Lyon 1

Mesh and Computational Geometry

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M2 ID3D
Image, Développement
et Technologie 3D
et 3A Centrale

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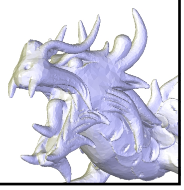

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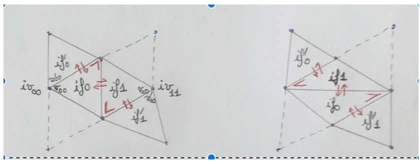
Operations for triangulating and flipping edges

- Updating the Mesh data structure
  - Division of a triangle into 3 triangles (FaceSplit)
  - Division of an edge into two edges (EdgeSplit)
  - Flip of an edge (Flip)



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Other use of the flip operation

- Insertion of a point P outside the convex-hull of a triangulation
  - New triangles joining P and the boundary edges that are visible from P
  - The infinite faces should be updated accordingly

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Other use of the flip operation

- Insertion of a 2D point P outside the convex-hull of a 2D triangulation
- A possible implementation using the infinite vertex and the flip operation
  - Let InfF be an infinite face incident to a visible edge of the convex-hull
  - Split InfF into 3
  - Iteratively flip the infinite edges bounding the modified area if they are incident to an other infinite face to be destroyed (starting from InfF)

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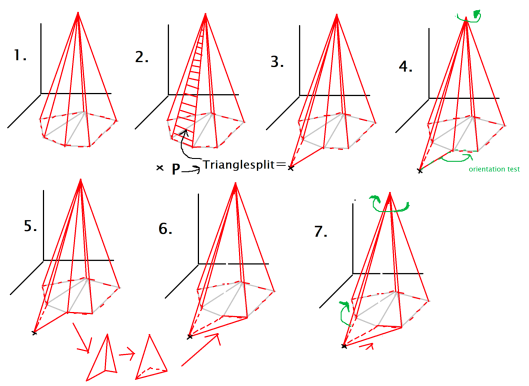

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Delaunay triangulation in any dimension n

- Paving the convex-hull of the points with n-simplices whose circumscribed sphere is empty
- Warning: Lawson's algorithm is only valid in 2D, because the notion of flip is more complicated in larger dimensions

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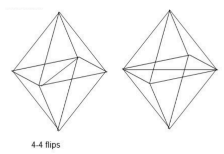
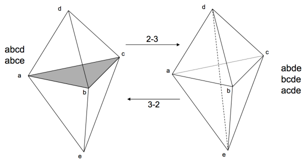
Flip in 3D

- 4-4 flip



- 2-3 and 3-2 flips



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Geometric algorithms implementation

- Distinction between exact, combinatorial vs. approximate objects
- Input data (considered as exacts) used to construct exact non combinatorial objects
  - points x,y or x,y,z
- Construction of combinatorial objects from the exact ones
- Approximate objets should be constructed for visualisation purpose only

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Algorithmic using predicates

- The progress of an algorithm should only depend on the sign of predicates evaluated accurately
  - Use of a controlled arithmetic
  - No use of inexact objects in the evaluation of predicates

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Algorithmic using predicates

- Without these precautions, there is a risk of aberrant behaviour of a geometric algorithm
- Example: How to express the simple insertion algorithm in a 2D triangulation according to these criteria?

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Algorithmic using predicates

- Algorithm of simple insertion in a 2D triangulation:
  - Using the three points orientation predicate
    - To perform inclusion tests in a triangle
    - To perform visibility tests on an edge of the convex envelope

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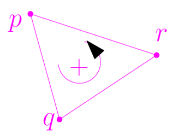
Algorithmic using predicates

- Three 2D points orientation predicate:

$$orientation(p,q,r) = sign(((q-p)\times(r-p)).Oz)$$

$$orientation(p,q,r) = sign((q_x-p_x)(r_y-p_y)-(q_y-p_y)(r_x-p_x))$$

$$= \begin{vmatrix} q_x - p_x & r_x - p_x \\ q_y - p_y & r_y - p_y \end{vmatrix}$$



$$orientation(p,q,r) = sign(\det\begin{bmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{bmatrix})$$

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Algorithmic using predicates

- How to evaluate this orientation predicate?
  - In the case where the input coordinates belong to the regular grid of integers?
  - In the case where the input coordinates are rationals?

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Algorithmic using predicates

- How to evaluate this orientation predicate?
  - In the case of the input coordinates belong tho the regular grid of integers?
  - In the case where the input coordinates are rationals?
  - In both cases the evaluation can be carried out accurately since we benefit from an exact multiplication, addition and subtraction for these types of numbers!

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Algorithmic using predicates

- How to evaluate this orientation predicate?
  - In the case where input coordinates are double?

$$\pm m 2^{e} \qquad -1023 < e < 1024$$
$$m = 1.m_1 m_2 \ldots m_{52} \ (m_i \in \{0,1\})$$

  - The result of the arithmetic operations is rounded to the nearest double

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Algorithmic using predicates

- How to evaluate this orientation predicate?
  - In the case where input coordinates are double?
  - It is only the sign of the predicates that matters
  - Is interval arithmetic enough to disambiguate the sign of a predicate?
  - The case where the three points are almost aligned can be error-prone

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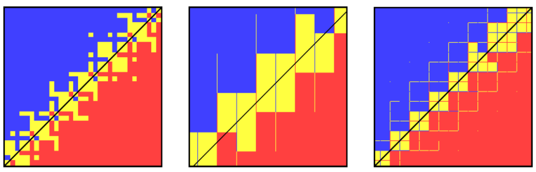
Algorithmic using predicates

- If the orientation predicate is evaluated using double arithmetic :
  - Result of orientation(p,q,r) with $p(p_x+Xu_x,\ p_y+Yu_y)$

$$0 \le X,Y \le 255 \qquad u_x = u_y = 2^{-53}$$



$$p:\begin{pmatrix}0.5\\0.5\end{pmatrix}\quad q:\begin{pmatrix}12\\12\end{pmatrix}\quad r:\begin{pmatrix}24\\24\end{pmatrix}$$

$$\begin{pmatrix}0.50000000000002531\\0.5000000000000171\end{pmatrix}\quad\begin{pmatrix}17.300000000000001\\17.300000000000001\end{pmatrix}\quad\begin{pmatrix}24.00000000000005\\24.0000000000000517765\end{pmatrix}$$

$$\begin{pmatrix}0.5\\0.5\end{pmatrix}\quad\begin{pmatrix}8.8000000000000007\\8.8000000000000007\end{pmatrix}\quad\begin{pmatrix}12.1\\12.1\end{pmatrix}$$

Images by S. Pion

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Algorithmic using predicates

- If the orientation predicate is evaluated using double arithmetic :
  - There are still values for which the sign is unambiguously certified (need to control the threshold)
  - Otherwise :
    - Consider the coordinates of p, q and r as rational (with a finer precision than that usually considered) and make the exact calculation

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Algorithmic unsing predicates

- Example : How to express Lawson algorithm using predicates?

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Algorithmic using predicates

- Lawson Algorithm :
  - An AB edge should be flipped if the circle circumscribed to one of its 2 incident triangles ABC contains point D located on the other side
  - **Do not** use an inaccurate temporary object (e. g. the centre of a circumscribed circle) when evaluating a predicate

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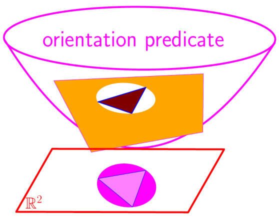
Algorithmic using predicates

- Reminder: the in-circle inclusion test can be expressed as an orientation test in a space of higher dimension (space of spheres)



in-sphere predicate

Images par O. Devillers

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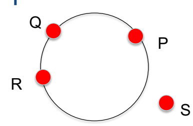
Algorithmic using predicates

- Predicate of inclusion of a point s in a circle circumscribed at p,q and r (orientation of the 4 points lifted on the paraboloid centered at p)

  Φ = lifting operator



- In_cercle(p,q,r,s)

  =-signe(((Φ(q)-Φ(p))x(Φ(r)-Φ(p))).(Φ(s)-Φ(p))

$$= - \mathrm{sign} \begin{vmatrix} q_x - p_x & r_x - p_x & s_x - p_x \\ q_y - p_y & r_y - p_y & s_y - p_y \\ (q_x - p_x)^2 + (q_y - p_y)^2 & (r_x - p_x)^2 + (r_y - p_y)^2 & (s_x - p_x)^2 + (s_y - p_y)^2 \end{vmatrix}$$

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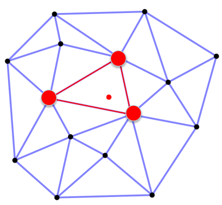
Back to Delaunay triangulation

- How to modify the incremental algorithm of insertion into a simple triangulation to obtain an incremental algorithm of insertion into a Delaunay triangulation?



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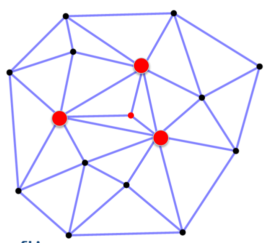
Incremental insertion into a Delaunay triangulation

- First perform a simple insertion :



  - Use Lawson flips
  - Is it necessary to test all the edges?

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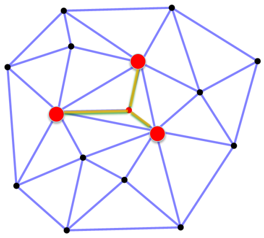
Delaunay incremental insertion

- Let Δ be the triangle in which P was inserted
  - After the simple insertion, the triangle is star-shaped with respect to P



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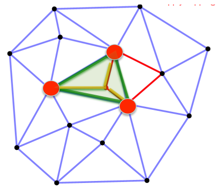
Incremental Delaunay insertion

- Let Δ be the triangle in which P was inserted
  - After the simple insertion, only the 3 edges of Δ could be candidates for flipping
  - The 3 new edges incident to P could not be flipped
  - The others have not changed their pair of incident triangles



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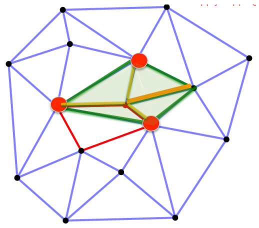
Incremental Delaunay insertion

- Each flip generates a new edge incident to P and two edges are added to the boundary of the modified area (in green)
- New green edges should be checked in turn (one of their incident triangles has been modified)



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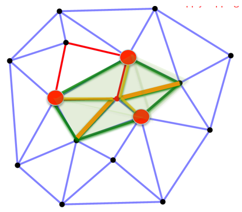

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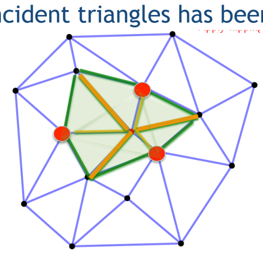

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Incremental Delaunay Algorithm

- The complexity of each insertion directly corresponds to the final number i of edges incident at the new vertex.
  - Every flipped edge gave birth to an edge incident to P ->there are i-3 flips
  - The flipping test was checked on each edge that was effectively flipped, and also on the green boundary of the resulting modified area (i edges)

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Incremental Delaunay Algorithm

- An insertion outside the convex envelope also starts as the simple insertion into a triangulation
  - Additional flips can be performed on the boundary of the modified area

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Delaunay incremental insertion (Alternative version)

- We just showed that the modified area of the triangulation is star-shaped around the inserted point P
- Alternative approach for incremental Delaunay :
  - Delete all the triangles whose circumscribed circle contains point P
  - Those triangles are said to be "in conflict" with P
  - Triangulate the conflict zone by star-shaping the conflict area around P

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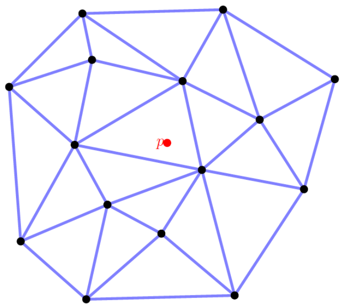
Incremental Delaunay Algorithm (Alternative Version)



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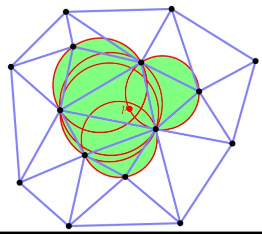
Delaunay incremental algorithm (Alternative Version)

- Determine the in-conflict triangles
  - Using breadth-first search (or deapth-first search) on the adjacency graph of triangles starting from Δ



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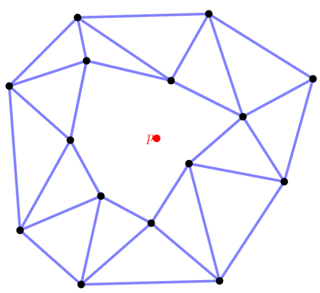
Delaunay incremental algorithm (Alternative Version)

- Conflict zone determination



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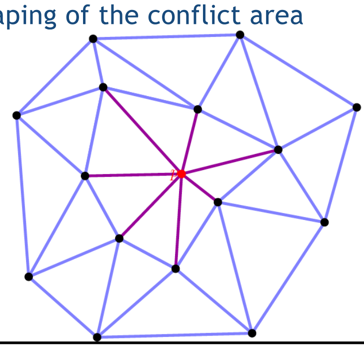
Delaunay incremental algorithm (Alternative Version)

- Star-shaping of the conflict area



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Delaunay incremental algorithm (Alternative Version)

- Star-shaping of the conflict area
- Validity of this algorithm in higher dimension
  - In 2D, the **edges of the boundary** of the conflict zone get connected to the inserted point by constructing new triangles
  - In 3D, the Delaunay triangulation is composed of tetrahedrons. The **triangles of the boundary** of the conflict zone get connected to the inserted point by constructing new tetrahedrons

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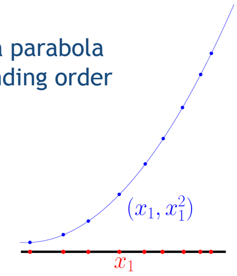
Incremental Delaunay Algorithm

- Complexity analysis
- Worst case :
  - Points distributed on a parabola and inserted in descending order of abscissa.



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Incremental Delaunay Algorithm

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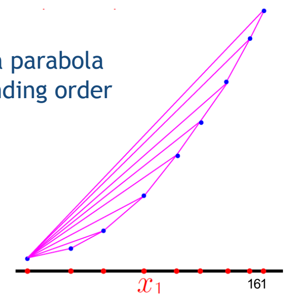

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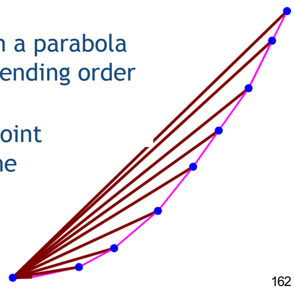
Incremental Delaunay Algorithm

- Complexity analysis
- Worst case :
  - Points distributed on a parabola and inserted in descending order of abscissa.
  - Each new inserted point conflicts with ALL the triangles

$\Omega(n^2)$



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