Lectures 9

Filtering : KF/EKF/PF

Goals of this lecture :

- KF, EKF
- PF

Discrete Kalman Filter

Assume that a random process to be estimated can be modeled, and the observation(measurement) of the process occurs linearly:

$$\begin{cases} x_{k+1} & = \Phi_k x_k + w_k \\ z_k & = C_k x_k + v_k \end{cases}$$

$$\mathbf{x_n} = \mathbf{F_n} \mathbf{x_{n\text{-}1}} + \mathbf{q_n}$$

where

Φ_k : the state transition matrix

C_k : the observation matrix

x_k : the process state vector

z_k : the measurement vector

w_k : white sequence with known covariance, $\sim N(0, Q_k)$

v_k : white sequence measurement error with known covariance, $\sim N(0, R_k)$

Discrete Kalman Filter

The covariance matrices for the w_k and v_k

$$E\left[w_k w_k^T\right] = Q_k, \quad E\left[w_k w_j^T\right] = 0\left(j \neq k\right)$$

$$E\left[v_k v_k^T\right] = R_k, \quad E\left[v_k v_j^T\right] = 0\left(j \neq k\right)$$

$$E\left[w_k v_j^T\right] = 0\left(\forall k, j\right)$$

Discrete Kalman Filter

Prediction (a priori) estimate

$$\bar{x}_k$$

Prediction (a priori) estimation error

$$\bar{e}_k = x_k - \bar{x}_k$$

Prediction (a priori) error covariance matrix

$$\bar{\Sigma}_k = E\left[\bar{e}_k \bar{e}_k^T\right] = E\left[\left(x_k - \bar{x}_k\right)\left(x_k - \bar{x}_k\right)^T\right]$$

How to use the measurement z_k to improve the prior estimate $\bar{x}_k$

Discrete Kalman Filter

We choose

$$\hat{x}_k = \overline{x}_k + L_k \left(z_k - C_k \overline{x}_k \right)$$

where

$\hat{x}_k$: the updated (a posteriori) estimate

L_k : a gain to be determined

How to find the gain L_k that yields an updated estimate that is optimal in some sense

- Minimum mean-square error as a performance criterion

Discrete Kalman Filter

- Discrete Kalman Filter Algorithm
  - Correction update (using measurement z_k) :

$$L_k = \bar{\Sigma}_k C_k^T \left(C_k \bar{\Sigma}_k C_k^T + R_k \right)^{-1}$$

$$\hat{x}_k = \bar{x}_k + L_k \left(z_k - C_k \bar{x}_k \right)$$

$$\Sigma_k = \left(I - L_k C_k \right) \bar{\Sigma}_k$$

  - Prediction update :

$$\bar{x}_{k+1} = \Phi_k \hat{x}_k$$

$$\bar{\Sigma}_{k+1} = \Phi_k \Sigma_k \Phi_k^T + Q_k$$

- Probabilistic Origin of Kalman Filter

$$E\left[x_k\right]=\hat{x}_k$$

$$E\left[e_k e_k^T\right]=\Sigma_k$$

$$P\left(x_k \middle| z_k\right)=N\left(E\left[x_k\right],E\left[e_k e_k^T\right]\right)=N\left(\hat{x}_k,\Sigma_k\right)$$

- Bayes Filter Remind
  - Prediction

$$\overline{Bel}\left(x_k\right)=\sum_{x_{k-1}}P\left(x_k \mid u_k,x_{k-1}\right)Bel\left(x_{k-1}\right)$$

  - Correction

$$Bel\left(x_k\right)=\eta\ P\left(z_k \mid x_k\right)\overline{Bel}\left(x_k\right)$$

- Probabilistic Kalman Filter

  - Discrete time process governed by the linear stochastic d.e. with control & measurement.

$$\begin{cases} x_k &= \Phi_k x_{k-1} + \Gamma_k u_k + w_k \\ z_k &= C_k x_k + v_k \end{cases}$$

  - Prediction

$$\overline{Bel}(x_k) = \begin{cases} \bar{x}_k &= \Phi_k \hat{x}_{k-1} + \Gamma_k u_k \\ \bar{\Sigma}_k &= \Phi_k \Sigma_{k-1} \Phi_k^T + Q_k \end{cases}$$

  - Correction

$$Bel(x_k) = \begin{cases} \hat{x}_k &= \bar{x}_k + L_k \left(z_k - C_k \bar{x}_k \right) \quad \text{with} \quad L_k = \bar{\Sigma}_k C_k^T \left(C_k \bar{\Sigma}_k C_k^T + R_k \right)^{-1} \\ \Sigma_k &= \left(I - L_k C_k \right) \bar{\Sigma}_k \end{cases}$$

- Probabilistic Kalman Filter
  - Prediction

$$
\begin{aligned}
\overline{Bel}(x_k) &= \sum_{x_{k-1}} P(x_k \mid u_k, x_{k-1}) Bel(x_{k-1}) \\
&= \eta \sum_{x_{k-1}} N(x_k; \Phi_k x_{k-1} + \Gamma_k u_k, Q_k) N(x_{k-1}; \hat{x}_{k-1}, \Sigma_{k-1}) \\
&= N(x_k; \bar{x}_k, \bar{\Sigma}_k)
\end{aligned}
$$

$$
\begin{aligned}
\bar{x}_{k+1} &= \Phi_k \hat{x}_k \\
\bar{\Sigma}_{k+1} &= \Phi_k \Sigma_k \Phi_k^T + Q_k
\end{aligned}
$$

  - asdfasd
  - Correction

$$
\begin{aligned}
Bel(x_k) &= \eta P(z_k \mid x_k) \overline{Bel}(x_k) \\
&= \eta N(z_k; C_k x_k, R_k) N(x_k; \bar{x}_k, \bar{\Sigma}_k) \\
&= N(x_k; \hat{x}_k, \Sigma_k)
\end{aligned}
$$

$$
\begin{aligned}
L_k &= \bar{\Sigma}_k C_k^T \left(C_k \bar{\Sigma}_k C_k^T + R_k \right)^{-1} \\
\hat{x}_k &= \bar{x}_k + L_k (z_k - C_k \bar{x}_k) \\
\Sigma_k &= (I - L_k C_k) \bar{\Sigma}_k
\end{aligned}
$$

- Steady State Kalman Filter

$$\bar{x}_{k+1} = \Phi_k \hat{x}_k$$
$$\bar{\Sigma}_{k+1} = \Phi_k \Sigma_k \Phi_k^T + Q_k$$

$$L_k = \bar{\Sigma}_k C_k^T \left(C_k \bar{\Sigma}_k C_k^T + R_k \right)^{-1}$$
$$\hat{x}_k = \bar{x}_k + L_k \left(z_k - C_k \bar{x}_k \right)$$
$$\Sigma_k = \left(I - L_k C_k \right) \bar{\Sigma}_k$$

- If $\left\| \bar{\Sigma}_{k+1} - \bar{\Sigma}_k \right\| < \varepsilon$, then Prediction error covariance exists a steady state value Σ_s

$$\bar{\Sigma}_{k+1} = \Phi_k \bar{\Sigma}_k \Phi_k^T + Q_k - \Phi_k \bar{\Sigma}_k C_k^T \left[C_k \bar{\Sigma}_k C_k^T + R \right]^{-1} C_k \bar{\Sigma}_k \Phi_k^T$$

- RICCATI Eq. Solve(using Matlab idare)

$\bar{\Sigma}_s$: Steady State Prediction error covariance

L_0 : Steady State-Feedback Gain

Examples

- Exercise

$$\begin{cases} x_k = \Phi_k x_{k-1} + \Gamma_k u_k + w_k \\ z_k = C_k x_k + v_k \end{cases}$$

$$\Phi_k = 0.7, \quad \Gamma_k = \frac{1}{\sqrt{2}}, \quad C_k = 1$$

$$Q_k = 0.5, \quad R_k = 0.15$$

$$u_k = 10$$

Examples

- Exercise

$$\begin{cases} x_k = \Phi_k x_{k-1} + w_k \\ z_k = C_k x_k + v_k \end{cases}$$

$$\Phi_k = e^{AT}, \quad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}, \quad C_k = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q_k = E\left[w_k w_k^T \right], \quad R_k = E\left[v_k v_k^T \right]$$

Examples

- Exercise

$$\begin{cases} x_k = \Phi_k x_{k-1} + \Gamma_k u_k + w_k \\ z_k = C_k x_k + v_k \end{cases}$$

$$\Phi_k = e^{AT}, \quad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}, \quad \Gamma_k = \begin{bmatrix} 1 \\ 1/\sqrt{2} \end{bmatrix}, \quad C_k = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q_k = E\left[w_k w_k^T \right], \quad R_k = E\left[v_k v_k^T \right]$$

$$u_k = \sin\left(0.1(k-1)\right)$$