Bayesian Statistics HW-2

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Exercise 1

k-out-of-n and Weibull Lifetime
$\mathbb{P}(T \geq t) = e^{-\lambda t^r}, t \geq 0, n = 8, r = \frac{3}{2}, \lambda = \frac{1}{10}$
(a) At time $t = 3$, for any one of the component, the probability that it works is given by

$$P_3 = \mathbb{P}(T \geq 3) = e^{-\frac{1}{10} \cdot 3^{\frac{3}{2}}}$$

The probability that the 4-out-of-8 system is still operational is given by

$$\begin{aligned} Q_1 &= \binom{8}{8} P_3^8 (1-P_3)^0 + \binom{8}{7} P_3^7 (1-P_3)^1 + \binom{8}{6} P_3^6 (1-P_3)^2 + \binom{8}{5} P_3^5 (1-P_3)^3 + \binom{8}{4} P_3^4 (1-P_3)^4 \\ &= 0.818094 \end{aligned}$$

(a) At time $t = 3$, the unconditional probability that there are exactly 5 components operational is given by

$$Q = \binom{8}{5} P_3^5 (1-P_3)^3 = 0.277351$$

Thus conditional on that the system was found operational at time t=3, the probability that exactly 5 components were operational is given by

$$Q_2 = \frac{Q}{Q_1} = 0.339021$$

Exercise 2

Precision of Lab Measurements

$$f(x) = \begin{cases} \frac{3x^2}{16}, & -2 \leq x \leq 2 \\ 0, & \text{else} \end{cases}$$

(a) The probability that a randomly chosen measurement can be classified as accurate is given by

$$Q_1 = \int_{-0.5}^{0.5} \frac{3x^2}{16} dx = 0.015625 = \frac{1}{64}$$

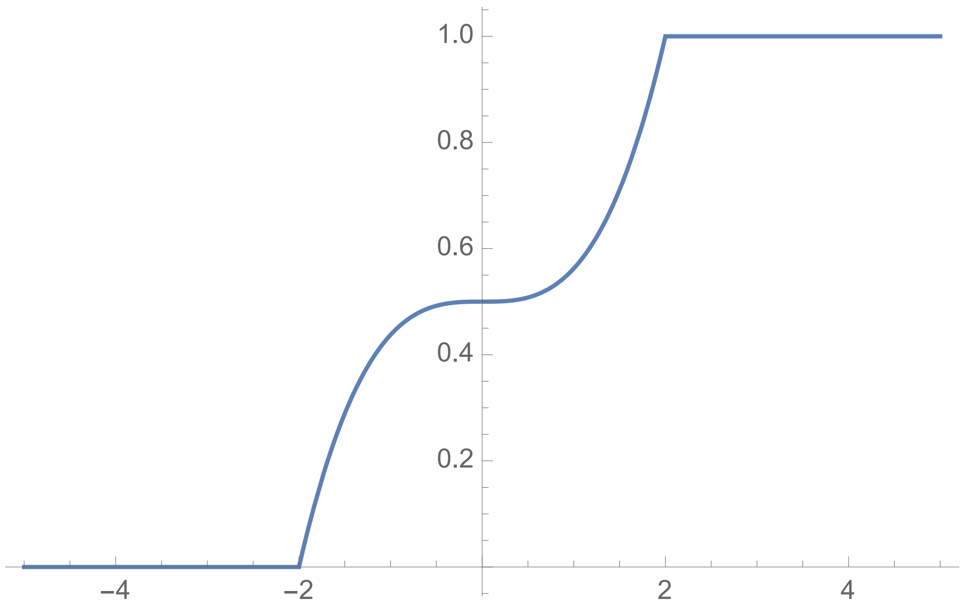
(b) For $x \in [-2, 2]$, the cumulative distribution is given by

$$F(x) = \int_{-2}^{x} f(u)du = \int_{-2}^{x} \frac{3u^2}{16} du = \frac{3}{16}(\frac{8}{3} + \frac{x^3}{3})$$

Thus the cumulative distribution function is given by

$$F(x) = \begin{cases} 0, & x \leq -2 \\ \frac{3}{16}(\frac{8}{3} + \frac{x^3}{3}), & -2 < x < 2 \\ 1, & x \geq 2 \end{cases}$$

Below is the plot of cumulative distribution function between [-5,5] (c) The mean of Y is given



by

$$\mathbb{E}[Y] = \int_{-2}^{2} x^2 f(x)dx = \int_{-2}^{2} \frac{3x^4}{16} dx = \frac{12}{5} = 2.4$$

Thus the mean of Y is $\frac{12}{5}$, representing an expected loss of $\frac{12}{5}$ thousand dollars. [1]

(d) The probability that the loss is less than \$3 is equivalent to the probability that $X^2 \leq \frac{3}{1000}$

$$\mathbb{P} = \int_{-\sqrt{\frac{3}{1000}}}^{\sqrt{\frac{3}{1000}}} \frac{3x^2}{16} dx = \frac{3\sqrt{\frac{3}{10}}}{80000} = 2.05396 \cdot 10^{-5}$$

Or if we mean less than \$3000 actually, the probability is equivalent to the probability that $X^2 \leq 3$, which is

$$\mathbb{P} = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{3x^2}{16} dx = \frac{3\sqrt{3}}{8} = 0.649519$$

[1] In case we still follow the definition from (a) (this is something brought up at edstem and I do not really like this definition...) and ignore the loss caused by measurement error when the measurement is considered to be accurate (i.e. $|X| < 0.5$), the mean will be given by

$$\mathbb{E}[Y] = \int_{-2}^{-0.5} \frac{3x^4}{16} dx + \int_{0.5}^{2} \frac{3x^4}{16} dx = 2.39766$$

Exercise 3

2-D Density Tasks

$$f(x,y) = \begin{cases} x+y, & 0 \le x \le 1; 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$

(a) For $x \in [0,1]$, the marginal distribution is given by

$$f_x(x) = \int_0^1 (x+y)dy = x + \frac{1}{2}$$

For $x \notin [0,1]$, the marginal distribution is 0. Thus

$$f_x(x) = \begin{cases} x + \frac{1}{2}, & 0 \le x, y \le 1 \\ 0, & \text{else} \end{cases}$$

(b) When $x \in [0,1]$, the conditional density $f(y|x) = \frac{x+y}{x+\frac{1}{2}}$

$$f(y|x) = \begin{cases} \text{Not possible}, & x \notin [0,1] \\ \frac{x+y}{x+\frac{1}{2}}, & x \in [0,1], y \in [0,1] \\ 0, & x \in [0,1], y \notin [0,1] \end{cases}$$

Exercise 4

From the first page of Rand's book

The likelihood of the sample is given by

$$f(u_1, ..., u_{34}) = \Pi_{i=1}^{34} \frac{1}{\theta} \mathbb{I}_{\theta > u_i} = \theta^{-34} \mathbb{I}_{\theta > M}, M = 0.54876$$

Prior on θ is given by $\pi(\theta) = \frac{1}{\theta} \mathbb{I}_{\theta > 0}$. Thus the joint distribution:

$$\begin{aligned} \pi(\theta | \{u_1, ..., u_{34}\}) &\propto \theta^{-34} \mathbb{I}_{\theta > M} \cdot \frac{1}{\theta} \mathbb{I}_{\theta > 0} \\ &= \theta^{-35} \mathbb{I}_{\theta > M} \end{aligned}$$

Given that posterior belongs to Pareto family with density $\frac{\alpha c^{\alpha}}{\theta^{\alpha+1}} \mathbb{I}_{\theta > c}$, we have $c = max\{M, 0\} = 0.54876$, $\alpha = 34$ and the posterior density is $\frac{34 \cdot 0.54876^{34}}{\theta^{35}} \mathbb{I}_{\theta > 0.54876}$

(b) As the expectation of the Pareto distribution is $\frac{\alpha \cdot c}{\alpha - 1} = \frac{34 \cdot 0.54876}{33} = 0.565389$. With CDF function to be $F(\theta) = [1 - (\frac{c}{\theta})^{\alpha}] \mathbb{I}_{\theta > c}$, the left cutoff of the equitailed credible set is given by $[1 - (\frac{c}{\theta_L})^{\alpha}] \mathbb{I}_{\theta_L > c} = 0.025$, thus $\theta_L = 0.549169$. Similarly, the right cutoff of the equitailed credible set is given by $[1 - (\frac{c}{\theta_H})^{\alpha}] \mathbb{I}_{\theta_H > c} = 0.975$, thus $\theta_H = 0.611648$. Thus the 95% credit set is given by [0.549169,0.611648] The true value of parameter is in the credible set.

Appendix Mathematica code for Q2:

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Plot[Piecewise[{{0, x < -2}, {3/16 (8/3 + x3̂/3), x >= -2 && x <= 2}, {1, x > 2}}], {x, -5, 5}]
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