1.
(b)

$$\underline{X} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{bmatrix} = \underline{\mathbf{A}}\underline{\mathbf{U}}$$

(i)

$$\underline{\boldsymbol{\mu}} = \underline{\mathbf{A}}\mathbf{E}[\underline{\mathbf{U}}] = \underline{\mathbf{0}}$$

$$\underline{\boldsymbol{\Sigma}} = \underline{\mathbf{A}}\,\mathbf{Cov}[\underline{\mathbf{U}}]\underline{\mathbf{A}}' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}' =$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

(ii) X_1 and X_2 are independent

2.

(a) $\hat{\underline{\boldsymbol{\mu}}} = \underline{[5.92\ ,\ 2.01]}^T$, $\quad \hat{\underline{\boldsymbol{\Sigma}}} = \begin{bmatrix} 0.86 & -0.13 \\ -0.13 & 0.32 \end{bmatrix}$

(b) $T^2 = n(\underline{\overline{X}} - \underline{\mu})^T \underline{S_n^{-1}}(\underline{\overline{X}} - \underline{\mu})$

$T^2 = n(TT) = 9 \times 4.2579 = 38.32$

$$\frac{(n-1)p}{n-p} F_{p,n-p} = \frac{8 \times 2}{7} F_{2,7}(0.95) = \frac{16}{7}(4.7374) = 10.83$$

$$T^2 > \frac{(n-1)p}{n-p} F_{p,n-p}$$

Reject H0 at 5% level of significance.

(c) 95% confidence region for $\underline{\boldsymbol{\mu}}$ is:

$$n(\underline{\overline{X}} - \underline{\mu})^T \underline{S_n^{-1}}(\underline{\overline{X}} - \underline{\mu}) \le \frac{(n-1)p}{n-p} F_{p,n-p} = 10.83$$

$$9\begin{bmatrix}5.92-\mu_1\\2.01-\mu_2\end{bmatrix}^T\begin{bmatrix}1.2389 & 0.5033\\0.5033 & 3.3295\end{bmatrix}\begin{bmatrix}5.92-\mu_1\\2.01-\mu_2\end{bmatrix}\le 10.83$$

$$9[(5.92-\mu_1)1.2389+(2.01-\mu_2)0.5033\ ,\ \ (5.92-\mu_1)0.5033+(2.01-\mu_2)3.3295]\begin{bmatrix}5.92-\mu_1\\2.01-\mu_2\end{bmatrix}\le 10.83$$

$$(5.92-\mu_1)^2 1.2389+(2.01-\mu_2)(5.92-\mu_1)0.5033+(5.92-\mu_1)(2.01-\mu_2)0.5033+(2.01-\mu_2)^2 3.3295\le 1.203$$

$$1.2389(5.92-\mu_1)^2+1.006(2.01-\mu_2)(5.92-\mu_1)+3.3295(2.01-\mu_2)^2-1.203\le 0$$

Major axes:

$$2\sqrt{\lambda_1}\sqrt{\frac{10.83}{9}}=2\sqrt{\frac{0.89\times 10.83}{9}}=2.07$$

Minor axes:

$$2\sqrt{\lambda_2}\sqrt{\frac{10.83}{9}}=2\sqrt{\frac{0.29\times 10.83}{9}}=1.18$$

Along $\underline{e}_1^*=\begin{bmatrix}0.97\\-0.22\end{bmatrix}$ and $\underline{e}_2^*=\begin{bmatrix}0.22\\0.97\end{bmatrix}$

(d) *Test* $\underline{\boldsymbol{\mu}}=\begin{bmatrix}6.5\\3\end{bmatrix}$ we can compute

$$1.2389(5.92-6.5)^2+1.006(2.01-3)(5.92-6.5)+3.3295(2.01-3)^2=4.1676+0.578+3.263=8.01>1.203$$

$\underline{\boldsymbol{\mu}}=\begin{bmatrix}6.5\\3\end{bmatrix}$ Lie outside the 95% confidence region, so we should reject H_0 at 5% level.

(e) $\underline{\mathbf{X}}$ follows a bivariate normal distribution

Q3

(a)

$Y_1=0.39X_1+0.48X_2+0.34X_3+0.71X_4$

$Y_2=0.52X_1-0.71X_2+0.42X_3+0.03X_4$

$Var(Y_1)=10.86,\qquad Var(Y_2)=4.77$

(b)

95% CI for λ_2:

$$\left(\frac{\hat{\lambda}_2}{1+z_{\alpha/2}\sqrt{\frac{2}{n}}},\frac{\hat{\lambda}_2}{1-z_{\alpha/2}\sqrt{\frac{2}{n}}}\right)=\left(\frac{4.77}{1+1.96\sqrt{\frac{2}{44}}},\frac{4.77}{1-1.96\sqrt{\frac{2}{44}}}\right)=(3.36,8.19)$$

Q4.

$$\begin{cases} 9 = \ell_{11}^2 + \psi_1, & 3.78 = \ell_1 \ell_2, \quad 6.75 = \ell_1 \ell_3 \\ & 4 = \ell_2^2 + \psi_2, \quad 3.5 = \ell_2 \ell_3 \\ & \qquad\qquad 25 = \ell_3^2 + \psi_3 \end{cases}$$

$$\frac{\ell_1}{\ell_2} = \frac{6.75}{3.5} \quad \Rightarrow \quad 3.78 = \frac{6.75}{3.5}\ell_2^2 \quad \Rightarrow \quad \ell_2^2 = \sqrt{\frac{(3.78)(3.5)}{6.75}} = 1.4$$

$$\ell_1 = \frac{6.75}{3.5}\ell_2 = 2.7 \quad , \quad \psi_1 = 9 - \ell_1^2 = 9 - 2.7^2 = 1.71$$

$$\ell_3 = \frac{6.75}{\ell_1} = \frac{6.75}{2.7} = 2.5 \quad , \quad \psi_2 = 4 - \ell_2^2 = 4 - 1.4^2 = 2.04$$

$$\psi_3 = 25 - \ell_3^2 = 25 - 2.5^2 = 18.75$$

$$\Sigma = \begin{bmatrix} 2.7 \\ 1.4 \\ 2.5 \end{bmatrix} \begin{bmatrix} 2.7 & 1.4 & 2.5 \end{bmatrix} + \begin{bmatrix} 1.71 & 0 & 0 \\ 0 & 2.04 & 0 \\ 0 & 0 & 18.75 \end{bmatrix}$$

Assumption: For the orthogonal factor model $\underline{X} - \underline{\mu} = \underline{L}\underline{F} - \underline{\varepsilon}$

(a) $E(\underline{F}) = 0, \quad \mathrm{cov}(\underline{F}) = \underline{I}$

(b) $E(\underline{\varepsilon}) = 0, \quad \mathrm{cov}(\underline{\varepsilon}) = \underline{\psi}$

(c) $\underline{F}$ and $\underline{\varepsilon}$ are independent

(b)

Step1: Merge A,E and D

	ADE	B	C
ADE	0		
B	3	0	
C	3	8	0

Step2: Merge (ADE) and B

	ADEB	C
ADEB	0	
C	8	0

Step3: Merge (ADE, B, C)

Q5:

(a) (i) Two multivariate Normal distributions $N_3(\underline{\mu}, \underline{\Sigma})$

(ii) $\Sigma_1 = \Sigma_2$

(b) $$\hat{Y} = \underline{\hat{a}}X = (164.86X_1 - 8.062X_2 + 25.43X_3 - 86.865) - (137.462X_1 - 12.647X_2 + 22.672X_3 - 63.696)$$
$$= 27.4X_1 - 4.58X_2 + 2.76X_3 - 25.17 \geq 0$$

(c) APER= $\frac{0+3}{18} = 0.17$

E(APER)= $\frac{1+3}{18} = 0.22$

(d)

A : $27.4(0.78) - 4.58(0.27) + 2.76(1.58) - 25.17 = 1.80 \geq 0,$ approve

B : $27.4(0.67) - 4.58(0.26) + 2.76(1.23) - 25.17 = 2.23 \geq 0,$ don't approve