

chapter 5 Performance Analysis

2021/6/27

Regret Analysis

- A player chooses an action $\theta^{(t)} \in K$ every t period, where K is a feasible set of actions.
- The cost function $f^{(t)}$ determines the cost $f^{(t)}(\theta^{(t)})$ for action $\theta^{(t)}$.
- The player decides his action based on the strategy.

Regret Analysis

- How does the player choose an action which minimizes a total cost $\sum f^{(t)}(\theta^{(t)})$?
- Can the cost function be minimized even if it is not unknown?
- We introduce a regret about the strategy.

definition (Regret)

The difference between the total cost of an action based on a strategy A and the total cost of the optimal strategy θ^ is defined as the regret $\text{Regret}(A)$ of strategy A .*

$$\text{Regret}(A) = \sum_{t=1}^T f^{(t)}(\theta^{(t)}) - \sum_{t=1}^T f^{(t)}(\theta^*)$$

Regret Analysis

Regret analysis in online learning

- Let action be the parameter of the online learner $\boldsymbol{\theta}^{(t)} \in \mathbb{R}^m$ given the training data $(\boldsymbol{x}^{(t)}, y^{(t)})$.
- Let the cost function be a loss function $f^{(t)} = (\boldsymbol{x}^{(t)}, y^{(t)}, \boldsymbol{\theta})$.
- In this case, the optimal strategy is the strategy that chooses the action that minimizes the cost function for all training data.

Follow the Leader

- At first, We consider strategy for choosing action that minimizes the total cost to date.

$$\boldsymbol{\theta}^{(t)} = \arg \min_{\boldsymbol{\theta} \in K} \sum_{i=1}^{t-1} f^{(i)}(\boldsymbol{\theta})$$

- This strategy is called Follow the Leader (FTL).

Follow the Leader

- However, there are cases where FTL doesn't work.
- Consider action $\theta \in [-1, 1]$ and cost function $f^{(t)}(\theta) = (1/2)(-1)^t\theta$.
- In this case, the action goes back and forth between -1 and 1 except for the first, as $\theta^{(1)} = 0, \theta^{(2)} = -1, \theta^{(3)} = 1, \dots$
- The cost function is $1/2$ except for the first, as $f^{(1)}(\theta^{(1)}) = 0, f^{(2)}(\theta^{(2)}) = 1/2, f^{(3)}(\theta^{(3)}) = 1/2, \dots$

Follow the Leader

- On the other hand, The optimal strategy is $\theta = 0$ and the total cost of it is 0.
- Therefore, FTL regret in this case doesn't approach 0.
- We have to expand FTL.

Regularized Follow the Leader

- Regularized Follow the Leader (RFTL)

$$\boldsymbol{\theta}^{(t)} = \arg \min_{\boldsymbol{\theta} \in K} \eta \sum_{i=1}^{t-1} f^{(i)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

- Let $R(\boldsymbol{\theta})$ be convex regularization function. Let $\eta \geq 0$ be parameter that determines the degree of regularization.
- When choosing the first action, the cost function is not presented, so action is determined only by the regularization term.

$$\boldsymbol{\theta}^{(1)} = \arg \min_{\boldsymbol{\theta} \in K} R(\boldsymbol{\theta})$$

Regularized Follow the Leader

- We introduce lemma and definitions to derive RFTL regret.

lemma

For any vector $\mathbf{u} \in K$, the following holds.

$$\sum_{t=1}^T \mathbf{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \mathbf{u}) \leq \sum_{t=1}^T \mathbf{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) + \frac{1}{\eta} \left\{ R(\mathbf{u}) - R(\boldsymbol{\theta}^{(1)}) \right\} \quad (1)$$

Regularized Follow the Leader

Proof.

For simplicity, let us assume that $\mathbf{f}^{(0)} = \frac{1}{\eta}R(\boldsymbol{\theta})$ and the algorithm starts at $t = 0$.

$$\sum_{t=0}^T \mathbf{f}^{(t)}(\boldsymbol{\theta}) = \sum_{t=1}^T \mathbf{f}^{(t)}(\boldsymbol{\theta}) + \frac{1}{\eta}R(\boldsymbol{\theta})$$

In this time, the lemma can be expressed as following.

$$\sum_{t=0}^T \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \mathbf{u}) \leq \sum_{t=0}^T \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})$$

Regularized Follow the Leader

Proof.

At $t = 0$,
by definition, $\boldsymbol{\theta}^{(1)} = \arg \min_{\boldsymbol{\theta}} R(\boldsymbol{\theta})$ and $\mathbf{f}^{(0)}(\boldsymbol{\theta}^{(1)}) \leq \mathbf{f}^{(0)}(\mathbf{u})$ holds.
therefore,

$$\mathbf{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \mathbf{f}^{(0)}(\mathbf{u}) \leq \mathbf{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \mathbf{f}^{(0)}(\boldsymbol{\theta}^{(1)})$$



Regularized Follow the Leader

Proof.

At $t > 0$,
assume that lemma holds for $t = T$.
In this time,

$$\theta^{(T+2)} = \arg \min_{\theta} \sum_{t=0}^{T+1} f^{(t)}(\theta) \quad (2)$$

$$\theta^{(T+1)} = \arg \min_{\theta} \sum_{t=0}^T f^{(t)}(\theta) \quad (3)$$



Regularized Follow the Leader

Proof.

Using equation (2) and (3),

$$\begin{aligned}& \sum_{t=0}^{T+1} \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \mathbf{u}) \\& \leq \sum_{t=0}^{T+1} \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \sum_{t=0}^{T+1} \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(T+2)}) \\& = \sum_{t=0}^T (\mathbf{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(T+2)})) + \mathbf{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \mathbf{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\& \leq \sum_{t=0}^T (\mathbf{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(T+1)})) + \mathbf{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \mathbf{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\& \quad - \sum_{t=0}^{T+1} \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(t)}) + \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(t+1)})\end{aligned}$$

Regularized Follow the Leader

definition (norm based on positive semi-definite matrix)

We define $\|\mathbf{x}\|_A = \sqrt{\mathbf{x}^T \mathbf{A} \mathbf{x}}$ as the norm of vector \mathbf{x} based on positive semi-definite matrix \mathbf{A} .

We also define $\|\mathbf{x}\|_{A^{-1}} = \|\mathbf{x}\|_A^*$ as a dual norm.

- In this time, from generalized Cauchy-Schwarz inequality, the following holds.

$$\mathbf{x}^T \mathbf{y} \leq \|\mathbf{x}\|_A \|\mathbf{y}\|_A^*$$

Regularized Follow the Leader

definition (norm of cost function)

A norm of cost function measured by the regularization function is defined as

$$\lambda = \max_{t, \boldsymbol{\theta} \in K} \mathbf{f}^{(t)T} \{\nabla^2 R(\boldsymbol{\theta})\}^{-1} \mathbf{f}^{(t)}$$

Regularized Follow the Leader

definition (diameter of feasible area)

A diameter measured by the regularization function is defined as

$$D = \max_{\boldsymbol{\theta} \in K} R(\boldsymbol{\theta}) - R(\boldsymbol{\theta}^{(1)})$$

Regularized Follow the Leader

theorem (regret of RFTL)

RFTL achieves the following regret for any vector $\mathbf{u} \in K$.

$$\text{Regret}(A) = \sum_{t=1}^T \mathbf{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \mathbf{u}) \leq 2\sqrt{2\lambda DT}$$

Regularized Follow the Leader

Proof.

At first, we define Φ as following.

$$\Phi^{(t)}(\boldsymbol{\theta}) = \eta \sum_{i=1}^t f^{(i)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

By Taylor expansion of $\Phi^{(t)}$ around $\boldsymbol{\theta}^{(t+1)}$ and using intermediate value theorem, we can show following.

$$\begin{aligned}\Phi^{(t)}(\boldsymbol{\theta}^{(t)}) &= \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + (\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})^T \nabla \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) \\ &\quad + \frac{1}{2} \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\mathbf{z}^{(t)}}^2 \\ &\geq \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + \frac{1}{2} \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\mathbf{z}^{(t)}}^2\end{aligned}$$

Regularized Follow the Leader

Proof.

Here, from intermediate value theorem, $\mathbf{z}^{(t)} \in [\boldsymbol{\theta}^{(t+1)}, \boldsymbol{\theta}^{(t)}]$

This inequality holds because $\boldsymbol{\theta}^{(t)}$ achieves the minimum value of $\Phi^{(t)}$.

By transforming the equation, we can get following.

$$\begin{aligned}\|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\mathbf{z}^{(t)}}^2 &\leq 2(\Phi^{(t)}(\boldsymbol{\theta}^{(t)}) - \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)})) \\ &= 2(\Phi^{(t-1)}(\boldsymbol{\theta}^{(t)}) - \Phi^{(t-1)}(\boldsymbol{\theta}^{(t+1)})) + 2\eta \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) \\ &\leq 2\eta \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})\end{aligned}$$



Regularized Follow the Leader

Proof.

Also, using generalized Cauchy-Schwarz inequality, the following holds.

$$\begin{aligned} \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) &\leq \|\mathbf{f}^{(t)}\|_z^* \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_z \\ &\leq \|\mathbf{f}^{(t)}\|_z^* \sqrt{2\eta \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})} \end{aligned}$$

furthermore,

$$\mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) \leq 2\eta \|\mathbf{f}^{(t)}\|_z^{*2} \leq 2\eta\lambda \quad (4)$$



Regularized Follow the Leader

Proof.

Using (1) and (4), adding up to T and letting η be $\eta = \sqrt{\frac{D}{2\lambda T}}$ which minimizes this equation, we can get

$$\text{Regret}(A) \leq \min_{\eta} \left[2\eta\lambda T + \frac{1}{\eta} \{R(\mathbf{u}) - R(\boldsymbol{\theta}^{(t)})\} \right] \leq 2\sqrt{2D\lambda T}$$



Regret with strong convex loss function

- a player chooses a point (action) from a set in Euclidean space denoted $K \subseteq \mathbb{R}^n$.
- We assume that the set K is non-empty, bounded, closed and convex.
- We denote the number of iterations by T which is unknown to the player.

Regret with strong convex loss function

- At iteration t , the player chooses $\theta_t \in K$.
- After committing to this choice, a convex cost function $f_t : K \mapsto \mathbb{R}$ is revealed.

Regret with strong convex loss function

- Consider a player using an algorithm for online game playing (strategy) A .
- At iteration t , the algorithm takes as input the history of cost function f_1, \dots, f_{t-1} and produces a feasible point $A(\{f_1, \dots, f_{t-1}\})$ in the domain K .
- When there is no ambiguity, we simply denote $\theta_t = A(\{f_1, \dots, f_{t-1}\})$.

Regret with strong convex loss function

- The regret of the online player using algorithm A at time T is defined to be the total cost minus the cost of the best single decision, where the best is chosen with the benefit of hindsight.

$$\text{Regret}(A, \{f_1, \dots, f_T\}) = E \left\{ \sum_{t=1}^T f_t(\boldsymbol{\theta}_t) \right\} - \min_{\boldsymbol{\theta} \in K} \sum_{t=1}^T f_t(\boldsymbol{\theta}).$$

- We are usually interested in an upper bound on the worst case guaranteed regret, denoted

$$\text{Regret}_T(A) = \sup_{\{f_1, \dots, f_T\}} \{ \text{Regret}(A, \{f_1, \dots, f_T\}) \}.$$

Regret with strong convex loss function

definition

We say that the cost functions have gradients upper bounded by a number G if the following holds:

$$\sup_{\boldsymbol{\theta} \in K, t \in \{1, \dots, T\}} \|\nabla f_t(\boldsymbol{\theta})\|_2 \leq G.$$

Regret with strong convex loss function

definition

We say that the Hessian of all cost functions is lower bounded by a number $\alpha > 0$, if the following holds:

$$\forall \boldsymbol{\theta} \in K, t \in \{f_1, \dots, f_T\} : \nabla^2 f_t(\boldsymbol{\theta}) \succeq \alpha \mathbf{I}_n.$$

α is a lower bound on eigenvalues of all the Hessians of the constraints at all points in the domain. Such function is called α -strong convex.

Regret with strong convex loss function

algorithm (Online Gradient Descent)

Inputs: convex set $K \subset \mathbb{R}^n$, step sizes $\eta_1, \eta_2, \dots \geq 0$, initial $\boldsymbol{\theta}_1 \in K$.

- In iteration 1, use $\boldsymbol{\theta}_1 \in K$.

- In iteration $t > 1$, use

$$\boldsymbol{\theta}_t = \pi_K(\boldsymbol{\theta}_{t-1} - \eta_{t-1} \nabla f_{t-1}(\boldsymbol{\theta}_{t-1})).$$

Here, π_K denotes the projection onto nearest point in K ,

$$\pi_K(\mathbf{y}) = \arg \min_{\boldsymbol{\theta} \in K} \|\boldsymbol{\theta} - \mathbf{y}\|_2.$$

Regret with strong convex loss function

definition

For a given positive semi-definite matrix \mathbf{A} , a generalized projection of $\mathbf{y} \in \mathbb{R}^n$ onto the convex set K is defined as

$$\pi_K^{\mathbf{A}}(\mathbf{y}) = \arg \min_{\boldsymbol{\theta} \in K} (\boldsymbol{\theta} - \mathbf{y})^\top \mathbf{A} (\boldsymbol{\theta} - \mathbf{y}).$$

Thus, the Euclidean projection can be seen to be a generalized projection with $\mathbf{A} = \mathbf{I}_n$.

Regret with strong convex function

lemma

Let $K \subseteq \mathbb{R}^n$ be a convex set, $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{z} = \pi_K^{\mathbf{A}}(\mathbf{y})$ be the generalized projection of \mathbf{y} onto K according to positive semi-definite matrix \mathbf{A} . Then for any point $\mathbf{a} \in K$ it holds that

$$(\mathbf{y} - \mathbf{a})^\top \mathbf{A}(\mathbf{y} - \mathbf{a}) \geq (\mathbf{z} - \mathbf{a})^\top \mathbf{A}(\mathbf{z} - \mathbf{a}).$$

Regret with strong convex loss function

Proof.

By definition of generalized projections, the point \mathbf{z} minimizes the function $f(\boldsymbol{\theta}) = (\boldsymbol{\theta} - \mathbf{y})^\top \mathbf{A}(\boldsymbol{\theta} - \mathbf{y})$ over the convex set. It is a well known fact in optimization that for the optimum \mathbf{z} the following holds.

$$\forall \mathbf{a} \in K : \nabla f(\mathbf{z})^\top (\mathbf{a} - \mathbf{z}) \geq 0$$

which implies

$$2(\mathbf{z} - \mathbf{y})^\top \mathbf{A}(\mathbf{a} - \mathbf{z}) \geq 0 \Rightarrow 2\mathbf{a}^\top \mathbf{A}(\mathbf{z} - \mathbf{y}) \geq 2\mathbf{z}^\top \mathbf{A}(\mathbf{z} - \mathbf{y}).$$



Regret with strong convex loss function

Proof.

Now by simple calculation:

$$\begin{aligned}(\mathbf{y} - \mathbf{a})^\top \mathbf{A}(\mathbf{y} - \mathbf{a}) - (\mathbf{z} - \mathbf{a})^\top \mathbf{A}(\mathbf{z} - \mathbf{a}) &= \mathbf{y}^\top \mathbf{A} \mathbf{y} - \mathbf{z}^\top \mathbf{A} \mathbf{z} + 2\mathbf{a}^\top \mathbf{A}(\mathbf{z} - \mathbf{y}) \\&\geq \mathbf{y}^\top \mathbf{A} \mathbf{y} - \mathbf{z}^\top \mathbf{A} \mathbf{z} + 2\mathbf{z}^\top \mathbf{A}(\mathbf{z} - \mathbf{y}) \\&= \mathbf{y}^\top \mathbf{A} \mathbf{y} - 2\mathbf{z}^\top \mathbf{A} \mathbf{y} + \mathbf{z}^\top \mathbf{A} \mathbf{z} \\&= (\mathbf{z} - \mathbf{y})^\top \mathbf{A}(\mathbf{z} - \mathbf{y}) \geq 0.\end{aligned}$$



Regret with strong convex loss function

theorem

Online Gradient Descent with step sizes $\eta_t = \frac{1}{\alpha t}$ achieves the following guarantee, for all $T \geq 1$

$$\text{Regret}_T(\text{OGD}) \leq \frac{G^2}{2\alpha}(1 + \log T).$$

Regret with strong convex loss function

Proof.

Let $\boldsymbol{\theta}^* = \arg \min \sum_{t=1}^T f_t(\boldsymbol{\theta})$. Recall the definition of regret

$$\text{Regret}_T(\text{OGD}) = \sum_{t=1}^T f_t(\boldsymbol{\theta}_t) - \sum_{t=1}^T f_t(\boldsymbol{\theta}^*).$$

Define $\nabla_t = \nabla f_t(\boldsymbol{\theta}_t)$. By using the Taylor series approximation, we have, for some point ξ_t on the line segment joining $\boldsymbol{\theta}_t$ to $\boldsymbol{\theta}^*$,

$$\begin{aligned} f_t(\boldsymbol{\theta}^*) &= f_t(\boldsymbol{\theta}_t) + \nabla_t^\top (\boldsymbol{\theta}^* - \boldsymbol{\theta}_t) + \frac{1}{2} (\boldsymbol{\theta}^* - \boldsymbol{\theta}_t)^\top \nabla^2 f_t(\xi_t) (\boldsymbol{\theta}^* - \boldsymbol{\theta}_t) \\ &\geq f_t(\boldsymbol{\theta}_t) + \nabla_t^\top (\boldsymbol{\theta}^* - \boldsymbol{\theta}_t) + \frac{\alpha}{2} \|\boldsymbol{\theta}^* - \boldsymbol{\theta}_t\|^2. \end{aligned}$$



Regret with strong convex loss function

Proof.

The inequality follows from α -strong convexity. Thus, we have

$$2\{f_t(\boldsymbol{\theta}_t - \boldsymbol{\theta}^*)\} \leq 2\nabla^\top(\boldsymbol{\theta}_t - \boldsymbol{\theta}^*) - \alpha\|\boldsymbol{\theta}^* - \boldsymbol{\theta}_t\|^2. \quad (5)$$

Using the update rule for $\boldsymbol{\theta}_{t+1}$ and lemma, we get

$$\begin{aligned} \|\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}^*\|^2 &= \|\pi(\boldsymbol{\theta}_t - \eta_t \nabla_t) - \boldsymbol{\theta}^*\|^2 \\ &\leq \|\boldsymbol{\theta}_t - \eta_t \nabla_t - \boldsymbol{\theta}^*\|^2. \end{aligned}$$



Regret with strong convex loss function

Proof.

Hence,

$$\begin{aligned}\|\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}^*\|^2 &\leq \|\boldsymbol{\theta}_t - \boldsymbol{\theta}^*\|^2 + \eta_t^2 \|\nabla_t\|^2 - 2\eta_t \nabla_t^\top (\boldsymbol{\theta}_t - \boldsymbol{\theta}^*), \\ 2\nabla_t^\top (\boldsymbol{\theta}_t - \boldsymbol{\theta}^*) &\leq \frac{\|\boldsymbol{\theta}_t - \boldsymbol{\theta}^*\|^2 - \|\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}^*\|^2}{\eta_t} + \eta_t G^2.\end{aligned}\tag{6}$$



Regret with strong convex loss function

Proof.

Sum up (6) from $t = 1$ to T and using (5), we have:

$$\begin{aligned} 2 \sum_{t=1}^T f_t(\boldsymbol{\theta}_t) - f_t(\boldsymbol{\theta}^*) &\leq \sum_{t=1}^T \|\boldsymbol{\theta}_t - \boldsymbol{\theta}^*\|^2 \left(\frac{1}{\eta_{t-1}} + \frac{1}{\eta_t} - \alpha \right) + G^2 \sum_{t=1}^T \eta_t \\ &= G^2 \sum_{t=1}^T \frac{1}{\alpha t} \\ &\leq \frac{G^2}{\alpha} (1 + \log T). \end{aligned}$$

