

# Homework 3

ISyE 6420

Spring 2022

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**1. Traffic.** Marietta Traffic Authority is concerned about the repeated accidents at the intersection of Canton and Piedmont Roads. Bayes-inclined city-engineer would like to estimate the accident rate, even better, find a credible set.

A well known model for modeling the number of road accidents in a particular location/time window is the Poisson distribution. Assume that  $X$  represents the number of accidents in a 3 month period at the intersection of Canton and Piedmont Roads.



Figure 1: The Intersection of Canton and Piedmont Roads, Marietta, East Cobb, GA

Assume that  $[X|\theta] \sim \mathcal{Poi}(\theta)$ . Nothing is known a priori about  $\theta$ , so it is reasonable to assume the Jeffreys' prior

$$\pi(\theta) = \frac{1}{\sqrt{\theta}} \mathbf{1}(0 < \theta < \infty).$$

In the four most recent three-month periods the following realizations for  $X$  are observed: 1, 2, 0, and 1.

- Compare the Bayes estimator for  $\theta$  with the MLE (For Poisson, recall,  $\hat{\theta}_{MLE} = \bar{X}$ ).
- Compute (numerically) a 95% equitailed credible set.
- Compute (numerically) a 95% HPD credible set.

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- (d) Numerically find the mode of the posterior, that is, MAP estimator of  $\theta$ .  
 (e) If you test the hypotheses

$$H_0 : \theta \geq 1 \quad \text{vs} \quad H_1 : \theta < 1,$$

based on the posterior, which hypothesis will be favored?

**2. Lady Guessing Coin Flips.** There is a famous story involving statistician (Sir Ronald Fisher) and a lady (Muriel Bristol). The lady claimed that, when tasting tea with milk, she can positively determine whether tea or milk had been poured in a cup first. Fisher designs an experiment with 8 cups, 4 in which milk was poured first and 4 in which tea was poured first. Muriel correctly guesses in all 8 cases. This has come to be known as the *Lady Tasting Tea Experiment* and the example is often used to illustrate Fisher's exact test.<sup>2</sup>

Now, imagine another lady who claims ESP and the ability to predict results in subsequent flips of a fair coin. A fair coin is flipped 16 times and the lady correctly predicts the outcome 15 times. Let  $p$  be the probability of the lady guessing correctly.

Test a precise hypothesis  $H_0 : p = 0.5$  (guessing) versus the alternative  $H_1 : p > 0.5$  (presence of ESP), in Bayesian fashion.

To test a precise hypothesis in Bayesian fashion, a prior with a point-mass is needed. Assume that the prior on  $p$  is

$$\pi(p) = \pi_0 \delta_{0.5} + \pi_1 \mathcal{U}(0.5, 1) = 0.95 \delta_{0.5} + 0.05 \cdot 2 \cdot \mathbf{1}(0.5 < p < 1),$$

where  $\delta_{0.5}$  is point-mass at 0.5, and  $\mathcal{U}(a, b)$  is uniform distribution on  $(a, b)$ . From the form of the prior we see that prior probabilities of the null and alternative hypotheses are  $P(H_0) = \pi_0 = 0.95$  and  $P(H_1) = \pi_1 = 1 - \pi_0 = 0.05$ . Thus, apriori we are sceptic of lady's ESP, and favor  $H_0$ .

(a) Find the posterior probabilities of hypotheses,  $p_0$  and  $p_1$ , and Bayes Factor.

(b) What is the assessment of  $H_1$  according to Jeffreys scale, i.e., is the experiment convincing that the lady possesses ESP?

*Hint.* Note that the likelihood is binomial and classical  $p$ -value of the test is  $P(X \geq 15|H_0) = \binom{16}{15} 0.5^{15} 0.5^1 + \binom{16}{16} 0.5^{16} = 17 \times 0.5^{16} = 0.0002594$ . Thus, given the evidence, a frequentist statistician is notoriously biased against  $H_0$ , and is strongly convinced of ESP.

Here

$$p_0 = P(H_0|X) = \left[ 1 + \frac{\pi_1}{\pi_0} \cdot \frac{m_1(x)}{f(x|0.5)} \right]^{-1} \quad \text{and} \quad B_{01} = \frac{f(x|0.5)}{m_1(x)},$$

where

$$f(x|p) = \binom{n}{x} p^x (1-p)^{n-x},$$

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<sup>2</sup>Fisher exact test is frequentist testing for homogeneity of 2 x 2 contingency tables with fixed marginals, see page 602 in the text at <http://statbook.gatech.edu>.

and

$$m_1(x) = \int_{0.5}^1 f(x|p) 2 dp.$$

To find  $m_1(x)$  you will need to numerically evaluate the incomplete beta function.