1 Metropolis for Correlation Coefficient.

(a) By observing n samples $(X_i, Y_i), i = 1, \dots, n$, the likelihood can be written as

$$\begin{aligned}\ell(\rho) &= \prod_{i=1}^{n} f(x_i, y_i|\rho) = \prod_{i=1}^{n} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}(x_i^2 - 2\rho x_i y_i + y_i^2) \right\} \\ &= \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left\{ -\frac{1}{2(1-\rho^2)} \sum_{i=1}^{n} (x_i^2 - 2\rho x_i y_i + y_i^2) \right\} \\ &= \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left\{ -\frac{1}{2(1-\rho^2)} \left(\sum_{i=1}^{n} x_i^2 - 2\rho \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \right) \right\}.\end{aligned}$$

By choosing prior on ρ as $\pi(\rho) = \frac{1}{(1-\rho^2)^{3/2}} \mathbf{1}(-1 \le \rho \le 1)$, we have the posterior as

$$\begin{aligned}\pi(\rho|\boldsymbol{x}, \boldsymbol{y}) &\propto \ell(\rho)\pi(\rho) \\ &= \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left\{ -\frac{1}{2(1-\rho^2)} \left(\sum_{i=1}^{n} x_i^2 - 2\rho \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \right) \right\} \\ &\quad \times \frac{1}{(1-\rho^2)^{3/2}} \mathbf{1}(-1 \le \rho \le 1). \\ &\propto (1-\rho^2)^{-\frac{1}{2}(n+3)} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left(\sum_{i=1}^{n} x_i^2 - 2\rho \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \right) \right\} \\ &\quad \times \mathbf{1}(-1 \le \rho \le 1).\end{aligned}$$

(b) By choosing the given proposal distribution, the algorithm for generating samples from posterior is shown below. Note that in the following algorithm, θ denotes the target parameter that we hope to obtain from posterior, which is ρ. To avoid confusion, we use $\gamma(\theta_n, \theta')$ to denote the acceptance ratio (which is same as what Brani used $\rho(\theta_n, \theta')$ in the lecture video).

- **Step 1.** Start with arbitrary θ_0.
- **Step 2.** At stage $n+1$, generate proposal θ' from uniform $\mathcal{U}(\theta_n - 0.1, \theta_n + 0.1)$.
- **Step 3.** Determine θ_{n+1} as
  - $\theta_{n+1} = \theta'$ with probability $\gamma(\theta_n, \theta')$;
  - $\theta_{n+1} = \theta_n$ with probability $1 - \gamma(\theta_n, \theta')$.
- **Step 4.** Set $n = n + 1$ and go to **Step 2**.

We now find the acceptance ratio $\gamma(\theta_n, \theta')$ as follows.

Based on the proposal distribution for θ, we know that

$$\begin{aligned} q(\theta'|\theta_n) &= \frac{1}{(\theta_n + 0.1) - (\theta_n - 0.1)} \mathbf{1}(\theta_n - 0.1 \le \theta' \le \theta_n + 0.1) \\ &= \frac{1}{0.2} \mathbf{1}(\theta_n - 0.1 \le \theta' \le \theta_n + 0.1). \end{aligned}$$

Similarly, we also have

$$q(\theta_n|\theta') = \frac{1}{0.2} \mathbf{1}(\theta' - 0.1 \le \theta_n \le \theta' + 0.1).$$

Notice that $\theta_n - 0.1 \le \theta' \le \theta_n + 0.1$ and $\theta' - 0.1 \le \theta_n \le \theta' + 0.1$ are indeed equivalent, and thus

$$\frac{q(\theta_n|\theta')}{q(\theta'|\theta_n)} = \frac{\frac{1}{0.2}\mathbf{1}(\theta' - 0.1 \le \theta_n \le \theta' + 0.1)}{\frac{1}{0.2}\mathbf{1}(\theta_n - 0.1 \le \theta' \le \theta_n + 0.1)} = 1.$$

Thus, the acceptance ratio can be computed as

$$\begin{aligned} \gamma(\theta_n, \theta') &= \min\left\{1, \frac{\pi(\theta')}{\pi(\theta_n)} \frac{q(\theta_n|\theta')}{q(\theta'|\theta_n)}\right\} \\ &= \min\left\{1, \frac{(1-\theta'^2)^{-\frac{1}{2}(n+3)} \exp\left\{-\frac{1}{2(1-\theta'^2)}\left(\sum_{i=1}^n x_i^2 - 2\theta' \sum_{i=1}^n x_i y_i + \sum_{i=1}^n y_i^2\right)\right\}\mathbf{1}(-1 \le \theta' \le 1)}{(1-\theta_n^2)^{-\frac{1}{2}(n+3)} \exp\left\{-\frac{1}{2(1-\theta_n^2)}\left(\sum_{i=1}^n x_i^2 - 2\theta_n \sum_{i=1}^n x_i y_i + \sum_{i=1}^n y_i^2\right)\right\}\mathbf{1}(-1 \le \theta_n \le 1)}\right\}. \end{aligned}$$

By plugging in the values for $\sum_{i=1}^n x_i^2, \sum_{i=1}^n y_i^2$ and $\sum_{i=1}^n x_i y_i$ based on $n = 100$ observations, the acceptance ratio can be computed as

$$\begin{aligned} \gamma(\theta_n, \theta') &= \min\left\{1, \frac{(1-\theta'^2)^{-\frac{103}{2}} \exp\left\{-\frac{1}{2(1-\theta'^2)}\left(114.9707 - 2\theta'(82.5247) + 105.9196\right)\right\}\mathbf{1}(-1 \le \theta' \le 1)}{(1-\theta_n^2)^{-\frac{103}{2}} \exp\left\{-\frac{1}{2(1-\theta_n^2)}\left(114.9707 - 2\theta_n(82.5247) + 105.9196\right)\right\}\mathbf{1}(-1 \le \theta_n \le 1)}\right\} \\ &= \min\left\{1, \frac{(1-\theta'^2)^{-\frac{103}{2}} \exp\{-\frac{1}{2(1-\theta'^2)}(220.8903 - 165.0494\theta')\}\mathbf{1}(-1 \le \theta' \le 1)}{(1-\theta_n^2)^{-\frac{103}{2}} \exp\{-\frac{1}{2(1-\theta_n^2)}(220.8903 - 165.0494\theta_n)\}\mathbf{1}(-1 \le \theta_n \le 1)}\right\}. \end{aligned}$$

(c) We attached the histogram of ρs and the realizations of the chain for the last 1000 simulations shown in Figure 1 and 2, respectively. The Bayes estimator can be computed from the mean of the ρs obtained. We have

$$\hat{\rho}_{Bayes} = 0.7224.$$

(d) By replacing the proposal distribution, we plot the histogram of ρs and the realizations of the chain for the last 1000 simulations shown in Figure 3 and 4, respectively. The Bayes estimator of ρ based on proposal distribution $\mathcal{U}(-1, 1)$ is computed as

$$\hat{\rho}_{Bayes} = 0.7227.$$

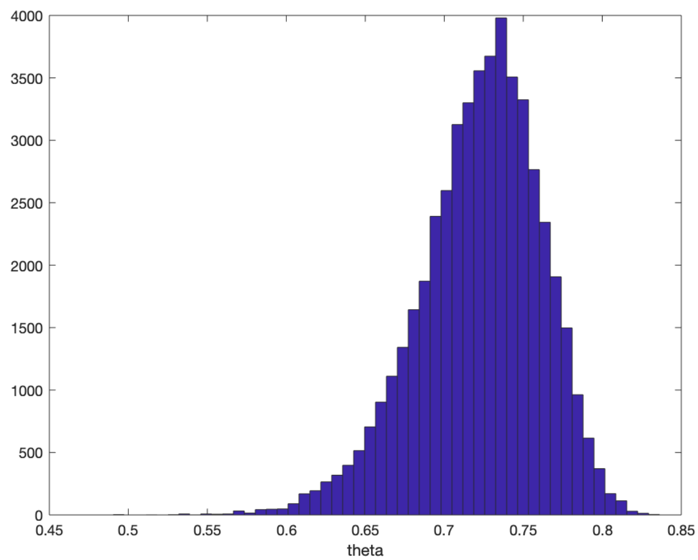

Figure 1: Histogram of ρs given proposal distribution $\mathcal{U}(\rho_{i-1} - 0.1, \rho_{i-1} + 0.1)$

We notice that the Bayes estimator with proposal distribution $\mathcal{U}(-1, 1)$ is similar to that with $\mathcal{U}(\rho_{i-1} - 0.1, \rho_{i-1} + 0.1)$. The acceptance ratio with proposal distribution $\mathcal{U}(-1, 1)$ is also lower (can be seen from the parallel segments from the behavior of the chain).

2 Gibbs Sampler and Lifetimes with Multiplicative Frailty.

(a)

$$
\begin{aligned}
p(\lambda|\mu, t_1, ..., t_n) &\propto \prod_{i=1}^{n} f(t_i|\lambda, \mu)\pi(\lambda) \\
&= \lambda^n \exp(-\lambda\mu \sum_{i=1}^{n} t_i)\lambda^{c-1}\exp(-\alpha\lambda) \\
&= \lambda^{n+c-1}\exp(-\lambda(\alpha + \mu \sum_{i=1}^{n} t_i)).
\end{aligned}
$$

Hence $\lambda|\mu, t_1, .., t_n$ respects to Gamma distribution with parameters $n+c$ and $\mu \sum_{i=1}^{n} t_1 + \alpha$. By symmetry, $\mu|\lambda, t_1, .., t_n$ respects to Gamma distribution with parameters $n + d$ and $\lambda \sum_{i=1}^{n} t_1 + \beta$.

(b) Gibbs sampler algorithm for sampling $(\lambda, \mu)|t_1, ..., t_n$ is as follows:

- **Step 1.** Start with $\mu_0 = 0.1$.

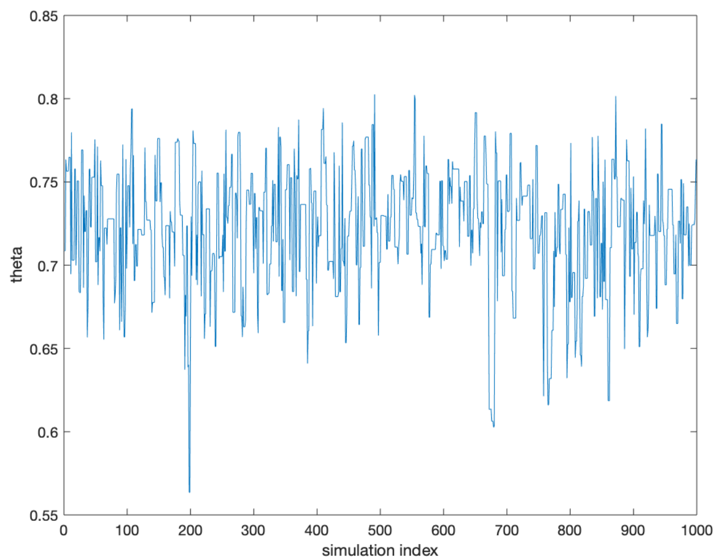

Figure 2: Realizations of the chain for the last 1000 simulations given given proposal distribution $\mathcal{U}(\rho_{i-1} - 0.1, \rho_{i-1} + 0.1)$

- **Step 2.** For $m = 1, ..., 51000$
  - λ_m is a sample generated from $\mathcal{G}a(n + c, \mu_{m-1} \sum_{i=1}^{n} t_1 + \alpha)$
  - μ_m is a sample generated from $\mathcal{G}a(n + d, \lambda_m \sum_{i=1}^{n} t_1 + \beta)$
- **Step 3.** Discard (λ_m, μ_m), $m = 1, ..., 1000$.

(c) Scatterplot of (λ, μ): Histograms of individual components λ and μ:

- Posterior means for $\lambda = 0.0547$ and for $\mu = 0.6911$.
- Posterior variances for $\lambda = 0.000358$ and for $\mu = 0.06579$.
- 95% equatailed credible sets for $\lambda = (0.0255, 0.0983)$ and for $\mu = (0.3185, 1.3139)$.

(d) $\hat{\mathbf{E}}[\lambda\mu|t_1, ..., t_n] = \frac{1}{50000} \sum_{m=1001}^{51000} \lambda_m \mu_m = 0.03425$

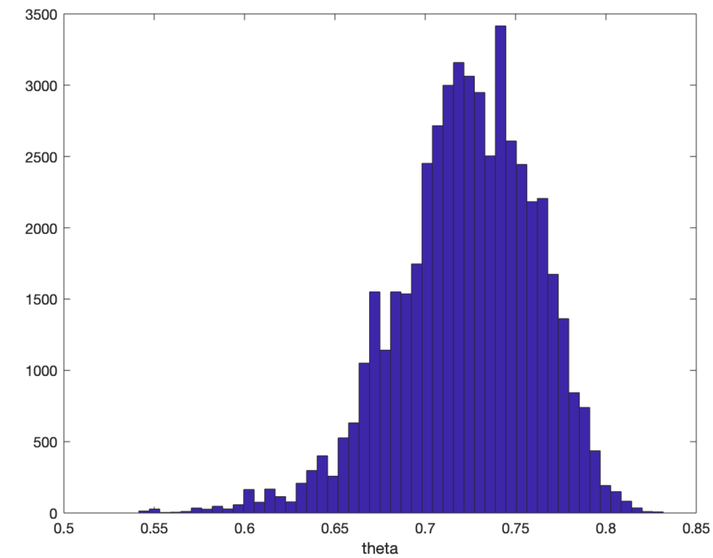

Figure 3: Histogram of ρs given proposal distribution $\mathcal{U}(-1, 1)$

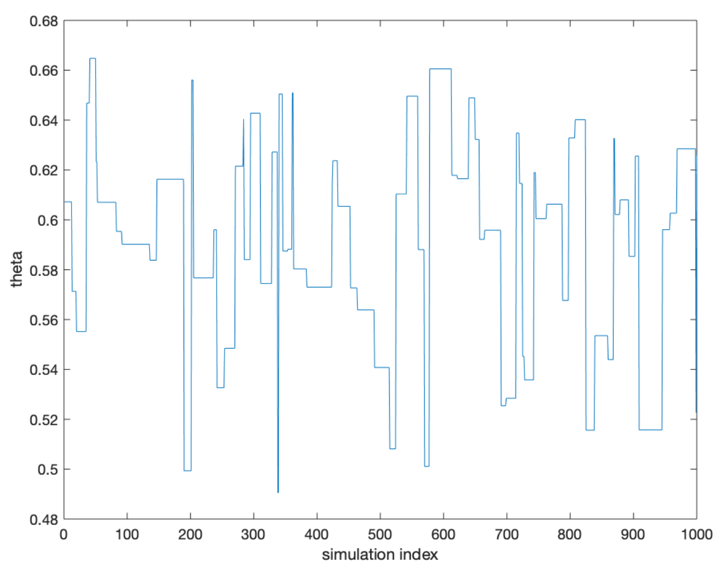

Figure 4: Realizations of the chain for the last 1000 simulations given given proposal distribution $\mathcal{U}(-1, 1)$

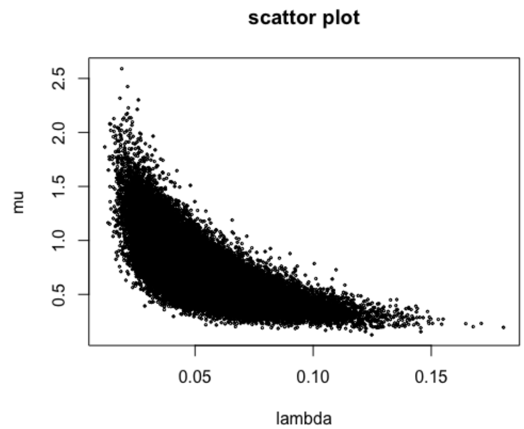

Figure 5: Scatter plot of (λ, μ)

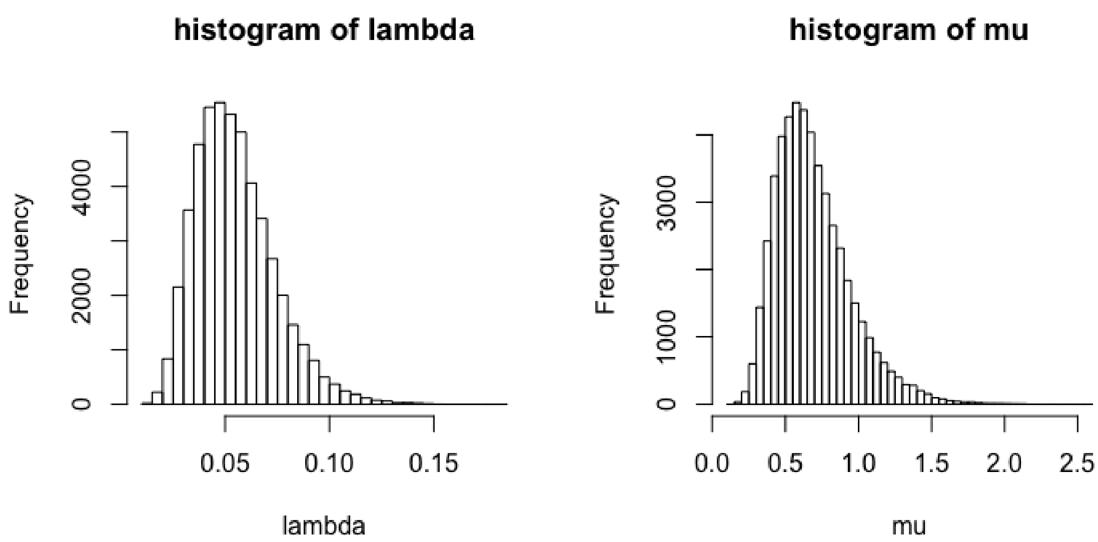

Figure 6: Histograms of individual components λ and μ

A MATLAB code for problem 1

```

close all force
clear all

theta = 0; %initial values
thetas = [theta];

sum_x2 = 114.9707;
sum_y2 = 105.9196;
sum_xy = 82.5247;

n = 100;

tic
 for i = 1:51000
    %theta_prop = theta-0.1+2*0.1*rand();
    theta_prop = -1+2*rand();
%----------------------------------------------------------------------------
    if (theta≥-1 && theta≤1)
        indicator_theta = 1;
    else
        indicator_theta = 0;
    end
    if (theta_prop≥-1 && theta_prop≤1)
        indicator_theta_prop = 1;
    else
        indicator_theta_prop = 0;
    end
%----------------------------------------------------------------------------
rr = ((1-theta_prop^2)^(-(n+3)/2)* ...
    exp(-(sum_x2-2*theta_prop*sum_xy+sum_y2)/(2*(1-theta_prop^2))) * ...
    indicator_theta_prop)/((1-theta^2)^(-(n+3)/2)* ...
    exp(-(sum_x2-2*theta*sum_xy+sum_y2)/(2*(1-theta^2)))*indicator_theta);
%----------------------------------------------------------------------------
rho = min( rr ,1);
   if (rand < rho)
       theta = theta_prop;
   end
thetas =  [thetas theta];
end
toc
%Burn in 1000
thetas = thetas(1000:end);
figure(1)
hist(thetas, 50)
xlabel('theta')
```

B R code for problem 2

```
alpha=100; beta=5
c=3; d=1
sumt=512
n=20
M=51000

mu=0.1
set.seed(1)
record = matrix(0, nrow=2, ncol=M)
for(m in 1:M){
lambda=rgamma(1, shape=(n+c), rate=(mu*sumt+alpha))
mu=rgamma(1, shape=(n+d), rate=(lambda*sumt+beta))
record[,m] = c(lambda, mu)
}
lambda=record[1,-(1:1000)]
mu=record[2,-(1:1000)]
## scatterplot
par(mfrow=c(1, 1))
plot(lambda, mu, main="scattor plot", xlab ="lambda", ylab="mu", cex=0.3)

## histogram of lambda and mu
par(mfrow=c(1, 2))
hist(lambda, xlab="lambda", main="histogram of lambda", breaks=50)
hist(mu, xlab="mu", main="histogram of mu", breaks=50)

## mean of lambda and mu
mean(lambda)
mean(mu)

var(lambda)
var(mu)

## 95% equitailed credible set of lambda and mu
slambda=sort(lambda)
smu=sort(mu)
slambda[c(1250, 48750)]
smu[c(1250, 48750)]

## mean of lambda*mu
mean(lambda*mu)
```