SIGNALS AND SYSTEMS USING MATLAB
Chapter — Frequency Analysis: The Fourier Transform

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From the Fourier Series to the Fourier Transform

Aperiodic signal $x(t)$ can be thought of as periodic signal $\tilde{x}(t)$ with infinite fundamental period. From Fourier series of $\tilde{x}(t)$ and limiting process we obtain Fourier transform pair

$$x(t) \quad \Leftrightarrow \quad X(\Omega)$$

$x(t)$ is transformed into $X(\Omega)$ in the frequency–domain by the

Fourier transform: $$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

while $X(\Omega)$ is transformed into $x(t)$ in the time–domain by the

Inverse Fourier Transform: $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$$

Existence of the Fourier Transform

- For $X(\Omega)$ to exist, $x(t)$ must be *absolutely integrable*

$$|X(\Omega)| \leq \int_{-\infty}^{\infty} |x(t)e^{-j\Omega t}|dt = \int_{-\infty}^{\infty} |x(t)|dt < \infty$$

- ROC of $X(s) = \mathcal{L}[x(t)]$ contains the $j\Omega$-axis then

$$\begin{aligned} \mathcal{F}[x(t)] &= \mathcal{L}[x(t)]|_{s=j\Omega} = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \\ &= X(s)\,|_{s=j\Omega} \end{aligned}$$

- Duality between time and frequency allows computation of Fourier transforms

Example: Fourier transform from Laplace transform

$$(a)\quad x_1(t) = u(t), \quad X_1(s) = \frac{1}{s},\ ROC: \sigma > 0,\ j\Omega\text{-axis not included}$$

$$X(\Omega)\ \text{ cannot be obtained}$$

$$(b)\quad x_2(t) = e^{-2t}u(t), \quad X_2(s) = \frac{1}{s+2},\ ROC: \sigma > -2$$

$$X_2(\Omega) = \frac{1}{s+2}\Big|_{s=j\Omega} = \frac{1}{j\Omega+2}$$

$$(c)\quad x_3(t) = e^{-|t|}, \quad X_3(s) = \frac{1}{s+1} + \frac{1}{-s+1},\ ROC: -1 < \sigma < 1$$

$$X_3(\Omega) = X_3(s)|_{s=j\Omega} = \frac{2}{1-(j\Omega)^2} = \frac{2}{1+\Omega^2}$$

Inverse proportionality of time and frequency

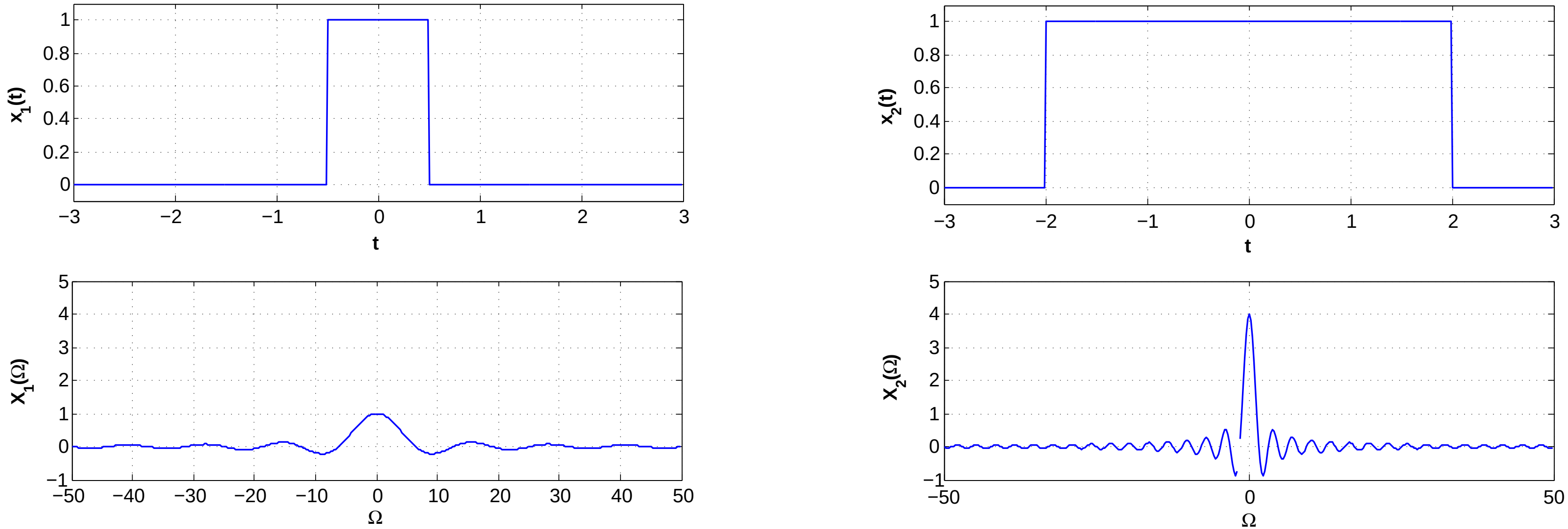
Support of $X(\Omega)$ is inversely proportional to the support of $x(t)$

If $x(t)$ has a Fourier transform $X(\Omega)$ and $\alpha \neq 0$ is a real number, then $x(\alpha t)$

- is a contracted signal when $\alpha > 1$;
- is a contracted and reflected signal when $(\alpha < -1)$;
- is an expanded signal when $0 < \alpha < 1$;
- is a reflected and expanded signal when $-1 < \alpha < 0$; or
- is a reflected signal when $\alpha = -1$

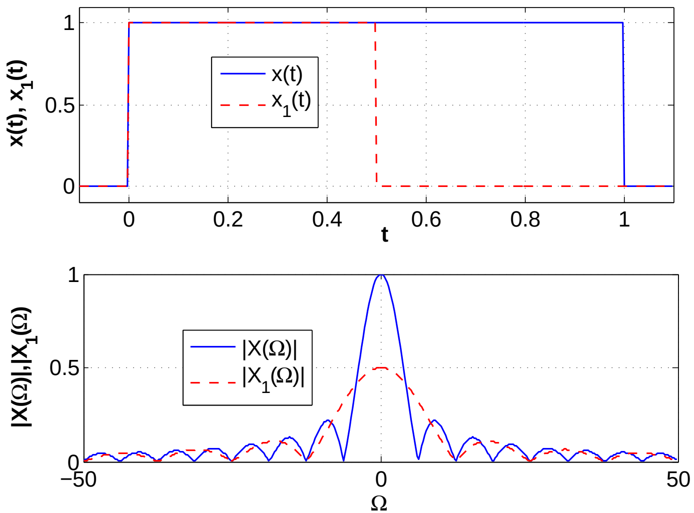
and

$$\boxed{x(\alpha t) \quad \Leftrightarrow \quad \frac{1}{|\alpha|} X\left(\frac{\Omega}{\alpha}\right)}$$



Fourier transform of pulses $x_1(t) = u(t+0.5) - u(t-0.5)$, *(left) and* $x_2(t) = u(t+2) - u(t-2)$ *(right). Notice the wider the pulse the more concentrated in frequency its Fourier transform*

Example: $x(t) = u(t) - u(t-1)$ vs $x_1(t) = x(2t)$



$$X(s) = \frac{1 - e^{-s}}{s}, \quad ROC: \text{ whole s-plane}$$

$$X(\Omega) = \frac{e^{-j\Omega/2}(e^{j\Omega/2} - e^{-j\Omega/2})}{2j\Omega/2} = \frac{\sin(\Omega/2)}{\Omega/2} e^{-j\Omega/2} \quad \text{infinite support}$$

$$x_1(t) = x(2t) = u(2t) - u(2t-1) = u(t) - u(t-0.5)$$

$$X_1(\Omega) = \frac{e^{-j\Omega/4}(e^{j\Omega/4} - e^{-j\Omega/4})}{j\Omega} = \frac{1}{2}\frac{\sin(\Omega/4)}{\Omega/4} e^{-j\Omega/4} = \frac{1}{2} X(\Omega/2)$$

Duality

$$\boxed{\begin{array}{rcr} x(t) & \Leftrightarrow & X(\Omega) \\ \\ X(t) & \Leftrightarrow & 2\pi x(-\Omega) \end{array}}$$

Example:

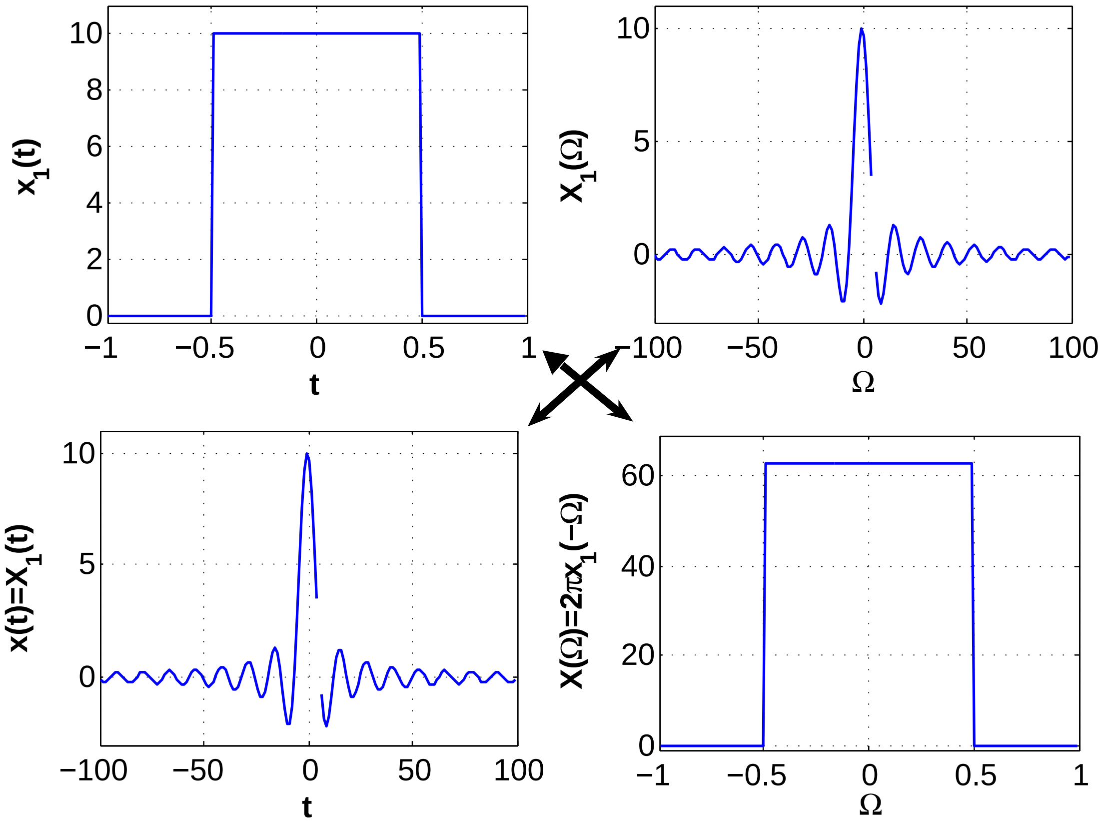
$$A\delta(t) \quad \Leftrightarrow \quad A$$

$$A \quad \Leftrightarrow \quad 2\pi A\delta(-\Omega) = 2\pi A\delta(\Omega)$$

Example:

$$\begin{aligned} \delta(t-\rho_0) + \delta(t+\rho_0) \quad &\Leftrightarrow \quad e^{-j\rho_0\Omega} + e^{j\rho_0\Omega} = 2\cos(\rho_0\Omega) \\ 2\cos(\rho_0 t) \quad &\Leftrightarrow \quad 2\pi[\delta(\Omega+\rho_0) + \delta(\Omega-\rho_0)] \end{aligned}$$

$$x(t) = \cos(\Omega_0 t) \quad \Leftrightarrow \quad X(\Omega) = \pi[\delta(\Omega+\Omega_0) + \delta(\Omega-\Omega_0)]$$



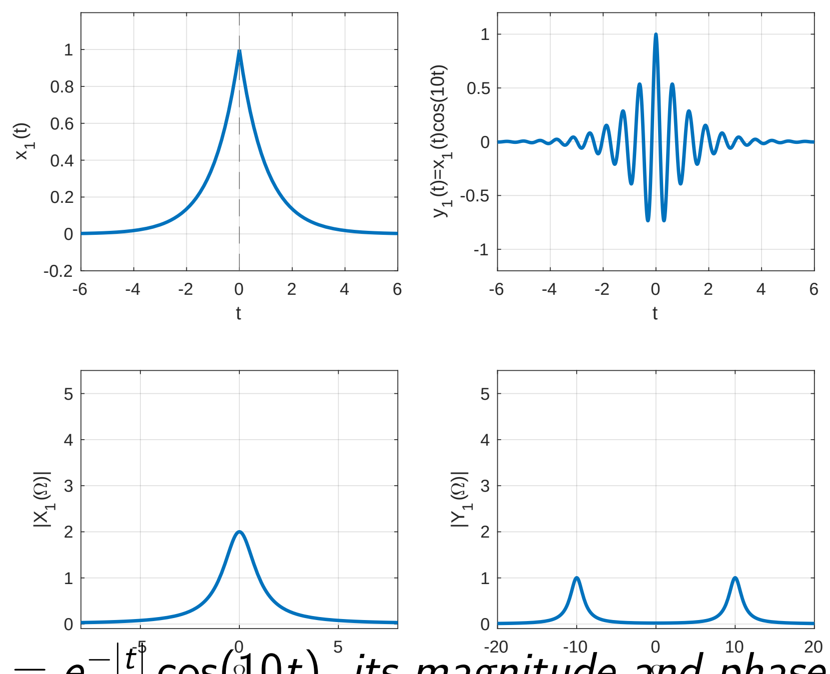
Duality to find Fourier transform of $x(t) = 10sinc(0.5t)$

- Frequency shift:

$$\boxed{\begin{array}{rcr} x(t) & \Leftrightarrow & X(\Omega) \\ x(t)e^{j\Omega_0 t} & \Leftrightarrow & X(\Omega - \Omega_0) \end{array}}$$

- Modulation:

modulated s … $\Omega_0) + X(\Omega + \Omega_0)]$



Modulated signal $y_1(t) = e^{-|t|}\cos(10t)$, *its magnitude and phase spectra*

Fourier transform of periodic signals

Represent periodic signal $x(t)$, of period T_0, by its Fourier series:

$$\boxed{x(t)=\sum_k X_k e^{jk\Omega_0 t} \quad \Leftrightarrow \quad X(\Omega)=\sum_k 2\pi X_k \delta(\Omega-k\Omega_0)}$$

Example: Periodic $x(t)$ with period $x_1(t)=r(t)-2r(t-0.5)+r(t-1)$, fundamental frequency $\Omega_0=2\pi$

$$X_1(s)=\frac{1}{s^2}\left(1-2e^{-0.5s}+e^{-s}\right)=\frac{e^{-0.5s}}{s^2}\left(e^{0.5s}-2+e^{-0.5s}\right)$$

Fourier coefficients :

$$X_k=\frac{1}{T_0}X_1(s)|_{s=j2\pi k}=(-1)^k\frac{\sin^2(\pi k/2)}{\pi^2k^2},\ \ k\neq 0,\ \ X_0=0.5$$

$$X(\Omega)=2\pi X_0\delta(\Omega)+\sum_{k=-\infty,\neq 0}^{\infty}2\pi X_k\delta(\Omega-2k\pi)$$

Parseval's energy relation

For aperiodic signal $x(t)$ with energy $E_x < \infty$:

- Energy conservation in time and frequency

$$\boxed{E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega}$$

- $|X(\Omega)|^2$ energy density: energy at each of the frequencies Ω. Plot $|X(\Omega)|^2$ vs Ω is called the energy spectrum of $x(t)$, and displays how the energy of the signal is distributed over frequency

Example: Impulse $x(t) = \delta(t)$ is not finite energy signal

$$X(\Omega) = \mathcal{F}[\delta(t)] = 1$$
$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega \to \infty$$

Symmetry of spectral representations

- $x(t)$ real–valued signal

$$\boxed{\begin{array}{l} X(\Omega) = \mathcal{F}[x(t)] = |X(\Omega)|e^{j\angle X(\Omega)} = \mathcal{R}e[X(\Omega)] + j\mathcal{I}m[X(\Omega)] \\ \\ |X(\Omega)| = |X(-\Omega)|, \qquad \mathcal{R}e[X(\Omega)] = \mathcal{R}e[X(-\Omega)] \;\; \text{(even functions of } \Omega) \\ \\ \angle X(\Omega) = -\angle X(-\Omega), \qquad \mathcal{I}m[X(\Omega)] = -\mathcal{I}m[X(-\Omega)] \;\; \text{(odd functions of } \Omega) \end{array}}$$

- Spectra

$$\begin{array}{ll} |X(\Omega)| \text{ vs } \Omega & \text{Magnitude Spectrum} \\ \angle X(\Omega) \text{ vs } \Omega & \text{Phase Spectrum} \\ |X(\Omega)|^2 \text{ vs } \Omega & \text{Energy/Power Spectrum.} \end{array}$$

Example:

$$(a)\quad x_1(t) = u(t) - u(t-1), \quad \text{let } z(t) = x_1(t+0.5)$$

$$Z(\Omega) = \frac{\sin(\Omega/2)}{\Omega/2} \text{ (real)}$$

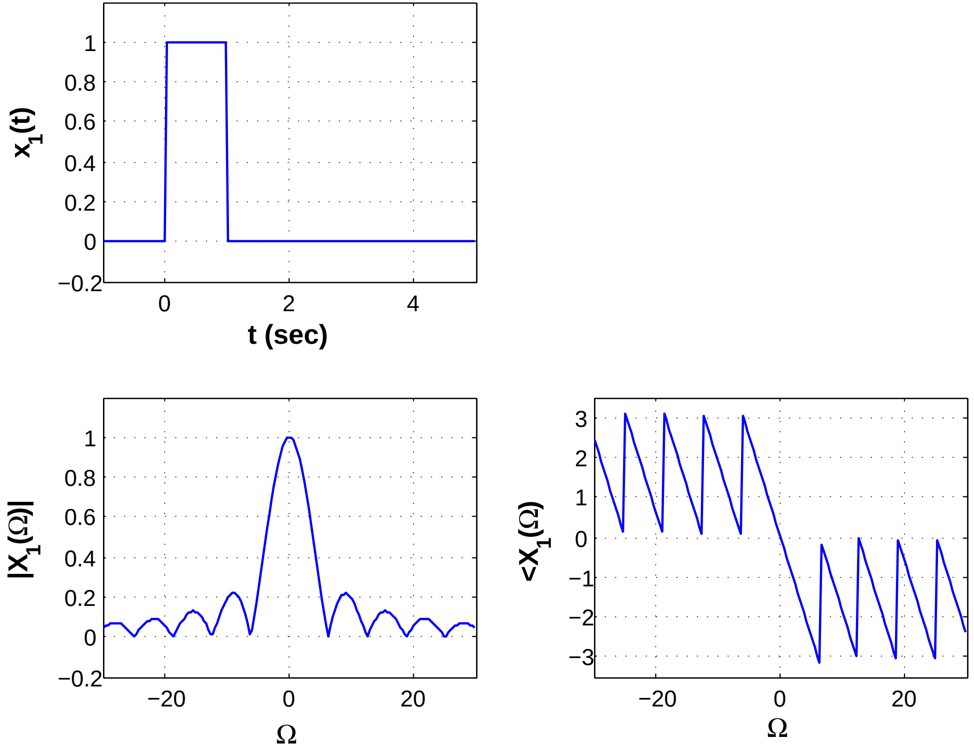
$$X_1(\Omega) = e^{-j0.5\Omega} Z(\Omega)$$

$$|X_1(\Omega)| = \left|\frac{\sin(\Omega/2)}{\Omega/2}\right|$$

$$\angle X_1(\Omega) = \angle Z(\Omega) - 0.5\Omega = \begin{cases} -0.5\Omega & Z(\Omega) \geq 0 \\ \pm\pi - 0.5\Omega & Z(\Omega) < 0 \end{cases}$$

$$(b)\quad x_2(t) = e^{-t}u(t), \quad X_2(\Omega) = \frac{1}{1+j\Omega}$$

$$|X_2(\Omega)| = \tfrac{1}{\sqrt{1+\Omega^2}}, \;\; \angle(X_2(\Omega)) = -\tan^{-1}\Omega,$$



Pulse $x_1(t) = u(t) - u(t-1)$ *and its magnitude and phase spectra.*

Convolution and filtering

- Input $x(t)$ (periodic or aperiodic) of stable LTI system has Fourier transform $X(\Omega)$
  system has frequency response $H(j\Omega) = \mathcal{F}[h(t)]$, $h(t)$ impulse response
  output is convolution integral $y(t) = (x * h)(t)$, with Fourier transform

$$\boxed{Y(\Omega) = X(\Omega)\ H(j\Omega)}$$

- If input $x(t)$ is periodic the output is also periodic of the same fundamental period, and with Fourier transform

$$\boxed{Y(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi\ X_k\ H(jk\ \Omega_0)\delta(\Omega - k\Omega_0)}$$

  where $\{X_k\}$ are the Fourier series coefficients of $x(t)$ and Ω_0 its fundamental frequency.

Example: Windowing

rectangular window $\mathrm{w}(t) = u(t+\Delta) - u(t-\Delta), \quad \Delta > 0$

windowed signal $y(t) = \mathrm{w}(t)x(t)$

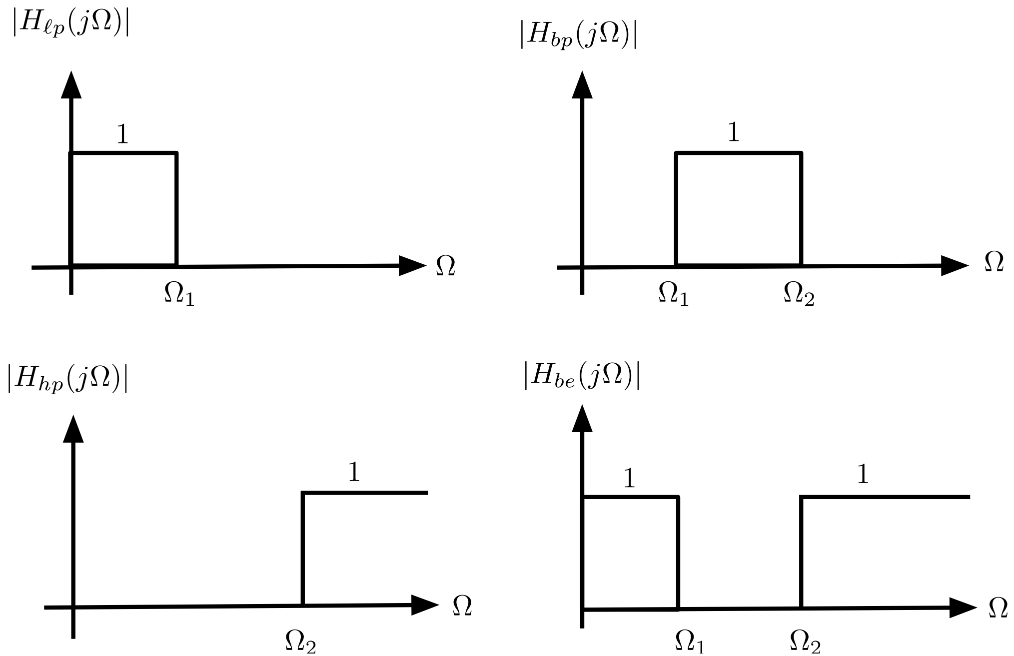
$$y(t) = \mathrm{w}(t)\ \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\rho)e^{j\rho t}d\rho}_{x(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\rho)\mathrm{w}(t)e^{j\rho t}d\rho$$

$$Y(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\rho)\mathcal{F}[\mathrm{w}(t)e^{j\rho t}]d\rho = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\rho)\mathrm{W}(\Omega-\rho)d\rho$$

$$y(t) = x(t)\mathrm{w}(t) \quad \Leftrightarrow \quad \frac{1}{2\pi}\text{convolution of } X(\Omega) \text{ and } W(\Omega) = \frac{2\sin(\Omega\Delta)}{\Omega}$$

Ideal filtering

Filtering: to pass desired frequency component and to attenuate undesirable components



Ideal filters: (top-left clockwise) low-pass, band-pass, band-eliminating and high-pass

Issues with ideal filters:

- Non–causal
- Paley-Wiener integral condition causal and stable filter with frequency response $H(j\Omega)$ should satisfy small

$$\int_{-\infty}^{\infty} \frac{|\log(H(j\Omega))|}{1+\Omega^2} d\Omega < \infty$$

Example: Gibb's phenomenon

Passing $x(t)$ through ideal low-pass filter

$$H(j\Omega) = \begin{cases} 1 & -\Omega_c \leq \Omega \leq \Omega_c, \quad N\Omega_0 < \Omega_c < (N+1)\Omega_0 \\ 0 & \text{otherwise} \end{cases}$$

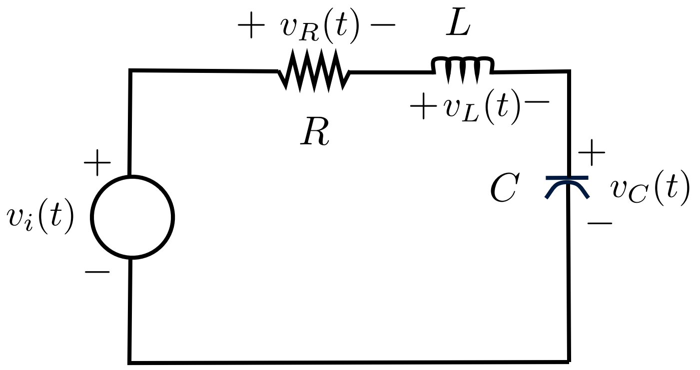
$$X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\Omega - k\Omega_0)$$

The output of the filter with $2N+1$ Fourier coefficients

$$\begin{aligned} x_N(t) &= \mathcal{F}^{-1}[X(\Omega)H(j\Omega)] = \mathcal{F}^{-1}\left[\sum_{k=-N}^{N} 2\pi X_k \delta(\Omega - k\Omega_0)\right] \\ &= [x * h](t), \qquad h(t) \;\; \text{sinc function} \end{aligned}$$

Convolution around the discontinuities of $x(t)$ causes ringing before and after them, independent of the value of N

Example: RLC circuit, $R = 1\ \Omega, L = 1\ H$, and C$=$ 1 F, and IC zero



$$\text{low–pass: output } v_c(t), \quad H_{lp}(s) = \frac{V_C(s)}{V_i(s)} = \frac{1}{s^2+s+1}$$

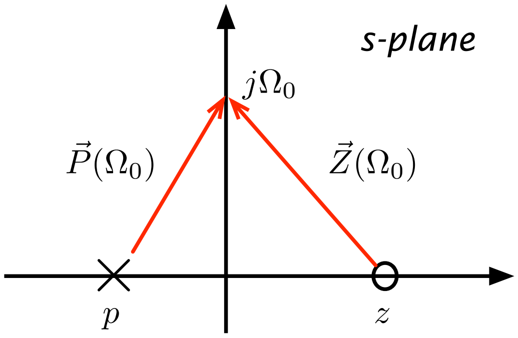
$$\text{high–pass: output } v_L(t), \quad H_{hp}(s) = \frac{V_L(s)}{V_i(s)} = \frac{s^2}{s^2+s+1}$$

$$\text{band–pass: output } v_R(t), \quad H_{bp}(s) = \frac{V_R(s)}{V_i(s)} = \frac{s}{s^2+s+1}$$

$$\text{band–stop: output } v_{cL}(t), \quad H_{bs}(s) = \frac{V_{cL}(s)}{V_i(s)} \frac{s^2+1}{s^2+s+1}$$

Frequency Response from Poles and Zeros

$$G(s) = K\frac{s-z}{s-p}, \quad \text{zero } z, \ \text{pole } p, \ \text{gain } K \neq 0$$



Frequency response of $G(s)$ at frequency Ω_0

$$G(j\Omega_0) \ = \ K\frac{\vec{Z}(\Omega_0)}{\vec{P}(\Omega_0)} = |K|e^{j\angle K}\frac{|\vec{Z}(\Omega_0)|}{|\vec{P}(\Omega_0)|}e^{j(\angle\vec{Z}(\Omega_0)-\angle\vec{P}(\Omega_0))}.$$

Magnitude response $\quad |G(j\Omega_0)| = |K|\dfrac{|\vec{Z}(\Omega_0)|}{|\vec{P}(\Omega_0)|}$

Phase response $\quad \angle G(j\Omega_0) = \angle K + \angle\vec{Z}(\Omega_0) - \angle\vec{P}(\Omega_0)$

Example: Frequency response of high-pass filter

$$H(s) = \frac{V_r(s)}{V_s(s)} = \frac{s}{s+1}$$

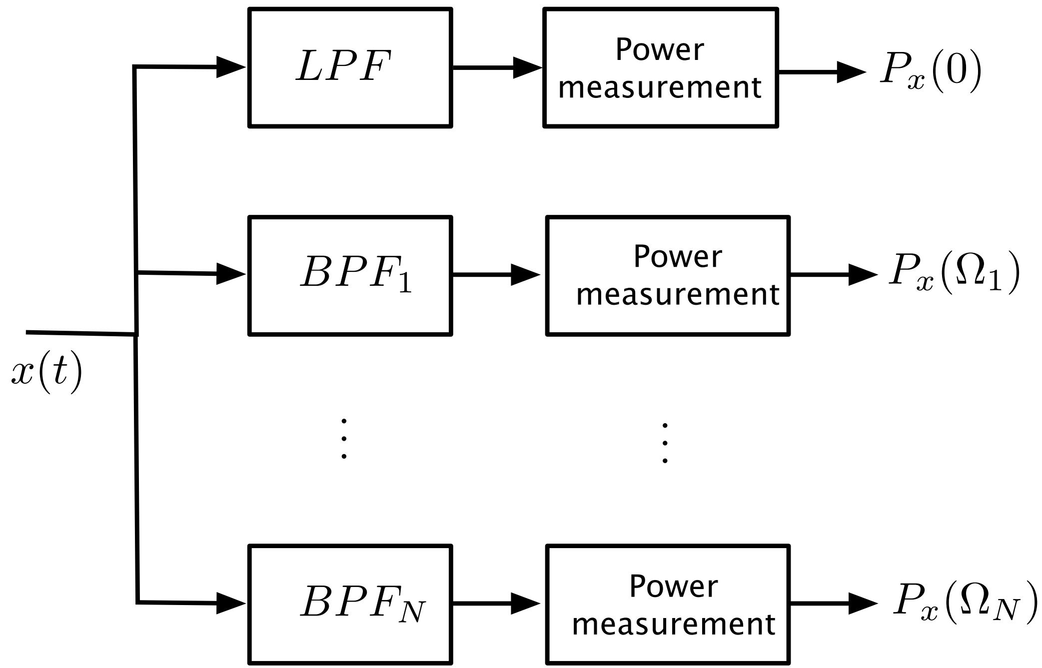
$$H(j\Omega) = \frac{j\Omega}{1+j\Omega} = \frac{\vec{Z}(\Omega)}{\vec{P}(\Omega)}$$

vector $\vec{Z}(\Omega)$ from $s=0$ to $j\Omega$

vector $\vec{P}(\Omega)$ from $s=-1$ to $j\Omega$

Ω	$\vec{Z}(\Omega)$	$\vec{P}(\Omega)$	$H(j\Omega) = \vec{Z}(\Omega)/\vec{P}(\Omega)$
0	$0e^{j\pi/2}$	$1e^{j0}$	$0e^{j\pi/2}$
1	$1e^{j\pi/2}$	$\sqrt{2}e^{j\pi/4}$	$0.707e^{j\pi/4}$
∞	$\infty\ e^{j\pi/2}$	$\infty\ e^{j\pi/2}$	$1e^{j0}$

Spectrum analyzer



Bank-of-filter spectrum analyzer: the frequency response of the bank–of–filters is that of an all–pass filter covering the desired range of frequencies

Table 5.1 Basic Properties of the Fourier Transform

Expansion/contraction	$x(\alpha t),\ \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Differentiation	$\frac{d^n x(t)}{dt^n},\ n \geq 1$	$(j\Omega)^n X(\Omega)$
Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Shifting	$x(t-\alpha),\ e^{j\Omega_0 t}x(t)$	$e^{-j\alpha\Omega}X(\Omega), X(\Omega-\Omega_0)$
Modulation	$x(t)\cos(\Omega_c t)$	$0.5[X(\Omega-\Omega_c)+X(\Omega+\Omega_c)]$
Periodic	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\Omega)\rvert = \lvert X(-\Omega)\rvert$, $\angle X(\Omega) = -\angle X(-\Omega)$
Convolution	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$

Table 5.2 Fourier Transform Pairs

$\delta(t), \quad \delta(t-\tau)$	$1, \quad e^{-j\Omega\tau}$
$u(t), \quad u(-t)$	$\dfrac{1}{j\Omega}+\pi\delta(\Omega), \quad \dfrac{-1}{j\Omega}+\pi\delta(\Omega)$
$\mathrm{sgn}(t)=2[u(t)-0.5]$	$\dfrac{2}{j\Omega}$
$A, \quad Ae^{-at}u(t), \; a>0$	$2\pi A\delta(\Omega), \quad \dfrac{A}{j\Omega+a}$
$Ate^{-at}u(t), \; a>0$	$\dfrac{A}{(j\Omega+a)^2}$
$e^{-a\|t\|}, \; a>0$	$\dfrac{2a}{a^2+\Omega^2}$
$\cos(\Omega_0 t), \quad -\infty<t<\infty$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$
$\sin(\Omega_0 t), \quad -\infty<t<\infty$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)]$
$p(t)=A[u(t+\tau)-u(t-\tau)],$	$2A\tau\dfrac{\sin(\Omega\tau)}{\Omega\tau}$