

1.

(b)

$$\underline{X} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{bmatrix} = \underline{A}\underline{U}$$

(i)

$$\underline{\mu} = \underline{A}\underline{E}[\underline{U}] = \underline{0}$$

$$\begin{aligned} \underline{\Sigma} &= \underline{A} \text{Cov}[\underline{U}] \underline{A}' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}' = \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \end{aligned}$$

(ii) X_1 and X_2 are independent

2.

$$(a) \quad \hat{\underline{\mu}} = [5.92, 2.01]^T, \quad \hat{\underline{\Sigma}} = \begin{bmatrix} 0.86 & -0.13 \\ -0.13 & 0.32 \end{bmatrix}$$

$$(b) \quad T^2 = n(\bar{\underline{X}} - \underline{\mu})^T \underline{S}_n^{-1} (\bar{\underline{X}} - \underline{\mu})$$

$$T^2 = n(TT) = 9 \times 4.2579 = 38.32$$

$$\frac{(n-1)p}{n-p} F_{p, n-p} = \frac{8 \times 2}{7} F_{2,7}(0.95) = \frac{16}{7} (4.7374) = 10.83$$

$$T^2 > \frac{(n-1)p}{n-p} F_{p, n-p}$$

Reject H_0 at 5% level of significance.

(c) 95% confidence region for $\underline{\mu}$ is:

$$n(\bar{\underline{X}} - \underline{\mu})^T \underline{S}_n^{-1} (\bar{\underline{X}} - \underline{\mu}) \leq \frac{(n-1)p}{n-p} F_{p, n-p} = 10.83$$

$$9 \begin{bmatrix} 5.92 - \mu_1 \\ 2.01 - \mu_2 \end{bmatrix}^T \begin{bmatrix} 1.2389 & 0.5033 \\ 0.5033 & 3.3295 \end{bmatrix} \begin{bmatrix} 5.92 - \mu_1 \\ 2.01 - \mu_2 \end{bmatrix} \leq 10.83$$

$$9[(5.92 - \mu_1)1.2389 + (2.01 - \mu_2)0.5033, (5.92 - \mu_1)0.5033 + (2.01 - \mu_2)3.3295] \begin{bmatrix} 5.92 - \mu_1 \\ 2.01 - \mu_2 \end{bmatrix} \leq 10.83$$

$$(5.92 - \mu_1)^2 1.2389 + (2.01 - \mu_2)(5.92 - \mu_1)0.5033 + (5.92 - \mu_1)(2.01 - \mu_2)0.5033 + (2.01 - \mu_2)^2 3.3295 \leq 1.203$$

$$1.2389(5.92 - \mu_1)^2 + 1.006(2.01 - \mu_2)(5.92 - \mu_1) + 3.3295(2.01 - \mu_2)^2 - 1.203 \leq 0$$

Major axes:

$$2\sqrt{\lambda_1} \sqrt{\frac{10.83}{9}} = 2\sqrt{\frac{0.89 \times 10.83}{9}} = 2.07$$

Minor axes:

$$2\sqrt{\lambda_2} \sqrt{\frac{10.83}{9}} = 2\sqrt{\frac{0.29 \times 10.83}{9}} = 1.18$$

$$\text{Along } \underline{e}_1^* = \begin{bmatrix} 0.97 \\ -0.22 \end{bmatrix} \text{ and } \underline{e}_2^* = \begin{bmatrix} 0.22 \\ 0.97 \end{bmatrix}$$

(d) *Test* $\underline{\mu} = \begin{bmatrix} 6.5 \\ 3 \end{bmatrix}$ we can compute

$$1.2389(5.92 - 6.5)^2 + 1.006(2.01 - 3)(5.92 - 6.5) + 3.3295(2.01 - 3)^2 = 4.1676 + 0.578 + 3.263 = 8.01 > 1.203$$

$\underline{\mu} = \begin{bmatrix} 6.5 \\ 3 \end{bmatrix}$ Lie outside the 95% confidence region, so we should reject H_0 at 5% level.

(e) $\underline{\mathbf{X}}$ follows a bivariate normal distribution

Q3

(a)

$$Y_1 = 0.39X_1 + 0.48X_2 + 0.34X_3 + 0.71X_4$$

$$Y_2 = 0.52X_1 - 0.71X_2 + 0.42X_3 + 0.03X_4$$

$$\text{Var}(Y_1) = 10.86, \quad \text{Var}(Y_2) = 4.77$$

(b)

95% CI for λ_2 :

$$\left(\frac{\hat{\lambda}_2}{1 + z_{\alpha/2} \sqrt{\frac{2}{n}}}, \frac{\hat{\lambda}_2}{1 - z_{\alpha/2} \sqrt{\frac{2}{n}}} \right) = \left(\frac{4.77}{1 + 1.96 \sqrt{\frac{2}{44}}}, \frac{4.77}{1 - 1.96 \sqrt{\frac{2}{44}}} \right) = (3.36, 8.19)$$

Q4.

$$\begin{cases} 9 = \ell_1^2 + \psi_1, & 3.78 = \ell_1 \ell_2, & 6.75 = \ell_1 \ell_3 \\ & 4 = \ell_2^2 + \psi_2, & 3.5 = \ell_2 \ell_3 \\ & & 25 = \ell_3^2 + \psi_3 \end{cases}$$

$$\frac{\ell_1}{\ell_2} = \frac{6.75}{3.5} \Rightarrow 3.78 = \frac{6.75}{3.5} \ell_2^2 \Rightarrow \ell_2^2 = \sqrt{\frac{(3.78)(3.5)}{6.75}} = 1.4$$

$$\ell_1 = \frac{6.75}{3.5} \ell_2 = 2.7, \quad \psi_1 = 9 - \ell_1^2 = 9 - 2.7^2 = 1.71$$

$$\ell_3 = \frac{6.75}{\ell_1} = \frac{6.75}{2.7} = 2.5, \quad \psi_2 = 4 - \ell_2^2 = 4 - 1.4^2 = 2.04$$

$$\psi_3 = 25 - \ell_3^2 = 25 - 2.5^2 = 18.75$$

$$\Sigma = \begin{bmatrix} 2.7 \\ 1.4 \\ 2.5 \end{bmatrix} \begin{bmatrix} 2.7 & 1.4 & 2.5 \end{bmatrix} + \begin{bmatrix} 1.71 & 0 & 0 \\ 0 & 2.04 & 0 \\ 0 & 0 & 18.75 \end{bmatrix}$$

Assumption: For the orthogonal factor model $\underline{X} - \underline{\mu} = \underline{L}\underline{F} - \underline{\varepsilon}$

- (a) $E(\underline{F}) = 0, \quad \text{cov}(\underline{F}) = \underline{I}$
- (b) $E(\underline{\varepsilon}) = 0, \quad \text{cov}(\underline{\varepsilon}) = \underline{\Psi}$
- (c) \underline{F} and $\underline{\varepsilon}$ are independent

(b)

Step1: Merge A,E and D

	ADE	B	C
ADE	0		
B	3	0	
C	3	8	0

Step2: Merge (ADE) and B

	ADEB	C
ADEB	0	
C	8	0

Step3: Merge (ADE, B, C)

Q5:

- (a) (i) Two multivariate Normal distributions $N_3(\underline{\mu}, \underline{\Sigma})$
- (ii) $\Sigma_1 = \Sigma_2$

$$(b) \quad \hat{Y} = \hat{a}X = (164.86X_1 - 8.062X_2 + 25.43X_3 - 86.865) - (137.462X_1 - 12.647X_2 + 22.672X_3 - 63.696) \\ = 27.4X_1 - 4.58X_2 + 2.76X_3 - 25.17 \geq 0$$

$$(c) \quad APER = \frac{0+3}{18} = 0.17 \\ E(APER) = \frac{1+3}{18} = 0.22$$

(d)

$$A: \quad 27.4(0.78) - 4.58(0.27) + 2.76(1.58) - 25.17 = 1.80 \geq 0, \quad \text{approve}$$

$$B: \quad 27.4(0.67) - 4.58(0.26) + 2.76(1.23) - 25.17 = 2.23 \geq 0, \quad \text{don't approve}$$