STA 6106 Homework 6: MLE

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1. Let $x_1, ..., x_n$ represent a random sample from each of the distributions having the following probability density function. In each case find the MLE of θ.

a. $f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!}$, $x = 0, 1, 2, ...$, and $0 \leq \theta$ where $f(0, 0) = 1$

The likelihood function is

$$L(\theta; x) = \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x_i!}$$

Taking the log, we obtain the log-likelihood

$$\begin{aligned}\ell(\theta; x) &= \sum_{i=1}^{n} [x_i \log(\theta) - \theta \log(e) - \log(x_i!)] \\ &= \log(\theta) \sum_{i=1}^{n} x_i - n\theta - \sum_{i=1}^{n} \log(x_i!)\end{aligned}$$

Take the derivative with respect to θ

$$\frac{d}{d\theta} \ell(\theta; x) = \frac{1}{\theta} \sum_{i=1}^{n} x_i - n$$

Now, set the derivative equal to 0 and solve for $\hat{\theta}$

$$\begin{aligned} & \frac{d}{d\theta} \ell(\hat{\theta}; x) = 0 \\ \Rightarrow \quad & \frac{1}{\hat{\theta}} \sum_{i=1}^{n} x_i - n = 0 \\ \Rightarrow \quad & \hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n}\end{aligned}$$

b. $f(x;\theta) = \theta x^{(\theta-1)}$, $0 < x < 1$, and $\theta > 0$

The likelihood function is

$$L(\theta; x) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$$

Taking the log, we obtain the log-likelihood

$$\begin{aligned}\ell(\theta; x) &= \sum_{i=1}^{n} [\log(\theta) + (\theta - 1) \log(x_i)] \\ &= n \log(\theta) + \theta \sum_{i=1}^{n} \log(x_i) - \sum_{i=1}^{n} \log(x_i)\end{aligned}$$

Take the derivative with respect to θ

$$\frac{d}{d\theta}\ell(\theta; x) = \frac{n}{\theta} + \sum_{i=1}^{n} \log(x_i)$$

Now, set the derivative equal to 0 and solve for $\hat{\theta}$

$$\begin{aligned}
&\frac{d}{d\theta}\ell(\hat{\theta}; x) = 0 \\
\Rightarrow\quad &\frac{n}{\hat{\theta}} + \sum_{i=1}^{n} \log(x_i) = 0 \\
\Rightarrow\quad &\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log(x_i)}
\end{aligned}$$

c. $f(x; \theta) = \frac{1}{\theta}e^{-x/\theta}$, $x > 0$, and $\theta > 0$
The likelihood function is

$$L(\theta; x) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x}{\theta}}$$

Taking the log, we obtain the log-likelihood

$$\begin{aligned}
\ell(\theta; x) &= \sum_{i=1}^{n} \left[-\frac{x_i}{\theta} - \log(\theta) \right] \\
&= -\frac{1}{\theta} \sum_{i=1}^{n} x_i - n\log(\theta)
\end{aligned}$$

Take the derivative with respect to θ

$$\frac{d}{d\theta}\ell(\theta; x) = \frac{\sum_{i=1}^{n} x_i}{\theta^2} - \frac{n}{\theta}$$

Now, set the derivative equal to 0 and solve for $\hat{\theta}$

$$\begin{aligned}
&\frac{d}{d\theta}\ell(\hat{\theta}; x) = 0 \\
\Rightarrow\quad &\frac{\sum_{i=1}^{n} x_i}{\hat{\theta}^2} - \frac{n}{\hat{\theta}} = 0 \\
\Rightarrow\quad &\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n}
\end{aligned}$$

d. $f(x; \theta) = \frac{1}{2}e^{-|x-\theta|}$, $-\infty < x < \infty$, and $-\infty < \theta < \infty$
The likelihood function is

$$L(\theta; x) = \prod_{i=1}^{n} \frac{1}{2} e^{-|x_i - \theta|}$$

Taking the log, we obtain the log-likelihood

$$\begin{aligned}
\ell(\theta; x) &= \sum_{i=1}^{n} \left[-\log(2) - |x_i - \theta| \right] \\
&= -n\log(2) - \sum_{i=1}^{n} |x_i - \theta|
\end{aligned}$$

The log-likelihood is maximized when $\sum_{i=1}^{n} |x_i - \theta|$ is minimized. It is well known that

$$\sum_{i=1}^{n} |x_i - \theta| \geq \sum_{i=1}^{n} |x_i - m|$$

where m is the median of the x_i's. Hence, the MLE is $\hat{\theta} = m$

e. $f(x;\theta) = e^{-(x-\theta)}$, $x \geq \theta$

The likelihood function is

$$\begin{aligned} L(\theta; x) &= \prod_{i=1}^{n} e^{-(x-\theta)} I_{(x_i \geq \theta)} \\ &= \prod_{i=1}^{n} e^{-(x-\theta)} I_{(\min(x_i) \geq \theta)} \end{aligned}$$

Taking the log, we obtain the log-likelihood

$$\ell(\theta; x) = \sum_{i=1}^{n} -(x_i - \theta), \quad \theta \leq \min(x_i)$$

To maximize $\ell(\theta; x)$, we need $(x_i - \theta)$ to be small. So, we pick the largest possible value for $\hat{\theta}$. Since $\hat{\theta}$ must satisfy $\hat{\theta} \leq \min(x_i)$, we choose $\hat{\theta}$ to be this upper bound. Hence, the MLE is $\hat{\theta} = min(x_i)$.