STA 6106 Homework 9

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1. Suppose that the random variables X and Y both take on values in the interval $(0, B)$. Suppose that the conditional density of X given that $Y = y$ is

$$f(x|y) = c(y)e^{-xy}, \quad 0 < x < B$$

and the conditional density of Y given that $X = x$ is

$$f(y|x) = c(x)e^{-xy}, \quad 0 < y < B$$

Give a method for simimately simulating the vector X, Y. Run a simulation to estimate (a) $E(X)$ and (b) $E(XY)$.

Algorithms

Since we are given conditional probabilities, Gibbs sampling can be used to generate observations from the marginal distributions. We will use Gibbs to generate values from $f(x)$, and then use $f(y|x)$ to generate a corresponding y. The following algorithm returns a random observation from the joint distribution of X and Y.

Gibbs sampling algorithm

1. Set x_0 to some initial value in (0, B), and set $i = 1$.
2. Generate $y_i \sim f(y|x_{i-1})$.
3. Generate $x_i \sim f(x|y_i)$.
4. Repeat 2 and 3, N times.
5. Return $(x, y) = (x_{N-1}, y_N)$

A method for generating observations from the conditional distributions is required. The inverse-cdf method will be used. First we consider generating a random variable from $y \sim f(y|x)$. The cdf of $Y|X$ is

$$\begin{aligned}
F(y|x) &= \int_0^y f(y|x)dx \\
&= \int_0^y c(x)e^{-xy}dx \\
&= \frac{-c(x)}{x}\left(e^{-xy} - 1\right), \quad 0 < y < B
\end{aligned}$$

The inverse of this cdf is

$$\begin{aligned}
u &= \frac{-c(x)}{x}\left(e^{-xy} - 1\right) \\
\Rightarrow 1 + \frac{-ux}{c(x)} &= e^{-xy} \\
\Rightarrow y &= \frac{-1}{x}\ln\left(1 - \frac{ux}{c(x)}\right)
\end{aligned}$$

Note that $c(x)$ is just a constant factor that allows the pdf to integrate to 1. Here, we have $\frac{1}{c(x)} = \int_0^B e^{-xy}dy = \frac{-1}{x}\left(e^{-By} - 1\right)$. Assuming some value for x and B are given, the following algorithm can be used to generate an observation from the conditional distribution.

Inverse-CDF method for $Y|X$

1. Generate a $u \sim U(0,1)$
2. Return $y = \frac{-1}{x} \ln\left(1 - u\left(1 - e^{-By}\right)\right)$.

And since the two conditional distributions have the same form, similar results are obtained for $X|Y$.

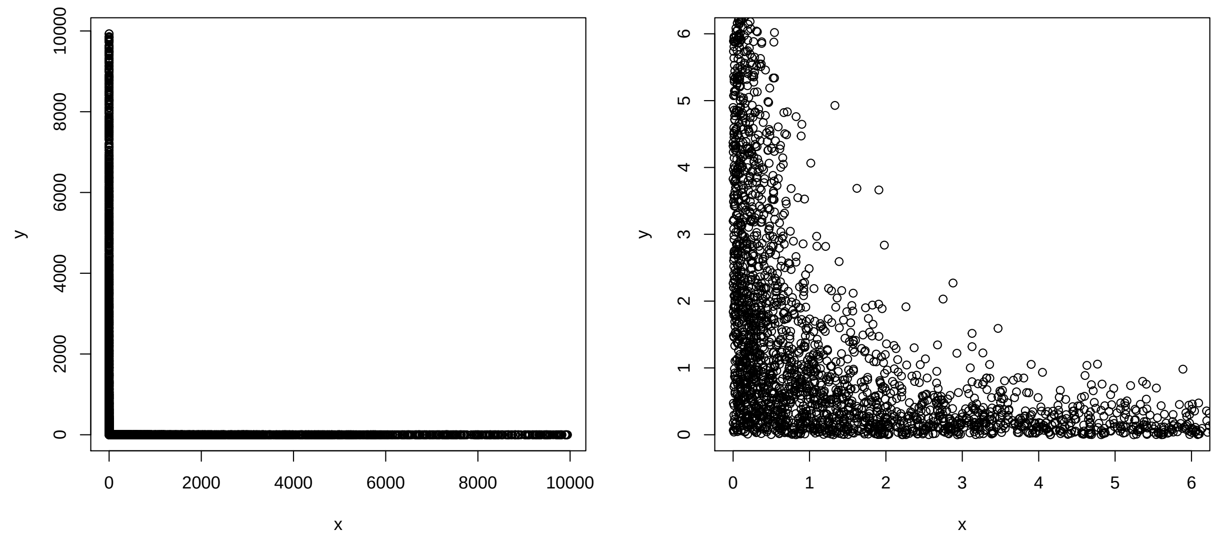
Inverse-CDF method for $X|Y$

1. Generate a $u \sim U(0,1)$
2. Return $x = \frac{-1}{y} \ln\left(1 - u(1 - e^{-Bx})\right)$.

Code

```
#Generate a random observation from the conditional Y/X.
fy_conditional <- function(x, B = 10000) {
  u <- runif(1)
  -1/x * log(1 - u*(1 - exp(-B*x)))
}

#Generate a random observation from the conditional X/Y.
fx_conditional <- function(y, B = 10000) {
  u <- runif(1)
  -1/y * log(1 - u*(1 - exp(-B*y)))
}

#Generate an observation from the joint X, Y.
fx_and_y <- function(x0 = 1, B = 10000, BURN = 1000) {
  #Initialize x and compute the first value for y.
  x <- x0
  y <- fy_conditional(x, B)

  #Generate BURN (x, y) pairs for the burn in period.
  for(i in 1:BURN) {
    x <- fx_conditional(y, B)
    y <- fy_conditional(x, B)
  }

  #Use the last (x, y) pair as the observation.
  return(c(x, y))
}

#Simulate n observations from the joint distribution of X and Y.
n <- 10000
x <- vector("numeric", n)
y <- vector("numeric", n)

for(i in 1:n) {
  temp <- fx_and_y()
  x[i] <- temp[1]
```

```
  y[i] <- temp[2]
}
```



We can estimate $E(X)$ by averaging the x values from our simulated observations.

```
E_X <- mean(x)
```

This gives the approximation $E(X) \approx 514.8$. Similarly, $E(XY)$ is estimated by averaging the products $x * y$ of each pair of observations from the simulated data.

```
E_XY <- mean(x*y)
```

The approximation is $E(XY \approx 0.9377$