Assignment 2 Multivariate

1 Bivariate normal distribution for `polution.csv` ¶

The data file `pollution.csv` (as on Canvas and SAS Studio) contains information on air pollution measurements. Using the file examine the pair of measurements `X5=Nitrious Oxide` and `X6=Ozone` for bivariate normality by completing the following:

In [1]:
```
/* Importing data */
Data pollutionData;
        infile "./data/pollution.csv" delimiter=",";
        input x1-x4 NITRIOUSOXIDE OZONE x7;
        run;

proc iml;
use pollutionData;
read all var { NITRIOUSOXIDE OZONE } into pollution;
```

```
SAS Connection established. Subprocess id is 28425
```

Out[1]:

```
34   ods listing close;ods html5 (id=saspy_internal) file=stdout options(bitmap_mode
='inline') device=svg; ods graphics on /
34 ! outputfmt=png;
NOTE: Writing HTML5(SASPY_INTERNAL) Body file: STDOUT
35
36   /* Importing data */
37   Data pollutionData;
38      infile "./data/pollution.csv" delimiter=",";
39      input x1-x4 NITRIOUSOXIDE OZONE x7;
40      run;
NOTE: The infile "./data/pollution.csv" is:
      Filename=/folders/myfolders/data/pollution.csv,
      Owner Name=sasdemo,Group Name=sas,
      Access Permission=-rw-r--r--,
      Last Modified=06Oct2018:12:44:17,
      File Size (bytes)=678

NOTE: 42 records were read from the infile "./data/pollution.csv".
      The minimum record length was 14.
      The maximum record length was 16.
NOTE: The data set WORK.POLLUTIONDATA has 42 observations and 7 variables.
NOTE: DATA statement used (Total process time):
      real time           0.00 seconds
      cpu time            0.00 seconds

41
NOTE: IML Ready
42   proc iml;
43   use pollutionData;
44   read all var { NITRIOUSOXIDE OZONE } into pollution;
45
46   ods html5 (id=saspy_internal) close;ods listing;

47
```

a) Calculate the distances of these observations from their means (2 marks)

In [2]:

```
centrePollution=mean(pollution); /*calclate mean vector*/
covariancePollution=cov(pollution); /*calculate cov matrix*/
correlationPollution=corr(pollution); /*calculate correlation matrix*/
/* print centrePollution covariancePollution; */
columnVectorPollution = t(pollution-centrePollution); /* col vector */
distancesPollution = t(columnVectorPollution)*inv(covariancePollution)*columnVectorPollu
tion; /* calculate distances */
```

Out[2]:

```
49   ods listing close;ods html5 (id=saspy_internal) file=stdout options(bitmap_mode
='inline') device=svg; ods graphics on /
49 ! outputfmt=png;
NOTE: Writing HTML5(SASPY_INTERNAL) Body file: STDOUT
50
51   centrePollution=mean(pollution);
51 !                                  /*calclate mean vector*/
52   covariancePollution=cov(pollution);
52 !                                     /*calculate cov matrix*/
53   correlationPollution=corr(pollution);
53 !                                       /*calculate correlation matrix*/
54   /* print centrePollution covariancePollution; */
55   columnVectorPollution = t(pollution-centrePollution);
55 !                                                       /* col vector */
56   distancesPollution = t(columnVectorPollution)*inv(covariancePollution)*columnVect
orPollution;
56 !
             /* calculate distances */
57
58   ods html5 (id=saspy_internal) close;ods listing;

59
```

In [3]:

```
mahalaPollution=vecdiag(distancesPollution); /*produce Mahalanobis vector*/
print mahalaPollution; /*produce Mahalanobis vector */
```

Out[3]:

The SAS System

mahalaPollution
0.4606524
0.6592206
2.377061
1.6282902
0.4135364
0.4760726
1.1848895
10.639179
0.1388339
0.8162468
1.3566301
0.6228096
5.6494392
0.3159498
0.4135364
0.1224973
0.8987982
4.7646873
3.0089122
0.6592206
2.7741416
1.0360061
0.7874152
3.4437748
6.1488606
1.0360061
0.1388339
0.8856041
0.1379719
2.2488867

mahalaPollution
0.1901188
0.4606524
1.1471939
7.0857237
1.4584229
0.1224973
1.8984708
2.7782596
8.4730649
0.6370218
0.7032485
1.8013611

b) Determine the proportion of the observations falling within the estimated `50% probability contour` of a bivariate normal distribution (1 mark)

Bivariate Contour to assess normality

$(X_i - \bar{X})\ S^{-1}\ (X_i - \bar{X}) \leq \chi 2_p(\alpha)$

chisquared: $\chi 2p(\alpha)$ -> $\chi 2_2(0.5)$ -> 1.39

In [4]:
```
pollutionInv = inv(covariancePollution);
pollutionMean = mean(pollution);
values=(pollution-pollutionMean)*pollutionInv*t(pollution-pollutionMean);
numbers=vecdiag(values);
inside=numbers<=1.39;
outside=numbers>1.39;
i_count=sum(inside);
o_count=sum(outside);
/* print i_count;
print o_count; */
proportion = i_count / (i_count + o_count);
print proportion;
```

Out[4]:

The SAS System

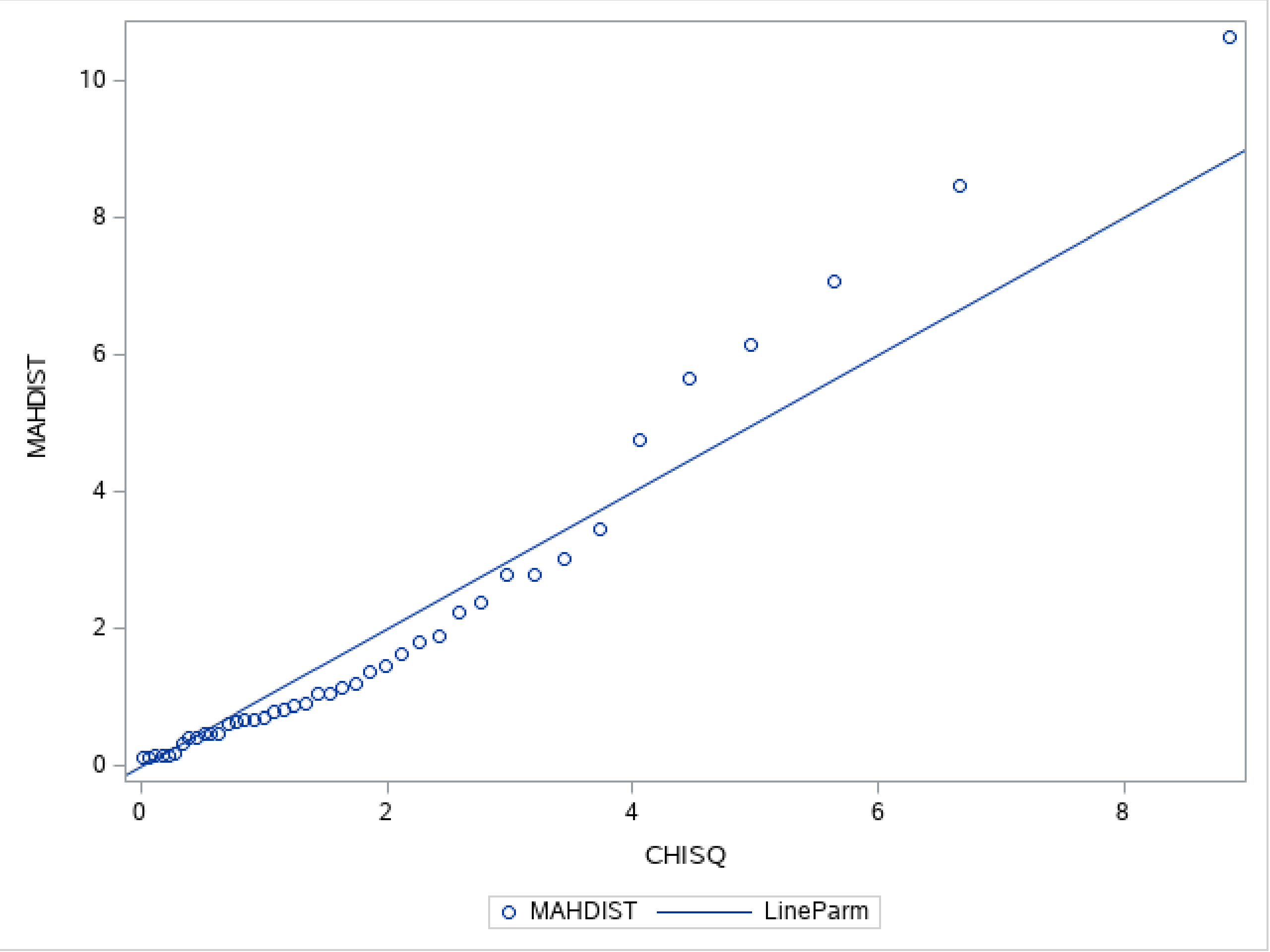
proportion
0.6190476

c) Construct a chi-square plot of your distances from part a) above (2 marks)

In [5]:

```
ranksPollution=rank(mahalaPollution); /*order distances*/
pPollution=ncol(pollution); /*number of columns*/
nPollution=nrow(pollution); /*number of rows*/
relativeFrequencyPollution=(ranksPollution-0.5)/nPollution; /* Compute the relative freq
uency */
chiSquaredPollution=cinv(relativeFrequencyPollution,pPollution); /* Quantile using Chi-s
quare distribution with p degrees of freedom */
chiplot=mahalaPollution||chiSquaredPollution;
create chiplot from chiplot[colname={'MAHDIST' 'CHISQ'}]; /*create a dataset to plot*/
append from chiplot;
proc sgplot data=chiplot;
    scatter y=MAHDIST x=CHISQ;
    lineparm x=0 y=0 slope=1;
run;
```

Out[5]:

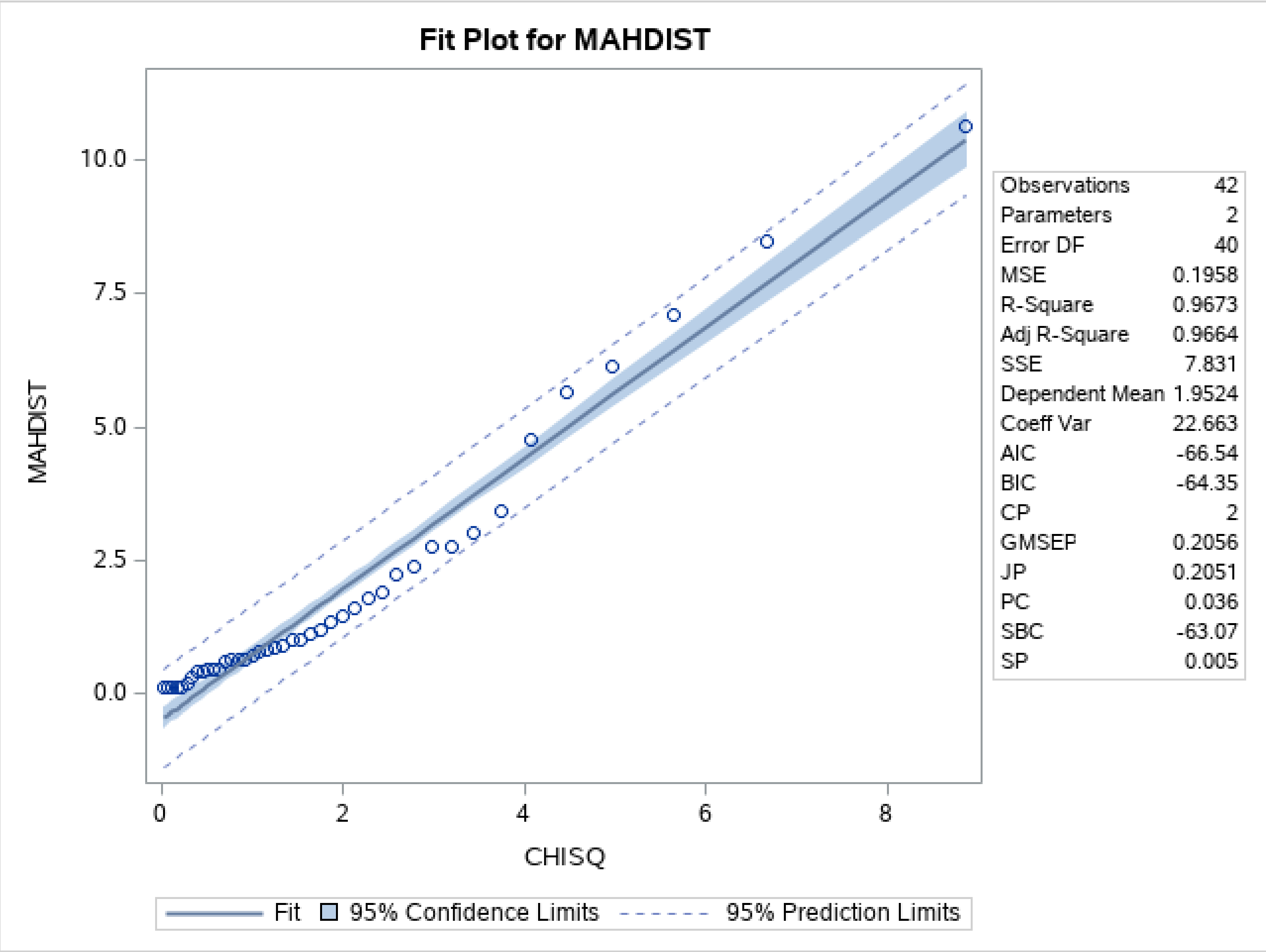


```
In [6]: proc reg data=chiplot plots=FitPlot(stats=all);
          model MAHDIST = CHISQ;
          ods select FitPlot;
        run;
```

Out[6]:

The SAS System

The REG Procedure
Model: MODEL1
Dependent Variable: MAHDIST



d) Given your results in part b) and part c) are these data approximately bivariate normal? Explain

For bivariate normality we require:

- Multivariate normal distribution (formula) $\mathbf{X} \sim \mathcal{N}_k(\mu, \mathbf{\Sigma})$ with a single mean
- Representative Sample
- independence of observations

Given the Chi Squared Plot results. And with only ~62% of the proportion falling within the estimated `50% probability contour` of a bivariate normal distribution.

- **Not bivariate normal**

Explain

- Because observation of qqplot and fit plot show two distinct curves
- Indicates either **bimodal** or light-tailed results
- $\mathbf{X} \sim \mathcal{N}_k(\boldsymbol{\mu}, \Sigma)$ bimodal
- This is even though the results fit within the 95% CI (95%) according to observation of fit plot

see 3. multivariate normal sect 5 / 6

2. Hypothesis testing - `results.csv`

In [7]:
```
/*read in data*/
Data temp;
infile "./data/results.csv" delimiter=",";
input x1 x2 x3;
run;

proc IML;
use temp;
read all var _all_ into X;
```

Out[7]:

The datafile `results.csv`, contains three test results assessing different types of intelligence.

Test the following hypothesis at:

```
α=0.02
H0 :μ′=[85 75 55]
```

a) Conduct the hypothesis test showing all steps required.

H_0: μ' = μ_0' = [85 75 55]
H_a: μ' ≠ μ_0' ≠ [85 75 55]

Which kind of test should we apply?

In [8]:
```
n=nrow(x); /* No. of observations */
p=ncol(x); /* No. of variables */
print n p;
/* n=nrow(typesOfIntelligence); /* No. of observations */ */
/* p=ncol(typesOfIntelligence); /* No. of variables */ */
```

Out[8]:

The SAS System

n	p
80	3

Since n is relatively large, compared to p, we will use the `Large Sample Theory:`

$n(\bar{X}_n - \mu)^T S_n^{-1} (\bar{X}_n - \mu) \approx \chi 2_p$

Hypothesis:

H_0: μ' = μ_0' = [85 75 55]
H_a: μ' ≠ μ_0' ≠ [85 75 55]

Get critical value from Chi Squared table

$\chi 2_p = \chi 2(80, 0.02) = 108.069$

Do maths to it

for $\mu_0' = [85\ 75\ 55]$

In [9]:
```
centre=t(mean(X));
cov_x=cov(X);
cor_x=corr(X);
incov=inv(cov_x); /* Inverse of the covariance matrix */
mu0={85,75,55}; /* Hypothesized values */
```

Out[9]:

Large sample:
$n(\bar{X}_n - \mu)^T S_n^{-1} (\bar{X}_n - \mu) \approx \chi 2_p$

In [10]:
```
tsq=n*t(centre-mu0)*incov*(centre-mu0);
cCriticalValue=cinv(0.98,p);
print tsq cCriticalValue; */
```

Out[10]:

The SAS System

tsq	cCriticalValue
51.308032	9.8374093

If the values given above are the average score for all college students over the last ten years, is there reason to believe the group in the datafile are scoring differently? Explain.(5 marks)

Hypothesis

H_0: $\mu' = \mu_0' = [85\ 75\ 55]$
H_a: $\mu' \neq \mu_0' \neq [85\ 75\ 55]$

Assumptions

$\mathbf{X} \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

- Representative
- Independence of observerations

We **reject** the null hypothesis because:

Reject if: $T^2 > \chi 2_p$

$T^2 > \chi 2_p(\alpha=0.02)$
$51.3 > 9.837$
$\therefore \mu \neq \mu_0$

- The test score sample mean is sufficiently different during this year than the mean (μ' [85 75 55]) of uni students during previous years.

b) Determine the lengths and directions for the axes of the `90%` confidence ellipsoid for $\boldsymbol{\mu}$ (2 marks)

notes from class /*Compute the lengths of axes of the confidence ellipsoid*/ lam=eigval(cov_x); E=eigvec(cov_x); length=2*sqrt(lam*ccri_90percent/n); print length; print E;

In [11]:
```
centre=t(mean(X));
cov_x=cov(X);
incov=inv(cov_x); /* Inverse of the covariance matrix */
mu0={85,75,55}; /* Hypothesized values */
cCriticalValue=cinv(0.98,p);

lambda=eigval(cov_x);
direction_aka_Eigenvector=eigvec(cov_x);
length=2*sqrt(lambda*cCriticalValue/n);

print length;
print direction_aka_Eigenvector;
```

Out[11]:

The SAS System

length
12.109934
6.633851
3.8863767

direction_aka_Eigenvector		
0.4928051	0.4178742	-0.763233
0.8135077	-0.53253	0.2337037
0.3087854	0.7360661	0.6023772

c) Construct the three possible scatter diagrams from the pairs of variables.

In [12]:
```
sd=vecdiag(cov_x);
ubound1=centre+sqrt(cCriticalValue)*sqrt(sd/n);
lbound1=centre-sqrt(cCriticalValue)*sqrt(sd/n);
print lbound1 ubound1;
```

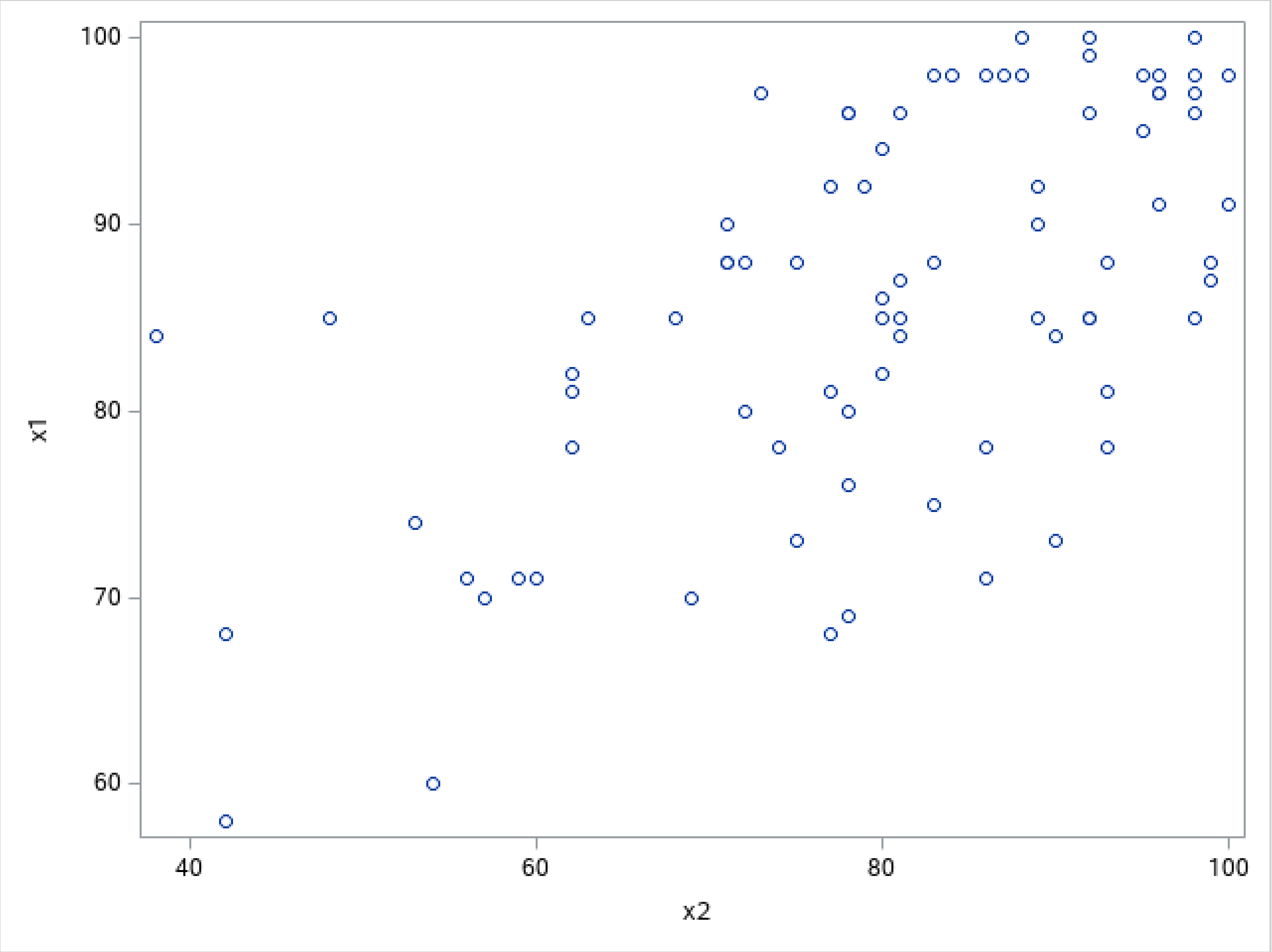
Out[12]:

The SAS System

lbound1	ubound1
82.216049	89.433951
74.684939	85.190061
46.634609	53.215391

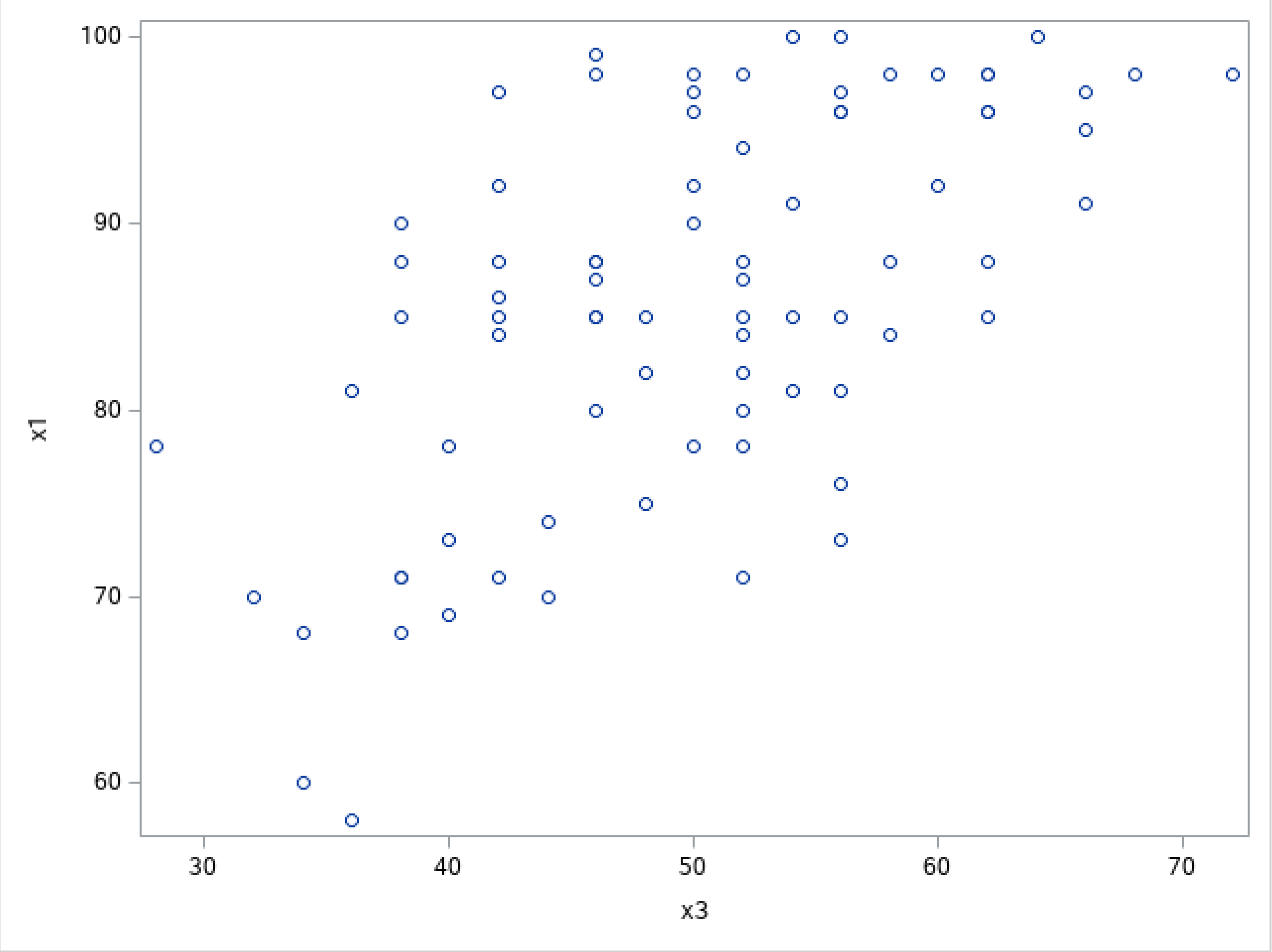
In [13]:
```
proc sgscatter data = temp;
PLOT X1 * X2; run;
```

Out[13]:



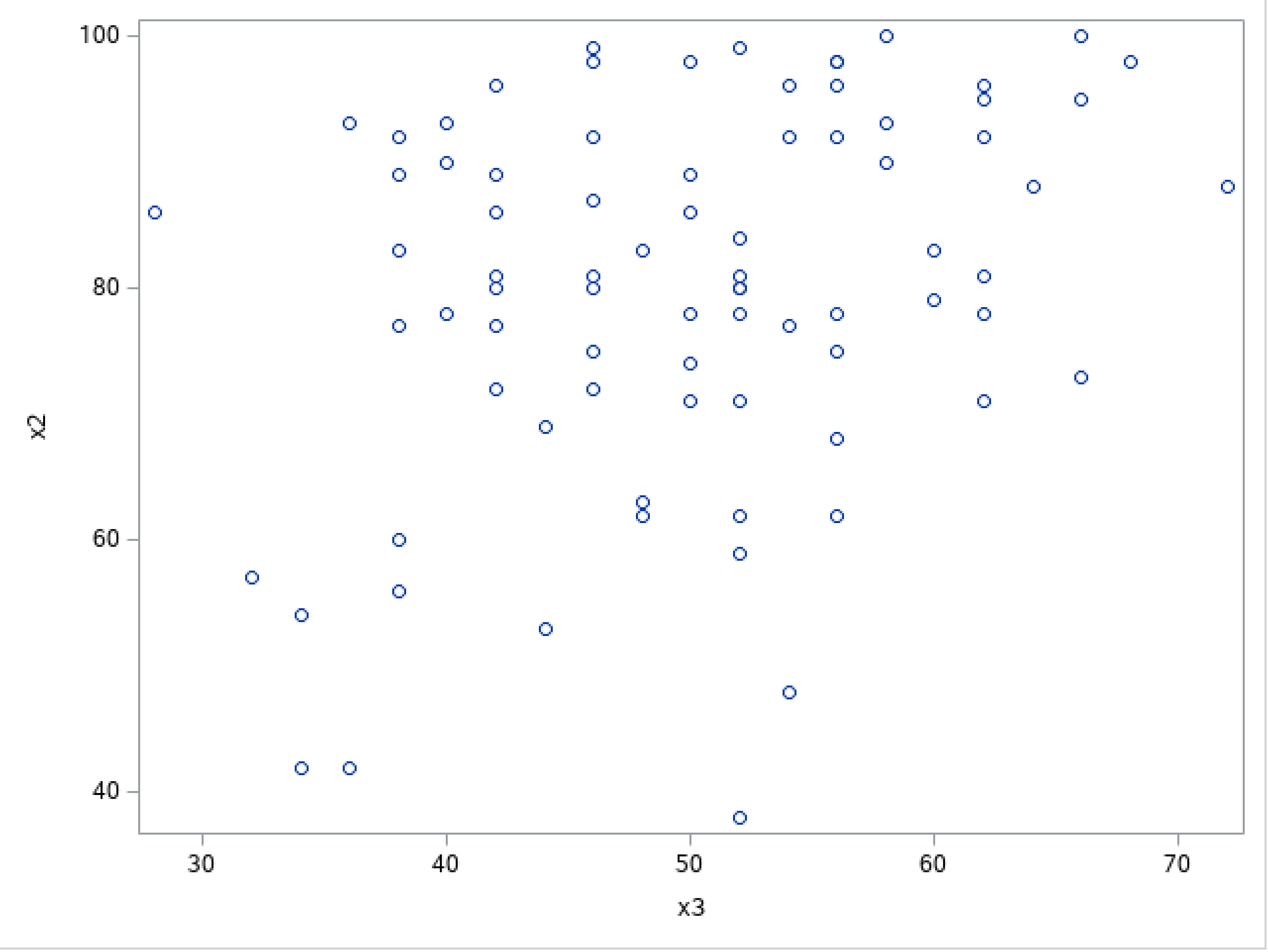
In [14]:
```
proc sgscatter data = temp;
PLOT X1 * X3; run;
```

Out[14]:



In [15]:
```
proc sgscatter data = temp;
PLOT X2 * X3; run;
```

Out[15]:



Do these data appear to be normally distributed?

Nope. This data does not appear to be normally distributed.

Discuss

(3 marks)

linear gradient = normally distributed

Question 3

In [17]:

```
proc iml;
/* reset print; */
dataQ3 = {
3 4 15 -6,
2 4 14 -7,
3 4 15 -5,
3 3 16 -6,
2 5 15 -7,
1 4 14 -4};
```

Out[17]:

```
224  ods listing close;ods html5 (id=saspy_internal) file=stdout options(bitmap_mode
='inline') device=svg; ods graphics on /
224! outputfmt=png;
NOTE: Writing HTML5(SASPY_INTERNAL) Body file: STDOUT
225
NOTE: IML Ready
226  proc iml;
227  /* reset print; */
228  dataQ3 = {
229  3 4 15 -6,
230  2 4 14 -7,
231  3 4 15 -5,
232  3 3 16 -6,
233  2 5 15 -7,
234  1 4 14 -4};
235
236  ods html5 (id=saspy_internal) close;ods listing;

237
```

In [18]:

```
nQ3=nrow(dataQ3); /* No. of observations */
pQ3=ncol(dataQ3); /* No. of variables */
alphaQ3 = 0.05;
muQ3 = mean(dataQ3);
degreesFreedomQ3 = nQ3 - 1;
mQ3 = pQ3;

covQ3=cov(dataQ3); /* covariance */
invCovQ3=inv(covQ3); /* Inverse of the covariance matrix - Might not need*/
sdQ3=vecdiag(CovQ3); /* get variance from diagonal of the covariance matrix */
sdQ3_T=T(vecdiag(CovQ3)); /* Transposed */
```

Out[18]:

```
239  ods listing close;ods html5 (id=saspy_internal) file=stdout options(bitmap_mode
='inline') device=svg; ods graphics on /
239! outputfmt=png;
NOTE: Writing HTML5(SASPY_INTERNAL) Body file: STDOUT
240
241  nQ3=nrow(dataQ3);
241!                    /* No. of observations */
242  pQ3=ncol(dataQ3);
242!                    /* No. of variables */
243  alphaQ3 = 0.05;
244  muQ3 = mean(dataQ3);
245  degreesFreedomQ3 = nQ3 - 1;
246  mQ3 = pQ3;
247
248  covQ3=cov(dataQ3);
248!                     /* covariance */
249  invCovQ3=inv(covQ3);
249!                       /* Inverse of the covariance matrix - Might not need*/
250  sdQ3=vecdiag(CovQ3);
250!                       /* get variance from diagonal of the covariance matrix */
251  sdQ3_T=T(vecdiag(CovQ3));
251!                            /* Transposed */
252
253  ods html5 (id=saspy_internal) close;ods listing;

254
```

a) The independent 95 % confidence intervals for each variable (1.5 marks)

$$\bar{X}_i \pm \quad t_{n-1, \quad 1-\alpha/2} \left(\frac{\sigma}{\sqrt{n}} \right)$$

In [19]:
```
t_Q3 = tinv((1-(alphaQ3/2)), degreesFreedomQ3);
```

Out[19]:
```
256  ods listing close;ods html5 (id=saspy_internal) file=stdout options(bitmap_mode
='inline') device=svg; ods graphics on /
256! outputfmt=png;
NOTE: Writing HTML5(SASPY_INTERNAL) Body file: STDOUT
257
258  t_Q3 = tinv((1-(alphaQ3/2)), degreesFreedomQ3);
259
260  ods html5 (id=saspy_internal) close;ods listing;

261
```

In [20]:
```
upperIndependentCIQ3 = muQ3 + t_Q3 * sdQ3_T/sqrt(nQ3);
lowerIndependentCIQ3 = muQ3 - t_Q3 * sdQ3_T/sqrt(nQ3);
independentConfidenceIntervalsQ3 = (lowerIndependentCIQ3`) || (upperIndependentCIQ3`);
print independentConfidenceIntervalsQ3[F=8.4 C={"Lower" "Upper"}];
```

Out[20]:

The SAS System

independentConfidenceIntervalsQ3	
Lower	**Upper**
1.6337	3.0330
3.5802	4.4198
14.2387	15.4280
-7.2676	-4.3991

b) The Bonferroni 95 % confidence intervals for each variable (1.5 marks)

$$\bar{X}_i \pm \quad t_{n-1} \left(\frac{\alpha}{2m} \right) \sqrt{\frac{s_{ii}}{n}}$$

$$t_{n-1} \left(\frac{\alpha}{2m} \right)$$

In [21]:
```
adjustedConfidenceIntervalQ3 = (1 - (alphaQ3 / (2 * mQ3)));
/* print adjustedConfidenceIntervalQ3; */
```

Out[21]:

t(df, alpha) <- how we write it
tinv(CI(aka 1-alpha), df) <- back to front

In [22]:
```
tCriticalValue=tinv(adjustedConfidenceIntervalQ3, degreesFreedomQ3);
/* print tCriticalValue; */
```
Out[22]:

$$\sqrt{\frac{s_{ii}}{n}}$$

In [23]:
```
sdQ3=vecdiag(CovQ3); /* this gets the values along the diagonal of the covariance matrix
 */
sdQ3_T=T(vecdiag(CovQ3)); /* Transposed */
```
Out[23]:

In [24]:
```
uboundQ3=muQ3+sqrt(tCriticalValue)*sqrt(sdQ3_T/nQ3);
lboundQ3=muQ3-sqrt(tCriticalValue)*sqrt(sdQ3_T/nQ3);
bonferroniConfidenceIntervalsQ3 = (lboundQ3`) || (uboundQ3`);
print bonferroniConfidenceIntervalsQ3[F=8.4 C={"Lower" "Upper"}];
```
Out[24]:

The SAS System

bonferroniConfidenceIntervalsQ3	
Lower	**Upper**
1.6827	2.9840
3.4960	4.5040
14.2335	15.4332
-6.7649	-4.9018

c) The simultaneous 95 % confidence intervals for each variable (1.5 marks)

$$\bar{X}_i \pm \sqrt{_{n-1}\left(\frac{p(n-1)}{n-p}\right) F_{p,n-p}(\alpha)\frac{S_{ii}}{n}}$$

In [25]:
```
firstPart = 1 - (nQ3 - pQ3) * alphaQ3;
tSquared = pQ3 * (nQ3 - 1)/(nQ3 - pQ3);
simultaneousCI = tinv(firstPart, degreesFreedomQ3);

upperSimultaneousCI = muQ3 + sqrt(tSquared * simultaneousCI) * sqrt(sdQ3_T / nQ3);
lowerSimultaneousCI = muQ3 - sqrt(tSquared * simultaneousCI) * sqrt(sdQ3_T / nQ3);

simultaneousConfidenceIntervalsQ3 = (lowerSimultaneousCI`) || (upperSimultaneousCI`);
print simultaneousConfidenceIntervalsQ3[F=8.4 C={"Lower" "Upper"}];
```

Out[25]:

The SAS System

simultaneousConfidenceIntervalsQ	
Lower	**Upper**
1.0528	3.6139
3.0081	4.9919
13.6527	16.0140
-7.6668	-3.9998

d) The 95 % confidence interval for the difference between μ2 and μ4. Are these means different? (1.5 marks)

In [26]:
```
muQ3X2 = mean(dataQ3[2]);
muQ3X4 = mean(dataQ3[4]);
diffXbar = (muQ3X2 - muQ3X4);
secondPartDiffEquation = sqrt(tSquared * sdQ3_T) * sqrt((covQ3[2,2] - 2 * covQ3[2,4] + c
ovQ3[4,4]) / nQ3);

upperDiffCIMu2Mu4Q3 = diffXbar + secondPartDiffEquation;
lowerDiffCIMu2Mu4Q3 = diffXbar - secondPartDiffEquation;


diffCIMu2Mu4Q3 = (lowerDiffCIMu2Mu4Q3`) || (upperDiffCIMu2Mu4Q3`);
print diffCIMu2Mu4Q3[F=8.4 C={"Lower" "Upper"}];


/* The CI for μ2 - μ4 does not include zero, therefore the muQ3s are different */
```

Out[26]:

The SAS System

diffCIMu2Mu4Q3	
Lower	**Upper**
8.4484	11.5516
8.7981	11.2019
8.5695	11.4305
7.7785	12.2215

e) Discuss the results from part a) to part d) above and explain any differences in the observed estimates. (2 marks)

In [27]:
```
muQ3_t = t(muQ3);
print muQ3_t independentConfidenceIntervalsQ3[F=8.4 C={"Lower" "Upper"}];
```

Out[27]:

The SAS System

muQ3_t	**independentConfidenceIntervalsQ3** **Lower**	**Upper**
2.3333333	1.6337	3.0330
4	3.5802	4.4198
14.833333	14.2387	15.4280
-5.833333	-7.2676	-4.3991

4 PCA's with - `track.csv`

see other file for full SAS code for question 4 due to small bug with jupyter with SAS.

- The data file `track.csv` contains information on female national track records.
  Using the file completing the following:

```
%web_drop_table(R);
FILENAME REFFILE '/folders/myshortcuts/ass2--multivariate/data/track.csv';
PROC IMPORT DATAFILE=REFFILE
    DBMS=CSV
    OUT=R; /* this is where we get the output with the PCA's */
    GETNAMES=YES;
RUN;
PROC CONTENTS DATA=R;
RUN;
%web_open_table(R);
```

a) pt1 Obtain the sample correlation matrix R

```
proc IML;
proc princomp data=R OUT=prinR;
     title3 '4 a) Obtain the sample correlation matrix R, and determine the eigenvalue/eigenvector
pairs.';
run;

proc IML; / fix printing error /
   ';"; run;
proc iml;
title3 '4 b) pt1) State the first two principal components for the standardized variables';
print 'PCA1 Y1 = 0.377766X1 + 0.383210X2 + 0.368036X3 + 0.394781X4 + 0.389261X5 + 0.376094X6 +
0.355203X7';
print 'PCA2 Y2 = -0.407176X1 - 0.413629X2 - 0.459353X3 + 0.161246X4 + 0.309088X5 + 0.423190X6 +
0.389215X7';
run;
</code>
```

100m__s	200m__s	400m__s	800m__min	1500m__min	3000m__min	Marathon_min	
100m__s	1.0000	0.9411	0.8708	0.8092	0.7816	0.7279	0.6690
200m__s	0.9411	1.0000	0.9088	0.8198	0.8013	0.7319	0.6800
400m__s	0.8708	0.9088	1.0000	0.8058	0.7198	0.6738	0.6769
800m__min	0.8092	0.8198	0.8058	1.0000	0.9051	0.8666	0.8540
1500m__min	0.7816	0.8013	0.7198	0.9051	1.0000	0.9734	0.7906
3000m__min	0.7279	0.7319	0.6738	0.8666	0.9734	1.0000	0.7987
Marathon_min	0.6690	0.6800	0.6769	0.8540	0.7906	0.7987	1.0000

a) pt2 determine the eigenvalue/eigenvector pairs.

		Eigenvalue	Eigen - Prin1	Eigen - Prin2	Eigen - Prin3	Eigen - Prin4	Eigen - Prin5	Eigen - Prin6	Eigen - Prin7
x1	*100m__s*	**5.80762446**	0.377766	-.407176	-.140580	0.587063	-.167069	0.539697	-.088939
x2	*200m__s*	**0.62869342**	0.383210	-.413629	-.100783	0.194075	0.093500	-.744931	0.265657
x3	*400m__s*	**0.27933457**	0.368036	-.459353	0.237026	-.645431	0.327273	0.240094	-.126604
x4	*800m__min*	**0.12455472**	0.394781	0.161246	0.147542	-.295208	-.819055	-.016507	0.195213
x5	*1500m__min*	**0.09097174**	0.389261	0.309088	-.421986	-.066690	0.026131	-.188988	-.730768
x6	*3000m__min*	**0.05451882**	0.376094	0.423190	-.406063	-.080157	0.351698	0.240500	0.571506
x7	Marathon_*min*	**0.01430226**	0.355203	0.389215	0.741061	0.321076	0.247008	-.048270	-.082084

b) State the first two principal components for the standardized variables

- PCA1 Y1 = 0.377766X1 + 0.383210X2 + 0.368036X3 + 0.394781X4 + 0.389261X5 + 0.376094X6 + 0.355203X7
- PCA2 Y2 = -0.407176X1 - 0.413629X2 - 0.459353X3 + 0.161246X4 + 0.309088X5 + 0.423190X6 + 0.389215X7

```
data Cumulative;
length CumulativePercent    $ 2;
input X $1-15 Eigenvalue $1-15 CumulativePercentCol;
CumulativePercent = substr(X,1,2);
datalines;
X   Eigenvalue  Cumulative
X1  5.80762446  82.97
X2  0.62869342  91.95
X3  0.27933457  95.94
X4  0.12455472  97.72
X5  0.09097174  99.02
X6  0.05451882  99.80
X7  0.01430226  100.00
;
proc print data=Cumulative;
title3 '4 b) pt2. Calculate the cumulative percentages of the total (standardized) sample variance
explained.';
run;
```

b) pt2. calculate the cumulative percentages of the total (standardized) sample variance explained. (2 marks)

PCA	Eigenvalue	CumulativePercent
X1	5.80762446	82.97
X2	0.62869342	91.95
X3	0.27933457	95.94
X4	0.12455472	97.72
X5	0.09097174	99.02
X6	0.05451882	99.80
X7	0.01430226	100.00

c) Prepare a table showing the correlation of the standardized variables with the first two components.

```
proc princomp data=R n=2 outstat=standardVariablesCorr noprint;
run;
proc print data = standardVariablesCorr;
    title3 "c) Prepare a table showing the correlation of the standardized variables with the first two
components.";
    where TYPE = 'SCORE';
run;

/ proc print data=R; /
/ run;  /
/ proc print data=prinR; /
/ run;  /


proc print data=prinR;
var Country Prin1 Prin2; /* filter to only display country and first Principle Component
run;
</code>
```

PCA	*100m__s*	*200m__s*	*400m__s*	*800m__min*	*1500m__min*	*3000m__min*	Marathon_*min*
Prin1	0.37777	0.38321	0.36804	0.39478	0.38926	0.37609	0.35520
Prin2	-0.40718	-0.41363	-0.45935	0.16125	0.30909	0.42319	0.38922

d) Interpret the two principal components from Part b). (2 marks) (3 marks)

- The first principle component accounts for ~82.97% of female national track records
- Second principle component accounts for an additonal ~8.98% of female national track records
- The dataset can be reduced to 2 principle components to explain ~91.95% of female national track records.

e) Rank the nations based on their score on the first principal component.

```
proc sort data=prinR; /* rank by principle component #1 */
by Prin1; run;
proc print data=prinR;
var Country Prin1; /* filter to only display country and first Principle Component */
title 'e) Rank the nations based on their score on the first principal component';
run;
```

Obs	Country	Prin1
1	ARG	0.39324
2	AUS	-1.93164
3	AUT	-1.26252
4	BEL	-1.29173
5	BER	1.39611
6	BRA	-1.00678
7	CAN	-1.73434
8	CHI	0.81184
9	CHN	-2.98947
10	COL	0.00193
11	COK	7.90623
12	CRC	2.16681
13	CZE	-2.40603
14	DEN	-0.08250
15	DOM	2.19241
16	FIN	-1.26673
17	FRA	-2.51835
18	GER	-3.04752
19	GBR	-2.44271
20	GRE	-1.19780
21	GUA	3.29412
22	HUN	-0.78825
23	INA	1.74194
24	IND	-0.35426
25	IRL	-1.03591
26	ISR	0.57416
27	ITA	-1.54745
28	JPN	-0.48166
29	KEN	-0.91774
30	KOR	0.83079
31	KOR	1.45535
32	LUX	1.72147
33	MAS	1.49521
34	MRI	1.74973
35	MEX	-0.99577
36	MYA	0.81598
37	NED	-1.54476
38	NZL	-0.75524
39	NOR	-0.55300
40	PNG	5.25745

Obs	Country	Prin1
41	PHI	1.76353
42	POL	-2.27377
43	POR	-1.17525
44	ROM	-2.12301
45	RUS	-3.04295
46	SAM	8.21342
47	SIN	3.09392
48	ESP	-1.88946
49	SWE	-0.83915
50	SUI	-1.11355
51	TPE	0.65909
52	THA	1.22381
53	TUR	-0.85013
54	USA	-3.29915

Discuss whether this meets your expectations. (2 marks)

Doesn't meet my expectations because current world records for track records are:

- Women's records are from USA, France, Soviet, Romania, Kenya, Ethiopia
- Men all from Africa (Jamaica, South Africa, Kenya, Ethiopia)

Possibly because date of the records isn't supplied.
Further investigation could compare current world records accross all countries

Ranked by Principle Component 1 (accross all track records)

Obs	Country	Prin1
1	ARG	0.39324
2	AUS	-1.93164
3	AUT	-1.26252
4	BEL	-1.29173
5	BER	1.39611
6	BRA	-1.00678
7	CAN	-1.73434
8	CHI	0.81184
9	CHN	-2.98947
10	COL	0.00193

Men's records (https://en.wikipedia.org/wiki/List_of_Olympic_records_in_athletics#Men's_records)

100 metres	0:9.63	Usain Bolt	Jamaica
200 metres	0:19.30	Usain Bolt	Jamaica
400 metres	0:43.03	Wayde van Niekerk	South Africa
800 metres	1:40.91	David Rudisha	Kenya
1500 metres	3:32.07	Noah Ngeny	Kenya
5000 metres	12:57.82	Kenenisa Bekele	Ethiopia
10000 metre	27:01.17	Kenenisa Bekele	Ethiopia
Marathon	2:06:32	Samuel Wanjiru	Kenya

Women's records (https://en.wikipedia.org/wiki/List_of_Olympic_records_in_athletics#Women's_records)

100 metres	10.62	Florence Griffith-Joyner	United States (USA)
200 metres	21.34	Florence Griffith-Joyner	United States (USA)
400 metres	48.25	Marie-José Pérec	France (FRA)
800 metres	1:53.43	Nadezhda Olizarenko	Soviet Union (URS)
1,500 metres	3:53.96	Paula Ivan	Romania (ROU)
5,000 metres	14:26.17	Vivian Cheruiyot	Kenya (KEN)
10,000 metres	29:17.45	Almaz Ayana	Ethiopia (ETH)
Marathon	2:23:07	Tiki Gelana	Ethiopia (ETH)

5 - Factor Analysis

The correlation matrix below is from the measurement of skeletal features of white leghorn fowl (Dunn, Storrs Agricultural Experimental Station Bulletin, 52, 1928). Where

X_1 = Skull length

X_2 = Skull breadth

X_3 = Femur length

X_4 = Tibia length

X_5 = Humerus length

X_6 = Ulna length

Using the **maximum likelihood procedure** the following estimated factor loadings were extracted:

Skeletal Feature	**Variable**	**Estimated-**	**-Loadings**	**Varimax-**	**-rotated-loadings**
		F1	F2	F1*	F2*
Skull length	1	0.602	0.200	0.484	0.411
Skull breadth	2	0.467	0.154	0.375	0.319
Femur length	3	0.926	0.143	0.603	0.717
Tibia length	4	1.000	0.000	0.519	0.855
Humerus length	5	0.874	0.476	0.861	0.499
Ulna length	6	0.894	0.327	0.744	0.594

Skeletal Feature	Variable	Estimated-	-Loadings			Varimax-	-rotated-loadings		
		F1	F1 Proportions	F2	F2 Proportions	F1*	F1*Proportions	F2*	F2*Proportions
Skull length	1	0.602	0.1263909	0.200	0.0419903	0.484	0.134969325	0.411	0.121060383
Skull breadth	2	0.467	0.0980474	0.154	0.0323326	0.375	0.104573341	0.319	0.093961708
Femur length	3	0.926	0.1944153	0.143	0.0300231	0.603	0.168153932	0.717	0.211192931
Tibia length	4	1.000	0.2099517	0.000	0	0.519	0.144729504	0.855	0.251840943
Humerus length	5	0.874	0.1834978	0.476	0.099937	0.861	0.24010039	0.499	0.146980854
Ulna length	6	0.894	0.1876968	0.327	0.0686542	0.744	0.207473508	0.594	0.174963181
Totals		4.763		1.3		3.586		3.395	

In [28]:

```
proc iml;
fowlCorr = {
1.000 0.505 0.569 0.602 0.621 0.603,
0.505 1.000 0.422 0.467 0.482 0.450,
0.569 0.422 1.000 0.926 0.877 0.878,
0.602 0.467 0.926 1.000 0.874 0.894,
0.621 0.482 0.877 0.874 1.000 0.937,
0.603 0.450 0.878 0.894 0.937 1.000
};
```

Out[28]:

In [29]:

```
/* data fowl (type=cov);
input _type_$ x1-x6;
datalines;
corr 1.000 0.505 0.569 0.602 0.621 0.603
corr 0.505 1.000 0.422 0.467 0.482 0.450
corr 0.569 0.422 1.000 0.926 0.877 0.878
corr 0.602 0.467 0.926 1.000 0.874 0.894
corr 0.621 0.482 0.877 0.874 1.000 0.937
corr 0.603 0.450 0.878 0.894 0.937 1.000
;

proc factor data=fowl method=principal corr p=80;

/* proc factor data=fowl method=principal cov rotate=varimax; */
run; */
```

Out[29]:

```
369  ods listing close;ods html5 (id=saspy_internal) file=stdout options(bitmap_mode='inline') device=svg; od
s graphics on /
369! outputfmt=png;
NOTE: Writing HTML5(SASPY_INTERNAL) Body file: STDOUT
370
371  /* data fowl (type=cov);
372  input _type_$ x1-x6;
373  datalines;
374  corr 1.000 0.505 0.569 0.602 0.621 0.603
375  corr 0.505 1.000 0.422 0.467 0.482 0.450
376  corr 0.569 0.422 1.000 0.926 0.877 0.878
377  corr 0.602 0.467 0.926 1.000 0.874 0.894
378  corr 0.621 0.482 0.877 0.874 1.000 0.937
379  corr 0.603 0.450 0.878 0.894 0.937 1.000
380  ;
381
382  proc factor data=fowl method=principal corr p=80;
383
384  /* proc factor data=fowl method=principal cov rotate=varimax; */
385  run;
NOTE: Module MAIN is undefined in IML; cannot be RUN.
385!      */
386
387  ods html5 (id=saspy_internal) close;ods listing;

388
```

d) Using the unrotated estimated factor loadings, obtain the maximum likelihood estimates of the following:

i. The specific variances. $\hat { \psi } $

In [30]:

```
proc iml;
fowlCorr = {
1.000 0.505 0.569 0.602 0.621 0.603,
0.505 1.000 0.422 0.467 0.482 0.450,
0.569 0.422 1.000 0.926 0.877 0.878,
0.602 0.467 0.926 1.000 0.874 0.894,
0.621 0.482 0.877 0.874 1.000 0.937,
0.603 0.450 0.878 0.894 0.937 1.000
};
estimatedFactorLoading = {
0.602 0.200,
0.467 0.154,
0.926 0.143,
1.000 0.000,
0.874 0.476,
0.894 0.327
};
estimatedFactorLoadingT = t(estimatedFactorLoading);
LLt = estimatedFactorLoading * estimatedFactorLoadingT;

psiWithoutZeros = fowlCorr - LLt;
psi = diag(psiWithoutZeros);
psiFowlEstimated = vecdiag(psi);
fowlCorrEstimated = LLt + psi;
redisualMatrixEstimated = fowlCorr - LLt - psi;

communalitiesFowlEstimated = vecDiag(1 - psi);
print psiFowlEstimated;
print fowlCorrEstimated;
print fowlCorr;
```

Out[30]:

The SAS System

psiFowlEstimated
0.597596
0.758195
0.122075
0
0.009548
0.093835

fowlCorrEstimated					
1	0.311934	0.586052	0.602	0.621348	0.603588
0.311934	1	0.454464	0.467	0.481462	0.467856
0.586052	0.454464	1	0.926	0.877392	0.874605
0.602	0.467	0.926	1	0.874	0.894
0.621348	0.481462	0.877392	0.874	1	0.937008
0.603588	0.467856	0.874605	0.894	0.937008	1

fowlCorr					
1	0.505	0.569	0.602	0.621	0.603
0.505	1	0.422	0.467	0.482	0.45
0.569	0.422	1	0.926	0.877	0.878
0.602	0.467	0.926	1	0.874	0.894
0.621	0.482	0.877	0.874	1	0.937
0.603	0.45	0.878	0.894	0.937	1

ii. The communalities. $ { h }_{ i }^{ 2 } = 1 -{ \psi }_{ i }$

In [31]:

```
print communalitiesFowlEstimated;
```

Out[31]:

The SAS System

communalitiesFowlEstimated
0.402404
0.241805
0.877925
1
0.990452
0.906165

iii. The proportion of variance explained by each factor.

Estimated-Loadings

Skeletal Feature	**Variable**				
		F1	F1 Proportions	F2	F2 Proportions
Skull length	x_1	0.602	0.1263909	0.200	0.0419903
Skull breadth	x_2	0.467	0.0980474	0.154	0.0323326
Femur length	x_3	0.926	0.1944153	0.143	0.0300231
Tibia length	x_4	1.000	0.2099517	0.000	0
Humerus length	x_5	0.874	0.1834978	0.476	0.099937
Ulna length	x_6	0.894	0.1876968	0.327	0.0686542
Totals		**4.763**		**1.300**	

iv. The residual matrix (2.5 marks)

$R-\widehat { L } \widehat { L } -\hat { \psi } $

In [32]:

```
print redisualMatrixEstimated;
```

Out[32]:

The SAS System

redisualMatrixEstimated					
0	0.193066	-0.017052	0	-0.000348	-0.000588
0.193066	0	-0.032464	0	0.000538	-0.017856
-0.017052	-0.032464	0	0	-0.000392	0.003395
0	0	0	0	0	0
-0.000348	0.000538	-0.000392	0	0	-8E-6
-0.000588	-0.017856	0.003395	0	-8E-6	0

e) Using the varimax rotated estimated factor loadings, obtain the maximum likelihood estimates of the following:

i. The specific variances.

In [33]:

```
proc iml;
fowlCorr = {
1.000 0.505 0.569 0.602 0.621 0.603,
0.505 1.000 0.422 0.467 0.482 0.450,
0.569 0.422 1.000 0.926 0.877 0.878,
0.602 0.467 0.926 1.000 0.874 0.894,
0.621 0.482 0.877 0.874 1.000 0.937,
0.603 0.450 0.878 0.894 0.937 1.000
};

VarimaxRotated = {
0.484 0.411,
0.375 0.319,
0.603 0.717,
0.519 0.855,
0.861 0.499,
0.744 0.594
};

redisualMatrixEstimated = {
0 0.193066 -0.017052 0 -0.000348 -0.000588,
0.193066 0 -0.032464 0 0.000538 -0.017856,
-0.017052 -0.032464 0 0 -0.000392 0.003395,
0 0 0 0 0 0,
-0.000348 0.000538 -0.000392 0 0 -8E-6,
-0.000588 -0.017856 0.003395 0 -8E-6 0
};

VarimaxRotatedT = t(VarimaxRotated);
LLt = VarimaxRotated * VarimaxRotatedT;

psiWithoutZeros = fowlCorr - LLt;
psi = diag(psiWithoutZeros);
psiFowlVarimax = vecdiag(psi);
fowlCorrSample = LLt + psi;
redisualMatrixRotated = fowlCorr - LLt - psi;

residualMatrixDiff = redisualMatrixEstimated - redisualMatrixRotated;

communalitiesFowlVarimax = vecDiag(1 - psi);
print psiFowlVarimax;
print fowlCorrSample;
print fowlCorr;
```

Out[33]:

The SAS System

psiFowlVarimax
0.596823
0.757614
0.122302
-0.000386
0.009678
0.093628

fowlCorrSample					
1	0.312609	0.586539	0.602601	0.621813	0.60423
0.312609	1	0.454848	0.46737	0.482056	0.468486
0.586539	0.454848	1	0.925992	0.876966	0.87453
0.602601	0.46737	0.925992	1	0.873504	0.894006
0.621813	0.482056	0.876966	0.873504	1	0.93699
0.60423	0.468486	0.87453	0.894006	0.93699	1

fowlCorr					
1	0.505	0.569	0.602	0.621	0.603
0.505	1	0.422	0.467	0.482	0.45
0.569	0.422	1	0.926	0.877	0.878
0.602	0.467	0.926	1	0.874	0.894
0.621	0.482	0.877	0.874	1	0.937
0.603	0.45	0.878	0.894	0.937	1

ii. The communalities.

In [34]:

```
print communalitiesFowlVarimax;
```

Out[34]:

The SAS System

communalitiesFowlVarimax
0.403177
0.242386
0.877698
1.000386
0.990322
0.906372

iii. The proportion of variance explained by each factor.

Varimax-rotated-loadings

		F1*	F1*Proportions	F2*	F2*Proportions
Skull length	1	0.484	0.134969325	0.411	0.121060383
Skull breadth	2	0.375	0.104573341	0.319	0.093961708
Femur length	3	0.603	0.168153932	0.717	0.211192931
Tibia length	4	0.519	0.144729504	0.855	0.251840943
Humerus length	5	0.861	0.24010039	0.499	0.146980854
Ulna length	6	0.744	0.207473508	0.594	0.174963181
Totals		3.586		3.395	

iv. Theresidualmatrix $R-\widehat { L } \widehat { L } -\hat { \psi } $ (2.5 marks)

In [35]:

```
print redisualMatrixRotated;
```

Out[35]:

The SAS System

redisualMatrixRotated					
0	0.192391	-0.017539	-0.000601	-0.000813	-0.00123
0.192391	0	-0.032848	-0.00037	-0.000056	-0.018486
-0.017539	-0.032848	0	8E-6	0.000034	0.00347
-0.000601	-0.00037	8E-6	0	0.000496	-6E-6
-0.000813	-0.000056	0.000034	0.000496	0	0.00001
-0.00123	-0.018486	0.00347	-6E-6	0.00001	0

f) Comment on the results using the two loading methods by comparing your results from part a) and part b) above. (2 marks)

Rotating the residual matrix

Everything appears to be quite similar, except for [2,1] in our corrolation matrix:

- Estimated loading 0.311934
- Varimax rotated   0.312609
- Corrolation matrix 0.505

Which seems to be slightly different based on our sample.

In [36]:

```
print residualMatrixDiff;
```

Out[36]:

The SAS System

residualMatrixDiff					
0	0.000675	0.000487	0.000601	0.000465	0.000642
0.000675	0	0.000384	0.00037	0.000594	0.00063
0.000487	0.000384	0	-8E-6	-0.000426	-0.000075
0.000601	0.00037	-8E-6	0	-0.000496	6E-6
0.000465	0.000594	-0.000426	-0.000496	0	-0.000018
0.000642	0.00063	-0.000075	6E-6	-0.000018	0

It's not obvious with the estimated loadings what is explained by F1, and F2 in the estimated loading

Ie.
F1 => mostly explained by Tibia, Femur, Ulna, Humerous, each ~20%
F2 => mostly explained by Ulna (0.07~)

Rotated is much easier to understand:

F1 => Mostly explained by Humerus (24%), Ulna (20.7%), Femur (17%)
F2 => Mostly explained by Tibia (25%), Femur(21%), Ulna (17%)

There is no part a) or part b)

Not rotated very far
rotating moves this away from psi of 0 and communality of 1
So psi[4] for Varimax gets rotated to -0.000386

```
psiFowlEstimated
0.597596
0.758195
0.122075
0                 this is strange
0.009548
0.093835

communalitiesFowlEstimated
0.402404
0.241805
0.877925
1                 this is strange
0.990452
0.906165

psiFowlVarimax
0.596823
0.757614
0.122302
-0.000386
0.009678
0.093628

communalitiesFowlVarimax
0.403177
0.242386
0.877698
1.000386
0.990322
0.906372
```