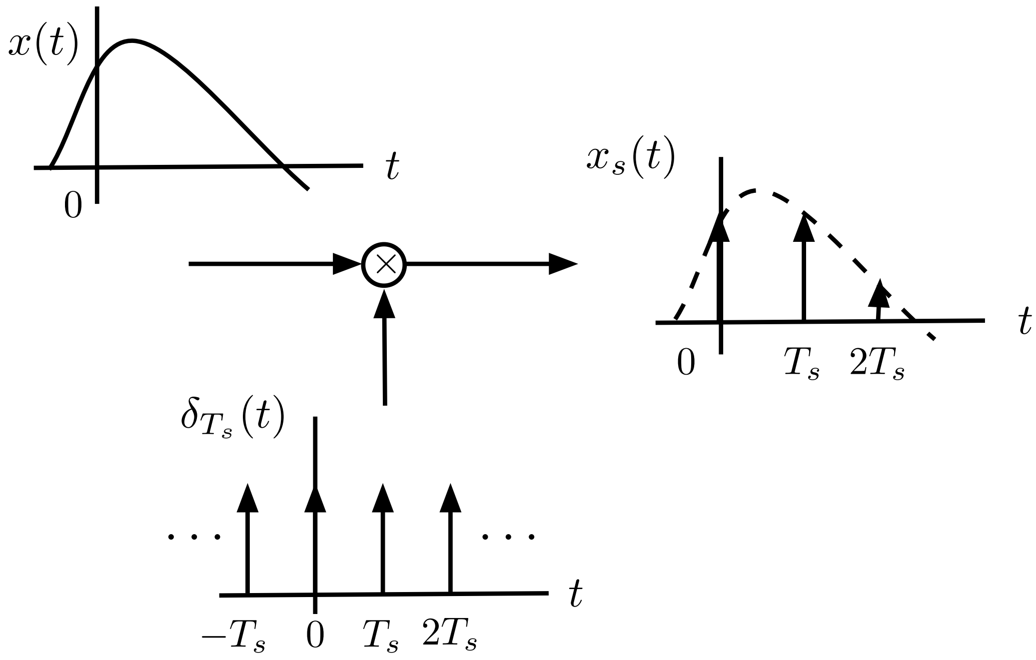
SIGNALS AND SYSTEMS USING MATLAB
Chapter 8 — Sampling Theory

L. F. Chaparro and A. Akan

Uniform sampling



$$\delta_{T_s}(t) = \sum_n \delta(t - nT_s)$$

$$x_s(t) = x(t)\delta_{T_s}(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} x(t) e^{jk\Omega_s t}$$

- Modulation

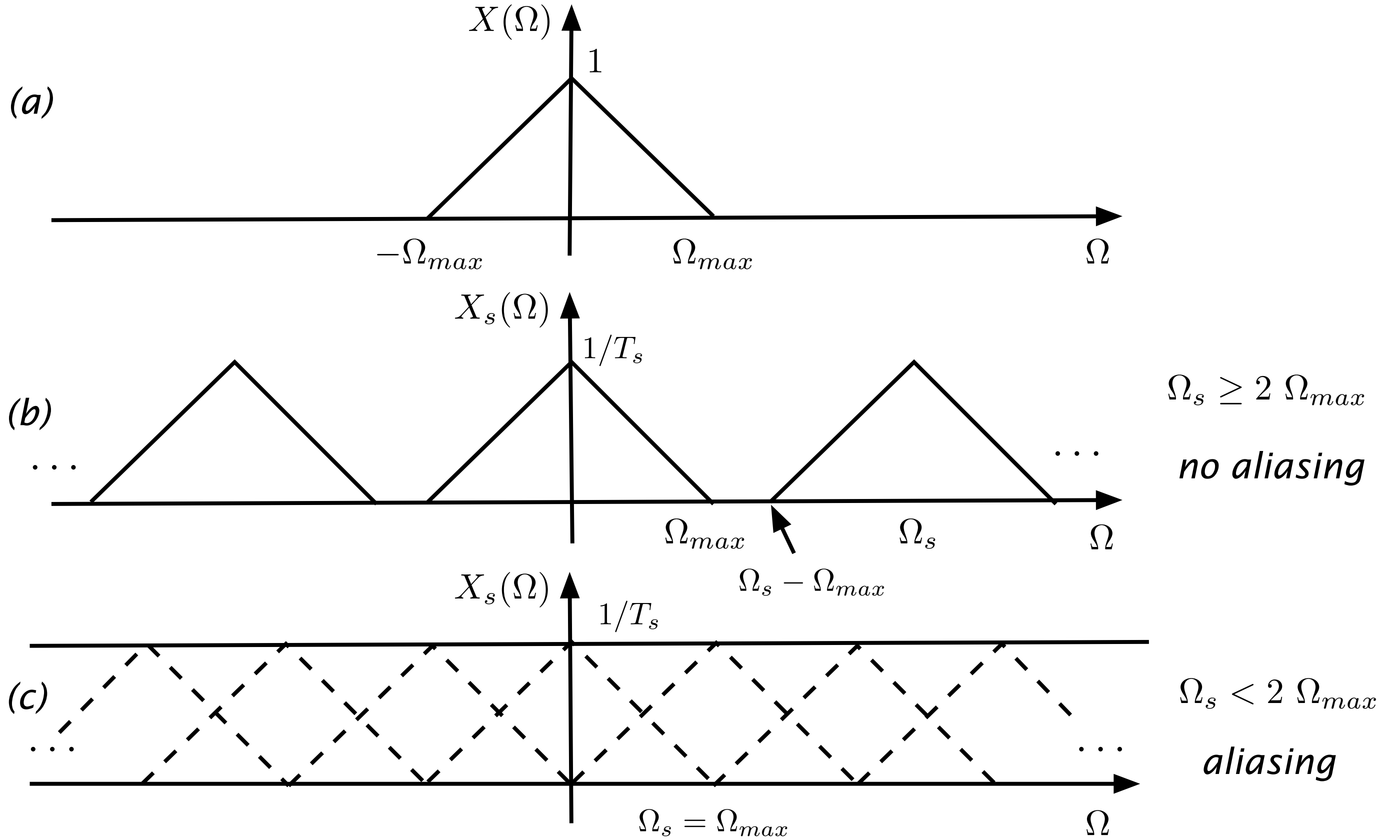
$$\delta_{T_s}(t),\ \text{periodic},\ \Omega_s = 2\pi/T_s, \quad \delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} D_k e^{jk\Omega_s t}$$

$$D_k = \frac{1}{T_s}, \quad x_s(t) = x(t)\delta_{T_s}(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} x(t)e^{jk\Omega_s t}$$

$$\boxed{X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)}$$

- Discrete–time Fourier transform

$$\boxed{\begin{aligned} x_s(t) &= \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \\ X_s(\Omega) &= \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\Omega T_s n} \end{aligned}}$$



(a) Spectrum of band–limited signal, (b) spectrum of sampled signal when satisfying the Nyquist sampling rate condition, (c) spectrum of sampled signal with aliasing (superposition of spectra, shown in dashed lines, gives a constant shown by continuous line)

Band–limited signals and Nyquist condition

A signal $x(t)$ is band–limited if its low–pass spectrum $X(\Omega)$ is such that

$$\boxed{|X(\Omega)| = 0 \text{ for } |\Omega| > \Omega_{max}, \quad \Omega_{max} : \text{ max frequency in } x(t)}$$

can be sampled uniformly and without frequency aliasing using a sampling frequency

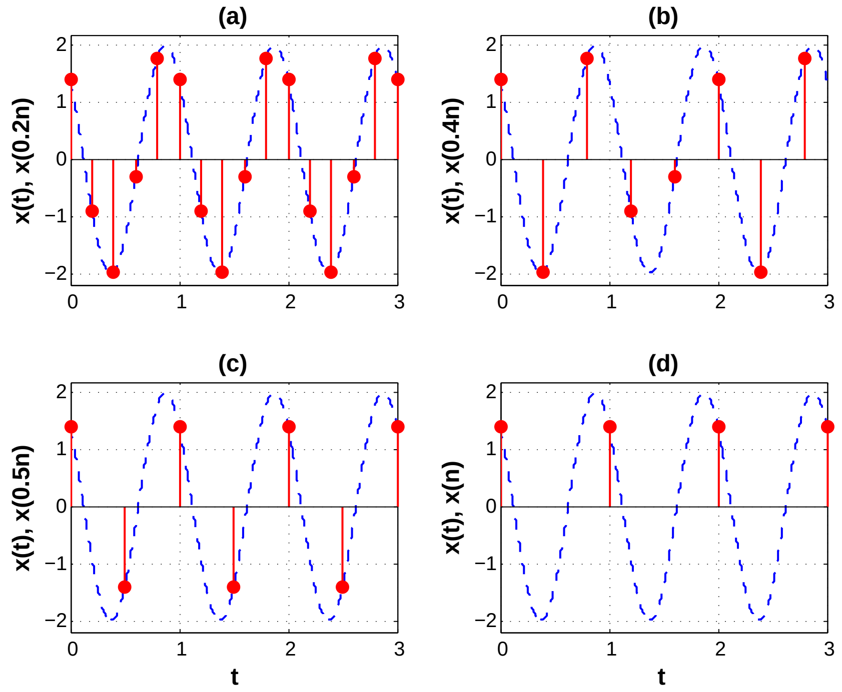
$$\boxed{\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_{max} \quad \text{Nyquist sampling rate condition}}$$

Example: $x(t) = 2\cos(2\pi t + \pi/4)$, $-\infty < t < \infty$, band–limited

$$T_s = 0.4, \quad \Omega_s = 2\pi/T_s = 5\pi > 2\Omega_{max} = 4\pi, \text{ satisfy Nyquist}$$

$$x(nT_s) = 2\cos(2\pi\ 0.4n + \pi/4) = 2\cos\left(\frac{4\pi}{5}n + \frac{\pi}{4}\right) \qquad -\infty < n < \infty$$

$$T_s = 1, \quad \Omega_s = 2\pi < 2\Omega_{max} = 4\pi \quad \text{aliasing}$$
$$x(nT_s) = 2\cos(2\pi n + \pi/4) = 2\cos(\pi/4) = \sqrt{2}.$$



Sampling of $x(t) = 2\cos(2\pi t + \pi/4)$*: (a)* $T_s = 0.2$*, (b)* $T_s = 0.4$*, (c)* $T_s = 0.5$ *and (d)* $T_s = 1$ *sec/sample*

Example: Causal exponential $x(t) = e^{-t}u(t)$ is not band–limited

$$X(\Omega) = \frac{1}{1 + j\Omega} \quad \text{so that} \quad |X(\Omega)| = \frac{1}{\sqrt{1 + \Omega^2}}$$

Frequency Ω_M so that 99% of the energy is in $-\Omega_M \le \Omega \le \Omega_M$:

$$\frac{1}{2\pi} \int_{-\Omega_M}^{\Omega_M} |X(\Omega)|^2 d\Omega = \frac{0.99}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega$$

$$2 \tan^{-1}(\Omega)|_0^{\Omega_M} = 2 \times 0.99 \tan^{-1}(\Omega)|_0^{\infty}$$

$$\Omega_M = \tan\left(\frac{0.99\pi}{2}\right) = 63.66 \text{ rad/sec}$$

Choose $\Omega_s = 2\pi/T_s = 5\Omega_M$ or $T_s = 2\pi/(5 \times 63.66) \approx 0.02$ sec/sample

Nyquist–Shannon sampling theorem

Low–pass signal $x(t)$ is band–limited (i.e., $X(\Omega) = 0$ for $|\Omega| > \Omega_{max}$)

- Information in $x(t)$ preserved by sampled signal $x_s(t)$, with samples $x(nT_s) = x(t)|_{t=nT_s}$, $n = 0, \pm 1, \pm 2, \cdots$, provided

  sampling frequency $\Omega_s \geq 2\Omega_{max}$ (Nyquist sampling rate condition), or sampling rate f_s (samples/sec) or sampling period T_s (sec/sample) are

  $$f_s = \frac{1}{T_s} \geq \frac{\Omega_{max}}{\pi}$$

- When Nyquist condition is satisfied, $x(t)$ can be reconstructed by ideal low–pass filtering $x_s(t)$:

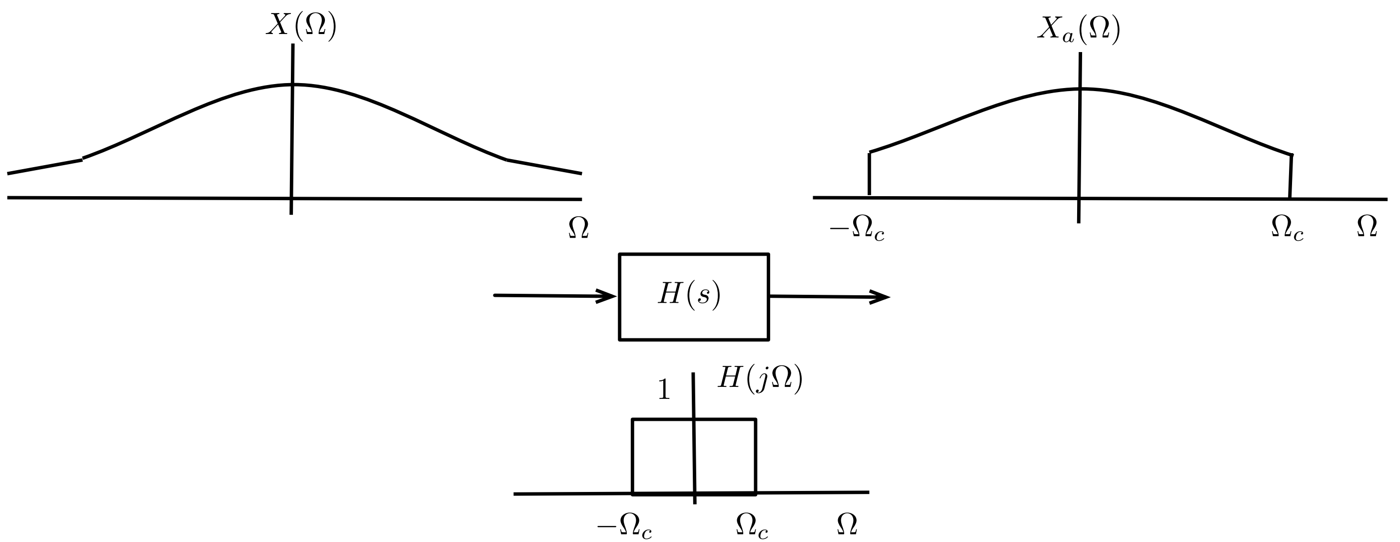
  frequency response ideal LPF $H(j\Omega) = \begin{cases} T_s & -\Omega_s/2 < \Omega < \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$

  Reconstructed (sinc interpolation)

  $$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin(\pi(t - nT_s)/T_s)}{\pi(t - nT_s)/T_s}$$

Antialiasing filtering

For signals that do not satisfy the band–limitedness condition



Anti–aliasing filtering of non band–limited signal

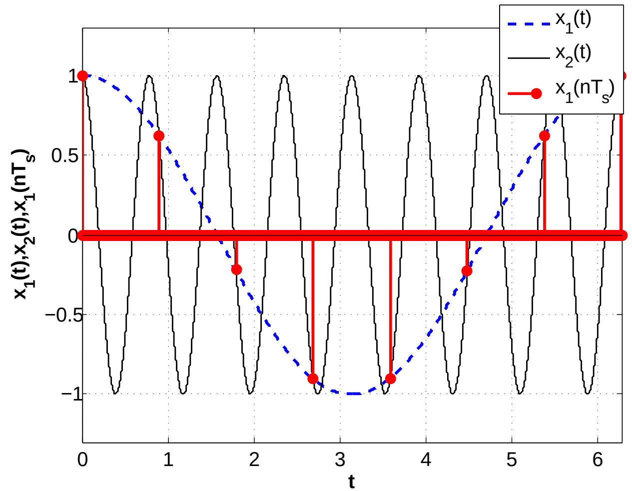
Example: Aliasing effects

$$x_1(t) = \cos(\Omega_0 t), \qquad x_2(t) = \cos((\Omega_0 + \Omega_1)t) \quad \Omega_1 > 2\Omega_0$$

sampling signals with $T_s = 2\pi/\Omega_1$

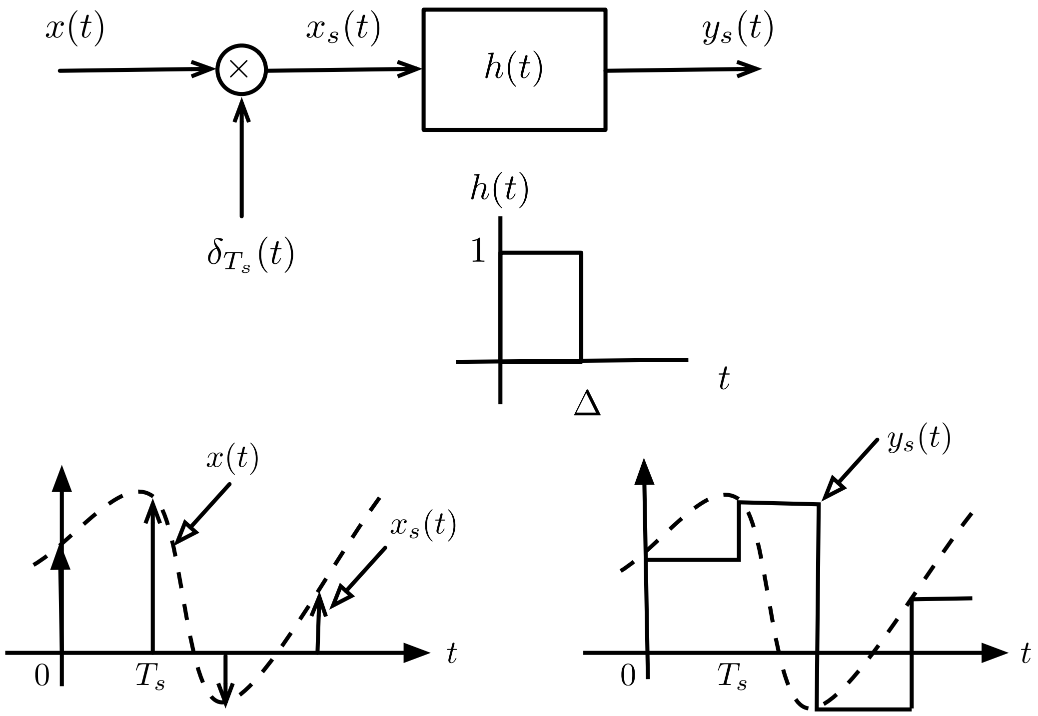
$$x_1(nT_s) = \cos(\Omega_0 nT_s), \qquad x_2(nT_s) = \cos((\Omega_0 + \Omega_1)nT_s) = \cos(\Omega_0 T_s n) = x$$

No frequency aliasing in $x_1(nT_s)$, frequency aliasing in $x_2(nT_s)$



Sampling sinusoids of frequencies $\Omega_0 = 1$ and $\Omega_0 + \Omega_1 = 8$ with $T_s = 2\pi/\Omega_1$. The higher frequency signal is under–sampled, causing aliasing and making the two sampled signals coincide

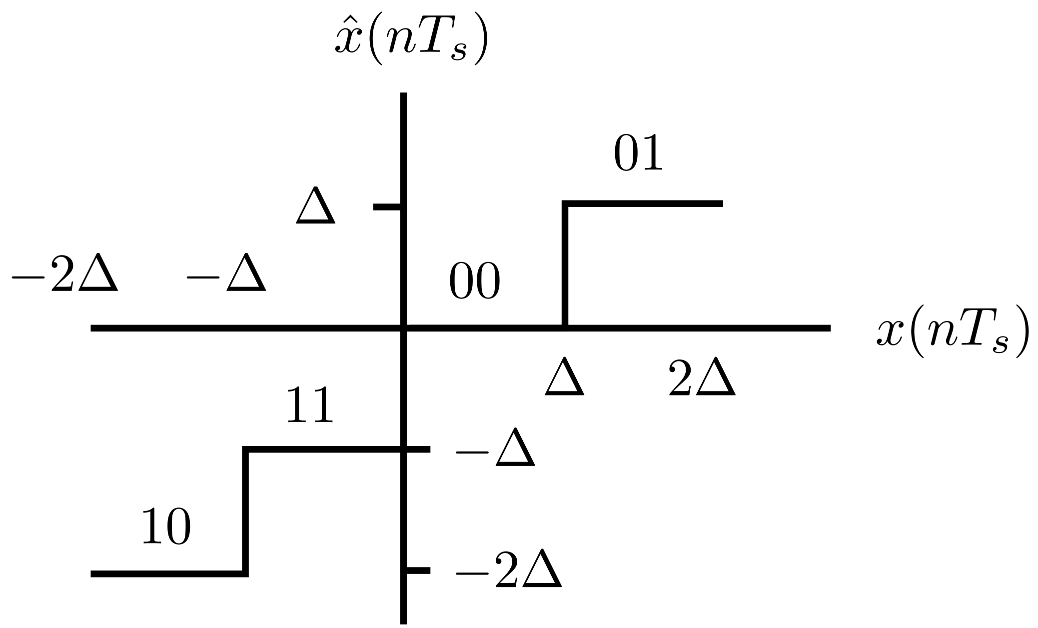
Practical aspects of sampling — Sample–and–hold sampling



Sample–and–hold sampling system for $\Delta = T_s$; $y_s(t)$ multi–level signal

$$y_s(t) = (x_s * h)(t) \quad \Rightarrow \quad Y_s(\Omega) = \left[\frac{1}{T_s} \sum_k X(\Omega - k\Omega_s) \right] \frac{\sin(\Delta\Omega/2)}{\Omega/2} e^{-j\Omega\Delta/2}$$

Practical aspects of sampling — Quantization and coding



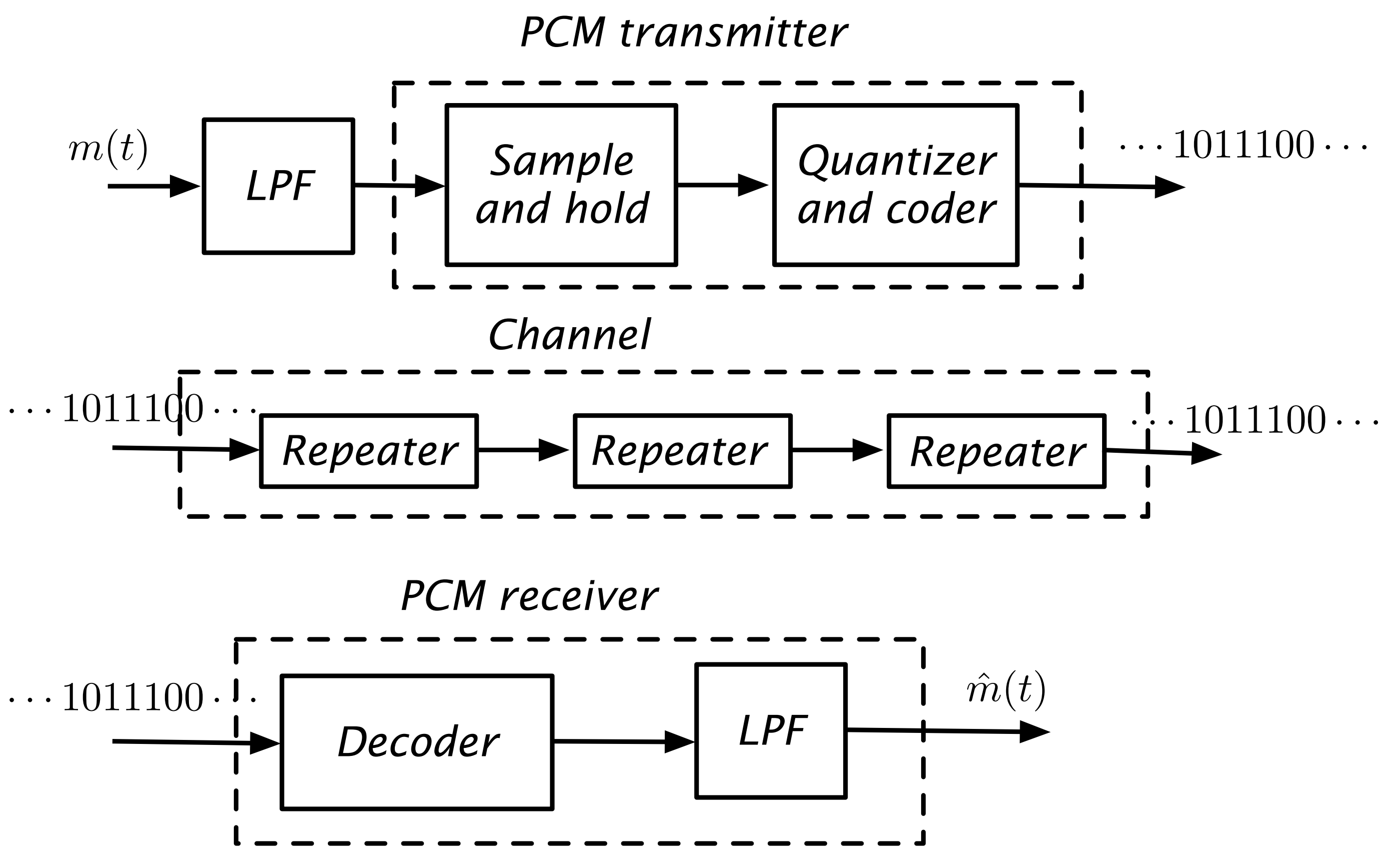
Four–level quantizer and coder.

Sampled signal $x(nT_s) = x(t)|_{t=nT_S}$

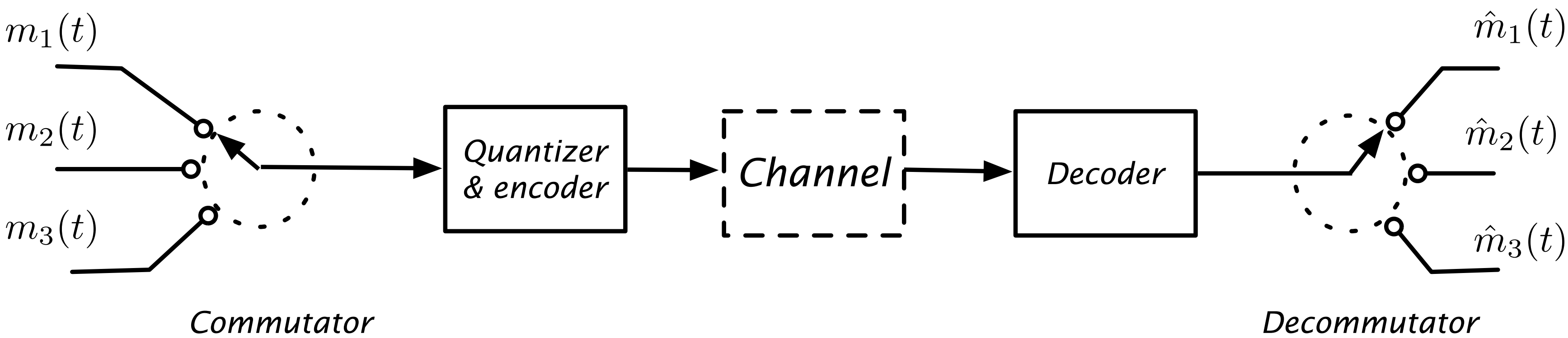
four–level quantizer: $k\Delta \leq x(nT_s) < (k+1)\Delta \;\; \Rightarrow \;\; \hat{x}(nT_s) = k\Delta, \quad k = -2, -1, 0, 1$

coder assigns binary number to each output level of quantizer

Application to digital communications



PCM system: transmitter, channel and receiver.



Time Division Multiplexin (TDM) system: transmitter, channel and receiver