SIGNALS AND SYSTEMS USING MATLAB
Chapter 4 — Frequency Analysis: The Fourier Series

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Eigenfunctions

$x(t) = e^{j\Omega_0 t}, \quad -\infty < t < \infty$, input to a causal and stable LTI system

$$\text{steady state output} \quad y(t) = e^{j\Omega_0 t} H(j\Omega_0)$$

$$H(j\Omega_0) = \int_0^\infty h(\tau) e^{-j\Omega_0 \tau} d\tau = H(s)|_{s=j\Omega_0}$$

frequency response at Ω_0

$x(t) = e^{j\Omega_0 t}$ is eigenfunction of LTI system

Example: RC circuit, voltage source be $v_s(t) = 4\cos(t + \pi/4)$, $R = 1\ \Omega$, $C = 1$F

$$\text{transfer function} \quad H(s) = \frac{V_c(s)}{V_s(s)} = \frac{1}{s+1}$$

$$H(j1) = \frac{\sqrt{2}}{2} \angle -\pi/4 \quad \text{frequency response at} \, \Omega_0 = 1$$

$$\text{steady-state} \quad v_c(t) = 4|H(j1)| \cos(t + \pi/4 + \angle H(j1)) = 2\sqrt{2}\cos(t)$$

Example: Low–pass filter using RC circuit

Input $v_s(t) = 1 + \cos(10,000t)$ to series RC circuit $(R = C = 1)$

$$v_s(t) = v_c(t) + \frac{dv_c(t)}{dt}$$

if input $v_s(t) = e^{j\Omega t}$ output $v_c(t) = e^{j\Omega t}H(j\Omega)$, then in o.d.e.

$$e^{j\Omega t} = e^{j\Omega t}H(j\Omega)(1 + j\Omega) \quad \Rightarrow \quad H(j\Omega) = \frac{1}{1 + j\Omega} = \frac{1}{\sqrt{1 + \Omega^2}}$$

$$v_s(t) = \cos(0t) + \cos(10,000t)$$

$$v_c(t) \approx 1 + \frac{1}{10,000}\cos(10,000t - \pi/2) \approx 1$$

attenuates higher frequency component (i.e., low-pass filter)

Complex exponential Fourier series

Fourier Series of periodic signal $x(t)$, of fundamental period T_0, is infinite sum of ortho–normal complex exponentials of frequencies multiples of fundamental frequency $\Omega_0 = 2\pi/T_0$ (rad/sec) of $x(t)$:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

$$\text{FS coefficients} \quad X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jk\Omega_0 t}dt$$

$\{e^{jk\Omega_0 t}\}$ are ortho-normalFourier basis

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} e^{jk\Omega_0 t}[e^{j\ell\Omega_0 t}]^* dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} e^{j(k-\ell)\Omega_0 t}dt$$

$$= \begin{cases} 0 & k \neq \ell \quad \text{orthogonal} \\ 1 & k = \ell \quad \text{normal} \end{cases}$$

Line spectrum

- Parseval's power relation

$$\boxed{\begin{array}{l} P_x : \text{ power of periodic signal } x(t) \text{ of fundamental period } T_0 \\ P_x = \dfrac{1}{T_0}\displaystyle\int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2, \qquad \text{for any } \ t_0 \end{array}}$$

- Periodic $x(t)$ is represented in frequency by
  - Magnitude line spectrum $|X_k|$ vs $k\Omega_0$
  - Phase line spectrum $\angle X_k$ vs $k\Omega_0$
  - Power line spectrum $|X_k|^2$ vs $k\Omega_0$
- Real-valued periodic signal $x(t)$, of fundamental period T_0,

$$\begin{array}{l} X_k = X^*_{-k} \ \text{ or equivalently} \\ (i) \ \ |X_k| = |X_{-k}|, \ \text{ i.e., magnitude } |X_k| \text{ is even function of } \ k\Omega_0. \\ (ii) \ \angle X_k = -\angle X_{-k}, \ \text{ i.e., phase } \angle X_k \text{ is odd function of } \ k\Omega_0 \end{array}$$

Trigonometric Fourier series

Real-valued, periodic signal $x(t)$, of fundamental period T_0,

$$\boxed{\begin{aligned} x(t) &= \underbrace{X_0}_{dc-component} + 2\sum_{k=1}^{\infty} \underbrace{|X_k|\cos(k\Omega_0 t + \theta_k)}_{k^{th}\,harmonic} \\ &= c_0 + 2\sum_{k=1}^{\infty}[c_k\cos(k\Omega_0 t) + d_k\sin(k\Omega_0 t)] \qquad \Omega_0 = \frac{2\pi}{T_0} \end{aligned}}$$

Fourier coefficients $\{c_k,\ d_k\}$

$$\begin{aligned} c_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\Omega_0 t)\ dt \qquad k = 0, 1, \cdots \\ d_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\Omega_0 t)\ dt \qquad k = 1, 2, \cdots \end{aligned}$$

Sinusoidal basis functions $\{\sqrt{2}\cos(k\Omega_0 t),\ \sqrt{2}\sin(k\Omega_0 t)\}$, $k = 0, \pm 1, \cdots$, are orthonormal in $[0,\ T_0]$

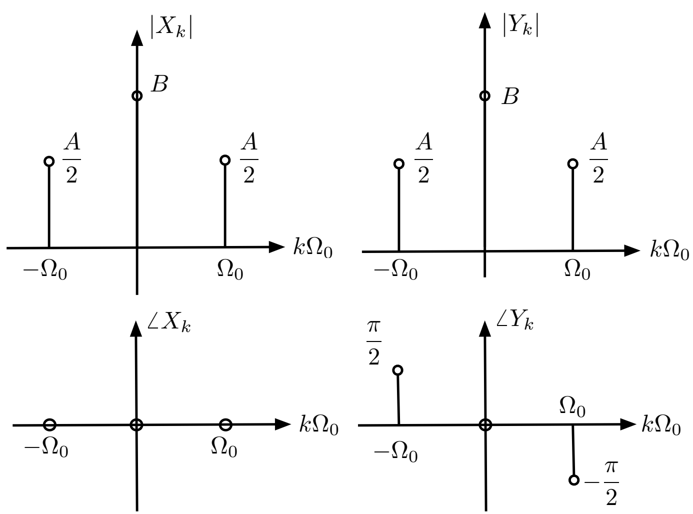
Example: $x(t) = B + A\cos(\Omega_0 t + \theta)$ periodic of fundamental period T_0

trigonometric Fourier series: $X_0 = B$; $|X_1| = A/2$, $\angle X_1 = \theta$

exponential Fourier series:

$$x(t) = B + \frac{A}{2}\left[e^{j(\Omega_0 t+\theta)} + e^{-j(\Omega_0 t+\theta)}\right]$$

$$X_0 = B, \quad X_1 = \frac{Ae^{j\theta}}{2}, \quad X_{-1} = X_1^* = \frac{Ae^{-j\theta}}{2}$$



Line spectrum of $x(t) = B + A\cos(\Omega_0 t)$ and of $y(t) = B + A\sin(\Omega_0 t)$ (right).

Fourier coefficients from Laplace

$x(t)$, periodic of fundamental period T_0

$$\boxed{\begin{array}{l}\text{period: } \; x_1(t) = x(t)[u(t-t_0) - u(t-t_0-T_0)], \;\; \text{any } t_0 \\ X_k = \dfrac{1}{T_0}\mathcal{L}\left[x_1(t)\right]_{s=jk\Omega_0} \quad \Omega_0 = \dfrac{2\pi}{T_0} \text{ (fundamental frequency)}, \; k = 0, \pm 1, \cdots \end{array}}$$

Example: $x(t)$ periodic, $T_0 = 2$, $x_1(t) = u(t) - u(t-1)$

$$x(t) = \sum_{m=-\infty}^{\infty} x_1(t-2m) = \sum_{k=-\infty}^{\infty} X_k e^{jk\pi t}$$

$$X_k = \frac{1}{2}\mathcal{L}\left[x_1(t)\right]_{s=jk\pi} = \frac{1 - e^{-jk\pi}}{jk\pi} = e^{-jk\pi/2} \; \frac{\sin(k\pi/2)}{k\pi/2}$$

Reflection and even and odd periodic signals

$x(t)$ periodic of fundamental period T_0, Fourier coefficients $\{X_k\}$

- Reflection: Fourier coefficients of $x(-t)$ are $\{X_{-k}\}$
- Even $x(t)$: $\{X_k\}$ are real

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} X_k \cos(k\Omega_0 t)$$

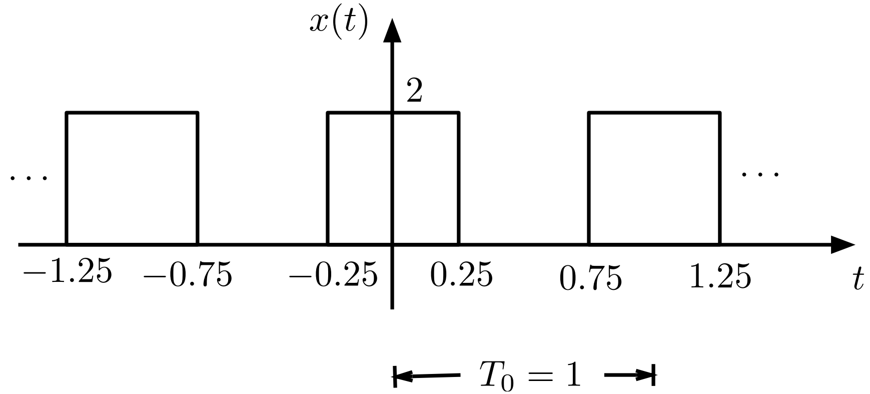
Odd $x(t)$: $\{X_k\}$ are imaginary

$$x(t) = 2\sum_{k=1}^{\infty} jX_k \sin(k\Omega_0 t)$$

- Any periodic signal $x(t)$ then $x(t) = x_e(t) + x_o(t)$, $x_e(t)$ and $x_o(t)$ even and odd components

$$\begin{aligned} X_k &= X_{ek} + X_{ok} \\ X_{ek} &= 0.5[X_k + X_{-k}] \\ X_{ok} &= 0.5[X_k - X_{-k}] \end{aligned}$$

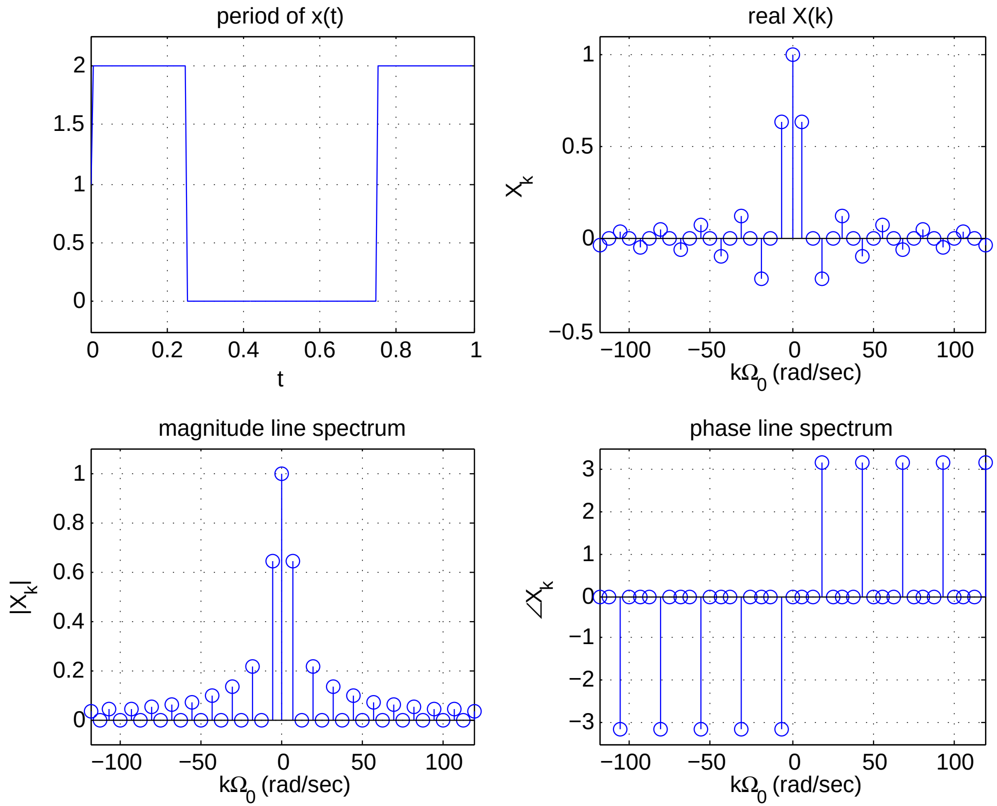
Example: periodic pulse train $x(t)$, of fundamental period $T_0 = 1$



Integral formula:
$$X_k = \frac{1}{T_0}\int_{-T_0/4}^{3T_0/4} x(t)e^{-j\Omega_0 kt}dt = \frac{\sin(\pi k/2)}{(\pi k/2)}, \quad k \neq 0$$
$$X_0 = \frac{1}{T_0}\int_{-T_0/4}^{3T_0/4} x(t)dt = \int_{-1/4}^{1/4} 2dt = 1$$

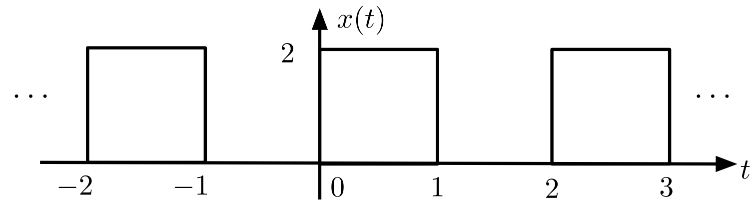
Laplace transform:
$$x_1(t-0.25) = 2[u(t) - u(t-0.5)], \quad X_1(s) = 2(e^{0.25s} - e^{-0.25s})$$
$$X_k = \frac{1}{T_0}\mathcal{L}\left[x_1(t)\right]|_{s=jk\Omega_0} = \frac{\sin(\pi k/2)}{\pi k/2} \qquad k \neq 0$$

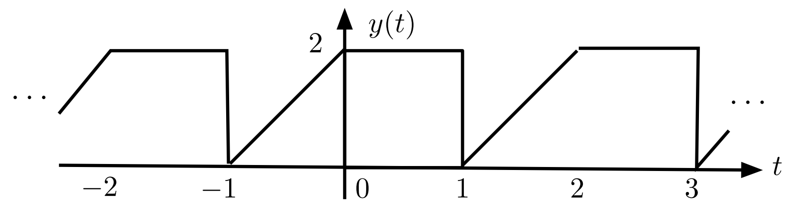
Fourier series:
$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\pi k/2)}{(\pi k/2)} e^{jk2\pi t}$$



Top: period of $x(t)$ and real X_k vs $k\Omega_0$; bottom magnitude and phase line spectra

Example: Non-symmetric periodic signals





$$z(t) = x(t+0.5), \quad \text{even}, \quad \text{period:} \;\; z_1(t) = 2[u(t+0.5) - u(t-0.5)]$$

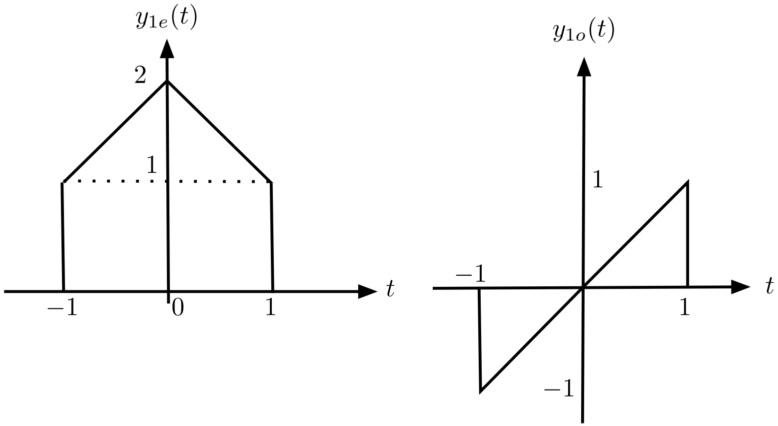
$$Z_1(s) = \frac{2}{s}[e^{0.5s} - e^{-0.5s}]$$

$$Z_k = \frac{1}{2}\,\frac{2}{jk\pi}[e^{jk\pi/2} - e^{-jk\pi/2}] = \frac{\sin(0.5\pi k)}{0.5\pi k} \quad \text{real–valued}$$

$$x(t) = z(t-0.5) = \sum_k Z_k e^{jk\Omega_0(t-0.5)} = \sum_k \underbrace{\left[Z_k e^{-jk\pi/2}\right]}_{X_k} e^{jk\pi t}$$

X_k complex since $x(t)$ neither even nor odd

Even and odd components of the period of $y(t)$, $-1 \leq t \leq 1$



$$y_{1e}(t) = \underbrace{[u(t+1) - u(t-1)]}_{\text{rectangular pulse}} + \underbrace{[r(t+1) - 2r(t) + r(t-1)]}_{\text{triangular pulse}}$$

$$y_{1o}(t) = \underbrace{[r(t+1) - r(t-1) - 2u(t-1)]}_{\text{triangular pulse}} - \underbrace{[u(t+1) - u(t-1)]}_{\text{rectangular pulse}}$$

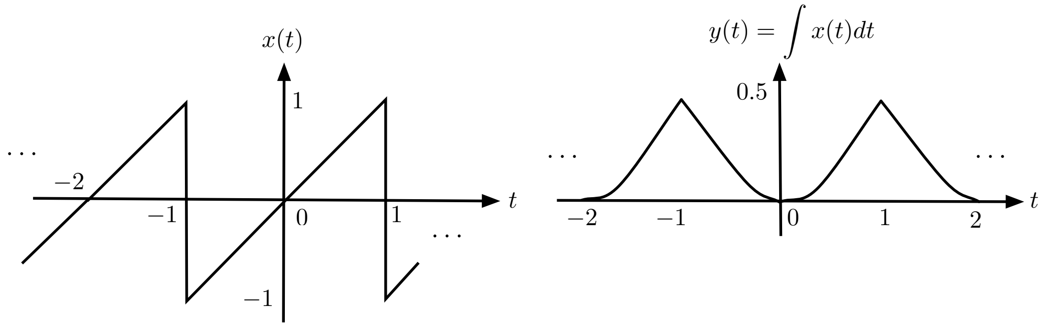
$$Y_{ek} = \frac{1}{T_0} Y_{1e}(s)\,|_{s=jk\Omega_0} = \frac{1-(-1)^k}{(k\pi)^2} \qquad k \neq 0, \quad Y_{e0} = 1.5$$

$$Y_{ok} = \frac{1}{T_0} Y_{1o}(s)\,|_{s=jk\Omega_0} = j\frac{(-1)^k}{k\pi} \qquad k \neq 0, \quad Y_{o0} = 0$$

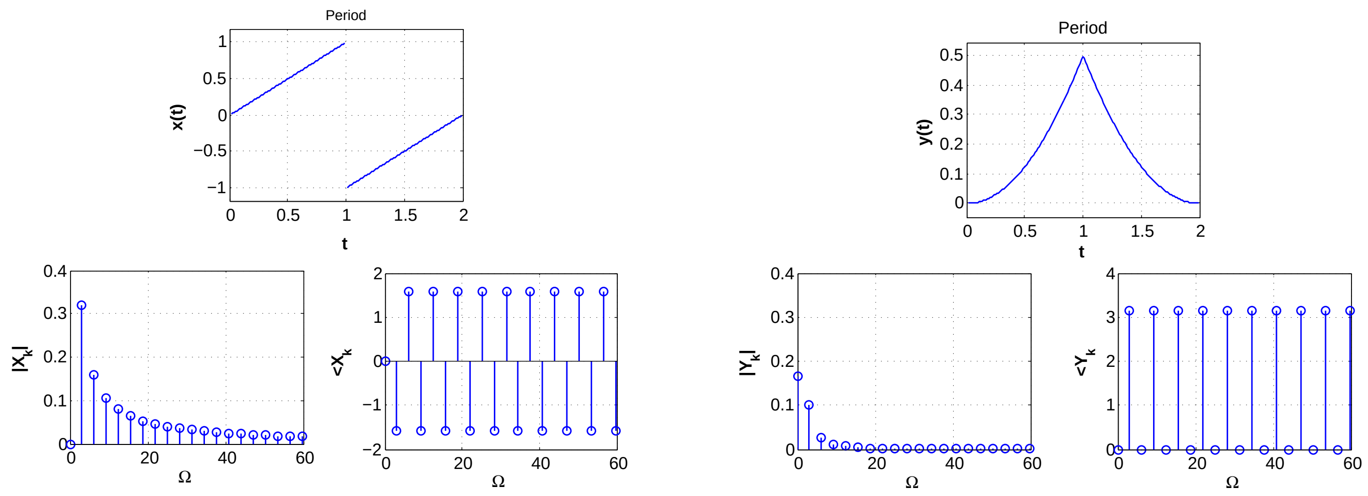
$$Y_k = \begin{cases} Y_{e0} + Y_{o0} = 1.5 + 0 = 1.5 & k = 0 \\ Y_{ek} + Y_{ok} = (1-(-1)^k)/(k\pi)^2 + j(-1)^k/(k\pi) & k \neq 0 \end{cases}$$

Example: Integration

$$y(t) = \int_{-\infty}^{t} x(t)dt$$



Integral does not exist if the dc is not zero



Convergence of Fourier series

For Fourier series of $x(t)$ to converge, it should:

- be absolutely integrable,
- have a finite number of maxima, minima and discontinuities.

FS equals $x(t)$ at every continuity point and $0.5[x(t+0_+)+x(t+0_-)]$ at every discontinuity point

Example: Approximate train of pulses with $x_2(t)=\alpha+\beta\cos(\Omega_0 t)$ by

$$\text{Minimize} \qquad E_2=\frac{1}{T_0}\int_{T_0}|x(t)-x_2(t)|^2dt, \text{ w.r.t. } \alpha,\ \beta$$

$$\frac{dE_2}{d\alpha}=-\frac{1}{T_0}\int_{T_0}2[x(t)-\alpha]dt=0$$

$$\frac{dE_2}{d\beta}=-\frac{1}{T_0}\int_{T_0}2[x(t)\cos(\Omega_0 t)-\beta\cos^2(\Omega_0 t)]dt=0$$

$$\alpha=\frac{1}{T_0}\int_{T_0}x(t)dt,$$

$$\beta=\frac{2}{T_0}\int_{T_0}x(t)\cos(\Omega_0 t)dt$$

Time and frequency shifting

Periodic signal $x(t)$

- Time-shifting: $x(\pm t_0)$ remains periodic of the same fundamental period

  $$x(t) \;\leftrightarrow\; \{X_k\} \;\Rightarrow\; x(t \mp t_0) \;\leftrightarrow\; X_k e^{\mp jk\Omega_0 t_0} = |X_k| e^{j(\angle X_k \mp k\Omega_0 t_0)}$$

  only change in phase
- Frequency-shifting:
  - $x(t)e^{j\Omega_1 t}$ is periodic of fundamental period T_0 if $\Omega_1 = M\Omega_0$, for an integer $M \geq 1$,
  - for $\Omega_1 = M\Omega_0$, $M \geq 1$, the Fourier coefficients X_k are shifted to frequencies $k\Omega_0 + \Omega_1 = (k + M)\Omega_0$
  - the modulated signal is real-valued by multiplying $x(t)$ by $\cos(\Omega_1 t)$.

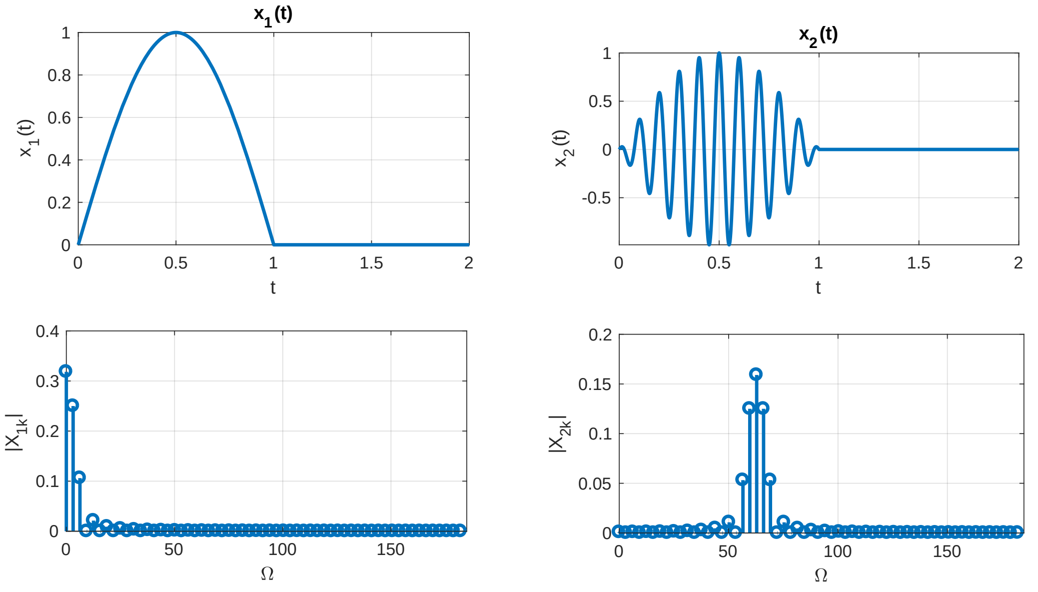
Example: Modulating $\cos(20\pi t)$ with

- a periodic train of square pulses

$$x_1(t) = 0.5[1 + \text{sign}(\sin(\pi t))] = \begin{cases} 1 & \sin(\pi t) \geq 0 \\ 0 & \sin(\pi t) < 0 \end{cases}$$

- with a sinusoid

$$x_2(t) = \sin(\pi t).$$



Modulated square-wave $x_1(t)\cos(20\pi t)$ (left) and modulated cosine $x_2(t)\cos(20\pi t)$

Response of LTI systems to periodic signals

Periodic input $x(t)$ of causal and stable LTI system, with impulse response $h(t)$, by eigenfunction property of LTI systems

Fourier series $x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\Omega_0 t + \angle X_k)$ $\qquad \Omega_0 = \dfrac{2\pi}{T_0}$

$$y_{ss}(t) = X_0|H(j0)| + 2\sum_{k=1}^{\infty} |X_k||H(jk\Omega_0)| \cos(k\Omega_0 t + \angle X_k + \angle H(jk\Omega_0))$$

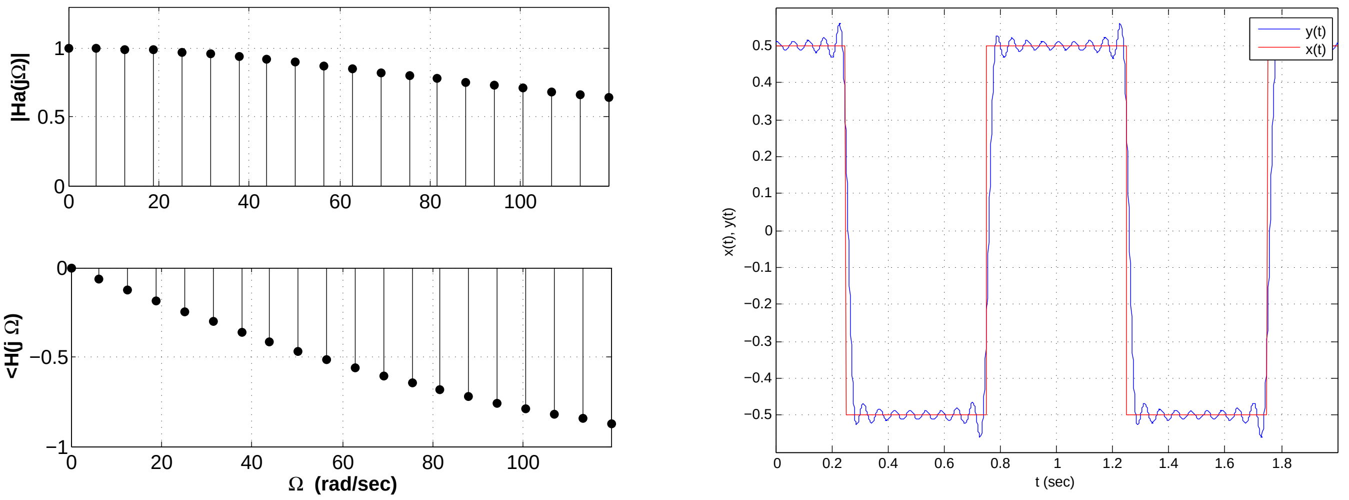
where $H(jk\Omega_0) = |H(jk\Omega_0)|e^{j\angle H(jk\Omega_0)} H(s)|_{s=jk\Omega_0}$

frequency response of the system at $k\Omega_0$

Example: Low–pass filtering using RC circuit with

$$\text{transfer function} \quad H(s) = \frac{1}{1 + s/100}$$

$$\text{input} \quad x(t) = \sum_{k=-\infty, \neq 0}^{\infty} \frac{\sin(k\pi/2)}{k\pi/2} e^{j2k\pi t}$$



Left: magnitude and phase response of the low-pass RC filter at harmonic frequencies. Right: response due to the train of pulses $x(t)$. Actual signal values are given by the dashed line, and the filtered signal is indicated by the continuous line

Derivatives and integrals of Periodic Signals

- Derivative: Derivative $dx(t)/dt$ of periodic signal $x(t)$ is periodic of the same fundamental period. If $\{X_k\}$ are the coefficients of the Fourier series of $x(t)$, the Fourier coefficients of $dx(t)/dt$ are

$$jk\Omega_0 X_k, \quad \Omega_0 \text{ fundamental frequency of} x(t)$$

- Integral: Zero-mean, periodic signal $y(t)$ with Fourier coefficients $\{Y_k\}$,

$$\text{integral} \;\; z(t) = \int_{-\infty}^{t} y(\tau)d\tau$$

$$Z_k = \frac{Y_k}{jk\Omega_0} \qquad k \text{ integer } \neq 0$$

$$Z_0 = -\sum_{m \neq 0} Y_m \frac{1}{jm\Omega_0} \qquad \Omega_0 = \frac{2\pi}{T_0}.$$

Example: Derivative

$$\text{period of } x(t): \; x_1(t) = 2r(t) - 4r(t-0.5) + 2r(t-1), \; 0 \le t \le 1, \; T_0 = 1$$

$$g(t) = \frac{dx(t)}{dt} \quad \Rightarrow \quad X_k = \frac{G_k}{jk\Omega_0} \quad k \neq 0$$

$$\text{period of } g(t): \; g_1(t) = dx_1(t)/dt = 2u(t) - 4u(t-0.5) + 2u(t-1)$$

$$X_k = \frac{G_k}{jk\Omega_0} = \frac{(-1)^{(k+1)}\,(\cos(\pi k) - 1)}{\pi^2 k^2} \qquad k \neq 0$$

$$X_0 = 0.5 \;\; \text{from plot of} \;\; x(t)$$

Integral

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau, \quad (G_0 = 0)$$

$$X_k = \frac{G_k}{j\Omega_0 k} = \frac{(-1)^{(k+1)}(\cos(\pi k) - 1)}{\pi^2 k^2} \qquad k \neq 0$$

$$X_0 = -\sum_{m=-\infty, m\neq 0}^{\infty} \frac{G_m}{j2m\pi} = 0.5 \sum_{m=-\infty, m\neq 0}^{\infty} (-1)^{m+1} \left[\frac{\sin(\pi m/2)}{(\pi m/2)}\right]^2$$

Table 4.1 Basic Properties of Fourier Series

Signals and constants	$x(t), y(t)$ periodic with period T_0, α, β	X_k, Y_k
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X_k + \beta Y_k$
Parseval's power relation	$P_x = \frac{1}{T_0} \int_{T_0} \lvert x(t)\rvert^2 dt$	$P_x = \sum_k \lvert X_k \rvert^2$
Differentiation	$\frac{dx(t)}{dt}$	$jk\Omega_0 X_k$
Integration	$\int_{-\infty}^{t} x(t')dt'$ only if $X_0 = 0$	$\frac{X_k}{jk\Omega_0} k \neq 0$
Time shifting	$x(t-\alpha)$	$e^{-j\alpha\Omega_0} X_k$
Frequency shifting	$e^{jM\Omega_0 t} x(t)$	X_{k-M}
Symmetry	$x(t)$ real	$\lvert X_k\rvert = \lvert X_{-k}\rvert$ even function of k $\angle X_k = -\angle X_{-k}$ odd function of k
Multiplication	$z(t) = x(t)y(t)$	$Z_k = \sum X_m Y_{k-m}$