Topic

Sparse Matrix and the Jacobi Method

The Jacobi Method

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n \end{aligned}$$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

The matrix A has no zeros on its diagonal: $a_{ii} \neq 0$.

If all off-diagonal elements are 0: $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

If not:

$$\boxed{\begin{aligned} x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n) \\ x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n) \\ &\vdots \\ x_n &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1}) \end{aligned}}$$

$$\boxed{\overbrace{\begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & & \\ a_{31} & a_{32} & 0 & & \\ \vdots & & & \ddots & \\ a_{n1} & & & & 0 \end{bmatrix}}^{R} \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}^{\boldsymbol{x}}}$$

$$A = D + R$$

$$\begin{aligned} \boldsymbol{x} &= D^{-1}\,\boldsymbol{b} + D^{-1} R\boldsymbol{x} \\ &= D^{-1}\,(\boldsymbol{b} + R\boldsymbol{x}) \end{aligned}$$

(Remember for R, all diagonal elements are 0).

If A is "sparse", i.e., most of $a_{ij} = 0$, for $i \neq j$ (i.e., R is mostly 0):

$$\boldsymbol{x} \cong D^{-1}\,\boldsymbol{b}$$

So you can start with ***x_guess*** $= D^{-1}$ **b**, which is close to the solution… then $D^{-1} R$ ***x_guess*** is approximately the correction you need to get even closer to the solution; then you can iterate:

$$\boldsymbol{x}^{(k+1)} \cong D^{-1}\,(\boldsymbol{b} + R\boldsymbol{x}^{(k)})$$

Making it concrete:

$$\boldsymbol{x}^{(k+1)} \cong D^{-1} \, (\boldsymbol{b} + R\boldsymbol{x}^{(k)})$$

$$x_1^{(k+1)} = \frac{1}{a_{11}} (b_1 - r_{12} \, x_2^{(k)} - r_{13} \, x_3^{(k)} - \cdots r_{1n} \, x_n^{(k)})$$

$$\boldsymbol{r}_1 = \{r_{1j}\}$$

(Remember: $r_{11} = 0$.)

$$x^{(k+1)}{}_1 \cong 1/a_{11} \, (b_1 + \boldsymbol{r}_1 \cdot \boldsymbol{x}^{(k)})$$

In general:

$$x^{(k+1)}{}_i \cong 1/a_{ii} \, (b_i + \boldsymbol{r}_i \cdot \boldsymbol{x}^{(k)})$$

Now, numerical implementation…