# MATH2023 Analysis

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# Contents

# 1 Number System & Foundations

## 1.1 Least Upper Bound Axiom/Completeness Axiom

Every non-empty set A $\subseteq \mathbb{R}$ which is bounded above has a supremum.

Remark: This axiom sets $\mathbb{R}$ apart from $\mathbb{Q}$. Basically this axiom completes the real number system by filling up the gaps between the rational numbers.

## 1.2 Archimedean Property of $\mathbb{N}$

$\mathbb{N}$ is not bounded above in $\mathbb{R}$.

## 1.3 Density of Rationals and Irrationals

Let a, b $\in \mathbb{R}$ with a $<$ b, Then

1. $\exists r \in \mathbb{Q}$ with a $<$ r $<$ b
2. $\exists x \in \mathbb{R} \backslash \mathbb{Q}$ with a $<$ x $<$ b

## 1.4 Boundedness

(Xn) in $\mathbb{R}$ is bounded above if $\exists$ M $\in \mathbb{R}$ with Xn $\leq$ M $\forall$ n $\in \mathbb{N}$.

(Xn) in $\mathbb{C}$ is bounded if $\exists$ M $>$ 0 s.t $|X| \leq$ M $\forall$ n $\in \mathbb{N}$.

Remark The notion of supremum and infimum has been introduced in the course but will be omitted in this note. For simpel recap, given a set of numbers or function, its supremum is the least upper bound and infimum is the most lower bound.

# 2 Sequence and Limits

## 2.1 Convergence of sequence

(Xn) in $\mathbb{R} \backslash \mathbb{C}$ converges to X as n $\to \infty$ if $\forall \epsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t $n > n_0 \implies |Xn - X| < \epsilon$

Remark: $n_0$ depends on $\epsilon$.

## 2.2 Proposition

A sequence is convergent $\implies$ it is bounded.

## 2.3 Lemma

If $(a_n)$ is bounded and $(b_n)$ is convergent with $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n b_n = 0$.

## 2.4 Limit Laws

Suppose $(X_n)(Y_n)$ are each convergent sequence with limits X and Y respectively, then

1. $(X_n + Y_n) \to X + Y$
2. $(X_n Y_n) \to XY$
3. If $(X_n)$ and $X \neq 0 \; \forall n$, $(\frac{1}{X_n}) \to \frac{1}{X}$

## 2.5 Convergence Theorem

A sequence is monotone and bounded $\implies$ it is convergent.

## 2.6 Ratio Test for sequences

If $(a_n)$ is a sequence in $\mathbb{R}$ with $a_n > 0 \forall n$. Then

1. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \implies \lim_{n\to\infty} a_n = 0$
2. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \implies \lim_{n\to\infty} a_n = \infty$

Remark: The test cannot determine for $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ or limit does not exist.

## 2.7 Important Limits

1. $\lim_{n\to\infty} \frac{c^n}{n!} = 0 \; \forall c \in \mathbb{R}$
2. $\lim_{n\to\infty} n^k a^n = \begin{cases} 0 \text{ if } 0 \leq a < 1, k > 0 \\ \infty \text{ if } a \geq 1 \end{cases}$
3. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
4. $\lim_{n\to\infty} \sqrt[n]{a} = 1, a > 0$
5. $\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$

## 2.8 Proposition

$(X_n)$ has a limit $\implies$ Its subsequence has a unique accumulation point in that limit.

### 2.9 Lemma

$\alpha \in \mathbb{R}$ is an accumulation point of $(X_n) \Leftrightarrow$ every $\epsilon$-neighborhood of $\alpha$ contains $X_n$ for infinitely many n.

### 2.10 Bolzano-Weierstrass Theorem

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

### 2.11 Cauchy Sequence

$(X_n)$ is Cauchy if $\forall \epsilon > 0 \exists n_0 \in \mathbb{N}$ such that m, n $> n_0 \implies |X_m - X_n| < \epsilon$

### 2.12 Cauchy Criterion for Sequence

A sequence is convergent $\Leftrightarrow$ it is Cauchy.

## 3 Series

### 3.1 Geometric Series

$\sum_{n=0}^{\infty} q^n$ converges $\Leftrightarrow |q| < 1$, and $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$.

### 3.2 Harmonic Series

$\sum_{n=0}^{\infty} \frac{1}{n} isdivergent$

### 3.3 p-series

$\sum_{n=0}^{\infty} \frac{1}{n^p}$ is convergent $\Leftrightarrow p > 1$

### 3.4 Alternating harmonic series

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ is convergent.

### 3.5 Leibniz Criterion for Series

If $\sum_{n=0}^{\infty} a_n$ converges $\implies \lim_{n \to \infty} a_n = 0$

### 3.6 Cauchy Criterion for Series

$\sum_{n=0}^{\infty} a_n converges \Leftrightarrow \forall \epsilon > 0 \exists n_0 \in \mathbb{N}$ such that $m > n > n_0$, $|\sum_{k=0}^{m} a_k - \sum_{k=0}^{n} a_k| < \epsilon$ which can be rewritten as $|\sum_{k=n+1}^{m}| < \epsilon$

### 3.7 Proposition

$\sum_{n=0}^{\infty} a_n$ with $a_n \geq 0 \forall n$ is convergent $\Leftrightarrow$ its partial sum (Sn) is bounded.

Remark: $\sum_{n=0}^{\infty} \frac{1}{n(n+1)}$ is a convergent telescoping series since its (Sn) is bounded.

### 3.8 Comparison test for Series

Let $(a_n)(b_n)$ be non-negative sequences, $\exists M > 0$ and $n_0 \in \mathbb{N}$ such that $n > n_0 \implies a_n \leq M \cdot b_n$.

1. $\sum_{n=0}^{\infty} b_n$ converges $\implies \sum_{n=0}^{\infty} a_n$ converges
2. $\sum_{n=0}^{\infty} a_n$ diverges $\implies \sum_{n=0}^{\infty} b_n$ diverges (contrapositive of point 1)

### 3.9 Corollary

$\sum_{n=0}^{\infty} \frac{a_n}{b_n} = \alpha > 0$, then $\sum_{n=0}^{\infty} a_n$ converges $\Leftrightarrow \sum_{n=0}^{\infty} b_n$ converges.

### 3.10 Leibniz test for alternating series

If $(a_n)$ satisfies the following conditions:

1. $a_n \geq 0 \forall n \in \mathbb{N}$.
2. $\lim_{n\to\infty} a_n = 0$.
3. $(a_n)$ is monotone decreasing for sufficiently large n .

Then $\sum_{n=0}^{\infty}(-1)^n a_n$ is convergent.

### 3.11 Absolute Convergence

$\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\sum_{n=0}^{\infty}|a_n|$ converges. $\sum_{n=0}^{\infty} a_n$ is conditionally convergent if it converges but $\sum_{n=0}^{\infty}|a_n|$ diverges.

Remark:If a series is absolutely convergent $\implies$ it is convergent.

### 3.12 Theorem

$\sum_{n=0}^{\infty} a_n$ is absolutely convergent and $\sum_{n=0}^{\infty} b_n$ is obtained by rearranging $(a_n)$, $\implies \sum_{n=0}^{\infty} b_n$ is absolutely convergent with $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n$.

## 3.13 Cauchy Product Formula

Let $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ be absolutely convergent, then $\sum_{m,n=0}^{\infty} a_m b_n$ is absolutely convergent and $(\sum_{m=0}^{\infty} a_m)(\sum_{n=0}^{\infty} b_n) = \sum_{k=0}^{\infty} \sum_{m+n=k} a_{k-n} b_n$.

## 3.14 Ratio Test for Series

(i) For $n_0 \in \mathbb{N}$ such that $n > n_0$ and $\frac{|a_{n+1}|}{|a_n|} < 1 \implies \sum_{n=0}^{\infty} a_n$ is absolutely convergent.

(ii) For $n_0 \in \mathbb{N}$ such that $n > n_0$ and $\frac{|a_{n+1}|}{|a_n|} \geq 1 \implies \sum_{n=0}^{\infty} a_n$ diverges.

## 3.15 Limit Ratio Test for Series

(i)$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1 \implies \sum_{n=0}^{\infty} a_n$ is absolutely convergent.

(ii)$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1 \implies \sum_{n=0}^{\infty} a_n$ is divergent.

(iii)$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$ or does not exist then the test is conclusive.

## 3.16 Power Series of Exponential Function

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

## 3.17 Root Test for Series

If $\exists 0 < r < 1$ and $n_0 \in \mathbb{N}$ such that $n > n_0 \implies \sqrt[n]{|a_n|} < r$, then $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

## 3.18 Limit Root Test for Series

(i)$\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1 \implies \sum_{n=0}^{\infty} a_n$ is absolutely convergent.

(ii)$\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1 \implies \sum_{n=0}^{\infty} a_n$ is divergent.

(iii)$\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1 \implies$ inconclusive.

## 3.19 Abel's Theorem for power series

If $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z = \alpha \neq 0$, then the series is absolutely convergent $\forall z \in \mathbb{C}$ with $|z| < |\alpha|$.

## 3.20 Radius of Convergence Theorem

(i) $\forall z \in \mathbb{C}$ with $|z| < \mathbb{R} \implies$ series is absolutely convergent.

(ii) $\forall z \in \mathbb{C}$ with $|z| > \mathbb{R} \implies$ series is divergent.

### 3.21 How to find radius of convergence R

For any power series $\sum_{n=0}^{\infty} a_n z^n$,

1. $R = \lim_{n\to\infty} \frac{|a_n|}{a_{n+1}}$ if the limit exists
2. $R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$ if the limit exists
3. Use the Cauchy Hadamard formula: $R = \frac{1}{\lim_{n\to\infty} sup \sqrt[n]{|a_n|}}$

Remark:If $\lim_{n\to\infty} sup \sqrt[n]{|a_n|} = 0$, we write R = $\infty$.

## 4 Limits and Continuity

### 4.1 Continuous Functions

f is continuous at $x_0$ if $\forall \epsilon > 0 \exists$ an $\epsilon > 0$ such that if $|x - x_n| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

### 4.2 Properties of continuous functions

Let f, g be continuous at $x_0$ in $\mathbb{R}$,

1. $f + g$ is continuous at $x_0$
2. $f \cdot g$ is continuous at $x_0$
3. If $g(x_0) \neq 0$, $\frac{f}{g}$ is continuous at $x_0$
4. $g \circ f$ or $f \circ g$ is continuous

### 4.3 Sequential Characterisation of Continuity

Let $f : A \to \mathbb{R}$, $x_0 \in A$. Then, f is continuous at $x_0 \Leftrightarrow x_n \to x_0$ such that $x_n \in A$, $f(x_n) \to f(x_0)$.

### 4.4 Weierstrass Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous $\implies f$ is bounded and has a minimum and maxmum.

Remark:f is bounded $\Leftrightarrow \exists\, c > 0$ such that $|f(x)| < c$ with $x \in [a, b]$.

### 4.5 Bolzano˙Cauchy Theorem

$f : [a, b] \to \mathbb{R}$ is continuous, assume $f(a) \cdot f(b) < 0 \Rightarrow \exists x_0 \in [a, b]$ such that $f(x_0) = 0$.

### 4.6 Corollary: Intermediate Value Theorem

Let $y \in [f(a), f(b)] \implies \exists x_0 \in [a, b]$ such that $f(x_0) = y$.

### 4.7 Proposition

f is continuous at $x_0 \Leftrightarrow \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = f(x_0)$

Remark:$f(x_0)$ must be defined in the question. Meaning it should not be a removable discontinuity at a point.

## 5 Uniform Convergence

### 5.1 Pointwise Convergence

If $\forall x \in$D, $f_n(x) \to f(x)$ as $n \to \infty$. In short, $\lim_{n\to\infty} f_n(x) = f(x)$.

### 5.2 Uniform Convergence

If $\forall \epsilon > 0, \exists\, n_\epsilon \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon\ \forall n > n_\epsilon$ and $\forall\ x \in D$.

Remark:Uniform convergence $\implies$ Pointwise Convergence

### 5.3 Theorem

If $f_n$ is a sequence of continuous function which converges uniformly to f $\implies$ f is continuous.

### 5.4 Weierstrass M-Test

Let $g_k(z) : D \to \mathbb{C}, D \subset \mathbb{C}$. Let $M_k = sup_{z\in D}|g_k(z)|$.

If $\sum_{k=0}^{\infty} M_k$ converges $\implies \sum_{k=0}^{\infty} g_k(z)$ converges uniformly and absolutely on D.

### 5.5 Corollary

If $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly and absolutely for $|z| < R \implies$ f(z) is a continuous function.

### 5.6 Uniform Convergence for Integration

Let $f_n : [a, b] \to \mathbb{R}$ be continuous and $f_n$ converges uniformly to f $\implies \lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$.

Remark:Uniform convergence preserves continuity in the limit and allows interchange between integral and limit.

### 5.7 Uniform convergence for differentiation

Let $f_n : [a, b] \to \mathbb{R}$ be continuously differentiable.

Assume $f_n' \to g$ uniformly locally and $f_n \to f$ pointwise on [a, b] $\implies f' = g$.

In short, $\lim_{n\to\infty} f_n' = (\lim_{n\to\infty} f_n)'$.

### 5.8 Theorem

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n, z \in \mathbb{C}$. Define $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{m=0}^{\infty}(m+1)a_{m+1}z^m$ and $F(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} a_n z^{n+1}$ where f' = g and F' = f.

If R is the radius of convergence for f(z), then R has the same value for F and g.

### 5.9 Lemma

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n, z \in \mathbb{C}$, f has derivatives at all orders for $|z| < \mathbb{R}$. $f^{(k)}(z) = k! \sum_{n=k}^{\infty} \binom{n}{k} a_n z^{n-k}$ and $a_n = \frac{1}{n!} f^{(n)}(0),\ n \geq 0$.

### 5.10 Theorem

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and assume $(z_n)$ with $|z_n| < R$ such that $z_n \to 0 \implies f(z) = 0 \forall |z| < R$.

### 5.11 Corollary (Uniqueness of expansion into power series)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n$, then $a_n = b_n \forall |z| < R \implies \sum_{n=0}^{\infty}(a_n - b_n)z^n = 0\ \forall z, a_n = b_n \forall n$.

## 6 Analytic Functions

### 6.1 Analytic Function

Let $f : D \to \mathbb{C}, D \subset \mathbb{C}$. f is analytic at $z_0 \in D$ if $\exists$ a disc $\subset D(z_0, \epsilon) = z \in D; |z - z_0| < \epsilon$ such that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, z \in D(z_0, \epsilon)$. In short, an analytic function is one that can be represented by a power series.

Remark:Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, then f(z) is analytic at $z_0$ within the radius of convergence.

### 6.2 Corollary

If f is analytic at every $z_0 \in D$, then f is analytic in D.

### 6.3 Uniqueness Theorem for analytic functions

Let D be a path connected domain. Let $x_n \in D, n \geq 1, a \in D$ and $x_n \to a$ but $x_n \neq a$. If f is analytic in D and $f(x_n) = g(x_n) \forall n$, then f(a) = g(a).

### 6.4 Differentiation

Let $f : \mathbb{R} \to \mathbb{R}, a \in \mathbb{R}$. $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ if the limit exists.

Remark:$f'(a)$ exists $\Leftrightarrow$ f is differentiable at x = a.

Remark:If $\lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} \implies f'(a)$ exists.

## 7 Complex Analysis

### 7.1 Cauchy Riemann Relations Theorem

Let $f : D \to \mathbb{C}, D \subset \mathbb{C}$. f = u + iv.

f'(z) exists at $z \in D \Leftrightarrow u_x = v_y$ and $u_y = -v_x$.

Remark:u, v are continuous and f is continuous.

### 7.2 Harmonic functions

$u_{xx} + u_{yy} = 0$

$v_{xx} + v_{yy} = 0$

### 7.3 Path Integral

Let D be path-connected, take $\gamma(a) = z_1$ and $\gamma(b) = z_2$ where $z_1 \& z_2$ in D. Let $a \leq t \leq b$ and assume path is continuously differentiable, then $\int_{z_1}^{z_2} f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$.

Remark:Path integral is well defined for piece wise differentiable paths. This means we can add multiple integrals to form a path integral.

### 7.4 Independence of Path Integral

Let $\gamma : [a, b]$ be a piece-wise differentiable path and F be primitive of f, F'(z) = f(z), $z \in D \Rightarrow \int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$.

Remark:f(z) has to be analytic in order to use this theorem.

Remark:f(z) is complex differentiable $\Leftrightarrow$ f(z) is analytic.

### 7.5 Corollary

Let $\gamma$ be a closed path/contour. If f = F'. then $\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)) = 0$.

Identity: $\int_\gamma \frac{dz}{z-z_0} = \begin{cases} 0 \text{ if } z_0 \text{ outside } \gamma \\ 2\pi i \text{ if } z_0 \text{ inside } \gamma \end{cases}$

### 7.6 Closed Path Integral Theorem

Let $D \subset \mathbb{C}$ be a disc and $D' = D \backslash z_1, z_2, ....z_n$

1. Assume $f : D' \to \mathbb{C}$ is differentiable
2. $\lim_{z \to z_j}(z - z_j)f(z) = 0$, j = 1 ... n

Then $\int_\gamma f(z)dz = 0$ for any closed path in D'.

Remark:$\lim_{z \to z_j}(z - z_j)f(z) = 0$ implies no singularities or poles in D'.

### 7.7 Cauchy Integral formula

Let $D \subset \mathbb{C}$ be a disc. Assume $f : D \to \mathbb{C}$ is differentiable. $\gamma$ is positively oriented circle in $D : \gamma \subset D$ closed with center a. Then $f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}dz$.

### 7.8 Cauchy Generalized Integral Formula

$f^{(n)}(z) = \frac{n!}{z\pi i} \int_\gamma \frac{\phi(w)}{(w-z)^{n+1}}dw$

### 7.9 Morera's Theorem

Let $f : D \to \mathbb{C}$ be continuous. Assume every closed path $\gamma \subset D$, $\int_\gamma f(z)dz = 0 \implies$ f is infinitely differentiable.

### 7.10 Liouville's Theorem

Let $f : \mathbb{C} \to \mathbb{C}$ be differentiable and bounded $\implies$ f must be constant.

### 7.11 Cauchy Integral Formula for Annulus

Let f be analytic in an annulus $R_1 < |z - z_0| < R_2$.

Let $c_1, c_2$ be 2 contours inside this annulus centered at $z_0$, positively oriented $c_1$ inside $c_2$.

Then $\forall z$ in between $c_1$ and $c_2$, $f(z) = \frac{1}{2\pi i} \int_{c_2} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i} \int_{c_1} \frac{f(w)}{w-z}dw$.

## 7.12 Laurent Series Theorem

Let f be analytic in the annulus $\implies$ $\forall z$ in the annulus, $f(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$. Where $\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$ is known as the singular part and $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is the regular part.

## 7.13 Singularities

$z_0$ is a singularity if $\exists$ a disc around $z_0$ with $\epsilon > 0$ such that f is analytic in the disc $\backslash z_0$. In short, $f : D(z_0, \epsilon)\backslash z_0 \to \mathbb{C}$ is analytic.

## 7.14 Residue Theorem

Let $D \subset \mathbb{C}$, $f : D\backslash z_1, ...z_n$ is analytic. Let $\gamma$ be a simple, closed curve in D $\implies$ $\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{\infty} Res(f, z_k)$.

Remark:Sum up residues of poles within the closed curve to get the integral of the function, ignore the poles that lie outside the closed curve.

## 7.15 Finding Residues

Let $z_0$ be an isolated singularity of f.

1. $z_0$ is removable $\implies$ Res(f, $z_0$) $= 0$.
2. $z_0$ is a simple pole $\implies$ Res(f, $z_0$) $= \lim_{z\to z_0}[(z-z_0)f(z)]$
3. $z_0$ is a pole of order $m \geq 1$ $\implies$ Res(f, $z_0$) $= \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{(m-1)}}[(z-z_0)^m f(z)]$