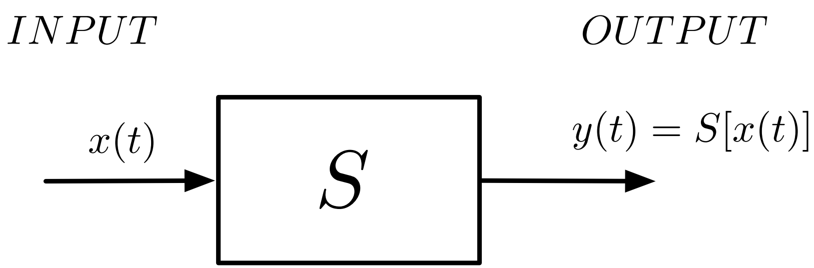
SIGNALS AND SYSTEMS USING MATLAB
Chapter 2 — Continuous-time Systems

L. F. Chaparro and A. Akan

System definition and types

- *System:* mathematical transformation of input signal (or signals) into output signal (or signals) resulting from idealized model of a physical device or process of interest
- *Types:*
  - Static or dynamic system
  - Lumped- or distributed-parameter system
  - Passive or active system
  - Continuous–time, discrete–time or hybrid system



Continuous-time system S with input $x(t)$ and output $y(t)$

Continuous–time system

$$\boxed{\begin{array}{lll} x(t) & \Rightarrow & y(t) = \mathcal{S}[x(t)] \\ \text{Input} & & \text{Output} \end{array}}$$

Properties

- Linearity
- Time-invariance
- Causality
- Stability

A system $\mathcal{S}$ is linear if for inputs $x(t)$ and $v(t)$, and constants α and β, superposition holds, i.e.,

$$\begin{aligned} \mathcal{S}[\alpha x(t) + \beta v(t)] &= \mathcal{S}[\alpha x(t)] + \mathcal{S}[\beta v(t)] \\ &= \alpha \mathcal{S}[x(t)] + \beta \mathcal{S}[v(t)] \end{aligned}$$

Examples

- Biased averager

$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)d\tau + B, \quad \text{linear if } B = 0$$

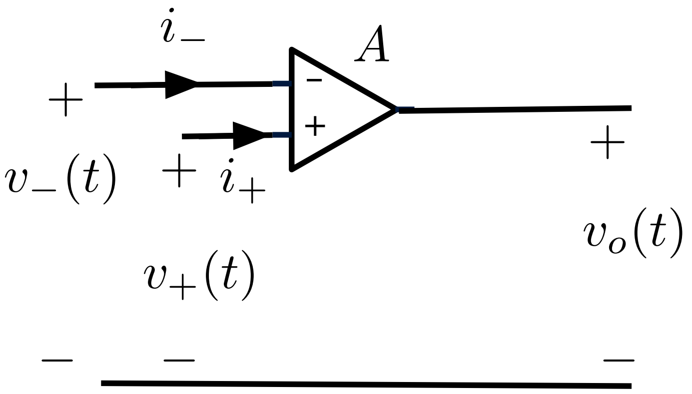
- Non–linear systems

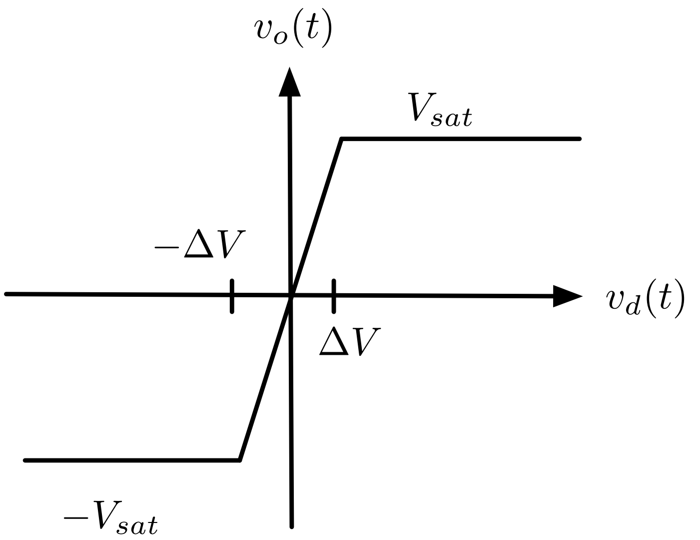
$$\begin{aligned} (i) &\quad y(t) = |x(t)| \\ (iii) &\quad v(t) = x^2(t) \end{aligned}$$

- RLC

$$\begin{aligned} &\text{resistor} \quad v(t) = Ri(t), \quad \text{linear} \\ &\text{capacitor} \quad v_c(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v_c(0), \quad \text{linear if } v_c(0) = 0 \\ &\text{inductor} \quad i_L(t) = \frac{1}{L}\int_0^t v(\tau)d\tau + i_L(0), \quad \text{linear if } i_L(0) = 0 \end{aligned}$$

Operational amplifier





Op amp: circuit diagram, and input-output voltage relation

Linear model

$$A \to \infty, \quad R_{in} \to \infty \quad \text{give}$$
$$\text{virtual short:} \quad i_-(t) = i_+(t) = 0, \quad v_d(t) = v_+(t) - v_-(t) = 0$$

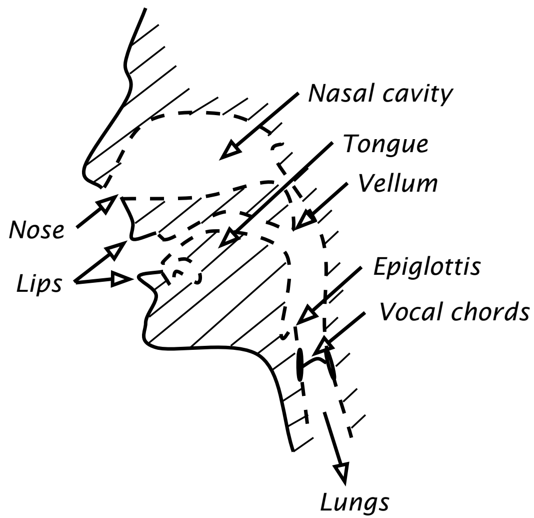
Time invariance

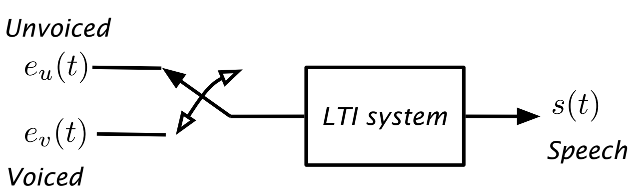
System $\mathcal{S}$ is time-invariant if

$$x(t) \quad \Rightarrow \quad y(t) = \mathcal{S}[x(t)]$$
$$x(t \mp \tau) \quad \Rightarrow \quad y(t \mp \tau) = \mathcal{S}[x(t \pm \tau)]$$

Examples

- Vocal system



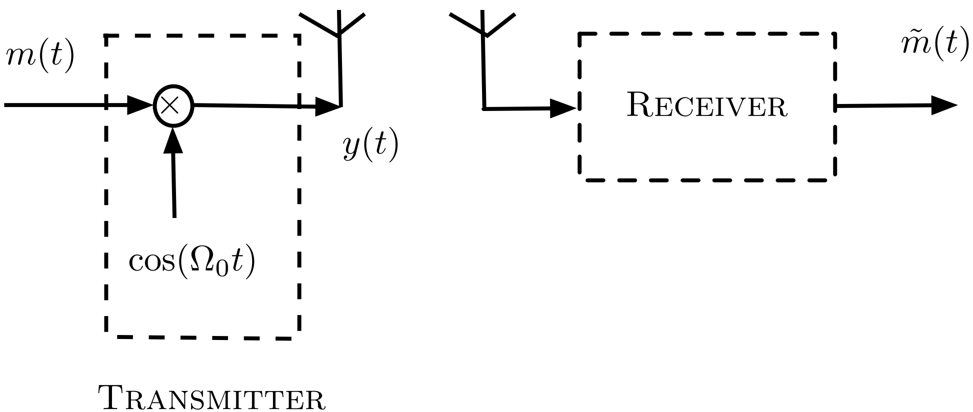


- Time–varying system

$$x(t),\ y(t) \ \text{ input and output of system defined by}$$
$$y(t) = f(t)x(t), \quad \text{TV if } f(t) \text{ not constant}$$

- Amplitude modulation (AM) communication system

$$y(t) = m(t)\cos(\Omega_0 t), \quad \text{LTV}$$



AM modulation: transmitter and receiver

- Frequency modulation (FM) communication system

$$z(t) = \cos\left(\Omega_c t + \int_{-\infty}^{t} m(\tau)d\tau\right), \quad m(t) \text{ message}$$

FM system non–linear

$$\text{scale message } \gamma m(t) \text{ then output is}$$

$$\cos\left(\Omega_c t + \gamma\int_{-\infty}^{t} m(\tau)d\tau\right) \neq \gamma z(t)$$

FM system time–varying

$$\text{delay message } m(t-\lambda) \text{ then output is}$$

$$\cos\left(\Omega_c t + \int_{-\infty}^{t} m(\tau-\lambda)d\tau\right) \neq z(t-\lambda)$$

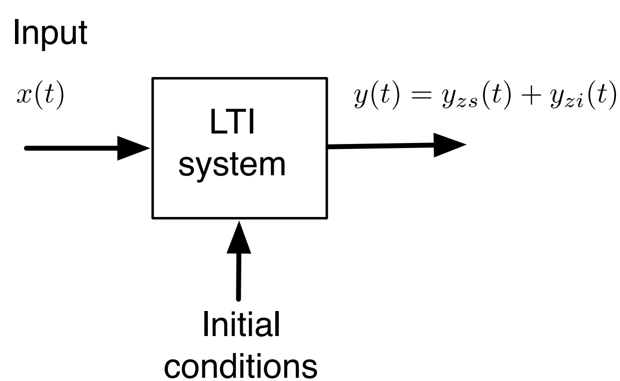
- System represented by linear, constant coefficient differential equation: System $\mathcal{S}$, with input $x(t)$ and output $y(t)$, represented by

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + \frac{d^N y(t)}{dt^N} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_M \frac{d^M x(t)}{dt^M} \qquad t \geq 0$$

  is linear time–invariant (LTI) if
  - IC are zero
  - input $x(t)$ is causal (i.e., zero for

  i.e., the system is not initially energized

If IC$\neq 0$, $x(t)$ causal consider superposition



LTI system with $x(t)$ and IC as inputs

- RL circuit: $R=1$, $L=1$ and voltage source $v(t)=Bu(t)$

$$v(t)=i(t)+\frac{di(t)}{dt}, \qquad t>0, \qquad i(0)=I_0$$

$$\text{solution} \quad i(t)=[I_0e^{-t}+B(1-e^{-t})]u(t)$$

IC$\neq 0$: (i) $I_0=1$ and $B=1$

complete response: $i_1(t)=[e^{-t}+(1-e^{-t})]u(t)=u(t)$

zero-state response: $i_{1zs}(t)=(1-e^{-t})u(t)$

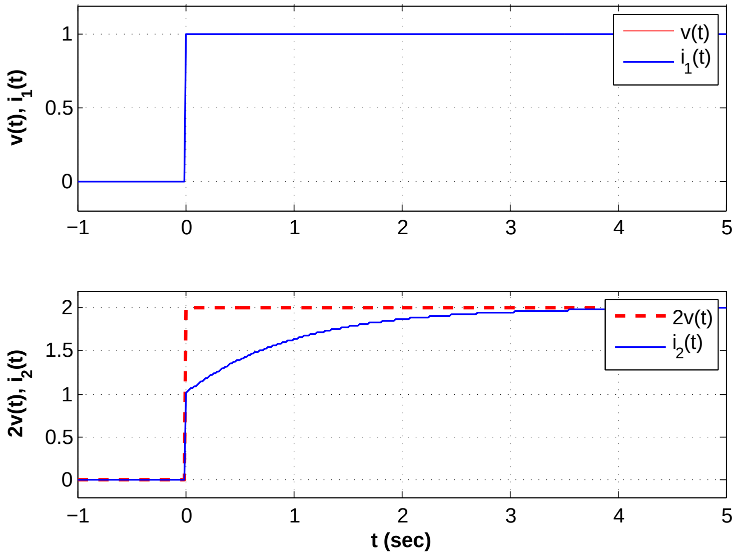
zero-input response: $i_{1zi}(t)=e^{-t}u(t)$

(ii) $I_0=1$ and $B=2$ (double input)

complete response: $i_2(t)=(2-e^{-t})u(t)\neq 2i_1(t)$

zero-state response: $i_{2zs}(t)=2(1-e^{-t})u(t)$, doubled

zero-input response: $i_{2zi}(t)=e^{-t}u(t)$, same



IC$= 0,\ B = 1, 2$, circuit is linear

$$IC = 0, B = 1: \quad i_1(t) = (1 - e^{-t})u(t)$$
$$IC = 0, B = 2: \quad i_2(t) = 2(1 - e^{-t})u(t) = 2i_1(t)$$

Time invariance: let $v(t) = u(t-1)$ and I_0 initial condition

$$i_3(t) = I_0 e^{-t} u(t) + (1 - e^{-(t-1)})u(t-1)$$

If $I_0 = 0$ then

$$i_3(t) = (1 - e^{-(t-1)})u(t-1) = i(t-1), \quad \text{time-invariant}$$

If $I_0 = 1$ then

$$i_3(t) = e^{-t} u(t) + (1 - e^{-(t-1)})u(t-1) \neq i(t-1), \quad \text{time-variant}$$

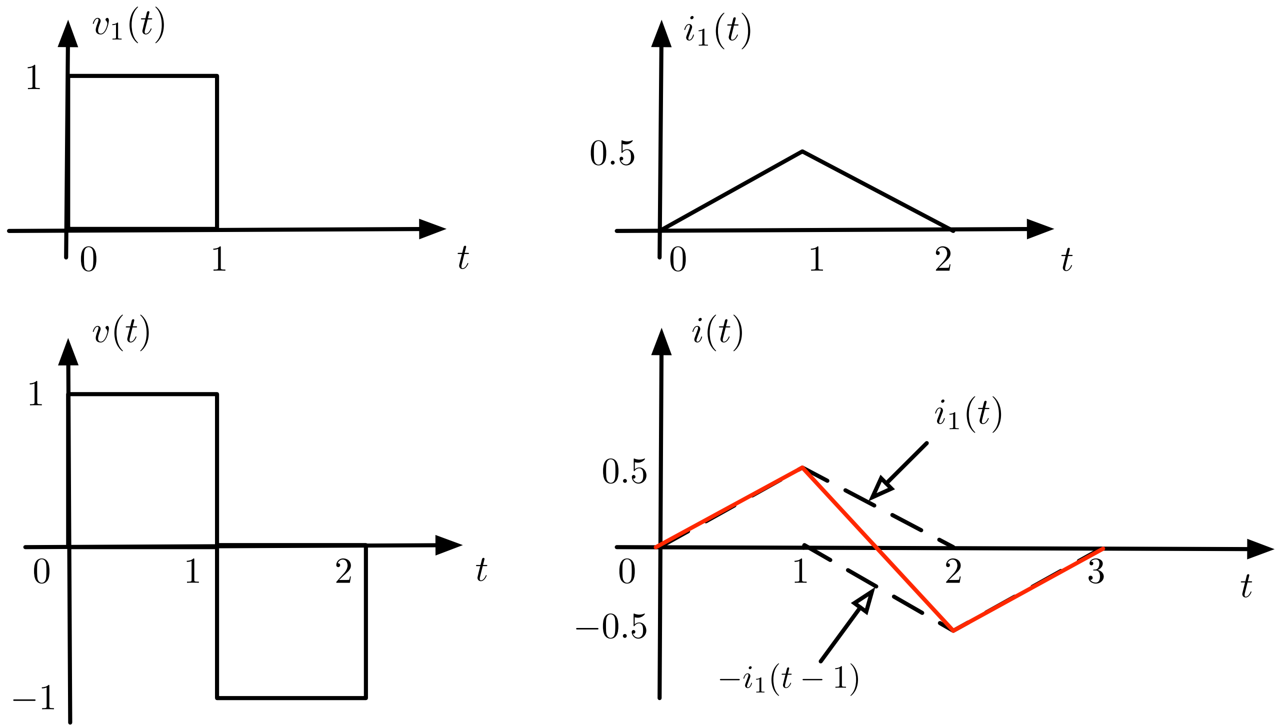
- Averager

$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)d\tau, \quad \text{(L)}$$

shifted input $x(t-\lambda)$, then output is

$$\frac{1}{T}\int_{t-T}^{t} x(\tau-\lambda)d\tau = \frac{1}{T}\int_{t-T-\lambda}^{t-\lambda} x(\sigma)d\sigma = y(t-\lambda), \quad \text{(TI)}$$

- Application of LTI
  If response of a LTI system to $v_1(t)$ is $i_1(t)$ the response to $v(t)$ applying LTI is $i(t)$.



Application of superposition and time invariance to find the response of a LTI system

- Impulse response of LTI system, $h(t)$, is output of the system corresponding to an impulse $\delta(t)$, and initial conditions of zero
- Convolution integral

$$
\begin{aligned}
\delta(t) &\rightarrow h(t) \quad \text{(definition)} \\
\delta(t-\tau) &\rightarrow h(t-\tau) \quad \text{(TI)} \\
x(\tau)h(t-\tau) &\rightarrow x(\tau)h(t-\tau) \quad \text{(L)} \\
x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau &\rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad \text{(L)}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\
&= [x*h](t) = [h*x](t)
\end{aligned}
}
$$

Example: for averager

$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)d\tau, \quad x(t) \text{ input}, \; y(t) \text{ output}$$

impulse response $$h(t) = \frac{1}{T}\int_{t-T}^{t} \delta(\tau)d\tau$$

$$= \begin{cases} 1/T & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

ramp response $$\rho(t) = \frac{1}{T}\int_{t-T}^{t} \sigma u(\sigma)d\sigma$$

$$= \begin{cases} 0 & t < 0 \\ t^2/(2T) & 0 \le t < T \\ t - T/2 & t \ge T \end{cases}$$

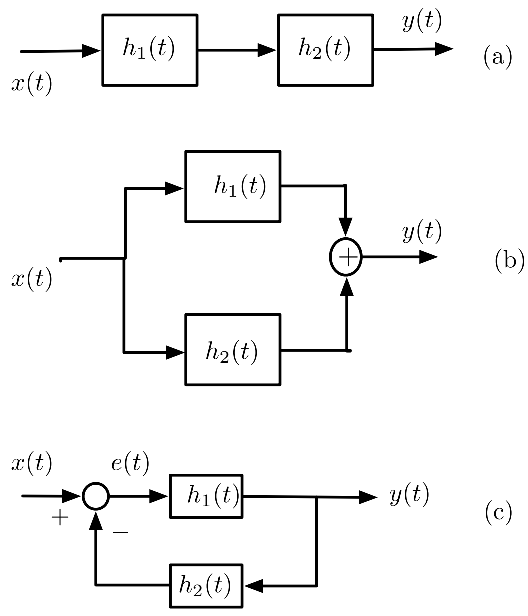
Note that

$$\frac{d^2\rho(t)}{dt^2} = h(t)$$

Impulse response $h(t)$, unit-step response $s(t)$, and ramp response $\rho(t)$ are related by

$$h(t) = \begin{cases} ds(t)/dt \\ \\ d^2\rho(t)/dt^2 \end{cases}$$

Interconnection of systems



Block diagrams of the connection of two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ in (a) cascade, (b) parallel, and (c) negative feedback

Cascade

$$y(t) = [[x * h_1] * h_2](t) = [x * [h_1 * h_2]](t) = [x * [h_2 * h_1]](t), \quad \text{(commute)}$$

Parallel

$$y(t) = [x * h_1](t) + [x * h_2](t) = [x * (h_1 + h_2)](t)$$

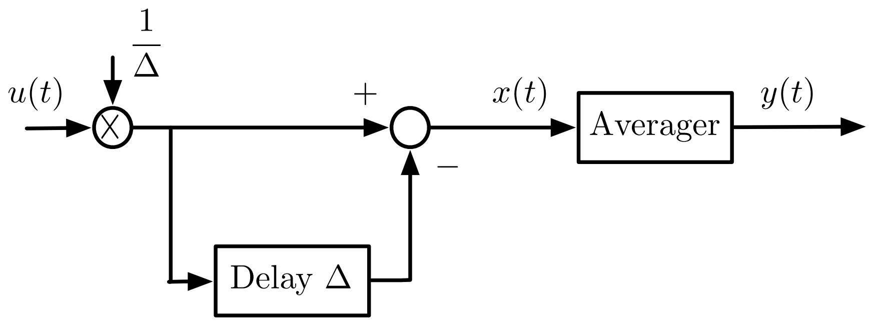
Negative feedback

$$y(t) = [h_1 * e](t)$$

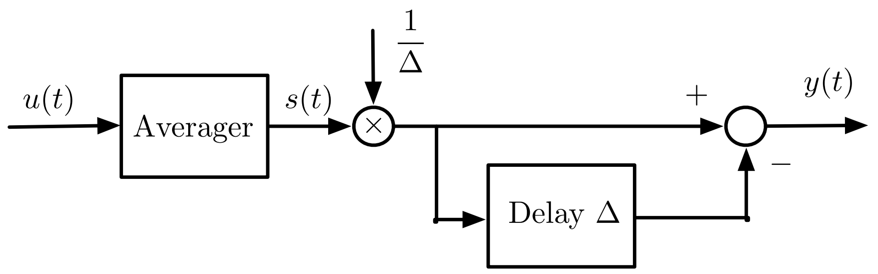
error signal $e(t) = x(t) - [y * h_2](t)$

closed loop impulse response $h(t) = [h_1 - h * h_1 * h_2](t)$, (implicit)

Example: cascading of two LTI systems



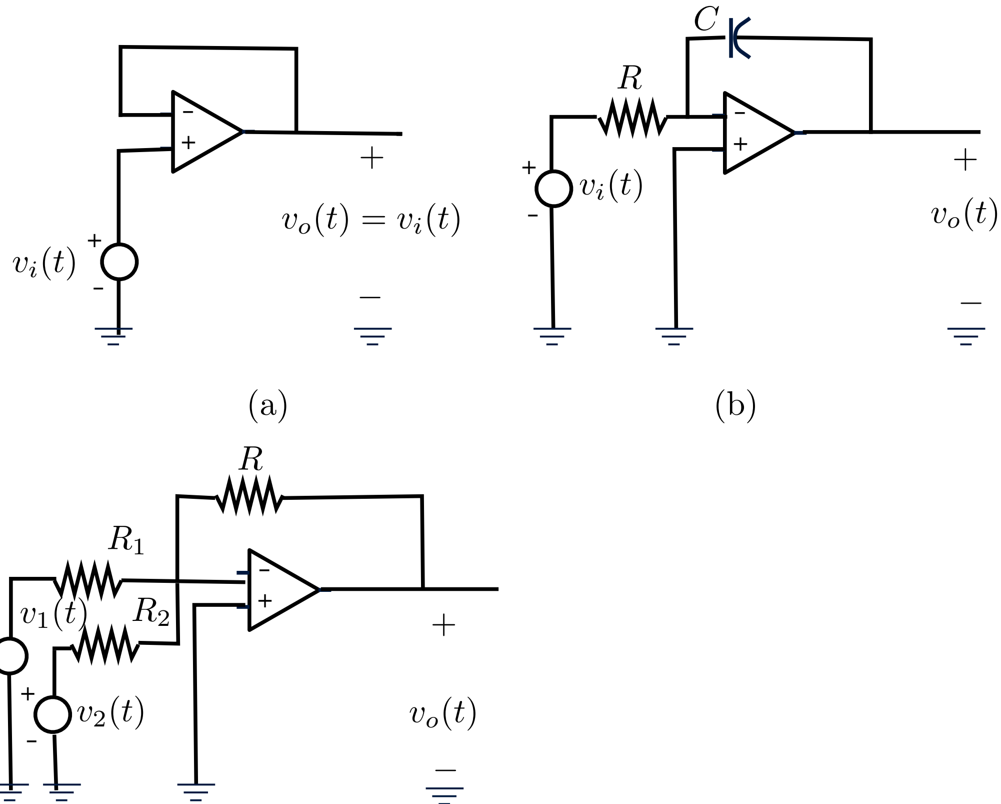
Equivalent block diagram



$$s(t) = \frac{1}{T}\int_{t-T}^{t} u(\tau)d\tau = \begin{cases} 0 & t < 0 \\ t/T & 0 \le t < T \\ 1 & t \ge T \end{cases}$$

$$y(t) = \frac{1}{\Delta}[s(t) - s(t-\Delta)]$$ approximate impulse response of averager

Example: negative feedback



Operational amplifier circuits: (a) voltage follower, (b) inverting integrator, and (c) adder with inversion

Causality

- Cause and effect relation between input and output
- For $\tau > 0$, when considering causality let
  - time t be the *present*
  - times $t - \tau$ be the *past*, and
  - times $t + \tau$ be the *future*
- System $\mathcal{S}$ is causal if
  - $x(t) = 0$, IC$= 0$, output $y(t) = 0$,
  - output $y(t)$ does not depend on future inputs
- LTI system $\mathcal{S}$ represented by its impulse response $h(t)$ is causal if

$$h(t) = 0 \qquad \text{for } t < 0$$

output of causal LTI system for causal input $x(t) = 0,\ t < 0$

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

Graphical computation of convolution

$\mathcal{S}$ is LTI and causal, $h(t) = 0, \; t < 0$, input is causal, $x(t) = 0, \; t < 0$, output

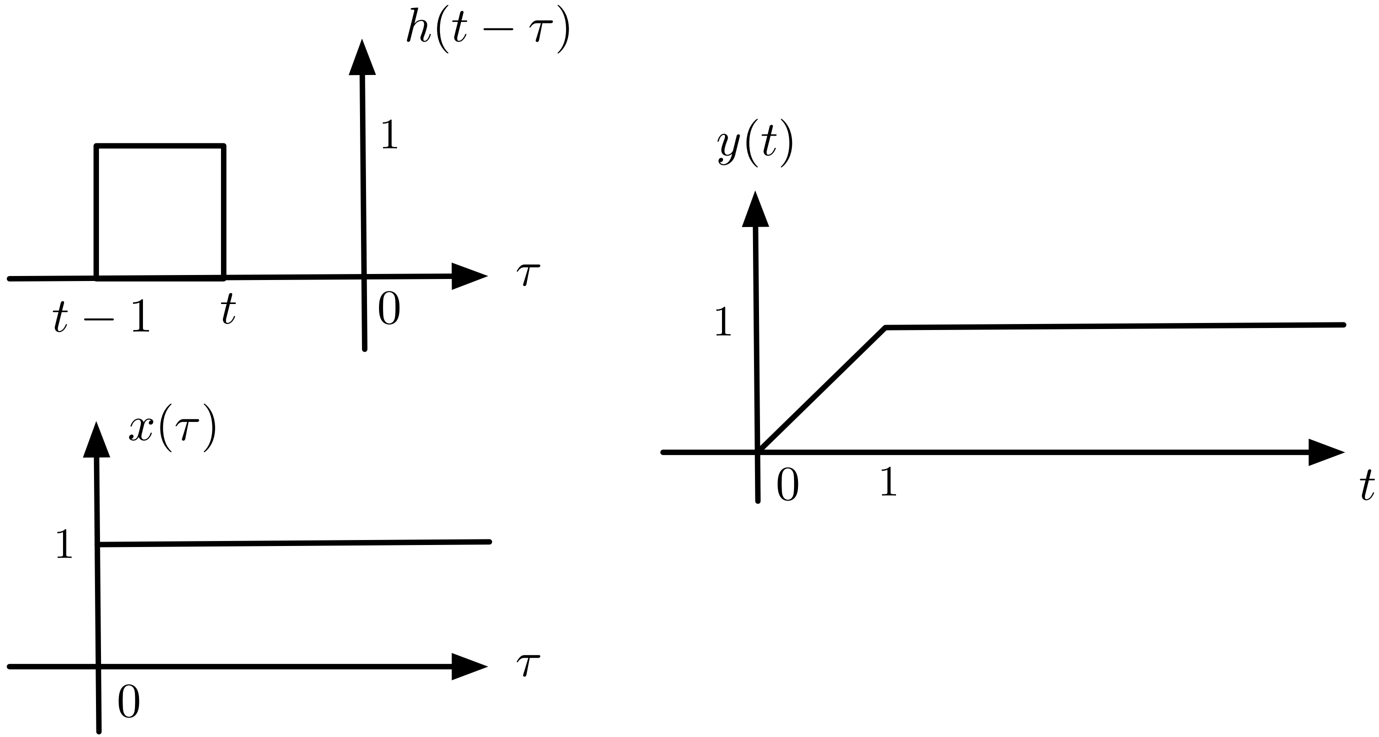
$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t h(\tau)x(t-\tau)d\tau$$

Graphical procedure

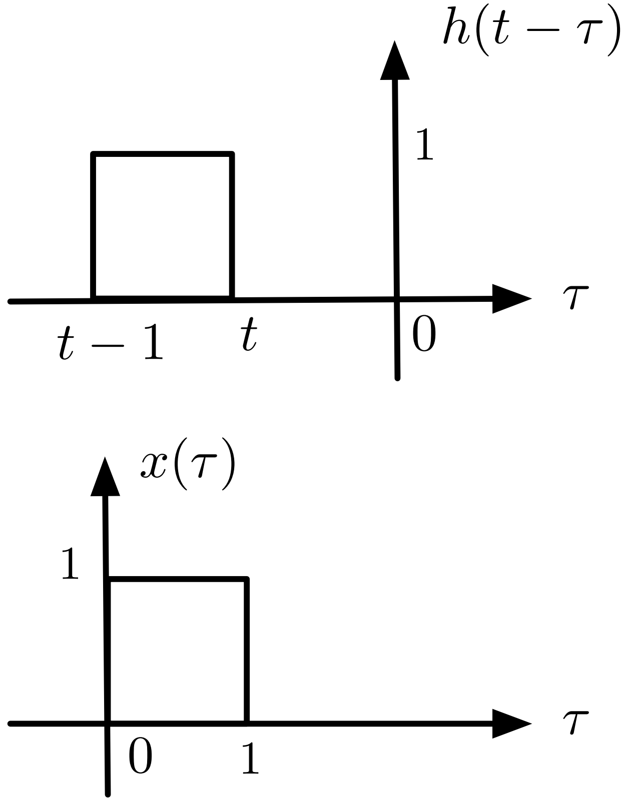
- Choose time t_0 to compute $y(t_0)$,
- Plot as functions of τ, $x(\tau)$ and the reflected and delayed t_0, $h(t_0 - \tau)$,
- Obtain $x(\tau)h(t_0 - \tau)$ and integrate it from 0 to t_0 to obtain $y(t_0)$.
- Increase t_0, move from $-\infty$ to ∞

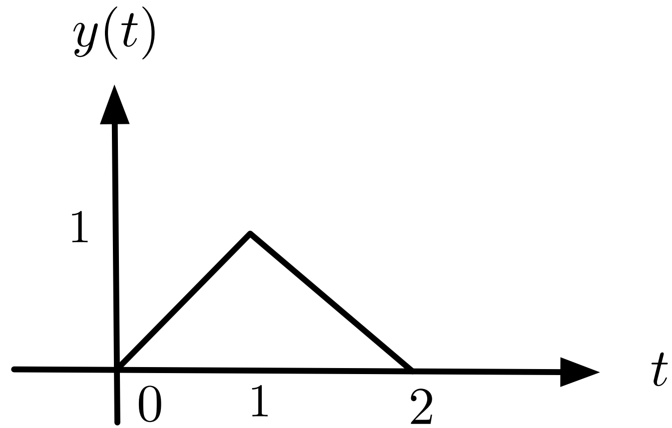
Equal results obtained if $x(t-\tau)$ and $h(\tau)$ used

Example: Unit-step response $y(t)$ of averager with impulse response $h(t) = u(t) - u(t-1)$



Example: Graphical computation of the convolution integral when $x(t) = h(t) = u(t) - u(t-1)$





BIBO stability

- Bounded-input-bounded-output (BIBO) stability: for a bounded $x(t)$ the output $y(t)$ *is also bounded*
- LTI $\mathcal{S}$ is BIBO stable if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty, \quad \text{(absolutely integrable)}$$

Indeed

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau \right| \leq M \int_{-\infty}^{\infty} |h(\tau)| d\tau \leq MK < \infty$$

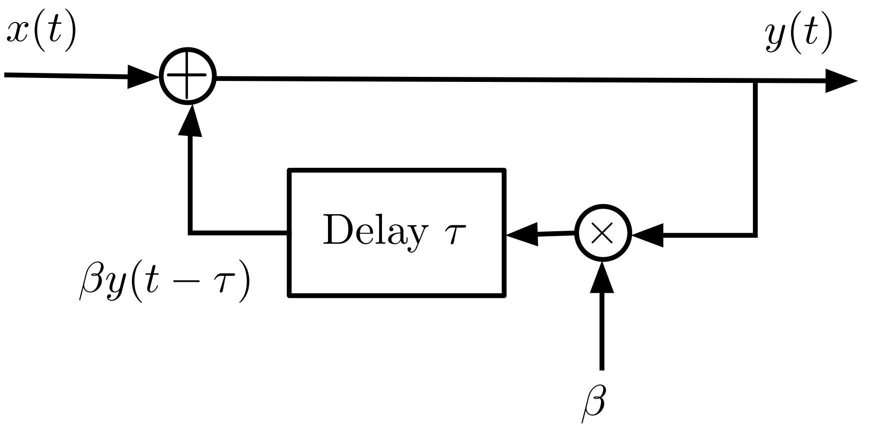
Example: RL circuit (R=L=1)

$$v_s(t) = i(t) + \frac{di(t)}{dt}$$

$$v_s(t) = \delta(t), i(0) = 0, \quad i(t) = h(t) = e^{-t} u(t)$$

$$\int_{-\infty}^{\infty} |h(t)| dt = -e^{-t}|_{t=0}^{\infty} = 1$$

Example: Positive feedback system (not BIBO stable)



$$
\begin{aligned}
y(t) &= x(t) + \beta y(t-\tau) \\
&= x(t) + \beta \underbrace{\left[x(t-\tau) + \beta y(t-2\tau)\right]}_{y(t-\tau)} \\
&\cdots \\
&= x(t) + \beta x(t-\tau) + \beta^2 x(t-2\tau) + \beta^3 x(t-3\tau) + \cdots
\end{aligned}
$$

If $x(t) = u(t)$, $\beta = 2$, then

$$y(t) = u(t) + 2u(t-1) + 4u(t-2) + 8u(t-3) + \cdots \to \infty$$