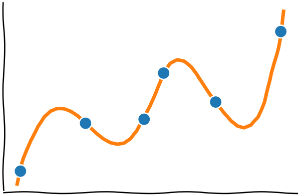
Interpolation and approximation

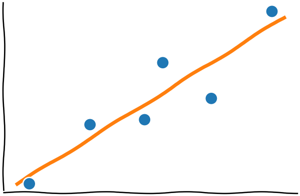
Given a set of points $\{(x_j, y_j), j = 1, \cdots, n\}$, and given a functional form $f(x; \vec{\beta})$, find *"best"* $\vec{\beta}$ so that $f(x; \vec{\beta})$ *"models"* the data.

Interpolation



$$f(x_j; \vec{\beta}) = y_j$$

Approximation

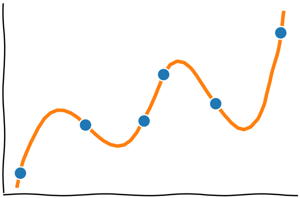


$$f(x_j; \vec{\beta}) + \varepsilon_j = y_j$$

ε_j is "noise", $\mathbb{E}(\varepsilon_j) = 0$

Interpolation

Given a set of points $\{(x_j, y_j), j = 1, \cdots, n\}$, and given a functional form $f(x; \vec{\beta})$, find *"best"* $\vec{\beta}$ so that $f(x; \vec{\beta})$ *interpolates* the data.



$$f(x_j; \vec{\beta}) = y_j$$

A system of n nonlinear equations for m unknowns $\beta_1, \ldots, \beta_m$

Interpolation

Let the model, $f(x;\vec{\beta})$, is a *linear* function of $\vec{\beta}$, a linear combination of m *basis functions*, $\varphi_k(x)$

$$f(x;\vec{\beta}) = \sum_{j=1}^{m} \beta_k \varphi_k(x)$$

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$$f(x;\vec{\beta}) = \sum_{j=1}^{m} \beta_k \varphi_k(x)$$

The interpolation conditions reduce to the system of *linear* equations

$$\mathbf{A}\vec{\beta} = \mathbf{y}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$.

Interpolation

Interpolation conditions:

$$\mathbf{A}\vec{\beta} = \mathbf{y}$$

where $\mathbf{A}$ is an $n \times m$ matrix

$$\mathbf{A} = \begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_m(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_m(x_2) \\ & & \cdots & \\ \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_m(x_n) \end{bmatrix}$$

The dimensions of $\mathbf{A}$ is *# of observations* $\times$ *# of parameters*

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When does this system have a unique solution $\vec{\beta}$?

Linear independence

Call $\vec{\varphi}_k$ the k-th column of $\mathbf{A}$:

$$\mathbf{A} = [\vec{\varphi}_1 \; \vec{\varphi}_2 \; \cdots \vec{\varphi}_m]$$

A system of functions $\varphi_1(x), \ldots, \varphi_m(x)$ is *linearly dependent* on the set of points $x_1, \ldots, x_n$ if at least one vector $\vec{\varphi}_k$ can be expressed as a linear combination of other $\vec{\varphi}$-s:

$$\vec{\varphi}_k = \sum_{s \neq k} \xi_s \vec{\varphi}_s$$

Linear independence: the Gram matrix

A set of vectors $\{\vec{\varphi}_k\}$ is linearly independent iff its *Grammian determinant* is non-zero, $\det \mathbf{\Gamma} \neq 0$.

The Gram matrix,

$$\mathbf{\Gamma} = \mathbf{A}^T \mathbf{A}$$

Global polynomial interpolation

Take $m = n$ and $\varphi_k(x) = x^{k-1}$ for $k = 1, \dots, m$.

$\mathbf{A}$ becomes a *Vandermonde* matrix

$$\mathbf{A} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{m-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{m-1} \\ & & \cdots & & \\ 1 & x_m & x_m^2 & \cdots & x_m^{m-1} \end{bmatrix}$$

The Vandermonde determinant

$$\det \mathbf{A} = \prod_{1 \leqslant p < q \leqslant m} (x_p - x_q) \neq 0$$

if all $\{x_k\}$ are distinct.

Global polynomial interpolation

Given n points x_k and y_k for $k = 1, \ldots, n$, construct a *unique* polynomial

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$$

which interpolates y_k, i.e. $P(x_k) = y_k$.

The coefficients of $P(x)$ satisfy

$$\mathbf{V}\vec{c} = \mathbf{y}$$

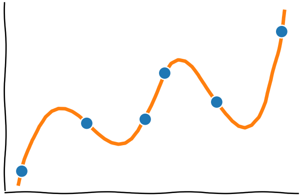
and $\mathbf{V}$ is the Vandermonde matrix.

Global polynomial interpolation

- Vandermonde matrices are poorly conditioned
- Evaluations of polynomials in terms of coefficients is poorly conditioned

Look for an alternative form of writing the interpolating polynomial.

Lagrange interpolating polinomial

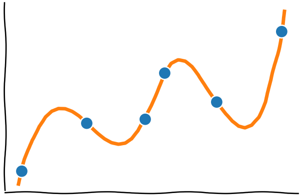


Consider $\ell_k(x)$, a polynomial of degree m,

$$\ell_k(x_j) = \begin{cases} 1, & k = j, \\ 0, & k \neq j \end{cases}$$

for $k = 0, \cdots, m$

Lagrange interpolating polinomial



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for $k = 0, \cdots, m$

Then

$$P(x) = \sum_{k=0}^{m} y_k \, \ell_k(x),$$

satisfies the interpolation conditions $P(x_j) = y_j$.

Lagrange interpolating polinomial

Explicit form of $\ell_k(x)$:

$$\ell_k(x) = \frac{x - x_0}{x_k - x_0} \cdots \frac{x - x_{k-1}}{x_k - x_{k-1}} \frac{x - x_{k+1}}{x_k - x_{k+1}} \cdots \frac{x - x_m}{x_k - x_m}$$

$$\equiv \prod_{j=0,\ j\neq k}^{m} \frac{x - x_j}{x_k - x_j}$$

For example, a linear Lagrange interpolator is

$$L_1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

Interpolation error

Given a smooth function $y = f(x)$, tabulate it on $x \in [a, b] \Longrightarrow$ have $\{x_j, y_j\}$, $j = 0, \cdots, m \Longrightarrow$ Construct the interpolator $P(x)$, $P(x_j) = y_j$

Interpolation error:

$$\Delta_m \equiv \max_{x \in [a,b]} |f(x) - P(x)| = ?$$

Interpolation error

Let $f(x)$ is continuously differentiable $m+1$ times on $[a, b]$.

Let $a \leqslant x_0 < x_1 < \cdots < x_m \leqslant b$

Then

$$f(x) - P(x) = \frac{f^{(m+1)}(\xi)}{(m+1)!}\omega_{m+1}(x)$$

where $\xi \in (a, b)$, and

$$\omega_{m+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_m)$$

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$$\Delta_m \leqslant \frac{M_{m+1}}{(m+1)!} \max_x |\omega_{m+1}(x)|,$$

$M_{m+1} = \max_x \left| f^{(m+1)}(x) \right|$

Interpolation error

Interpolation: $a < x < b$

Let $h \equiv \max_k |x_{k+1} - x_k|$. Then,

$$\Delta_m \leqslant \text{const} \times h^{m+1}$$

Interpolation error

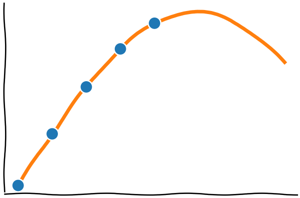
Interpolation: $a < x < b$

Let $h \equiv \max_k |x_{k+1} - x_k|$. Then,

$$\Delta_m \leqslant \text{const} \times h^{m+1}$$

Extrapolation: $x \notin [a, b]$

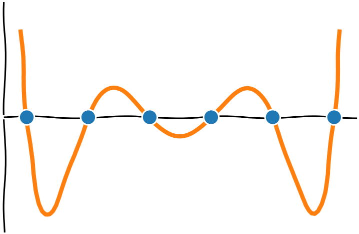
$$\omega_{m+1}(x \to \infty) \to \infty$$



Runge phenomenon

$$f(x) - P(x) \propto w_{m+1}(x)$$

$$\omega_{m+1}(x) = (x - x_0)(x - x_1) \cdots (x - x_m)$$



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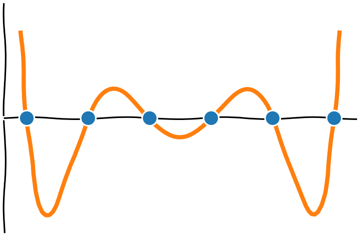

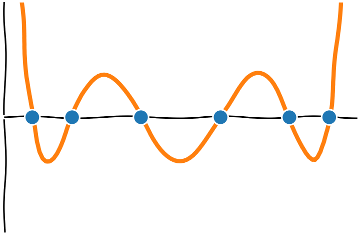

Goal: Select $\{x_k\}$ s.t. $\max\limits_{x \in [a,b]} |\omega_{m+1}(x)| \Rightarrow \min$

Chebyshev polynomials

Consider a family of polynomials $T_n(x)$, with the domain $[-1, 1]$:

$$
\begin{aligned}
T_0(x) &= 1, \\
T_1(x) &= x, \\
T_n(x) &= 2x\,T_{n-1}(x) - T_{n-2}(x), \qquad n \geqslant 2.
\end{aligned}
$$

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Properties of Chebyshev polynomials

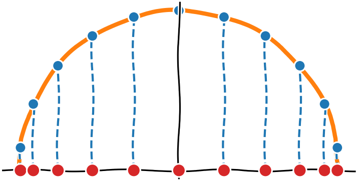
1. $T_n(x) = 2^{n-1}x^n + \cdots, \quad n \geqslant 1$
2. $T_n(x) = \cos(n \arccos(x)), \quad x \in [-1, 1]$
3. $\max\limits_{x \in [-1,1]} |T_n(x)| = 1, \quad n \geqslant 0$
4. $T_n(x)$ has exactly n distinct real roots on $[-1, 1]$

Chebyshev nodes

5. The roots x_k of $T_n(x)$ are given by

$$x_k = \cos \frac{2k+1}{2n}\pi, \qquad k = 0, \cdots, n-1$$

NB: Chebyshev nodes are not equidistant.



Chebyshev polynomials: deviation from zero

6. Any polynomial of degree n with the leading coefficient equal to unity,

$$\xi(x) = x^n + a_{n-1}x^{n-1} + \cdots ,$$

has larger deviation from zero than the (scaled) Chebyshev polynomial of the same degree

$$\max_{x\in[-1,1]} |T_n(x)| \leqslant \max_{x\in[-1,1]} \left|2^{n-1}\xi(x)\right|$$

Global polynomial interpolation

- Do not use uniform grids
- Use Chebyshev nodes

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- If the interpolation interval is $[a, b] \neq [-1, 1]$, scale

$$x = \frac{a+b}{2} + \frac{b-a}{2} t$$

where $t \in [-1, 1]$, and use Chebyshev nodes.

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Faber's theorem:
$\forall$ set of nodes $\{x_k\}$, $\exists$ a continuous function $f(x)$, such that

$$\max_x |f(x) - P(x)| \to \infty \quad \text{as} \quad m \to \infty$$