Chapter 1

Discrete Time Analysis

1.1 Discrete Models of Sampled-Data Systems

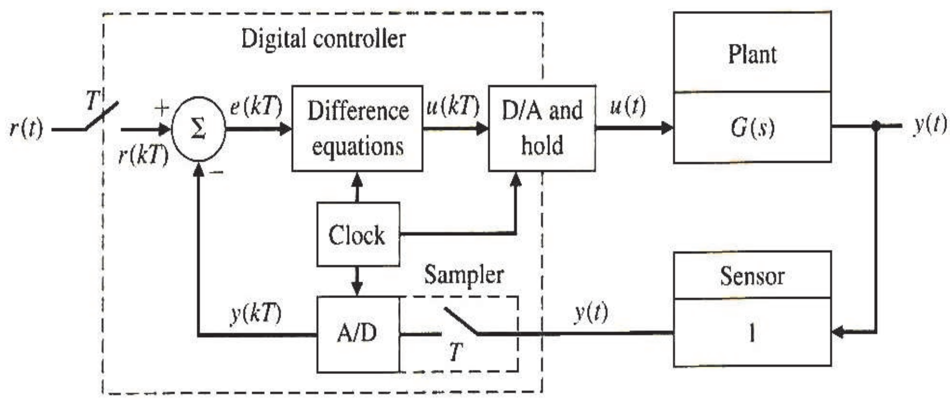

Figure 1.1: Basic control system block diagram with a digital computer.

Continuous-Time Signal :

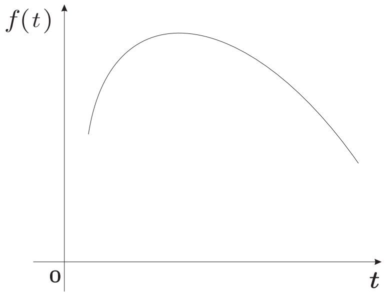

Figure 1.2: Continuous-time signal and continuous signal.

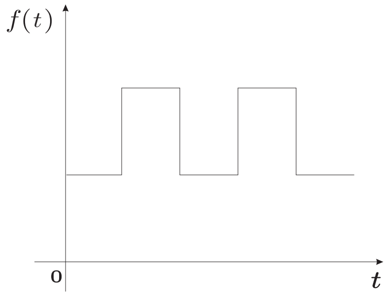

Figure 1.3: Continuous-time signal but not continuous signal, but piecewise continuous signal.

Discrete Time Signal : may not be equally spaced

Sampled Signal : equally spaced

Digital Signal : Discrete-time quantized signal

Digital signal $\approx$ Discrete-Time signal $\approx$ Sampled signal

Hold Methods :

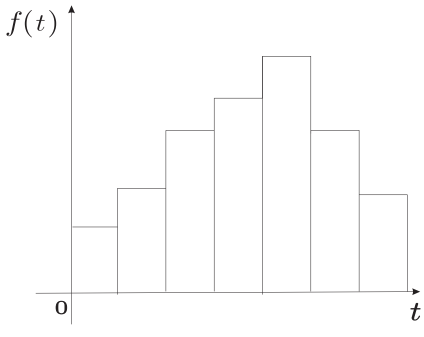

Figure 1.4: Zero-Order Hold (ZOH) Method (causal).

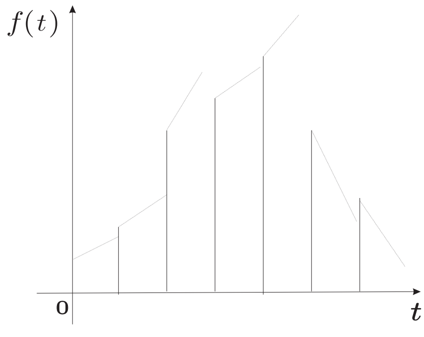

Figure 1.5: First-Order Hold (FOH) Method (causal).

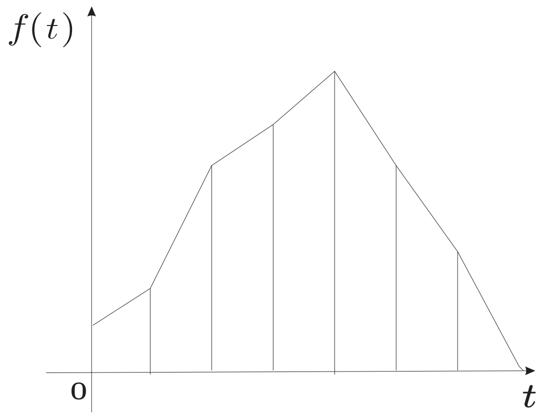

Figure 1.6: Triangular Hold. Connect previous point to the next point (non-causal).

(1) Time-Domain Model

Systems : LTI, SISO(Single Input Single Output) Systems with $\mathbf{G}(s)$ or $(\mathbf{A}, \mathbf{b}, \mathbf{c})$.

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) = \mathbf{c}^T\mathbf{x}(t) \end{cases} \quad : \text{ State equation}$$

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau$$

$$y(t) = \mathbf{c}^T e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{c}^T e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau$$

$$\begin{cases} \text{y(t) : total response} \\ \mathbf{c}^T e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) \ \text{ : zero input response, or homogeneous solution} \\ \mathbf{c}^T e^{\mathbf{A}(t-\tau)}\mathbf{b} : h(t-\tau), \text{ zero state response, or particular solution} \\ h(t) = \mathbf{c}^T e^{\mathbf{A}t}\mathbf{b} \ : \text{ impulse response} \end{cases}$$

Proof

$$
\begin{aligned}
& \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\
\Rightarrow\ & \dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{b}u(t) \\
\Rightarrow\ & \frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - \mathbf{A}e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t}\left(\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)\right) \\
\Rightarrow\ & e^{-\mathbf{A}t}\left(\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\mathbf{b}u(t) \\
\Rightarrow\ & \frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\mathbf{b}u(t) \\
\Rightarrow\ & e^{-\mathbf{A}t}\mathbf{x}(t)\Big|_{t_0}^{t} = \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau \\
\Rightarrow\ & e^{-\mathbf{A}t}\mathbf{x}(t) - e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau \\
\Rightarrow\ & e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t_0}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau \\
\Rightarrow\ & \begin{cases} \mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau \\ y(t) = \mathbf{c}^T e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{c}^T e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau \end{cases}
\end{aligned}
$$

$\square$

(2) About $e^{\mathbf{A}t}$

Definition of $e^{\mathbf{A}t}$:

$$
\begin{aligned}
e^{at} &= \sum_{k=0}^{\infty}\frac{(at)^k}{k!} = 1 + at + \frac{a^2t^2}{2!} + \cdots \\
e^{\mathbf{A}t} &\triangleq \sum_{k=0}^{\infty}\frac{(\mathbf{A}t)^k}{k!}
\end{aligned}
$$

Properties of $e^{\mathbf{A}t}$:

$$
\begin{aligned}
\frac{d}{dt}(e^{\mathbf{A}t}) &= \frac{d}{dt}\sum_{k=0}^{\infty}\frac{(\mathbf{A}t)^k}{k!} \\
&= \sum_{k=0}^{\infty}\frac{d}{dt}\frac{(\mathbf{A}t)^k}{k!} \\
&= \sum_{k=0}^{\infty}\frac{\mathbf{A}^k k t^{k-1}}{k!} \\
&= \sum_{k=1}^{\infty}\frac{\mathbf{A}^{k-1}t^{k-1}}{(k-1)!}\mathbf{A} \quad (\Leftarrow \text{Let } k-1=j) \\
&= \sum_{j=0}^{\infty}\frac{(\mathbf{A}t)^j}{j!}\mathbf{A} \\
&= e^{\mathbf{A}t}\mathbf{A} \\
&= \mathbf{A}e^{\mathbf{A}t} \\
\therefore\ \frac{d}{dt}(e^{\mathbf{A}t}) &= \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}
\end{aligned}
$$

Solutions of $e^{\mathbf{A}t}$:

1. Cayley-Hamilton Method (time-domain method)

   ⋇**Cayely-Hamilton Theorem :**

Suppose that $C(\lambda)$ be the characteristic equation of $\mathbf{A}$ (or $C(\lambda) = |\lambda\mathbf{I} - \mathbf{A}|$), then $C(\mathbf{A}) = \mathbf{0}$.

Using Cayely-Hamilton Theorem, we have

$$
e^{\mathbf{A}} = \sum_{k=0}^{n-1} C_k\mathbf{A}^k \qquad \text{or} \qquad e^{\mathbf{A}t} = \sum_{k=0}^{n-1} C_k(t)\mathbf{A}^k
$$

Example 1.1 When $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, find the solution of $e^{\mathbf{A}t}$ by using Cayely-Hamilton Theorem.

(solution)

$$\begin{aligned} e^{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} t} &= c_0(t)\mathbf{I} + c_1(t)\mathbf{A} \\ e^{\lambda_1 t} &= c_0(t) \cdot 1 + c_1(t) \cdot \lambda_1 \\ e^{\lambda_2 t} &= c_0(t) \cdot 1 + c_1(t) \cdot \lambda_2 \end{aligned}$$

a) $\lambda_1 = 0$

$$\begin{aligned} e^{0 \cdot t} &= c_0(t) \cdot 1 + c_1(t) \cdot 0 \\ \therefore c_0(t) &= 1 \end{aligned}$$

b) $\lambda_2 = 1$

$$\begin{aligned} e^{1 \cdot t} &= c_0(t) \cdot 1 + c_1(t) \cdot 1 \\ \therefore c_1(t) &= e^t - 1 \end{aligned}$$

$$\begin{aligned} \therefore e^{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} t} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (e^t - 1) \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^t & 0 \\ e^t - 1 & 1 \end{pmatrix} \end{aligned}$$

$\square$

Example 1.2 When $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, find the solution of $e^{\mathbf{A}t}$ by using Cayely-Hamilton Theorem.

(solution)

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 1 & 0 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2$$

$$\therefore\ \lambda_1 = \lambda_2 = 1$$

$$e^{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} t} = c_0(t)\mathbf{I} + c_1(t)\mathbf{A}$$

a) $\lambda_1 = 1$

$$e^{1 \cdot t} = c_0(t) \cdot 1 + c_1(t) \cdot 1$$

b) Differentiate the equation $e^{\lambda t} = c_0(t) \cdot 1 + c_1(t) \cdot \lambda$ with respect to λ

$$\begin{aligned} t \cdot e^{\lambda t} &= c_1(t) \\ t \cdot e^{\lambda t}\big|_{\lambda=1} &= c_1(t) \end{aligned}$$

$$\therefore\ \begin{cases} c_1(t) = t \cdot e^t \\ c_0(t) = (1 - t) \cdot e^t \end{cases}$$

$$\begin{aligned} \therefore\ e^{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} t} &= (1 - t) \cdot e^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \cdot e^t \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^t & 0 \\ t \cdot e^t & e^t \end{pmatrix} \end{aligned}$$

$\square$

2. Resolvent Method

$$\begin{aligned}
\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) &\Rightarrow \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) \\
&\Rightarrow \mathscr{L}[\dot{\mathbf{x}}(t)] = \mathscr{L}[\mathbf{A}\mathbf{x}(t)] \\
&\Rightarrow s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) \\
&\Rightarrow [s\mathbf{I} - \mathbf{A}]\mathbf{X}(s) = \mathbf{x}(0) \\
&\Rightarrow \mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) \\
&\Rightarrow \mathbf{x}(t) = \mathscr{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}\mathbf{x}(0) \\
\therefore\ e^{\mathbf{A}t} &= \mathscr{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}
\end{aligned}$$

Example 1.3 When $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, find the solution of $e^{\mathbf{A}t}$ by using Resolvent Method.

(solution)

$$e^{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}t} = \mathscr{L}^{-1}\left\{\left(s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right)^{-1}\right\}$$

$$= \mathscr{L}^{-1}\left\{\left(\left(\begin{array}{cc} s-1 & 0 \\ -1 & s \end{array}\right)\right)^{-1}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{s(s-1)}\left(\begin{array}{cc} s & 0 \\ 1 & s-1 \end{array}\right)\right\}$$

$$= \mathscr{L}^{-1}\left\{\left(\left(\begin{array}{cc} \frac{1}{s-1} & 0 \\ \frac{1}{s(s-1)} & \frac{1}{s} \end{array}\right)\right\}$$

$$= \left(\begin{array}{cc} e^t & 0 \\ e^t - 1 & 1 \end{array}\right)$$

$\square$

Example 1.4 When $\mathbf{A} = \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right)$, find the solution of $e^{\mathbf{A}t}$ by using Resolvent Method.

(solution)

$$e^{\left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right)t} = \mathscr{L}^{-1}\left\{\left(s\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) - \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right)\right)^{-1}\right\}$$

$$= \mathscr{L}^{-1}\left\{\left(\left(\begin{array}{cc} s-1 & 0 \\ -1 & s-1 \end{array}\right)\right)^{-1}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\left(\begin{array}{cc} s-1 & 0 \\ 1 & s-1 \end{array}\right)\right\}$$

$$= \mathscr{L}^{-1}\left\{\left(\begin{array}{cc} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{array}\right)\right\}$$

$$= \begin{pmatrix} e^t & 0 \\ t \cdot e^t & e^t \end{pmatrix}$$

$\square$

Now let's derive the time domain model for the sampled-data system.

1. Continuous state and output from LTI solution

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}(t-to)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau \\ y(t) &= \mathbf{c}^{\mathbf{T}}e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{c}^{\mathbf{T}}e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau \end{aligned}$$

2. The sampled solution(discrete state and output after sampling)

$$\mathbf{x}(kT) = e^{\mathbf{A}(kT-to)}\mathbf{x}(t_0) + \int_{t_0}^{kT} e^{\mathbf{A}(kT-\tau)}\mathbf{b}u(\tau)d\tau$$

Choosing $t_0 = (k-1)T$,

$$\mathbf{x}(kT) = e^{\mathbf{A}T}\mathbf{x}\left((k-1)T\right) + \int_{(k-1)T}^{kT} e^{\mathbf{A}(kT-\tau)}\mathbf{b} \underbrace{u(\tau)}_{u((k-1)T)} d\tau$$

If we choose k instead of kT in the argument, then

$$\mathbf{x}(k) = e^{\mathbf{A}T}\mathbf{x}(k-1) + \int_{(k-1)T}^{kT} e^{\mathbf{A}(kT-\tau)}\mathbf{b}d\tau \cdot u(k-1)$$

Let $v = \tau - (k-1)T$, then

$$\begin{aligned} \int_{(k-1)T}^{kT} e^{\mathbf{A}(kT-\tau)}\mathbf{b}d\tau &= \int_{0}^{T} e^{\mathbf{A}(T-v)}\mathbf{b}dv \qquad (T-v=\xi, \ -dv = d\xi) \\ &= \int_{T}^{0} e^{\mathbf{A}\xi}\mathbf{b}(-d\xi) \\ &= \int_{0}^{T} e^{\mathbf{A}\xi}\mathbf{b}d\xi \end{aligned}$$

In conclusion, we have

$$\mathbf{x}(k) = \underbrace{e^{\mathbf{A}T}}_{\mathbf{F}}\mathbf{x}(k-1) + \underbrace{\int_0^T e^{\mathbf{A}\xi}\mathbf{b}d\xi}_{\mathbf{g}} \cdot u(k-1)$$

or

$$\mathbf{x}(k+1) \quad = \quad \mathbf{F}\mathbf{x}(k) + \mathbf{g}u(k)$$

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{g}u(k) \\ y(k) = \mathbf{c}^T\mathbf{x}(k) \end{cases} \quad : \text{ discrete} - \text{time state equation}$$

where

$$\begin{cases} \mathbf{F} = e^{\mathbf{A}T} \\ \mathbf{g} = \int_0^T e^{\mathbf{A}\xi}\mathbf{b}d\xi \end{cases}$$

Example 1.5 (Satellite system)

When $G(s) = 1/s^2$, find the $G(z)$.

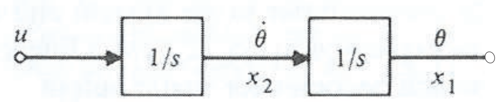

Figure 1.7: Satellite attitude control in classical representation.

(solution)

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{b}} u(t)$$

$$y(t) = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{c}^T} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

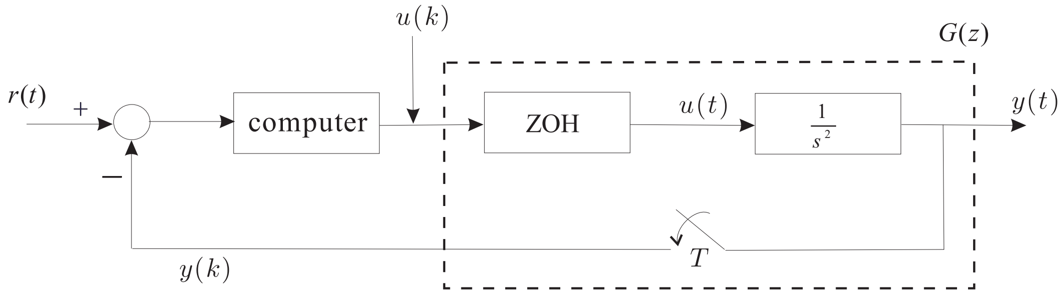

Figure 1.8: Simulation diagrams for Example 1.5.

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix}}_{\mathbf{g}} u(t)$$

$$y(k) = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{c}^T} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}$$

$$\mathbf{F} = e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} T} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g} = \int_0^T e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xi} \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\xi = \begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix}$$

and

$$
\begin{aligned}
G(z) &= \mathbf{c}^T(z\mathbf{I}-\mathbf{F})^{-1}\mathbf{g} \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix} \\
&= \frac{T^2}{2} \cdot \frac{z+1}{(z-1)^2}
\end{aligned}
$$

$\square$

1.2 Discrete-Time Systems

Continuous-Time System : Using differential equation

$$
\begin{aligned}
\frac{d^n y}{dt^n} &+ a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_n y \\
&= b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_m u
\end{aligned}
$$

Laplace Transform :

$$
X(s) = \int_0^\infty x(t) e^{-st} dt
$$

State Equation of Continuous Time Systems :

$$
\begin{cases}
\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\
y(t) = \mathbf{c}^T \mathbf{x}(t)
\end{cases}
$$

and if initial value is zero, then

$$\mathbf{G}(s) = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

Discrete-Time System : Using difference equation

$$\begin{aligned} y(k) &+ a_1 y(k-1) + a_2 y(k-2) + \cdots + a_n y(k-n) \\ &= b_1 u(k-1) + b_2 u(k-2) + \cdots + b_m u(k-m) \qquad (n > m) \end{aligned}$$

Here, representing

$$\begin{aligned} y(k+1) &= q \cdot y(k) \\ y(k-1) &= q^{-1} \cdot y(k) \end{aligned}$$

we have

$$\begin{aligned} (1 + a_1 q^{-1} + \; & a_2 q^{-2} \; + a_3 q^{-3} + \cdots + a_n q^{-n}) y(k) \\ &= (b_0 + b_1 q^{-1} + b_2 q^{-2} + \cdots + b_m q^{-m}) u(k) \end{aligned}$$

$$\frac{y(k)}{u(k)} = \frac{b_0 + b_1 q^{-1} + b_2 q^{-2} + \cdots + b_m q^{-m}}{1 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3} + \cdots + a_n q^{-n}}$$

z-Transform :

$$\mathscr{Z}[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k) \cdot z^{-k}$$

Then

$$\begin{aligned}
\mathscr{Z}[x(k+1)] &= z \cdot X(z) - zx(0) \\
\mathscr{Z}[x(k-1)] &= z^{-1} \cdot X(z) - z^{-1}x(0)
\end{aligned}$$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots + a_n z^{-n}}$$

State Equation of Discrete Time Systems :

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{g}u(k) \\ y(k) = \mathbf{c}^T\mathbf{x}(k) \end{cases}$$

z-Transform for this state equation

$$\begin{aligned}
& z\mathbf{X}(z) - \mathbf{x}(0) = \mathbf{F}\mathbf{X}(z) + \mathbf{g}U(z) \\
\Rightarrow \quad & (z\mathbf{I} - \mathbf{F})\mathbf{X}(z) = \mathbf{x}(0) + \mathbf{g}U(z)
\end{aligned}$$

$$\begin{cases} \mathbf{X}(z) = (z\mathbf{I} - \mathbf{F})^{-1}\mathbf{x}(0) + (z\mathbf{I} - \mathbf{F})^{-1}\mathbf{g}U(z) \\ Y(z) = \mathbf{c}^T(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{x}(0) + \mathbf{c}^T(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{g}U(z) \end{cases}$$

and if $\mathbf{x}(0) = 0$, then

$$\mathbf{G}(z) = \mathbf{c}^T(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{g}$$

Example 1.6 Find $\mathbf{F}$, $\mathbf{g}$, $\mathbf{c}$.

$$
\begin{aligned}
& y(k) + a_1 y(k-1) = b_1 u(k-1) \\
\Rightarrow\ & y(k) = -a_1 y(k-1) + b_1 u(k-1) \\
\Rightarrow\ & y(k+1) = -a_1 y(k) + b_1 u(k) \\
\Rightarrow\ & \begin{cases} x(k+1) = -a_1 x(k) + b_1 u(k) \\ y(k) = x(k) \end{cases}
\end{aligned}
$$

$$
\therefore \begin{cases} \mathbf{F} = -a_1 \\ \mathbf{g} = b_1 \\ \mathbf{c} = 1 \end{cases}
$$

□

Example 1.7 Find the state equation and the output equation.

(Solution)

$$
\begin{aligned}
& y(k) + a_1 y(k-1) + a_2 y(k-2) = b_1 u(k-1) + b_2 u(k-2) \\
\Rightarrow\ & y(k+2) + a_1 y(k+1) + a_2 y(k) = b_1 u(k+1) + b_2 u(k)
\end{aligned}
$$

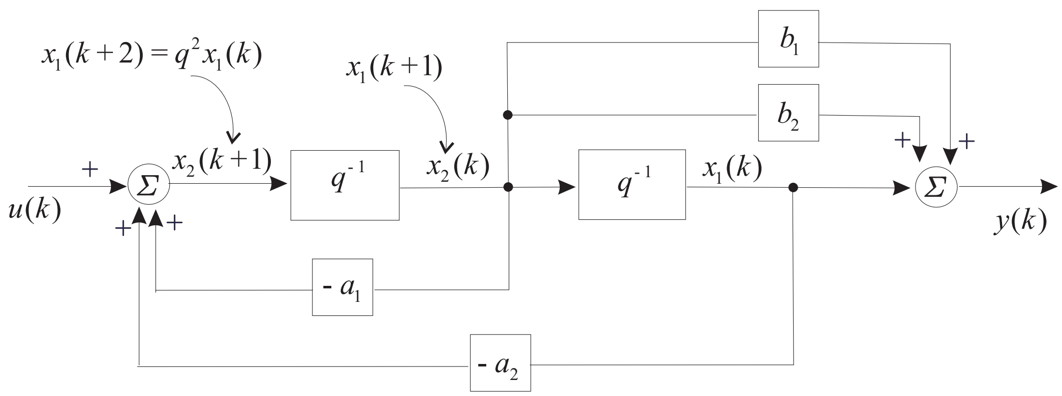

Figure 1.9: Simulation diagrams for Example 1.7.(controllable canonical form)

Choose $x_1(k)$ and $x_2(k)$ such that

$$\begin{aligned}(q^2 + a_1 q + a_2)x_1(k) &= u(k) \\ (q^2 + a_1 q + a_2)x_2(k) &= u(k+1)\end{aligned}$$

Then

$$\begin{aligned}y(k) &= b_1 x_2(k) + b_2 x_1(k) \\ &= \underbrace{\begin{pmatrix} b_2 & b_1 \end{pmatrix}}_{\mathbf{c}^T} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \qquad : \text{ output equation}\end{aligned}$$

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) \qquad : \text{ state equation}$$

□

Solution of the state equation :

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{g}u(k) \\ y(k) = \mathbf{c}^T\mathbf{x}(k) \end{cases}$$

$$\begin{aligned}
\mathbf{x}(k) &= \mathbf{F}\mathbf{x}(k-1) + \mathbf{g}u(k-1) \\
&= \mathbf{F}(\mathbf{F}\mathbf{x}(k-2) + \mathbf{g}u(k-2)) + \mathbf{g}u(k-1) \\
&= \mathbf{F}^2\mathbf{x}(k-2) + \mathbf{F}\mathbf{g}u(k-2) + \mathbf{g}u(k-1) \\
&= \mathbf{F}^3\mathbf{x}(k-3) + \mathbf{F}^2\mathbf{g}u(k-3) + \cdots \\
&\vdots \\
&= \mathbf{F}^k\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{F}^{k-j-1}\mathbf{g}u(j), \qquad k = 0, 1, 2, \cdots \\
y(k) &= \mathbf{c}^T\mathbf{x}(k) \\
&= \underbrace{\mathbf{c}^T\mathbf{F}^k\mathbf{x}(0)}_{\text{zero input solution}} + \underbrace{\sum_{j=0}^{k-1} \overbrace{\mathbf{c}^T\mathbf{F}^{k-j-1}\mathbf{g}}^{h(k-j)}\, u(j)}_{\text{zero state solution}}
\end{aligned}$$

Example 1.8 When $\mathbf{F} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, find the solution of $\mathbf{F}^{100}$ by using Cayely-Hamilton Theorem.

(solution)

$$\begin{aligned}
\Delta(\lambda) &= \lambda^2 - (a+d)\lambda + (ad - bc) = 0 \\
&= \lambda^2 - \lambda = 0 \\
\Delta(\mathbf{F}) &= \mathbf{F}^2 - \mathbf{F} = 0 \\
\Rightarrow \mathbf{F}^{100} &= \mathbf{F}
\end{aligned}$$

□

Stability :

As the continuous time system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is asymptotically stable iff all the eigenvalues of $\mathbf{A}$ stay in R_o^-(open left half plane).

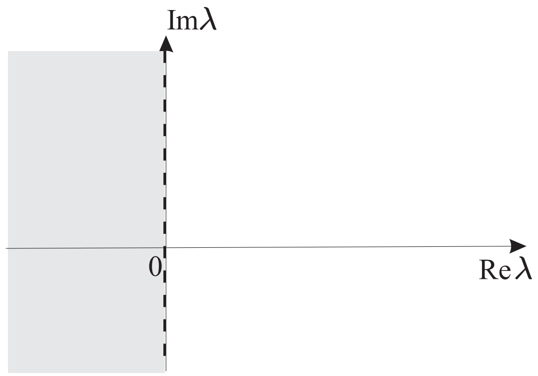

Figure 1.10: The stable region in the continuous time system(R_o^-).

[Z transform] can be derived from the Laplace Transform.

$$\begin{aligned}
x_T(t) &= \sum_{k=0}^{\infty} x(kT)\delta(t-kT) \qquad (\textit{sampling}) \\
\mathscr{L}\{x_T(t)\} &= \int_0^{\infty} e^{-st} x_t(t)dt = \int_0^{\infty} e^{-st} \sum_{k=0}^{\infty} x(kT)\delta(t-kT)dt \\
&= \sum_{k=0}^{\infty} x(kT)\int_0^{\infty} e^{-st}\delta(t-kT)dt = \sum_{k=0}^{\infty} x(kT)e^{-st\cdot k} = \sum_{k=0}^{\infty} x(kT)z^{-k} \\
\Rightarrow \quad & Z = e^{-st}
\end{aligned}$$

$$\begin{aligned}
Z &= e^{-st} = e^{(\sigma+j\omega)t} = e^{\sigma t}\cdot e^{j\omega t} \qquad (\sigma < 0 \qquad \textit{stable}) \\
|Z| &= |e^{\sigma t}|\cdot|e^{j\omega t}| = |e^{\sigma t}| < 1 \\
&\Rightarrow |Z| < 1 \qquad \textit{stable}
\end{aligned}$$

The discrete time system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k)$ is stable if all the eigenvalues of $\mathbf{F}$ stay in $|\lambda_i(\mathbf{F})| < 1$ for all i = 1,2,$\cdots$,n.

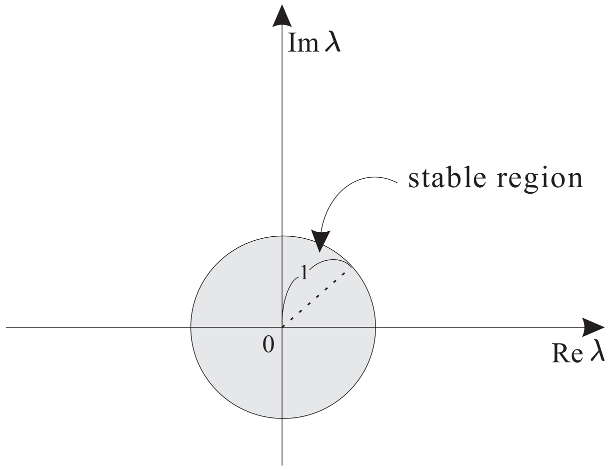

Figure 1.11: The stable region in the discrete time system.

Proof

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}$$
$$\Rightarrow \mathbf{x} = \mathbf{F}^k\mathbf{x}(0)$$

If this system is stable, then $|\lambda|^k \to 0$ as $k \to \infty$. And
$|\lambda|^{\mathbf{k}} \to \mathbf{0}$ **as** $\mathbf{k} \to \infty$ **if and only if** $|\lambda| < \mathbf{1}$**.**