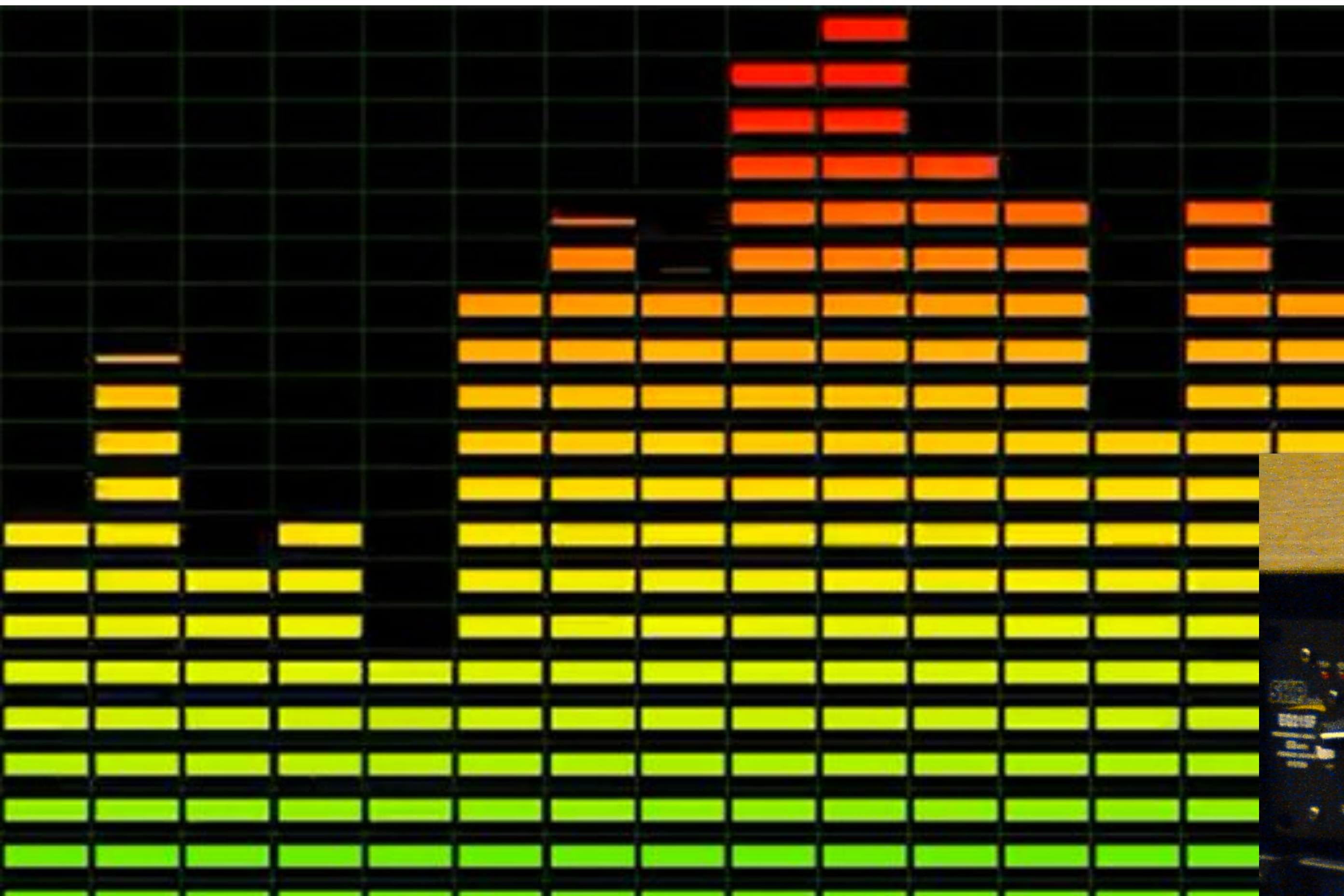
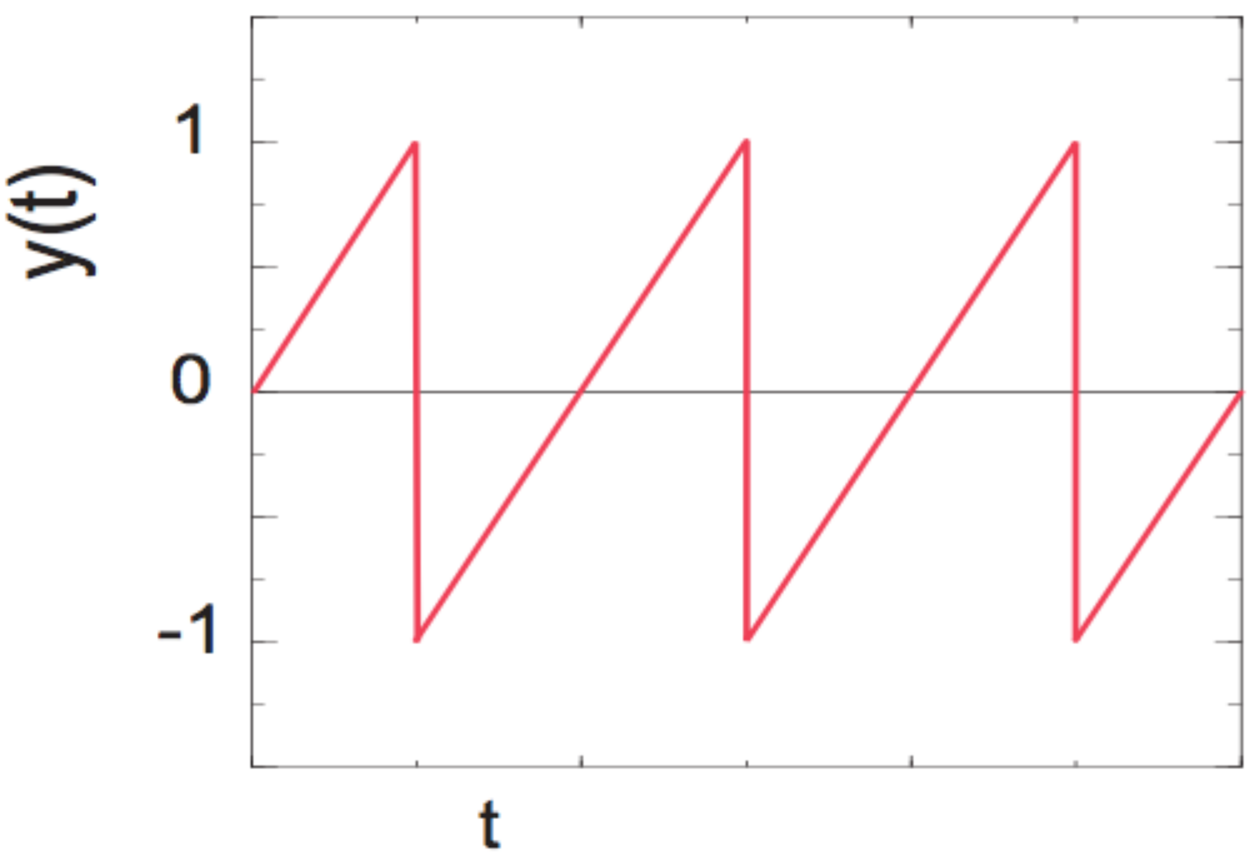
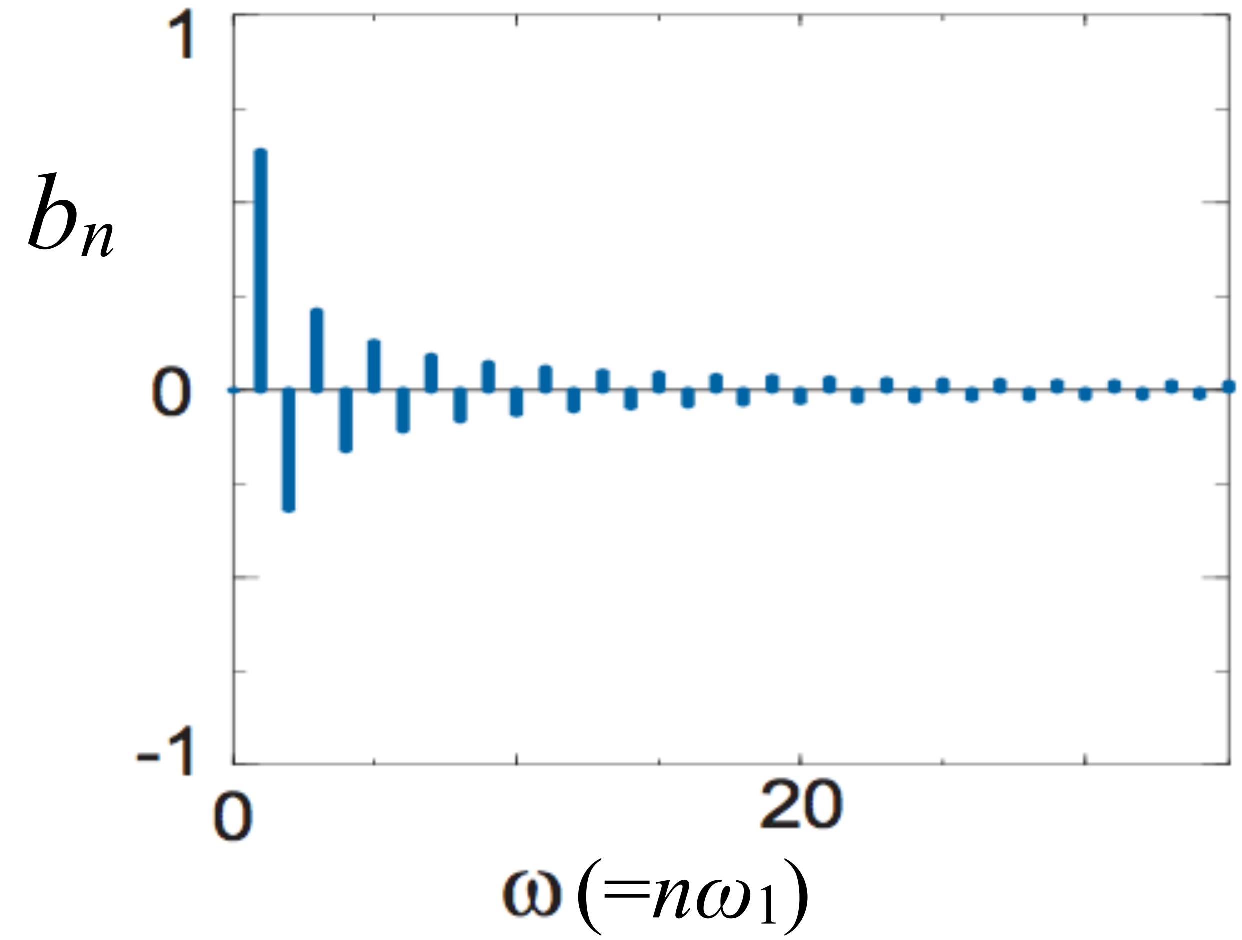
Sound Equalizer



Fourier Series





$$y(t+T) = y(t)$$

$$\omega_1 = \frac{2\pi}{T}$$

$$y(t) = b_1 sin(\omega_1 t) + b_2 sin(2\omega_1 t) + \ldots$$

In general,

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_1 t) + b_n \sin(n\omega_1 t)$$

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{2}{T}\int_0^T dt \begin{pmatrix} \cos n\omega t \\ \sin n\omega t \end{pmatrix} y(t), \quad \omega \stackrel{\text{def}}{=} \frac{2\pi}{T}$$

Fourier Series

$$a_0 = 2\,\langle y(t) \rangle \qquad\qquad a_n = \frac{4}{T}\int_0^{T/2} dt\; y(t)\cos n\omega t$$

$$b_n = \frac{4}{T}\int_0^{T/2} dt\; y(t)\sin n\omega t$$

Example:

$$y(t) = \begin{cases} \frac{t}{T/2}, & \text{for } 0 \le t \le \frac{T}{2}, \\ \frac{t-T}{T/2}, & \text{for } \frac{T}{2} \le t \le T. \end{cases}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} dt\; \sin n\omega t \frac{t}{T/2} = \frac{\omega}{\pi}\int_{-\pi/\omega}^{+\pi/\omega} dt\; \sin n\omega t \frac{\omega t}{\pi} = \frac{2}{n\pi}(-1)^{n+1}$$

$$y(t) = \frac{2}{\pi}\left[\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \cdots\right].$$

Fourier Transform

$$y(t) = \int_{-\infty}^{+\infty} d\omega\, Y(\omega)\, \frac{e^{i\omega t}}{\sqrt{2\pi}} \qquad Y(\omega) = \int_{-\infty}^{+\infty} dt\, \frac{e^{-i\omega t}}{\sqrt{2\pi}} y(t)$$

Discrete Fourier Transform

$$y_k \stackrel{\text{def}}{=} y(t_k), \qquad k = 1, 2, \ldots, N,$$

$$t_k \stackrel{\text{def}}{=} kh, \qquad h = \Delta t.$$

$$T \stackrel{\text{def}}{=} Nh, \quad s = \frac{N}{T} = \frac{1}{h} \qquad \omega_1 = \frac{2\pi}{T},$$

(sampling rate)

Discrete Fourier Transform

$$\omega_1 = \frac{2\pi}{T}, \qquad \omega_n = n\omega_1 = n\frac{2\pi}{Nh}, \qquad n = 1, \ldots, N$$

$$Y(\omega_n) = \int_{-\infty}^{+\infty} dt\, \frac{e^{-i\omega_n t}}{\sqrt{2\pi}} y(t) \simeq \int_0^T dt\, \frac{e^{-i\omega_n t}}{\sqrt{2\pi}} y(t),$$

$$\simeq \sum_{k=1}^{N} h\, y(t_k) \frac{e^{-i\omega_n t_k}}{\sqrt{2\pi}} = h \sum_{k=1}^{N} y_k \frac{e^{-2\pi i k n/N}}{\sqrt{2\pi}}.$$

$$Y_n \stackrel{\text{def}}{=} \frac{1}{h} Y(\omega_n) = \sum_{k=1}^{N} y_k \frac{e^{-2\pi i k n/N}}{\sqrt{2\pi}}$$

Remember:

$$\omega_n = n\frac{2\pi}{T} = n\frac{2\pi}{Nh}$$

$$t_k = kh$$

$$y(t) = \int_{-\infty}^{+\infty} d\omega\, \frac{e^{i\omega t}}{\sqrt{2\pi}} Y(\omega) \simeq \sum_{n=1}^{N} \frac{2\pi}{Nh} \frac{e^{i\omega_n t}}{\sqrt{2\pi}} Y(\omega_n)$$

$$= \sum_{n=1}^{N} \frac{2\pi}{N} \frac{Y(\omega_n)}{h} \frac{e^{2\pi i k n/N}}{\sqrt{2\pi}} \qquad = \frac{1}{N} \sum_{n=1}^{N} \sqrt{2\pi}\, Y_n e^{2\pi i k n/N} = y_k$$

Two (Sensible) Changes

Change #1

$$Y_n = \sum_{k=1}^{N} y_k \frac{e^{-2\pi ikn/N}}{\sqrt{2\pi}} = \sum_{k=0}^{N-1} y_k \frac{e^{-2\pi ikn/N}}{\sqrt{2\pi}}$$

The legitimacy: $e^{-2\pi ikn/N} = 1$, for $k = 0$ or N,

and $y_0 = y_N$

Remember: $y(0) = y(T)$, because $y(t)$ is (implicitly) assumed to be a periodic function, and therefore,

In general $\boxed{y_k = y_{k+N}}$ $\left(y(t) = y(t+T)\right)$

It's also clear: $\boxed{Y_n = Y_{n+N}}$ $\left(e^{-2\pi ikn/N} = e^{-2\pi i(k+N)n/N}\right)$

Two (Sensible) Changes

Change #2

$$y_k = \frac{1}{N}\sum_{n=1}^{N}\sqrt{2\pi}Y_n e^{2\pi ikn/N}$$

We know $e^{-2\pi ikn/N} = e^{-2\pi i(k+N)n/N}$ and $Y_n = Y_{n+N}$

→ The terms in the summation sign have a periodicity of *N*.
→ We can start the sum from anywhere as long as we sum over *N* consecutive terms.

Assume *N* is even $$y_k = \frac{1}{N}\sum_{n=-(N/2-1)}^{N/2}\sqrt{2\pi}Y_n e^{2\pi ikn/N}$$

Accounting: $N/2$ +ve terms, ($N/2 - 1$) -ve terms, plus $k = 0$ term:
A total of N terms!

Final Cosmetic Change

$$Y_n = \sum_{k=0}^{N-1} y_k \frac{e^{-2\pi ikn/N}}{\sqrt{2\pi}} \qquad y_k = \frac{1}{N} \sum_{n=-(N/2-1)}^{N/2} \sqrt{2\pi} Y_n e^{2\pi ikn/N}$$

$$Y_n' = \frac{\sqrt{2\pi} Y_n}{N} \qquad Y_n = \frac{N Y_n'}{\sqrt{2\pi}}$$

$$Y_n' = \frac{1}{N} \sum_{k=0}^{N-1} y_k e^{-2\pi ikn/N} \qquad y_k = \sum_{n=-(N/2-1)}^{N/2} Y_n' e^{2\pi ikn/N}$$

Dropping the prime:

$$\boxed{Y_n = \frac{1}{N} \sum_{k=0}^{N-1} y_k e^{-2\pi ikn/N} \qquad y_k = \sum_{n=-(N/2-1)}^{N/2} Y_n e^{2\pi ikn/N}}$$

DFT Inverse DFT

The Meaning of the 0-Frequency Component

$$\boxed{Y_n = \frac{1}{N}\sum_{k=0}^{N-1} y_k e^{-2\pi ikn/N} \qquad y_k = \sum_{n=-(N/2-1)}^{N/2} Y_n e^{2\pi ikn/N}}$$

$$Y_0 = \frac{1}{N}\sum_{k=0}^{N-1} y_k$$

The zeroth Fourier component is the average of the signal!

Switching from angular frequency to frequency:

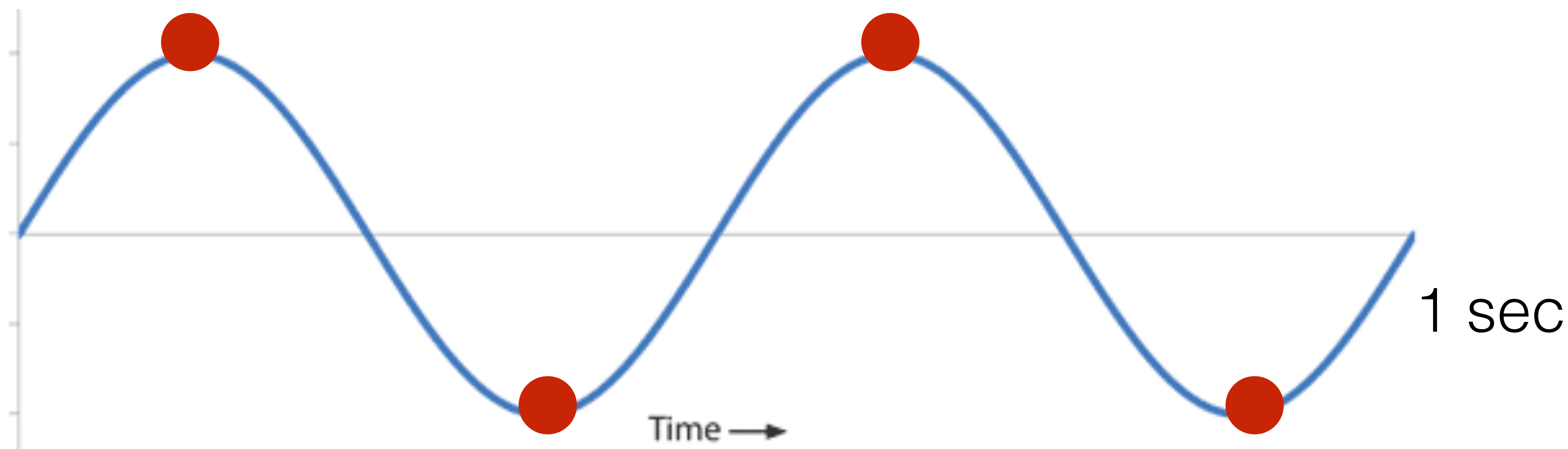
$$\omega_n = n\frac{2\pi}{T} = n\frac{2\pi}{Nh} \qquad \rightarrow f_n = n\frac{1}{T} = n\frac{1}{Nh}$$

Given sampling interval h, you may think the highest detectable frequency is $f_N = 1/h$ … and you'd be wrong!

Highest Detectable Frequency—Nyquist Frequency

- To detect a sine wave with frequency f, you need a sample rate of $2f$:

An oscillation has to have one up and one down.



E.g., for a **2 Hz** signal, one needs to sample **4** times per second!

- Conversely, if the sampling rate is $1/h$, the highest frequency you can possibly detect is $1/(2h)$; this is called the *Nyquist Frequency*

- Thus for DFT, given a sampling rate $s = 1/h$, the highest frequency you need to include is the Nyquist frequency, $\boldsymbol{f_{Ny} = 1/(2h)}$.

Nyquist Frequency

$$\boxed{Y_n = \frac{1}{N}\sum_{k=0}^{N-1} y_k e^{-2\pi ikn/N} \qquad y_k = \sum_{n=-(N/2-1)}^{N/2} Y_n e^{2\pi ikn/N}}$$

In y_k the exponent really is $\quad \underbrace{i(kh)}_{t_k}\underbrace{\left(2\pi\frac{n}{Nh}\right)}_{\omega_n}$

The highest frequency (at $n = N/2$) is $\quad f_{N/2} = \frac{1}{2h} = f_{Ny}$

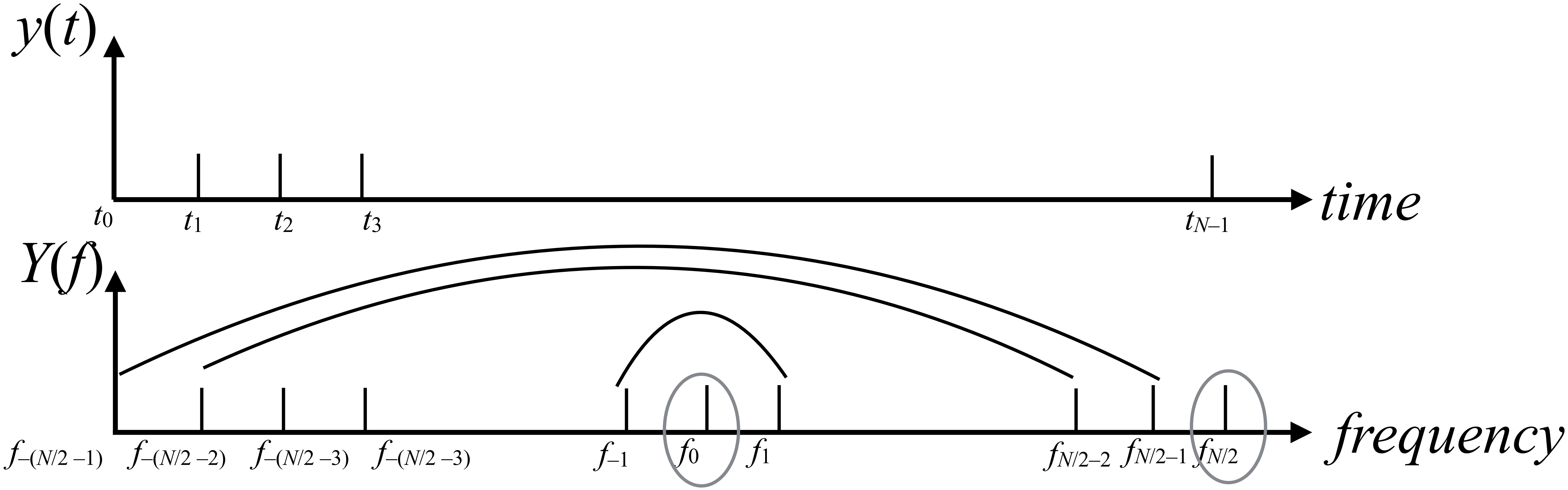
Because the Nyquist term ($Y_{N/2}$) is unpaired, it has to be real for a real signal.

Preservation of the Degrees of Freedom

Total number of t_k's: N

Total number of f_n's: N

DFT can NOT add or lose information!



FFT in NUMPY: numpy.fft

https://docs.scipy.org/doc/numpy-1.14.0/reference/routines.fft.html

The values in the result follow so-called “standard” order: If `A = fft(a, n)`, then `A[0]` contains the zero-frequency term (the sum of the signal), which is always purely real for real inputs. Then `A[1:n/2]` contains the positive-frequency terms, and `A[n/2+1:]` contains the negative-frequency terms, in order of decreasingly negative frequency. For an even number of input points, `A[n/2]` represents both positive and negative Nyquist frequency, and is also purely real for real input.

Thus, we expect $N/2$ +ve frequency terms, and $(N/2 - 1)$ -ve frequency terms, and 1 term with zero-frequency $\rightarrow$ N terms in total.