SIGNALS AND SYSTEMS USING MATLAB
Chapter 10 — The Z–transform

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$$x(t) = \sum_n x(nT_s)\delta(t - nT_s) \quad \text{(sampled signal)}$$

$$X(s) = \sum_n x(nT_s)\mathcal{L}[\delta(t - nT_s)] = \sum_n x(nT_s)e^{-nsT_s}$$

Letting $z = e^{sT_s}$

$$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]|_{z=e^{sT_s}} = \sum_n x(nT_s)z^{-n} \quad \text{Z–transform}$$

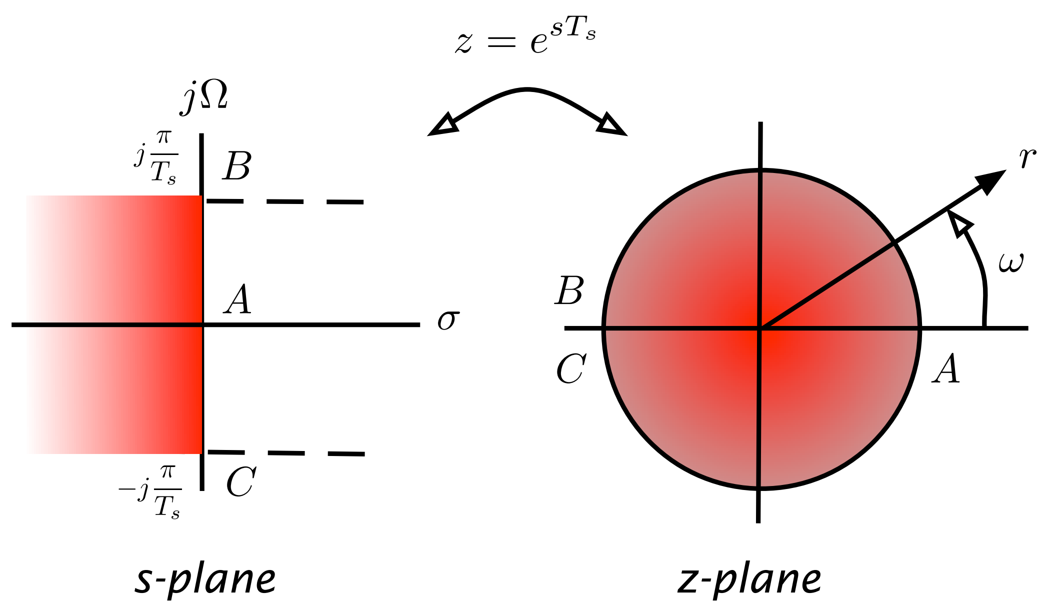


Figure: *Mapping of the Laplace plane into the Z–plane*

Two-sided/ One-sided Z-transforms

- Two-sided Z-transform

$$\boxed{\begin{array}{l}\text{discrete-time signal} \quad x[n], -\infty < n < \infty \\ \displaystyle X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad ROC: \ \mathcal{R}\end{array}}$$

- One–sided Z–transform

$$\boxed{\begin{array}{l}\text{causal signal} \quad x[n]u[n] \\ \displaystyle X_1(z) = \mathcal{Z}(x[n]u[n]) = \sum_{n=0}^{\infty} x[n]u[n]z^{-n}, \quad ROC: \ \mathcal{R}_1\end{array}}$$

- Two–sided in terms of one–sided Z–transform

$$\boxed{\begin{array}{l} x[n] = x[n]u[n] + x[n]u[-n] - x[0] \\ \\ X(z) = \mathcal{Z}\left(x[n]u[n]\right) + \mathcal{Z}\left(x[-n]u[n]\right)|_z - x[0], \quad \mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2 \\ \\ \mathcal{R}_1 = ROC[\mathcal{Z}\left(x[n]u[n]\right)], \quad \mathcal{R}_2 = ROC[\mathcal{Z}\left(x[-n]u[n]\right)|_z] \end{array}}$$

Poles/Zeros, ROC

- Z-transform $X(z)$
  - pole p_k such that $X(p_k) \to \infty$
  - zero z_k such that $X(z_k) = 0$
- ROC of finite-support signal

$$x[n], \quad \text{finite support } -\infty < N_0 \le n \le N_1 < \infty$$

$$X(z) = \sum_{n=N_0}^{N_1} x[n] z^{-n}$$

ROC : whole Z–plane, excluding 0 and/or $\pm\infty$ depending on $N_0,\ N_1$

Examples:

$$(i) \quad X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} = \frac{z^3 + 2z^2 + 3z + 4}{z^3} = \frac{N_1(z)}{D_1(z)}$$

zeros: roots of $N_1(z) = 0$, $z_1 = -1.65$, $z_2 = -0.175 \pm j1.547$

poles: roots of $D_1(z) = 0$ $z = 0$ triple

$$(ii) \quad X_2(z) = \frac{(z^{-1} - 1)(z^{-1} + 2)^2}{z^{-1}(z^{-2} + \sqrt{2}z^{-1} + 1)} = \frac{(1 - z)(1 + 2z)^2}{1 + \sqrt{2}z + z^2} = \frac{N_2(z)}{D_2(z)}$$

zeros: roots of $N_2(z) = 0$, $z_1 = 1$, $z_{2,3} = -0.5$

poles: roots of $D_2(z) = 0$, $p_{1,2} = -0.707 \pm j0.707$

Example: Discrete-time pulse $x[n] = u[n] - u[n-10]$

$$X(z) = \sum_{n=0}^{9} 1\ z^{-n} = \frac{1 - z^{-10}}{1 - z^{-1}} = \frac{z^{10} - 1}{z^{9}(z-1)}$$

zeros: roots of $z^{10} - 1 = 0$, or $z_k = e^{j2\pi k/10}$, $k = 0 \cdots 9$

$z_0 = 1$ cancels pole $p = 1$ $\Rightarrow X(z) = \frac{\prod_{k=1}^{9}(z - e^{j\pi k/5})}{z^{9}}$,

ROC whole z–plane excluding the origin

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7} + z^{-8} + z^{-9}$$

only tends to infinity when $z = 0$

ROC of Z-transform of infinite–support signals

- causal signal $x[n]$, ROC: $|z| > R_1$, R_1 the largest radius of poles of $X(z)$
- anti-causal signal $x[n]$, ROC: $|z| < R_2$, R_2 smallest radius of poles of $X(z)$
- non-causal signal $x[n]$, ROC: $R_1 < |z| < R_2$, or inside a torus of inside radius R_1 and outside radius R_2

Example: Possible regions of convergence of $X(z)$ with poles $z = 0.5$ and $z = 2$

- $\{\mathcal{R}_1 : \ |z| > 2\}$, outside of circle of radius 2, $X(z)$ associated with causal signal $x_1[n]$
- $\{\mathcal{R}_2 : \ |z| < 0.5\}$, inside of circle of radius 0.5, $X(z)$ associated with anti-causal signal $x_2[n]$
- $\{\mathcal{R}_3 : \ 0.5 < |z| < 2\}$, torus of radii 0.5 and 2, $X(z)$ associated with non-causal signal $x_3[n]$

Example:Noncausal $c[n] = \alpha^{|n|}$, $0 < \alpha < 1$, (autocorrelation function related to the power spectrum of a random signal)

$$\mathcal{Z}(c[n]u[n]) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}, \quad ROC: \; |z| > \alpha$$

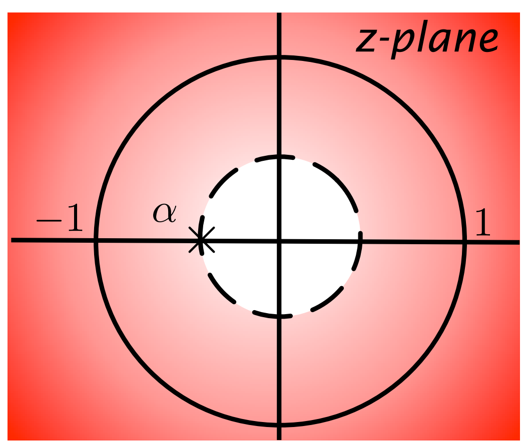
$$\mathcal{Z}(c[-n]u[n])_z = \sum_{n=0}^{\infty} \alpha^n z^{n} = \frac{1}{1-\alpha z}, \quad ROC: \; |z| < 1/\alpha$$

$$C(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\alpha z} - 1 = \frac{(\alpha - 1/\alpha)z}{(z-\alpha)(z-1/\alpha)}$$

$$ROC: \; \alpha < |z| < \frac{1}{\alpha}$$

Example: Causal $x[n] = \alpha^n u[n]$

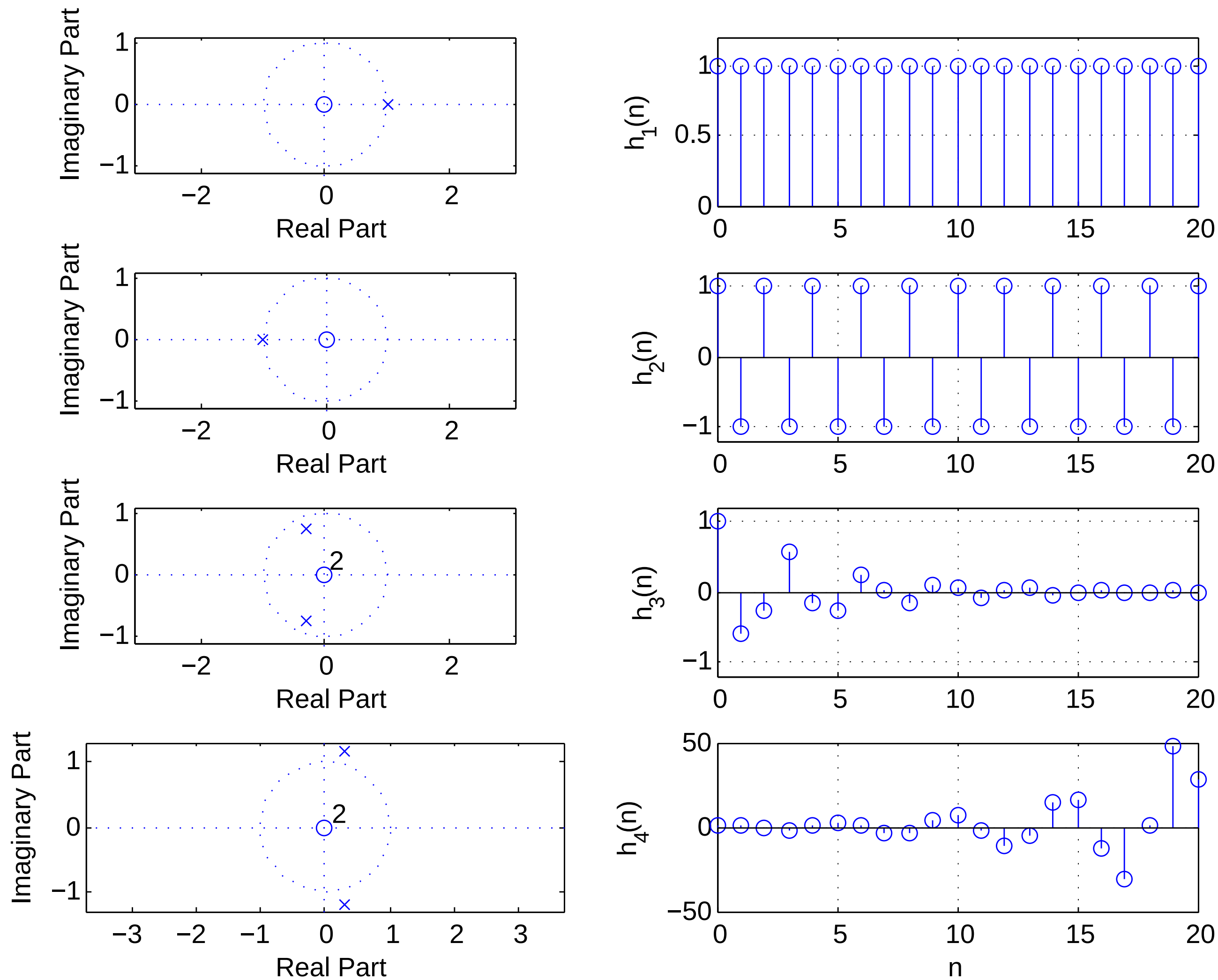
$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha} \qquad \text{ROC: } |z| > |\alpha|$$



Region of convergence (shaded area) of $X(z)$ with a pole at $z = \alpha$, $\alpha < 0$

Table 10.1 One-sided Z-transforms

$\delta[n]$	$1, \quad \text{whole z-plane}$
$u[n]$	$\dfrac{1}{1-z^{-1}}, \quad \lvert z\rvert > 1$
$nu[n]$	$\dfrac{z^{-1}}{(1-z^{-1})^2}, \quad \lvert z\rvert > 1$
$n^2u[n]$	$\dfrac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}, \quad \lvert z\rvert > 1$
$\alpha^n u[n], \ \lvert\alpha\rvert < 1$	$\dfrac{1}{1-\alpha z^{-1}}, \quad \lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n u[n], \ \lvert\alpha\rvert < 1$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}, \quad \lvert z\rvert > \lvert\alpha\rvert$
$\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}, \quad \lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}, \quad \lvert z\rvert > 1$
$\alpha^n \cos(\omega_0 n)u[n], \ \lvert\alpha\rvert < 1$	$\dfrac{1-\alpha\cos(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}, \quad \lvert z\rvert > \lvert\alpha\rvert$
$\alpha^n \sin(\omega_0 n)u[n], \ \lvert\alpha\rvert < 1$	$\dfrac{\alpha\sin(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}, \quad \lvert z\rvert > \lvert\alpha\rvert$



Effect of pole location on the inverse Z-transform (from top to bottom): if pole is at $z = 1$ *the signal is* $u(n)$*, constant for* $n \geq 0$*; if pole is at* $z = -1$ *the signal is a cosine of frequency* π *continuously changing, constant amplitude; when poles are complex, if inside the unit circle the signal is a decaying modulated exponential, and if outside the unit circle the signal is a growing modulated exponential*

Table 10.2 Basic Properties of One-sided Z-transform

Causal signals	$\alpha x[n],\ \beta y[n]$	$\alpha X(z),\ \beta Y(z)$
Linearity	$\alpha x[n]+\beta y[n]$	$\alpha X(z)+\beta Y(z)$
Convolution sum	$\sum_k x[n]y[n-k]$	$X(z)Y(z)$
Time shifting	$x[n-N],\ N>0$	$z^{-N}X(z)+x[-1]z^{-N+1}$ $+x[-2]z^{-N+2}+\cdots+x[-N]$
Time reversal	$x[-n]$	$X(z^{-1})$
Multiplication	$n\ x[n]$	$-z\dfrac{dX(z)}{dz}$
	$n^2\ x[n]$	$z^2\dfrac{d^2X(z)}{dz^2}+z\dfrac{dX(z)}{dz}$
Finite difference	$x[n]-x[n-1]$	$(1-z^{-1})X(z)-x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\dfrac{X(z)}{1-z^{-1}}$
Initial value	$x[0]$	$\lim_{z\to\infty} X(z)$
Final value	$\lim_{n\to\infty} x[n]$	$\lim_{z\to 1}(z-1)X(z)$

Convolution sum and transfer Function

output of causal LTI system

$$y[n] = [x * h][n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} h[k]x[n-k]$$

$x[n]$ causal input, $h[n]$ impulse response of system

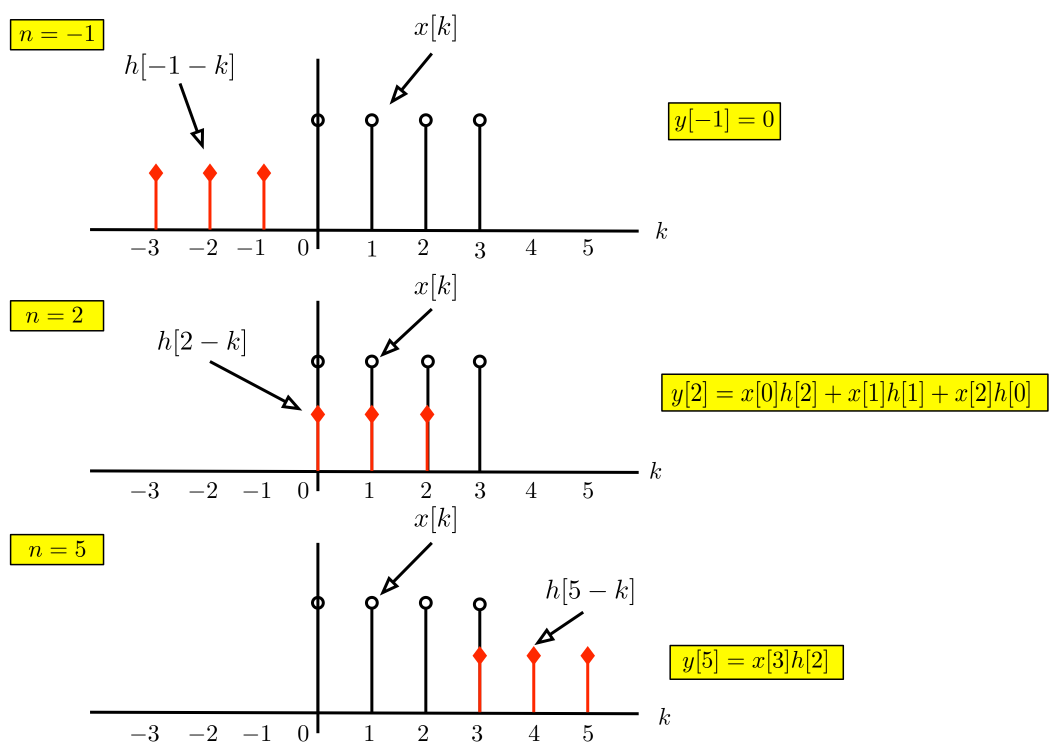
$$Y(z) = \mathcal{Z}\{[x * h][n]\} = \mathcal{Z}\{x[n]\}\mathcal{Z}\{h[n]\} = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\mathcal{Z}[\text{ output } y[n]]}{\mathcal{Z}[\text{ input } x[n]]} \quad \text{transfer function}$$

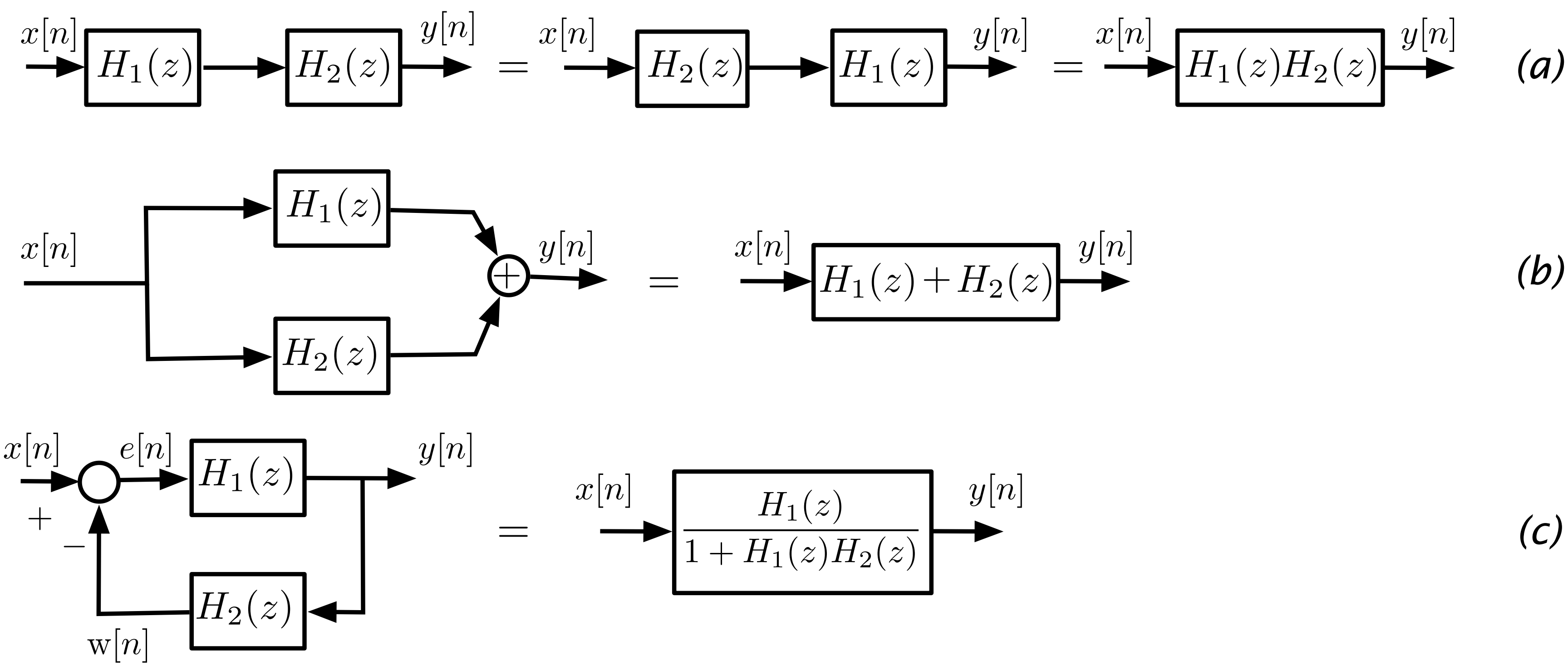
- Convolution gives coefficients of multiplication of polynomials
- FIR systems implemented using convolution
- Length of convolution of two sequences of lengths M and N is $M + N - 1$

Example: FIR filter

$$y[n] = \frac{1}{2}(x[n] + x[n-1] + x[n-2])$$
$$x[n] = u[n] - u[n-4], \quad h[n] = 0.5(\delta[n] + \delta[n-1] + \delta[n-2])$$
$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3}, \quad H(z) = \frac{1}{2}[1 + z^{-1} + z^{-2}]$$
$$Y(z) = X(z)H(z) = \frac{1}{2}(1 + 2z^{-1} + 3z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5})$$
$$y[0] = 0.5,\ y[1] = 1,\ y[2] = 1.5,\ y[3] = 1.5,\ y[4] = 1,\ y[5] = 0.5, \cdots$$



Graphical approach: $x[k]$ and $h[n-k]$ are plotted as functions of k for a given value of n. The signal $x[k]$ remains stationary, while $h[n-k]$ moves linearly from left to right



Connections of LTI systems: (a) cascade, (b) parallel, and (c) negative feedback.

- Long–division Rational function $X(z) = \mathcal{Z}[x[n]] = B(z)/A(z)$, $x[n]$ causal. By division

$$\begin{aligned} X(z) &= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \\ \text{inverse} \quad x[n] &= x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots \end{aligned}$$

- Partial fraction expansion

$$X(z) = \mathcal{Z}[x[n]] = B(z)/A(z), \quad x[n] \text{ causal}$$

  - proper rational $X(z)$: degree $N(z) <$ degree $D(z)$
  - $N(z)$, $D(z)$ polynomials with real coefficients poles/zeros are
    - (i) real
    - (ii) complex conjugate pairs
    - (iii) simple
    - (iv) multiple

Example: Non-proper rational function

$$X(z) = \frac{2 + z^{-2}}{1 + 2z^{-1} + z^{-2}}$$

By division

$$X(z) = 1 + \frac{1 - 2z^{-1}}{1 + 2z^{-1} + z^{-2}} \quad \Rightarrow \quad x[n] = \delta[n] + \mathcal{Z}^{-1}\left[\frac{1 - 2z^{-1}}{1 + 2z^{-1} + z^{-2}}\right]$$

Example:

$$X(z) = \frac{1 + z^{-1}}{(1 + 0.5z^{-1})(1 - 0.5z^{-1})} = \frac{z(z+1)}{(z+0.5)(z-0.5)} \qquad |z| > 0.5$$

Partial fraction expansion in z^{-1} terms

$$\begin{aligned} X(z) &= \frac{1 + z^{-1}}{(1 + 0.5z^{-1})(1 - 0.5z^{-1})} = \frac{A}{1 + 0.5z^{-1}} + \frac{B}{1 - 0.5z^{-1}} \\ A &= X(z)(1 + 0.5z^{-1})|_{z^{-1}=-2} = -0.5 \\ B &= X(z)(1 - 0.5z^{-1})|_{z^{-1}=2} = 1.5 \end{aligned}$$

Partial fraction expansion in positive powers of z

$$\begin{aligned} \frac{X(z)}{z} &= \frac{z+1}{(z+0.5)(z-0.5)} = \frac{C}{z+0.5} + \frac{D}{z-0.5} \\ C &= \frac{X(z)}{z}(z+0.5)|_{z=-0.5} = -0.5 \\ D &= \frac{X(z)}{z}(z-0.5)|_{z=0.5} = 1. \end{aligned}$$

Either gives $x[n] = [-0.5(-0.5)^n + 1.5(0.5)^n]u[n]$

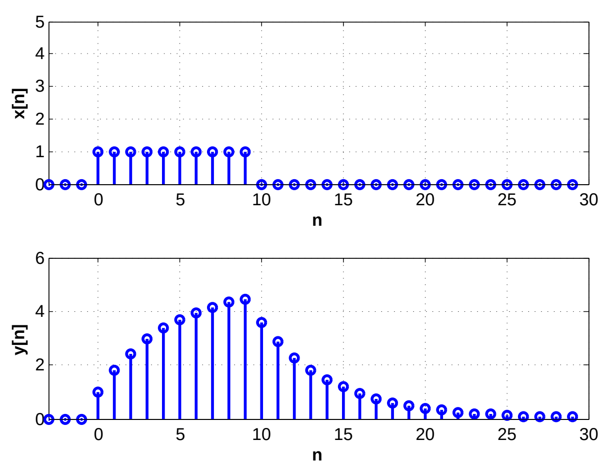
$$\boxed{\begin{array}{l} x[n] \;\leftrightarrow\; X(z) \\ \mathcal{Z}[x[n-N]] = z^{-N}X(z) + x[-1]z^{-N+1} + x[-2]z^{-N+2} + \cdots + x[-N] \end{array}}$$

Example: IIR system with input $x[n]$, $y[n]$ output, is represented by

$$y[n] = 0.8y[n-1] + x[n] \qquad n \geq 0, \quad IC: \; y[-1]$$

Closed-form solution

$$\begin{aligned} \mathcal{Z}(y[n]) &= \mathcal{Z}(0.8y[n-1]) + \mathcal{Z}[x[n]) \\ Y(z) &= 0.8(z^{-1}Y(z) + y[-1]) + X(z) \\ Y(z) &= \underbrace{\frac{X(z)}{1-0.8z^{-1}}}_{y_{zs}[n]} + \underbrace{\frac{0.8y[-1]}{1-0.8z^{-1}}}_{y_{zi}[n]} \end{aligned}$$



Solution of difference equation (bottom) with input $x[n] = u[n] - u[n-11]$, $y[-1] = 0$

Example: Steady–state response

$$y[n] + y[n-1] - 4y[n-2] - 4y[n-3] = 3x[n], \quad n \geq 0,$$
$$y[-1] = 1,\ y[-2] = y[-3] = 0, \quad x[n] = u[n]$$

$$Y(z) = 3\frac{X(z)}{A(z)} + \frac{-1 + 4z^{-1} + 4z^{-2}}{A(z)}, \quad |z| > 2,\ A(z) = (1 + z^{-1})(1 + 2z^{-1})(1 - 2z^{-1})$$

BIBO stability: transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3}{A(z)}, \text{ poles } z = -1,\ z = -2,\ z = 2 \text{ (on and outside UC)}$$

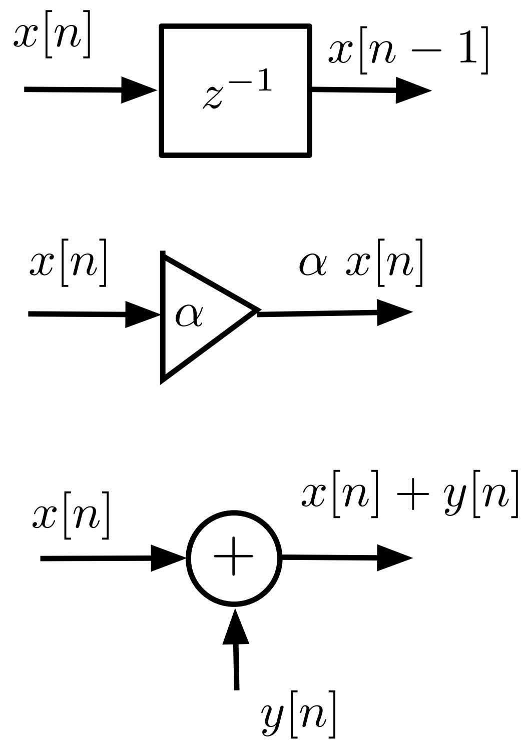
$h[n] = \mathcal{Z}^{-1}[H(z)]$ not absolutely summable, so system is not BIBO stable

$$\begin{aligned} Y(z) &= \frac{2 + 5z^{-1} - 4z^{-3}}{(1 - z^{-1})(1 + z^{-1})(1 + 2z^{-1})(1 - 2z^{-1})} \\ &= \frac{B_1}{1 - z^{-1}} + \frac{B_2}{1 + z^{-1}} + \frac{B_3}{1 + 2z^{-1}} + \frac{B_4}{1 - 2z^{-1}} \end{aligned}$$

$$\begin{aligned} B_1 &= Y(z)(1 - z^{-1})|_{z^{-1}=1} = -\tfrac{1}{2}, \quad & B_2 &= Y(z)(1 + z^{-1})|_{z^{-1}=-1} = -\tfrac{1}{6}, \\ B_3 &= Y(z)(1 + 2z^{-1})|_{z^{-1}=-1/2} = 0, & B_4 &= Y(z)(1 - 2z^{-1})|_{z^{-1}=1/2} = \tfrac{8}{3}, \end{aligned}$$

$$y[n] = \left(-0.5 - \frac{1}{6}(-1)^n + \frac{8}{3}2^n\right) u[n] \ \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \text{no steady-state}$$

- Used in modern control theory
- State variables are memory of a system
- State variable representation is non–unique internal representation

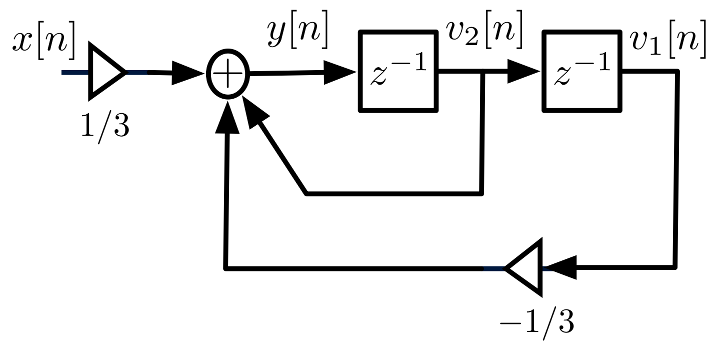


Different components used to represent discrete–time systems (top to bottom): delay, constant multiplier and adder.

Example: A continuous–time system is represented by

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t) \qquad t \geq 0$$

discretized to $y[n] - y[n-1] + \frac{1}{3}y[n-2] = \frac{1}{3}x[n]$



State variables $v_1[n] = y[n-2]$ *and* $v_2[n] = y[n-1]$*.*

$$v_1[n] = y[n-2], \quad v_2[n] = y[n-1]$$
$$\begin{bmatrix} v_1[n+1] \\ v_2[n+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1/3 & 1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/3 \end{bmatrix}}_{\mathbf{b}} x[n]$$

Output equation

$$y[n] = -\frac{1}{3}v_1[n] + v_2[n] + \frac{1}{3}x[n] \quad \text{or in matrix form}$$
$$y[n] = \underbrace{\begin{bmatrix} -\frac{1}{3} & 1 \end{bmatrix}}_{\mathbf{c}^T} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{3} \end{bmatrix}}_{d} x[n]$$

- State variables are not unique: invertible transformation matrix $\mathbf{F}$ defines a new set of state variables

$$\mathrm{w}[n] = \mathbf{Fv}[n]$$

Matrix representation

$$\begin{aligned} \mathrm{w}[n+1] &= \mathbf{Fv}[n+1] = \mathbf{FAv}[n] + \mathbf{Fb}x[n] \\ &= \mathbf{FAF}^{-1}\mathrm{w}[n] + \mathbf{Fb}x[n] \end{aligned}$$

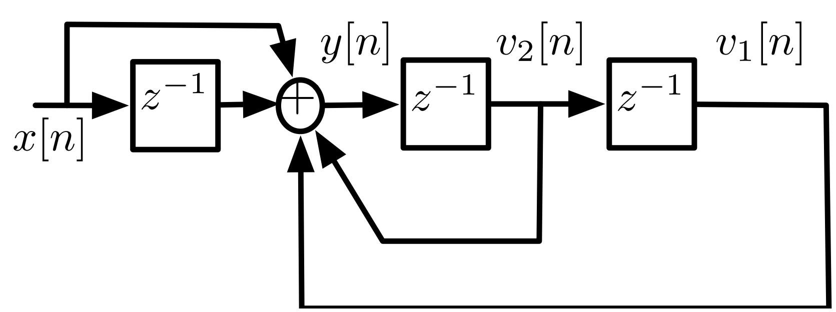
$$y[n] = \mathbf{c}^T\mathbf{v}[n] + \mathbf{d}x[n] = \mathbf{c}^T\mathbf{F}^{-1}\mathrm{w}[n] + \mathbf{d}x[n]$$

- Minimal realizations

$$y[n] - y[n-1] - y[n-2] = x[n] - x[n-1], \quad \text{input } x[n], \text{ output } y[n]$$

Transfer function is not an "constant–numerator"

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - z^{-1} - z^{-2}}$$



Non–minimal realization (3 delays for second–order system) displaying the state variables $v_1[n]$ and $v_2[n]$

Solution of the state and output equations

Recursive solution of state equations

$$\mathbf{v}[n+1] = \mathbf{A}\mathbf{v}[n] + \mathbf{B}\mathbf{x}[n], \qquad n \geq 0$$

$$\begin{aligned}
\mathbf{v}[1] &= \mathbf{A}\mathbf{v}[0] + \mathbf{B}\mathbf{x}[0] \\
\mathbf{v}[2] &= \mathbf{A}\mathbf{v}[1] + \mathbf{B}\mathbf{x}[1] = \mathbf{A}^2\mathbf{v}[0] + \mathbf{A}\mathbf{B}\mathbf{x}[0] + \mathbf{B}\mathbf{x}[1] \\
&\vdots \\
\mathbf{v}[n] &= \mathbf{A}^n\mathbf{v}[0] + \sum_{k=0}^{n-1} \mathbf{A}^{n-1-k}\mathbf{B}\mathbf{x}[k]
\end{aligned}$$

complete solution

$$y[n] = \underbrace{\mathbf{c}^T\mathbf{A}^n\mathbf{v}[0]}_{\text{zero-input response}} + \underbrace{\sum_{k=0}^{n-1} \mathbf{c}^T\mathbf{A}^{n-1-k}\mathbf{B}\mathbf{x}[k] + \mathbf{d}\mathbf{x}[n]}_{\text{zero–state response}}$$

initial conditions

$$v_1[0] = y[-1], \quad v_2[0] = y[-2], \quad \cdots \quad v_N[0] = y[-N]$$

Z-transform solution of state and output equations

State and output equations

$\mathbf{v}[n+1] = \mathbf{A}\mathbf{v}[n] + \mathbf{B}\mathbf{x}[n]$

$y[n] = \mathbf{c}^T\mathbf{v}[n] + \mathbf{d}^T\mathbf{x}[n], \qquad n \geq 0$

$V_i(z) = \mathcal{Z}(v_i[n]),\ i = 1, \cdots, N;\ X_m(z) = \mathcal{Z}(x[n-m]),\ m = 0, \cdots, M,$

$Y(z) = \mathcal{Z}(y[n])$

$(z\mathbf{I} - \mathbf{A})\mathbf{V}(z) = z\mathbf{v}[0] + \mathbf{B}\mathbf{X}(z), \ \ \det(z\mathbf{I} - \mathbf{A}) \neq 0 \ \Rightarrow \ (z\mathbf{I} - \mathbf{A})^{-1}$ exists

$$\mathbf{V}(z) = \frac{\mathrm{Adj}(z\mathbf{I} - \mathbf{A})}{\det(z\mathbf{I} - \mathbf{A})} z\mathbf{v}[0] + \frac{\mathrm{Adj}(z\mathbf{I} - \mathbf{A})}{\det(z\mathbf{I} - \mathbf{A})}\mathbf{B}\mathbf{X}(z)$$

$$Y(z) = \frac{\mathbf{c}^T\mathrm{Adj}(z\mathbf{I} - \mathbf{A})}{\det(z\mathbf{I} - \mathbf{A})} z\mathbf{v}[0] + \left[\frac{\mathbf{c}^T\mathrm{Adj}(z\mathbf{I} - \mathbf{A})}{\det(z\mathbf{I} - \mathbf{A})}\mathbf{B} + \mathbf{d}\right]\mathbf{X}(z)$$

transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\mathbf{c}^T\mathrm{Adj}(z\mathbf{I} - \mathbf{A})}{\det(z\mathbf{I} - \mathbf{A})}\mathbf{b} + d$$

Example: System represented by state/output equations with matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1/3 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$$

$$\mathbf{c}^T = \begin{bmatrix} -\frac{1}{3} & 1 \end{bmatrix}, \qquad d = \begin{bmatrix} \frac{1}{3} \end{bmatrix}.$$

To find transfer function use Cramer's rule:

$$\underbrace{\begin{bmatrix} z & -1 \\ 1/3 & z-1 \end{bmatrix}}_{(z\mathbf{I}-\mathbf{A})} \underbrace{\begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix}}_{\mathbf{V}(z)} = \underbrace{\begin{bmatrix} 0 \\ X(z)/3 \end{bmatrix}}_{\mathbf{b}X(z)}$$

$$V_1(z) = \frac{X(z)/3}{\Delta(z)}, \quad V_2(z) = \frac{zX(z)/3}{\Delta(z)}, \quad \Delta(z) = z^2 - z + 1/3$$

$$Y(z) = \frac{-V_1(z)}{3} + V_2(z) = \frac{z^2/3}{z^2 - z + 1/3} X(z), \qquad \text{so that}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1/3}{1 - z^{-1} + z^{-2}/3}$$

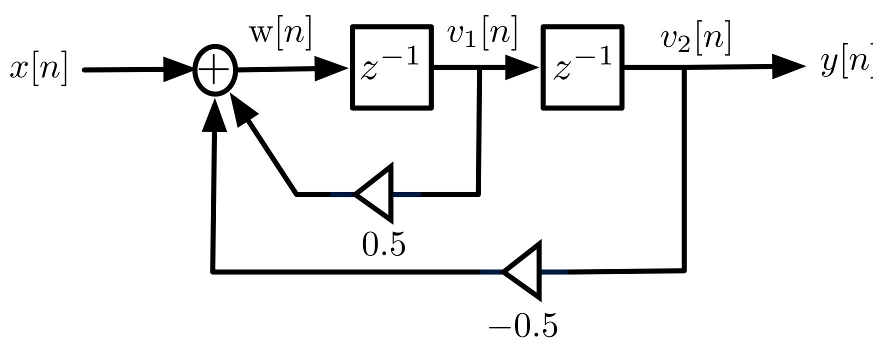
Example: Minimal realization of transfer function

$$H(z) = \frac{z^{-2}}{1 - 0.5z^{-1} + 0.5z^{-2}} \quad \text{not "constant–numerator"}$$

$$H(z) = \frac{Y(z)}{X(z)} = \underbrace{z^{-2}}_{Y(z)/W(z)} \times \underbrace{\frac{1}{1 - 0.5z^{-1} + 0.5z^{-2}}}_{W(z)/X(z)}$$

$$\mathrm{w}[n] = 0.5\mathrm{w}[n-1] - 0.5\mathrm{w}[n-2] + x[n]$$

$$y[n] = \mathrm{w}[n-2]$$



$$\mathbf{v}[n+1] = \begin{bmatrix} 1/2 & -1/2 \\ 1 & 0 \end{bmatrix} \mathbf{v}[n] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n]$$
$$y[n] = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{v}[n]$$

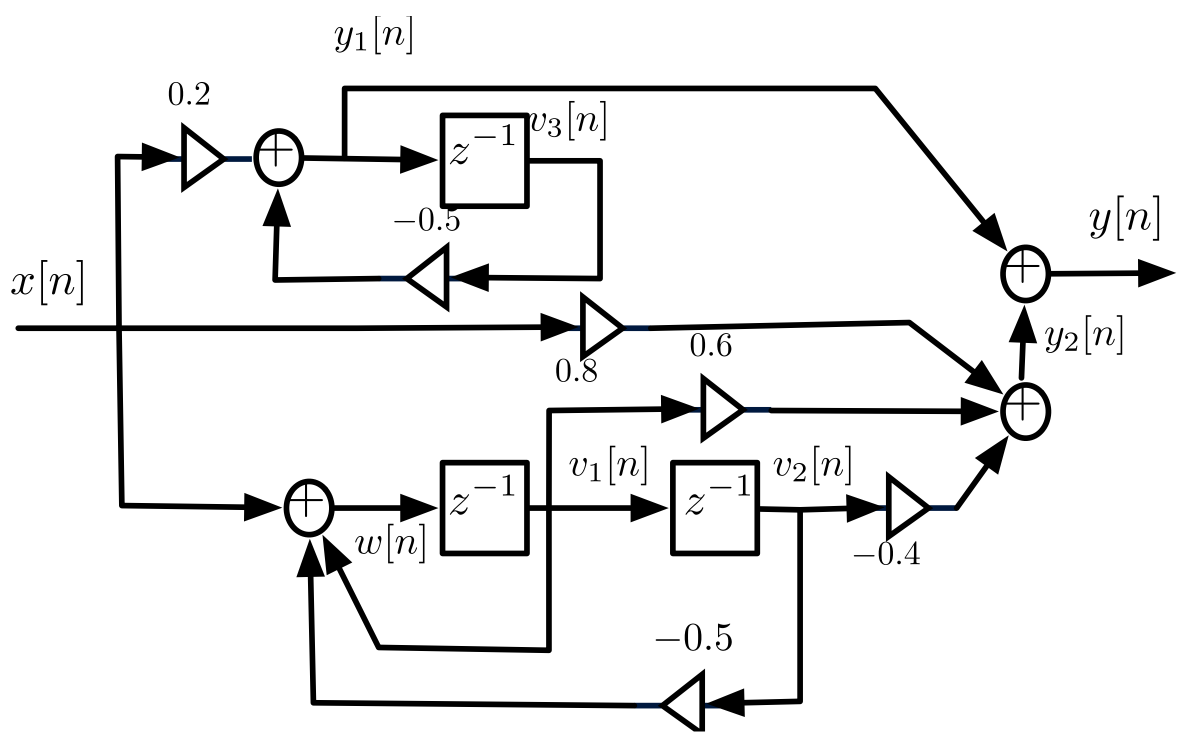
Example:

$$H(z) = \frac{z^3}{(z+0.5)[(z-0.5)^2+0.25]}$$

Partial fraction expansion

$$H(z) = \frac{1/5}{1+0.5z^{-1}} + \frac{0.8-0.2z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

$$Y(z) = \underbrace{\frac{0.2X(z)}{1+0.5z^{-1}}}_{Y_1(z)} + \underbrace{\frac{(0.8-0.2z^{-1})X(z)}{1-z^{-1}+0.5z^{-2}}}_{Y_2(z)}$$



Minimum realization (3 delays corresponding to the third–order system)

Two-Dimensional Z-Transform

Discrete signal $x[m,n], -\infty < m < \infty, \quad -\infty < n < \infty$

$$X(z_1, z_2) = \mathcal{Z}_2(x[m,n]) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[m,n] z_1^{-m} z_2^{-n}, \qquad z_1, z_2 \in \text{ROC}$$

- ROC: region of convergence: region where $X(z_1, z_2)$ is analytic
- $X(z_1, z_2)$ defined in a 2D space spanned by the two complex variables

Example: Zeros and poles of

$$X(z_1, z_2) = \frac{z_1^{-1} + z_2^{-1}}{1 + z_1^{-1} z^{-2}} \qquad (z_1, z_2) = (r_1 e^{j\theta_1}, r_2 e^{j\theta_2})$$

Zeros: $z_1^{-1} + z_2^{-1} = 0 \;\Rightarrow\; z_1 = -z_2 \;\Rightarrow\; r_1 = r_2$ and $\theta_1 = \theta_2 + \pi$

Poles: $1 + z_1^{-1} z_2^{-1} = 0 \;\Rightarrow\; z_1 = -1/z_2 \;\Rightarrow\; r_1 = 1/r_2$ and $\theta_1 = -\theta_2 + \pi$

Example: Z-transform and ROC of $x[m,n] = \alpha^m u_1[m,m]$. For which of $\alpha = 0.5$ and $\alpha = 2$, is $X(e^{j\omega_1}, e^{j\omega_2})$ defined?

$$X(z_1, z_2) = \sum_{k=0}^{\infty} \alpha^k z_1^{-k} z_2^{-k} = \frac{1}{1 - \alpha z_1^{-1} z_2^{-1}}, \quad \text{ROC: } |\alpha z_1^{-1} z_2^{-1}| < 1, \quad \text{or } |z_1| > \frac{|\alpha|}{|z_2|}$$

- The ROC is expressed in terms of $|z_1|$ and $|z_2|$ in the 2D plane $(|z_1|, |z_2|)$
- The unit-bidisc $|z_1| = 1, |z_2| = 1$ becomes a point
- $\alpha = 0.5$, the unit-bidisc at $(1, 1)$ is inside the ROC $\rightarrow X(e^{j\omega_1}, e^{j\omega_2})$ defined
- $\alpha = 2$, the ROC does not contain the unit-bidisc $\rightarrow X(e^{j\omega_1}, e^{j\omega_2})$ not defined

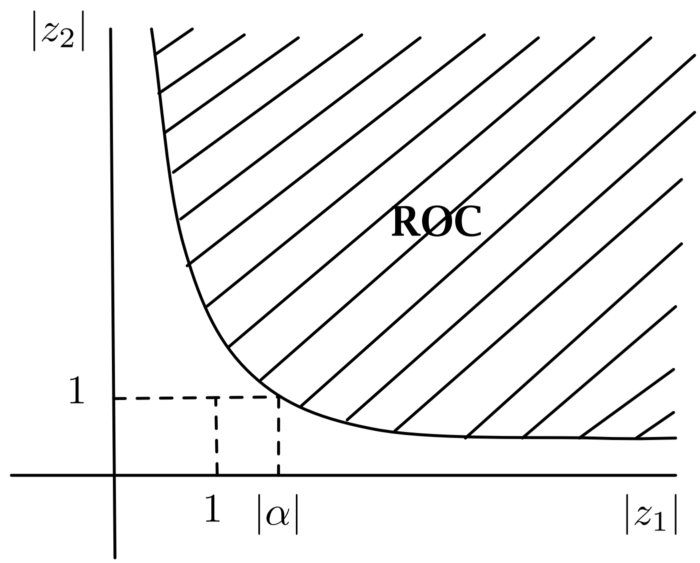


Figure: *Region of convergence when* $|\alpha| > 1$ *for* $X(z_1, z_2) = 1/(1 - \alpha z_1^{-1} z_2^{-1})$

- 2D Z-transform is linear
- Given that

$$\mathcal{Z}_2(\delta[m-k, n-\ell]) = z_1^{-k} z_2^{-\ell}$$

by linearity

$$\begin{aligned}\mathcal{Z}_2(x[m,n]) &= \mathcal{Z}_2\left(\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} x[k,\ell]\delta[m-k, n-\ell]\right) \\ &= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} x[k,\ell] \underbrace{\mathcal{Z}_2(\delta[m-k, n-\ell])}_{z_1^{-k} z_2^{-\ell}} = X(z_1, z_2)\end{aligned}$$

- Using above makes possible in some cases to find the inverse transform, but it might not be in close form

Example: The inverse transform of

$$\begin{aligned}X(z_1, z_2) &= \frac{1}{1-\alpha z_1^{-1} z_2^{-1}} = \sum_{k=0}^{\infty} \alpha^k z_1^{-k} z_2^{-k}, \qquad |z_1 z_2| > |\alpha| \\ &= \underbrace{1}_{x[0,0]} + \underbrace{\alpha}_{x[1,1]} z_1^{-1} z_2^{-1} + \underbrace{\alpha^2}_{x[2,2]} z_1^{-2} z_2^{-2} + \cdots + \underbrace{\alpha^k}_{x[k,k]} z_1^{-k} z_2^{-k} + \cdots \\ x[m,n] &= \begin{cases} \alpha^m & 0 \le m < \infty, \; n = m \\ 0 & \text{otherwise} \end{cases} = \alpha^m u_1[m,m]\end{aligned}$$

Application of 2D Z-transform to Systems

Example: Find transfer function

$$H(z_1, z_2) = \frac{Y(z_1, z_2)}{X(z_1, z_2)}$$

using impulse response $h[m, n]$ of the system with recursive equation

$$y[m, n] = x[m, n] + y[m-1, n] + y[m, n-1], \qquad m \geq 0, n \geq 0$$

where $x[m, n]$ is the input and $y[m, n]$ the output

- Let $x[m, n] = \delta[m, n]$ be the only input (zero boundary conditions) then

$$h[m, n] = \delta[m, n] + h[m-1, n] + h[m, n-1], \qquad m \geq 0, n \geq 0$$

- Use linearity and shifting properties, $H(z_1, z_2) = \mathcal{Z}_2(h[m, n])$, and $\mathcal{Z}_2(\delta[m, n]) = 1$

$$H(z_1, z_2) = 1 + H(z_1, z_2)z_1^{-1} + H(z_1, z_2)z_2^{-1} \quad \Rightarrow \quad H(z_1, z_2) = \frac{1}{1 - z_1^{-1} - z_2^{-1}}$$

- The ROC is obtained by

$$H(z_1, z_2) = \frac{1}{1 - (z_1^{-1} + z_2^{-1})} = \sum_{k=0}^{\infty} (z_1^{-1} + z_2^{-1})^k \qquad |z_1^{-1} + z_2^{-1}| < 1$$

thus ROC: $|z_1^{-1} + z_2^{-1}| < 1$

For an LSI system, if for the input $x[m,n]$, $Z_2(x[m,n]) = X(z_1,z_2)$, and for the impulse response $h[m,n]$,$Z_2(h[m,n]) = H(z_1,z_2)$ (transfer function) then the output $y[m,n]$ and its transform $Y(z_1,z_2)$ are such that

$$\boxed{y[m,n] = (h*x)[m,n] \quad \Leftrightarrow \quad Y(z_1,z_2) = H(z_1,z_2)X(z_1,z_2)}$$

Example: The input $x[m,n]$, and the impulse response $h[m,n]$ of a 2D-LSI system are

$$x[m,n] = \delta[m,n] + 2\delta[m-1,n] + 3\delta[m,n-1]$$
$$h[m,n] = \delta[m,n] + \delta[m-1,n].$$

find the output $y[m,n]$

- 2D Z-transforms:

$$X(z_1,z_2) = 1 + 2z_1^{-1} + 3z_2^{-1}$$
$$H(z_1,z_2) = 1 + z_1^{-1} \quad \text{then}$$
$$Y(z_1,z_2) = H(z_1,z_2)X(z_1,z_2) = 1 + 3z_1^{-1} + 2z_1^{-2} + 3z_2^{-1} + 3z_1^{-1}z_2^{-1}$$

- The coefficients of $Y(z_1,z_2)$ are the values of two-dimensional convolution

$$\begin{aligned} y[m,n] &= (x*h)[m,n] \\ &= \delta[m,n] + 3\delta[m-1,n] + 2\delta[m-2,n] + 3\delta[m,n-1] \\ &\quad +3\delta[m-1,n-1] \end{aligned}$$

2D Z-transform and BIBO Stability

- A 2D LSI system is BIBO stable if

$$\sum_{k,\ell} |h[k,\ell]| < \infty.$$

- $H(z_1, z_2) = \mathcal{Z}_2(h[m,n])$ at $z_1 = e^{j\omega_1}$ and $z_2 = e^{j\omega_2}$, on the unit bidisc

$$|H(e^{j\omega_1}, e^{j\omega_2})| = \left| \sum_{k,\ell} h[k,\ell] e^{-jk\omega_1} e^{-j\ell\omega_2} \right| \leq \sum_{k,\ell} |h[k,\ell]| < \infty$$

- $z_1 = e^{j\omega_1}$ and $z_2 = e^{j\omega_2}$ or $|z_1| = |z_2| = 1$, is in the ROC $\Rightarrow H(e^{j\omega_1}, e^{j\omega_2})$ is defined

Example: Determine if this system is BIBO stable;

$$y[m,n] = x[m,n] + y[m-1,n] + y[m,n-1] \qquad m \geq 0,\ n \geq 0.$$

- Impulse response:

$$h[m,n] = \binom{m+n}{n} u_1[m,n]$$

- Not absolutely summable $\Rightarrow$ not BIBO stable
- Transfer function of this system

$$H(z_1,z_2) = \frac{Y(z_1,z_2)}{X(z_1,z_2)} = \frac{1}{1-(z_1^{-1}+z_2^{-1})}, \quad \text{ROC: } |z_1^{-1}+z_2^{-1}| < 1.$$

- $z_1 = e^{j\omega_1}$ and $z_2 = e^{j\omega_1}$, not in the ROC
- $\omega_1 = \omega_2 = 0$; $z_1 = e^{j0} = 1$ and $z_2 = e^{j0} = 1$ and $|z_1^{-1} + z_2^{-1} = 1 + 1| = 2 > 1$