Iterative methods for linear systems

Seidel iteration, Successive over-relaxation

Seidel iteration

$$\mathbf{Ax} = \mathbf{b} \quad \longrightarrow \quad \mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$$

lower diagonal upper

Rewrite

$$\mathbf{Dx} + (\mathbf{L} + \mathbf{U})\mathbf{x} = \mathbf{b}$$

Seidel's and Jacobi iteration

- Jacobi iteration: $\mathbf{D}\mathbf{x}^{(n+1)} + (\mathbf{L}+\mathbf{U})\mathbf{x}^{(n)} = \mathbf{b}$

- Seidel iteration: $\mathbf{D}\mathbf{x}^{(n+1)} + \mathbf{L}\mathbf{x}^{(n+1)} + \mathbf{U}\mathbf{x}^{(n)} = \mathbf{b}$

Seidel's iteration in components

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ & & \cdots & & \end{pmatrix}$$

$$a_{11}x_1^{(n+1)} + \qquad a_{12}x_2^{(n)} + a_{13}x_3^{(n)} + \cdots + a_{1m}x_m^{(n)} = b_1$$

$$a_{22}x_2^{(n+1)} + a_{21}x_1^{(n+1)} + \qquad a_{23}x_3^{(n)} + \cdots + a_{2m}x_m^{(n)} = b_2$$

$$\cdots$$

At each iteration, sweep down the system of equations; at each step use $\mathbf{x}^{(n+1)}$-s from previous steps: for $x_i^{(n+1)}$ use $x_1^{(n+1)} \cdots x_{i-1}^{(n+1)}$(Jacobi iteration uses $x_1^{(n)} \cdots x_{i-1}^{(n)}$)

Seidel iterations: Sufficient conditions for convergence

Write the original system as

$$\mathbf{x} = \mathbf{B}\mathbf{x} + \mathbf{c}$$

with $\mathbf{B} = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$

Theorem S_1

Let $\|\mathbf{B}\|_1 < 1$ or $\|\mathbf{B}\|_\infty < 1$.
Then, Seidel's iterations converge for any initial $\mathbf{x}^{(0)}$ with the rate of a geometric series,

$$\|\mathbf{x}^{(n)} - \widehat{\mathbf{x}}\| \leqslant q^n \|\mathbf{x}^{(0)} - \widehat{\mathbf{x}}\|,$$

with $q < \|\mathbf{B}\|$

A weaker convergence condition

Define $\mathbf{B}_L = -\mathbf{D}^{-1}\mathbf{L}$ and $\mathbf{B}_U = -\mathbf{D}^{-1}\mathbf{U}$.

Theorem S$_2$

Let $\|\mathbf{B}_U\| + \|\mathbf{B}_U\| < 1$. Then, Seidel's iterations converge for any initial $\mathbf{x}^{(0)}$, and

$$\|\mathbf{x}^{(n)} - \widehat{\mathbf{x}}\| \leqslant q^n \|\mathbf{x}^{(0)} - \widehat{\mathbf{x}}\|,$$

with

$$q = \frac{\|\mathbf{B}_U\|}{1 - \|\mathbf{B}_L\|}$$

Symmetric positive definite matrices

In practice, we often have symmetric positive definite $\mathbf{A}$-s

Theorem S_3

Let $\mathbf{A}$ is symmetric positive definite. Then for any initial $\mathbf{x}^{(0)}$, Seidel's iteration converges with the rate of a geometric series.

Note: No conditions on the norm of $\mathbf{A}$ or $\mathbf{B}$

Necessary and sufficient conditions for convergence

Rewrite Seidel's algorithm as a simple iteration:

$$\mathbf{x}^{(n+1)} = \widetilde{\mathbf{B}}\mathbf{x}^{(n)} + \widetilde{\mathbf{c}}$$

with

$$\widetilde{\mathbf{B}} = (\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}$$

Theorem S$_4$

Seidel's iterations converge for any initial $\mathbf{x}^{(0)}$ if and only if

$$\rho(\widetilde{\mathbf{B}}) < 1$$

Jacobi vs Seidel

Loosely speaking,

- Jacobi method is suitable for $\mathbf{A} \approx$ diagonal.
- Seidel's method is suitable for $\mathbf{A} \approx$ triangular (or symmetric).

Successive over-relaxation method (SOR)

At step j of iteration $n+1$,

1. Do a Seidel's iteration step: compute $\widetilde{x}_j^{(n+1)}$
2. Shift $x_i^{(n+1)} = \omega\widetilde{x}_i^{(n+1)} + (1-\omega)x_i^{(n)}$

Here ω is an arbitrary parameter $\omega \in [0, 2]$. Tune ω to speed up convergence.

In the matrix form

Seidel iteration: $$\mathbf{x}^{(n+1)} = -\mathbf{D}^{-1}\mathbf{L}\mathbf{x}^{(n+1)} - \mathbf{D}^{-1}\mathbf{U}\mathbf{x}^{(n)} + \mathbf{c}$$

SOR iteration: $$\begin{aligned}\mathbf{x}^{(n+1)} &= (1-\omega)\mathbf{x}^{(n)} \\ &\quad + \omega\left(-\mathbf{D}^{-1}\mathbf{L}\mathbf{x}^{(n+1)} - \mathbf{D}^{-1}\mathbf{U}\mathbf{x}^{(n)} + \mathbf{c}\right)\end{aligned}$$

Further variations of SOR

- Use j-dependent ω_j
- Symmetric successive over-relaxation
- ...