Exercise 1:

$$\det(A) = \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2\det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3\det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$= 45 - 48 - 2(36 - 42) + 3(32 - 35)$$

$$= -3 + 2.6 - 3.3 = 0$$

A is singular

$$\det(B) = 2\det\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} + 2\det\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

$$= 2.1 + 2.5 + 1 = 13$$

B is well conditioned

$$\det(C) = \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2.0001\det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3\det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$= -3 + 2.0001.6 - 9$$

$$= 0.0006$$

$\det(C) \ll \max(c_{ij}) \Rightarrow$ C is ill-conditioned

<u>Exercise 2:</u>

$$A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 0 & -1 \\ 4 & 1 & 1 \end{pmatrix}$$

Dolittle's decomposition is obtained through Gauss elimination by storing the multipliers in the lower part of A.

$$(2) \leftarrow (2) - \frac{1}{2}(1)$$

$$\begin{pmatrix} 2 & -2 & 1 \\ \boxed{\frac{1}{2}} & 1 & -3/2 \\ 4 & 1 & 1 \end{pmatrix}$$

$$(3) \leftarrow (3) - 2(1)$$

$$\begin{pmatrix} 2 & -2 & 1 \\ \boxed{\frac{1}{2}} & 1 & -3/2 \\ \boxed{2} & 5 & -1 \end{pmatrix}$$

$$(3) \leftarrow (3) - 5(1)$$

$$[L\backslash U] = \begin{pmatrix} 2 & -2 & 1 \\ \boxed{\frac{1}{2}} & 1 & -3/2 \\ \boxed{2} & \boxed{5} & 13/2 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix} ; \quad U = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & -3/2 \\ 0 & 0 & 13/2 \end{pmatrix}$$

Exercise 3:

1. Solve $Ly = b$ by forward substitution:

$$y_1 = 1$$

$$\frac{1}{2} y_1 + y_2 = 2 \Rightarrow y_2 = \frac{3}{2}$$

$$2 y_1 + 5 y_2 + y_3 = 0 \Rightarrow y_3 = -\frac{19}{2}$$

2. Solve $Ux = y$ by back substitution:

$$\frac{13}{2} x_3 = -\frac{19}{2} \Rightarrow x_3 = -\frac{19}{13}$$

$$x_2 - \frac{3}{2} x_3 = \frac{3}{2} \Rightarrow x_2 = -\frac{9}{13}$$

$$2x_1 - 2x_2 + x_3 = 1 \Rightarrow x_1 = \frac{1}{2}\left(1 - \frac{18}{13} + \frac{19}{13}\right) = \frac{7}{13}$$

Exercise 4:

$$[A|B] = \left(\begin{array}{ccc|cc} 0 & 1 & 4 & -1 & 3 \\ 2 & 0 & 7 & 6 & 2 \\ 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

row 1 needs to be moved, for instance swapped with row 2:

$$[A|B] = \left(\begin{array}{ccc|cc} 2 & 0 & 7 & 6 & 2 \\ 0 & 1 & 4 & -1 & 3 \\ 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$(3) \leftarrow (3) - \frac{1}{2}(1)$

$$\left(\begin{array}{ccc|cc} 2 & 0 & 7 & 6 & 2 \\ 0 & 1 & 4 & -1 & 3 \\ 0 & 4 & -\frac{7}{2} & -3 & 0 \end{array}\right)$$

$(3) \leftarrow (3) - 4 \cdot (2)$

$$\left(\begin{array}{ccc|cc} 2 & 0 & 7 & 6 & 2 \\ 0 & 1 & 4 & -1 & 3 \\ 0 & 0 & -\frac{39}{2} & 1 & -12 \end{array}\right)$$

Backward substitution:

$-\frac{39}{2} b_{31} = 1 \Rightarrow b_{31} = \frac{-2}{39}$

$b_{21} + 4 b_{31} = -1 \Rightarrow b_{21} = -\frac{31}{39}$

$2 b_{11} + 7 b_{31} = 6 \Rightarrow b_{11} = \frac{124}{39}$

$-\frac{39}{2} b_{32} = -12 \Rightarrow b_{32} = \frac{8}{13}$

$b_{22} + 4 b_{32} = 3 \Rightarrow b_{22} = \frac{7}{13}$

$2 b_{12} + 7 b_{32} = 2 \Rightarrow b_{12} = -\frac{15}{13}$

Exercise 5: $A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 0 & -1 \\ 4 & 1 & 1 \end{pmatrix}$

$$\text{cond}(A) = \|A\|_\infty . \|A^{-1}\|_\infty$$
$$= 6 . \|A^{-1}\|_\infty$$

We need to find A^{-1}. We could do it by solving $AX = I$ by Gauss elimination. But we know the LU decomposition of A from exercise 2:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix} \quad ; \quad U = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & -3/2 \\ 0 & 0 & 13/2 \end{pmatrix}$$

~~and in exercise 3 we have solved L~~

So we will solve $LUX = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$:

1. Solve $LUx = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$:

1.1 Solve $Ly = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$:

By forward substitution: $y_1 = 1$

$y_2 = -\frac{1}{2}$

$y_3 = \frac{1}{2}$

1.2 Solve $Ux = y = \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}$

By back substitution: $x_3 = \frac{1}{13}$

$x_2 = -\frac{5}{13}$

$x_1 = \frac{1}{13}$

2. Solve $LUx = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

2.1. Solve $Ly = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$y_1 = 0$

$y_2 = 1$

$y_3 = -5$

2.2 Solve $Ux = y = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$

$x_3 = -\frac{10}{13}$; $x_2 = -\frac{2}{13}$; $x_1 = \frac{3}{13}$

3. Solve $LUx = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

3.1. Solve $Ly = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$y_1 = 0$

$y_2 = 0$

$y_3 = 1$

3.2. Solve $Ux = y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$x_3 = \frac{2}{13}$; $x_2 = \frac{3}{13}$; $x_1 = \frac{2}{13}$

Finally: $A^{-1} = \begin{pmatrix} \frac{1}{13} & \frac{3}{13} & \frac{2}{13} \\ -\frac{5}{13} & -\frac{2}{13} & \frac{3}{13} \\ \frac{1}{13} & -\frac{10}{13} & \frac{2}{13} \end{pmatrix}$ and $\|A^{-1}\|_\infty = 1$

$\Rightarrow \underline{\underline{\text{cond}(A) = 6}}$

Exercise 6:

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 5 & 8 & 9 \end{pmatrix}$$

We will solve $AX = I$ by Gauss elimination.

$$[A|I] = \left(\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 5 & 8 & 9 & 0 & 0 & 1 \end{array}\right)$$

$(2) \longleftarrow (2) - \frac{1}{3}(1)$ and $(3) \longleftarrow (3) - \frac{5}{3}(1)$

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2/3 & -2/3 & -1/3 & 1 & 0 \\ 0 & \frac{19}{3} & \frac{17}{3} & -5/3 & 0 & 1 \end{array}\right)$$

$(3) \longleftarrow (3) - \frac{19}{2}(2)$

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2/3 & -2/3 & -1/3 & 1 & 0 \\ 0 & 0 & 12 & 3/2 & -\frac{19}{2} & 1 \end{array}\right)$$

Back substitution:

$12\, x_{31} = \frac{3}{2}$; $x_{31} = \frac{1}{8}$

$\frac{2}{3} x_{21} = -\frac{1}{3} + \frac{2.1}{3.8}$; $x_{21} = -\frac{3}{8}$

$x_{11} = \frac{1}{3}\left(1 + \frac{3}{8} - \frac{2}{8}\right) = \frac{3}{8}$

$12\, x_{32} = -\frac{19}{2}$; $x_{32} = -\frac{19}{24}$

$x_{22} = \frac{3}{2}\left(1 - \frac{2}{3}\frac{19}{24}\right) = \frac{17}{24}$

$x_{12} = \frac{1}{3}\left(0 - \frac{17}{24} + 2 \cdot \frac{19}{24}\right) = \frac{7}{24}$

$12\, x_{33} = 1$; $x_{33} = \frac{1}{12}$

$x_{23} = \frac{3}{2}\left(0 + \frac{2}{3} \cdot \frac{1}{12}\right) = \frac{1}{12}$

$x_{13} = \frac{1}{3}\left(0 - \frac{1}{12} - \frac{2}{12}\right) = -1/12$

Finally: $A^{-1} = \begin{pmatrix} \frac{3}{8} & \frac{7}{24} & -\frac{1}{12} \\ -\frac{3}{8} & \frac{17}{24} & \frac{1}{12} \\ \frac{1}{8} & -\frac{19}{24} & \frac{1}{12} \end{pmatrix}$

Exercice 7: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

$A = LL^T = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

identification of column 1

$\Rightarrow L_{11} = 1$

$L_{11}L_{21} = 1 \Rightarrow L_{21} = 1$

$L_{11}L_{31} = 1 \Rightarrow L_{31} = 1$

column 2:

$L_{21}^2 + L_{22}^2 = 2 \Rightarrow L_{22} = 1$

$1 + L_{22}L_{32} = 2 \Rightarrow L_{32} = 1$

column 3:

$\underbrace{L_{31}^2}_{=1} + \underbrace{L_{32}^2}_{=1} + L_{33}^2 = 3$

$\Rightarrow L_{33} = 1$

$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$