Homework 4

ISyE 6420

Spring 2022

Use of unsolicited electronic and printed resources is allowed except the communication that violates Georgia Tech Academic Integrity Rules (e.g., direct communication about solutions with a third party, use of HW-solving sites, and similar). [1]

1. Metropolis for Correlation Coefficient. Pairs $(X_i, Y_i), i = 1, \dots, n$ consist of correlated standard normal random variables (mean 0, variance 1) forming a sample from a bivariate normal $\mathcal{MVN}_2(\mathbf{0}, \Sigma)$ distribution, with covariance matrix

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

The density of $(X, Y) \sim \mathcal{MVN}_2(\mathbf{0}, \Sigma)$ is[2]

$$f(x, y|\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2) \right\},$$

with ρ as the only parameter. Take prior on ρ by assuming Jeffreys' prior on Σ as $\pi(\Sigma) = \frac{1}{|\Sigma|^{3/2}} = \frac{1}{(1-\rho^2)^{3/2}}$, since the determinant of Σ is $1 - \rho^2$. Thus

$$\pi(\rho) = \frac{1}{(1-\rho^2)^{3/2}} \mathbf{1}(-1 \le \rho \le 1).$$

(a) If $(X_i, Y_i), i = 1, \dots, n$ are observed, write down the likelihood for ρ. Write down the expression for the posterior, up to the proportionality constant (that is, un-normalized posterior as the product of likelihood and prior).

(b) Since the posterior for ρ is complicated, develop a Metropolis-Hastings algorithm to sample from the posterior. Assume that $n = 100$ observed pairs (X_i, Y_i) gave the following summaries:

$$\sum_{i=1}^{100} x_i^2 = 114.9707, \quad \sum_{i=1}^{100} y_i^2 = 105.9196, \quad \text{and} \quad \sum_{i=1}^{100} x_i y_i = 82.5247.$$

In forming a Metropolis-Hastings chain take the following proposal distribution for ρ: At step i generate a candidate ρ' from the uniform $\mathcal{U}(\rho_{i-1} - 0.1, \rho_{i-1} + 0.1)$ distribution. Why the proposal distribution cancels in the acceptance ratio expression?

[1] Course Material for ISyE6420 by Brani Vidakovic is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License.

[2] See (6.1) on page 243 in `http://statbook.gatech.edu`.

(c) Simulate 51000 samples from the posterior of ρ and discard the first 1000 samples (burn in). Plot two figures: the histogram of ρs and the realizations of the chain for the last 1000 simulations. What is the Bayes estimator of ρ based on the simulated chain?

(d) Replace the proposal distribution from (b) by the uniform $\mathcal{U}(-1,1)$ (independence proposal). Comment on the results of MCMC.

2. Gibbs Sampler and Lifetimes with Multiplicative Frailty. Exponentially distributed lifetimes have constant hazard rate equal to the rate parameter λ. When λ is a constant hazard rate, a simple way to model heterogeneity of hazards is to introduce a multiplicative frailty parameter μ, so that lifetimes T_i have distribution[3]

$$T_i \sim f(t_i|\lambda,\mu) = \lambda\mu \exp\{-\lambda\mu t_i\},\ t_i > 0,\ \lambda,\mu > 0.$$

The prior on (λ,μ) is

$$\pi(\lambda,\mu) \propto \lambda^{c-1}\mu^{d-1}\exp\{-\alpha\lambda - \beta\mu\},$$

that is, λ and μ are apriori independent with distributions $\mathcal{G}a(c,\alpha)$ and $\mathcal{G}a(d,\beta)$, respectively. The hyperparameters c, d, α and β are known (elicited) and positive.

Assume that lifetimes $t_1, t_2, \ldots, t_n$ are observed.

(a) Show that full conditionals for λ and μ are gamma,

$$[\lambda|\mu, t_1,\ldots,t_n] \sim \mathcal{G}a\left(n+c, \mu\sum_{i=1}^{n} t_i + \alpha\right),$$

and by symmetry,

$$[\mu|\lambda, t_1,\ldots,t_n] \sim \mathcal{G}a\left(n+d, \lambda\sum_{i=1}^{n} t_i + \beta\right).$$

(b) Using the result from (a) develop Gibbs Sampler algorithm that will sample 51000 pairs (λ,μ) from the posterior and burn-in the first 1000 simulations. Assume that $n = 20$ and that the sum of observed lifetimes is $\sum_{i=1}^{20} t_i = 512$.

Assume further that the priors are specified by hyperparameters $c = 3, d = 1, \alpha = 100$, and $\beta = 5$. Start the chain with $\mu = 0.1$.

(c) From the produced chain, plot the scatterplot of (λ,μ) as well as histograms of individual components, λ, and μ. Estimate posterior means and variances for λ and μ. Find 95% equitailed credible sets for λ and μ.

(d) A frequentist statistician estimates the product $\lambda\mu$ as $\frac{n}{\sum_{i=1}^{n} t_i} = 20/512 = 0.0391$. What is the Bayes estimator of this product? (Hint: It is not the product of averages, it is the average of products, so you will need to save products in the MCMC loop).

[3] This is a toy example. In realistic applications the multiplicative frailty depends on the i, or on a subclass of population as $\mu_{j(i)}$.