SIGNALS AND SYSTEMS USING MATLAB
Chapter 7 — Fourier Analysis in Communications and Filtering

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Modulation systems

- Given lowpass nature of most message signals, it is necessary to shift in frequency the spectrum of the message to avoid using a very large antenna
- This is attained by modulation: changing either the magnitude $A(t)$ or the phase $\theta(t)$ of a carrier

$$A(t)\cos(2\pi f_c + \theta(t)).$$

giving

  - Amplitude Modulation (AM): $A(t)$ proportional to message, for constant phase
  - Frequency Modulation (FM), Phase modulation (PM): $\theta(t)$ changes with the message

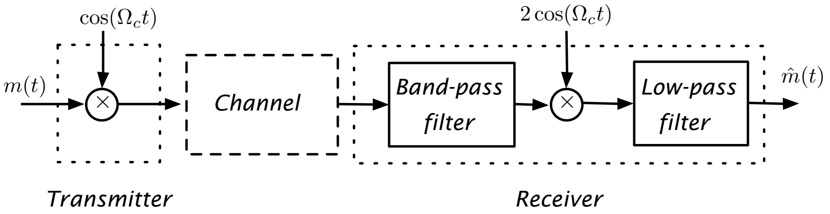
- Communication system: cascade of transmitter, channel and receiver none LTI

- AM Suppressed Carrier

$$\text{message } m(t), \quad \text{carrier } \cos(\Omega_c t)$$

$$\text{modulated signal } s(t) = m(t)\cos(\Omega_c t)$$
$$\Omega_c >> 2\pi f_0, \quad f_0 \text{ maximum frequency in } m(t)$$

Fourier spectrum
$$S(\Omega) = \frac{1}{2}\left[M(\Omega - \Omega_c) + M(\Omega + \Omega_c)\right]$$
$M(\Omega)$ spectrum of $m(t)$



AM-SC transmitter, channel and receiver

- Demodulation: assuming output of band–pass filter is $s(t)$

$$
\begin{aligned}
R(\Omega) &= S(\Omega - \Omega_c) + S(\Omega + \Omega_c) \\
&= M(\Omega) + \frac{1}{2}\left[M(\Omega - 2\Omega_c) + M(\Omega + 2\Omega_c)\right]
\end{aligned}
$$

Output of LPF is $M(\Omega)$ or $m(t)$

- Demodulation requires exact carrier frequency:
if demodulator uses $\Omega_c + \Delta$, $\Delta > 0$:

$$\tilde{r}(t) = s(t)\cos((\Omega_c + \Delta)t)$$

$$
\begin{aligned}
\tilde{R}(\Omega) = S(\Omega - \Omega_c - \Delta) + S(\Omega + \Omega_c + \Delta) \\
= \frac{1}{2}\left[M(\Omega + \Delta) + M(\Omega - \Delta)\right] \\
+ \frac{1}{2}\left[M(\Omega - 2(\Omega_c + \Delta/2)) + M(\Omega + 2(\Omega_c + \Delta/2)\right].
\end{aligned}
$$

Output of LPF is distorted message

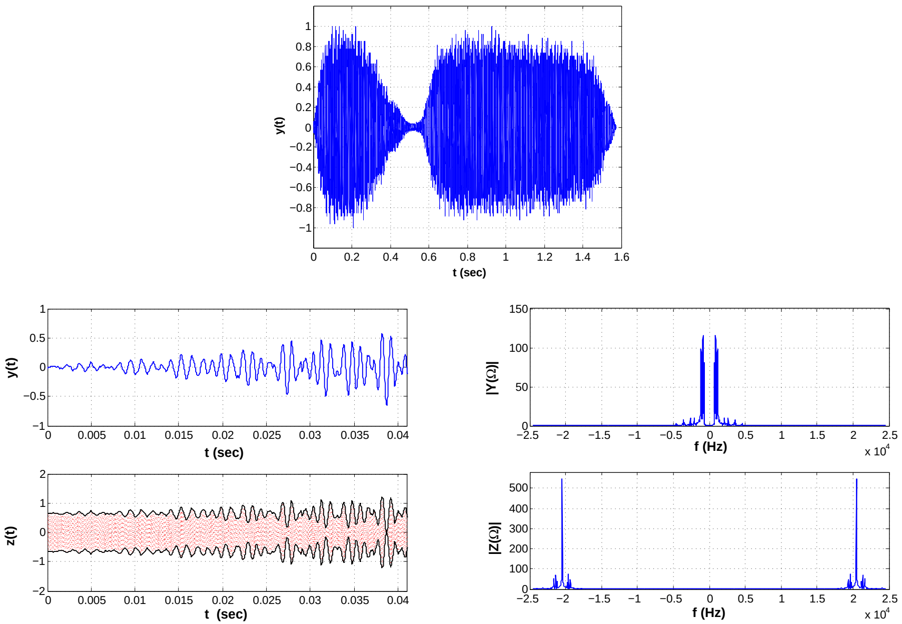
- Commercial AM

$$s(t) = [K + m(t)]\cos(\Omega_c t) \quad \text{modulated signal}$$

K chosen so that $K + m(t) > 0$

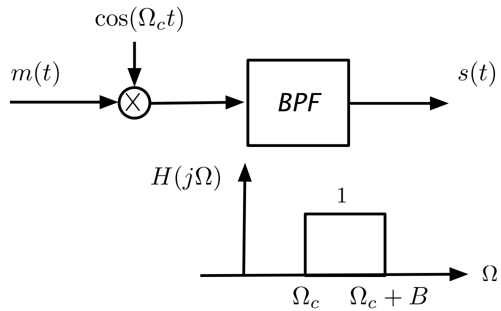
$$S(\Omega) = K\pi\,[\delta(\Omega - \Omega_c) + \delta(\Omega + \Omega_c)] + \frac{1}{2}\,[M(\Omega - \Omega_c) + M(\Omega + \Omega_c)]\,.$$

Example: Commercial AM modulation



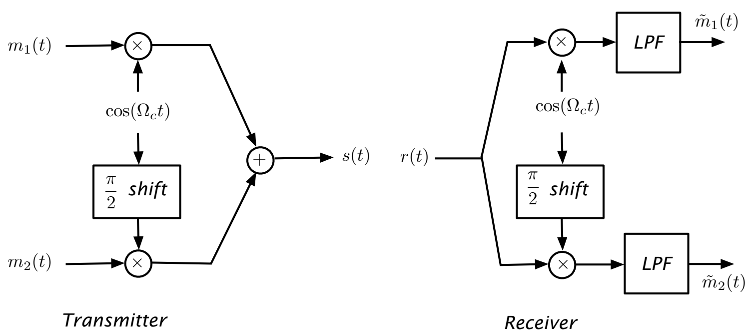
Message (top), part of original signal and corresponding AM modulated signal (bottom left), spectrum of the original signal and of the modulated signal (top and bottom right).

- Single side–band modulation – efficient use of spectrum by reducing bandwidth of modulate signal



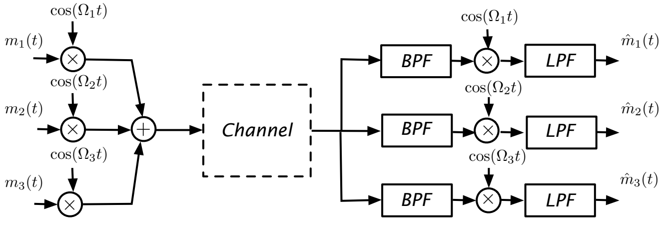
Upper side-band AM transmitter. Ω_c is the carrier frequency and B the bandwidth of the message

- Quadrature AM –efficient use of spectrum by sending two messages on the same band



QAM transmitter and receiver: $s(t)$ is the transmitted signal and $r(t)$ the received signal

- Frequency division multiplexing – sharing the spectrum



Frequency division multiplexing (FDM) system

- Frequency modulation (FM) – non–linear, time–varying modulation system

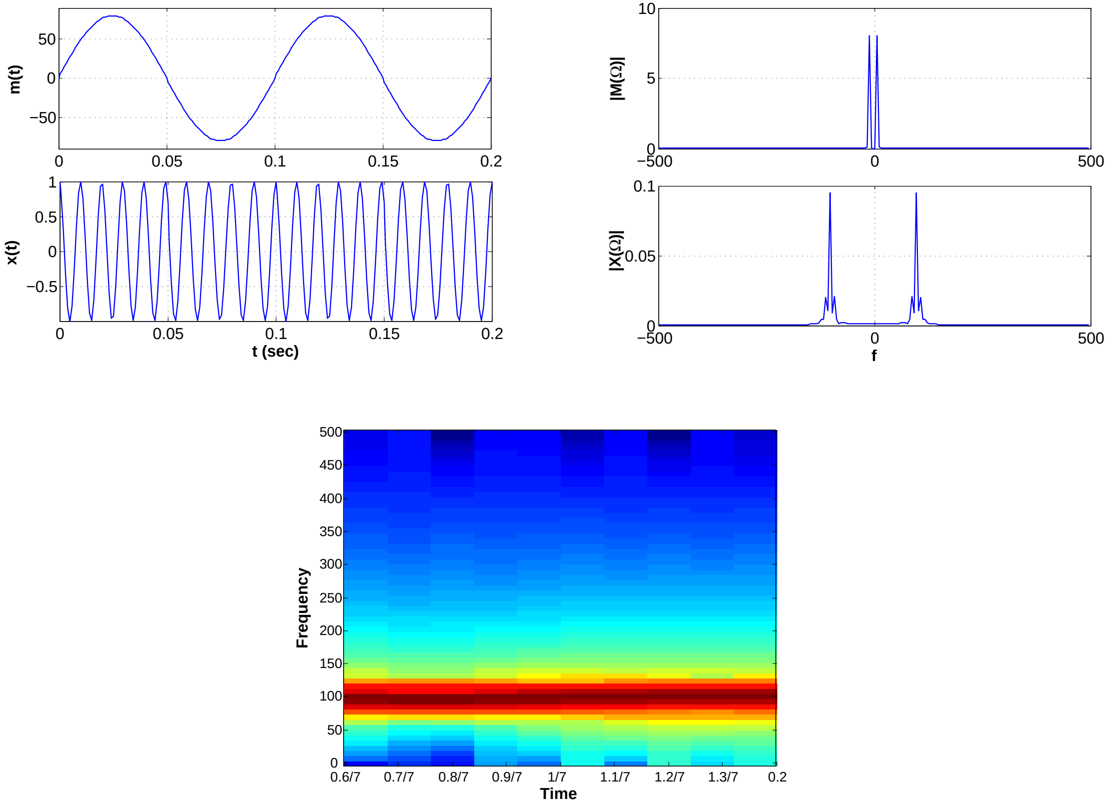
$$s_{FM}(t) = \cos(\Omega_c t + \Delta\Omega \int_{-\infty}^{t} m(\tau)d\tau)$$

  - Narrow–band FM: angle $\theta(t)$ small so that $\cos(\theta(t)) \approx 1, \;\; \sin(\theta(t)) \approx \theta(t)$

$$\frac{d\theta(t)}{dt} = \Delta\Omega \; m(t), \quad IF(t) = \frac{d[\Omega_c t + \theta(t)]}{dt} = \Omega_c + \Delta\Omega \; m(t)$$

$$S(\Omega) = \mathcal{F}\left[\cos(\Omega_c t + \theta(t))\right] = \mathcal{F}\left[\cos(\Omega_c t)\cos(\theta(t)) - \sin(\Omega_c t)\sin(\theta(t))\right]$$
$$\approx \pi\left[\delta(\Omega - \Omega_c) + \delta(\Omega + \Omega_c)\right] + \frac{1}{2j}\left[\Theta(\Omega - \Omega_c) - \Theta(\Omega + \Omega_c)\right]$$
$$\Theta(\Omega) = \frac{\Delta\Omega}{j\Omega} M(\Omega)$$

Example: Narrow–band FM



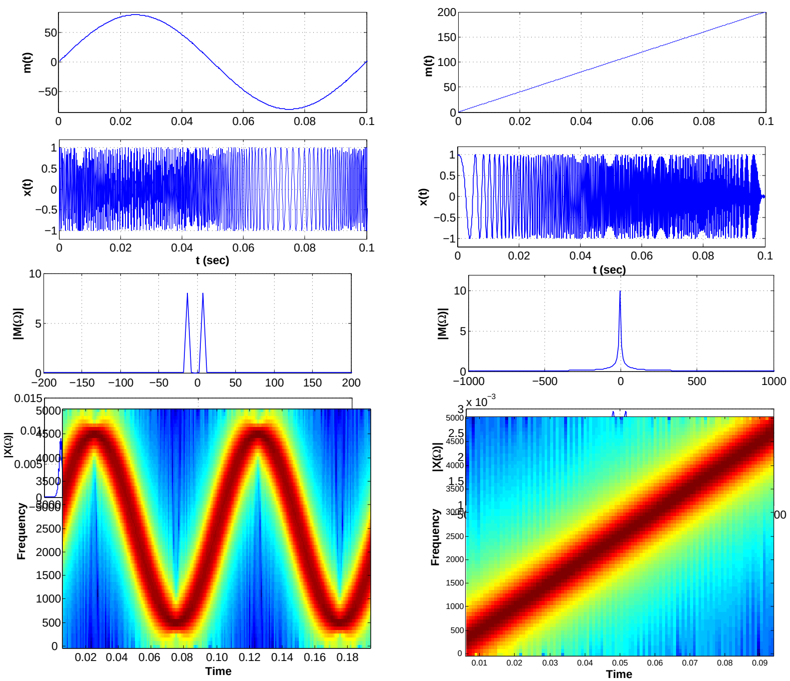
Top left— message $m(t) = 80\sin(20\pi t)u(t)$ *and narrow-band FM signal* $x(t) = \cos(2\pi f_c t + 0.1\pi\int_{-\infty}^{t} m(\tau)d\tau)$; *top-right— magnitude spectra of* $m(t)$ *and* $x(t)$. *Spectrogram of* $x(t)$ *displaying evolution of its Fourier transform with respect to time.*

- Wide–band FM

  messages: $m_1(t) = 80\sin(20\pi t)u(t), \quad m_2(t) = 2000tu(t)$

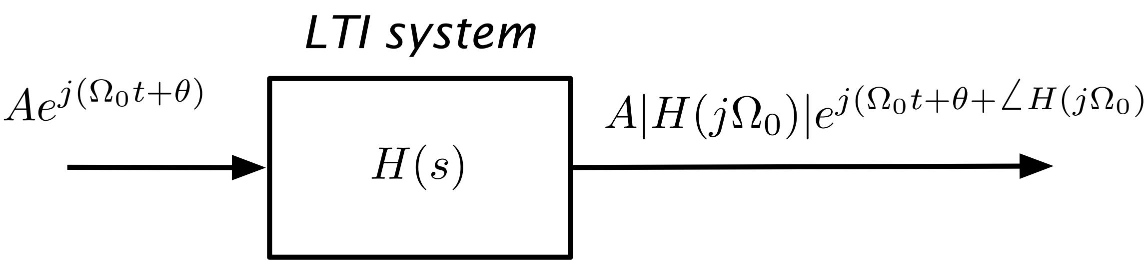
  instantaneous frequencies $IF_i(t) = 2\pi f_{ci} + 50\pi m_i(t)\ i = 1, 2$

  $f_{c1} = 2500, \ f_{c2} = 25Hz$



Left sinusoidal message and right ramp message: messages, FM modulated signals, spectra of messages, spectra of FM signals, and spectrograms of FM signals

- Use of eigenfunction property of LTI systems – periodic and aperiodic signals have Fourier representations consisting of sinusoids of different frequencies, the frequency components of any signal can be modified by appropriately choosing the frequency response of the LTI system or filter



Eigenfunction property of continuous LTI systems

- Appropriate filter for a certain application is specified using the spectral characterization of the input and the desired spectral characteristics of the output

- Classical approach in filter design is to consider lowpass prototypes, with normalized frequency and magnitude responses, which may be transformed into other filters with the desired frequency response

Filter transfer function $H(s) = \dfrac{B(s)}{A(s)}$ (LTI system with specific frequency response)

filter output $Y(\Omega) = X(\Omega)H(j\Omega)$

- Low–pass filter design

$$\text{Choose magnitude squared function } |H(j\Omega)|^2 = \frac{1}{1 + f(\Omega^2)}$$

such that for low frequencies $f(\Omega^2) << 1 \quad \Rightarrow \quad |H(j\Omega)|^2 \approx 1$,

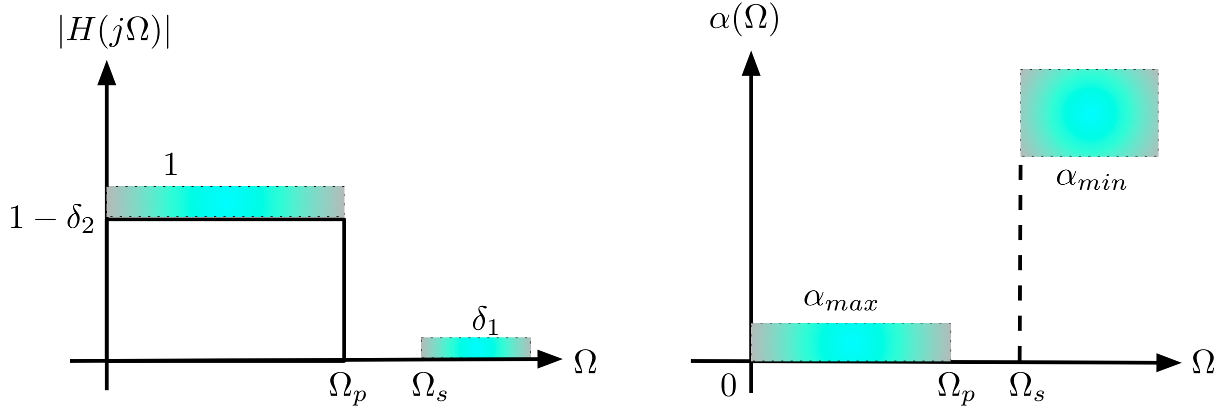
for high frequencies $f(\Omega^2) >> 1 \quad \Rightarrow |H(j\Omega)|^2 \rightarrow 0$

Issues to consider:

- selection of the appropriate function $f(.)$,
- the factorization needed to get $H(s)$ from the magnitude squared function
- frequency transformation to convert LPF into other filters

- Magnitude specifications

$$\begin{aligned} 1 - \delta_2 \; &\leq \; |H(j\Omega)| \leq 1 \qquad 0 \leq \Omega \leq \Omega_p \quad \text{(passband)} \\ 0 \; &\leq \; |H(j\Omega)| \leq \delta_1 \qquad \Omega \geq \Omega_s \quad \text{(stopband)} \end{aligned}$$



Equivalent magnitude specifications for a lowpass filter

- Loss specifications

$$\text{Loss function } \alpha(\Omega) = -10\log_{10}|H(j\Omega)|^2 = -20\log_{10}|H(j\Omega)| \text{ dBs}$$

$$0 \leq \alpha(\Omega) \leq \alpha_{max} \qquad 0 \leq \Omega \leq \Omega_p \quad \text{(passband)}$$
$$\alpha(\Omega) \geq \alpha_{min} \qquad \Omega \geq \Omega_s \quad \text{(stopband)}$$
$$\alpha_{max} = -20\log_{10}(1-\delta_2), \quad \alpha_{min} = -20\log_{10}(\delta_1)$$

- General case: $\alpha(0) = \alpha_1$, α_2 in the passband and α_3 in the stopband

$$\alpha(0) = \alpha_1 \quad \text{dc loss}$$
$$\alpha_{max} = \alpha_2 - \alpha_1 \quad \text{maximum attenuation in passband}$$
$$\alpha_{min} = \alpha_3 - \alpha_1 \quad \text{minimum attenuation in stopband}$$

- Magnitude response

$$N^{th}\text{–order lowpass Butterworth filter}$$
$$|H_N(j\Omega')|^2 = \frac{1}{1+[\Omega']^{2N}} \qquad \Omega' = \frac{\Omega}{\Omega_{hp}}$$
$$\Omega_{hp} \text{ half-power or } -3dB \text{ frequency}$$

- Factorization

$$S = s/\Omega_{hp} \;\Rightarrow\; S/j = \Omega' = \Omega/\Omega_{hp}$$
$$H_N(S)H_N(-S) = \frac{1}{1+(-S^2)^N}$$
$$D(S)D(-S) = 1+(-S^2)^N \;\Rightarrow\; H_N(S) = 1/D(S)$$
$$\text{Poles: } (-1)^N S_k^{2N} = e^{j(2k-1)\pi} \quad\Rightarrow\quad S_k^{2N} = \frac{e^{j(2k-1)\pi}}{e^{-j\pi N}} = e^{j(2k-1+N)\pi}$$
$$S_k = e^{j(2k-1+N)\pi/(2N)} \qquad k = 1,\cdots,2N$$

- Poles in circle of radius 1
- No poles on $j\Omega$–axis
- Consecutive poles separated by π/N radians

- Filter design

$$\alpha(\Omega) = -10\log_{10}|H_N(\Omega/\Omega_{hp})|^2 = 10\log_{10}(1+(\Omega/\Omega_{hp})^{2N})$$

$$0 \le \alpha(\Omega) \le \alpha_{max} \qquad 0 \le \Omega \le \Omega_p$$

$$\alpha_{min} \le \alpha(\Omega) < \infty \qquad \Omega \ge \Omega_s$$

$$\Omega = \Omega_p \;\Rightarrow\; \alpha(\Omega_p) = 10\log_{10}(1+(\Omega_p/\Omega_{hp})^{2N}) \le \alpha_{max} \text{ so that}$$

$$\frac{\Omega_p}{\Omega_{hp}} \le \left(10^{0.1\alpha_{max}} - 1\right)^{1/2N}$$

$$\Omega = \Omega_s \;\Rightarrow\; \alpha(\Omega_s) = 10\log_{10}(1+(\Omega_s/\Omega_{hp})^{2N}) \ge \alpha_{min} \text{ so that}$$

$$\frac{\Omega_s}{\Omega_{hp}} \ge (10^{0.1\alpha_{min}} - 1)^{1/2N}$$

half-power frequency:

$$\frac{\Omega_p}{(10^{0.1\alpha_{max}} - 1)^{1/2N}} \le \Omega_{hp} \le \frac{\Omega_s}{(10^{0.1\alpha_{min}} - 1)^{1/2N}}$$

minimum order:

$$N \ge \frac{\log_{10}[(10^{0.1\alpha_{min}} - 1)/(10^{0.1\alpha_{max}} - 1)]}{2\log_{10}(\Omega_s/\Omega_p)}$$

- Normalized magnitude squared function

$$|H_N(j\Omega')|^2 = \frac{1}{1+\varepsilon^2 C_N^2(\Omega')}, \quad \Omega' = \frac{\Omega}{\Omega_p}$$

N order of filter, ε ripple factor, $C_N(.)$ Chebyshev polynomials

- Chebyshev polynomials

$$C_N(\Omega') = \begin{cases} \cos(N\cos^{-1}(\Omega')) & |\Omega'| \le 1 \\ \cosh(N\cosh^{-1}(\Omega')) & |\Omega'| > 1. \end{cases}$$

Three term difference equation:

$$C_{N+1}(\Omega') + C_{N-1}(\Omega') = 2\Omega' C_N(\Omega'), \quad N \ge 0$$

initial conditions

$$C_0(\Omega') = \cos(0) = 1$$
$$C_1(\Omega') = \cos(\cos^{-1}(\Omega')) = \Omega'$$

$$C_0(\Omega') = 1,$$
$$C_1(\Omega') = \Omega',$$
$$C_2(\Omega') = -1 + 2\Omega'^2,$$
$$C_3(\Omega') = -3\Omega' + 4\Omega'^3, \quad \cdots$$

- Filter design

$$\alpha(\Omega') = 10\log_{10}\left[1+\varepsilon^2 C_N^2(\Omega')\right] \qquad \Omega' = \frac{\Omega}{\Omega_p}$$

Ripple factor

$$\varepsilon = \sqrt{10^{0.1\alpha_{max}}-1}, \qquad RW = 1 - \frac{1}{\sqrt{1+\varepsilon^2}}$$

Minimum order

$$N \geq \frac{\cosh^{-1}\left(\left[\frac{10^{0.1\alpha_{min}}-1}{10^{0.1\alpha_{max}}-1}\right]^{0.5}\right)}{\cosh^{-1}(\frac{\Omega_s}{\Omega_p})}$$

Half-power frequency :

$$\alpha(\Omega_{hp}) = 10\log_{10}(1+\varepsilon^2 C_N^2(\Omega'_{hp})) = 3 \text{ dB}, \text{ then}$$

$$1+\varepsilon^2 C_N^2(\Omega'_{hp}) = 10^{0.3} \approx 2$$

$$C_N(\Omega'_{hp}) = \frac{1}{\varepsilon} = \cosh(N\cosh^{-1}(\Omega'_{hp}))$$

$$\Omega_{hp} = \Omega_p \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\varepsilon}\right)\right]$$

- Factorization

$$\Omega' = S/j, \quad S = s/\Omega_p$$

$$H(S)H(-S) = \frac{1}{1+\varepsilon^2 C_N^2(S/j)} = \frac{1}{D(S)D(-S)}$$

Guillemin's

$$a = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)$$

$$\sigma_k = -\sinh(a)\cos(\psi_k) \qquad \text{real part of pole}$$

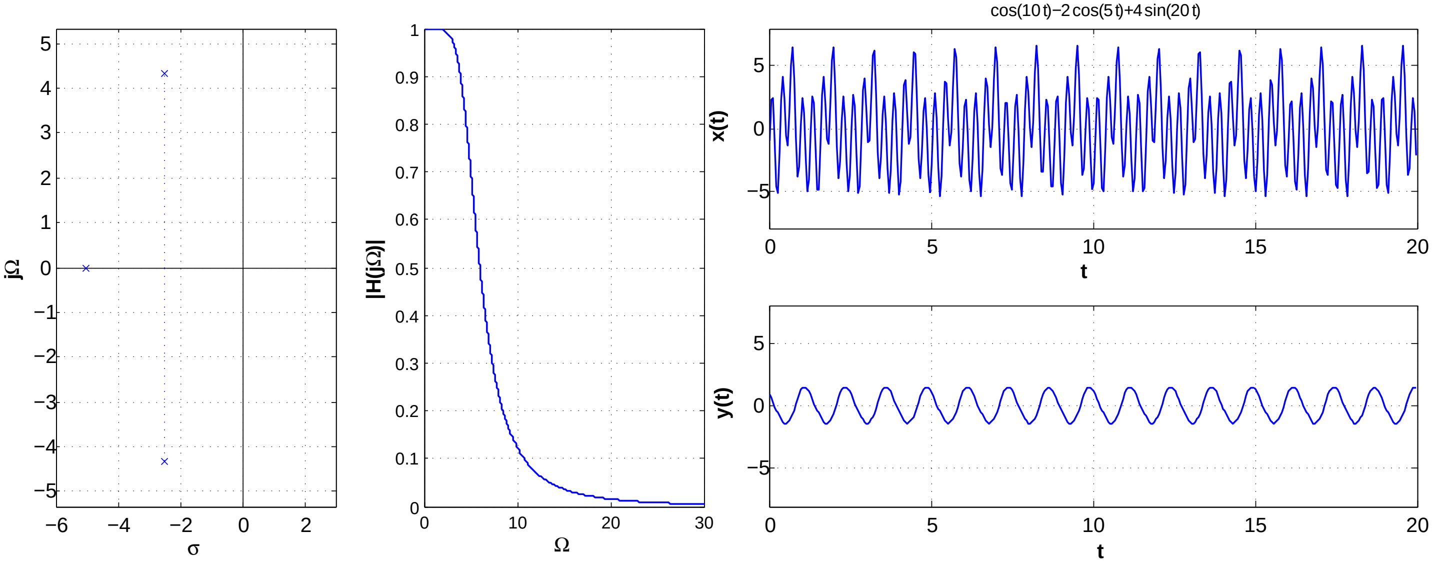
$$\Omega'_k = \pm\cosh(a)\sin(\psi_k) \qquad \text{imaginary part of pole}$$

where $0 \le \psi_k < \pi/2$ are the angles corresponding to the Butterworth filters (measured with respect to the negative real axis of the S plane)

Example: Lowpass filtering

$$x(t) = [-2\cos(5t) + \cos(10t) + 4\sin(20t)]u(t)$$

Design third-order lowpass Butterworth filter with a half-power frequency $\Omega_{hp} = 5$ rad/sec, to attenuate frequency components of frequency 10 and 20



Signal $x(t)$, top right figure; lowpass Butterworth filter with poles and magnitude response shown on the left. Filtered signal, bottom right, is approximately the low–frequency component of $x(t)$

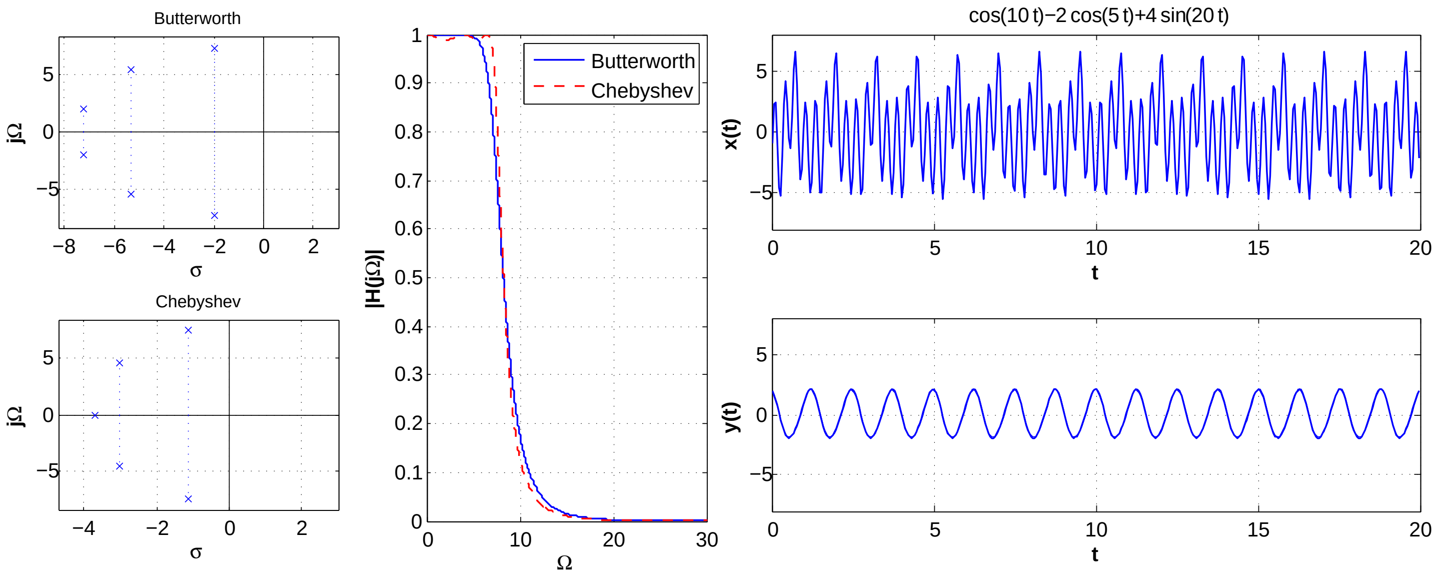
Example: Butterworth vs Chebyshev lowpass filters with $\Omega_{hp} = 5$ rad/sec
Filtering $x(t) = [-2\cos(5t) + \cos(10t) + 4\sin(20t)]u(t)$

Specifications

$\alpha(0) = 0$ dB

$\alpha_{max} = 0.1$ dB, $\Omega_p = 5$ rad/sec

$\alpha_{min} = 15$ dB, $\Omega_s = 10$ rad/sec

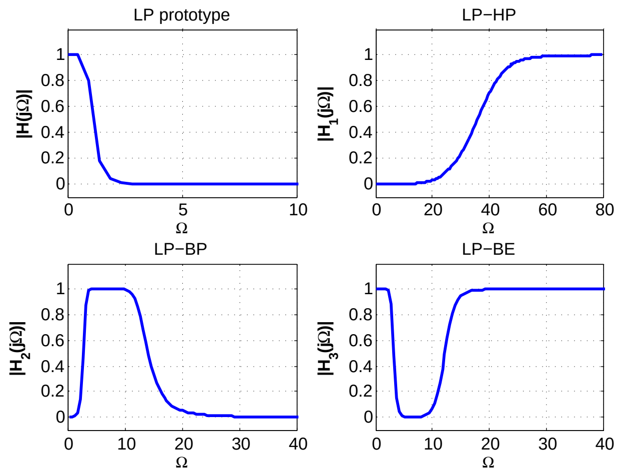


Frequency transformations

Lowpass-Lowpass	$S = \dfrac{s}{\Omega_0}$
Lowpass-Highpass	$S = \dfrac{\Omega_0}{s}$
Lowpass-Bandpass	$S = \dfrac{s^2 + \Omega_0}{s\ BW}$
Lowpass-Bandstop	$S = \dfrac{s\ BW}{s^2 + \Omega_0}$

S is the normalized and s the final variables, Ω_0 is a desired cut-off frequency and BW a desired bandwidth

Example:Lowpass prototype filter (Butterworth), $\Omega_0 = 40$ and $BW = 10$



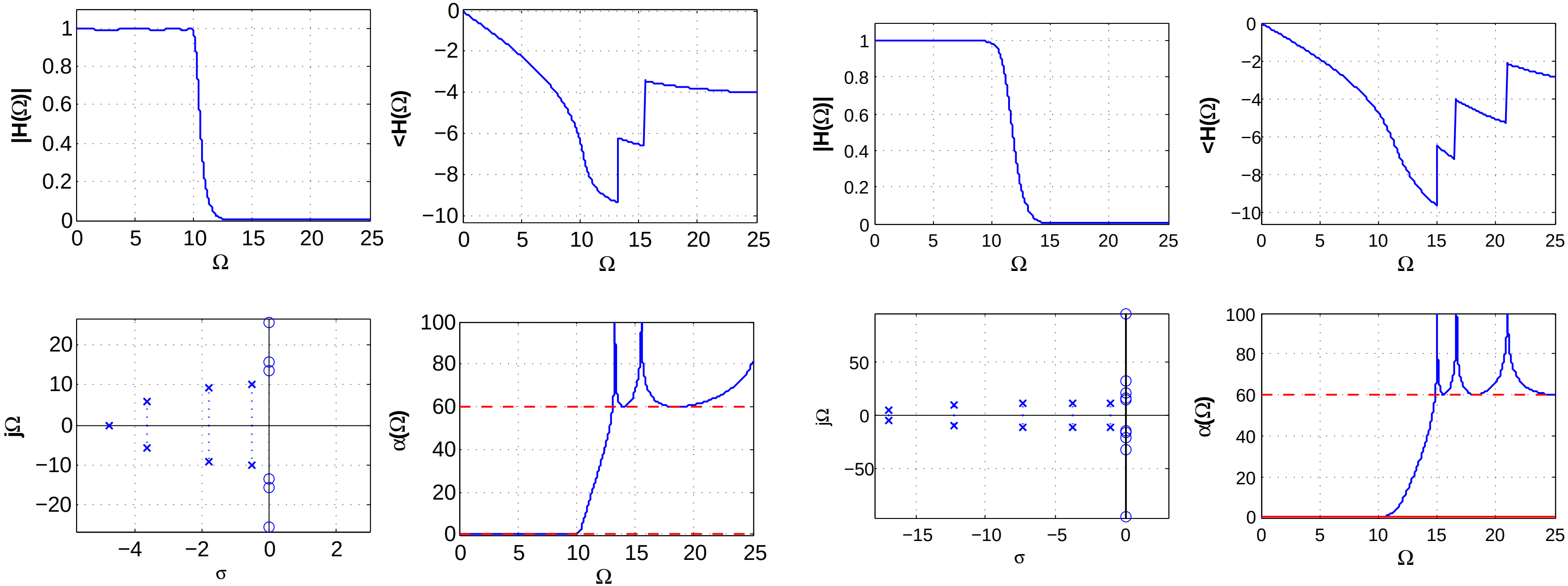
Top-left: prototype LP; top-right LP-HP transformation; bottom-left LP-BP transformation; bottom-right LP-BE transformation

Example: General filter design

Specifications

$$\alpha(0) = 0, \quad \alpha_{max} = 0.1, \quad \alpha_{min} = 60 \quad \text{dB}$$

$$\Omega_p = 10, \quad \Omega_s = 15 \quad \text{rad/sec}$$



Elliptic (left) and Chebyshev2 (right) lowpass filter designs using analogfil *function. Clockwise, magnitude, phase, loss function and poles and zeros are shown for each design.*