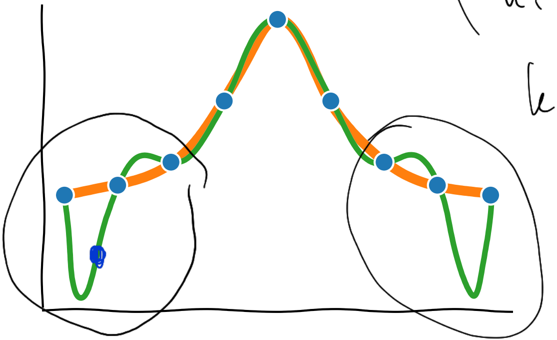
Interpolation. Splines.

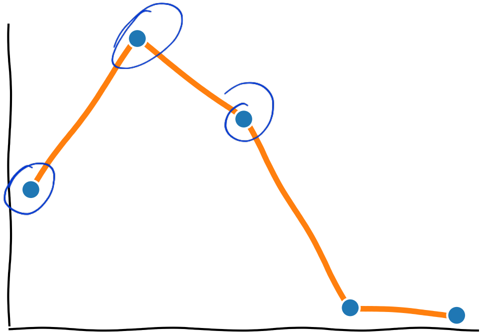
Interpolation with higher order polynomials is prone to the Runge phenomenon.

(x_k, y_k)

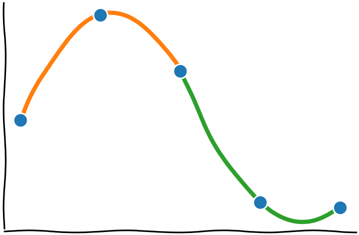
$k = \overline{0, n}$



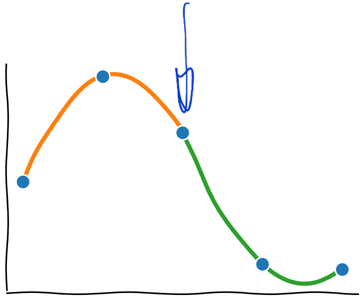
Piecewise linear interpolation

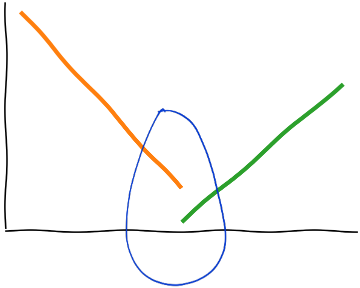


Piecewise parabolic interpolation



Piecewise parabolic interpolation

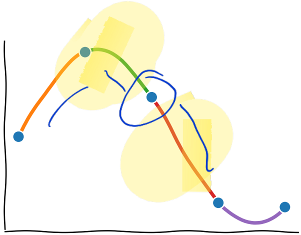




Piecewise polynomials. Splines.

Condider a set of *breakpoints* on the interval $x \in [a, b]$

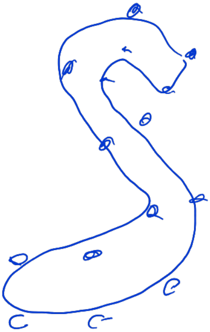
$$a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{m-1} \leqslant x_m = b,$$



A *spline function*, $S(x)$, is a piecewise polynomial on $[a, b]$.

- $S(x)$ has *degree* n if the max degree of all polynomial pieces is n.
- $S(x) \in C^p$ if $S(x)$ and its p derivatives, $S'(x)$, $S''(x)$, $\cdots$, $S^{(p)}(x)$ are continuous on $x \in [a, b]$.

Cubic splines

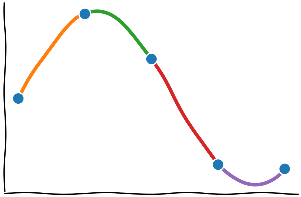


Hermite form of a cubic polynomial

Piecewise cubic polynomials

Given

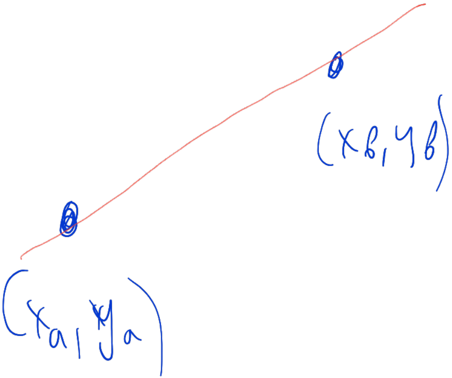
$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b,$$



A cubic polynomial, $p_{3,k}(x)$, on each interval $[x_{k-1}, x_k]$, for $k = 1, \cdots, m$

A single interval: a linear function

Construct $p_1(x)$ with $p_1(x_a) = y_a$ and $p_1(x_b) = y_b$



$$p_1(x) = y_b \frac{x - x_a}{h} + y_a \frac{x_b - x}{h}$$

where $h = x_b - x_a$.

$p_1(x_a) = y_a$

$p_1(x_b) = y_b$

A single interval: a cubic function

Construct $p_3(x)$ with

$$p_3(x_a) = y_a \qquad p_3'(x_a) = s_a$$
$$p_3(x_b) = y_b \qquad p_3'(x_b) = s_b$$

A single interval: a cubic function

Construct $p_3(x)$ with

$$p_3(x_a) = y_a \qquad p_3'(x_a) = s_a$$
$$p_3(x_b) = y_b \qquad p_3'(x_b) = s_b$$

$x_a \leq x \leq x_b$

$$p_3(x) = p_1(x) + \underbrace{\frac{(x - x_a)}{h}\frac{(x_b - x)}{h}}\left[A\,(x - x_a) + B\,(x_b - x)\right]$$

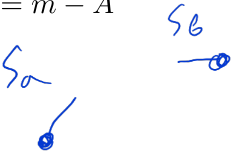
And find A and B given s_a and s_b.

A single interval: a cubic function

$$x = x_a : \qquad s_a = p_3'(x_a) = m + B$$
$$x = x_b : \qquad s_b = p_3'(x_b) = m - A$$

with

$$m = \frac{y_b - y_a}{h}.$$



So that we have an explicit form of a (unique) cubic parabola on $[x_a, x_b]$ with given tangents (s_a, s_b) and values (y_a, y_b) at the edges:

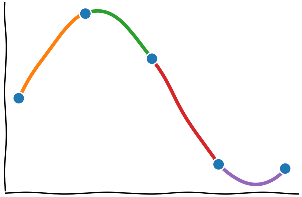
$$p_3(x) = p_1(x) + \frac{(x - x_a)}{h}\frac{(x_b - x)}{h}\left[(m - s_b)\,(x - x_a) + (s_a - m)\,(x_b - x)\right]$$

Cubic splines

Multiple intervals: a piecewise cubic function

Given

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b,$$



use $p_{3,k}(x)$ on each interval $[x_{k-1}, x_k]$, for $k = 1, \cdots, m$

$$p_3(x) = p_1(x) + \frac{(x - x_a)}{h}\frac{(x_b - x)}{h}\Big[(m - s_b)\,(x - x_a) + (s_a - m)\,(x_b - x)\Big]$$

Multiple intervals: a piecewise cubic function

continuity : by construction

$$p_3(x) = p_1(x) + \dots$$

Multiple intervals: a piecewise cubic function

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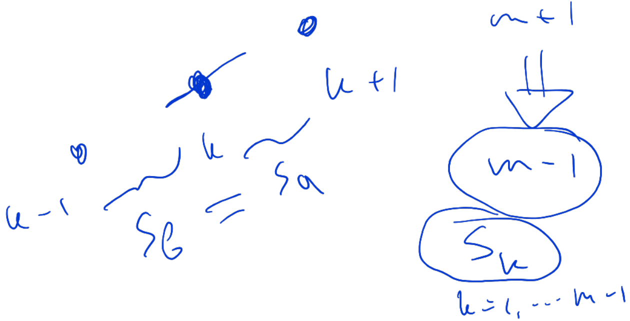
first derivative : use consistent tangents at internal breakpoints

$$\Rightarrow S(x) \in C^1 \text{ on } x \in [a, b]$$



Multiple intervals: a piecewise cubic function

continuity : by construction

$$p_3(x) = p_1(x) + \dots$$

first derivative : use consistent tangents at internal breakpoints

$\Rightarrow S(x) \in C^1$ on $x \in [a, b]$

second derivative : select internal tangents, s_k, so that

$$p''_{3,k-1}(x_k) = p''_{3,k}(x_k)$$

for $k = 1, \cdots, m$.

NB: $m - 1$ conditions for $m + 1$ tangents.
Need two more *boundary conditions.*

Work out second derivatives on intervals

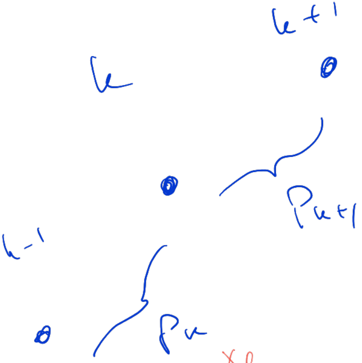
On a single interval $[x_a, x_b]$:

$$
\begin{aligned}
p_3''(x) = &-2\frac{s_b - m}{h^2}[(x_b - x) - 2(x - x_a)] \\
&-2\frac{s_a - m}{h^2}[2(x_b - x) - (x - x_a)]
\end{aligned}
$$

Thus

$$
\begin{aligned}
p_3''(x_a) &= -2\frac{s_b - m}{h} - 4\frac{s_a - m}{h} \\
p_3''(x_b) &= 4\frac{s_b - m}{h} + 2\frac{s_a - m}{h}
\end{aligned}
$$

Match second derivatives at breakpoints



For $p_{3,k}(x),\ x \in [x_{k-1}, x_k]$, for $k = 1, \cdots, m$ define

$$h_k = x_k - x_{k-1}$$

$$m_k = \frac{y_k - y_{k-1}}{h_k}$$

xb

$$p''_{3,k}(x_k) = p''_{3,k+1}(x_k)$$

xa

$$4\frac{s_k - m_k}{h_k} + 2\frac{s_{k-1} - m_k}{h_k} = -2\frac{s_{k+1} - m_{k+1}}{h_{k+1}} - 4\frac{s_k - m_{k+1}}{h_{k+1}}$$

Sb Sa Sb Sa

Cubic splines

A C^2 continuous spline interpolator with breakpoints

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b,$$

is defined by a solution of an (almost) tridiagonal system of linear equations

$$s_k(\frac{2}{h_k} + \frac{2}{h_{k+1}}) + \frac{1}{h_k}s_{k-1} + \frac{1}{h_k}s_{k+1} = 3(\frac{m_k}{h_k} + \frac{m_{k+1}}{h_{k+1}}),$$

where $k = 1, \cdots, m-1$.

Cubic splines

A C^2 continuous spline interpolator with breakpoints

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b,$$

is defined by a solution of an (almost) tridiagonal system of linear equations

$$s_k\left(\frac{2}{h_k} + \frac{2}{h_{k+1}}\right) + \frac{1}{h_k}s_{k-1} + \frac{1}{h_k}s_{k+1} = 3\left(\frac{m_k}{h_k} + \frac{m_{k+1}}{h_{k+1}}\right),$$

where $k = 1, \cdots, m-1$.

$A + uv^T$

Need two more equations:

- $m+1$ unknowns s_k for $k = 0, \cdots, m$
- $m-1$ equations for $k = 1, \cdots, m-1$

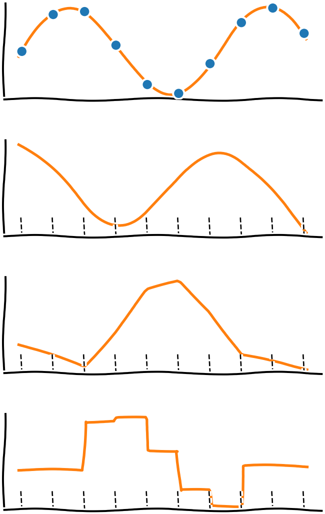
Cubic splines: boundary conditions

"Fundamental spline" Known $f'(a)$ and $f'(b)$ or $f''(a)$ and $f''(b)$.

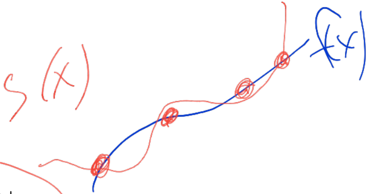
"Natural spline" Set $f''(a) = f''(b) = 0$.

"Not-a-knot" Require continuous *third* derivative at second and second-to-last breakpoints.

Cubic spline interpolation: continuity



Approximation errors



Let $f(x) \in C^4$ on $x \in [a, b]$ and

$$M_4 = \max_{x \in [a,b]} |f^{(4)}(x)|;$$

Then, for an interpolating spline $S_3(x)$ with *not-a-knot* or *fundamental* boundary conditions,

$$\max_{x \in [a,b]} |f(x) - S_3(x)| \leqslant C\, M_4\, h^4,$$

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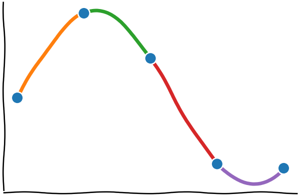
where C is some constant and $h = \max_k |x_k - x_{k-1}|$.

However, the *natural* spline has the approximation error $\propto h^2$. !

Local piecewise interpolation

Piecewise polynomial interpolation

Given $a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$,



A cubic interpolating polynomial in the Hermite form

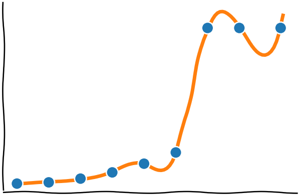
$$p_3(x) = p_1(x) + \frac{(x - x_a)}{h}\frac{(x_b - x)}{h}\Big[(m - s_b)\,(x - x_a) + (s_a - m)\,(x_b - x)\Big]$$

Cubic spline interpolation

- C^2 continuous on $[a, b]$
- Need two boundary conditions
- Construction is $O(N)$

Cubic spline interpolation

- C^2 continuous on $[a, b]$
- Need two boundary conditions
- Construction is $O(N)$
- Not guaranteed to be monotonic



May or may not need the C^2 continuity.

Local interpolating schemes

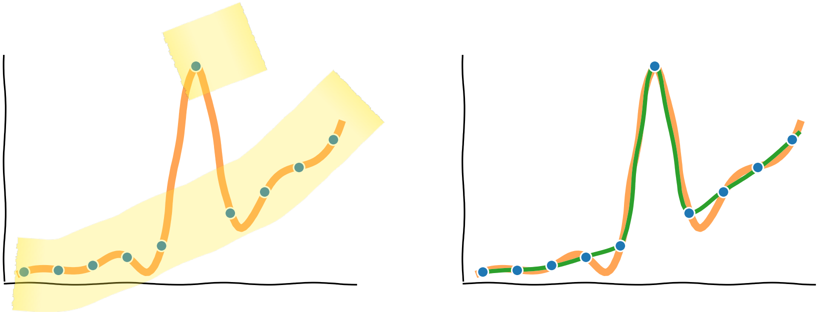
Other, local, prescriptions for s_k are possible. For example,

$$s_k = \frac{y_{k+1} - y_{k-1}}{x_{k+1} - x_{k-1}}$$

Produces a *local* C^1 continuous interpolant.

Monotone interpolants

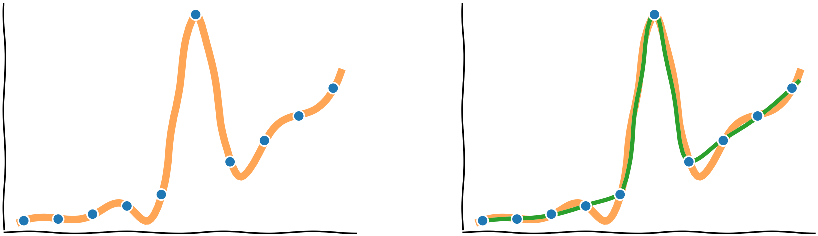
Outliers or sharp features in data: C^2 splines may overshoot.



Use a local scheme for s_k, with weighting and clipping.

Monotone interpolants

Outliers or sharp features in data: C^2 splines may overshoot.



Use a local scheme for s_k, with weighting and clipping.

Most popular schemes: Akima splines and PCHIP algorithm.

Monotone piecewise cubic interpolant: PCHIP

PCHIP algorithm

Let $h_k = x_{+1} - x_k$, and

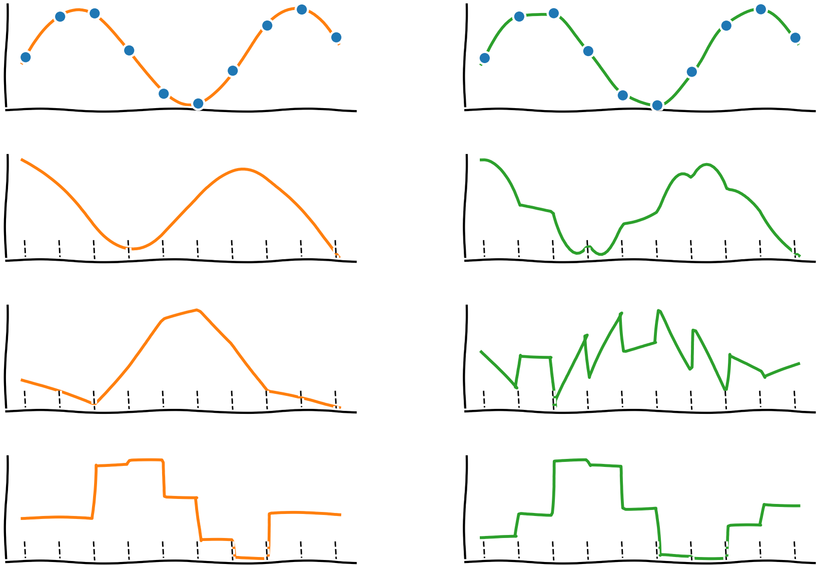
$$m_k = (y_{k+1} - y_k)/h_k$$

- if $\operatorname{sign} m_{k-1} \neq \operatorname{sign} m_k$, then $s_k = 0$
- if $m_{k-1} = 0$ or $m_k = 0$, then $s_k = 0$
- otherwise,

$$\frac{w_1 + w_2}{s_k} = \frac{w_1}{m_{k-1}} + \frac{w_2}{m_k}$$

where $w_1 = 2h_k + h_{k-1}$ and $w_2 = h_k + 2h_{k-1}$

Monotone piecewise cubic interpolant: PCHIP



Interpolation

- Lagrange interpolating polynomial
- Piecewise interpolators: splines
- Local piecewise interpolators

- Other forms of the interpolating polynomial: Newton, Krogh
- Spline *curves*
- Bezier curves
- B-splines, rational interpolation

- Smoothing splines
- Computer graphics, CAD, animation
- Image processing, computer vision