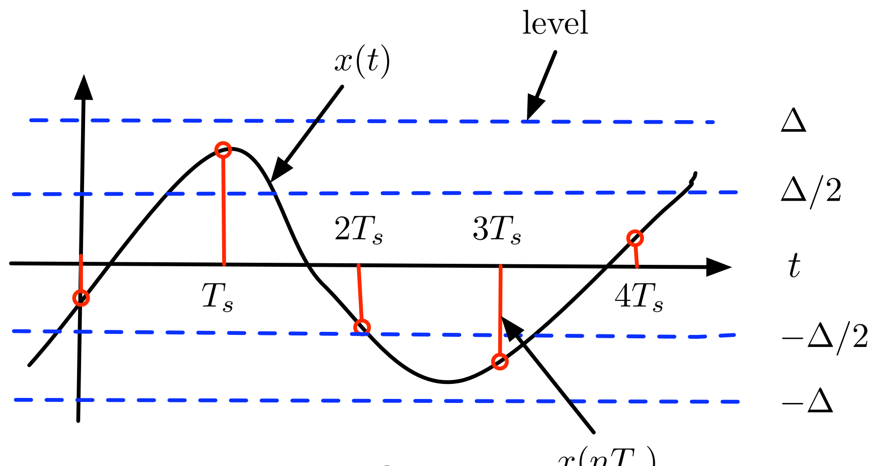
SIGNALS AND SYSTEMS USING MATLAB
Chapter 1 — Continuous–time Signals

L. F. Chaparro and A. Akan

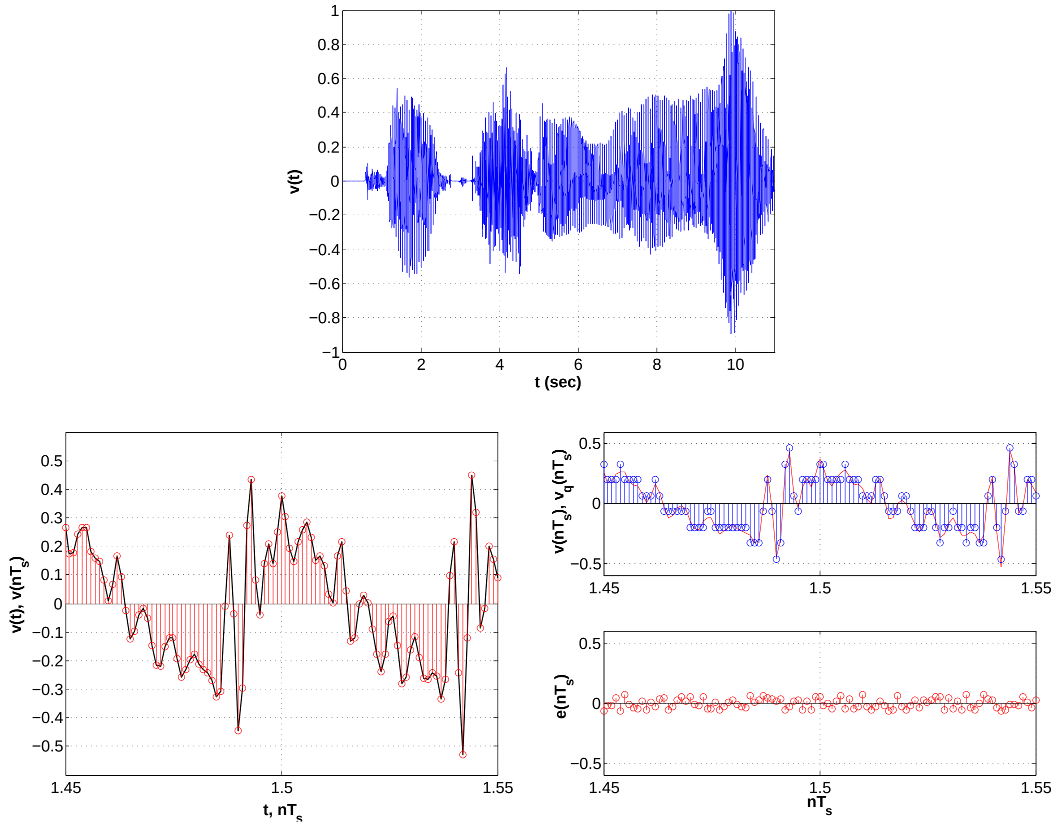
Classification of time–dependent signals

- *Predictability:* random or deterministic
- *Variation of time and amplitude:* continuous-time, discrete–time, or digital
- *Energy/power:* finite or infinite energy/power
- *Repetitive behavior:* periodic or aperiodic
- *Symmetry with respect to time origin:* even or odd
- *Support:* Finite or infinite support (outside support signal is always zero)

- Analog to digital converter (ADC or A/D converter): converts analog signals into digital signals
- Digital to analog converter (DAC or D/A converter): converts digital to analog signals



Discretization in time and in amplitude of analog signal using sampling period T_s and quantization level Δ. In time, samples are taken at uniform times $\{nT_s\}$, and in amplitude the range of amplitudes is divided into a finite number of levels so that each sample value is approximated by one of them



Segment of voice signal on top is sampled and quantized. Bottom left: voice segment (continuous line) and the sampled signal (vertical samples) using a sampling period $T_s = 10^{-3}$ *sec. Bottom-right: sampled and quantized signal at the top, and quantization error, difference between the sampled and the quantized signals, at the bottom.*

$$\boxed{\begin{array}{rl} x(.) : \mathcal{R} \rightarrow \mathcal{R} & (\mathcal{C}) \\ t \rightarrow & x(t) \end{array}}$$

Example: complex signal $y(t) = (1+j)e^{j\pi t/2}, \quad 0 \leq t \leq 10, \quad 0$ otherwise

$$\begin{aligned} y(t) &= \sqrt{2}e^{j(\pi t/2+\pi/4)} \\ &= \begin{cases} \sqrt{2}\left[\cos(\pi t/2+\pi/4)+j\sin(\pi t/2+\pi/4)\right], & 0 \leq t \leq 10, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$$\text{If} \quad \begin{aligned} & x(t) = \sqrt{2}\cos(\pi t/2+\pi/4), -\infty < t < \infty \\ & p(t) = 1, \quad 0 \leq t \leq 10, \quad 0 \ \text{ otherwise then} \end{aligned}$$

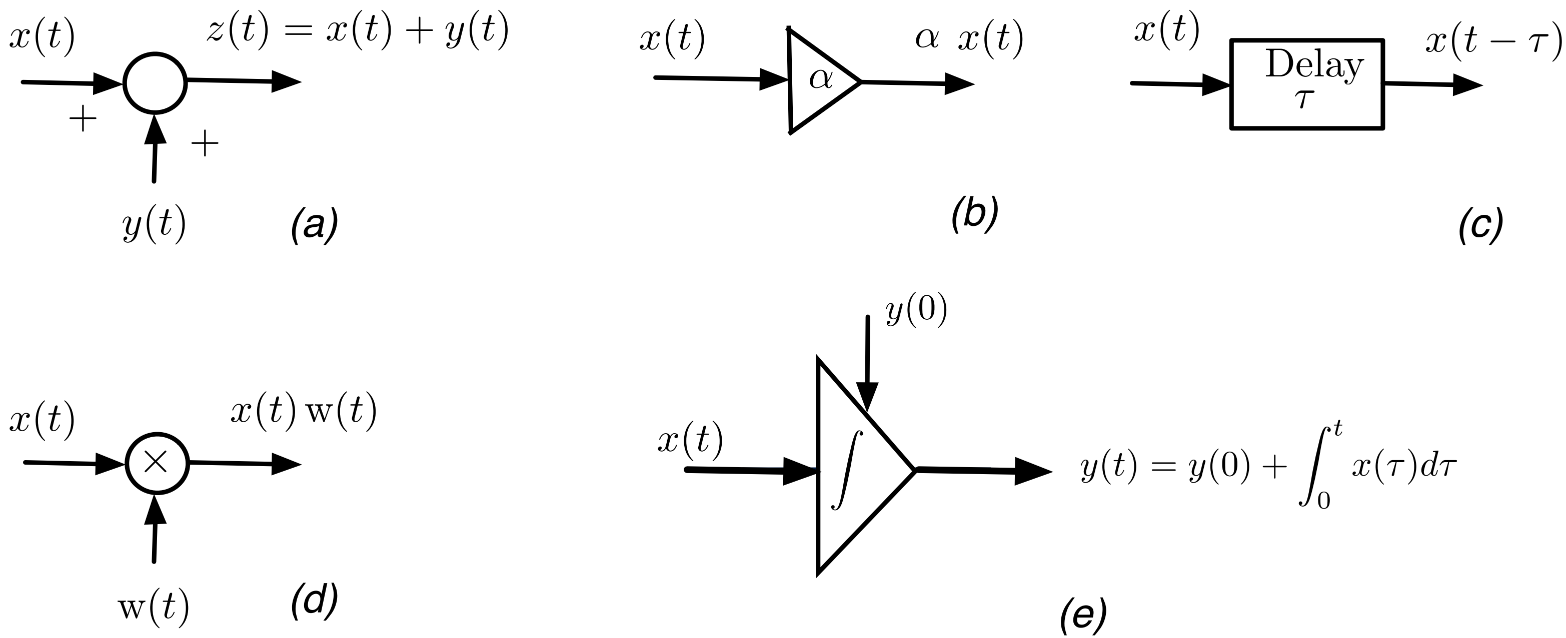
$$y(t) = [x(t) + jx(t-1)]p(t)$$

Basic signal operations

Given signals $x(t)$, $y(t)$, constants α and τ, and function $\mathrm{w}(t)$:

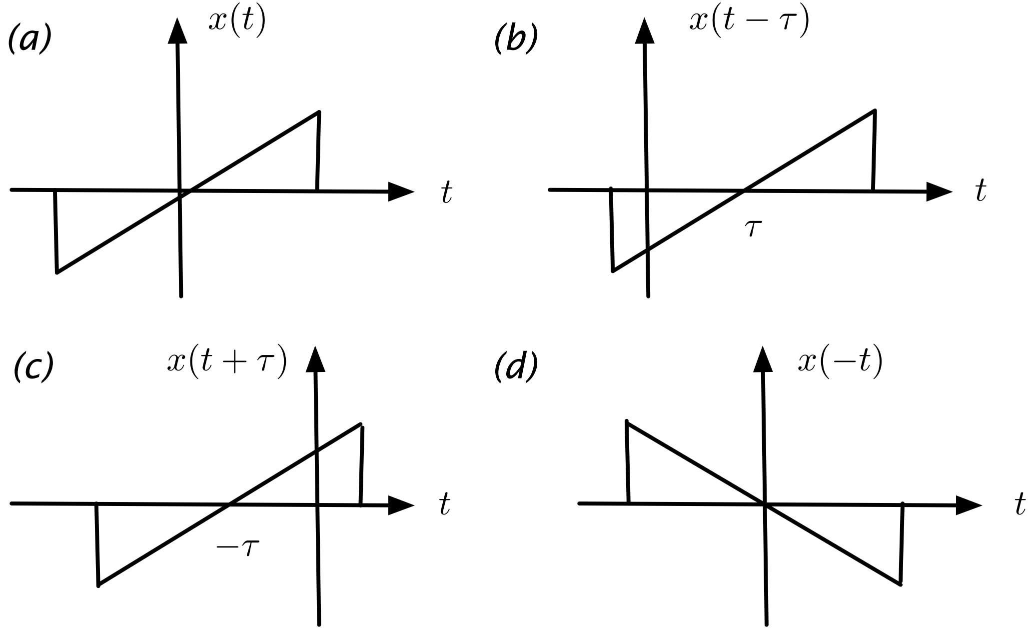
- *Signal addition/subtraction:* $x(t) + y(t)$, $x(t) - y(t)$
- *Constant multiplication:* $\alpha x(t)$
- *Time shifting*
  - $x(t - \tau)$ is $x(t)$ *delayed* by τ
  - $x(t + \tau)$ is $x(t)$ *advanced* by τ
- *Time scaling* $x(\alpha t)$
  - $\alpha = -1$, $x(-t)$ reversed in time or *reflected*
  - $\alpha > 1$, $x(\alpha t)$ is $x(t)$ *compressed*
  - $\alpha < 1$, $x(\alpha t)$ is $x(t)$ *expanded*
- Time windowing $x(t)\mathrm{w}(t)$, $\mathrm{w}(t)$ *window*
- Integration

$$y(t) = \int_{t_0}^{t} x(\tau)d\tau + y(t_0)$$



Basic signal operations: (a) adder, (b) constant multiplier, (c) delay, (d) time-windowing, (e) integrator

Delayed, advanced and reflected signals



Continuous-time signal (a), and its delayed (b), advanced (c), and reflected (d) versions.

Example

$$x(t) = \begin{cases} t & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

delayed by 1: $$x(t-1) = \begin{cases} t-1 & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

advanced by 1: $$x(t+1) = \begin{cases} t+1 & -2 \le t \le 0 \\ 0 & \text{otherwise} \end{cases}$$

reflected: $$x(-t) = \begin{cases} -t & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

reflected and delayed by 1: $$x(-t+1) = \begin{cases} -t+1 & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

reflected and advanced by 1: $$x(-t-1) = \begin{cases} -t-1 & -2 \le t \le 0 \\ 0 & \text{otherwise} \end{cases}$$

compressed by 2: $$x(2t) = \begin{cases} 2t & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

expanded by 2: $$x(t/2) = \begin{cases} t/2 & -2 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Even and odd signals

$$\boxed{\begin{array}{lll} x(t) & \text{even}: & x(t) = x(-t) \\ x(t) & \text{odd}: & x(t) = -x(-t) \end{array}}$$

- *Even and odd decomposition:* For any signal $y(t)$

$$y(t) = y_e(t) + y_o(t)$$

$$\begin{aligned} y_e(t) &= 0.5\,[y(t) + y(-t)] \quad \text{even component} \\ y_o(t) &= 0.5\,[y(t) - y(-t)] \quad \text{odd component} \end{aligned}$$

Example $x(t) = \cos(2\pi t + \theta), \quad -\infty < t < \infty$

$$\text{even} \quad x(t) = x(-t) \to \cos(2\pi t + \theta) = \cos(-2\pi t + \theta) = \cos(2\pi t - \theta)$$

$$\theta = -\theta, \text{or } \theta = 0, \pi$$

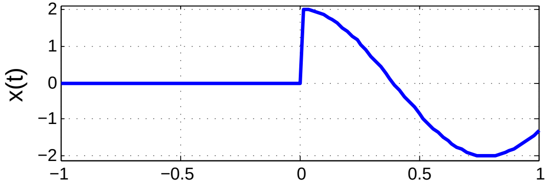
$$\begin{aligned} \text{odd} \quad x(t) = -x(-t) \to \cos(2\pi t + \theta) &= -\cos(-2\pi t + \theta) = \cos(-2\pi t + \theta \pm \pi) \\ &= \cos(2\pi t - \theta \mp \pi) \end{aligned}$$

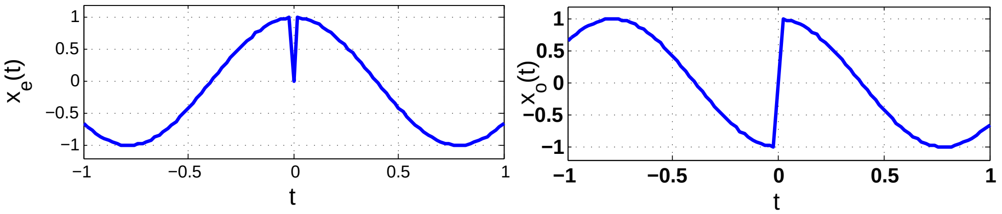
$$\theta = -\theta \mp \pi, \text{or } \theta = \mp\pi/2$$

Example Given

$$x(t) = \begin{cases} 2\cos(4t) & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

not even or odd, its even and odd components are





If signal is 2 at $t = 0$

$$x_1(t) = \begin{cases} 2\cos(4t) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

the odd component is same as before, and the even component is 2 at $t = 0$ and same as before otherwise

$$\boxed{\begin{array}{l} x(t) \text{ is periodic if} \\ (i)\ x(t) \text{ defined in } -\infty < t < \infty, \text{ and} \\ (ii) \text{ there is } T_0 > 0, \text{ the fundamental period of } x(t), \\ \qquad \text{such that } x(t + kT_0) = x(t), \text{ integer} k \end{array}}$$

Example $x(t) = e^{j2t}$ and $y(t) = e^{j\pi t}$

- $x(t) = \cos(2t) + j\sin(2t)$ periodic with $T_0 = 2\pi/2 = \pi$
- $y(t) = \cos(\pi t) + j\sin(\pi t)$ periodic with $T_1 = 2\pi/\pi = 2$
- $z(t) = x(t) + y(t)$ is not periodic as $T_0/T_1 \neq M/N$ where M, N integers
- $w(t) = x(t)y(t) = e^{j(2+\pi)t} = \cos(\Omega_2 t) + j\sin(\Omega_2 t)$, $\Omega_2 = 2 + \pi \rightarrow w(t)$ periodic with $T_2 = 2\pi/(2 + \pi)$
- $p(t) = (1 + x(t))(1 + y(t)) = 1 + x(t) + y(t) + x(t)y(t)$ not periodic

Finite–energy and finite-power signals

$$\boxed{\begin{aligned}&\text{Energy of } x(t): \quad E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt,\\ &\text{Power of } x(t): \quad P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt\end{aligned}}$$

- $x(t)$ is *finite–energy, or square integrable,* if $E_x < \infty$
- $x(t)$ is *finite–power* if $P_x < \infty$

Example

- $x(t) = e^{-at}$, $a > 0, t \geq 0$ and 0 otherwise is finite energy and zero power
- $y(t) = (1+j)e^{j\pi t/2}$, $0 \leq t \leq 10$, and 0 otherwise is finite energy and zero power

$$E_y = \int_0^{10} |(1+j)e^{j\pi t/2}|^2 dt = 2\int_0^{10} dt = 20$$

Power of periodic signal

$x(t)$ period of fundamental period T_0 is

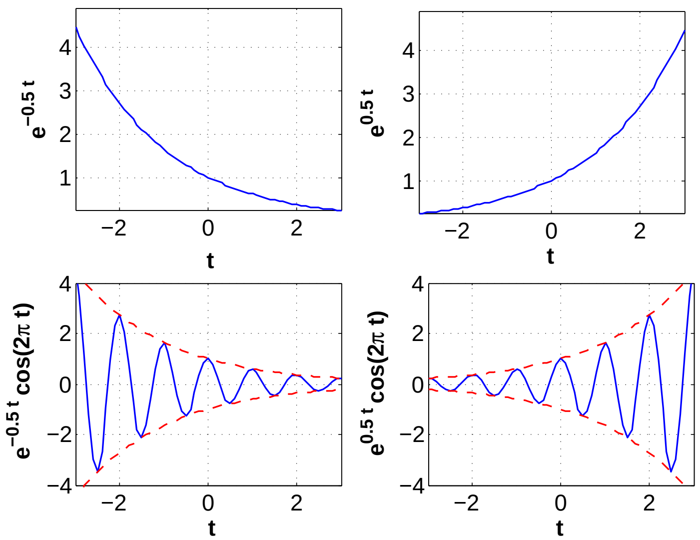
$$\boxed{P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^2(t)dt}$$

for any t_0, i.e., the average energy in a period of the signal Let $T = NT_0$, integer $N > 0$:

$$\begin{aligned}
P_x &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x^2(t)dt = \lim_{N\to\infty}\frac{1}{2NT_0}\int_{-NT_0}^{NT_0} x^2(t)dt \\
&= \lim_{N\to\infty}\frac{1}{2NT_0}\left[N\int_{-T_0}^{T_0} x^2(t)dt\right] = \frac{1}{2T_0}\int_{-T_0}^{T_0} x^2(t)dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^2(t)dt
\end{aligned}$$

- Complex exponential

$$\boxed{\begin{aligned} x(t) &= Ae^{at} = |A|e^{j\theta}e^{(r+j\Omega_0)t} \\ &= |A|e^{rt}\left[\cos(\Omega_0 t+\theta)+j\sin(\Omega_0 t+\theta)\right] \quad -\infty < t < \infty \end{aligned}}$$



Analog exponentials: decaying exponential (top left), growing exponential (top right), modulated exponential decaying and growing (bottom left and right).

- Sinusoid

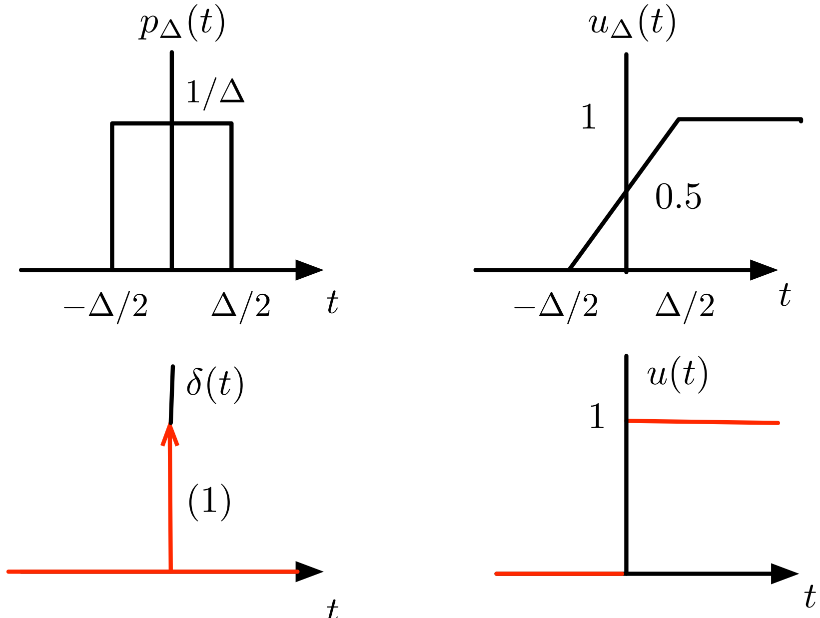
$$\boxed{A\cos(\Omega_0 t+\theta)=A\sin(\Omega_0 t+\theta+\pi/2) \qquad -\infty<t<\infty}$$

Modulation systems

$$A(t)\cos(\Omega(t)t+\theta(t))$$

- *Amplitude modulation or AM:*$A(t)$ changes according to the message, frequency and phase constant,
- *Frequency modulation or FM:* $\Omega(t)$ changes according to the message, amplitude and phase constant,
- *Phase modulation or PM:* $\theta(t)$ changes according to the message, amplitude and frequency constant

- Unit-impulse signal



Unit-impulse $\delta(t)$ and unit–step $u(t)$ as $\Delta \to 0$ in pulse $p_\Delta(t)$ and its integral $u_\Delta(t)$.

Unit-impulse

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$$

$$\int_{-\infty}^{t} \delta(\tau) d\tau = \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Unit–step signal

$$\boxed{u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}}$$

- Ramp signal

$$\boxed{r(t) = tu(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}}$$

- Relations

$$\frac{dr(t)}{dt} = u(t), \quad \frac{d^2r(t)}{dt^2} = \delta(t)$$

$$\frac{du(t)}{dt} = \delta(t)$$

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t), \quad \int_{-\infty}^{t} u(\tau)d\tau = r(t)$$

Example Triangular pulse

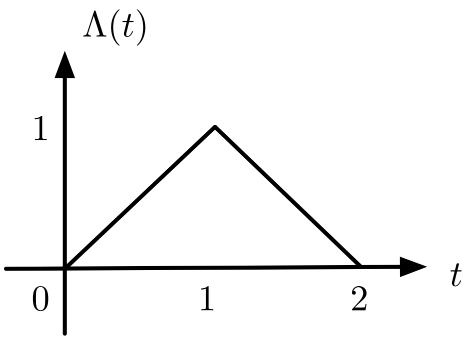
$$\Lambda(t) = \begin{cases} t & 0 \le t \le 1 \\ -t+2 & 1 < t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

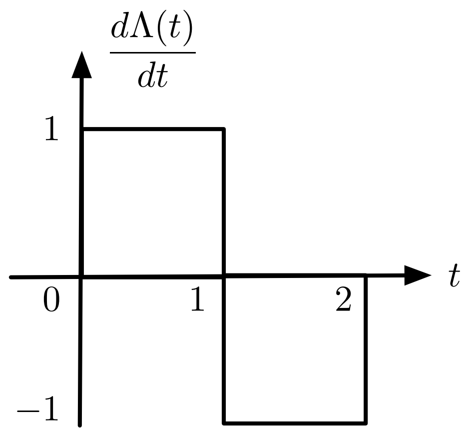
$$= r(t) - 2r(t-1) + r(t-2)$$

Derivative

$$\frac{d\Lambda(t)}{dt} = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 < t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= u(t) - 2u(t-1) + u(t-2)$$



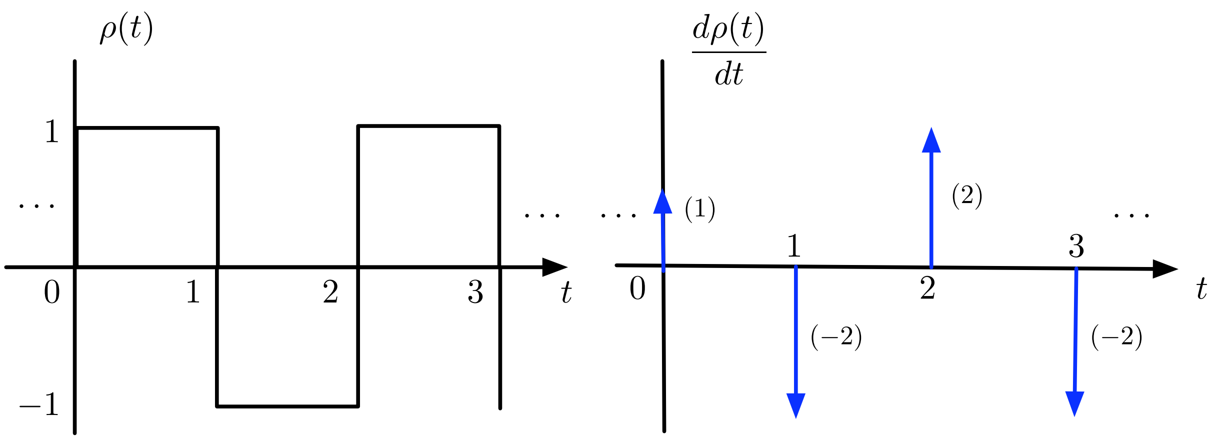


Example Causal train of pulses

$$\rho(t) = \sum_{k=0}^{\infty} s(t - 2k), \qquad s(t) = u(t) - 2u(t-1) + u(t-2)$$

Derivative

$$\frac{d\rho(t)}{dt} = \delta(t) + 2\sum_{k=1}^{\infty} \delta(t - 2k) - 2\sum_{k=1}^{\infty} \delta(t - 2k + 1)$$

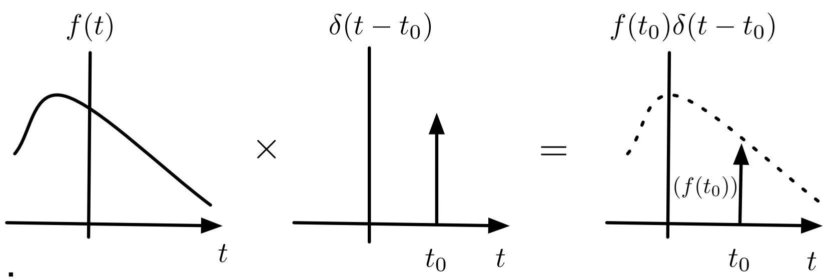


The number in () is area of the corresponding delta signal and it indicates the jump at the particular discontinuity, positive when increasing and negative when decreasing

Generic representation of signals

- Sifting property of $\delta(t)$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)dt = f(\tau)\int_{-\infty}^{\infty} \delta(t-\tau)dt = f(\tau), \quad \text{for any } \tau$$



- Generic representation

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$