chapter 5 Performance Analysis

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Regret Analysis

- A player chooses an action $\theta^{(t)} \in K$ every t period, where K is a feasible set of actions.
- The cost function $f^{(t)}$ determines the cost $f^{(t)}(\theta^{(t)})$ for action $\theta^{(t)}$.
- The player decides his action based on the strategy.

Regret Analysis

- How does the player choose an action which minimizes a total cost $\Sigma f^{(t)}(\theta^{(t)})$?
- Can the cost function be minimized even if it is not unknown?
- We introduce a regret about the strategy.

definition (Regret)

The difference between the total cost of an action based on a strategy A and the total cost of the optimal strategy θ^* is defined as the regret $Regret(A)$ of strategy A.

$$Regret(A) = \Sigma_{t=1}^{T} f^{(t)}(\theta^{(t)}) - \Sigma_{t=1}^{T} f^{(t)}(\theta^*)$$

Regret Analysis

Regret analysis in online learning

- Let action be the parameter of the online learner $\boldsymbol{\theta}^{(t)} \in \mathbb{R}^m$ given the training data $(\boldsymbol{x}^{(t)}, y^{(t)})$.
- Let the cost function be a loss function $f^{(t)} = (\boldsymbol{x}^{(t)}, y^{(t)}, \boldsymbol{\theta})$.
- In this case, the optimal strategy is the strategy that chooses the action that minimizes the cost function for all training data.

Follow the Leader

- At first, We consider strategy for choosing action that minimizes the total cost to date.

$$\boldsymbol{\theta}^{(t)} = \underset{\boldsymbol{\theta} \in K}{\arg\min}\, \Sigma_{i=1}^{t-1} f^{(t)}(\boldsymbol{\theta})$$

- This strategy is called Follow the Leader (FTL).

Follow the Leader

- However, there are cases where FTL doesn't work.
- Consider action $\theta \in [-1, 1]$ and cost function $f^{(t)}(\theta) = (1/2)(-1)^t\theta$.
- In this case, the action goes back and forth between -1 and 1 except for the first, as $\theta^{(1)} = 0, \theta^{(2)} = -1, \theta^{(2)} = 1, ...$
- The cost function is $1/2$ except for the first, as $f^{(1)}(\theta^{(1)}) = 0, f^{(2)}(\theta^{(2)}) = 1/2, f^{(3)}(\theta^{(3)}) = 1/2, ...$

Follow the Leader

- On the other hand, The optimal strategy is $\theta = 0$ and the total cost of it is 0.
- Therefore, FTL regret in this case doesn't approach 0.
- We have to expand FTL.

Regularized Follow the Leader

- Regularized Follow the Leader (RFTL)

$$\boldsymbol{\theta}^{(t)} = \underset{\boldsymbol{\theta} \in K}{\arg\min}\, \eta \Sigma_{i=1}^{t-1} f^{(t)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

- Let $R(\boldsymbol{\theta})$ be convex regularization function. Let $\eta \geq 0$ be parameter that determines the degree of regularization.
- When choosing the first action, the cost function is not presented, so action is determined only by the regularization term.

$$\boldsymbol{\theta}^{(1)} = \underset{\boldsymbol{\theta} \in K}{\arg\min}\, R(\boldsymbol{\theta})$$

Regularized Follow the Leader

- We introduce lemma and definitions to derive RFTL regret.

lemma

For any vector $\boldsymbol{u} \in K$, the following holds.

$$\Sigma_{t=1}^{T} \boldsymbol{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq \Sigma_{t=1}^{T} \boldsymbol{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) + \frac{1}{\eta} \left\{ R(\boldsymbol{u}) - R(\boldsymbol{\theta}^{(1)}) \right\} \tag{1}$$

Regularized Follow the Leader

Proof.

For simplicity, let us assume that $\boldsymbol{f}^{(0)} = \frac{1}{\eta}R(\boldsymbol{\theta})$ and the algorithm starts at $t = 0$.

$$\Sigma_{t=0}^{T}\boldsymbol{f}^{(t)}(\boldsymbol{\theta}) = \Sigma_{t=1}^{T}\boldsymbol{f}^{(t)}(\boldsymbol{\theta}) + \frac{1}{\eta}R(\boldsymbol{\theta})$$

In this time, the lemma can be expressed as following.

$$\Sigma_{t=0}^{T}\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq \Sigma_{t=0}^{T}\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})$$

Regularized Follow the Leader

Proof.

At $t = 0$,

by definition, $\boldsymbol{\theta}^{(1)} = \arg\min_{\boldsymbol{\theta}} R(\boldsymbol{\theta})$ and $\boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(1)}) \leq \boldsymbol{f}^{(0)}(\boldsymbol{u})$ holds.

therefore,

$$\boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \boldsymbol{f}^{(0)}(\boldsymbol{u}) \leq \boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(1)})$$

□

Regularized Follow the Leader

Proof.

At $t > 0$,
assume that lemma holds for $t = T$.
In this time,

$$\boldsymbol{\theta}^{(T+2)} = \arg\min_{\boldsymbol{\theta}} \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}) \tag{2}$$

$$\boldsymbol{\theta}^{(T+1)} = \arg\min_{\boldsymbol{\theta}} \Sigma_{t=0}^{T} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}) \tag{3}$$

□

Regularized Follow the Leader

Proof.

Using equation (2) and (3),

$$
\begin{aligned}
&\Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \\
&\le \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+2)}) \\
&= \Sigma_{t=0}^{T}(\boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+2)})) + \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\
&\le \Sigma_{t=0}^{T}(\boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+1)})) + \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\
&= \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t+1)}) \\
&= \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})
\end{aligned}
$$

$\square$

Regularized Follow the Leader

definition (norm based on positive semi-definite matrix)

We define $\|\boldsymbol{x}\|_{\boldsymbol{A}} = \sqrt{\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}}$ *as the norm of vector* $\boldsymbol{x}$ *based on positve semi-definite matrix* $\boldsymbol{A}$*.*
We also define $\|\boldsymbol{x}\|_{\boldsymbol{A}^{-1}} = \|\boldsymbol{x}\|_{\boldsymbol{A}}^*$ *as a dual norm.*

- In this time, from generalized Cauchy-Schwarz inequality, the following holds.

$$\boldsymbol{x}^T \boldsymbol{y} \leq \|\boldsymbol{x}\|_{\boldsymbol{A}} \|\boldsymbol{y}\|_{\boldsymbol{A}}^*$$

Regularized Follow the Leader

definition (norm of cost function)

A norm of cost function measured by the regularization function is difined as

$$\lambda = \max_{t, \boldsymbol{\theta} \in K} \boldsymbol{f}^{(t)T} \{\nabla^2 R(\boldsymbol{\theta})\}^{-1} \boldsymbol{f}^{(t)}$$

Regularized Follow the Leader

definition (diameter of feasible area)

A diameter measured by the regularization function is defined as

$$D = \max_{\boldsymbol{\theta} \in K} R(\boldsymbol{\theta}) - R(\boldsymbol{\theta}^{(1)})$$

Regularized Follow the Leader

theorem (regret of RFTL)

RFTL achives the following regret for any vector $\boldsymbol{u} \in K$.

$$Regret(A) = \Sigma_{t=1}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq 2\sqrt{2\lambda DT}$$

Regularized Follow the Leader

Proof.

At first, we define Φ as following.

$$\Phi^{(t)}(\boldsymbol{\theta}) = \eta \Sigma_{i=1}^{t} f^{(i)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

By Taylor expansion of $\Phi^{(t)}$ around $\boldsymbol{\theta}^{(t+1)}$ and using intermediate value theorem, we can show following.

$$\begin{aligned}\Phi^{(t)}(\boldsymbol{\theta}^{(t)}) &= \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + (\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})^T \nabla \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) \\ &+ \frac{1}{2}\|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\boldsymbol{z}^{(t)}}^2 \\ &\geq \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + \frac{1}{2}\|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\boldsymbol{z}^{(t)}}^2\end{aligned}$$

$\square$

Regularized Follow the Leader

Proof.

Here, from intermediate value theorem, $\boldsymbol{z}^{(t)} \in [\boldsymbol{\theta}^{(t+1)}, \boldsymbol{\theta}^{(t)}]$
This inequality holds because $\boldsymbol{\theta}^{(t)}$ achieves the minimum value of $\Phi^{(t)}$.
By transforming the equation, we can get following.

$$
\begin{aligned}
\|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|^2_{\boldsymbol{z}^{(t)}} &\leq 2(\Phi^{(t)}(\boldsymbol{\theta}^{(t)}) - \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)})) \\
&= 2(\Phi^{(t-1)}(\boldsymbol{\theta}^{(t)}) - \Phi^{(t-1)}(\boldsymbol{\theta}^{(t+1)})) + 2\eta \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) \\
&\leq 2\eta \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})
\end{aligned}
$$

□

Regularized Follow the Leader

Proof.

Also, using generalized Cauchy-Schwarz inequality, the following holds.

$$
\begin{aligned}
\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) &\leq \|\boldsymbol{f}^{(t)}\|_{\boldsymbol{z}}^{*}\|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\boldsymbol{z}} \\
&\leq \|\boldsymbol{f}^{(t)}\|_{\boldsymbol{z}}^{*}\sqrt{2\eta\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})}
\end{aligned}
$$

furthermore,

$$
\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) \leq 2\eta\|\boldsymbol{f}^{(t)}\|_{\boldsymbol{z}}^{*2} \leq 2\eta\lambda \tag{4}
$$

□

Regularized Follow the Leader

Proof.

Using (1) and (4), adding up to T and letting η be $\eta = \sqrt{\frac{D}{2\lambda T}}$ which minimizes this equation, we can get

$$Regret(A) \leq min_{\eta} \left[2\eta\lambda T + \frac{1}{\eta}\{R(\boldsymbol{u}) - R(\boldsymbol{\theta}^{(t)})\} \right] \leq 2\sqrt{2D\lambda T}$$

$\square$

Regret with strictly convex loss function

- If the loss function is a strictly convex function, even smaller regret can be obtained by gradient descent.
- Gradient descent method updates the parameters based on the following update rule.

$$
\begin{aligned}
\boldsymbol{\theta}^{'(t)} &= \boldsymbol{\theta}^{(t-1)} - \eta_{t-1} \nabla \boldsymbol{f}^{(t-1)}(\boldsymbol{t}heta^{(t-1)}) \\
\boldsymbol{\theta}^{(t)} &= \arg\min_{\boldsymbol{\theta} \in K} \|\boldsymbol{\theta} - \boldsymbol{\theta}^{'(t)}\|_2
\end{aligned}
$$

- This algorithm updates the parameters based on the gradient of the current position and maps it to the nearest point on the feasible area.

Regret with strictly convex loss function

theorem (Regret with gradient descent for strictly convex loss functions)

Gradient descent method with a learning rate of $\eta^{(t)} = \frac{1}{\alpha t}$ achieves the following regret for any vector $\boldsymbol{u} \in K$.

$$Regret(A) = \Sigma_{t=1}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq \frac{G^2}{2\alpha}(1 + \log T)$$

where, G is an upper bound on the Euclidean norm of the gradient.