1 Traffic.

(a) As we know that $X|\theta \sim \mathcal{P}oi(\theta)$ (i.e., $f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}$), and the likelihood can be found as

$$f(\boldsymbol{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \left(\prod_{i=1}^{n} \frac{1}{x_i!}\right) e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}.$$

Since we have

$$\begin{aligned}\pi(\theta|\boldsymbol{x}) &\propto f(\boldsymbol{x}|\theta)\pi(\theta) = \left(\prod_{i=1}^{n} \frac{1}{x_i!}\right) e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}\frac{1}{\sqrt{\theta}} = \left(\prod_{i=1}^{n} \frac{1}{x_i!}\right) e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i - \frac{1}{2}} \\ &\propto e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i - \frac{1}{2}},\end{aligned}$$

we know that the posterior follows $\mathcal{G}amma(\sum_{i=1}^{n} x_i + \frac{1}{2}, n)$. Thus the Bayes estimator for θ can be computed as

$$\mathbb{E}_{\theta|\boldsymbol{x}}[\theta] = \frac{\sum_{i=1}^{n} x_i + \frac{1}{2}}{n}.$$

Based on the four realizations 1,2,0, and 1, we know that the Bayes estimator for θ is

$$\widehat{\theta}_{Bayes} = \frac{1+2+0+1+\frac{1}{2}}{4} = 1.125.$$

MLE for θ is $\widehat{\theta}_{MLE} = \overline{X} = 1$. This shows that $\widehat{\theta}_{Bayes} > \widehat{\theta}_{MLE}$.

(b) We use the following code to find the 95% equitailed credible set

```
> gaminv(0.025, 4+0.5, 1/4)
[1] 0.3375
> gaminv(0.975, 4+0.5, 1/4)
[1] 2.3778
```

The 95% equitailed credible set is computed as [0.3375, 2.3778].

(c) We find the 95% HPD as [0.23781, 2.17410]. The sketch of HPD and the distribution of posterior is shown in Figure 1. The MATLAB code for finding HPD is attached in appendix.

(d) We compute the mode of posterior, which is also MAP estimator of θ, as

$$\widehat{\theta}_{MAP} = \frac{\sum_{i=1}^{n} x_i + \frac{1}{2} - 1}{n} = \frac{1+2+0+1+\frac{1}{2}-1}{4} = 0.875.$$

(e) We use the following code to compute the probability for H_1.

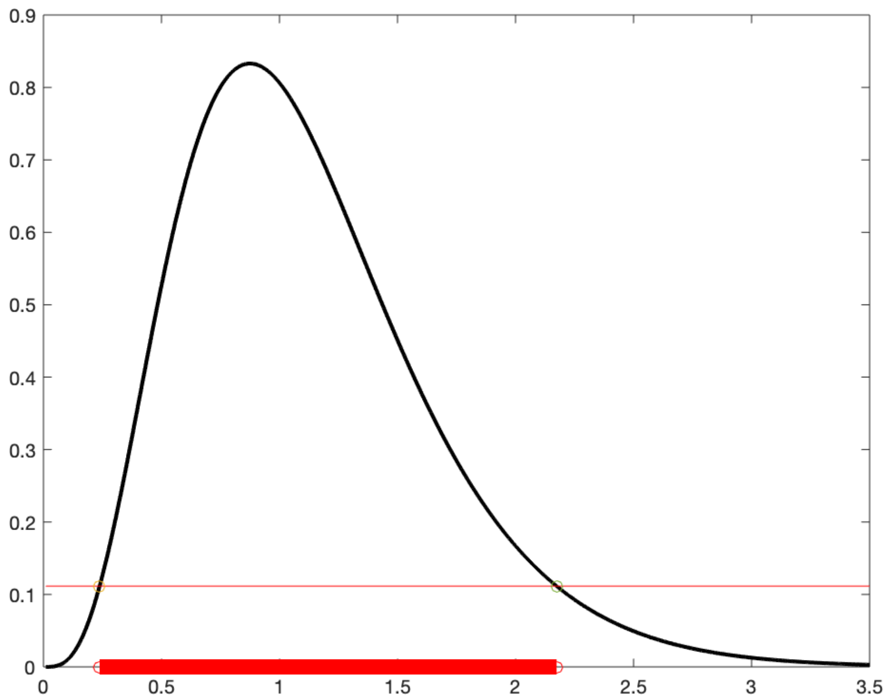

Figure 1: Sketch of HPD

```
> gamcdf(1, 4+0.5, 1/4)
[1] 0.4659
```

This means that the probability for H_0 is $1 - 0.4659 = 0.5341$. Hence, hypothesis H_0 is more favored.

2 Lady Guessing Coin Flips.

(a) According to Bayes formula,

$$
\begin{aligned}
p_0 = p(H_0|X=15) &= \frac{p(X=15|H_0)p(H_0)}{p(X=15|H_0)p(H_0)+p(X=15|H_1)p(H_1)} \\
&= \frac{f(15|0.5)\pi(0.5)}{f(15|p=.5)\pi(0.5)+\int_{0.5}^{1} f(15|p)\pi(p)dp} \\
&= \left[1+\frac{\pi_1}{\pi_0}\frac{m_1(15)}{f(15|0.5)}\right]^{-1} \\
&= \left[1+\frac{0.05}{0.95}\frac{0.1176}{16\times 2^{-16}}\right]^{-1} \\
&= 0.0379
\end{aligned}
$$

and $p_1 = 1 - p_0 = 0.9621$. The Bayes factor is

$$
K = \frac{p(H_0|X\geq 15)p(H_1)}{p(H_1|X\geq 15)p(H_0)} = \frac{0.0379\times 0.05}{(1-0.0379)\times 0.95} = 0.0021
$$

Since $K < 1$, the result is negative (supports H_1)

(b) Since the Bayes factor K is less than 1, the result is negative (supports H_1). The experiment is convincing that the lady possesses ESP.

A MATLAB for finding HPD in problem 1 (c)

```matlab
%%
% search for k
format long
a = 4.5; b = 4;
for k=0.1:0.000001:0.8
    ff=@(x) 1/gamma(a) * x.^(a-1) .* b.^a .* exp(- b * x) - k;
    a1=fzero(ff, 0.5);
    a2=fzero(ff, 2);
    c=gamcdf(a2, a, 1/b) - gamcdf(a1, a, 1/b);
    if (abs(c-0.95)<0.00001)
        break;
    end
end
k

%%
% find HPD
format long
k=0.111507;
a=4.5; b=4;
ff=@(x) 1/gamma(a) * x.^(a-1) .* b.^a .* exp(- b * x) - k
a1=fzero(ff, 0.5) % 0.23781
a2=fzero(ff, 2)   % 2.17410
gamcdf(a2, a, 1/b) - gamcdf(a1, a, 1/b) % 0.95
format short
lengthhpd = a2 - a1  % 1.9363

xx=0.01:0.001:3.5 ;
plot(xx, f(xx, a, b),'k-','linewidth',2)
hold on
plot(xx, k*ones(size(xx)),'r-')
plot(0.23781, f(0.23781, a, b), 'o')
plot(0.23781, 0, 'ro')
plot(2.17410, f(2.17410, a, b), 'o')
plot(2.17410, 0, 'ro')
plot([0.23781  2.17410],[0  0], 'r-','linewidth',8)
hold off
```