Bayesian Statistics Midterm

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Caution:

- I keep 6 digits for all decimal numbers. For simulated results, since it involves random sampling from certain non-degenerative distribution, the results may be slightly different in different trials (and thus from the official solution)
- There is some ambiguity in Q3, see detailed explanation page 4

Exercise 1

Quality Control
10 products were inspected and none of them were rejected

(a) Assume the prior distribution of rejecting a product , θ is $U(0,1)$, denote these observations by $\{x_i\}_{i=1}^{10}$, then the posterior distribution is given by

$$\begin{aligned}\pi(\theta|\{x_i\}_{i=1}^{10}) &\propto \pi(\theta)\cdot\Pi_{i=1}^{10}f(x_i|\theta)\\ &= 1\cdot(1-\theta)^{10}\end{aligned}$$

To find the normalizing constant, we integrate the expression and get

$$\int_0^1(1-\theta)^{10}d\theta = \frac{1}{11}$$

So the posterior distribution is given by

$$\pi(\theta|\{x_i\}_{i=1}^{10}) = \begin{cases}11(1-\theta)^{10}, & 0\le\theta\le 1\\ 0, & \text{else}\end{cases}$$

Thus the posterior mean is given by $\int_0^1\pi(\theta|\{x_i\}_{i=1}^{10})\cdot\theta d\theta = \frac{1}{12}$ (Actually the posterior distribution is a special case of $Beta$ distribution $Beta(\alpha=1,\beta=11)$ if we look closely at its kernel $(1-\theta)^{10}$, we can directly apply the mean of $Beta$ distribution $\frac{\alpha}{\alpha+\beta}=\frac{1}{12}$)

(b) For the $(1-\alpha)$ equitailed credible interval, we need to find θ_L and θ_H such that

$$\frac{\alpha}{2} = \int_0^{\theta_L}11(1-\theta)^{10}d\theta = (\theta-1)^{11}|_0^{\theta_L} = (\theta_L-1)^{11}-(-1)^{11} = (\theta_L-1)^{11}+1$$
$$\frac{\alpha}{2} = \int_{\theta_H}^{1}11(1-\theta)^{10}d\theta = (\theta-1)^{11}|_{\theta_H}^{1} = 0^{11}-(\theta_H-1)^{11} = -(\theta_H-1)^{11}$$

Solve the two equations listed above, we have $\theta_L = 1 - (1 - \frac{\alpha}{2})^{\frac{1}{11}}$ and $\theta_H = 1 - (\frac{\alpha}{2})^{\frac{1}{11}}$, thus the $(1 - \alpha)$ equitailed credible interval is given by $[1 - (1 - \frac{\alpha}{2})^{\frac{1}{11}}, 1 - (\frac{\alpha}{2})^{\frac{1}{11}}]$

(c) For the $(1 - \alpha)$ HPD credible interval, notice that the posterior distribution is strictly decreasing over it domain $0 \le \theta \le 1$, thus we need to find θ_L and θ_H such that

$$
\begin{aligned}
0 &= \theta_L \\
\alpha &= \int_{\theta_H}^{1} 11(1-\theta)^{10} d\theta = (\theta - 1)^{11}|_{\theta_H}^{1} = 0^{11} - (\theta_H - 1)^{11} = -(\theta_H - 1)^{11}
\end{aligned}
$$

Solve the two equations listed above, we have $\theta_L = 0$ and $\theta_H = 1 - \alpha^{\frac{1}{11}}$, thus the $(1 - \alpha)$ HPD credible interval is given by $[0, 1 - \alpha^{\frac{1}{11}}]$

Exercise 2

Waiting Time
Waiting time for a bus has a $U(0,\theta)$ distribution and want to test $H_0 : 0 \le \theta \le 15$ v.s. $H_1 : \theta > 15$, θ has a prior $Pareto(5,3)$ where $Pareto(\xi,\gamma)$ has density $p(x;\xi,\gamma) = \frac{\gamma\xi^\gamma}{x^{\gamma+1}}\mathbb{I}_{(\xi,\infty)}(x)$. Observations are 10, 3, 2, 5, 14.

We know that the prior distribution of θ is given by

$$\pi(\theta) = \frac{3\cdot 5^3}{\theta^{3+1}}\mathbb{I}_{(5,\infty)}(\theta) = \frac{375}{\theta^4}\mathbb{I}_{(5,\infty)}(\theta)$$

Denote the observations by $\{x_i\}_{i=1}^5$, notice that the domain of observations is $[0,\theta]$, we have $\theta \ge x_i, \forall x_i$, we have the posterior distribution given by

$$\begin{aligned}
p(\theta|\{x_i\}_{i=1}^5) &\propto \pi(\theta)\cdot \Pi_{i=1}^5 f(x_i|\theta) \\
&= \frac{375}{\theta^4}\mathbb{I}_{(5,\infty)}(\theta)\cdot \Pi_{i=1}^5(\frac{1}{\theta}\mathbb{I}_{\theta>x_i}) \\
&= \frac{375}{\theta^9}\mathbb{I}_{(5,\infty)}(\theta)\cdot\mathbb{I}_{(10,\infty)}(\theta)\cdot\mathbb{I}_{(3,\infty)}(\theta)\cdot\mathbb{I}_{(2,\infty)}(\theta)\cdot\mathbb{I}_{(5,\infty)}(\theta)\cdot\mathbb{I}_{(14,\infty)}(\theta) \\
&= \frac{375}{\theta^9}\cdot\mathbb{I}_{(14,\infty)}(\theta)
\end{aligned}$$

This is still a kernel of the Pareto distribution and the parameters are given by $\gamma+1 = 9$, $\xi = 14$, i.e. $\gamma = 8$, $\xi = 14$, thus the posterior distribution is given by

$$p(\theta|\{x_i\}_{i=1}^5) = \frac{8\cdot 14^8}{\theta^9}\mathbb{I}_{(14,\infty)}(\theta)$$

The posterior odds is given by

$$\frac{p_0}{p_1} = \frac{\int_0^{15}\frac{8\cdot 14^8}{\theta^9}\mathbb{I}_{(14,\infty)}(\theta)d\theta}{\int_{15}^{\infty}\frac{8\cdot 14^8}{\theta^9}\mathbb{I}_{(14,\infty)}(\theta)d\theta} = 0.736624$$

Recall that the prior odds is given by

$$\frac{\pi_0}{\pi_1} = \frac{\int_0^{15}\frac{375}{\theta^4}\mathbb{I}_{5,\infty}(\theta)d\theta}{\int_{15}^{\infty}\frac{375}{\theta^4}\mathbb{I}_{(5,\infty)}(\theta)d\theta} = 26$$

Thus the Bayes factor in favor of H_0 is given by $B_{01} = \frac{p_0/p_1}{\pi_0/\pi_1} = \frac{0.736624}{26}$ and the Bayes factor against H_0 is given by $B_{10} = \frac{26}{0.736624} = 35.296164$, since $\log_{10} B_{10} = 1.547728$, we have very strong evidence against H_0

Exercise 3

Casting Defects

Number of defects follow Poisson distribution with mean θ, the density is given by $\frac{\theta^k e^{-\theta}}{k!}$. Observations are 0,2,2,3,3,1,2,1,1. θ has a prior distribution $Gamma(2, b)$ where density of $Gamma(a, b)$ is $\frac{b^a \theta^{a-1} e^{-b\theta}}{\Gamma(a)}$. The hyperparameter b has a distribution $Exp(1)$ where the density of $Exp(\lambda)$ is $\lambda e^{-\lambda b}$

As the prior distribution of θ is given by $\frac{b^2 x^1 e^{-bx}}{\Gamma(2)}$, denote the observations by $\{x_i\}_{i=1}^9$ (the number/frequency of value 0 is 1, the number/frequency of value 1 is 3, the number/frequency of value 2 is 3, the number/frequency of value 3 is 2), we have the joint distribution of θ, b given by

$$
\begin{aligned}
f(\theta, b, \{x_i\}_{i=1}^n) &\propto f(b)\pi(\theta|b) \cdot \Pi_{i=1}^9 f(x_i|\theta) \\
&= \lambda e^{-\lambda b} \frac{b^2 \theta^1 e^{-b\theta}}{\Gamma(2)} \cdot \Pi_{i=1}^9 (\frac{\theta^{x_i} e^{-\theta}}{x_i!}) \\
&= \lambda e^{-\lambda b} \frac{b^2 \theta^1 e^{-b\theta}}{\Gamma(2)} \cdot (\frac{\theta^0 e^{-\theta}}{0!}) \cdot (\frac{\theta^1 e^{-\theta}}{1!})^3 \cdot (\frac{\theta^2 e^{-\theta}}{2!})^3 \cdot (\frac{\theta^3 e^{-\theta}}{3!})^2 \\
&= \lambda e^{-\lambda b} \frac{b^2 \theta^{1+0+1\cdot 3+2\cdot 3+3\cdot 2} e^{-\theta(b+1+1\cdot 3+1\cdot 3+1\cdot 2)}}{\Gamma(2) \cdot 1 \cdot 1^3 \cdot 2^3 \cdot 6^2} \\
&= \lambda e^{-\lambda b} \frac{b^2 \cdot \theta^{16} e^{-\theta(b+9)}}{\Gamma(2) \cdot 288} \\
&= \frac{\lambda b^2}{\Gamma(2) \cdot 288} \theta^{16} e^{-\theta(b+9)-\lambda b} \\
&\propto b^2 \theta^{16} e^{-\theta(b+9)-\lambda b}
\end{aligned}
$$

For the posterior distribution of $\theta|(b, \{x_i\}_{i=1}^n)$, we only focus on the term with θ, which is $\theta^{16} e^{-\theta(b+9)}$, this is the kernel of $Gamma(17, b+9)$ (i.e. $Gamma(2 + \sum_{i=1}^n x_i, b+n)$, where n is the number of observations).

Notice that there is some ambiguity in the question whether we should treat the distribution of $b \sim Exp(1)$ as a prior distribution or as something we are already sure about/there is no uncertainty about it (i.e. whether the distribution of b is updatable or not). I include both cases below though the first case seems to be more aligned with the example questions in lecture notes (but the lecture note is explicit that a certain distribution is a **prior**)

- In the first case, we treat $b \sim Exp(1)$ as a prior distribution, then the posterior distribution of b is given by only focusing on terms with b, that is $b|(\theta, \{x_i\}_{i=1}^n) \sim b^2 e^{-(\theta+\lambda)b}$, which is the kernel of $Gamma(3, \theta + \lambda) = Gamma(3, \theta + 1)$ as $\lambda = 1$
- In the second case, we treat $b \sim Exp(1)$ as something fixed and not updatable, then we still sample b from $Exp(1)$

For the first case, the Gibbs sampling is given by the following steps

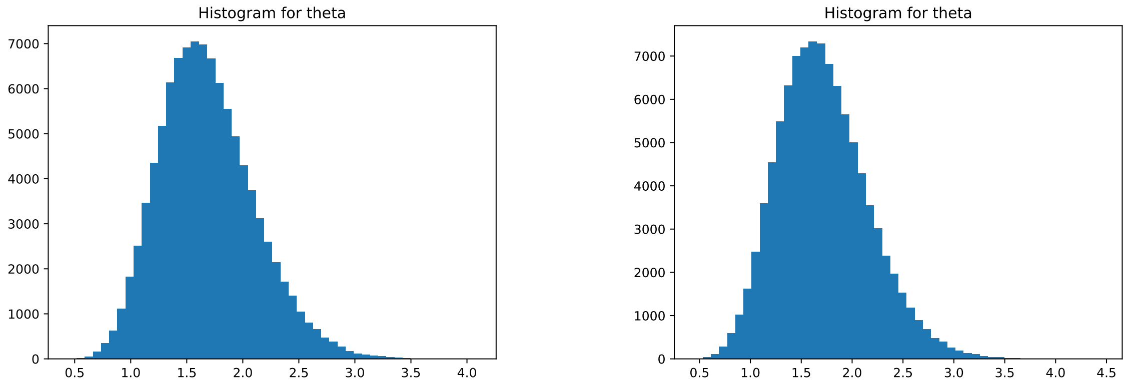
- Initialize θ_0
- Sample b from $Gamma(3, \theta + 1)$
- Sample θ from $Gamma(17, b + 9)$

- With the newly obtained θ, go back to step (2), repeat the steps (2) and (3) above until we get enough observations (ignore the first 1000 observations)

For the second case, the Gibbs sampling is given by the following steps[1]

- Sample b from $Exp(1)$
- Sample θ from $Gamma(17, b + 9)$
- Repeat the two steps above until we get enough observations (ignore the first 1000 observations)

(a) See the following graph Notice that for this question and the questions below, the results



(a) Case 1: When distribution of b is a prior (updatable)

(b) Case 2: When distribution of b is fixed (not updatable)

Figure 1: Histogram of θ

for both cases are very close mainly because the posterior distribution of b does not really shift too much from its prior.

(b)

- In case 1: The posterior mean of θ is given by 1.683211 when we assume the distribution of $b \sim Exp(1)$ is a prior distribution and we update it along the way. The posterior mean of b is 1.142747
- In case 2: 1.717011 when we assume the distribution of $b \sim Exp(1)$ is a distribution that we already know and we fix it, not update it along the way. The mean of b is 0.995968

(c)

- Case 1: The 95% equitailed credible interval is given by (0.961401, 2.616482) when we assume the distribution of $b \sim Exp(1)$ is a prior distribution and we update it along the way
- Case 2: The 95% equitailed credible interval is given by (0.957683, 2.690481) when we assume the distribution of $b \sim Exp(1)$ is a distribution that we already know and we fix it, not update it along the way

[1] Note that this is actually a degenerated Gibbs sampler as we only have one way dependence in the distributional assumption (i.e. θ is dependent on b but not the other way around)

```python
import numpy as np
import matplotlib.pyplot as plt

np.random.seed(1234)
#statistics
obs=[0,2,2,3,3,1,2,1,1]
n_sample=100000
n_burn=1000
a=2
lam=1

#posterior density function
a_delta=sum(obs)
b_delta=len(obs)


#first case: we assume the distribution of b is a prior and
#            update it in the process
theta_0=0.5
b_collector=[]
theta_collector=[theta_0]
for i in range(n_sample+n_burn):
    #sample b
    cur_b=np.random.gamma(3,1/(1+theta_collector[-1]))
    b_collector.append(cur_b)
    #sample theta
    cur_theta=np.random.gamma(a+a_delta,1/(cur_b+b_delta))
    theta_collector.append(cur_theta)
#part a: density
plt.hist(theta_collector[n_burn+1:],bins=50)
plt.title('Histogram for theta')
plt.savefig('q3_parta_hist_theta_case1.pdf')
plt.show()
plt.hist(b_collector[n_burn:],bins=50)
plt.title('Histogram for b')
plt.savefig('q3_parta_hist_b_case1.pdf')
plt.show()
#part b: posterior mean
print('Posterior mean of theta is '+
      str(np.mean(theta_collector[n_burn+1:])))
print('Posterior mean of b is '+
      str(np.mean(b_collector[n_burn:])))
#part c: 95% equitailed credible interval
print('95% equitailed credible interval is given by ('+
      str(np.percentile(theta_collector,2.5))+', '+
      str(np.percentile(theta_collector,97.5))+')')
```

```
#second case: we assume the distribution of b is fixed/knowned without
#           uncertainty and we do not update it in the process
b_collector=[]
theta_collector=[]
for i in range(n_sample+n_burn):
    #sample b
    cur_b=np.random.exponential(1/lam)
    b_collector.append(cur_b)
    #sample theta
    cur_theta=np.random.gamma(a+a_delta,1/(cur_b+b_delta))
    theta_collector.append(cur_theta)
#part a: density
plt.hist(theta_collector[n_burn:],bins=50)
plt.title('Histogram for theta')
plt.savefig('q3_parta_hist_theta_case2.pdf')
plt.show()
plt.hist(b_collector[n_burn:],bins=50)
plt.title('Histogram for b')
plt.savefig('q3_parta_hist_b_case2.pdf')
plt.show()
#part b: posterior mean
print('Posterior mean of theta is '+
      str(np.mean(theta_collector[n_burn:])))
print('Posterior mean of b is '+
      str(np.mean(b_collector[n_burn:])))
#part c: 95% equitailed credible interval
print('95% equitailed credible interval is given by ('+
      str(np.percentile(theta_collector,2.5))+', '+
      str(np.percentile(theta_collector,97.5))+')')
```