Lyon 1

Mesh and Computational Geometry

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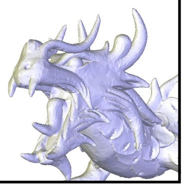
M2 ID3D
Image, Développement
et Technologie 3D
et 3A Centrale

2024-25



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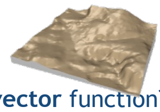
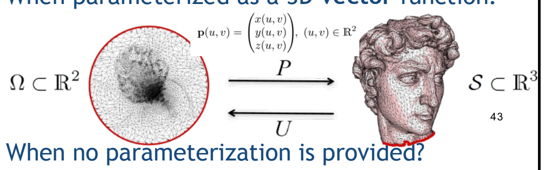
Local surface variations

- How to study and characterize surface variations?
  - When defined as a height function over a plane?
    z=f(x,y)
    Multivariate **scalar** function
  - When parameterized as a 3D **vector** function?





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  - When no parameterization is provided?

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Interest

- Local surface variations
  - Used to compute geometric information such as normal vectors, curvature, and higher order information …
- Useful for :
  - Surface analysis
  - Surface rendering
  - Surface texturing
  - Constructing a better parameterization with less distortion

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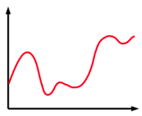
Differential operators

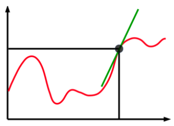
- Reminder :
  - Given a univariate function $f : \mathbb{R} \to \mathbb{R}$



  - The derivative of f is another function $\frac{\partial f}{\partial x}$ that describes the growing speed of f

$f' : \mathbb{R} \to \mathbb{R}$

$f'(x) = \lim_{a \to 0} \frac{f(x+a) - f(x)}{a}$



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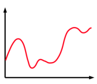
Differential operators

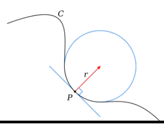
- Reminder :
  - The Laplacian (second derivative) of a function $f : \mathbb{R} \to \mathbb{R}$ is a measure of the difference between the value of f at any point P and the average value of f in the vicinity of P



  - It is linked to the curvature of the curve (inverse of the osculating circle radius)

$\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$

Since the curve is oriented, the norm | | can be removed to get a signed curvature



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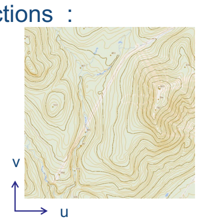
Differential operators

- Derivatives of a multivariate functions :
  - Gradient of a scalar function f(u,v)

$$grad(f) = \begin{bmatrix} \frac{\delta f}{\delta u} \\ \frac{\delta f}{\delta v} \end{bmatrix}$$

- Orthogonal to iso-lines
- Scalar product with a direction U,V gives the derivative in direction U,V



  - Laplacian of a scalar function f(u,v)

$$\Delta(f) = \frac{\delta^2 f}{\delta u^2} + \frac{\delta^2 f}{\delta v^2}$$

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Differential operators

- Laplace equation

$$f : U \in \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\Delta f = 0$$

- Solutions of Laplace equation :
  - Harmonic functions = kernel of the Laplacian operator
  - No local maxima or minima in U
  - Minimizing the Dirichlet energy that measures the smoothness of a function

$$E(f) = \int_U < \nabla f, \nabla f > dA$$

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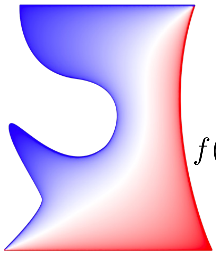
Differential operators

- Solutions of Laplace equation :
  - Used for interpolation



$$\Delta f(x) = 0 \qquad x \in U$$

$$f(x) = f_0(x) \qquad x \in \delta U$$

[K. Crane]

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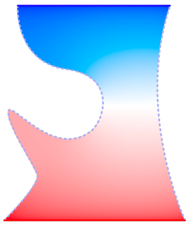
Differential operators

- Solutions of Laplace equation :
  - Used for interpolation



$$\Delta f(x) = 0 \quad x \in U$$

$$f(x) = f_0(x) \quad x \in \delta U_D \; (Dirichlet\, boundary)$$

$$\nabla f.n = g_0(x) \quad x \in \delta U_N \; (Neumann\, boundary)$$

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Differential operators

- Harmonic functions not to be misunderstood with the eigenvectors and eigen values of the Laplacian operator

$$\Delta f = \lambda f$$

When $U = \mathbb{R}$ eigenvectors are the sines and cosines functions that are used within Fourier framework

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Differential operators

- In a multivariate context, differential operators are often used in differential equations
- Generally expressed in $\mathbb{R}^k$
- Spatial notation "nabla" : Think of it as a vector of partial derivative operators! $\nabla = \begin{bmatrix} \delta_u \\ \delta_v \end{bmatrix}$
  - Gradient operator of a **scalar** function f : ∇ f
  - Divergence operator of a **vector** function **V** : $\nabla . \mathbf{V}$
  - Laplacian operator of a **scalar** function f : $\Delta f = \nabla . \nabla f$
  - Curl (rotational) operator of a **vector** function **V** : $\nabla \times \mathbf{V}$
- Expression of nabla depending on the coordinate system of the input domain

Cartesian coordinates u,v

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Gradients, divergence et rotationnels

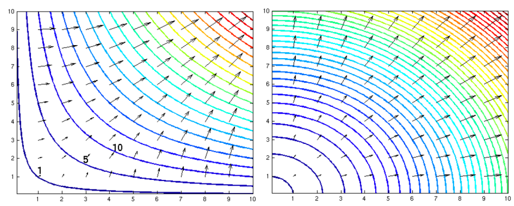

FIGURE 1 – Exemples de champs scalaires (couleur, valeurs élevées en rouge) et de leur gradient (flèches). À gauche : $f(x, y) = xy$, à droite $f(x, y) = x^2 + y^2$.

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Gradients, divergence et rotationnels

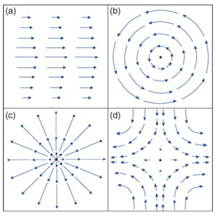


FIGURE 2 – Exemples de champs de vecteurs bidimensionnels : (a,b) champs rotationnels ; cisaillement pur (a) et rotation solide (b) (imaginer la rotation d'une petite roue à aubes insérée dans l'écoulement). (c) champ central divergent. (d) champ avec déformation, mais à divergence et rotationnel nuls (compression dans une direction, dilatation dans l'autre à surface constante).

Expression of nabla on different coordinates systems

Opération	Coordonnées cartésiennes (x, y, z)	Coordonnées cylindriques (r, θ, z)	Coordonnées sphériques (r, θ, φ)
Définition des coordonnées		$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$	$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$
$\vec{A}$	$A_x\vec{u}_x + A_y\vec{u}_y + A_z\vec{u}_z$	$A_r\vec{u}_r + A_\theta\vec{u}_\theta + A_z\vec{u}_z$	$A_r\vec{u}_r + A_\theta\vec{u}_\theta + A_\varphi\vec{u}_\varphi$
$\vec{\nabla}$	$\frac{\partial}{\partial x}\vec{u}_x + \frac{\partial}{\partial y}\vec{u}_y + \frac{\partial}{\partial z}\vec{u}_z$	$\frac{\partial}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\vec{u}_\theta + \frac{\partial}{\partial z}\vec{u}_z$	$\frac{\partial}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\vec{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{u}_\varphi$

[Wikipedia]

Differential operator

- A few reminders :
  - Gradient : direction (u_{max},v_{max}) of maximal slope
  - Divergence : flow traversing a unit element around (u,v)
  - Laplacian : measures the difference between the function and its mean value in a small neighborhood
    - Useful for several geometry processing tasks (interpolation by heat diffusion, spectral analysis, mean-curvature, smoothing)
  - Curl (Rotational) : does the vector field locally turn around one vector?
- Questions :
  - How could you define the "gradient" of a vector field? Jacobian matrix
  - How could you define the "laplacian" of a vector field? Vector with the Laplacian of each component

Other use of Nabla

- Nabla transpose

  ${}_x\delta$: Derivative of the scalar field on the left

$$\nabla^t = \left[{}_x\delta, {}_y\delta, {}_z\delta\right]$$

- Jacobian matrix of a vector field **V**

$$\mathbb{J}_{\mathbf{V}} = \mathbf{V}\nabla^{\mathbf{t}} = \begin{bmatrix} \delta_x v_x, \delta_y v_x, \delta_z v_x \\ \delta_x v_y, \delta_y v_y, \delta_z v_y \\ \delta_x v_z, \delta_y v_z, \delta_z v_z \end{bmatrix}$$

Differential operator

- Functions defining a curve

  $\mathbf{x} : [0, L] \to \mathrm{R}^3$ de classe C^3

  Tangent, normal and binormal vectors?

Vecteur tangent : $\cancel{\mathbf{t}(s) = \frac{\mathbf{x}'(s)}{\|\mathbf{x}'(s)\|}}$ $\quad \mathbf{t}(t) = \frac{\mathbf{X}'(t)}{\|\mathbf{X}'(t)\|}$

Vecteur normal : $\cancel{\mathbf{m}(s) = \frac{\mathbf{x}''(s)}{\|\mathbf{x}''(s)\|}}$ $\quad \mathbf{m}(t) = \frac{\mathbf{t}'(t)}{\|\mathbf{t}'(t)\|}$

Vecteur binormal : $\mathbf{b}(s) \cancel{= \mathbf{t}(s)} \times \mathbf{m}(s)$ $\quad \mathbf{b}(t) = \mathbf{b}(t) \times \mathbf{b}(m)$

Repère de Serret-Frénet : $(\mathbf{x}(s); \cancel{\mathbf{t}(s), \mathbf{m}(s)}, \mathbf{b}(s))$

$(\mathbf{X}(t); \mathbf{t}(t), \mathbf{m}(t), \mathbf{b}(t))$

Ici on n'a pas encore parlé de s, absisse curviligne, d'où le correctif du slide qui était ambigu

Differential operator

- Let s be the curvilinear abscissa
  - s : length of curve between 0 and t

$$s(t) = \int_0^t \|\mathbf{X}'(t)\| dt$$

  - parameterize the curve wrt s
  - t(s)=x'(s) is a unit vector

Courbure : $\kappa(s) = \mathbf{x}''(s)$

- Mesure la déviation par rapport à une droite

Torsion : $\tau(s) = \frac{det[\mathbf{x}'(s), \mathbf{x}''(s), \mathbf{x}'''(s)]}{\kappa^2(s)}$

- Mesure le défaut de planarité

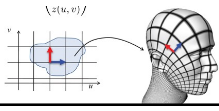
Differential operator

- Functions defining a surface

$\mathbf{X} : \Omega \subset \mathrm{R}^2 \rightarrow \mathrm{R}^3$ de classe C^r

Analogues vecteurs tangent/normal/binormal ?

On note $\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}$ et $\mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v}$
Plan tangent en p : $T_p\mathbf{X}$ = plan passant par p et engendré par les vecteurs $\mathbf{X}_u(p)$ et $\mathbf{X}_v(p)$
Vecteur normal : $\mathbf{n}(p) = \frac{\mathbf{X}_u(p) \times \mathbf{X}_v(p)}{\|\mathbf{X}_u(p) \times \mathbf{X}_v(p)\|}$
$(\mathbf{X}(p); \mathbf{X}_u(p), \mathbf{X}_v(p), \mathbf{n}(p))$ forme aussi un repère local



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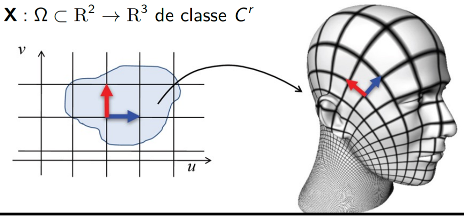
Differential operator

- Functions defining a surface
- How do unitary vectors are transformed by the parameterization?
  - Length and angles distorsion?

$\mathbf{X} : \Omega \subset \mathrm{R}^2 \rightarrow \mathrm{R}^3$ de classe C^r



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Differential operator

- Functions defining a surface?

On note de manière analogue $\mathbf{X}_{uu} = \frac{\partial \mathbf{X_u}}{\partial u} = \frac{\partial^2 \mathbf{X}}{\partial u^2}$, etc.

Première forme fondamentale :

$$\mathbf{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_u^T\mathbf{x}_u & \mathbf{x}_u^T\mathbf{x}_v \\ \mathbf{x}_u^T\mathbf{x}_v & \mathbf{x}_v^T\mathbf{x}_v \end{bmatrix}$$

Angle change

Length change

Seconde forme fondamentale :

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T\mathbf{n} & \mathbf{x}_{uv}^T\mathbf{n} \\ \mathbf{x}_{uv}^T\mathbf{n} & \mathbf{x}_{vv}^T\mathbf{n} \end{bmatrix}$$

Opérateur de forme/Application de Weingarten :

$$\mathbf{W} \quad := \quad \frac{1}{EG - F^2} \begin{bmatrix} eG - fF & fG - gF \\ fE - eF & gE - fF \end{bmatrix} = \begin{pmatrix} D_u n & D_v n \end{pmatrix}$$

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I = outil géométrique (**tenseur métrique**)

- Permet de mesurer aires locales, longueurs de courbes sur la surface, angles, . . . $dA = \sqrt{EG - F^2}\, dudv$
- Exemple : anisotropie locale de la surface : décomposition spectrale de **I**

Propriétés différentielles ne dépendant que de **I** sont dites **intrinsèques**

- Ne dépendent pas de la paramétrisation
- Ne dépendent pas de l'espace 3D

Eigenvalues of I : maximal/minimal stretching of a tangent vector

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II = propriétés **extrinsèques** de la surface

- Dépendent du plongement dans l'espace ambiant R^3

W détermine les directions de courbure locale de la surface

- Valeurs propres = courbures principales
- Vecteurs propres = directions principales de courbure

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Courbures principales et directions principales de courbure :

$$\mathbf{W} \quad = \quad \begin{bmatrix} \bar{\mathbf{t}}_1 & \bar{\mathbf{t}}_2 \end{bmatrix} \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{t}}_1 & \bar{\mathbf{t}}_2 \end{bmatrix}^{-1}$$

Courbure moyenne : $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2} trace(\mathbf{W})$

Courbure de Gauss : $K = \kappa_1 \cdot \kappa_2 = det(\mathbf{W})$

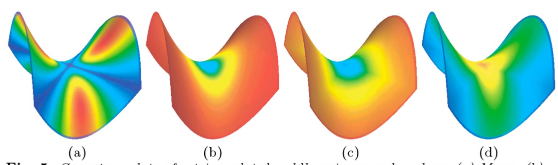

Fig. 5. *Curvature plots of a triangulated saddle using pseudo-colors: (a) Mean, (b) Gaussian, (c) Minimum, (d) Maximum.*

[Meyer et al. 2003]

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Implicit surfaces

2.1.2. Differential Operators for implicit Surfaces
Assume that

$$\Gamma_c := \left\{ \mathbf{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^3 : \phi(\mathbf{x}) = c \right\}$$

where $\phi(\mathbf{x})$ is a properly smooth function defined on $\mathbb{R}^3$, c is an arbitrarily given constant. Suppose that $\|\nabla\phi\|$ on Γ_c , thus, according to the implicit function theorem, the level-set surface could be locally parameterized. For each point on the surface, we can obtain the unit normal vector

$$\mathbf{n} = \frac{\nabla\phi}{\|\nabla\phi\|}.$$

The mean curvature H for the level-set surface can be deduced as

$$H = -\frac{1}{2}\,\mathrm{div}\left(\frac{\nabla\phi}{\|\nabla\phi\|}\right) \qquad (3)$$

where ∇ and div denote the classical gradient and divergence operatorts, respectively.

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Differential operators

- Functions defined on a surface
  - Let u be a scalar function defined on a surface
  - We would like to express local variations of u
- Let consider the discrete case of a surface being approximated by a simplicial mesh
  - Function u discretized on vertices
  - Gradient of u discretized on triangles
  - Divergence of a vector (defined on triangles) discretized on vertices
- Bibliography : Keenan Crane

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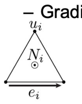
Differential operators

- Gradient d'une fonction définie sur les sommets d'un triangle
  - Gradient de l'interpolation linéaire de la fonction sur le triangle
  - Gradient ∇ of u inside a triangle = sum over the 3 vertices



$$\boxed{\nabla u = \frac{1}{2A_f} \sum_i u_i (N \times e_i)}$$

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Differential operators

- Laplacien en un sommet d'une fonction définie sur les sommets
  - Différence entre la moyenne des valeurs sur les voisins et la valeur au sommet
  - Quels coefficients choisir pour faire la moyenne?
  - Choisir des coefficients cohérents avec le fait que le laplacien de la fonction de position des points sur une surface doit être liée à la courbure moyenne et à la normale à la surface (et correspond au gradient de l'aire locale de la surface quand la position du point varie).

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Differential geometry

- How to generalize normal, curvatures?
- Consider u corresponding to each coordinate fonction in turn
  - u = ^{t}(x,y,z)

$$\triangle_{\mathbf{x}} \mathbf{u} = -2H\mathbf{n}$$

H : mean curvature
n : normal vector

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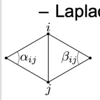
Differential operators

- Laplacien en un sommet d'une fonction définie sur les sommets
  - Différence entre la moyenne des valeurs sur les voisins et la valeur au sommet
  - Quels coefficients choisir pour faire la moyenne?

– Laplacian $\triangle$ of u at vertex i = sum over neighbor vertices j



$$(Lu)_i = \frac{1}{2A_i} \sum_j (\cot\alpha_{ij} + \cot\beta_{ij})(u_j - u_i)$$

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Differential operators

- Fait au tableau
  - Gradient de la fonction qui donne l'aire d'un triangle quand son sommet s0 varie

- Sur la surface regardons la fonction qui a un point associe ses coordonnées (x,y,z)
  Laplacien de la position (x,y,z) en un point P
  = gradient de l'aire locale de surface autour du point P quand P varie.

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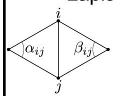
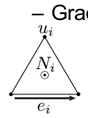
Differential operators

- Simplicial meshes
  - Laplacian Δ of u at vertex i = sum over neighbor vertices j



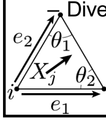
$$(Lu)_i = \frac{1}{2A_i} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i)$$

  - Gradient ∇ of u inside a triangle = sum over the 3 vertices I



$$\nabla u = \frac{1}{2A_f} \sum_i u_i (N \times e_i)$$

  - Divergence at vertex i of a vector X defined on faces = sum over incident faces j



$$2A_f \nabla \cdot X = \frac{1}{2} \sum_j \cot \theta_1 (e_1 \cdot X_j) + \cot \theta_2 (e_2 \cdot X_j)$$

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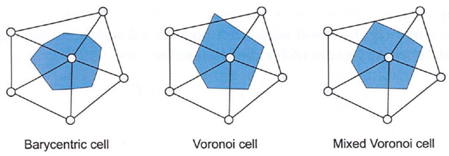
Area A_i

- Computed by duality to a vertex



Courtesy M. Botsch et al.

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Signals being studied in Computer Graphics

- Signals defined on $\mathbb{R}^2$
  - Scalar signals :
    - height value (terrain)
    - density of some fluid flowing in the plane
  - Vector fields
    - Surface parameterization $\Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$
    - Displacement field of some fluid flowing in the plane
- Signals defined in $\mathbb{R}^3$
    - Density of a volume material (scalar)
    - Displacement field of some fluid (vector)

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Signals being studied in Computer Graphics

- Signals defined on a surface $\mathbb{S}$
  - Scalar values :
    - Temperature, grey color …
    - Position coordinates (x, y or z)
  - Vector fields
    - Normal vector
    - Maximal/minimal curvature direction
    - Displacement field
    - …

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