chapter 5 Performance Analysis

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Regret Analysis

- A player chooses an action $\theta^{(t)} \in K$ every t period, where K is a feasible set of actions.
- The cost function $f^{(t)}$ determines the cost $f^{(t)}(\theta^{(t)})$ for action $\theta^{(t)}$.
- The player decides his action based on the strategy.

Regret Analysis

- How does the player choose an action which minimizes a total cost $\sum f^{(t)}(\theta^{(t)})$?
- Can the cost function be minimized even if it is not unknown?
- We introduce a regret about the strategy.

definition (Regret)

The difference between the total cost of an action based on a strategy A and the total cost of the optimal strategy θ^* is defined as the regret $Regret(A)$ of strategy A.

$$Regret(A) = \sum_{t=1}^{T} f^{(t)}(\theta^{(t)}) - \sum_{t=1}^{T} f^{(t)}(\theta^*)$$

Regret Analysis

Regret analysis in online learning

- Let action be the parameter of the online learner $\boldsymbol{\theta}^{(t)} \in \mathbb{R}^m$ given the training data $(\boldsymbol{x}^{(t)}, y^{(t)})$.
- Let the cost function be a loss function $f^{(t)} = (\boldsymbol{x}^{(t)}, y^{(t)}, \boldsymbol{\theta})$.
- In this case, the optimal strategy is the strategy that chooses the action that minimizes the cost function for all training data.

Follow the Leader

- At first, We consider strategy for choosing action that minimizes the total cost to date.

$$\boldsymbol{\theta}^{(t)} = \arg\min_{\boldsymbol{\theta} \in K} \sum_{i=1}^{t-1} f^{(t)}(\boldsymbol{\theta})$$

- This strategy is called Follow the Leader (FTL).

Follow the Leader

- However, there are cases where FTL doesn't work.
- Consider action $\theta \in [-1, 1]$ and cost function $f^{(t)}(\theta) = (1/2)(-1)^t\theta$.
- In this case, the action goes back and forth between -1 and 1 except for the first, as $\theta^{(1)} = 0, \theta^{(2)} = -1, \theta^{(2)} = 1, ...$
- The cost function is $1/2$ except for the first, as $f^{(1)}(\theta^{(1)}) = 0, f^{(2)}(\theta^{(2)}) = 1/2, f^{(3)}(\theta^{(3)}) = 1/2, ...$

Follow the Leader

- On the other hand, The optimal strategy is $\theta = 0$ and the total cost of it is 0.
- Therefore, FTL regret in this case doesn't approach 0.
- We have to expand FTL.

Regularized Follow the Leader

- Regularized Follow the Leader (RFTL)

$$\boldsymbol{\theta}^{(t)} = \arg\min_{\boldsymbol{\theta} \in K} \eta \sum_{i=1}^{t-1} f^{(t)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

- Let $R(\boldsymbol{\theta})$ be convex regularization function. Let $\eta \geq 0$ be parameter that determines the degree of regularization.
- When choosing the first action, the cost function is not presented, so action is determined only by the regularization term.

$$\boldsymbol{\theta}^{(1)} = \arg\min_{\boldsymbol{\theta} \in K} R(\boldsymbol{\theta})$$

Regularized Follow the Leader

- We introduce lemma and definitions to derive RFTL regret.

lemma

For any vector $\boldsymbol{u} \in K$*, the following holds.*

$$\sum_{t=1}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq \sum_{t=1}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) + \frac{1}{\eta}\left\{R(\boldsymbol{u}) - R(\boldsymbol{\theta}^{(1)})\right\} \tag{1}$$

Regularized Follow the Leader

Proof.

For simplicity, let us assume that $\boldsymbol{f}^{(0)} = \frac{1}{\eta}R(\boldsymbol{\theta})$ and the algorithm starts at $t = 0$.

$$\sum_{t=0}^{T} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}) = \sum_{t=1}^{T} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}) + \frac{1}{\eta}R(\boldsymbol{\theta})$$

In this time, the lemma can be expressed as following.

$$\sum_{t=0}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq \sum_{t=0}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})$$

Regularized Follow the Leader

Proof.

At $t = 0$,
by definition, $\boldsymbol{\theta}^{(1)} = \arg\min_{\boldsymbol{\theta}} R(\boldsymbol{\theta})$ and $\boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(1)}) \leq \boldsymbol{f}^{(0)}(\boldsymbol{u})$ holds.
therefore,

$$\boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \boldsymbol{f}^{(0)}(\boldsymbol{u}) \leq \boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(1)})$$

□

Regularized Follow the Leader

Proof.

At $t > 0$,
assume that lemma holds for $t = T$.
In this time,

$$\boldsymbol{\theta}^{(T+2)} = \arg\min_{\boldsymbol{\theta}} \sum_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}) \tag{2}$$

$$\boldsymbol{\theta}^{(T+1)} = \arg\min_{\boldsymbol{\theta}} \sum_{t=0}^{T} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}) \tag{3}$$

□

Regularized Follow the Leader

Proof.

Using equation (2) and (3),

$$
\begin{aligned}
&\sum_{t=0}^{T+1} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \\
&\leq \sum_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \sum_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+2)}) \\
&= \sum_{t=0}^{T} (\boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+2)})) + \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\
&\leq \sum_{t=0}^{T} (\boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+1)})) + \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\
&= \sum_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t+1)})
\end{aligned}
$$

Regularized Follow the Leader

definition (norm based on positive semi-definite matrix)

We define $\|\boldsymbol{x}\|_{\boldsymbol{A}} = \sqrt{\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}}$ as the norm of vector $\boldsymbol{x}$ based on positve semi-definite matrix $\boldsymbol{A}$.
We also define $\|\boldsymbol{x}\|_{\boldsymbol{A}^{-1}} = \|\boldsymbol{x}\|_{\boldsymbol{A}}^*$ as a dual norm.

- In this time, from generalized Cauchy-Schwarz inequality, the following holds.

$$\boldsymbol{x}^T \boldsymbol{y} \leq \|\boldsymbol{x}\|_{\boldsymbol{A}} \|\boldsymbol{y}\|_{\boldsymbol{A}}^*$$

Regularized Follow the Leader

definition (norm of cost function)

A norm of cost function measured by the regularization function is difined as

$$\lambda = \max_{t, \boldsymbol{\theta} \in K} \boldsymbol{f}^{(t)T} \{\nabla^2 R(\boldsymbol{\theta})\}^{-1} \boldsymbol{f}^{(t)}$$

Regularized Follow the Leader

definition (diameter of feasible area)

A diameter measured by the regularization function is defined as

$$D = \max_{\boldsymbol{\theta} \in K} R(\boldsymbol{\theta}) - R(\boldsymbol{\theta}^{(1)})$$

Regularized Follow the Leader

theorem (regret of RFTL)

RFTL achives the following regret for any vector $\boldsymbol{u} \in K$.

$$Regret(A) = \sum_{t=1}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq 2\sqrt{2\lambda DT}$$

Regularized Follow the Leader

Proof.

At first, we define Φ as following.

$$\Phi^{(t)}(\boldsymbol{\theta}) = \eta \sum_{i=1}^{t} f^{(i)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

By Taylor expansion of $\Phi^{(t)}$ around $\boldsymbol{\theta}^{(t+1)}$ and using intermediate value theorem, we can show following.

$$\begin{aligned}
\Phi^{(t)}(\boldsymbol{\theta}^{(t)}) &= \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + (\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})^T \nabla \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) \\
&\quad + \frac{1}{2} \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|^2_{\boldsymbol{z}^{(t)}} \\
&\geq \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + \frac{1}{2} \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|^2_{\boldsymbol{z}^{(t)}}
\end{aligned}$$

Regularized Follow the Leader

Proof.

Here, from intermediate value theorem, $\boldsymbol{z}^{(t)} \in [\boldsymbol{\theta}^{(t+1)}, \boldsymbol{\theta}^{(t)}]$
This inequality holds because $\boldsymbol{\theta}^{(t)}$ achieves the minimum value of $\Phi^{(t)}$.
By transforming the equation, we can get following.

$$
\begin{aligned}
\|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|^2_{\boldsymbol{z}^{(t)}} &\leq 2(\Phi^{(t)}(\boldsymbol{\theta}^{(t)}) - \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)})) \\
&= 2(\Phi^{(t-1)}(\boldsymbol{\theta}^{(t)}) - \Phi^{(t-1)}(\boldsymbol{\theta}^{(t+1)})) + 2\eta \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) \\
&\leq 2\eta \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})
\end{aligned}
$$

□

Proof.

Also, using generalized Cauchy-Schwarz inequality, the following holds.

$$\begin{aligned} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) &\leq \|\boldsymbol{f}^{(t)}\|_{\boldsymbol{z}}^{*}\|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\boldsymbol{z}} \\ &\leq \|\boldsymbol{f}^{(t)}\|_{\boldsymbol{z}}^{*}\sqrt{2\eta \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})} \end{aligned}$$

furthermore,

$$\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) \leq 2\eta\|\boldsymbol{f}^{(t)}\|_{\boldsymbol{z}}^{*2} \leq 2\eta\lambda \tag{4}$$

□

Regularized Follow the Leader

Proof.

Using (1) and (4), adding up to T and letting η be $\eta = \sqrt{\frac{D}{2\lambda T}}$ which minimizes this equation, we can get

$$Regret(A) \leq min_\eta \left[2\eta\lambda T + \frac{1}{\eta}\{R(\boldsymbol{u}) - R(\boldsymbol{\theta}^{(t)})\}\right] \leq 2\sqrt{2D\lambda T}$$

□

Regret with strong convex loss function

- a player chooses a point (action) from a set in Euclidean space denoted $K \subseteq \mathbb{R}^n$.
- We assume that the set K is non-empty, bounded, closed and convex.
- We denote the number of iterations by T which is unknown to the player.

Regret with strong convex loss function

- At iteration t, the player chooses $\boldsymbol{\theta}_t \in K$.
- After committing to this choice, a convex cost function $f_t : K \mapsto \mathbb{R}$ is revealed.

Regret with strong convex loss function

- Consider a player using an algorithm for online game playing (strategy) A.
- At iteration t, the algorithm takes as input the history of cost function $f_1, ..., f_{t-1}$ and produces a feasible point $A(\{f_1, ..., f_{t-1}\})$ in the domain K.
- When there is no ambiguity, we simply denote $\boldsymbol{\theta}_t = A(\{f_1, ..., f_{t-1}\})$.

Regret with strong convex loss function

- The regret of the online player using algorithm A at time T is defined to be the total cost minus the cost of the best single decision, where the best is chosen with the benefit of hindsight.

$$Regret(A, \{f_1, ..., f_T\}) = E\left\{\sum_{t=1}^{T} f_t(\boldsymbol{\theta}_t)\right\} - \min_{\boldsymbol{\theta}\in K}\sum_{t=1}^{T} f_t(\boldsymbol{\theta}).$$

- We are usually interested in an upper bound on the worst case guaranteed regret, denoted

$$Regret_T(A) = \sup_{\{f_1,...,f_T\}}\{Regret(A, \{f_1, ..., f_T\})\}.$$

Regret with strong convex loss function

definition

We say that the cost functions have gradients upper bounded by a number G *if the following holds:*

$$\sup_{\boldsymbol{\theta} \in K, t \in \{1,\ldots,T\}} \|\nabla f_t(\boldsymbol{\theta})\|_2 \leq G.$$

Regret with strong convex loss function

definition

We say that the Hessian of all cost functions is lower bounded by a number $\alpha > 0$, if the following holds:

$$\forall \boldsymbol{\theta} \in K, t \in \{f_1, ..., f_T\} : \nabla^2 f_t(\boldsymbol{\theta}) \succeq \alpha \boldsymbol{I}_n.$$

α is a lower bound on eigenvalues of all the Hessians of the constraints at all points in the domain. Such function is called α-strong convex.

Regret with strong convex loss function

algorithm (Online Gradient Descent)

Inputs: convex set $K \subset \mathbb{R}^n$*, step sizes* $\eta_1, \eta_2, ... \geq 0$*, inital* $\boldsymbol{\theta}_1 \in K$*.*

- In iteration 1*, use* $\boldsymbol{\theta}_1 \in K$*.*

- In iteration $t > 1$*, use*

$$\boldsymbol{\theta}_t = \pi_K(\boldsymbol{\theta}_{t-1} - \eta \nabla f_{t-1}(\boldsymbol{\theta}_{t-1})).$$

Here, π_K *denotes the projection onto nearest point in* K*,*
$\pi_K(\boldsymbol{y}) = \arg\min_{\boldsymbol{\theta} \in K} \|\boldsymbol{\theta} - \boldsymbol{y}\|_2$*.*

Regret with strong convex loss function

definition

For a given positive semi-definite matrix $\boldsymbol{A}$, *a generalized projection of* $\boldsymbol{y} \in \mathbb{R}^n$ *onto the convex set* K *is defined as*

$$\pi_K^{\boldsymbol{A}}(\boldsymbol{y}) = \arg\min_{\boldsymbol{\theta} \in K} (\boldsymbol{\theta} - \boldsymbol{y})^\top \boldsymbol{A} (\boldsymbol{\theta} - \boldsymbol{y}).$$

Thus, the Euclidean projection can be seen to be a generalized projection with $\boldsymbol{A} = \boldsymbol{I}_n$.

Regret with strong convex function

lemma

Let $K \subseteq \mathbb{R}^n$ *be a convex set,* $\boldsymbol{y} \in \mathbb{R}^n$ *and* $\boldsymbol{z} = \pi_K^{\boldsymbol{A}}(\boldsymbol{y})$ *be the generalized projection of* $\boldsymbol{y}$ *onto* K *according to positive semi-definite matrix* $\boldsymbol{A}$*. Then for any point* $\boldsymbol{a} \in K$ *it holds that*

$$(\boldsymbol{y}-\boldsymbol{a})^\top \boldsymbol{A}(\boldsymbol{y}-\boldsymbol{a}) \geq (\boldsymbol{z}-\boldsymbol{a})^\top \boldsymbol{A}(\boldsymbol{z}-\boldsymbol{a}).$$

Regret with strong convex loss function

Proof.

By definition of generalized projections, the point $\boldsymbol{z}$ minimizes the function $f(\boldsymbol{\theta}) = (\boldsymbol{\theta} - \boldsymbol{y})^\top \boldsymbol{A}(\boldsymbol{\theta} - \boldsymbol{y})$ over the convex set. It is a well known fact in optimization that for the optimum $\boldsymbol{z}$ the following holds.

$$\forall \boldsymbol{a} \in K : \nabla f(\boldsymbol{z})^\top (\boldsymbol{a} - \boldsymbol{z}) \geq 0$$

which implies

$$2(\boldsymbol{z} - \boldsymbol{y})^\top \boldsymbol{A}(\boldsymbol{a} - \boldsymbol{z}) \leq 0 \Rightarrow 2\boldsymbol{a}^\top \boldsymbol{A}(\boldsymbol{z}^{\boldsymbol{y}}) \geq 2\boldsymbol{z}^\top \boldsymbol{A}(\boldsymbol{z}^{\boldsymbol{y}}).$$

□

Regret with strong convex loss function

Proof.

Now by simple calculation:

$$\begin{aligned}(\boldsymbol{y}-\boldsymbol{a})^\top \boldsymbol{A}(\boldsymbol{y}-\boldsymbol{a})-(\boldsymbol{z}-\boldsymbol{a})^\top \boldsymbol{A}(\boldsymbol{z}-\boldsymbol{a}) &= \boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y}-\boldsymbol{z}^\top \boldsymbol{A}\boldsymbol{z}+2\boldsymbol{a}^\top \boldsymbol{A}(\boldsymbol{z}-\boldsymbol{y}) \\ &\geq \boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y}-\boldsymbol{z}^\top \boldsymbol{A}\boldsymbol{z}+2\boldsymbol{z}^\top \boldsymbol{A}(\boldsymbol{z}-\boldsymbol{y}) \\ &= \boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y}-2\boldsymbol{z}^\top \boldsymbol{A}\boldsymbol{y}+\boldsymbol{z}^\top \boldsymbol{A}\boldsymbol{z} \\ &= (\boldsymbol{z}-\boldsymbol{y})^\top \boldsymbol{A}(\boldsymbol{z}-\boldsymbol{y}) \geq 0.\end{aligned}$$

□

Regret with strong convex loss function

theorem

Online Gradient Descent with step sizes $\eta_t = \frac{1}{\alpha_t}$ *achives the following guarantee, for all* $T \geq 1$

$$Regret_T(OGD) \leq \frac{G^2}{2\alpha}(1 + \log T).$$

Regret with strong convex loss function

Proof.

Let $\boldsymbol{\theta}^* = \arg\min \sum_{t=1}^{T} f_t(\boldsymbol{\theta})$. Recall the difinition of regret

$$Regret_T(OGD) = \sum_{t=1}^{T} f_t(\boldsymbol{\theta}_t) - \sum_{t=1}^{T} f_t(\boldsymbol{\theta}^*).$$

Define $\nabla_t = \nabla f_t(\boldsymbol{\theta}_t)$. By using the Taylor series approximation, we have, for some point ξ_t on the line segment joining $\boldsymbol{\theta}_t$ to $\boldsymbol{\theta}^*$,

□