SIGNALS AND SYSTEMS USING MATLAB
Chapter 6 — Application of Laplace Analysis to Control

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- Classical control: uses frequency–domain methods
  Modern control: uses time–domain methods
- LTI system connections and block diagrams
  - Cascade: isolated systems with overall transfer function

$$H(s) = H_1(s)H_2(s)$$

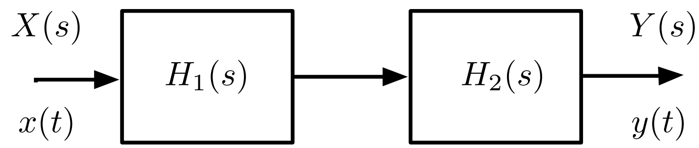
  - Parallel: same input into systems, overall transfer function

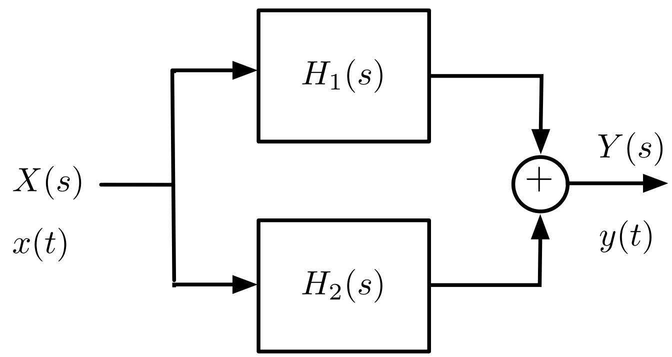
$$H(s) = H_1(s) + H_2(s)$$

  - Negative Feedback: output is fed back into input and subtracted from it. Overall transfer function

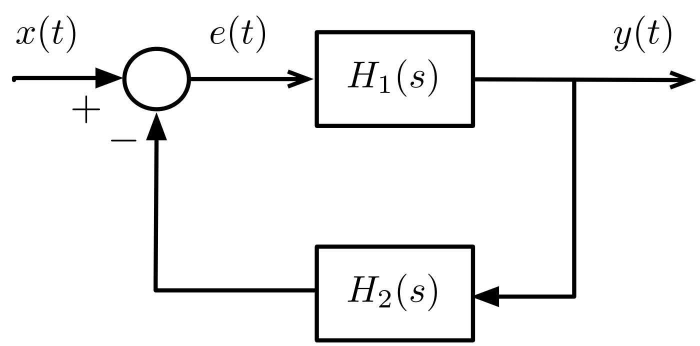
$$H(s) = \frac{H_1(s)}{1 + H_2(s)H_1(s)}$$

Open-loop transfer function: $H_{o\ell}(s) = H_1(s)$
Closed-loop transfer function: $H_{c\ell}(s) = H(s)$

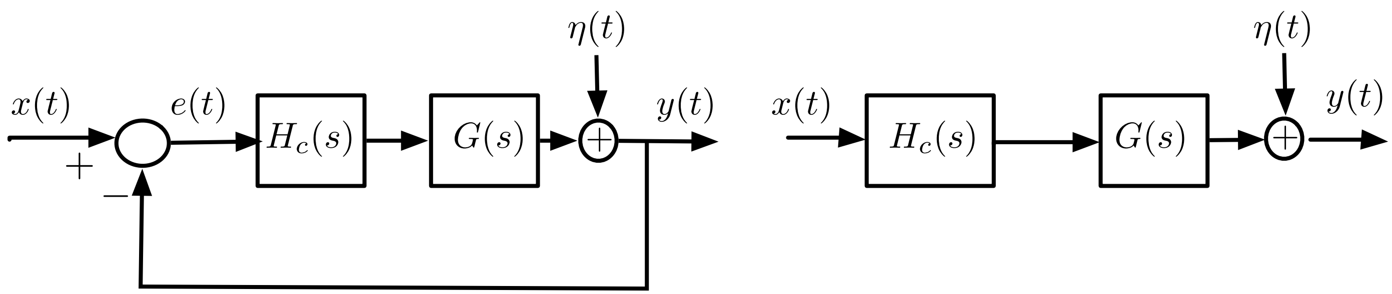




Cascade (top) and parallel (bottom) connections of systems with transfer function $H_1(s)$ and $H_2(s)$. The input/output are given in the time or frequency domains



Negative feedback connection of systems with transfer function $H_1(s)$ and $H_2(s)$. The input and the output are $x(t)$ and $y(t)$, $e(t)$ is the error signal



Closed-loop system *Open-loop system*

Closed- and open-loop control of systems. The transfer function of the plant is $G(s)$ and the transfer function of the controller is $H_c(s)$

- Open–loop control: Controller cascaded with plant. So output $y(t)$ follows reference signal $x(t)$, minimize error signal

$$e(t) = y(t) - x(t)$$

No disturbance, $\eta(t) = 0$

$$Y(s) = \mathcal{L}[y(t)] = H_c(s)G(s)X(s)$$

$$E(s) = Y(s) - X(s) = [H_c(s)G(s) - 1]X(s) \to 0 \;\Rightarrow\; H_c(s) = 1/G(s)$$

Disturbance $\eta(t)$

$$Y(s) = H_c(s)G(s)X(s) + \eta(s), \quad \eta(s) = \mathcal{L}[\eta(t)]$$

$$E(s) = [H_c(s)G(s) - 1]X(s) + \eta(s) \text{ cannot be made zero}$$

Closed–loop control: Assume $y(t)$ and $x(t)$ same type of signals,

No disturbance $\eta(t) = 0$

$$E(s) = X(s) - Y(s), \quad Y(s) = H_c(s)G(s)E(s)$$
$$E(s) = \frac{X(s)}{1 + G(s)H_c(s)}$$

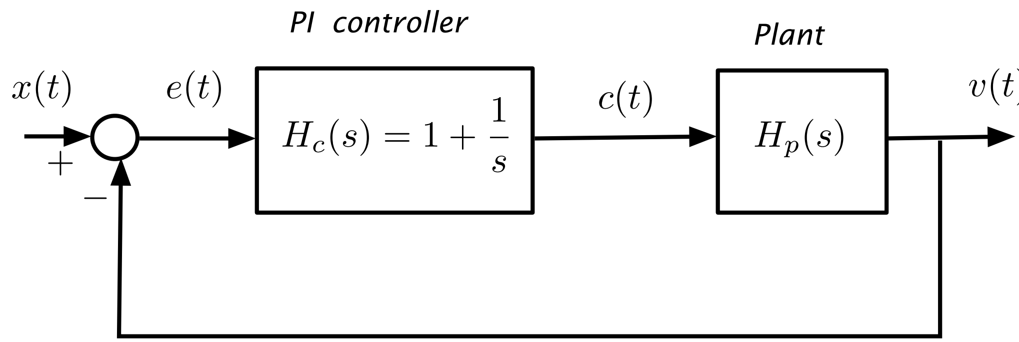
steady–state $e(t) \to 0$, poles $E(s)$ in open left–hand s–plane

Disturbance $\eta(t)$

$$E(s) = X(s) - Y(s), \quad Y(s) = H_c(s)G(s)E(s) + \eta(s)$$
$$E(s) = \frac{X(s)}{1 + G(s)H_c(s)} - \frac{\eta(s)}{1 + G(s)H_c(s)} = E_1(s) + E_2(s)$$

steady–state $e(t) \to 0$, poles $E_1(s)$, $E_2(s)$ in open left–hand s–plane

Example: Cruise control of speed of car using controller $H_c(s) = 1 + 1/s$



Cruise control system: reference speed $x(t) = V_0 u(t)$, output speed of car $v(t)$. Car model $H_p(s) = \beta/(s+\alpha)$, mass $\beta > 0$ and friction coefficient $\alpha > 0$

$$V(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)} X(s) = \frac{V_0}{s(s+1)}$$

$$V(s) = \frac{B}{s+1} + \frac{V_0}{s} \quad \text{steady–state} \quad \lim_{t\to\infty} v(t) = V_0$$

Error in steady–state

$$\text{final–value theorem} \quad E(s) = X(s) - V(s) = \frac{V_0}{s}\left[1 - \frac{1}{s+1}\right]$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} V_0 \left[1 - \frac{1}{s+1}\right] = 0$$

Controlling signal $c(t)$

$$c(t) = e(t) + \int_0^t e(\tau)d\tau$$

even if $e(t) = 0$ at some point, the value of $c(t) \neq 0$

Causal LTI system with transfer function $H(s) = B(s)/A(s)$ exhibiting no pole–zero cancellation is

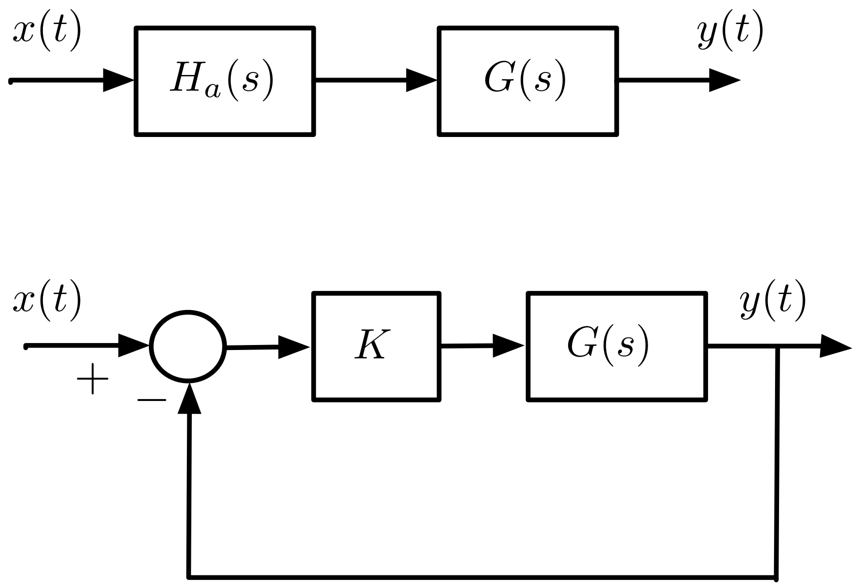
- Asymptotically stable if all-pole transfer function $H_1(s) = 1/A(s)$, used to determine the zero–input response, has all its poles in the open left–hand s–plane (the $j\Omega$–axis excluded), or

$$A(s) \neq 0 \;\text{ for }\; \mathcal{R}e[s] \geq 0$$

- BIBO stable if all the poles of $H(s)$ are in the open left–hand s-plane (the $j\Omega$-axis excluded), or equivalently

$$A(s) \neq 0 \;\text{ for }\; \mathcal{R}e[s] \geq 0$$

- If $H(s)$ exhibits pole–zero cancellations, the system may be BIBO stable but not asymptotically stable.



Stabilization of an unstable plant $G(s) = 1/(s-2)$ using (top) an all-pass filter, and (bottom) a proportional controller of gain K.

<ins>All–pass system</ins> $H_a(s) = \dfrac{s-2}{s+2}$

$$H_a(j\Omega)H_a(-j\Omega) = H_a(j\Omega)H_a^*(j\Omega) = |H_a(j\Omega)|^2 = 1$$

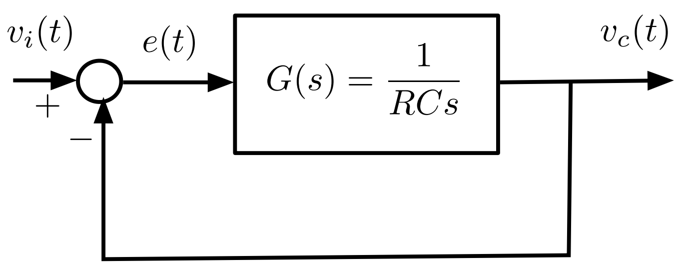
stabilized system $H(s) = G(s)H_a(s) = \dfrac{1}{s+2}$

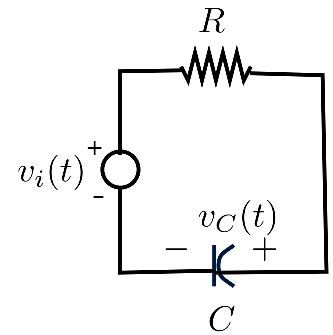
same magnitude as $G(s)$: $|H(j\Omega)| = |G(j\Omega)||H_a(j\Omega)| = |G(j\Omega)|$

<ins>Negative feedback system</ins> $H(s) = \dfrac{KG(s)}{1+KG(s)} = \dfrac{K}{s+(K-2)}$

gain $K > 2$, stable feedback system

Transient analysis



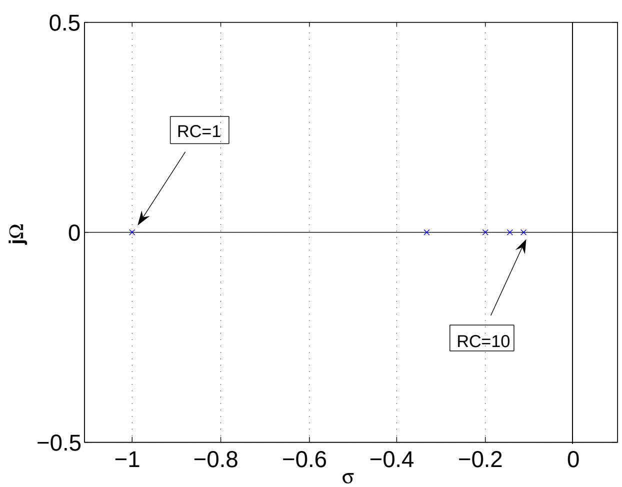


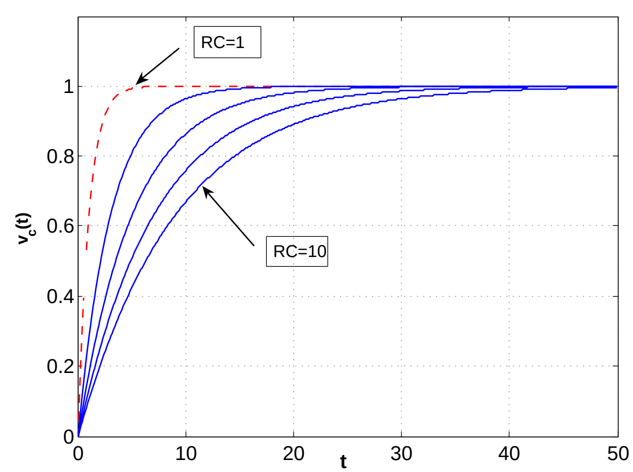
Feedback modeling of RC circuit as first–order system with transfer function
$H(s) = V_c(s)/V_i(s) = 1/(1 + RCs) = G(s)/(1 + G(s)),\ G(s) = 1/RCs$

Unit–step response

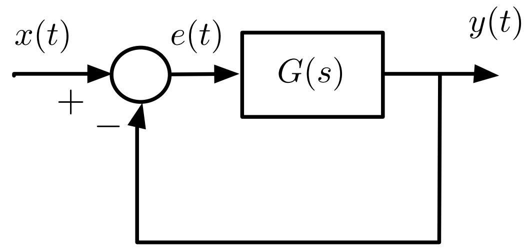
$$v_i(t) = u(t) \ \Rightarrow\ V_i(s) = 1/s, \quad V_c(s) = \frac{1}{s(sRC+1)} = \frac{1}{s} - \frac{1}{s + 1/RC}$$

$$v_c(t) = (1 - e^{-t/RC})u(t)$$





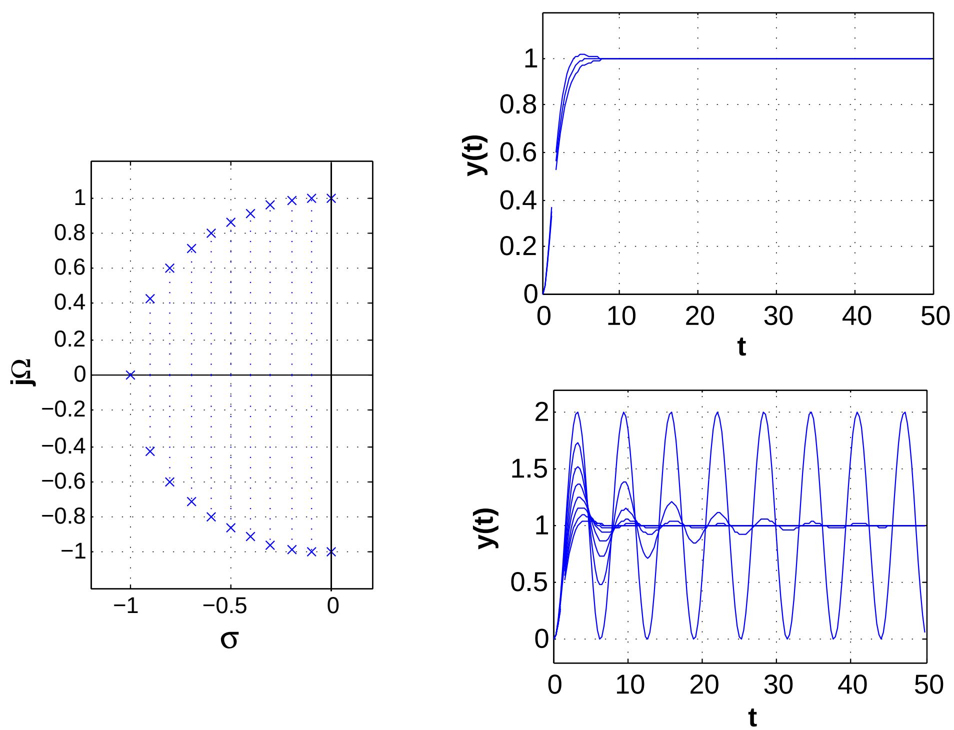
Clustering of poles (left) and transient responses $v_c(t)$ for $1 \leq RC \leq 10$



RLC circuits modeled as second–order feedback system

$$\frac{V_c(s)}{V_s(s)} = \frac{1/LC}{s^2 + (R/L)s + 1/LC} = \frac{G(s)}{1 + G(s)}, \quad G(s) = \frac{\Omega_n^2}{s(s + 2\psi\Omega_n)}$$

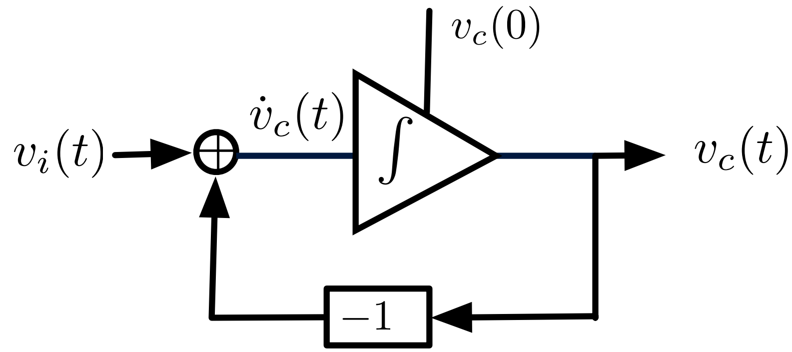
where $\Omega_n = 1/\sqrt{CL}$ natural frequency $\psi = 0.5R\sqrt{C/L}$ damping ratio



Clustering of poles (left) and time responses of second-order feedback system for $\sqrt{2}/2 \leq \psi \leq 1$ (top right) and $0 \leq \psi \leq \sqrt{2}/2$ (bottom right)

State variable representation of LTI systems

- State–variable representation— non–unique internal representation of system. State variables are memory of system
- Transfer function representation —external representation of system



RC circuit represented by $v_C(t) + dv_C(t)/dt = v_i(s)$, $t \geq 0$, *and initial condition* $v_C(0)$

state–variable $v_C(t)$

$$\text{for } t_1 > t_0 > 0, \;\; v_C(t_1) = e^{-(t_1 - t_0)} v_C(t_0) + \int_{t_0}^{t_1} e^{-(t_1 - \tau)} v_i(\tau) d\tau$$

i.e., given the state at t_0 and the input $v_i(t)$, we can compute future value $v_C(t_1)$ for $t_1 > t_0$ independent of how $v_C(t_0)$ is attained

$$\text{State–equation:} \quad \left[\frac{dv_C(t)}{dt}\right] = [-1][v_C(t)] + [1][v_i(t)]$$

State $\{x_k(t)\}$, $k = 1, \cdots, N$, of LTI system: the smallest set of variables that if known at a certain time t_0 allows us to compute the response of the system at times $t > t_0$ for specified inputs $\{w_i(t)\}$, $i = 1, \cdots, M$

Multiple-input multiple-output state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{w}(t)$$

$\mathbf{x}^T(t) = [x_1(t)\ x_2(t) \cdots x_N(t)]$, state vector

$\dot{\mathbf{x}}^T(t) = [\dot{x}_1(t)\ \dot{x}_2(t) \cdots \dot{x}_N(t)]$

$\mathbf{A} = [a_{ij}]$ $N \times N$ matrix

$\mathbf{B} = [b_{ij}]$ $N \times M$ matrix

$\mathbf{w}^T(t) = [w_1(t)\ w_2(t) \cdots w_M(t)]$ input vector

Output equation

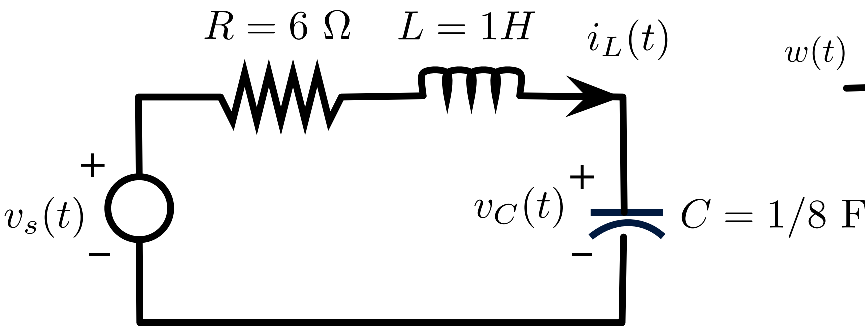
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{w}(t)$$

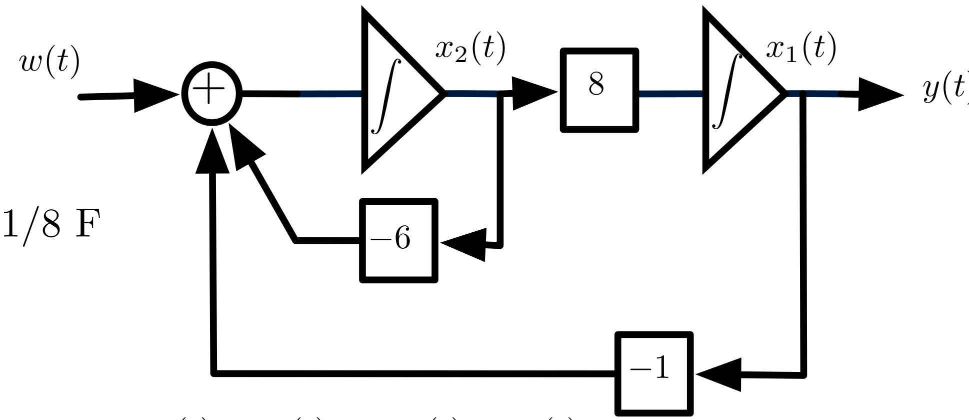
$\mathbf{y}(t) = [y_1(t)\ y_2(t) \cdots y_L(t)]$, output vector

$\mathbf{C} = [c_{ij}]$ $L \times N$ matrix

$\mathbf{D} = [d_{ij}]$ $L \times M$ matrix

Example:





$w(t) = v_s(t) \qquad x_2(t) = i_L(t)$

$x_1(t) = v_C(t) \qquad y(t) = v_C(t)$

$$\frac{dv_C(t)}{dt} = \frac{1}{C} i_L(t)$$
$$\frac{di_L(t)}{dt} = -\frac{1}{L} v_C(t) - \frac{R}{L} i_L(t) + \frac{1}{L} v_s(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ -1 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]\ w(t)$$

Initial conditions

$$y(t) = x_1(t)$$
$$\dot{y}(t) = \dot{x}_1(t) = 8x_2(t)$$

Another set of state equations

$$V_C(s) = \frac{8}{s^2+6s+8} V_s(s) = \underbrace{\frac{4V_s(s)}{s+2)}}_{\hat{X}_1(s)} + \underbrace{\frac{-4V_s(s)}{s+4}}_{\hat{X}_2(s)}$$

Letting $x_1(t), x_2(t)$ state variables, $w(t) = v_s(t)$ input, and $y(t) = v_C(t)$ output:

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \end{bmatrix} w(t)$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}$$

Connection between the first and second set of state equations

$$\mathbf{x}(t) = \mathbf{F}\hat{\mathbf{x}}(t),\ \mathbf{F} \text{ invertible} \quad \Rightarrow \quad \begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathbf{F}^{-1}\mathbf{A}\mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{F}^{-1}\mathbf{b}w(t) \\ y(t) &= \mathbf{c}^T\mathbf{F}\hat{\mathbf{x}}(t) \end{aligned}$$

$$\text{where } \ \mathbf{F} = \begin{bmatrix} 1 & 1 \\ -0.25 & -0.5 \end{bmatrix}, \qquad \mathbf{F}^{-1} = \begin{bmatrix} 2 & 4 \\ -1 & -4 \end{bmatrix} \qquad \text{so that}$$
$$\mathbf{F}^{-1}\mathbf{A}\mathbf{F} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}, \qquad \mathbf{F}^{-1}\mathbf{b} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}, \qquad \mathbf{c}^T\mathbf{F} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

State variables of a system are not unique

Given state and output equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}w(t)$$

$$y(t) = \mathbf{c}^T\mathbf{x}(t) + dw(t)$$

New set of state variables $\{z_i(t)\}$ obtained using invertible transformation matrix $\mathbf{F}$

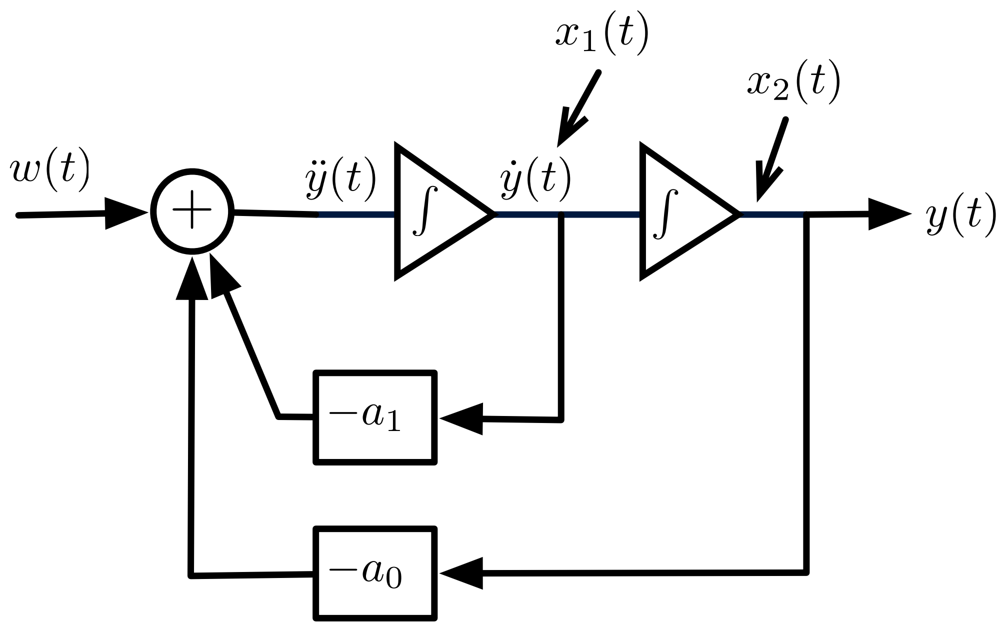
$$\mathbf{x}(t) = \mathbf{F}\mathbf{z}(t)$$

New state variable representation is

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{A_1}\mathbf{z}(t) + \mathbf{b_1}w(t) \\ y(t) &= \mathbf{c_1}^T\mathbf{z}(t) + d_1 w(t) \end{aligned}$$

$$\mathbf{A}_1 = \mathbf{F}^{-1}\mathbf{A}\mathbf{F}, \quad \mathbf{b}_1 = \mathbf{F}^{-1}\mathbf{b}, \quad \mathbf{c}_1^T = \mathbf{c}^T\mathbf{F}, \quad d_1 = d$$

- Direct minimal realization



Minimal direct realization of $d^2y(t)/dt^2 + a_1\, dy(t)/dt + a_0 y(t) = w(t)$ *(all-pole system) with state variables* $x_1(t) = \dot{y}(t)$ *and* $x_2(t) = y(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -a_1 & -a_0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Minimal realization: Number of integrators equals order of the system

General transfer function: input $w(t)$, output $y(t)$

$$H(s) \;=\; \frac{Y(s)}{W(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}, \quad m < n$$

$$Y(s) \;=\; \underbrace{\frac{W(s)}{D(s)}}_{Z(s)} N(s)$$

allow us to define all-pole and only–zeros transfer functions

$$(i)\ \frac{Z(s)}{W(s)} = \frac{1}{D(s)}, \qquad (ii)\ \frac{Y(s)}{Z(s)} = N(s)$$

From which

$$D(s)Z(s) = W(s) \;\Rightarrow\; \frac{d^n z(t)}{dt^n} + a_{n-1}\frac{d^{n-1}z(t)}{dt^{n-1}} + \cdots + a_0 z(t) = w(t)$$

requiring n integrators (minimal realization)

$$N(s)Z(s) = Y(s) \;\Rightarrow\; b_m\frac{d^m z(t)}{dt^m} + b_{m-1}\frac{d^{m-1}z(t)}{dt^{m-1}} + \cdots + b_0 z(t) = y(t)$$

requiring n integrators (minimal realization)

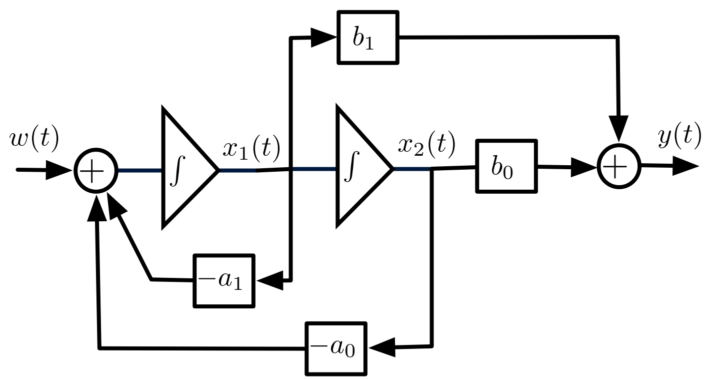
Example:

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_1\frac{dw(t)}{dt} + b_0w(t)$$

$$H(s) = \left[\frac{Y(s)}{Z(s)}\right]\left[\frac{Z(s)}{W(s)}\right] = [b_0 + b_1s]\left[\frac{1}{a_0 + a_1s + a_2s^2}\right]$$

Realizations of

$$\begin{aligned} w(t) &= a_0z(t) + a_1\frac{dz(t)}{dt} + \frac{d^2z(t)}{dt^2} \quad \text{and of} \\ y(t) &= b_0z(t) + b_1\frac{dz(t)}{dt} \end{aligned}$$



State variables $x_1(t) = \dot{z}(t)$ and $x_2(t) = z(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1 & -a_0 \\ 1 & 0 \end{bmatrix}}_{\mathbf{A}_c} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{b}_c} w(t), \;\; y(t) = \underbrace{\begin{bmatrix} b_1 & b_0 \end{bmatrix}}_{\mathbf{c}_c^T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Parallel and cascade realizations

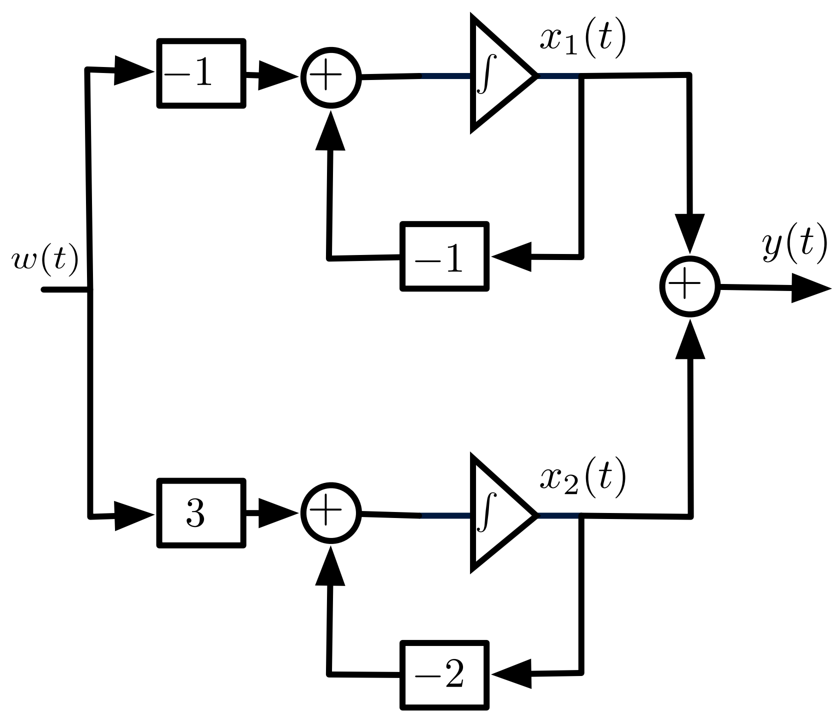
$$H(s) = \frac{N(s)}{D(s)} \text{ proper rational}$$

$$\text{parallel realization} \quad H(s) = \sum_{i=1}^{N} H_i(s)$$

$$H_i(s) \text{ proper rational with real coefficients}$$

Example:

$$H(s) = \frac{1+2s}{2+3s+s^2} = \frac{1+2s}{(s+1)(s+2)} = \frac{-1}{s+1} + \frac{3}{s+2}$$



$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} w(t)$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Example: Transfer function with repeated poles

$$G(s) = \frac{s+2}{(s+1)^2}$$

- parallel realization:

$$G(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} \quad \text{not minimal}$$

- cascade and parallel realization:

$$G(s) = \frac{1}{s+1}\left[1 + \frac{1}{s+1}\right] \quad \text{minimal}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

State and output equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\mathrm{w}(t)$$
$$y(t) = \mathbf{c}^T\mathbf{x}(t), \qquad t > 0$$

Complete solution

$$y(t) = \mathbf{c}^T e^{\mathbf{A}(t)}\mathbf{x}(0) + \mathbf{c}^T \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}\mathrm{w}(\tau)d\tau \quad t \geq 0$$

exponential matrix $e^{\mathbf{A}t} = [\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!}\cdots]$

Impulse response

$$h(t) = \mathbf{c}^T e^{\mathbf{A}t}\mathbf{b}$$

Example: Input $w(t) = u(t)$, IC: $x_1(0) = 1$ and $x_2(0) = 0$.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \end{bmatrix} \mathrm{w}(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Laplace transform of the state equation, non-zero initial conditions:

$$\begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) + 8W(s) \end{bmatrix}$$

$$Y(s) = X_1(s)$$

Cramer's rule

$$\begin{aligned} Y(s) &= X_1(s) = \frac{\det \begin{bmatrix} x_1(0) & -1 \\ x_2(0) + 8W(s) & s+6 \end{bmatrix}}{s(s+6)+8} \\ &= \frac{x_1(0)(s+6) + x_2(0) + 8W(s)}{s^2+6s+8} = \underbrace{\frac{s+6}{s^2+6s+8}}_{Y_{zi}(s)} + \underbrace{\frac{8}{s(s^2+6s+8)}}_{Y_{zs}(s)} \end{aligned}$$