chapter 5 Performance Analysis

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Regret Analysis

- A player chooses an action $\theta^{(t)} \in K$ every t period, where K is a feasible set of actions.
- The cost function $f^{(t)}$ determines the cost $f^{(t)}(\theta^{(t)})$ for action $\theta^{(t)}$.
- The player decides his action based on the strategy.

Regret Analysis

- How does the player choose an action which minimizes a total cost $\Sigma f^{(t)}(\theta^{(t)})$?
- Can the cost function be minimized even if it is not unknown?
- We introduce a regret about the strategy.

definition (Regret)

The difference between the total cost of an action based on a strategy A and the total cost of the optimal strategy θ^* is defined as the regret $Regret(A)$ of strategy A.

$$Regret(A) = \Sigma_{t=1}^{T} f^{(t)}(\theta^{(t)}) - \Sigma_{t=1}^{T} f^{(t)}(\theta^*)$$

Regret Analysis

Regret analysis in online learning

- Let action be the parameter of the online learner $\boldsymbol{\theta}^{(t)} \in \mathbb{R}^m$ given the training data $(\boldsymbol{x}^{(t)}, y^{(t)})$.
- Let the cost function be a loss function $f^{(t)} = (\boldsymbol{x}^{(t)}, y^{(t)}, \boldsymbol{\theta})$.
- In this case, the optimal strategy is the strategy that chooses the action that minimizes the cost function for all training data.

Follow the Leader

- At first, We consider strategy for choosing action that minimizes the total cost to date.

$$\boldsymbol{\theta}^{(t)} = \arg\min_{\boldsymbol{\theta} \in K} \Sigma_{i=1}^{t-1} f^{(t)}(\boldsymbol{\theta})$$

- This strategy is called Follow the Leader (FTL).

Follow the Leader

- However, there are cases where FTL doesn't work.
- Consider action $\theta \in [-1, 1]$ and cost function $f^{(t)}(\theta) = (1/2)(-1)^t\theta$.
- In this case, the action goes back and forth between -1 and 1 except for the first, as $\theta^{(1)} = 0, \theta^{(2)} = -1, \theta^{(2)} = 1, ...$
- The cost function is $1/2$ except for the first, as $f^{(1)}(\theta^{(1)}) = 0, f^{(2)}(\theta^{(2)}) = 1/2, f^{(3)}(\theta^{(3)}) = 1/2, ...$

Follow the Leader

- On the other hand, The optimal strategy is $\theta = 0$ and the total cost of it is 0.
- Therefore, FTL regret in this case doesn't approach 0.
- We have to expand FTL.

Regularized Follow the Leader

- Regularized Follow the Leader (RFTL)

$$\boldsymbol{\theta}^{(t)} = \underset{\boldsymbol{\theta} \in K}{\arg\min}\, \eta \Sigma_{i=1}^{t-1} f^{(t)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

- Let $R(\boldsymbol{\theta})$ be convex regularization function. Let $\eta \geq 0$ be parameter that determines the degree of regularization.
- When choosing the first action, the cost function is not presented, so action is determined only by the regularization term.

$$\boldsymbol{\theta}^{(1)} = \underset{\boldsymbol{\theta} \in K}{\arg\min}\, R(\boldsymbol{\theta})$$

Regularized Follow the Leader

- We introduce lemma and definitions to derive RFTL regret.

lemma

For any vector $\boldsymbol{u} \in K$, the following holds.

$$\Sigma_{t=1}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq \Sigma_{t=1}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{t+1}) + \frac{1}{\eta}\left\{R(\boldsymbol{u}) - R(\boldsymbol{\theta}^{(1)})\right\} \tag{1}$$

Regularized Follow the Leader

Proof.

For simplicity, let us assume that $\boldsymbol{f}^{(0)} = \frac{1}{\eta}R(\boldsymbol{\theta})$ and the algorithm starts at $t = 0$.

$$\Sigma_{t=0}^{T}\boldsymbol{f}^{(t)}(\boldsymbol{\theta}) = \Sigma_{t=1}^{T}\boldsymbol{f}^{(t)}(\boldsymbol{\theta}) + \frac{1}{\eta}R(\boldsymbol{\theta})$$

In this time, the lemma can be expressed as following.

$$\Sigma_{t=0}^{T}\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq \Sigma_{t=0}^{T}\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})$$

Regularized Follow the Leader

Proof.

At $t = 0$,
by definition, $\boldsymbol{\theta}^{(1)} = \arg\min_{\boldsymbol{\theta}} R(\boldsymbol{\theta})$ and $\boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(1)}) \leq \boldsymbol{f}^{(0)}(\boldsymbol{u})$ holds.

therefore,

$$\boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \boldsymbol{f}^{(0)}(\boldsymbol{u}) \leq \boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(1)})$$

$\square$

Regularized Follow the Leader

Proof.

At $t > 0$,
assume that lemma holds for $t = T$.
In this time,

$$\boldsymbol{\theta}^{(T+2)} = \arg\min_{\boldsymbol{\theta}} \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}) \tag{2}$$

$$\boldsymbol{\theta}^{(T+1)} = \arg\min_{\boldsymbol{\theta}} \Sigma_{t=0}^{T} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}) \tag{3}$$

□

Regularized Follow the Leader

Proof.

Using equation (2) and (3),

$$
\begin{aligned}
&\Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \\
&\leq \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+2)}) \\
&= \Sigma_{t=0}^{T}(\boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+2)})) + \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\
&\leq \Sigma_{t=0}^{T}(\boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+1)})) + \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\
&= \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t+1)}) \\
&= \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})
\end{aligned}
$$

□

Regularized Follow the Leader

definition (norm based on positive semi-definite matrix)

We define $\|\boldsymbol{x}\|_{\boldsymbol{A}} = \sqrt{\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}}$ *as the norm of vector* $\boldsymbol{x}$ *based on positve semi-definite matrix* $\boldsymbol{A}$.
We also define $\|\boldsymbol{x}\|_{\boldsymbol{A}^{-1}} = \|\boldsymbol{x}\|_{\boldsymbol{A}}^{*}$ *as a dual norm.*

- In this time, from generalized Cauchy-Schwarz inequality, the following holds.

$$\boldsymbol{x}^T \boldsymbol{y} \leq \|\boldsymbol{x}\|_{\boldsymbol{A}} \|\boldsymbol{y}\|_{\boldsymbol{A}}^{*}$$

Regularized Follow the Leader

definition (norm of cost function)

A norm of cost function measured by the regularization function is difined as

$$\lambda = \max_{t,\boldsymbol{\theta} \in K} \boldsymbol{f}^{(t)T} \{\nabla^2 R(\boldsymbol{\theta})\}^{-1} \boldsymbol{f}^{(t)}$$

Regularized Follow the Leader

definition (diameter of feasible area)

A diameter measured by the regularization function is defined as

$$D = \max_{\boldsymbol{\theta} \in K} R(\boldsymbol{\theta}) - R(\boldsymbol{\theta}^{(1)})$$

Regularized Follow the Leader

theorem (regret of RFTL)

RFTL achives the following regret for any vector $\boldsymbol{u} \in K$.

$$Regret(A) = \Sigma_{t=1}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq 2\sqrt{2\lambda DT}$$

Regularized Follow the Leader

Proof.

At first, we define Φ as following.

$$\Phi^{(t)}(\boldsymbol{\theta}) = \eta \Sigma_{i=1}^{t} f^{(i)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

By Taylor expansion of $\Phi^{(t)}$ around $\boldsymbol{\theta}^{(t+1)}$ and using intermediate value theorem, we can show following.

$$\begin{aligned}
\Phi^{(t)}(\boldsymbol{\theta}^{(t)}) &= \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + (\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})^T \nabla \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) \\
&+ \frac{1}{2} \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\boldsymbol{z}^{(t)}}^2 \\
&\geq \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + \frac{1}{2} \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\boldsymbol{z}^{(t)}}^2
\end{aligned}$$

□

Regularized Follow the Leader

Proof.

Here, from intermediate value theorem, $\boldsymbol{z}^{(t)} \in [\boldsymbol{\theta}^{(t+1)}, \boldsymbol{\theta}^{(t)}]$
This inequality holds because $\boldsymbol{\theta}^{(t)}$ achieves the minimum value of $\Phi^{(t)}$.
By transforming the equation, we can get following.

$$
\begin{aligned}
\|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|^2_{\boldsymbol{z}^{(t)}} &\leq 2(\Phi^{(t)}(\boldsymbol{\theta}^{(t)}) - \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)})) \\
&= 2(\Phi^{(t-1)}(\boldsymbol{\theta}^{(t)}) - \Phi^{(t-1)}(\boldsymbol{\theta}^{(t+1)})) + 2\eta \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) \\
&\leq 2\eta \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})
\end{aligned}
$$

□

Regularized Follow the Leader

Proof.

Also, using generalized Cauchy-Schwarz inequality, the following holds.

$$\begin{aligned} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) &\leq \|\boldsymbol{f}^{(t)}\|_{\boldsymbol{z}}^{*}\|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\boldsymbol{z}} \\ &\leq \|\boldsymbol{f}^{(t)}\|_{\boldsymbol{z}}^{*}\sqrt{2\eta\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})} \end{aligned}$$

furthermore,

$$\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) \leq 2\eta\|\boldsymbol{f}^{(t)}\|_{\boldsymbol{z}}^{*2} \leq 2\eta\lambda \tag{4}$$

□

Regularized Follow the Leader

Proof.

Using (1) and (4), adding up to T and letting η be $\eta = \sqrt{\frac{D}{2\lambda T}}$ which minimizes this equation, we can get

$$Regret(A) \leq min_\eta \left[2\eta\lambda T + \frac{1}{\eta}\{R(\boldsymbol{u}) - R(\boldsymbol{\theta}^{(t)})\} \right] \leq 2\sqrt{2D\lambda T}$$

$\square$

Regret with strictly convex loss function

- If the loss function is a strictly convex function, even smaller regret can be obtained by gradient descent.
- Gradient descent method updates the parameters based on the following update rule.

$$
\begin{aligned}
\boldsymbol{\theta}^{'(t)} &= \boldsymbol{\theta}^{(t-1)} - \eta_{t-1} \nabla \boldsymbol{f}^{(t-1)}(\boldsymbol{t}heta^{(t-1)}) \\
\boldsymbol{\theta}^{(t)} &= \arg\min_{\boldsymbol{\theta} \in K} \|\boldsymbol{\theta} - \boldsymbol{\theta}^{'(t)}\|_2
\end{aligned}
$$

- This algorithm updates the parameters based on the gradient of the current position and maps it to the nearest point on the feasible area.