Lectures 8 ~10

Filtering : KF/EKF/PF

Goals of this lecture :

- KF, EKF
- PF

- Discretization of Linear Dynamic Models
  - Goal : Solve the first order ODE ➔ Scalar Model

$$\dot{x}(t) = ax(t) + bu(t)$$

on the interval $\left(t_{n-1}, t_n\right]$

$$\begin{aligned}
\frac{d}{dt} e^{-at} x(t) &= e^{-at} \dot{x}(t) - a e^{-at} x(t) \\
&= e^{-at} \left(ax(t) + bu(t) \right) - a e^{-at} x(t) \\
&= e^{-at} bu(t)
\end{aligned}$$

$$\int_{t_{n-1}}^{t_n} d\left[e^{-at} x(t) \right] = \int_{t_{n-1}}^{t_n} e^{-at} bu(t)\, dt$$

$$\left[e^{-at} x(t) \right]_{t=t_{n-1}}^{t=t_n} = \int_{t_{n-1}}^{t_n} e^{-at} bu(t)\, dt$$

$$e^{-at_n} x(t_n) - e^{-at_{n-1}} x(t_{n-1}) = \int_{t_{n-1}}^{t_n} e^{-at} bu(t)\, dt$$

- Discretization of Linear Dynamic Models

$$e^{-at_n}x(t_n)-e^{-at_{n-1}}x(t_{n-1})=\int_{t_{n-1}}^{t_n} e^{-at}bu(t)\,dt$$

$$\begin{aligned}x(t_n) &= e^{at_n-at_{n-1}}x(t_{n-1})+e^{at_n}\int_{t_{n-1}}^{t_n} e^{-at}bu(t)\,dt \\ &= e^{a(t_n-t_{n-1})}x(t_{n-1})+\int_{t_{n-1}}^{t_n} e^{a(t_n-t)}bu(t)\,dt\end{aligned}$$

$$\therefore x_n = e^{a\vartriangle t}x_{n-1}+\int_{t_{n-1}}^{t_n} e^{a(t_n-t)}bu(t)\,dt$$

- Discretization of Linear Dynamic Models
  - Goal : Solve the first order ODE ➔ Vector Model

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}_\mathbf{u}\mathbf{u}(\mathbf{t})$$

$$\begin{aligned}\frac{d}{dt}e^{-At}x(t) &= e^{-At}\dot{x}(t) - e^{-At}Ax(t) \\ &= e^{-At}\left(Ax(t) + B_u u(t)\right) - e^{-At}Ax(t) \\ &= e^{-At}B_u u(t)\end{aligned}$$

$$\int_{t_{n-1}}^{t_n} d\left[e^{-At}x(t)\right] = \int_{t_{n-1}}^{t_n} e^{-At}B_u u(t)\,dt$$

$$\left[e^{-At}x(t)\right]_{t=t_{n-1}}^{t=t_n} = \int_{t_{n-1}}^{t_n} e^{-At}B_u u(t)\,dt$$

$$e^{-At_n}x(t_n) - e^{-At_{n-1}}x(t_{n-1}) = \int_{t_{n-1}}^{t_n} e^{-At}B_u u(t)\,dt$$

- Discretization of Linear Dynamic Models

$$e^{-At_n}x(t_n)-e^{-At_{n-1}}x(t_{n-1})=\int_{t_{n-1}}^{t_n} e^{-At}B_u u(t)\,dt$$

$$\begin{aligned} x(t_n) &= e^{At_n-At_{n-1}}x(t_{n-1})+e^{At_n}\int_{t_{n-1}}^{t_n} e^{-At}B_u u(t)\,dt \\ &= e^{A(t_n-t_{n-1})}x(t_{n-1})+\int_{t_{n-1}}^{t_n} e^{A(t_n-t)}B_u u(t)\,dt \end{aligned}$$

$$\therefore x_n = e^{A\triangle t}x_{n-1}+u_{n-1}\int_{t_{n-1}}^{t_n} e^{A(t_n-t)}B_u\,dt$$

$$\dot{\mathbf{x}}(\mathbf{t})=\mathbf{A}\mathbf{x}(\mathbf{t})+\mathbf{B_u}\mathbf{u}(\mathbf{t}) \quad \Rightarrow \quad \mathbf{x_n}=\mathbf{F_n}\mathbf{x_{n-1}}+\mathbf{L_n}\mathbf{u_{n-1}}$$

- Stochastic Linear Dynamic Models

  - Stochastic linear dynamic model :

$$\dot{x}(t) = Ax(t) + B_w w(t)$$

  - The only difference to the deterministic model is the $u(t)\left(w(t)\right)$

  - $w(t)$ is a white stochastic process

  - Auto-correlation function :

$$R_{ww}(\tau) = \mathrm{E}\left\{w(t+\tau)\,w(\tau)\right\} = \Sigma_w \delta(\tau)$$

  - Hence :

$$x_n = e^{A\triangle t} x_{n-1} + \int_{t_{n-1}}^{t_n} e^{A(t_n - t)} B_w w(t)\,dt$$

$$\mathbf{x_n} = \mathbf{F_n x_{n-1}} + \mathbf{q_n}$$

- Stochastic Linear Dynamic Models
  - Discretized stochastic dynamic model :

$$\mathbf{x_n} = \mathbf{F_n}\mathbf{x_{n-1}} + \mathbf{q_n}$$

where

$$\mathbf{F_n} = e^{A\left(t_n - t_{n-1}\right)}$$

$$\mathbf{q_n} \sim N\left(0, \mathbf{Q_n}\right)$$

$$\mathbf{Q_n} = \int_{t_{n-1}}^{t_n} e^{A\left(t_n - \tau\right)} B_w \Sigma_w B_w^T e^{A^T\left(t_n - \tau\right)} d\tau$$

- Example : Deterministic 1D Motion Model

  - Dynamic model :

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

  - Recall :

$$F_n = e^{A\triangle t} = \sum_{j=0}^{\infty} \frac{1}{j!} A^j (\triangle t)^j$$

$$= \frac{1}{0!} I (\triangle t)^0 + \frac{1}{1!} A(\triangle t)^1 + \frac{1}{2!} 0(\triangle t)^2 + \cdots$$

$$= I + A\triangle t$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \triangle t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \triangle t \\ 0 & 1 \end{bmatrix}$$

$$A^0 = I$$

$$A^1 = A$$

$$A^2 = 0$$

$$A^j = 0 (j \geq 2)$$

- Example : Deterministic 1D Motion Model
  - Input matrix

$$L_n = \int_{t_{n-1}}^{t_n} e^{A(t_n - t)} B_u dt$$

where

$$e^{A(t_n - t)} = \begin{bmatrix} 1 & t_n - t \\ 0 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L_n = \int_{t_{n-1}}^{t_n} \begin{bmatrix} 1 & t_n - t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt = \int_{t_{n-1}}^{t_n} \begin{bmatrix} t_n - t \\ 1 \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}(\triangle t)^2 \\ \triangle t \end{bmatrix}$$

- Recap : Average Filter

$$\bar{x}_k = \frac{x_1 + x_2 + \cdots x_k}{k}$$

$$\bar{x}_k = \frac{k-1}{k}\bar{x}_{k-1} + \frac{1}{k}x_k$$

$$\bar{x}_k = \alpha\bar{x}_{k-1} + (1-\alpha)x_k$$

- Recap : Moving Average Filter

$$\overline{x}_k = \frac{x_{k-n+1} + x_{k-n+2} + \cdots x_k}{n}$$

$$\overline{x}_k = \overline{x}_{k-1} + \frac{x_k - x_{k-n}}{n}$$

- Recap : Low Pass Filter

$$\bar{x}_k = \alpha \bar{x}_{k-1} + (1-\alpha) x_k$$

Discrete Kalman Filter

Assume that a random process to be estimated can be modeled, and the observation(measurement) of the process occurs linearly:

$$\begin{cases} x_{k+1} & = \Phi_k x_k + w_k \\ z_k & = C_k x_k + v_k \end{cases}$$

$$\mathbf{x_n} = \mathbf{F_n x_{n\text{-}1}} + \mathbf{q_n}$$

where

Φ_k : the state transition matrix

C_k : the observation matrix

x_k : the process state vector

z_k : the measurement vector

w_k : white sequence with known covariance, $\sim N\left(0, Q_k\right)$

v_k : white sequence measurement error with known covariance, $\sim N\left(0, R_k\right)$

Discrete Kalman Filter

The covariance matrices for the w_k and v_k

$$E\left[w_k w_k^T\right] = Q_k, \quad E\left[w_k w_j^T\right] = 0\left(j \neq k\right)$$

$$E\left[v_k v_k^T\right] = R_k, \quad E\left[v_k v_j^T\right] = 0\left(j \neq k\right)$$

$$E\left[w_k v_j^T\right] = 0\left(\forall k, j\right)$$

Discrete Kalman Filter

Prediction (a priori) estimate

$$\bar{x}_k$$

Prediction (a priori) estimation error

$$\bar{e}_k = x_k - \bar{x}_k$$

Prediction (a priori) error covariance matrix

$$\bar{\Sigma}_k = E\left[\bar{e}_k \bar{e}_k^T\right] = E\left[\left(x_k - \bar{x}_k\right)\left(x_k - \bar{x}_k\right)^T\right]$$

How to use the measurement z_k to improve the prior estimate $\bar{x}_k$

Discrete Kalman Filter

We choose

$$\hat{x}_k = \overline{x}_k + L_k \left(z_k - C_k \overline{x}_k \right)$$

where

$\hat{x}_k$: the updated (a posteriori) estimate

L_k : a gain to be determined

How to find the gain L_k that yields an updated estimate that is optimal in some sense

- Minimum mean-square error as a performance criterion

Discrete Kalman Filter

Updated (a posteriori) estimation error :

$$e_k = x_k - \hat{x}_k$$

The covariance associated with the updated estimate error :

$$\Sigma_k = E\left[e_k e_k^T\right] = E\left[\left(x_k - \hat{x}_k\right)\left(x_k - \hat{x}_k\right)^T\right]$$

Using

$$\hat{x}_k = \overline{x}_k + L_k\left(z_k - C_k\overline{x}_k\right) = \left(I - L_kC_k\right)\overline{x}_k + L_kC_kx_k + L_kv_k \left(\because z_k = C_kx_k + v_k\right)$$

$$e_k = x_k - \hat{x}_k = \left(I - L_kC_k\right)\overline{e}_k - L_kv_k$$

We have

$$\Sigma_k = E\left[e_k e_k^T\right] = \left(I - L_kC_k\right)\overline{\Sigma}_k\left(I - L_kC_k\right)^T + L_kR_kL_k^T$$

(The a priori estimation error $\overline{e}_k$ uncorrelated with v_k , $E\left[e_kv_k^T\right] = 0$)

Joseph form, If L_k is the optimal Kalman gain, we can be simplified

Discrete Kalman Filter

Optimization

Seek to minimize the expected value of the square of the magnitude of

$$E\left[\left\|e_k\right\|^2\right] = E\left[\left\|x_k - \hat{x}_k\right\|^2\right]$$

Equivalent to minimizing the trace of the posteriori estimate covariance matrix

$$\min_{L_k} \operatorname{tr}\left[\Sigma_k\right]$$

The trace is minimized when its matrix derivative with respect to the gain matrix zero.

$$\frac{d\, \operatorname{tr}\left[\Sigma_k\right]}{dL_k} = 0$$

Discrete Kalman Filter

- Matrix differentiation formulas : ➔ see matrix cookbook

$$\frac{d\,\mathrm{tr}\left[AB\right]}{dA} = B^T \quad \left(AB \quad \text{square}\right)$$

$$\frac{d\,\mathrm{tr}\left[BA^T\right]}{dA} = B \quad \left(AB \quad \text{square}\right)$$

$$\frac{d\,\mathrm{tr}\left[ACA^T\right]}{dA} = 2AC \quad \left(C = C^T \quad \text{symmetric}\right)$$

Discrete Kalman Filter

The posteriori estimate covariance matrix

$$
\begin{aligned}
\Sigma_k &= E\left[e_k e_k^T\right] = \left(I - L_k C_k\right)\bar{\Sigma}_k\left(I - L_k C_k\right)^T + L_k R_k L_k^T \\
&= \bar{\Sigma}_k - L_k C_k \bar{\Sigma}_k - \bar{\Sigma}_k C_k^T L_k^T + L_k\left(C_k \bar{\Sigma}_k C_k^T + R_k\right) L_k^T
\end{aligned}
$$

$$
\frac{d\,\mathrm{tr}\left[\Sigma_k\right]}{dL_k} = -2\left(C_k \bar{\Sigma}_k\right)^T + 2L_k\left(C_k \bar{\Sigma}_k C_k^T + R_k\right) = 0
$$

We get Optimal Kalman Gain

$$
L_k = \bar{\Sigma}_k C_k^T\left(C_k \bar{\Sigma}_k C_k^T + R_k\right)^{-1}
$$

Discrete Kalman Filter

- Simplification of the posteriori error covariance formula

$$L_k = \bar{\Sigma}_k C_k^T \left(C_k \bar{\Sigma}_k C_k^T + R_k \right)^{-1} \qquad \left\{ \times \left(C_k \bar{\Sigma}_k C_k^T + R_k \right) L_k^T \right\}$$

$$L_k \left(C_k \bar{\Sigma}_k C_k^T + R_k \right) L_k^T = \bar{\Sigma}_k C_k^T L_k^T$$

$$\begin{aligned} \Sigma_k &= \bar{\Sigma}_k - L_k C_k \bar{\Sigma}_k - \bar{\Sigma}_k C_k^T L_k^T + L_k \left(C_k \bar{\Sigma}_k C_k^T + R_k \right) L_k^T \\ &= \bar{\Sigma}_k - L_k C_k \bar{\Sigma}_k - \bar{\Sigma}_k C_k^T L_k^T + \bar{\Sigma}_k C_k^T L_k^T \\ &= \left(I - L_k C_k \right) \bar{\Sigma}_k \end{aligned}$$

Discrete Kalman Filter

Prediction(a priori) model

$$\bar{x}_{k+1} = \Phi_k \hat{x}_k$$

The error covariance matrix associated with $\bar{x}_{k+1}$

$$\bar{e}_{k+1} = x_{k+1} - \bar{x}_{k+1} = \Phi_k x_k + w_k - \Phi_k \hat{x}_k = \Phi_k e_k + w_k$$

The prediction error covariance matrix $\bar{\Sigma}_{k+1} = E\left[\bar{e}_{k+1}\bar{e}_{k+1}^T\right]$

$$\begin{aligned}\bar{\Sigma}_{k+1} &= E\left[\bar{e}_{k+1}\bar{e}_{k+1}^T\right] = \left(\Phi_k e_k + w_k\right)\left(\Phi_k e_k + w_k\right)^T \\ &= \Phi_k \Sigma_k \Phi_k^T + Q_k\end{aligned}$$

($E\left[e_k w_k^T\right] = 0$, e_k uncorrelated with w_k)

Discrete Kalman Filter

- Discrete Kalman Filter Algorithm
  - Correction update (using measurement z_k) :

$$L_k = \bar{\Sigma}_k C_k^T \left(C_k \bar{\Sigma}_k C_k^T + R_k \right)^{-1}$$

$$\hat{x}_k = \bar{x}_k + L_k \left(z_k - C_k \bar{x}_k \right)$$

$$\Sigma_k = \left(I - L_k C_k \right) \bar{\Sigma}_k$$

  - Prediction update :

$$\bar{x}_{k+1} = \Phi_k \hat{x}_k$$

$$\bar{\Sigma}_{k+1} = \Phi_k \Sigma_k \Phi_k^T + Q_k$$

Examples

- Exercise

$$\begin{cases} x_k = \Phi_k x_{k-1} + \Gamma_k u_k + w_k \\ z_k = C_k x_k + v_k \end{cases}$$

$$\Phi_k = 0.7, \quad \Gamma_k = \frac{1}{\sqrt{2}}, \quad C_k = 1$$

$$Q_k = 0.5, \quad R_k = 0.15$$

$$u_k = 10$$

Examples

- Exercise

$$\begin{cases} x_k = \Phi_k x_{k-1} + w_k \\ z_k = C_k x_k + v_k \end{cases}$$

$$\Phi_k = e^{AT}, \quad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}, \quad C_k = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q_k = E\left[w_k w_k^T \right], \quad R_k = E\left[v_k v_k^T \right]$$

Examples

- Exercise

$$\begin{cases} x_k = \Phi_k x_{k-1} + \Gamma_k u_k + w_k \\ z_k = C_k x_k + v_k \end{cases}$$

$$\Phi_k = e^{AT}, \quad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}, \quad \Gamma_k = \begin{bmatrix} 1 \\ 1/\sqrt{2} \end{bmatrix}, \quad C_k = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q_k = E\left[w_k w_k^T \right], \quad R_k = E\left[v_k v_k^T \right]$$

$$u_k = \sin\left(0.1\left(k-1\right)\right)$$