Iterative methods for systems of linear equations

$\mathbf{A}\mathbf{x} = \mathbf{b}$, $\qquad \mathbf{x} \in \mathbb{R}^m, \mathbf{b} \in \mathbb{R}^m, \mathbf{A}$ is $m \times m$

Jacobi iteration: $\mathbf{D}\mathbf{x}^{(n+1)} + (\mathbf{L} + \mathbf{U})\mathbf{x}^{(n)} = \mathbf{b}$

Seidel's iteration: $\mathbf{D}\mathbf{x}^{(n+1)} + \mathbf{L}\mathbf{x}^{(n+1)} + \mathbf{U}\mathbf{x}^{(n)} = \mathbf{b}$

S.O.R.: $\mathbf{D}\mathbf{x}^{(n+1)} = (1-\omega)\mathbf{D}\mathbf{x}^{(n)} + \omega(-\mathbf{L}\mathbf{x}^{(n+1)} - \mathbf{U}\mathbf{x}^{(n)} + \mathbf{b})$

Canonical form of iterative methods

A *canonical form* of a single-step method of solving

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

is defined by $\mathbf{P}_{n+1} \in \mathbb{R}^{m,m}$ and $\tau_{n+1} \in \mathbb{R}$:

$$\mathbf{P}_{n+1} \frac{\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}}{\tau_{n+1}} + \mathbf{A}\mathbf{x}^{(n)} = \mathbf{b}$$

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- If $\mathbf{P}_{n+1} = \mathbf{P}$ and $\tau_{n+1} = \tau$ are n-independent, the method is *stationary*
- If $\mathbf{P} = \widehat{\mathbf{1}}$, the method is *explicit*.

Iterative methods for systems of linear equations

For an explicit method, the iteration is

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \tau_{n+1}\left[\mathbf{b} - \mathbf{A}\mathbf{x}^{(n)}\right]$$

For an implicit method, we have to solve an additional system of equations at each step:

$$\mathbf{P}_{n+1}\mathbf{x}^{(n+1)} = \mathbf{P}_{n+1}\mathbf{x}^{(n)} + \tau_{n+1}\left[\mathbf{b} - \mathbf{A}\mathbf{x}^{(n)}\right]$$

In practice

- Compute the residual $\mathbf{r}^{(n)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(n)}$
- Solve $\mathbf{P}_{n+1}\mathbf{z} = \mathbf{r}^{(n)}$
- Then $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \tau_{n+1}\mathbf{z}$

Canonical form of iterative methods

Jacobi iteration

$$\mathbf{D}\mathbf{x}^{(n+1)} - \mathbf{D}\mathbf{x}^{(n)} + \underbrace{\mathbf{D}\mathbf{x}^{(n)} + (\mathbf{L} + \mathbf{U})\mathbf{x}^{(n)}}_{\mathbf{A}\mathbf{x}^{(n)}} = \mathbf{b}$$

$\Rightarrow \quad \mathbf{P} = \mathbf{D}$ and $\tau = 1$

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Seidel iteration

$$(\mathbf{D}+\mathbf{L})\mathbf{x}^{(n+1)} - (\mathbf{D}+\mathbf{L})\mathbf{x}^{(n)} + \underbrace{(\mathbf{D}+\mathbf{L})\mathbf{x}^{(n)} + \mathbf{U}\mathbf{x}^{(n)}}_{\mathbf{A}\mathbf{x}^{(n)}} = \mathbf{b}$$

$\Rightarrow \quad \mathbf{P} = \mathbf{D}+\mathbf{L}$, and $\tau = 1$

Canonical form of iterative methods

S.O.R.

$\mathbf{P} = \mathbf{D} + \omega \mathbf{L}$, and $\tau = \omega$

Convergence analysis of iterative methods

If iterations converge, then $\mathbf{x}^{(n)} \to \widehat{\mathbf{x}}$, the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$

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Define

$$\mathbf{e}^{(n)} = \mathbf{x}^{(n)} - \widehat{\mathbf{x}}$$

Then, iterations are

$$\mathbf{e}^{(n+1)} = \mathbf{S}_{n+1}\mathbf{e}^{(n)}$$

where the *transition matrix*

$$\mathbf{S}_{n+1} = \mathbf{1} - \tau_{n+1}\mathbf{P}_{n+1}^{-1}\mathbf{A}$$

Convergence analysis of iterative methods

For symmetric positive definite $\mathbf{A}$, a stationary iteration ($\mathbf{S}_n = \mathbf{S}$) converges if and only if

$$\rho(\mathbf{S}) < 1$$

where $\rho(\mathbf{S}) = \max_{1 \leqslant j \leqslant m} |\lambda_j|$

Variational approaches

Let $\widehat{\mathbf{x}}$ is the solution of $\mathbf{A}\widehat{\mathbf{x}} = \mathbf{b}$.

Consider

$$\frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{\tau_{n+1}} + \mathbf{A}\mathbf{x}_n = \mathbf{b}$$

The main idea

Choose τ_{n+1} to minimize $\|\mathbf{x}_{n+1} - \widehat{\mathbf{x}}\|$ given $\|\mathbf{x}_n - \widehat{\mathbf{x}}\|$.

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Alternatively, minimize the *residual* $\mathbf{r}_n = \mathbf{A}\mathbf{x}_n - \mathbf{b}$:

$$\|\mathbf{r}_{n+1}\| \Rightarrow \min \quad \text{given } \|\mathbf{r}_n\|$$

Minimum residual method

Rewrite the iteration as

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \tau_{n+1}\mathbf{r}_n$$

Multiply by $\mathbf{A}$ and subtract $\mathbf{b}$:

$$\mathbf{r}_{n+1} = \mathbf{r}_n - \tau_{n+1}\mathbf{A}\mathbf{r}_n$$

Minimize the 2-norm of $\mathbf{r}_{n+1}$:

$$\begin{aligned}\|\mathbf{r}_{n+1}\|^2 &= \|\mathbf{r}_n - \tau_{n+1}\mathbf{A}\mathbf{r}_n\|^2 \\ &= \|\mathbf{r}_n\|^2 + \tau_{n+1}^2\|\mathbf{A}\mathbf{r}_n\|^2 - 2\tau_{n+1}\langle\mathbf{r}_n \cdot \mathbf{A}\mathbf{r}_n\rangle\end{aligned}$$

Minimum residual method

So that the n-th iteration of the algorithm is

1. Compute $\mathbf{r}_n = \mathbf{A}\mathbf{x}_n - \mathbf{b}$
2. Compute
$$\tau_{n+1} = \frac{\langle \mathbf{r}_n \cdot \mathbf{A}\mathbf{r}_n \rangle}{\|\mathbf{A}\mathbf{r}_n\|^2}$$
3. Compute $\mathbf{x}_{n+1} = \mathbf{x}_n - \tau_{n+1}\mathbf{r}_n$