DISCRIMINANT ANALYSIS

1 Introduction

future prediction

exploratory

<u>Reference:</u> Chapter 11, Johnson and Wichern

Discriminant analysis is also called *classification analysis*. Suppose we have number of observations from several populations and we have a new observation that is known to come from one of these populations. If the population of the new observation is unknown, the rules in discriminant analysis will enable to identify the most likely population for the new object.

2 Classification for Two Populations

Consider two p-dimensional multivariate populations as follows:

probability density functions

Population 1: Π_1 with pdf $f_1(\boldsymbol{x})$

Population 2: Π_2 with pdf $f_2(\boldsymbol{x})$.

Suppose a new observation vector $\boxed{\boldsymbol{x}_0}$ is known to come from either Π_1 or Π_2, we need a rule to classify $\boldsymbol{x}_0$ into population Π_1 or Π_2.

The cost of misclassification of $\boldsymbol{x}_0$ can be defined by the following table:

		Classified as:	
		Π_1	Π_2
True	Π_1	0 ✓	$\boxed{c(2\|1)}$
population	Π_2	$\boxed{c(1\|2)}$	0 ✓

misclassification costs

where $c(j|i)$ is the cost of incorrectly classifying $\boldsymbol{x}_0$ as Π_j when it is from Π_i $(i \neq j)$ for $i, j = 1, 2$. Note that the cost of correct classification is 0.

Let us assume $p(j|i)$ be the conditional probability of incorrectly classifying $\boldsymbol{x}_0$ as Π_j when it is from Π_i $(i, j = 1, 2)$ and $\boxed{p_i}$ be the prior probability of

prior probabilities

$\boldsymbol{x}_0$ from population Π_i for $i = 1, 2$ such that $p_1 + p_2 = 1$. The expected cost of misclassification (ECM) of $\boldsymbol{x}_0$ is given by

$$\boxed{\text{ECM} = \underbrace{c(2|1)p(2|1)p_1}_{\text{pop1}} + \underbrace{c(1|2)p(1|2)p_2}_{\text{pop2}}.}$$

The rule that minimizes the ECM are as follows:

comparison of prob. density function

$$\text{Allocate } \boldsymbol{x}_0 \text{ to } \Pi_1 \text{ if } \frac{f_1(\boldsymbol{x}_0)}{f_2(\boldsymbol{x}_0)} \geq \frac{c(1|2)}{c(2|1)} \frac{p_2}{p_1} \tag{1}$$

otherwise allocate $\boldsymbol{x}_0$ to Π_2.

Special Case:

In general, $c(j|i)$ and p_i's are unknown, however we can assume that the misclassification cost and also the prior probabilities are equal. That is, $\underbrace{c(2|1) = c(1|2)}_{\text{costs}}$ and $\underbrace{p_1 = p_2}_{\text{priors}}$ then the above rule becomes:

Allocate $\boldsymbol{x}_0$ to Π_1 if $f_1(\boldsymbol{x}_0) \geq f_2(\boldsymbol{x}_0)$, otherwise allocate $\boldsymbol{x}_0$ to Π_2.

Note that the likelihood rule can be applied to non-normal populations.

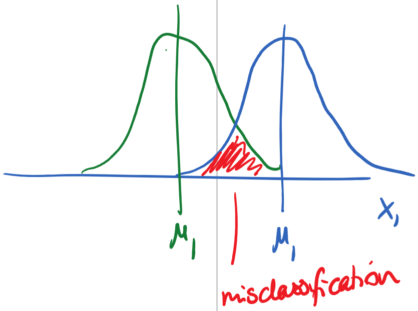
Example 1: Allocate the following observations, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ to most suitable exponential population among Π_1: Exp(λ_1) and Π_2: Exp(λ_2), where

$$\lambda_1 = 2, \quad \text{and } \lambda_2 = 1$$

Observations are:

$$x_1 = 2.0 \quad \text{and} \quad x_2 = 2.5.$$

Assume misclassification costs, $c(2|1) = 2c(1|2)$ and, prior probabilities $p_1 = 0.25$ and $p_2 = 0.75$.



3 Classification for Two Normal Populations When $\Sigma_1 = \Sigma_2 = \Sigma$

multivariate normal pdf

Consider two multivariate normal populations Π_1 : $N_p(\boldsymbol{\mu}_1, \boxed{\Sigma})$ and Π_2 : $N_p(\boldsymbol{\mu}_2, \boxed{\Sigma})$. That is

$$f_i(\boldsymbol{x}) = (2\pi)^{-\frac{p}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_i)\right\}, \ i = 1, 2$$

and

$$\frac{f_1(\boldsymbol{x})}{f_2(\boldsymbol{x})} = \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_1) + \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_2)^T \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_2)\right\}.$$

Now using the allocation rule given in (1):

Allocate $\boldsymbol{x}_0$ to Π_1 if

$$\exp\left\{-\tfrac{1}{2}(\boldsymbol{x}_0-\boldsymbol{\mu}_1)^T \Sigma^{-1} (\boldsymbol{x}_0-\boldsymbol{\mu}_1) + \tfrac{1}{2}(\boldsymbol{x}_0-\boldsymbol{\mu}_2)^T \Sigma^{-1} (\boldsymbol{x}_0-\boldsymbol{\mu}_2)\right\} \geq \frac{c(1|2)}{c(2|1)} \frac{p_2}{p_1}$$

otherwise allocate $\boldsymbol{x}_0$ to Π_2.

Equivalently we can write the allocate rule as:

Allocate $\boldsymbol{x}_0$ to Π_1 if

$$(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^T \Sigma^{-1} \boldsymbol{x}_0 - \tfrac{1}{2}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^T \Sigma^{-1} (\boldsymbol{\mu}_1+\boldsymbol{\mu}_2) \geq \ln\left[\frac{c(1|2)}{c(2|1)} \frac{p_2}{p_1}\right]$$

otherwise allocate $\boldsymbol{x}_0$ to Π_2.

Let $\boldsymbol{b} = \Sigma^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)$ and $k = \frac{1}{2}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^T\Sigma^{-1}(\boldsymbol{\mu}_1+\boldsymbol{\mu}_2) + \ln\left[\frac{c(1|2)}{c(2|1)} \frac{p_2}{p_1}\right]$.
Now we can write the above rule as:

positive

$$\text{Allocate } \boldsymbol{x}_0 \text{ to } \Pi_1 \text{ if } \ \boldsymbol{b}^T\boxed{\boldsymbol{x}_0} - k \geq 0, \quad \text{otherwise allocate } \boldsymbol{x}_0 \text{ to } \Pi_2. \tag{2}$$

negative

This is also called *Fisher's linear discriminant rule.* The function $\boldsymbol{b}^T\boldsymbol{x}$ is called the *linear discriminant function* of $\boldsymbol{x}$.

Special Case: For equal misclassification costs and equal prior probabilities, that is, $c(2|1) = c(1|2)$ and $p_1 = p_2$, the above rule becomes:

Allocate $\boldsymbol{x}_0$ to Π_1 if $\boldsymbol{b}^T\boldsymbol{x}_0 - k \geq 0$, otherwise allocate $\boldsymbol{x}_0$ to Π_2.

where $\boldsymbol{b} = \Sigma^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)$ and $k = \frac{1}{2}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^T\Sigma^{-1}(\boldsymbol{\mu}_1+\boldsymbol{\mu}_2)$.

Example 2: Allocate the following observations, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ to most suitable population among $\Pi_1 : N_3(\boldsymbol{\mu}_1, \Sigma)$ and $\Pi_2 : N_3(\boldsymbol{\mu}_2, \Sigma)$, where

$$\boldsymbol{\mu}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{\mu}_2 = \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}, \quad \text{and} \ \Sigma = \begin{pmatrix} 9 & 4 & -2 \\ 4 & 4 & 3 \\ -2 & 3 & 16 \end{pmatrix}.$$

Observations are: $\boldsymbol{x}_1^T = (1, 1, 0)$ and $\boldsymbol{x}_2^T = (0, 2, -3)$.

Assume equal misclassification costs and prior probabilities.

4 Sample Discriminant Rule for Two Normal Populations When $\Sigma_1 = \Sigma_2 = \Sigma$

more common to have sample

If any or all the parameters $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ and Σ are unknown then estimate those unknown parameters using random samples from each of the two populations and apply the rule using the estimated values of the parameters.

Let $\underbrace{\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n_1}}_{\text{pop1}}$ be a random sample from population Π_1 and $\underbrace{\boldsymbol{y}_1, \dots, \boldsymbol{y}_{n_2}}_{\text{pop2}}$ be a random sample from population Π_2. Then estimators of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are respectively given by

$$\underbrace{\widehat{\boldsymbol{\mu}}_1 = \overline{\boldsymbol{x}}_{n_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} \boldsymbol{x}_i}_{\text{pop1}}, \quad \text{and} \quad \underbrace{\widehat{\boldsymbol{\mu}}_2 = \overline{\boldsymbol{y}}_{n_2} = \frac{1}{n_2}\sum_{i=1}^{n_2} \boldsymbol{y}_i}_{\text{pop2}}.$$

Consider the two sample covariance matrices

unbiased

$$\mathcal{S}_x = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1} (\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n_1})(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n_1})^T$$

and

$$\mathcal{S}_y = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2} (\boldsymbol{y}_i - \overline{\boldsymbol{y}}_{n_2})(\boldsymbol{y}_i - \overline{\boldsymbol{y}}_{n_2})^T.$$

Since the two populations have the same covariance matrix Σ, the estimate of Σ is given by the pooled sample covariance matrix $\mathcal{S}_{pooled}$.

$$\widehat{\Sigma} = \mathcal{S}_{pooled} = \frac{(n_1 - 1)\mathcal{S}_x + (n_2 - 1)\mathcal{S}_y}{n_1 + n_2 - 2}.$$

Using the above estimates, the allocation rule given in (2) can be written as:

Allocate $\boldsymbol{x}_0$ to Π_1 if
$$\left(\overline{\boldsymbol{x}}_{n_1} - \overline{\boldsymbol{y}}_{n_2}\right)^T \mathcal{S}_{pooled}^{-1}\boldsymbol{x}_0 - \tfrac{1}{2}\left(\overline{\boldsymbol{x}}_{n_1} - \overline{\boldsymbol{y}}_{n_2}\right)^T \mathcal{S}_{pooled}^{-1}\left(\overline{\boldsymbol{x}}_{n_1} + \overline{\boldsymbol{y}}_{n_2}\right) \geq \ln\left[\frac{c(1|2)}{c(2|1)}\frac{p_2}{p_1}\right]$$
otherwise allocate $\boldsymbol{x}_0$ to Π_2.

Equivalently

$$\text{Allocate } \boldsymbol{x}_0 \text{ to } \Pi_1 \text{ if } \quad \widehat{\boldsymbol{b}}^T\boldsymbol{x}_0 - \widehat{k} \geq 0, \text{ otherwise allocate } \boldsymbol{x}_0 \text{ to } \Pi_2 \qquad (3)$$

where $\widehat{\boldsymbol{b}} = \mathcal{S}_{pooled}^{-1}(\overline{\boldsymbol{x}}_{n_1} - \overline{\boldsymbol{y}}_{n_2})$ and

$$\widehat{k} = \frac{1}{2}(\overline{\boldsymbol{x}}_{n_1} - \overline{\boldsymbol{y}}_{n_2})^T\mathcal{S}_{pooled}^{-1}(\overline{\boldsymbol{x}}_{n_1} + \overline{\boldsymbol{y}}_{n_2}) + \ln\left[\frac{c(1|2)}{c(2|1)}\frac{p_2}{p_1}\right].$$

This is also called *Fisher's sample linear discriminant rule*. The function $\widehat{\boldsymbol{b}}^T\boldsymbol{x}$ is called the *sample linear discriminant function* of $\boldsymbol{x}$.

Example 3: Let $\boldsymbol{X}^T = (X_1, X_2, X_3)$ be a random vector representing important characteristics to distinguish between genuine and forged bank notes. A random sample of 50 genuine bank notes gives the mean $\overline{\boldsymbol{x}}_1^T = (2.1, 5.3, 4.0)$ and covariance matrix

$$\mathcal{S}_1 = \begin{pmatrix} 3.1 & 2.2 & 5.1 \\ 2.2 & 4.1 & 2.4 \\ 5.1 & 2.4 & 15.1 \end{pmatrix}.$$

Also the mean and covariance matrix of a random sample of 26 forged bank notes are as follows:

$$\overline{\boldsymbol{x}}_2 = \begin{pmatrix} 8.0 \\ 10.1 \\ 5.0 \end{pmatrix} \quad \text{and} \quad \mathcal{S}_2 = \begin{pmatrix} 2.9 & 2.8 & 5.1 \\ 2.8 & 4.0 & 2.6 \\ 5.1 & 2.6 & 14.9 \end{pmatrix}$$

(a) Identify the following two suspected bank notes as genuine or forged bank notes using Linear discriminant function.

$$\text{Bank note 1:} = \begin{pmatrix} 6.0 \\ 9.0 \\ 4.1 \end{pmatrix} \quad \text{and} \quad \text{Bank note 2:} = \begin{pmatrix} 2.1 \\ 4.9 \\ 4.9 \end{pmatrix}.$$

(b) List the assumptions you used for the above analysis.

5 Classification for Two Normal Populations When $\boxed{\Sigma_1 \neq \Sigma_2}$

Let $\Pi_1 : N_p(\boldsymbol{\mu}_1, \boxed{\Sigma_1})$ and $\Pi_2 : N_p(\boldsymbol{\mu}_2, \boxed{\Sigma_2})$ two multivariate normal populations where $\Sigma_1 \neq \Sigma_2$. That is

$$f_i(\boldsymbol{x}) = (2\pi)^{-\frac{p}{2}} |\Sigma_i|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T \Sigma_i^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_i)\right\}, \; i = 1, 2$$

and

$$\frac{f_1(\boldsymbol{x})}{f_2(\boldsymbol{x})} = \frac{|\Sigma_2|^{\frac{1}{2}}}{|\Sigma_1|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1) + \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_2)^T \Sigma_2^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_2)\right\}.$$

Now using allocation rule given in (1):

Allocate $\boldsymbol{x}_0$ to Π_1 if

$$\frac{|\Sigma_2|^{\frac{1}{2}}}{|\Sigma_1|^{\frac{1}{2}}} \exp\left\{-\tfrac{1}{2}(\boldsymbol{x}_0-\boldsymbol{\mu}_1)^T \Sigma_1^{-1}(\boldsymbol{x}_0-\boldsymbol{\mu}_1) + \tfrac{1}{2}(\boldsymbol{x}_0-\boldsymbol{\mu}_2)^T \Sigma_2^{-1}(\boldsymbol{x}_0-\boldsymbol{\mu}_2)\right\} \geq \frac{c(1|2)}{c(2|1)}\frac{p_2}{p_1}$$

otherwise allocate $\boldsymbol{x}_0$ to Π_2.

Equivalently we can write the allocate rule as:

Allocate $\boldsymbol{x}_0$ to Π_1 if

$$-\tfrac{1}{2}\boldsymbol{x}_0^T\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right)\boldsymbol{x}_0 + \left(\boldsymbol{\mu}_1^T\Sigma_1^{-1} - \boldsymbol{\mu}_2^T\Sigma_2^{-1}\right)\boldsymbol{x}_0 - K \geq 0, \qquad (4)$$

otherwise allocate $\boldsymbol{x}_0$ to Π_2

where

$$K = \frac{1}{2}\left(\boldsymbol{\mu}_1^T\Sigma_1^{-1}\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2^T\Sigma_2^{-1}\boldsymbol{\mu}_2\right) + \frac{1}{2}\ln\left(\frac{|\Sigma_1|}{|\Sigma_2|}\right) + \ln\left[\frac{c(1|2)}{c(2|1)}\frac{p_2}{p_1}\right]$$

Example 4: Allocate the following observations, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ to most suitable population among $\Pi_1 : N_2(\boldsymbol{\mu}_1, \Sigma_1)$ and $\Pi_2 : N_2(\boldsymbol{\mu}_2, \Sigma_2)$, where

$$\boldsymbol{\mu}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \boldsymbol{\mu}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \ \Sigma_1 = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \ \text{and} \ \Sigma_2 = \begin{pmatrix} 4 & -2 \\ -2 & 16 \end{pmatrix}.$$

Observations are:

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \text{and} \ \boldsymbol{x}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Assume misclassification costs, $c(2|1) = 2c(1|2)$ and, prior probabilities $p_1 = 0.25$ and $p_2 = 0.75$.

6 Sample Discriminant Rule for Two Normal Populations When $\Sigma_1 \neq \Sigma_2$

If any or all the parameters $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ Σ_1 and Σ_2 are unknown then estimate those unknown parameters using random samples from each of the two populations and apply the rule using the estimated values of the parameters. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n_1}$ be a random sample from population Π_1 and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{n_2}$ be a random sample from population Π_2. Then estimators of $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$, Σ_1 and Σ_2 are respectively given by

$$\widehat{\boldsymbol{\mu}}_1 = \overline{\boldsymbol{x}}_{n_1} = \frac{1}{n_1} \sum_{i=1}^{n_1} \boldsymbol{x}_i, \quad \widehat{\boldsymbol{\mu}}_2 = \overline{\boldsymbol{y}}_{n_2} = \frac{1}{n_2} \sum_{i=1}^{n_2} \boldsymbol{y}_i,$$

$$\widehat{\Sigma}_1 = \mathcal{S}_x = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n_1}) (\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n_1})^T$$

and

$$\widehat{\Sigma}_1 = \mathcal{S}_y = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (\boldsymbol{y}_i - \overline{\boldsymbol{y}}_{n_2}) (\boldsymbol{y}_i - \overline{\boldsymbol{y}}_{n_2})^T.$$

Using the above estimates, the allocation rule given in (4) can be written as:

Allocate $\boldsymbol{x}_0$ to Π_1 if

$$-\tfrac{1}{2}\boldsymbol{x}_0^T\left(\mathcal{S}_x^{-1}-\mathcal{S}_y^{-1}\right)\boldsymbol{x}_0+\left(\overline{\boldsymbol{x}}_{n_1}^T\mathcal{S}_x^{-1}-\overline{\boldsymbol{y}}_{n_2}T\mathcal{S}_y^{-1}\right)\boldsymbol{x}_0-\widehat{K}\geq 0 \tag{5}$$

otherwise allocate $\boldsymbol{x}_0$ to Π_2

where

$$\widehat{K}=\frac{1}{2}\ln\left(\frac{|\mathcal{S}_x|}{|\mathcal{S}_y|}\right)+\frac{1}{2}\left(\overline{\boldsymbol{x}}_{n_1}^T\mathcal{S}_x^{-1}\overline{\boldsymbol{x}}_{n_1}-\overline{\boldsymbol{y}}_{n_2}^T\mathcal{S}_y^{-1}\overline{\boldsymbol{y}}_{n_2}\right)+\ln\left[\frac{c(1|2)}{c(2|1)}\frac{p_2}{p_1}\right].$$

Example 5: Let $\boldsymbol{X}^T=(X_1,X_2)$ be a random vector representing important characteristics to distinguish between two normal populations Π_1 and Π_2. A random sample of 10 observations from Π_1, gives the mean $\overline{\boldsymbol{x}}_1^T=(-1,3)$ and the sample covariance matrix

$$\mathcal{S}_1=\begin{pmatrix}1 & -1\\ -1 & 4\end{pmatrix}.$$

Also the mean and covariance matrix of a random sample of 15 from Π_2 are as follows:

$$\overline{\boldsymbol{x}}_2=\begin{pmatrix}0\\ -2\end{pmatrix}\quad\text{and}\quad\mathcal{S}_2=\begin{pmatrix}4 & 1\\ 1 & 9\end{pmatrix}$$

Given the prior probability $p_1=0.4$, identify the following two observations assuming equal misclassification costs.

Observations are: $\boldsymbol{x}_1^T=(0.5,1)$ and $\boldsymbol{x}_2^T=(-1,-3)$.

7 Evaluating Discriminant Functions

The way to evaluate the discriminant functions is to calculate their error rate, that is probability of misclassification.

The total probability of misclassification (TPM) is given by

$$\begin{aligned}\text{TMP} &= \mathcal{P}(\text{misclassifiying observations})\\ &= \mathcal{P}(\text{observations from } \Pi_1 \text{ is misclassified})\\ &\quad +\mathcal{P}(\text{observations from } \Pi_2 \text{ is misclassified})\\ &= p_1\mathcal{P}(\text{misclassifing an observation from } \Pi_1)\\ &\quad +p_2\mathcal{P}(\text{misclassifing an observation from } \Pi_2).\end{aligned}$$

The smallest value of TPM is called the optimum error rate (OER). Thus

$$\begin{aligned}\text{OER} &= p_1\mathcal{P}(\text{misclassifing an observation from } \Pi_1)\\ &\quad +p_2\mathcal{P}(\text{misclassifing an observation from } \Pi_2).\end{aligned}$$

Apparent Error Rate (APER)

The APER is the fraction of the misclassified observations in the training sample. This can be easily calculated from the following confusion matrix. Let n_1 be the number of observations in the training sample from population Π_1 and n_2 be the number of observations in the training sample from population Π_2.

Confusion Matrix

	Population	Predicted membership Π_1	Π_2	Number of Observations
Actual	Π_1	n_{1c} ✓	$\boxed{n_{1m}}$	n_1
membership	Π_2	$\boxed{n_{2m}}$	n_{2c} ✓	n_2
				N total obs.

where n_{1c} = number of correctly classified observations in Π_1

n_{1m}= number of misclassified observations in Π_1

n_{2c} = number of correctly classified observations in Π_2

n_{2m}= number of misclassified observations in Π_2.

Note that $n_1 = n_{1c} + n_{1m}$ and $n_2 = n_{2c} + n_{2m}$. Then, the proportion of the misclassified observations in the training sample is given by

$$\text{APER} \;= \frac{n_{1m} + n_{2m}}{n_1 + n_2}.$$

$\frac{\text{misclassification}}{\text{total N}}$

Example 6: Refer Example 11.5, page 602, Johnson and Wichern

Actual Error Rate(AER)

The AER indicate how the sample discriminant function will perform in the future. In general it cannot be calculated. However using cross-validation method we can estimate the expected AER.

Estimation of Expected AER

iterative process

Step 1: Start with the observations in Π_1. Remove (holdout) one observation and obtain the discriminant function using remaining $(n_1 - 1)$ observations from Π_1 and n_2 from Π_2.

remove one obs.

AKA
- Holdout method Lachenbruch
- Jack knifing
- $\boxed{\text{Cross validation in SAS}}$

Step 2: Classify the removed observation.

Step 3: Repeat Steps 1 and 2 until all the Π_1 observations are classified. Let $n_{1m}^{(H)}$ be the number of misclassified observations.

Step 4: Repeat Steps 1 and 3 for the Π_2 observations. Let $n_{2m}^{(H)}$ be the number of misclassified observations.

Now the estimate of the actual error rate is given by

$$\widehat{\mathbf{E}}(\text{AER}) = \frac{n_{1m}^{(H)} + n_{2m}^{(H)}}{n_1 + n_2}.$$

$\frac{\text{misclassification}}{\text{total N}}$

Example 7: Refer Example 11.6, page 603, Johnson and Wichern

Example 1

Two populations

$\pi_1 \quad Exp(\lambda_1) \quad \lambda_1 = 2 \quad p_1 = 0.25$

$\pi_2 \quad Exp(\lambda_2) \quad \lambda_2 = 1 \quad p_2 = 0.75$

$\} \rightarrow 1$

Misclassification costs

		Classify π_1	Classify π_2
True	π_1	0	$\boxed{2} = c(2\|1)$
	π_2	1	0

Minimise the cost

$$\frac{\text{pdf } f_1(x_0)}{\text{pdf } f_2(x_0)} \geq \frac{c(1|2)}{c(2|1)} \cdot \frac{p_2}{p_1}$$

allocate to π_1, otherwise π_2

$$\text{pdf} = \lambda e^{-\lambda x}$$

First obs

$x_1 = \boxed{2} = x_0$

$$\frac{2 \times e^{-2 \times \boxed{2}}}{1 \times e^{-1 \times \boxed{2}}} = 0.27$$

Second obs

$x_2 = \boxed{2.5}$

$$\frac{2 \times e^{-2 \times \boxed{2.5}}}{1 \times e^{-1 \times \boxed{2.5}}} = 0.16$$

both of these are less than RHS of 1.5 $\Rightarrow$ assigned to π_2

RHS

$$\frac{c(1|2)}{c(2|1)} \times \frac{p_2}{p_1} = \frac{1}{2} \times \frac{0.75}{0.25} = 1.5$$

Example 2

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$\pi_1 \quad N_3(\mu_1, \Sigma) \qquad \mu_1' = (0, 0, 0)$

$\pi_2 \quad N_3(\mu_2, \Sigma) \qquad \mu_2' (5, 6, 1)$

$$\Sigma = \begin{bmatrix} 9 & 4 & -2 \\ 4 & 4 & 3 \\ -2 & 3 & 16 \end{bmatrix}$$

assuming equal costs and equal prior $\quad p_1 = p_2$

$c(1|2) = c(2|1)$

$b^T x_0 - k \geq 0$ allocate π_1 otherwise π_2

$b = \Sigma^{-1}(\mu_1 - \mu_2)$

$$= \begin{bmatrix} 11/35 & -2/5 & 4/35 \\ -2/5 & 4/5 & -1/5 \\ 4/35 & -1/5 & 4/35 \end{bmatrix} \begin{bmatrix} -5 \\ -6 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/7 \\ -13/5 \\ 18/35 \end{bmatrix}$$

$k = \frac{1}{2}(\mu_1 - \mu_2)' \Sigma^{-1} (\mu_1 + \mu_2)$

$$= \frac{1}{2}[-5, -6, -1] \begin{bmatrix} 11/35 & -2/5 & 4/35 \\ -2/5 & 4/5 & -1/5 \\ 4/35 & -1/5 & 4/35 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix} = -5.7571$$

First obs

$x_1' = (1, 1, 0)$

$b^T x_0 - k$

$$\left[5/7, -13/5, 18/35\right] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - -5.7571 = \boxed{3.871}$$

positive allocate π_1

Second obs

$x_2' = (0, 2, -3)$

$b^T x_0 - k$

$$\left[5/7, -13/5, 18/35\right] \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} - -5.7571 = \boxed{-0.985}$$

negative

∴ allocate to

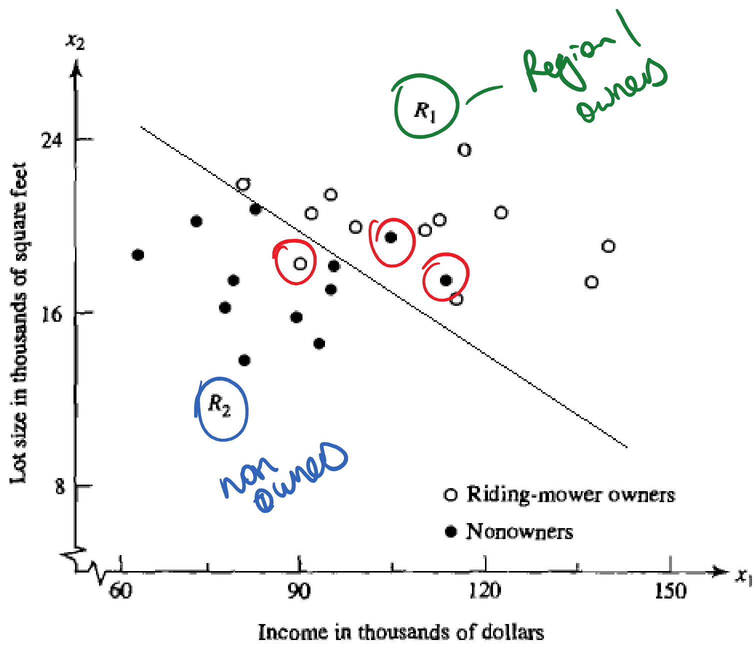
$\therefore$ allocate to
π_2

Example 11.1

Example 11.1 (Discriminating owners from nonowners of riding mowers) Consider two groups in a city: π_1, riding-mower owners, and π_2, those without riding mowers—that is, nonowners. In order to identify the best sales prospects for an intensive sales campaign, a riding-mower manufacturer is interested in classifying families as prospective owners or nonowners on the basis of x_1 = income and x_2 = lot size. Random samples of $n_1 = 12$ current owners and $n_2 = 12$ current nonowners yield the values in Table 11.1.

Table 11.1

π_1: Riding-mower owners		π_2: Nonowners	
x_1 (Income in \$1000s)	x_2 (Lot size in 1000 ft^2)	x_1 (Income in \$1000s)	x_2 (Lot size in 1000 ft^2)
90.0	18.4	105.0	19.6
115.5	16.8	82.8	20.8
94.8	21.6	94.8	17.2
91.5	20.8	73.2	20.4
117.0	23.6	114.0	17.6
140.1	19.2	79.2	17.6
138.0	17.6	89.4	16.0
112.8	22.4	96.0	18.4
99.0	20.0	77.4	16.4
123.0	20.8	63.0	18.8
81.0	22.0	81.0	14.0
111.0	20.0	93.0	14.8



Example 11.6

π_1 owners

π_2 non-owners

		Prediction π_1	Prediction π_2	
Actual	π_1	$n_{1C} =$ 11	$n_{1M} =$ 1	$n_1 = 12$
	π_2	$n_{2M} =$ 2	$n_{2C} =$ 10	$n_2 = 12$
				24 Total

Apparent Error Rate APER $= \dfrac{n_{1M} + n_{2M}}{n_1 + n_2} = \dfrac{1+2}{24} = 0.125$

$= 12.5\%$

Easy to calculate!

BUT it tends to underestimate actual error rate