-1}$.

(ii) A preserves inner product, i.e. $< x, y > = (Ax, Ay)$ for all vectors $x, y \in \mathbb{C}^n$.

(iii) $|\det(A)| = 1$, since $\det(A^*) = \overline{\det(A)}$ and $\det(A^*A) = |\det(A)|^2 = \det(I_n) = 1$.

Example 2.9 The circle group $\mathbb{T}$ is isomorphic to $U(1)$, i.e $\mathbb{T} \cong U(1)$

Example 2.10 The unitary group $\mathbf{U(2)}$ is written as

$$U(2) = \left\{ A = \begin{pmatrix} \alpha & \beta \\ -e^{i\theta}\bar{\beta} & e^{i\theta}\bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}$$

with $\det(A) = e^{i\theta}$.

Definition 2.22 The **special orthogonal group** is written as

$$SU(n) = \{A \in U(n) : \det(A) = 1\}$$

Example 2.11 The special unitary group $\mathbf{SU(2)}$ is written as

$$SU(2) = \left\{ A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}$$

with $\det(A) = 1$.

2.5 Compactness and Connectedness

Matrix groups are subsets of $\mathbb{R}^{n^2}$, thus we could use the **Heine-Borel theorem** to see if they are compact sets. This theorem says that in R^n, compact sets are exactly the closed and bounded sets.

Proposition 2.5 The circle group $\mathbb{T} \cong U(1)$ is compact.

Proof:

We know that

$$\mathbb{T} \cong A = \{(x, y) | x^2 + y^2 = 1\} \subset \mathbb{R}^2$$

where $\mathbb{R}^2$ is equiped with standard topology such that the open set (open ball) is defined by $B_r(x) = \{y \mid d(y, x) < r\}$. A is clearly closed and bounded.

Proposition 2.6 The special unitary group $SU(2)$ is compact.

Proof: We know that

$$SU(2) = \left\{ A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}$$

Let $\alpha = a + ib$ and $\beta = c + id$ for $a, b, c, d \in \mathbb{R}$, then

$$a^2 + b^2 + c^2 + d^2 = 1$$

This is S^3, the 3-sphere in $\mathbb{R}^4$, therefore it is compact.

Hence, $SU(2) \cong S^3$ is compact.

In fact, we have a criterion for compactness for matrix Lie groups.

Definition 2.23 A matrix Lie group G is said to be **compact** if these two conditions are satisfied:

1. Any matrix sequence A_n in G converges to a matrix A in G.
2. There exists a constant c such that for all $A \in G$, $|A_{ij}| \leq c$ for all $1 \leq i, j \leq n$.

Proposition 2.7 The groups $O(n)$, $SO(n)$, $U(n)$ and $SU(n)$ are compact.

Proof: For orthogonal groups,

1. The limit of a sequence orthogonal matrices is orthogonal and the limit of a sequence orthogonal matrices with $\det(A) = 1$ has determinant 1.
2. if A is orthogonal, then the column vectors of A has norm 1. Hence, $|A_{ij}| \leq 1$ for all $1 \leq i, j \leq n$.

Similar argument can be made to unitary groups.

Thus, $O(n)$, $SO(n)$, $U(n)$ and $SU(n)$ are compact.

Definition 2.24 A matrix Lie group is **connected** if given any two matrices A and B, there exists a continuous path $\psi(x)$, for $0 \leq x \leq 1$, on G such that $\psi(0) = A$ and $\psi(1) = B$.

From Definition A.18, this is **path-connectedness** in topology, and in fact, a matrix Lie group is connected if and only if it is path-connected.

Definition 2.25 If a matrix Lie is not connected, then it can be decomposed into union of pieces, called **connected components**. The connected components are the largest sets where each two elements can be joined by a continuous path.

Proposition 2.8 The groups $O(n)$ is not connected, but $SO(n)$ is connected.

Proof:
From Definition 2.*, the determinant of $O(n)$ is defined by

$$\det : O(n) \to \{-1, 1\}$$

which is surjective and continuous since $O(n)$ and $\{-1, 1\}$ in their topologies. Since

$$O(n) = \det^{-1}(\{-1\}) \cup \det^{-1}(\{-1\})$$

so $O(n)$ is decomposed into the union of two components. Thus, $O(n)$ is not connected.

However, for $SO(n)$, $\det : SO(n) \to 1$. Thus $SO(n)$ is connected.

Proposition 2.9 The groups $U(n)$ and $SU(n)$ are connected.

Proof:

From linear algebra, every unitary matrix has eigenvalues of the form $e^{i\theta}$ and orthogonal eigenvectors. It follows that every unitary matrix A can be decomposed as

$$A = A_1 \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} A_1^*$$

where A_1 is unitary and $\theta_n \in \mathbb{R}$. Now define

$$A(t) = A_1 \begin{pmatrix} e^{i(1-t)\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{(1-t)i\theta_n} \end{pmatrix} A_1^*$$

with $t \in [0, 1]$, to be a continuous path on $U(n)$ joining A and I. This shows $U(n)$ is connected. Since $SU(n) \subset U(n)$, so $SU(n)$ is also connected.

Definition 2.26 A connected matrix Lie group G is **simply connected** if every loop in G can be continuously shrunk to a single point on G.

That is, G is simply connected if given any continuous path $\psi(x)$, for $0 \leq x \leq 1$, on G with $\psi(0) = \psi(1)$, there exists a continuous function $\psi(\gamma, x)$, for $0 \leq \gamma, x \leq 1$, taking values in G.

We can think of $\psi(\gamma, x)$ as a family of loops varying with respect to γ. It has following properties:

(i) $\psi(\gamma, 0) = \psi(\gamma, 1)$, means there is a loop for each γ.

(ii) $\psi(0, x) = \psi(x)$, means it is the specified loop when $\gamma = 0$.

(iii) $\psi(1, x) = \psi(1, 0)$, means the loop is a point when $\gamma = 1$.

This is actually a homotopy.

Proposition 2.10 The group $SU(2)$ is simply connected.

Proof:

We have $SU(2) \cong S^3$, the 3-sphere in $\mathbb{R}^4$, S^3 is simply connected, i.e. every loop on S^3 can be shrunk to a point without leaving S^3.

Proposition 2.11 The group $U(1)$ is not simply connected.

Proof:

The group $U(1)$ is isomorphic to the circle group, $\mathbb{T}$, and every loop on $\mathbb{T}$ is just the circle group itself. However, we can not shrink every loop to a point on the circle without leaving the circle. So, $U(1)$ is not simply connected.

2.6 Structure of Compact Lie Groups

Proposition 2.12 If G is a compact Lie group, and G_0 is the connected component of e (identity component). Then G_0 is a closed normal subgroup of G, G_0 is also open whenever G is a Lie group amd G/G_0 is a finite group.

Proof:

Since G_0 is the maximal connected subset of G containing the identity e, thus $g^{-1}G_0g$ is connected and $e \in g^{-1}G_0g$, and $G_0 = g^{-1}G_0g$. Hence, $G_0 \triangleleft G_0$.

G is a Lie group, for each $x \in G_0$, the left translation $xG_0 \mapsto G_0$ is continuous. Also, the inverse x^{-1} is continuous, and x^{-1} is in G_0. So G_0 is open.

Also, if G is a compact Lie group, then G_0 is an open subset of G. So, the cosets xG_0 exactly cover G. Since G is compact, this cover is finite. Hence, G/G_0 is finite.

Theorem 2.1 Let G be compact connected group, then there exists a family $(G_i)_{i\in I}$ of compact simply connected simple Lie groups, a compact connected abelian group A, and a totally disconnected subgroup of the center of $A \times \prod_{i\in I} G_i$ such that

$$G \cong (A \times \prod_{i\in I} G_i) \,/\, Z$$

If G is a connected compact Lie group, then we may assume A is $\mathbb{T}^n$ (n-dimensional torus), which is

$$\mathbb{T}^n \cong \mathbb{T} \times \mathbb{T} \times ......\mathbb{T} \quad (n \text{ times})$$

We omit the proof. See [Pri77] p145.

Definition 2.27 The *centre* of a group G is a normal subgroup Z, defined by

$$Z = \{x \in G : xy = yx \text{ for all } y \in G\}$$

Here are some examples of centre of compact Lie groups:

(i) $O(n) : \{I_n, -I_n\}$.

(ii) $SO(n) : \{I_n, -I_n\}$ if n is even and $\{I_n\}$ otherwise.

(iii) $U(n) : \{e^{i\theta} \cdot I_n : \theta \in [0, 2\pi)\}$

(iv) $SU(n) : \{e^{i\theta} \cdot I_n : \theta = 2k\pi/n, k = 0, 1, ...n-1\}$

For example, $SU(2)$, with $\theta = 0, \pi$, so the centre of $SU(2)$ is $\{I_2, -I_2\}$.

Proposition 2.13 $U(2)$ can be written as

$$U(2) \cong (\mathbb{T} \times SU(2)) \,/\, \{(1, I), (-1, -I)\}$$

Proof:

We want to find the centre Z.

If $A \in U(2)$, then $|\det(A)| = 1$, and $\det(A) \in \mathbb{T}$.

Let $z \in \mathbb{T}$ and $B \in SU(2)$, then $A = zB$.

Since $AA' = zBz'B' = zz'BB'$, so $f((zz', BB')) = f((z, B))f((z', B')) = AA'$ is a homomorphism from $\mathbb{T} \times SU(2)$ to $U(2)$.

We know that $ker(f) = \{(z, B) : zB = I\}$, so B is in the form of

$$B = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \qquad \text{and} \quad |\alpha|^2 = 1$$

and

$$zB = \begin{pmatrix} z\alpha & 0 \\ 0 & z\bar{\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which implies $z\alpha = z\bar{\alpha}$, then $\alpha = \bar{\alpha}$. This means α is real, and z is real. We know $|z| = 1$, thus $z = \pm 1$ and $\alpha = \pm 1$. Then

$$ker(f) = \left\{ \pm 1, \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{(1, I), (-1, -I)\} = Z$$

and we can conclude that

$$U(2) \cong (\mathbb{T} \times SU(2)) \,/\, \{(1, I), (-1, -I)\}$$

Proposition 2.14 $SU(2)$ is simply connected, and

$$SU(2)/\{I, -I\} \cong SO(3)$$

Proof:

Consider a 3-dimensional real vector space V containing all 2×2 complex matrices which are self-adjoint and have trace zero.

Let $\{A_1, A_2, A_3\}$ be a basis of V

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We can define an inner product on V

$$< A \,,\, B >= \frac{1}{2}\mathrm{Tr}(AB)$$

e.g

$$< A_1\,,\,A_2 >= 0 \qquad < A_1\,,\,A_1 >= 1$$

So, $\{A_1, A_2, A_3\}$ is an orthonormal basis of V. In fact, V represents $\mathbb{R}^3$.

To see this, for any $(x_1, x_2, x_3) \in \mathbb{R}^3$

$$x_1A_1 + x_2A_2 + x_3A_3 = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix}$$

Also, if $U \in SU(2)$ and $A \in V$, then

$$UAU^{-1} \in V$$

For any $U \in SU(2)$, we define a map $\phi_U : V \to V$ such that $\phi_U(A) = UAU^{-1}$.

Notice that for any $A, B \in V$,

$$\begin{aligned} < \phi(A)\,,\,\phi(B) > &= \frac{1}{2}\mathrm{Tr}(UAU^{-1}UBU^{-1}) \\ &= \frac{1}{2}\mathrm{Tr}(U^{-1}UAU^{-1}UB) \\ &= \frac{1}{2}\mathrm{Tr}(AB) \\ &=< A\,,\,B > \end{aligned}$$

Thus, ϕ_U is an orthogonal transformation of $\mathbb{R}^3$, and we conclude that

$$\phi : SU(2) \to O(3)$$

ϕ is homomorphism and continuous, i.e. $\phi_{U_1U_2} = \phi_{U_1}\phi_{U_2}$.

However, $\det(O(3)) = \pm 1$, and $O(3)$ is <u>not</u> connected, but $SU(2)$ is connected, so ϕ maps $SU(2)$ to $SO(3)$.

Since $\phi_U = \phi_{-U}$, so it is a two-to-one mapping, then $\ker(\phi) = \{I, -I\}$.

Thus, $SU(2)/\{I, -I\} \cong SO(3)$.

Chapter 3

Representations of Compact Lie Groups

In this chapter, we begin with an introduction to the characters of an abelian group and their relationship to Fourier series. Then we will introduce the unitary representations of compact groups on a Hilbert space $\mathcal{H}$. If G is compact, then an irreducible representation of G on $\mathcal{H}$ is finite dimensional. If we choose a basis on $\mathcal{H}$, then the irreducible representation on $\mathcal{H}$ is in the form of a matrix. Once we have these irreducible representations, we need to find a way to pass these representations to $L^2(G)$ to perform Fourier analysis. To do so, we have a notion for this, called matrix coefficients. Also, we will introduce the Plancherel theorem for a non-abelian group. Most importantly, we will provide a concrete example on finding the irreducible representations of the special unitary group $SU(2)$. Eventually, we will briefly talk about the induced representations of compact groups.

The contents in this chapter are developed from these references:

[Fol94], [Doo00], [HR70]

3.1 Characters of Abelian Groups and Fourier Series

<u>**Definition 3.8**</u> Let G be a locally compact Abelian group, let σ be an irreducible representation of G on $\mathcal{H}_\sigma = \mathbb{C}$, then $\sigma(g)z = \chi(g)z$ for $z \in \mathbb{C}$, where $\chi(g)$ is defined by

$$\chi : G \to \mathbb{T}$$

which is a continuous homomorphism of G into the circle group $\mathbb{T}$.

These homomorphisms are called **characters** of G, and the set of all characters of G is called **dual** of G, denoted by $\widehat{G}$.

By Definition 3.*, $\chi(g) = <\sigma(g)1\,,\,1>$, so $\chi(g)$ is in $L^2(G)$. By Proposition 3.*, if G is compact, then $\widehat{G}$ is an orthonormal basis for $L^2(G)$.

<u>**Proposition 3.5**</u>

Here are some duals of locally compact Abelian groups.

(i) $\widehat{\mathbb{R}} \cong \mathbb{R}$

(ii) $\widehat{\mathbb{T}} \cong \mathbb{Z}$

(iii) $\widehat{\mathbb{Z}} \cong \mathbb{T}$

Proof:

(i) Let $\chi \in \widehat{\mathbb{R}}$, then $\chi(e) = \chi(0) = 1$. Since χ is continuous, then there exists $a > 0$ such that
$$\int_0^a \chi(u)du \neq 0$$
Let $A = \int_0^a \chi(u)du$, then
$$A\chi(x) = \int_0^a \chi(x)\chi(u)du = \int_0^a \chi(x+u)du = \int_x^{a+x} \chi(t)dt$$
We can see $\chi(x)$ is differentiable and
$$\chi'(x) = A^{-1}(\chi(a+x) - \chi(x)) = A^{-1}\chi(x)(\chi(a) - 1) = c\chi(x)$$
where $c = A^{-1}(\chi(a) - 1)$ is some constant.

$\chi'(x) = c\chi(x)$ is just an ODE, solve it we get $\chi(x) = e^{cx}$, since $|\chi| = 1$, and $c = i\lambda$ for some $\lambda \in \mathbb{R}$, i.e,
$$\chi(x) = e^{i\lambda x}$$
Thus, $\widehat{\mathbb{R}} \cong \mathbb{R}$

(ii) $\widehat{\mathbb{T}}$ can be derived from $\widehat{\mathbb{R}}$. By Example 1.22, $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$, i.e, $x \in [0, 2\pi)$, we can then write

$$\xi(x) = e^{i\lambda x}$$

for some $\lambda \in \mathbb{R}$ and $x \in [0, 2\pi)$.

However, consider $e^{i\lambda x}e^{i\lambda y} = e^{i\lambda(x+y)}$ where $x + y = 2\pi$, then $e^{i\lambda(x+y)} = 1$, the only way to make this possible is that $\lambda \in \mathbb{Z}$, so we rewrite $\chi(x)$ as

$$\chi(x) = e^{inx}$$

for $n \in \mathbb{Z}$ and $x \in [0, 2\pi)$.

Thus, $\widehat{\mathbb{T}} \cong \mathbb{Z}$

(iii) Let $\chi \in \widehat{\mathbb{Z}}$, then $\alpha = \chi(1) \in \mathbb{T}$, and

$$\chi(n) = \chi(1 + 1 + ...1) = \chi(1)\chi(1)....\chi(1) = \chi(1)^n = \alpha^n$$

Thus, $\widehat{\mathbb{T}} \cong \mathbb{Z}$

Example 3.3

Here we will work out the characters of the compact Abelian group $\mathbb{T}$ on $L^2(\mathbb{T})$.

Let $G = \mathbb{T}$, $g = e^{i\theta} \in \mathbb{T}$ and $\mathcal{H} = L^2(\mathbb{T})$.

For $f(z) \in L^2(\mathbb{T})$, we define the regular representation $\rho(g)f(z) = f(g \cdot z) = f(e^{i\theta}z)$.

We also define inner product on $L^2(\mathbb{T})$ by

$$< f\, ,\, k >= \int_{\mathbb{T}} f(z)\overline{k(z)}dz$$

This inner product is invariant under $\rho(g)$ since

$$\begin{aligned} < \rho(g)f\, ,\, \rho(g)k > &= \int_{\mathbb{T}} f(e^{i\phi}z)\overline{k(e^{i\phi}z)}dz \\ &= \int_{\mathbb{T}} f(z)\overline{k(z)}dz \quad \text{invariant Haar measure} \\ &=< f\, ,\, k > \end{aligned}$$

Since $\widehat{\mathbb{T}} \cong \mathbb{Z}$, then $\rho(e^{i\theta}) = e^{in\theta}$, and by Peter-Weyl's theorem, for $n \in \mathbb{Z}$, we have

$$L^2(\mathbb{T}) = \oplus_{n\in\widehat{T}}\mathcal{H}_n$$

where $\mathcal{H}_n$ is

$$\mathcal{H}_n = \{f \in L^2(\mathbb{T}) : f(e^{i\theta}z) = e^{in\theta}f(z)\}$$

Let $z = e^{i\phi} \in \mathbb{T}$, then $f(z) = f(e^{i\phi}) = e^{in\phi}$, since

$$f(e^{i\theta}z) = f(e^{i(\theta+\phi)}) = e^{in(\theta+\phi)} = e^{in\theta}f(z)$$

In fact, $L^2(\mathbb{T}) = \oplus_{n\in\widehat{T}}\mathcal{H}_n$ is the Fourier series. It turns out that the Fourier series is hiding in the group representation theory!

For example, suppose f is 2π-periodic and $f \in L^2(\mathbb{T})$.

The measure of $\mathbb{T}$ is normalized to be 1,

$$\lambda(\mathbb{T}) = \int_{\mathbb{T}} d\lambda = \frac{1}{2\pi}\int_0^{2\pi} dx = 1$$

Then the Fourier series of f is

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx < \infty$$

and the Fourier coefficient is

$$a_n = \widehat{f}(n) =< f \, , \, e_n >= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$$

In the next sections, we will deal with non-abelian groups, and the characters will be replaced by irreducible representations.

3.2 Compact Group Representation

Definition 3.1 Let $\mathcal{H}$ be Hilbert space, then the **unitary representation** of G on $\mathcal{H}$ is a homomorphism defined by

$$\sigma : G \to \mathcal{U}(\mathcal{H})$$

such that it satisfies following properties:

(i) $\sigma(gh) = \sigma(g)\sigma(h)$.

(ii) $\sigma(g^{-1}) = \sigma(g)^{-1} = \sigma(g)^*$.

(iii) $g \mapsto \sigma(g)\xi$ is continuous for any $\xi \in \mathcal{H}$

(iv) $< \sigma(g)\xi \, , \, \sigma(g)\eta >=< \xi \, , \, \eta >$ for any $\xi, \eta \in \mathcal{H}$

Theorem 3.1 If G is a compact group, ρ is an irreducible representation of G on $\mathcal{H}$, then $\dim(\mathcal{H}) < \infty$.

Proof:

Let $\xi \in \mathcal{H}$ with $||\xi|| = 1$, and there is a continuous map $g \mapsto \sigma(g)\xi$.

The image of this map is compact, and also a compact subset of the unit ball $\mathcal{H}$, and by irreducibility, this compact subset spans $\mathcal{H}$. Thus, $\mathcal{H}$ is finite dimensional.

Hence, an irreducible representation of a compact group must be finite dimensional.

Remark:
If σ is a representation of a compact group G on a separable Hilbert space $\mathcal{H}$, we have

$$\mathcal{H} = \oplus_\sigma \mathcal{H}_\sigma$$

where each $\mathcal{H}_\rho$ is a finite-dimensional space and carrying an irreducible representation $\rho|_{\mathcal{H}_\sigma}$ of G.

Proposition 3.1 Let G be a compact group, and σ is a representation (not necessarily unitary) of G on $\mathcal{H}$ with inner product $(\cdot\,,\,\cdot)$. Then we define a new inner product

$$<\xi\,,\,\eta>=\int_G (\sigma(g)\xi,\,\sigma(g)\eta)\,dg$$

for any $\xi, \eta \in \mathcal{H}$, and $|(\sigma(g)\xi,\,\sigma(g)\eta)| \leq K||\xi||||\eta||$, such that

$$<\sigma(g_0)\xi\,,\,\sigma(g_0)\eta>=<\xi\,,\,\eta>$$

for all $g_0 \in G$, which makes σ unitary.

Proof:

First notice that since G is compact, so integrating a continuous function on a compact set is finite. Then,

$$\begin{aligned}
<\sigma(g_0)\xi\,,\,\sigma(g_0)\eta> &= \int_G (\sigma(g_0)\sigma(g)\xi,\,\sigma(g_0)\sigma(g)\eta)\,dg \\
&= \int_G (\sigma(g_0 g)\xi,\,\sigma(g_0 g)\eta)\,dg \\
&= \int_G (\sigma(g)\xi,\,\sigma(g)\eta)\,dg \qquad \text{Invariant Haar measure} \\
&=<\xi\,,\,\eta>
\end{aligned}$$

(This is known as Weyl's trick.)

Definition 3.2 Let G be a compact group, and $L^2(G)$ is a Hilbert space such that

$$L^2(G) = \{f : G \to \mathbb{C} \,:\, ||f||_2 = \left(\int |f(g)|^2 dg\right)^{\frac{1}{2}} < \infty\}$$

Then, the **regular representation** of G on $L^2(G)$ is defined by

$$(\rho(g)f)(x) = f(g^{-1}x) \quad \text{and} \quad (\rho(g)f)(x) = f(xg)$$

are the **left** and **right** regular representations.

Proposition 3.2 The regular representations of G on $L^2(G)$ are unitary.
i.e. $< \rho(g_0)f \, , \, \rho(g_0)k >=< f \, , \, k >$

Proof:
The inner product on $L^2(G)$ is defined by

$$< f \, , \, k >= \int_G f(g)\overline{k(g)} \, dg$$

Then,

$$\begin{aligned} < \rho(g_0)f \, , \, \rho(g_0)k > &= \int_G \rho(g_0)f(g)\overline{\rho(g_0)k(g)} \, dg \\ &= \int_G f(g_0^{-1}g)\overline{k(g_0^{-1}g)} \, dg \\ &= \int_G f(g)\overline{k(g)} \, dg \quad \text{left invariant Haar measure} \\ &=< f \, , \, k > \end{aligned}$$

Theorem 3.2 **(Schur's Lemma)**

If σ_1 and σ_2 are unitary representations of G, then an **intertwining operator** for σ_1 and σ_2 is a bounded linear map $T : \mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2}$, such that $T\sigma_1(g) = \sigma_2(g)T$ for all $g \in G$. The set of all such operators is defined by

$$P(\sigma_1, \sigma_2) = \{T : \mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2} : T\sigma_1(g) = \sigma_2(g)T \text{ for all } g \in G\}$$

and σ_1, σ_2 are called **equivalent** if $P(\sigma_1, \sigma_2)$ contains a unitary operator.

Let $P(\sigma) = P(\sigma, \sigma)$, then this is the space of bounded operators on $\mathcal{H}_\sigma$ that commute with $\sigma(g)$ for all $g \in G$.

The statements of Schur's lemma are:

(i) A unitary representation σ of G is irreducible if and only if $P(\sigma)$ contains scalar multiples of the identity.

(ii) Suppose σ_1 and σ_2 are irreducible unitary representations of G. If σ_1 and σ_2 are equivalent, then $P(\sigma_1, \sigma_2)$ is one-dimensional; otherwise, $P(\sigma_1, \sigma_2) = \{0\}$.

Proposition 3.3 If G is an **abelian** group, then every irreducible representation of G is **one-dimensional**.

Proof:

If σ is an irreducible representation of G, then all $\sigma(g)$ commute with each other, so they belong $P(\sigma)$. Thus we have $\sigma(g) = c_g I$.

Since H_σ is invariant under $\sigma(g)$, so $\dim(\sigma) = 1$.

3.3 Matrix Coefficients

Definition 3.3 Let V and W be finite dimensional vector spaces over $\mathbb{F}$ and

$$\psi : V \to W$$

is a linear operator such that

$$\psi(\alpha v + \beta w) = \alpha\psi(v) + \beta\psi(w)$$

for $v \in V, w \in W$ and $\alpha, \beta \in \mathbb{F}$.

Choose a basis $\{v_1, ..., v_n\}$ for V and a basis $\{w_1, ..., w_m\}$ for W, then

$$\psi(v_i) = \sum_{j=1}^{m} \alpha_{j,i} \cdot w_j$$

is the **change of basis**, given a $m \times n$ matrix $A = (\alpha_{ij})_{i=1,j=1}^{m,n}$.

Let $v = \sum_{i=1}^{n} \beta_i \cdot v_i$, then $\psi(v) = \sum_{i=1}^{n} \beta_i \cdot \psi(v_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_i \cdot \alpha_{j,i} \cdot w_j$.

For example, $\psi(v) = Av$.

In addition, if we want to choose different bases in V, W, and there are change of basis matrices $C : V \to V$ and $D : W \to W$, and the new change of basis matrix $\tilde{A}$ is $D^{-1}AC$.

Definition 3.4 If V, W are Hilbert spaces, and the map $\psi : V \to W$ is linear and continuous, also there are orthonormal bases $\{v_i : i \in \mathbb{N}\}$ of V and $\{w_i : i \in \mathbb{N}\}$ of W. Then,

$$< \psi \cdot v_i \, , \, w_j >_W$$

is a kind of infinite matrix which represents ψ.

Example 3.1 Let $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, then

$$< \psi \cdot v_i \, , \, w_j >_W = < \sum_{j=1}^{m} \alpha_{j,i} \cdot w_j \, , \, w_j >_W = \alpha_{j,i}$$

which is the $j - i$ th entry of matrix form of ψ.

Definition 3.5 In fact, every irreducible representation of G on H_σ is equivalent to a subrepresentation of $L^2(G)$. With the intuition of change of basis, we can define a operator that maps H_σ onto $L^2(G)$.

Let σ be the irreducible representation of $\mathcal{H}_\sigma$, and $\xi \in H_\sigma, \eta \in H_\sigma \backslash \{0\}$, then the map

$$t^{\sigma}_{\eta,\xi} : g \mapsto < \eta \, , \, \sigma(g)\xi >$$

is continuous and belongs to $L^2(G)$.

Thus, the mapping

$$\psi : \xi \mapsto t^{\sigma}_{\eta,\xi} : \mathcal{H}_\rho \to L^2(G)$$

intertwines σ and the right regular representation ρ.

That is, $\psi(\sigma(g)\xi)(g_0) = \rho(g)\psi(\xi)(g_0)$

To show this, since $\xi = \sum_i \beta_i \xi_i$, where $\{\zeta_i\}$ is the basis of $\mathcal{H}_\sigma$, then it is sufficient to show for $\eta = \zeta_i$ and $\xi = \zeta_j$.

Thus, we have

$$\begin{aligned}
\psi(\sigma(g)\zeta_i)(g_0) &= < \zeta_i \, , \, \sigma(g_0)\sigma(g)\zeta_j > \\
&= < \zeta_i \, , \, \sigma(g_0 g)\zeta_j > \\
&= \psi(\xi)(g_0 g) \\
&= \rho(g)\psi(\xi)(g_0)
\end{aligned}$$

The functions $t^{\rho}_{\eta,\xi}$ are called **matrix coefficients** of G.

Definition 3.6 Let σ be a unitary representation of the compact group G. Fix a unit vector $\xi_1 \in \mathcal{H}_\sigma$, then for any $\eta \in \mathcal{H}_\sigma$, we define the operator T on $\mathcal{H}_\sigma$ by

$$T\eta = \int_G < \sigma(x)\xi_1 \, , \, \eta > \sigma(x)\xi_1 \, dx$$

where T is positive, non-zero and compact, and $T \in P(\sigma)$.

Notice that $\eta \mapsto < \sigma(x)\xi_1 \, , \, \eta > \sigma(x)\xi_1$ is the **orthogonal projection** of η onto the line through $\sigma(x)\xi_1$, and T is the average over G of all these projections.

Also, we define

$$< T\eta \, , \, \eta > = \int_G < \sigma(x)\xi_1 \, , \, \eta > \overline{< \sigma(x)\xi_1 \, , \, \eta >} \, dx$$

Proposition 3.4 Let σ and $\sigma^{'}$ be the irreducible representations of G, then fix the unit vectors $\xi_1 \in \mathcal{H}_\sigma$ and $\eta_1 \in \mathcal{H}_{\sigma^{'}}$. Also let $\xi \in \mathcal{H}_\sigma$ and $\eta \in \mathcal{H}_{\sigma^{'}}$, then we have

$$\int_G t^{\sigma}_{\xi_1,\xi} \overline{t^{\sigma^{'}}_{\eta_1,\eta}} \, dg = \begin{cases} \dfrac{1}{d_\sigma} < \xi_1 \, , \, \eta_1 >< \xi \, , \, \eta > & \text{if } \sigma = \sigma^{'} \\ 0 & \text{otherwise} \end{cases}$$

These equations are called **Schur's orthogonality relations**.

Proof:

We define $T : \mathcal{H}_\sigma \to \mathcal{H}_{\sigma'}$, then

$$< T\xi \,,\, \eta >= \int_G \; < \sigma(x)\xi_1 \,,\, \xi > \overline{< \sigma'(x)\eta_1 \,,\, \eta >}\, dx$$

Also, T intertwines ρ and $\rho^{'}$. Notice that

$$\begin{aligned}
< T\sigma(x_0)\xi \,,\, \eta > &= \int_G \; < \sigma(x)\xi_1 \,,\, \sigma(x_0)\xi > \overline{< \sigma'(x)\eta_1 \,,\, \eta >}\, dx \\
&= \int_G \; < \sigma(x_0)^*\sigma(x)\xi_1 \,,\, \xi > \overline{< \sigma'(x)\eta_1 \,,\, \eta >}\, dx \\
&= \int_G \; < \sigma(x_0^{-1}x)\xi_1 \,,\, \xi > \overline{< \sigma'(x)\eta_1 \,,\, \eta >}\, dx \\
&= \int_G \; < \sigma(y)\xi_1 \,,\, \xi > \overline{< \sigma'(x_0y)\eta_1 \,,\, \eta >}\, dy \qquad (y = x_0^{-1}x) \\
&= \int_G \; < \sigma(y)\xi_1 \,,\, \xi > \overline{< \sigma'(y)\eta_1 \,,\, \sigma'(x_0^{-1})\eta >}\, dy \\
&=< T\xi \,,\, \sigma^{'}(x_0^{-1})\eta > \\
&=< \sigma^{'}(x_0)T\xi \,,\, \eta >
\end{aligned}$$

Thus, we have $T\sigma(x_0) = \sigma^{'}(x_0)T$.

By Schur's lemma, if $\sigma \neq \sigma^{'}$, then they are not equivalent, so $T = 0$. However, if $\sigma = \sigma^{'}$, then $T = cI$, a scalar multiple of identity.

Since G is compact, then it is unimodular (left and right Haar measure are the same), so we have

$$\begin{aligned}
< T\xi \,,\, \eta > &= \int_G \; < \sigma(x)\xi_1 \,,\, \xi > \overline{< \sigma(x)\eta_1 \,,\, \eta >}\, dx \\
&= \int_G \; < \xi_1 \,,\, \sigma(x^{-1})\xi > \overline{< \eta_1 \,,\, \sigma(x^{-1})\eta >}\, dx \\
&= \int_G \; < \xi_1 \,,\, \sigma(x)\xi > \overline{< \eta_1 \,,\, \sigma(x)\eta >}\, dx \\
&= c < \xi \,,\, \eta >
\end{aligned}$$

where c is a scalar which depends on ξ_1, η_1, so $c = c(\xi_1, \eta_1)$.

Same argument gives

$$c(\xi_1, \eta_1) < \xi \,,\, \eta >= c(\xi, \eta) < \xi_1 \,,\, \eta_1 >$$

Thus, $c(\xi, \eta) =< \xi \,,\, \eta >$, and the result follows.

Remarks:

Therefore, for each fixed unit vector η in H_σ, the mapping $\xi \mapsto t^\sigma_{\eta,\xi}$ is a map from H_σ into $L^2(G)$. In fact, it is an isometry (distance preserving map).

If $\{\xi_1, ...\xi_d\}$ is an orthonormal basis for $\mathcal{H}_\sigma$, then the subspaces $\mathcal{H}^\sigma_{\xi_1} ... \mathcal{H}^\sigma_{\xi_d}$ are orthogonal to each other. Hence, there are d copies of $\mathcal{H}_\sigma$ in $L^2(G)$.

From the orthogonality relations, it can be easily shown that the maps $t^{\sigma}_{\xi_i,\xi_j}$ are actually an orthonormal basis for $L^2(G)$. Linear combinations of these are called **trigonometric polynomials**. This result is one of the bases of harmonic analysis on compact groups.

Theorem 3.3 **(Peter-Weyl Theorem)**

Suppose that G is a compact group. Then,

$$L^2(G) = \oplus_{\sigma} d_{\sigma} \mathcal{H}_{\sigma}$$

This direct sum is over all irreducible representations of G.

3.4 The Plancherel Theorem for Compact Groups

Definition 3.7 Suppose that G is compact, $f \in L^2(G)$, and ρ is an irreducible representation of G on $\mathcal{H}_{\sigma}$. Choose an orthonormal basis for $\mathcal{H}_{\sigma}$. Then we have

$$< \widehat{f}(\sigma)\xi_i \, , \, \xi_j >_{\mathcal{H}_{\sigma}} = \int_G f(g) \, \overline{< \sigma(g)\xi_i \, , \, \xi_j >} \, dg =< f \, , \, t^{\sigma}_{\xi_i,\xi_j} >_{L^2(G)}$$

This is actually the **Fourier transform** for the compact group G.

By the orthogonality relations, if f is a trigonometric polynomial, then we have

$$f(x) = \sum_{\sigma} d_{\sigma} \sum_{i,j=1}^{d_{\sigma}} < \widehat{f}(\sigma)\xi_i \, , \, \xi_j > t^{\sigma}_{\xi_i,\xi_j}(x) = \sum_{\sigma} d_{\sigma} \mathrm{Tr}(\widehat{f}(\sigma)\sigma(x))$$

Theorem 3.5 (**The Plancherel Theorem**)
If $f, g \in L^2(G)$, then

$$\int_G f(x)\overline{g(x)} \, dx = \sum_{\sigma} d_{\sigma} \mathrm{Tr}(\widehat{f}(\sigma)\widehat{g}(\sigma)^*)$$

This version of the Plancherel theorem holds for a non-abelian compact group. An equivalent statement of the theorem is that for $f \in L^2(G)$, we have

$$\int_G |f(x)|^2 \, dx = \sum_{\sigma} d_{\sigma} \mathrm{Tr}(\widehat{f}(\sigma)\widehat{f}(\sigma)^*)$$

Example 3.2 Let G be a finite group and $f = 1$, then

$$\int_G 1^2 dg = |G| = \sum_{\sigma} d_{\sigma} \widehat{1}(\sigma)$$

where

$$\widehat{1}(\sigma) = \int_G \sigma(g)\,dg = \chi_\sigma(e) = d_\sigma$$

and $\chi_\sigma(e)$ is the character of irreducible representation σ at the identity element.

$$|G| = \sum_\sigma d_\sigma^2$$

This is exactly Frobenius theorem (Theorem 1.1) for finite groups.

The Plancherel theorem shows that all irreducible representations occur as subrepresentations of the regular representations.

3.5 Representations of $SU(2)$

In physics, the representations of $SU(2)$ describes the spin of particles.

Recall that the special unitary group of degree 2 is

$$SU(2) = \left\{ u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| u^{-1} = u^*, det(u) = 1, \alpha, \beta \in \mathbb{C} \right\}$$

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix} \quad \text{and} \quad |\alpha|^2 + |\beta|^2 = 1$$

Let ℓ be a non-negative half-integer in the set $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, ...\}$, we define a finite dimensional linear space, H_ℓ, of all complex polynomials $f(z)$, of degree $2\ell + 1$, i.e.

$$f(z) = \sum_{j=0}^{2\ell} a_j z^j$$

e.g.

$$\begin{aligned} \ell = 0 : &\quad f(z) = a_0 \\ \ell = \frac{1}{2} : &\quad f(z) = a_0 + a_1 z \\ \ell = 1 : &\quad f(z) = a_0 + a_1 z + a_2 z^2 \end{aligned}$$

For all $f(z) \in H_\ell$, we let $\sigma^{(\ell)}$ be the representation of $SU(2)$ of dimension $2\ell + 1$:

$$\sigma^{(\ell)}(u) f(z) = (\beta z + \bar{\alpha})^{2\ell} f\left(\frac{\alpha z - \bar{\beta}}{\beta z + \bar{\alpha}} \right)$$

This representation is of dimension $2\ell + 1$.

For $f(z) = z^k,\ k \in \{0, 1, 2, ..., 2\ell\}$, we obtain:

$$\sigma^{(\ell)}(u) z^k = (\beta z + \bar{\alpha})^{2\ell - k} (\alpha z - \bar{\beta})^k$$

This shows $\sigma^{(\ell)}(u) : H_\ell \mapsto H_\ell$. Also, $\sigma^{(\ell)}(u)$ is a homomorphism since

$$\sigma^{(\ell)}(u_1 u_2)^\ell f(z) = \sigma^{(\ell)}(u_1)(\sigma^{(\ell)}(u_2)f)(z)$$

To derive the representation $\sigma^{(\ell)}(u)$, we realise it by choosing a **basis** for H_ℓ and defining an **inner product** $< \; , \; >$, such that all $\sigma^{(\ell)}(u)$ are unitary.

Also, to show $\{\sigma^{(0)}, \sigma^{(\frac{1}{2})}, \sigma^{(1)}, ...\}$ is a set of continuous irreducible representations of $SU(2)$, we need some 'tools'.

Consider matrices in in the form of

$$m_1(t) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & i\sin\left(\frac{t}{2}\right) \\ i\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{pmatrix}$$

$$m_2(t) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & -\sin\left(\frac{t}{2}\right) \\ \sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{pmatrix}$$

$$m_3(t) = \begin{pmatrix} \exp\left(i\frac{t}{2}\right) & 0 \\ 0 & \exp\left(-i\frac{t}{2}\right) \end{pmatrix}$$

for $t \in \mathbb{R}$. We can see $m_i(t)m_i(t^{'}) = m_i(t+t^{'})$, so each $m_i(t)$ forms a closed subgroup of $SU(2)$, and actually they are topologically isomorphic with circle group $\mathbb{T}$.

By these subgroups of $SU(2)$, we can define the **derivative of the representations** $\sigma^{(\ell)}$. Consider

$$\frac{1}{t}(\sigma^{(\ell)}_{m_i(t)} - I)$$

If we take the limit of $t \to 0$, then

$$\lim_{t\to 0} \frac{1}{t}(\sigma^{(\ell)}_{m_i(t)} - I)z^k = A_i^{(\ell)} z^k$$

Let $\phi_i(t,z) = \sigma^{(\ell)}_{m_i(t)} z^k$, and

$$\lim_{t\to 0} \frac{1}{t}(\sigma^{(\ell)}_{m_i(t)} - I)z^k = \lim_{t\to 0} \frac{1}{t}(\phi_i(t,z) - \phi_i(0,z)) = \frac{\partial \phi_i}{\partial t}(t,z)\bigg|_{t=0}$$

We can write out $A_i^{(\ell)} z^k$ for $m_i(t)$. For example

$$\phi_1(t,z) = \left(i\sin\left(\frac{t}{2}\right)z + \cos\left(\frac{t}{2}\right)\right)^{2\ell-k}\left(\cos\left(\frac{t}{2}\right)z + i\sin\left(\frac{t}{2}\right)\right)^k$$

$$A_1^{(\ell)} z^k = \frac{\partial \phi_1}{\partial t}(t,z)\bigg|_{t=0} = i\ell z \cdot z^k - \frac{i}{2}kz^2 \cdot z^{k-1} + \frac{i}{2}kz^{k-1} = \left[i\ell z + \frac{i}{2}(1-z^2)\frac{d}{dz}\right] z^k$$

Also,

$$A_2^{(\ell)} z^k = \left[-\ell z + \frac{1}{2}(1+z^2)\frac{d}{dz}\right] z^k$$

$$A_3^{(\ell)} z^k = i\left[z\frac{d}{dz} - l\right] z^k$$

Let $\xi_j = z^{\ell-j}$ for $j = \{-\ell, -\ell+1, -\ell+2, ...\ell\}$, e.g. $\ell = \frac{1}{2} : \xi_{-\frac{1}{2}} = z^1$ and $\xi_{\frac{1}{2}} = z^0$

Then, the basis element defined by $\eta_j = c_j^{(\ell)} \xi_j$ is to satisfy

$$< \eta_j \,,\, \eta_k >= \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases}$$

To derive the value of $c_j^{(\ell)}$, first notice that

$$\sigma_{m_3(t)}^{(\ell)} \xi_j = e^{-itj} \xi_j$$

Then

$$\begin{aligned} < \xi_j \,,\, \xi_k > &=< \sigma_{m_3(t)}^{(\ell)} \xi_j \,,\, \sigma_{m_3(t)}^{(\ell)} \xi_k > \\ &= < e^{-itj} \xi_j \,,\, e^{-itk} \xi_k > \\ &= e^{i(k-j)t} < \xi_j \,,\, \xi_k > \end{aligned}$$

Since $e^{i(k-j)t} \neq 0$, so $< \xi_j \,,\, \xi_k >= 0$ for $j \neq k$.

Also, for non-zero real t, we can write

$$\begin{aligned} 0 &= \frac{1}{t} < \xi_j \,,\, \xi_{j-1} >= \frac{1}{t} < \sigma_{m_2(t)}^{(\ell)} \xi_j \,,\, \sigma_{m_2(t)}^{(\ell)} \xi_{j-1} > \\ &=< \frac{1}{t} \sigma_{m_2(t)}^{(\ell)} \xi_j \,,\, \sigma_{m_2(t)}^{(\ell)} \xi_{j-1} > - < \frac{1}{t} \xi_j \,,\, \sigma_{m_2(t)} \xi_{j-1} > + < \xi_j \,,\, \frac{1}{t} \sigma_{m_2(t)} \xi_{j-1} > - < \xi_j \,,\, \frac{1}{t} \xi_{j-1} > \\ &=< \frac{1}{t} (\sigma_{m_2(t)}^{(\ell)} - I) \xi_j \,,\, \sigma_{m_2(t)}^{(\ell)} \xi_{j-1} > + < \xi_j \,,\, \frac{1}{t} (\sigma_{m_2(t)}^{(\ell)} - I) \xi_{j-1} > \end{aligned}$$

Taking the limit as $t \to 0$, and by the derivative of representation, $A_2^{(\ell)}$, we find

$$0 =< A_2^{(\ell)} \xi_j \,,\, \xi_{j-1} > + < \xi_j \,,\, A_2^{(\ell)} \xi_{j-1} >$$

Using $A_2^{(\ell)}$ that we derived earlier, we have

$$\begin{aligned} 0 = (\ell - j) < \xi_{j+1} \,,\, \xi_{j-1} > -(\ell + j) < \xi_{j-1} \,,\, \xi_{j-1} > \\ +(\ell - j + 1) < \xi_j \,,\, \xi_j > -(\ell + j - 1) < \xi_j \,,\, \xi_{j-2} > \end{aligned}$$

For $j = \ell$, the coefficient of $< \xi_{\ell+1} \,,\, \xi_{\ell-1} >$ is 0 and for $j = -\ell + 1$, the coefficient of $< \xi_{-\ell+1} \,,\, \xi_{-\ell-1} >$ is 0. Thus, we conclude that

$$< \xi_{j-1} \,,\, \xi_{j-1} >= \frac{\ell - j + 1}{\ell + j} < \xi_j \,,\, \xi_j >$$

for all $j \in \{-\ell + 1, -\ell + 2, ...\ell\}$.

At this point, it is convenient to let

$$< \xi_\ell \, , \, \xi_\ell >= (2\ell)!$$

then work out the other $< \xi_j \, , \, \xi_j >$ from backwards. A simple induction gives

$$< \xi_j \, , \, \xi_j >= (\ell - j)!(\ell + j)!$$

Now, let $\eta_j = [(\ell - j)!(\ell + j)!]^{-\frac{1}{2}}\xi_j$, and $< \eta_j \, , \, \eta_j >= 1$. Then $\{\eta_{-\ell}, \eta_{-\ell+1}, ...\eta_\ell\}$ is an orthonormal basis of H_ℓ.

Let $\sigma^{(\ell)}_{j,k}(u)$ be the (j,k)-th entry in the matrix form of the representation $\sigma^{(\ell)}(u)$ in this basis. To determine $\sigma^{\ell}_{j,k}(u)$, we write

$$\sigma^{(\ell)}(u)\, \eta_k = [(\ell - j)!(\ell + j)!]^{-\frac{1}{2}}(\beta z + \bar{\alpha})^{\ell+k}(\alpha z - \bar{\beta})^{\ell-k} = \sum_{j=-\ell}^{\ell} p_j \eta_j$$

and after some computations, we get

$$\sigma^{(\ell)}_{j,k}(u) = (-1)^{j-k}\left(\frac{(\ell - j)!(\ell + j)!}{(\ell - k)!(\ell + k)!}\right)^{\frac{1}{2}} \times$$

$$\sum_{max\{0,k-\ell\}}^{min\{\ell+k,\ell-j\}} (-1)^s \binom{\ell + k}{s}\binom{\ell - k}{\ell - j - s}\alpha^{\ell-j-s}\bar{\alpha}^{\ell+k-s}\beta^s\bar{\beta}^{j-k+s}$$

We can write down the matrix form of $\sigma^{(\ell)}(u)$ for small values of ℓ

(i) $\ell = 0$: $H_0 = span\{\eta_0 = 1\}$, and $\sigma^{(0)}(u)1 = 1$ for all $u \in SU(2)$. $\sigma^{(0)}(u)$ is the trivial representation.

(ii) $\ell = \frac{1}{2}$: $H_{\frac{1}{2}} = span\{\eta_{-\frac{1}{2}} = z, \, \eta_{-\frac{1}{2}} = 1\}$

$$[\sigma^{(\frac{1}{2}}(u))]^{\frac{1}{2}}_{j,k=-\frac{1}{2}} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

(iii) $\ell = 1$: $H_1 = span\{\eta_{-1} = \frac{1}{\sqrt{2}}z^2, \, \eta_0 = z, \, \eta_1 = \frac{1}{\sqrt{2}}\}$

$$[\sigma^{(1)}(u)]^{1}_{j,k=-1} = \begin{pmatrix} \alpha^2 & \sqrt{2}\alpha\beta & \beta^2 \\ -\sqrt{2}\alpha\bar{\beta} & \alpha\bar{\alpha} - \beta\bar{\beta} & \sqrt{2}\bar{\alpha}\beta \\ \bar{\beta}^2 & -\sqrt{2}\bar{\alpha}\beta & \bar{\alpha}^2 \end{pmatrix}$$

(iv) $\ell = \frac{3}{2}$: $H_{\frac{3}{2}} = span\{\eta_{-\frac{3}{2}} = \frac{1}{\sqrt{6}}z^3, \, \eta_{-\frac{1}{2}} = \frac{1}{\sqrt{2}}z^2, \, \eta_{\frac{1}{2}} = \frac{1}{\sqrt{2}}z, \, \eta_{\frac{3}{2}} = \frac{1}{\sqrt{6}}\}$

$$[\sigma^{(\frac{3}{2})}(u)]^{\frac{3}{2}}_{j,k=-\frac{3}{2}} = \begin{pmatrix} \alpha^3 & \sqrt{3}\alpha^2\beta & \sqrt{3}\alpha\beta^2 & \beta^3 \\ -\sqrt{3}\alpha^2\bar{\beta} & \alpha^2\bar{\alpha} - 2\alpha\beta\bar{\beta} & 2\alpha\bar{\alpha}\beta - \beta^2\bar{\beta} & \sqrt{3}\bar{\alpha}\beta^2 \\ \sqrt{3}\alpha\bar{\beta}^2 & -2\alpha\bar{\alpha}\bar{\beta}^2 + \beta\bar{\beta}^2 & \alpha\bar{\alpha}^2 - 2\bar{\alpha}\beta\bar{\beta} & \sqrt{3}\bar{\alpha}^2\beta \\ -\bar{\beta}^3 & \sqrt{3}\bar{\alpha}\bar{\beta}^2 & -\sqrt{3}\bar{\alpha}^2\bar{\beta} & \bar{\alpha}^3 \end{pmatrix}$$

We can then calculate $\sigma^{(\ell)}(u)$ for $\ell = 2, \frac{5}{2}, 3, ...$ etc.

Proposition 3.6 All of representations $\{\sigma^{(0)}, \sigma^{(\frac{1}{2})}, \sigma^{(1)}, ...\}$ are irreducible.

Proof:

Let H_s be a non-zero subspace of H_ℓ of dimension $2\ell + 1$, and H_s that is invariant under all representations $\sigma_u^{(\ell)}$.

H_s is invariant under $\frac{1}{t}(\sigma^{(\ell)}_{m_i(t)} - I)$ for $i = 1, 2, 3$, thus is also invariant under derivatives of representations $A_i^{(\ell)}$ for $i = 1, 2, 3$.

It follows that H_s contains all monomials $1, z, z^2, ...z^{2\ell}$, so $H_s = H_l$.

As seen in the Plancherel theorem, we can use these representations to do Fourier analysis on $SU(2)$.

(i) By the Peter-Weyl theorem,

$$L^2(SU(2)) = \oplus_{\sigma^{(\ell)}} d_{\sigma^{(\ell)}} H_{\sigma^{(\ell)}}$$

(ii) If f is integrable on $SU(2)$, and if $\{\eta_{-\ell}, ...\eta_\ell\}$ is an orthonormal basis for $H_{\sigma^{(\ell)}}$, then we have

$$< \widehat{f}(\sigma^{(\ell)})\eta_i \,,\, \eta_j >_{\mathcal{H}_{\sigma^{(\ell)}}} = \int_G f(g)\, \overline{< \sigma^{(\ell)}\eta_i \,,\, \eta_j >}\, dg =< f \,,\, t^{\sigma^{(\ell)}}_{\eta_i,\eta_j} >_{L^2(SU(2))}$$

(iii) Also, the Plancherel theorem

$$\int_{SU(2)} |f(g)|^2\, dg = \sum_{\sigma^{(\ell)}} d_{\sigma^{(\ell)}} \operatorname{Tr}(\widehat{f}(\sigma^{(\ell)})\widehat{f}(\sigma^{(\ell)})^*)$$

We now describe the irreducible representations of $U(2)$. We will use the structure of $U(2)$ given in Proposition 2.13. In fact, it is easy to describe the irreducible representations of product groups.

Definition 3.9 Let G_1 and G_2 be two compact Lie groups. Let σ_1 be a representation of G_1 on $\mathcal{H}_1$ and σ_2 be a representation of G_2 on $\mathcal{H}_2$ respectively. Then the representations of $G_1 \times G_2$ is defined by $\sigma_1 \times \sigma_2$, and $\sigma_1 \times \sigma_2$ acts on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by

$$(\sigma_1 \times \sigma_2)(g_1, g_2)(v_1 \times v_2) = (\sigma_1(g_1)v_1) \otimes (\sigma_2(g_2)v_2)$$

for $g_1 \in G_1, g_2 \in G_2$ and $v_1 \in \mathcal{H}_1, v_2 \in \mathcal{H}_2$.

In fact, all the irreducible representations of $G_1 \times G_2$ have the form of $\sigma_1 \times \sigma_2$ above.

Definition 3.10 Let G/Z be a compact Lie group, and Z is a normal subgroup of G, e.g. center of G, then the irreducible representations of G/Z are in one-to-one correspondence with the set of irreducible representations, σ, of G, where $\sigma|_Z = I$.

Example 3.4 Recall from Proposition 2.13, we have

$$U(2) \cong (\mathbb{T} \times SU(2)) \,/\, \{(1, I), (-1, -I)\}$$

with $Z = \{(1, I), (-1, -I)\}$.

The irreducible representations of $\mathbb{T} \times SU(2)$ are

$$\chi_n(z)\sigma^{(\ell)}(u)$$

with $z \in \mathbb{T}$ and $u \in SU(2)$. χ_n is the character of $\mathbb{T}$, and $\sigma^{(\ell)}$ is the irreducible representation of $SU(2)$ we derived earlier.

We need $\chi_n(z)\sigma^{(\ell)}(u)$ to be trivial on Z, therefore, for $n \in \mathbb{Z}$ and $\ell \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, ...\}$, we need

$$\chi_n(-1)\sigma^{(\ell)}(-I) = 1$$

which equals to

$$(-1)^n \begin{pmatrix} (-1)^{2\ell} & & 0 \\ & \ddots & \\ 0 & & (-1)^{2\ell} \end{pmatrix} = 1$$

Thus, $(-1)^n(-1)^{2\ell} = 1$, so we can conclude that if n is even, then ℓ is an integer, and if n is odd, then ℓ is a half-integer.

Similarly, from Proposition 2.14,

$$SU(2)/\{I, -I\} \cong SO(3)$$

we see that the irreducible representations $\sigma^{(\ell)}$ of $SU(2)$ must have ℓ to be an integer.

Using the structure theorem of compact Lie groups, we see that we can describe the irreducible representations of compact Lie groups. Once we understand the representations of simple simply connected compact Lie groups. These are to be obtained from Lie algebra methods.

3.6 Induced Representations of Compact Lie Groups

Definition 3.9 Let G be a compact lie group and H is a subgroup of G. Let ρ be a representation of H on a Hilbert space $\mathcal{H}_\rho$. We will define $\sigma = \rho \uparrow_H^G$ to be the **induced representation**. The space $\mathcal{H}_\sigma$ is defined by

$$\mathcal{H}_\sigma = \{f : G \to \mathbb{C} : f(gh) = \rho(h)f(g) \,\text{and}\, f \in L^2(G/H)\}$$

The meaning of $f \in L^2(G/H)$ will be discussed below.

The induced representation σ acts by

$$\sigma(g_0)f(g) = f(gg_0)$$

so the induced representations are kind of generalization of regular representations.

Definition 3.10 We need to define an **inner product** on $\mathcal{H}_\sigma$, to do so, first notice that

$$< f(g)\, ,\, k(g) >_{\mathcal{H}_\rho} = < f(gh)\, ,\, k(gh) >_{\mathcal{H}_\rho}$$

To see this, we have

$$< f(gh)\, ,\, k(gh) >_{\mathcal{H}_\rho} = < \rho(h)f(g)\, ,\, \rho(h)k(g) >_{\mathcal{H}_\rho} = < f(g)\, ,\, k(g) >_{\mathcal{H}_\rho}$$

since ρ is a unitary representation.

Therefore, $< f(hg)\, ,\, k(hg) >$ is constant on cosets G/H.

Now, we want to define

$$< f(g)\, ,\, k(g) >_{\mathcal{H}_\sigma} = \int_{G/H} < f(gH)\, ,\, k(gH) > \; d\mu(gH)$$

for a suitable measure μ on G/H, and assume that f satisfies an L^2 condition, i.e. $||f||^2 < \infty$. If there is an invariant measure on G/H, we take μ to be this measure. Otherwise, we choose a **quasi-invariant** measure.

Definition 3.10 Let $A = G/H$ and $g \in G$, then the quasi-invariant measure on G/H is defined by

$$\mu \circ g(A) = \mu(Ag) = \int_A \phi(g)\, d\mu$$

where ϕ is a Radon–Nikodym derivative, which is a measurable function almost everywhere, and $\frac{d\mu\circ g}{d\mu} = \phi$

Then

$$\begin{aligned}
\int_{G/H} ||f(gHg_0)||^2 d\mu(gH) &= \int_{G/H} ||f(gH)||^2 d\mu \circ g_0(gH) \\
&= \int_{G/H} ||f(gH)||^2 \frac{d\mu \circ g_0}{d\mu}\, d\mu(gH) \\
&= \int_{G/H} \left|\left| \sqrt{\frac{d\mu \circ g_0}{d\mu}} f(gH) \right|\right|^2 d\mu(gH)
\end{aligned}$$

Thus, we conclude that the induced representation σ acts on $\mathcal{H}_\sigma$ by

$$\sigma(g_0)f(gH) = \sqrt{\frac{d\mu \circ g_0}{d\mu}} f(gHg_0)$$

In fact, it can be shown that for compact groups, a quasi-invariant measure μ always exists.

<u>**Example 3.4**</u> Let $G = SU(2)$ and $H = \mathbb{T}$, $\mathbb{T}$ is a subgroup of $SU(2)$. Recall that

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \,\text{and}\, (\alpha, \beta) \in \mathbb{C}^2 \right\}$$

$$\mathbb{T} = \left\{ \begin{pmatrix} \alpha_1 & 0 \\ 0 & \bar{\alpha_1} \end{pmatrix} : |\alpha_1| = 1 \right\}$$

with $\alpha_1 = e^{i\theta} \in \mathbb{T}$.

The irreducible representations of $\mathbb{T}$ are just the characters of $\mathbb{T}$, which is $\rho_n = \alpha_1^n$ on $\mathcal{H}_{\rho_n}$.

Now, to find the induced representation $\sigma_n = \rho_n \uparrow_{\mathbb{T}}^{SU(2)}$, we define

$$\mathcal{H}_{\sigma_n} = \left\{ f : \mathbb{C}^2 \to \mathbb{C} : f\left(\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \bar{\alpha_1} \end{pmatrix} \right) = \alpha_1^n f\left(\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \right) \right\}$$

with $f \in L^2(SU(2)/\mathbb{T})$ such that

$$\int_{SU(2)/\mathbb{T}} |f(gH)|^2 d\mu(gH) < \infty$$

The measure $\mu(gH)$ is the G invariant measure on the Riemann sphere $SU(2)/\mathbb{T}$.

So, we can conclude that f satisfies

$$f(e^{i\theta}\alpha, e^{-i\theta}\beta) = e^{in\theta} f(\alpha, \beta)$$

and f is a homogeneous complex polynomial of degree n on $\mathbb{C}^2$.

Chapter 4

From Lie Group to Lie Algebra

In Chapter 4, we will briefly look at the concept of Lie algebra and representations of Lie algebra. We will cover the concept of tangent vector space, Lie bracket and adjoint representation.

The contents in this chapter are developed from these references:

[Doo00], [Sch13], [Hal00]

4.1 Tangent Vector Space

Definition 4.1 Suppose we have two smooth manifolds M and N, and there's map $\phi : M \to N$, and there are charts (U, f) and (V, g) where $U \in M$ and $V \in N$, and smooth maps $f : U \to \mathbb{R}^{dimM}$ and $g : V \to \mathbb{R}^{dimN}$, then we can define:

$$g \circ \phi \circ f^{-1} : \mathbb{R}^{dimM} \to \mathbb{R}^{dimN}$$

Notice that ϕ is differetiable if each $g \circ \phi \circ f^{-1}$ is differentiable map from $\mathbb{R}^{dimM}$ to $\mathbb{R}^{dimN}$.

$$\begin{tikzcd}
U \subseteq M \arrow[r, "\phi"] \arrow[d, "f"] & V \subseteq N \arrow[d, "g"] \\
f(U) \subseteq \mathbb{R}^{dimM} \arrow[r, "g\circ\phi\circ f^{-1}"] & g(V) \subseteq \mathbb{R}^{dimN}
\end{tikzcd}$$

Now, consider there is a path in M passing through a point $x \in U \subset M$, and it is a differentiable map $\gamma : \mathbb{R} \to M$, $t \mapsto \gamma(t)$. It is convenient to define $\gamma(0) = x$. Notice that $\gamma(t) \in \text{domain}(f)$ for all t (in a neighbourhood of $t = 0$, i.e U). Then, we have a differentiable map $f \circ \gamma : \mathbb{R} \to \mathbb{R}^{dimM}$. We know well about the differentiation in n-Euclidean space, so we can define a derivative at $t = 0$, which is:

$$\left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma)$$

This is actually an element of $\mathbb{R}^{dimM}$.

Example 4.1 Let $M = \mathbb{R}^3$, the chart map $f = Id_{\mathbb{R}^3}$ and the there's a curve

$$\gamma(t) = \{(1 + t^2, \sin(t), e^t) \mid t \in \mathbb{R}\}$$

and the derivative of curve at 0 is:

$$\left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma) = (2t, \cos(t), e^t)|_{t=0} = (0, 1, 1)$$

which is an element (a vector) of $\mathbb{R}^3$.

If we have many different paths passing through x, and by differentiating these paths, we can get the set of vectors in $\mathbb{R}^{dimM}$. This set forms a tangent vector space to M at x, and it is denoted as $T_x(M)$. We could also say this tangent vector space is a subspace of $\mathbb{R}^{dimM}$ (possibly the entire $\mathbb{R}^{dimM}$).

T_xM is a vector space "attached" to M at x. Bundle $T_x(M)$ is a vector space over 'M - Bundle'. We can define a map $\rho : T_x(M) \to x$, which is a projection of the bundle vector space $\bigcup_x T_x(M)$ on the Manifold.

Proposition 4.1 Given $\phi : M \to N$, $x \mapsto \phi(x) = y$, where $y \in N$, we say $\phi_{*,x}$ is the derivative of ϕ at x, and it is written as:

$$\phi_{*,x} : T_x(M) \to T_y(N)$$

Proof: We know that $g \circ \phi \circ f^{-1}$ maps a path in M to a path in N, which is

$$(g \circ \phi \circ f^{-1}) \circ (f \circ \gamma) = (g \circ \phi \circ \gamma)$$

and it maps from $\mathbb{R}$ to $\mathbb{R}^{dimN}$, and it's derivative is a tangent vector to N 'attached' at $y = \phi(x)$. The derivative of $f \circ \gamma$ is a tangent vector $v \in T_x(M)$ and the derivative of $g \circ \phi \circ \gamma$ is a tangent vector $w \in T_y(N)$.

Now, we can define $\phi_{*,x} v = w$, $T_x(M) \to T_y(N)$. If we have a bases in them, then $\phi_{*,x}$ is a matrix, so it is a linear map. Also, if dim(M) = m, and dim(N) = n, then $\phi_{*,x}$ is a $n \times m$ matrix.

We also can have all the essential properties of differentiation applying to diffeomorphism on between multiple manifolds, such like chain rule:

$$(\phi_2 \circ \phi_1)_{*,x} = \phi_{2\,*,\phi_1(x)} \circ \phi_{1\,*,x}$$

4.2 From Lie Group to Lie Algebra

Definition 4.2 Let G be lie group, so it is a n-dimensional manifold with smooth group translation and inversion, i.e. the (right) translation:

$$R_g : G \to G, \quad x \mapsto xg$$

for $x \in G$, and any $g \in G$, is smooth and invertible. So it is a diffeomorphism maps G to itself, and by previous assertions, we can define the its derivative at x to be:

$$(R_g)_{*,x} : T_x(G) \to T_{xg}(G)$$

and this is an invertible map.

With the amazing group properties, we can find that:

$$(R_g)_{*,e} : T_e(G) \to T_g(G)$$

is an invertible linear map so these vector spaces are isomorphic, i.e. $T_e(G) \cong T_g(G)$. This means the vector space at any $g \in G$ is "same" as the one at identity element.

Definition 4.3 Let $\mathfrak{g} = T_e(G)$ be a n-dimensional vector space, and $T_g(G) = (R_g)_{*,e}$. We can find a invertible chart map f in the neighbourhood of identity element such that $f^{-1} : \mathfrak{g} \to G$.

We propose that exponential map is the wisest chart map

$$exp : \mathfrak{g} \to G$$

this is a path passing through identity element in G, i.e. $exp(0 \cdot X) = I$, and its derivative is

$$\left.\frac{d}{dt}\right|_{t=0} exp(tX) = X \cdot exp(tX)|_{t=0} = X$$

for any $X \in \mathfrak{g}$.

The exponential mapping is really natural, and the derivative of exp at $t = 0$ is a tangent (velocity) vector at identity element.

The tangent vector space at other $g \in G$ is a invertible linear map $(R_g)_{*,e}(\mathfrak{g})$. The curve through other $g \in G$ is $(exp(tX)) \cdot g$ for $X \in \mathfrak{g}$.

Definition 4.4 Now, we want to give the tangent vector space $\mathfrak{g}$ the structure of a **Lie algebra** by defining a **Lie bracket**

$$[X, Y] = XY - YX, \qquad X, Y \in \mathfrak{g}$$

such that $[X, Y] = -[Y, X]$, and it has to satisfy Jacobi Identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Definition 4.5 The conjugation mapping

$$Ag : x \to gxg^{-1}$$

is a differentiable map from G to G, and $Ag(e) = e$, and we can have

$$Ag_{*,e} : T_e(G) \to T_e(G)$$

which is a linear map from $\mathfrak{g}$ to $\mathfrak{g}$.

Thus, we can define a map:

$$Ad(g) : \mathfrak{g} \to \mathfrak{g}$$

which is called the adjoint map, it is a representation of G on $\mathfrak{g}$ because it is a homomorphism:

$$Ad(gh) = Ad(g) \circ Ad(h)$$

since

$$A_g \circ A_h(x) = ghxh^{-1}g^{-1} = (gh)x(gh)^{-1} = A_{gh}$$

Note that $g(\exp(X))g^{-1} = \exp(Ad(g)X)$, thus

$$Ad : G \to GL(\mathfrak{g})$$

maps G onto a general linear group acting on $\mathfrak{g}$, it is also a Lie group. The map

$$(Ad)_{*,e} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = M_n(\mathfrak{g})$$

The adjoint representation of $\mathfrak{g}$ can be rewritten as:

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

given by $ad(X)(Y) = [X, Y]$.

Future Work

Using the Cartan-Weyl theory of representations of Lie algebras, we can describe all the irreducible representations of compact simple simply connected Lie groups by passing to the Lie group.

This is beyond the scope of this thesis.

Appendix A

Topological Space

The fundamental concepts in point-set topology are continuity, compactness, and connectedness. The words 'nearby', 'arbitrarily small', and 'far apart' can all be made precise by using the concept of open sets. Each choice of definition for 'open set' is called a topology. A set with a topology is called a topological space.

Definition of Topological Space

Definition A.1 Let X be a set, and let τ be a family of subsets of X, then τ is called a topology on X if

(i) $\varnothing \in \tau$ and $X \in \tau$.

(ii) Any intersection of finitely many elements of τ is an element of τ.

(iii) Any union of elements of τ is an element of τ.

The pair (X, τ) is called a topological space.

Remarks:

(1) The elements of τ are called open sets in X.

(2) A subset $A \in X$, is said to be closed if its complement $X \backslash A$ is in the τ (elements of τ are open).

(3) A subset of X can both be open and close (Clopen), the empty set $\varnothing$ and X are always both open and closed. (Since $\varnothing^c = X$, and $X \in \tau$ is open , so $\varnothing$ is closed. In other hand, $X^c = \varnothing$, and $\varnothing \in \tau$ is open, so X is closed.)

Example A.1 Let $X = \{1, 2, 3\}$, then $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$ is a topology on X. Note τ is a discrete topology, it means that every subset in X is an open set in its topology.

Example A.2 By the concept of topology, we can construct the topology on n-dimensional Euclidean space, i.e. $\mathbb{R}^n$.

Let $S = \mathbb{R}^n := \underbrace{\mathbb{R} \times \mathbb{R} ... \times \mathbb{R}}_{\text{n times}}$.

Then, for every $x \in \mathbb{R}^n$ and every $r \in \mathbb{R}^+$, we can define open balls to be

$$B_r(x) := \left\{ y \in \mathbb{R}^n | \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} < r \right\}$$

Thus, let A be a subset of $\mathbb{R}^n$, and $A \in \tau$, and for every $x \in A$, there exists a $r \in \mathbb{R}^+$ such that $B_r(x) \subseteq A$.

We need to show that $(\mathbb{R}^n, \tau)$ is indeed a topology.

Proof:

(1) Show $\varnothing \in \tau$. That is for every $x \in \varnothing$, there exists a $r \in \mathbb{R}^+$ such that $B_r(x) \subseteq \varnothing$. By the concept of vacuous truth, the "$x \in \varnothing$" is obviously false, since empty set does not contain any points. But no matter what it implies, the whole statement is vacuous true, so $\varnothing \in \tau$.

(2) Show $\mathbb{R}^n \in \tau$. That is for every $x \in \mathbb{R}^n$, there exists a $r \in \mathbb{R}^+$ such that $B_r(x) \subseteq \mathbb{R}^n$. This one is obvious.

(3) Show that if $A, B \in \tau$, then $A \cap B \in \tau$. Consider if there's a $x \in A \cap B$, then it implies that $x \in A$ and $x \in B$, and we can find a $r \in \mathbb{R}^+$ such that $B_r(x) \subseteq A$ and also find a $s \in \mathbb{R}^+$ such that $B_s(x) \subseteq B$. Thus, $B_{min(r,s)}(x) \subseteq A$ and $B_{min(r,s)}(x) \subseteq B$, which implies $B_{min(r,s)}(x) \subseteq A \cap B$.

(4) Show that if E is a collection of subsets of $\mathbb{R}^n$ and $E \subseteq \tau$, then $\bigcup_i E_i \in \tau$.

Let $x \in \bigcup_i E_i$ and suppose $x \in A$ for $A \in E$, there exists a $r \in \mathbb{R}^+$ such that $B_r(x) \subseteq A$, and $B_r(x) \subseteq \bigcup_i E_i$.

The concept of topology certainly can let us construct many useful spaces to work with.

Induced Topological Space

We can construct a new topology from given a topology.

Definition A.2 Let (X, τ) be a topological space, and $E \subset X$, then

$$\tau|_E := \{A \cap E \,|\, A \in \tau\} \subseteq \mathcal{P}(E)$$

is a topology on E, and $\tau|_E$ is called *induced topology* of E from topology τ of X.

Proof:

(1) Show $\varnothing \in \tau|_E$. Since $\varnothing = \varnothing \cap E$, with $\varnothing \in \tau$, thus $\varnothing \in \tau|_E$.

(2) Show $E \in \tau|_E$. Since $E = X \cap E$, with $X \in \tau$, thus $E \in \tau|_E$.

(3) Let $A, B \in \tau|_E$, show $A \cap B \ \in \tau|_E$. Suppose there exists a $U \in \tau$ such that $A = U \cap E$ and there exists a $V \in \tau$ such that $B = V \cap E$. Then, $A \cap B = (U \cap E) \cap (V \cap E) = (U \cap V) \cap E$. Since $U \cap V \in \tau$, thus $A \cap B \ \in \tau|_E$.

(4) Let S be a collection of subsets of X and $S \in \tau|_E$, then show $\bigcup_i S_i \in \tau|_E$. Suppose there exists a collection of subset U of X, and $U \in \tau$, such that $S_i = U_i \cap E$, then $\bigcup_i S_i = \bigcup_i (U_i \cap E) = (\bigcup_i U_i) \cap E$. Since $(\bigcup_i U_i) \in \tau$, thus $\bigcup_i S_i \in \tau|_E$.

Example A.3 Given the topological space $(\mathbb{R}, \tau)$, and a subset E of $\mathbb{R}$,

$$E = [-1, 1] := \{x \in \mathbb{R} | -1 \leq x \leq 1\}$$

Then we have a induced topology $(E, \tau|_E)$.

An extreme case is that it is obvious that $(0, 1] \notin \tau$, because any open ball defined at 1 is not fully contained in $(0, 1]$, so it is not an open set, so it's not in τ. However, $(0, 1] = (0, 2) \cap [-1, 1]$, where $(0, 2) \in \tau$, thus $(0, 1] \in \tau|_E$, and it is an open set.

Definition A.3 Let (X, τ) be a topological space and let $\sim$ be an equivalence relation on X. Then the quotient set

$$X/\sim := \{[x] \in \mathcal{P}(X) \,|\, x \in X\}$$

which can be given a *quotient topology* $\tau_{X/\sim}$ defined as

$$\tau_{X/\sim} := \{E \in X/\sim \ | \ \bigcup_{[x] \in E} [x] \in \tau\}$$

where E is collection of quotient sets in X.

Example A.4 The circle in $\mathbb{R}^2$ is defined as the set $S^1 := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$. The topology on S^1 is induced from $(\mathbb{R}^2, \tau)$. The open sets of the circle are clearly the unions of open arcs (arcs without endpoints). Every single point is not open since there is no open set of $\mathbb{R}^2$ whose interaction with the circle is a single on the circle (open set has no boundary in $\mathbb{R}^2$). However, every single point on the circle forms a closed set since its complement is an open arc.

Alternatively, we can define circle by equivalence relation on $\mathbb{R}$, which is

$$x \sim y : x = y + 2\pi n$$

if there exists a $n \in \mathbb{Z}$.

Remarks:

We have a group $\mathbb{R}$ under group operation addition, and $\mathbb{Z}$ is a normal subgroup of $\mathbb{R}$ (all subgroups of an abelian group are normal), then we have the quotient group $\mathbb{R}/\mathbb{Z}$, and $\mathbb{R}/\mathbb{Z} = [0, 1)$, to illustrate this:

$$0 + \mathbb{Z} = \{......, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5......\}$$

$$...$$

$$...$$

$$...$$

$$0.1 + \mathbb{Z} = \{......, -4.9, -3.9, -2.9, -1.9, -0.9, 0.1, 1.1, 2.1, 3.1, 4.1, 5.1\}$$

$$...$$

$$...$$

$$...$$

$$0.99 + \mathbb{Z} = \{......, -4.01, -3.01, -2.01, -1.01, -0.01, 0.99, 1.99, 2.99, 3.99, 4.99, 5.99\}$$

$$...$$

$$...$$

$$...$$

These are the cosets of $\mathbb{Z}$, adding such cosets is done by adding the corresponding real numbers, and subtracting 1 if the result is greater than or equal to 1. The quotient group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the circle group S^1, since we can define a homeomorphism, $f(x+\mathbb{Z}) = e^{i2\pi x}$, for $x \in [0, 1)$, and $\mathbb{Z}$ is the kernel of this homeomorphism, since integers maps to 1.

Definition A.4 Let (A, τ_A) and (B, τ_B) be two topological spaces, then we can equip $A \times B$ (Cartesian product) with so called *product topology*, $\tau_{A\times B}$. It is defined as: if there's an open set $E \in \tau_{A\times B}$, and for every $x = (a, b) \in A \times B$ and $x \in E$, there exists a $S \in \tau_A$ and $T \in \tau_B$, such that $S \times T \subseteq E$.

Remarks:

(1) Cartesian product of any finite sets can equip product topology.
i.e. $A_1 \times A_2 \times A_3 \times A_4 \times ... \times A_n$.

(2) The standard topology on $\mathbb{R}^n$, denoted as $\tau_{\mathbb{R}^n}$, can be written as $\tau_{\mathbb{R}\times\mathbb{R}\times...\times\mathbb{R}}$.

Convergence

Definition A.5 A sequence ϕ (i.e. a map $\phi : \mathbb{N} \to X$) on a topological space (X, τ) is said to converge to a limit point $a \in X$, if for every open set $E \in \tau$ that contains a (every open neighborhood of a), then there exists a $N \in \mathbb{N}$, such that for every $n \geq N$, $\phi(n) \in E$.

Example A.5 Consider the *chaotic topology* $(X, \{\varnothing, X\})$, let $\phi : \mathbb{N} \to X$ be some sequence. Then, we claim that any sequence converges against every point. This is obvious true, since every point is contained in the open set X of the chaotic topology, and every sequence maps into X, so each point in X is fully independent of how exactly the sequence is defined.

Example A.6 Consider the topological space $(X, \mathcal{P}(X))$ has discrete topology. Then only all almost constant sequences converge.

Example A.7 Let $X = \mathbb{R}^n$ and topology τ (standard topology), then the sequence $\phi : \mathbb{N} \to \mathbb{R}^n$ converges to a limit point $a \in X$ if for any $\epsilon > 0$, there exists a $N \in \mathbb{N}$, such that for every $n \geq N$, $||\phi(n) - a|| < \epsilon$.

Let $X = \mathbb{R}$, and define a sequence $q(n) = 1 - \frac{1}{n+1}$. Then, $q(n)$ does not converge in $(\mathbb{R}, \mathcal{P}(R))$ because would not be constant at each n. However, $q(n)$ converges in $(\mathbb{R}, \tau)$.

Continuity

Definition A.6 Let (X, τ_X) and (Y, τ_Y) be two topological spaces, and let $f : X \to Y$ be a map. Then, f is called continuous if for every open set $U \in \tau_Y$, the pre-image of U, denoted $preim(U)$, such that $preim(U) \in \tau_X$. In other words, f is continuous if pre-image of open set is open. There $preim(U) := \{x \in X | \phi(x) \in U\}$.

Example A.8 We list out a few examples which reflect continuity between topological spaces:

(i) Let X have discrete topology $\mathcal{P}(X)$ and Y have any topology τ_Y, and if there's a map $f : X \to Y$ then it is continuous, since every set in X is an open set in its discrete topology $\mathcal{P}(X)$, so f is continuous. In general, if the domain of a map is equipped with discrete topology, then it's always continuous. Indeed, this is not quite interesting.

(ii) Let X have any topology τ_X and Y have chaotic topology $\{\varnothing, Y\}$, and if there's a map $f : X \to Y$, then it is continuous. Since $\varnothing$ and Y are open sets in τ_X, which gives $preim(\varnothing) = \varnothing$ and $preim(Y) = X \in \tau_X$, these are both open sets.

(iii) Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ equip standard topology, and if there's a map $f : \mathbb{R}^n \to \mathbb{R}^m$, then it is continuous if it satisfies the standard δ, ϵ definition of continuity.

Definition A.7 Let $(X, \tau_X) and (Y, \tau_Y)$ be two topological spaces. A bijection (one to one mapping) $f : X \to Y$ is called a *homeomorphism* if both $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous. Homeomorphisms are structure preserving maps in topology.

Definition A.8 If there exists a hoemomorphism between two topologcial spaces (X, τ_X) and (Y, τ_Y), then we say two spaces are homeomorphic or topoglocally isomorphic (existence of bijection between two sets.) and we write $(X, \tau_X) \cong_{top} (Y, \tau_Y)$, and it implies two underlying sets are isomorphic, i.e. $X \cong_{set} Y$. The reverse is not true, since we could have different topology on a set.

Separation

We start by talking about the *separation* properties of topology.

Definition A.9 A topological space (X, τ) is called T_1 (Fréchet), if for any distinct points, i.e. $x \neq y$, there exists a $U \in \tau$ and $x \in U$, such that $y \notin U$.

Definition A.10 A topological space (X, τ) is called T_2 (Hausdorff), if for any distinct points, i.e. $x \neq y$, there exists a $U \in \tau$ and a $V \in \tau$, and $x \in U$ and $y \in V$, such that $U \cap V = \varnothing$. T_2 is stronger than T_1, that is, if a topological space is T_2, then it is also T_1.

Example A.9 Obviously, R^n with standard topology is Hausdorff. However, a counter example is that if we have a chaotic topology on X, which is $(X, \{\varnothing, X\})$, then it's not T_2 or T_1, since every distinct point is contained in the open set X of $\{\varnothing, X\}$.

Compactness and Paracompactness

Definition A.11 A topological space (X, τ) is called *compact*, if every open cover C of X has a finite subcover $\tilde{C}$. That is, X is compact if for every collection C of open subsets of X, such that

$$X = \bigcup_{E \in C} E$$

there is a finite subset $\tilde{C}$ of C such that

$$X = \bigcup_{E \in \tilde{C}} E$$

Definition A.12 Let (X, τ) be a topological space and $U \subseteq X$. Then U is compact if (U, τ_U) (induced topology) is compact.

Definition A.13 Given a set X, a map $d : X \times X \to \mathbb{R}_0^+$ is called a *metric* on X. For any $x, y, z \in X$, it has to satisfy following properties:

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) \geq 0$, if $d(x, y) = 0$, then $x = y$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ (triangular inequality)

Theorem A.1 (Heine-Borel) In a metric space (X, d) (equipped with metric induced topology), every closed and bounded subset of X is compact. The standard topology of $\mathbb{R}$ can be induced by metric, which is the definition of open ball (open set) in its topology can be rewritten as $B_r(x) = \{y \mid d(y, x) < r\}$.

Theorem A.2 Let (X, τ_X) and (Y, τ_Y) be two compact topological spaces, then the product topological space $(X \times Y, \tau_{X \times Y})$ is compact.

Definition A.14 Let (X, τ) be a topological space and let C be a cover of X. A *refinement* of C is a cover D, then for every $U \in D$, there exists a $V \in C$, such that $U \subseteq V$.

Any subcover of a cover is a refinement of that cover, but the converse is not true. A refinement D has following properties:

(i) D is open if $D \subseteq \tau$.

(ii) D is locally finite if for any $x \in X$, there exists a neighborhood $U(x)$ such that the set
$$\{U \in D \,|\, U \cap U(x) \neq \varnothing\}$$
is a finite set.

Definition A.15 A topological space (X, τ) is called *paracompact* if every open cover C has a *open refinement* D which is *locally finite*.

Corollary A.1 If a topological space is compact, then it is also paracompact.

Theorem A.3 (A.H.Stone) Every metrisable space is paracompact.

Example A.11

(i) The topological space $(\mathbb{R}^n, \tau)$ with standard topology is metrisable since τ is induced by a metric which is induced by the 2-norm (Example A.2).

(ii) Alexandroff long line is not paracompact. Intuitively, real number line consists of countable number of line segments $[0, 1)$ end to end, that is $\mathbb{R} = \mathbb{Z} \times [0, 1)$, this is paracompact. The long line is defined similarly, which is $L : \omega \times [0, 1)$, where ω is an uncountable infinite set, then the resulting L is not paracomapct.

Remarks: It turns out that paracompactness is rather a natural property since every example of non-paracompact space looks artificial.

Theorem A.4 Let (X, τ_X) be paracompact space and let (Y, τ_Y) be compact, then $(M \times N, \tau_{M\times N})$ is paracompact.

Definition A.16 Let (X, τ) be a topological space. A *partition of unity* of X is a set $\mathcal{F}$ of continuous maps from X to $[0, 1]$ such that for each $x \in X$ the following conditions hold:

(i) there exists a neighbourhood $U(x)$ such that the set
$\{f \in \mathcal{F} \,|\, \text{for every } y \in U(x) : f(y) \neq 0\}$ is finite;

(ii) $\sum_{f\in\mathcal{F}} f(x) = 1$.

If C is an open cover of X, then $\mathcal{F}$ is said to be subordinate to the cover C if for every $f \in \mathcal{F}$, there exists an element $U \in C$, such that $f(x) \neq 0$, which implies $x \in U$.

Theorem A.5 Let (X, τ_X) be a Hausdorff (disjoint neighbourhoods for distinct points) topological space, then (X, τ_X) is paracompact if and only if, every open cover admits a partition of unity subordinate to that cover.

Example A.12 Let $\mathbb{R}$ be equipped with the standard topology, then $\mathbb{R}$ is metrisable and it is paracompact by Theorem A.3 (Stone). Thus, every open cover of $\mathcal{R}$ admits a partition of unity subordinate to that cover. For example, consider $\mathcal{F} = \{f, g\}$, $(f, g : \mathbb{R} \to [0,1])$ where:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 1 - x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

Then, $\mathcal{F}$ is a partition of unity on $\mathbb{R}$. Since f, g are both continuous maps, $\mathcal{F}$ is finite, and also for every $x \in \mathbb{R}$, $f(x) + g(x) = 1$.

Let $C = \{(-\infty, 1), (0, \infty)\}$, then C is an open cover of $\mathbb{R}$ and since $f(x) \neq 0$ if $x \in (0, \infty)$, and $g(x) \neq 0$ if $x \in (-\infty, 1)$, thus, the partition of unity $\mathcal{F}$ is subordinate to open cover C.

Connectedness and Path-connectedness

Definition A.17 A topological space (X, τ_X) is said to be *connected* unless there exists two non-empty, disjoint open sets U and V such that $X = U \cup V$, then it's *disconnected*.

Example A.13

(i) Let $(\mathbb{R}, \tau)$ be a topological space on $\mathbb{R}$ with standard topology, then let $X = [0,1]$ and let $(X, \tau|_X)$ be the subset topology induced from $(\mathbb{R}, \tau)$. Then, $(X, \tau|_X)$ is connected.

(ii) Follow from the previous example, let $Y = \mathbb{R}\backslash\{0\} = (-\infty, 0) \cup (0, \infty)$, and let $(Y, \tau|_Y)$ be the subset topology induced from $(\mathbb{R}, \tau)$. Then, $(-\infty, 0)$ and $(0, \infty)$ are two non-empty and disjoint open sets in $\tau|_Y$, thus $\mathbb{R}\backslash\{0\} = (-\infty, 0) \cup (0, \infty)$, so it is disconnected.

Theorem A.6 A topological space (X, τ_X) is connected if and only if, the only subsets that are both open and closed are $\varnothing$ and X.

Proof:

Prove by contradiction. Suppose there exists $U \subset X$ such that is open and closed and $U \notin \{\varnothing, X\}$. Now, consider the sets U and $X\backslash U$. Clearly, U and $X\backslash U$ are disjoint. Also, $X\backslash U$ is open since U is closed. Therefore, U and $X\backslash U$ are two non-empty, disjoints such that $X = U \cup X\backslash U$, which contradicts the connectedness of (X, τ_X).

Alternatively, suppose (X, τ_X) is not connected, then there exists two non-empty, disjoint open sets U and V such that $U, V \subset X$ and $X = U \cup V$. Since U is open, so $V = X\backslash U$ is closed. Now, V is both open and closed, however, this contradicts that $\varnothing$ and X are the only two both open and closed sets.

Definition A.18 A topological space (X, τ_X) is said to be *path-connected* if for every pair of points $a, b \in X$, there exists a continuous curve $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$.

Example A.14 The topological space (R^n, τ_{std}) is path-connected.

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