## **Topic**

Sparse Matrix and the Jacobi Method

## The Jacobi Method

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
  
 $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$   
 $\vdots$   
 $a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$ 

$$Ax = b$$

The matrix A has no zeros on its diagonal:  $a_{ii} \neq 0$ .

If all off-diagonal elements are 0:  $\mathbf{x} = A^{-1}\mathbf{b}$ 

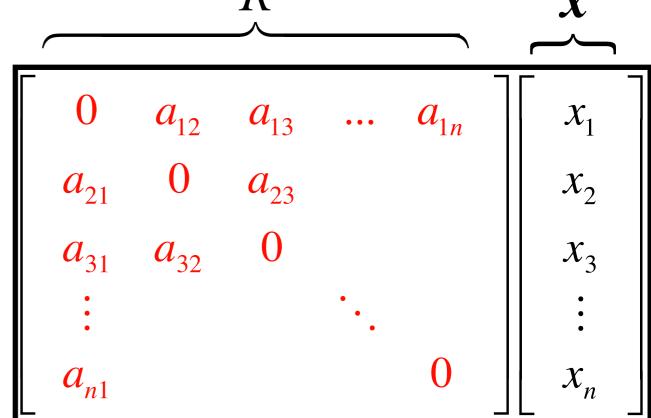
If not:

$$x_{1} = \frac{1}{a_{11}} (b_{1} - a_{12}x_{2} - a_{13}x_{3} - \cdots a_{1n}x_{n})$$

$$x_{2} = \frac{1}{a_{22}} (b_{2} - a_{21}x_{1} - a_{23}x_{3} - \cdots a_{2n}x_{n})$$

$$\vdots$$

$$x_{n} = \frac{1}{a_{nn}} (b_{n} - a_{n1}x_{1} - a_{n2}x_{2} - \cdots a_{n,n-1}x_{n-1})$$



$$A = D + R$$
  $x = D^{-1} b + D^{-1} Rx$   
=  $D^{-1} (b + Rx)$ 

(Remember for *R*, all diagonal elements are 0).

If *A* is "sparse", i.e., most of  $a_{ij} = 0$ , for  $i \neq j$  (i.e., *R* is mostly 0):

$$\boldsymbol{x} \cong D^{-1} \boldsymbol{b}$$

So you can start with  $x_guess = D^{-1}$  b, which is close to the solution... then  $D^{-1}Rx_guess$  is approximately the correction you need to get even closer to the solution; then you can iterate:

$$x^{(k+1)} \cong D^{-1} (b + Rx^{(k)})$$

## Making it concrete:

$$\mathbf{x}^{(k+1)} \cong D^{-1} (\mathbf{b} + R\mathbf{x}^{(k)})$$

$$x_1^{(k+1)} = \frac{1}{a_{11}} (b_1 - r_{12} x_2^{(k)} - r_{13} x_3^{(k)} - \cdots r_{1n} x_n^{(k)})$$

$$r_1 = \{r_{1j}\}$$
 (Remember:  $r_{11} = 0$ .)

$$x^{(k+1)}_1 \cong 1/a_{11} (b_1 + r_1 \cdot x^{(k)})$$

In general:

$$x^{(k+1)}_i \cong 1/a_{ii} \left(b_i + \mathbf{r}_i \cdot \mathbf{x}^{(k)}\right)$$

Now, numerical implementation...