

## chapter 5 Performance Analysis

2021/6/27

# Regret Analysis

- A player chooses an action  $\theta^{(t)} \in K$  every  $t$  period, where  $K$  is a feasible set of actions.
- The cost function  $f^{(t)}$  determines the cost  $f^{(t)}(\theta^{(t)})$  for action  $\theta^{(t)}$ .
- The player decides his action based on the strategy.

# Regret Analysis

- How does the player choose an action which minimizes a total cost  $\sum f^{(t)}(\theta^{(t)})$ ?
- Can the cost function be minimized even if it is not unknown?
- We introduce a regret about the strategy.

## definition (Regret)

*The difference between the total cost of an action based on a strategy  $A$  and the total cost of the optimal strategy  $\theta^*$  is defined as the regret  $\text{Regret}(A)$  of strategy  $A$ .*

$$\text{Regret}(A) = \sum_{t=1}^T f^{(t)}(\theta^{(t)}) - \sum_{t=1}^T f^{(t)}(\theta^*)$$

# Regret Analysis

## Regret analysis in online learning

- Let action be the parameter of the online learner  $\theta^{(t)} \in \mathbb{R}^m$  given the training data  $(\mathbf{x}^{(t)}, y^{(t)})$ .
- Let the cost function be a loss function  $f^{(t)} = (\mathbf{x}^{(t)}, y^{(t)}, \theta)$ .
- In this case, the optimal strategy is the strategy that chooses the action that minimizes the cost function for all training data.

# Follow the Leader

- At first, We consider strategy for choosing action that minimizes the total cost to date.

$$\boldsymbol{\theta}^{(t)} = \arg \min_{\boldsymbol{\theta} \in K} \sum_{i=1}^{t-1} f^{(i)}(\boldsymbol{\theta})$$

- This strategy is called Follow the Leader (FTL).

# Follow the Leader

- However, there are cases where FTL doesn't work.
- Consider action  $\theta \in [-1, 1]$  and cost function  $f^{(t)}(\theta) = (1/2)(-1)^t\theta$ .
- In this case, the action goes back and forth between  $-1$  and  $1$  except for the first, as  $\theta^{(1)} = 0, \theta^{(2)} = -1, \theta^{(3)} = 1, \dots$
- The cost function is  $1/2$  except for the first, as  $f^{(1)}(\theta^{(1)}) = 0, f^{(2)}(\theta^{(2)}) = 1/2, f^{(3)}(\theta^{(3)}) = 1/2, \dots$

# Follow the Leader

- On the other hand, The optimal strategy is  $\theta = 0$  and the total cost of it is 0.
- Therefore, FTL regret in this case doesn't approach 0.
- We have to expand FTL.

# Regularized Follow the Leader

- Regularized Follow the Leader (RFTL)

$$\boldsymbol{\theta}^{(t)} = \arg \min_{\boldsymbol{\theta} \in K} \eta \sum_{i=1}^{t-1} f^{(i)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

- Let  $R(\boldsymbol{\theta})$  be convex regularization function. Let  $\eta \geq 0$  be parameter that determines the degree of regularization.
- When choosing the first action, the cost function is not presented, so action is determined only by the regularization term.

$$\boldsymbol{\theta}^{(1)} = \arg \min_{\boldsymbol{\theta} \in K} R(\boldsymbol{\theta})$$



# Regularized Follow the Leader

- We introduce lemma and definitions to derive RFTL regret.

## lemma

*For any vector  $\mathbf{u} \in K$ , the following holds.*

$$\sum_{t=1}^T \mathbf{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \mathbf{u}) \leq \sum_{t=1}^T \mathbf{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) + \frac{1}{\eta} \left\{ R(\mathbf{u}) - R(\boldsymbol{\theta}^{(1)}) \right\} \quad (1)$$

# Regularized Follow the Leader

## Proof.

For simplicity, let us assume that  $\mathbf{f}^{(0)} = \frac{1}{\eta}R(\boldsymbol{\theta})$  and the algorithm starts at  $t = 0$ .

$$\sum_{t=0}^T \mathbf{f}^{(t)}(\boldsymbol{\theta}) = \sum_{t=1}^T \mathbf{f}^{(t)}(\boldsymbol{\theta}) + \frac{1}{\eta}R(\boldsymbol{\theta})$$

In this time, the lemma can be expressed as following.

$$\sum_{t=0}^T \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \mathbf{u}) \leq \sum_{t=0}^T \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})$$

# Regularized Follow the Leader

Proof.

At  $t = 0$ ,  
by definition,  $\boldsymbol{\theta}^{(1)} = \arg \min_{\boldsymbol{\theta}} R(\boldsymbol{\theta})$  and  $\mathbf{f}^{(0)}(\boldsymbol{\theta}^{(1)}) \leq \mathbf{f}^{(0)}(\mathbf{u})$  holds.  
therefore,

$$\mathbf{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \mathbf{f}^{(0)}(\mathbf{u}) \leq \mathbf{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \mathbf{f}^{(0)}(\boldsymbol{\theta}^{(1)})$$



# Regularized Follow the Leader

Proof.

At  $t > 0$ ,  
assume that lemma holds for  $t = T$ .  
In this time,

$$\theta^{(T+2)} = \arg \min_{\theta} \sum_{t=0}^{T+1} f^{(t)}(\theta) \quad (2)$$

$$\theta^{(T+1)} = \arg \min_{\theta} \sum_{t=0}^T f^{(t)}(\theta) \quad (3)$$



# Regularized Follow the Leader

## Proof.

Using equation (2) and (3),

$$\begin{aligned}& \sum_{t=0}^{T+1} \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \mathbf{u}) \\& \leq \sum_{t=0}^{T+1} \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \sum_{t=0}^{T+1} \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(T+2)}) \\& = \sum_{t=0}^T (\mathbf{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(T+2)})) + \mathbf{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \mathbf{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\& \leq \sum_{t=0}^T (\mathbf{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(T+1)})) + \mathbf{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \mathbf{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\& \quad - \sum_{t=0}^{T+1} \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(t)}) + \mathbf{f}^{(t)}(\boldsymbol{\theta}^{(t+1)})\end{aligned}$$

# Regularized Follow the Leader

definition (norm based on positive semi-definite matrix)

We define  $\|\mathbf{x}\|_A = \sqrt{\mathbf{x}^T \mathbf{A} \mathbf{x}}$  as the norm of vector  $\mathbf{x}$  based on positive semi-definite matrix  $\mathbf{A}$ .

We also define  $\|\mathbf{x}\|_{A^{-1}} = \|\mathbf{x}\|_A^*$  as a dual norm.

- In this time, from generalized Cauchy-Schwarz inequality, the following holds.

$$\mathbf{x}^T \mathbf{y} \leq \|\mathbf{x}\|_A \|\mathbf{y}\|_A^*$$

# Regularized Follow the Leader

definition (norm of cost function)

*A norm of cost function measured by the regularization function is defined as*

$$\lambda = \max_{t, \boldsymbol{\theta} \in K} \mathbf{f}^{(t)T} \{\nabla^2 R(\boldsymbol{\theta})\}^{-1} \mathbf{f}^{(t)}$$

# Regularized Follow the Leader

definition (diameter of feasible area)

*A diameter measured by the regularization function is defined as*

$$D = \max_{\boldsymbol{\theta} \in K} R(\boldsymbol{\theta}) - R(\boldsymbol{\theta}^{(1)})$$



# Regularized Follow the Leader

theorem (regret of RFTL)

*RFTL achieves the following regret for any vector  $\mathbf{u} \in K$ .*

$$\text{Regret}(A) = \sum_{t=1}^T \mathbf{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \mathbf{u}) \leq 2\sqrt{2\lambda DT}$$

# Regularized Follow the Leader

## Proof.

At first, we define  $\Phi$  as following.

$$\Phi^{(t)}(\boldsymbol{\theta}) = \eta \sum_{i=1}^t f^{(i)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

By Taylor expansion of  $\Phi^{(t)}$  around  $\boldsymbol{\theta}^{(t+1)}$  and using intermediate value theorem, we can show following.

$$\begin{aligned}\Phi^{(t)}(\boldsymbol{\theta}^{(t)}) &= \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + (\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})^T \nabla \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) \\ &\quad + \frac{1}{2} \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\mathbf{z}^{(t)}}^2 \\ &\geq \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + \frac{1}{2} \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\mathbf{z}^{(t)}}^2\end{aligned}$$

# Regularized Follow the Leader

Proof.

Here, from intermediate value theorem,  $\mathbf{z}^{(t)} \in [\boldsymbol{\theta}^{(t+1)}, \boldsymbol{\theta}^{(t)}]$

This inequality holds because  $\boldsymbol{\theta}^{(t)}$  achieves the minimum value of  $\Phi^{(t)}$ .

By transforming the equation, we can get following.

$$\begin{aligned}\|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\mathbf{z}^{(t)}}^2 &\leq 2(\Phi^{(t)}(\boldsymbol{\theta}^{(t)}) - \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)})) \\ &= 2(\Phi^{(t-1)}(\boldsymbol{\theta}^{(t)}) - \Phi^{(t-1)}(\boldsymbol{\theta}^{(t+1)})) + 2\eta \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) \\ &\leq 2\eta \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})\end{aligned}$$



# Regularized Follow the Leader

Proof.

Also, using generalized Cauchy-Schwarz inequality, the following holds.

$$\begin{aligned} \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) &\leq \|\mathbf{f}^{(t)}\|_z^* \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_z \\ &\leq \|\mathbf{f}^{(t)}\|_z^* \sqrt{2\eta \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})} \end{aligned}$$

furthermore,

$$\mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) \leq 2\eta \|\mathbf{f}^{(t)}\|_z^{*2} \leq 2\eta\lambda \quad (4)$$



# Regularized Follow the Leader

## Proof.

Using (1) and (4), adding up to  $T$  and letting  $\eta$  be  $\eta = \sqrt{\frac{D}{2\lambda T}}$  which minimizes this equation, we can get

$$\text{Regret}(A) \leq \min_{\eta} \left[ 2\eta\lambda T + \frac{1}{\eta} \{R(\mathbf{u}) - R(\boldsymbol{\theta}^{(t)})\} \right] \leq 2\sqrt{2D\lambda T}$$



# Regret with strong convex loss function

- a player chooses a point (action) from a set in Euclidean space denoted  $K \subseteq \mathbb{R}^n$ .
- We assume that the set  $K$  is non-empty, bounded, closed and convex.
- We denote the number of iterations by  $T$  which is unknown to the player.

# Regret with strong convex loss function

- At iteration  $t$ , the player chooses  $\theta_t \in K$ .
- After committing to this choice, a convex cost function  $f_t : K \mapsto \mathbb{R}$  is revealed.

# Regret with strong convex loss function

- Consider a player using an algorithm for online game playing (strategy)  $A$ .
- At iteration  $t$ , the algorithm takes as input the history of cost function  $f_1, \dots, f_{t-1}$  and produces a feasible point  $A(\{f_1, \dots, f_{t-1}\})$  in the domain  $K$ .
- When there is no ambiguity, we simply denote  $\theta_t = A(\{f_1, \dots, f_{t-1}\})$ .



## Regret with strong convex loss function

- The regret of the online player using algorithm  $A$  at time  $T$  is defined to be the total cost minus the cost of the best single decision, where the best is chosen with the benefit of hindsight.

$$\text{Regret}(A, \{f_1, \dots, f_T\}) = E \left\{ \sum_{t=1}^T f_t(\boldsymbol{\theta}_t) \right\} - \min_{\boldsymbol{\theta} \in K} \sum_{t=1}^T f_t(\boldsymbol{\theta}).$$

- We are usually interested in an upper bound on the worst case guaranteed regret, denoted

$$\text{Regret}_T(A) = \sup_{\{f_1, \dots, f_T\}} \{ \text{Regret}(A, \{f_1, \dots, f_T\}) \}.$$

# Regret with strong convex loss function

## definition

*We say that the cost functions have gradients upper bounded by a number  $G$  if the following holds:*

$$\sup_{\boldsymbol{\theta} \in K, t \in \{1, \dots, T\}} \|\nabla f_t(\boldsymbol{\theta})\|_2 \leq G.$$

# Regret with strong convex loss function

## definition

*We say that the Hessian of all cost functions is lower bounded by a number  $\alpha > 0$ , if the following holds:*

$$\forall \boldsymbol{\theta} \in K, t \in \{f_1, \dots, f_T\} : \nabla^2 f_t(\boldsymbol{\theta}) \succeq \alpha \mathbf{I}_n.$$

*$\alpha$  is a lower bound on eigenvalues of all the Hessians of the constraints at all points in the domain. Such function is called  $\alpha$ -strong convex.*

# Regret with strong convex loss function

## algorithm (Online Gradient Descent)

*Inputs: convex set  $K \subset \mathbb{R}^n$ , step sizes  $\eta_1, \eta_2, \dots \geq 0$ , initial  $\boldsymbol{\theta}_1 \in K$ .*

*- In iteration 1, use  $\boldsymbol{\theta}_1 \in K$ .*

*- In iteration  $t > 1$ , use*

$$\boldsymbol{\theta}_t = \pi_K(\boldsymbol{\theta}_{t-1} - \eta \nabla f_{t-1}(\boldsymbol{\theta}_{t-1})).$$

*Here,  $\pi_K$  denotes the projection onto nearest point in  $K$ ,*

$$\pi_K(\mathbf{y}) = \arg \min_{\boldsymbol{\theta} \in K} \|\boldsymbol{\theta} - \mathbf{y}\|_2.$$

# Regret with strong convex loss function

## definition

*For a given positive semi-definite matrix  $\mathbf{A}$ , a generalized projection of  $\mathbf{y} \in \mathbb{R}^n$  onto the convex set  $K$  is defined as*

$$\pi_K^{\mathbf{A}}(\mathbf{y}) = \arg \min_{\boldsymbol{\theta} \in K} (\boldsymbol{\theta} - \mathbf{y})^\top \mathbf{A} (\boldsymbol{\theta} - \mathbf{y}).$$

*Thus, the Euclidean projection can be seen to be a generalized projection with  $\mathbf{A} = \mathbf{I}_n$ .*

## Regret with strong convex function

### lemma

*Let  $K \subseteq \mathbb{R}^n$  be a convex set,  $\mathbf{y} \in \mathbb{R}^n$  and  $\mathbf{z} = \pi_K^{\mathbf{A}}(\mathbf{y})$  be the generalized projection of  $\mathbf{y}$  onto  $K$  according to positive semi-definite matrix  $\mathbf{A}$ . Then for any point  $\mathbf{a} \in K$  it holds that*

$$(\mathbf{y} - \mathbf{a})^\top \mathbf{A}(\mathbf{y} - \mathbf{a}) \geq (\mathbf{z} - \mathbf{a})^\top \mathbf{A}(\mathbf{z} - \mathbf{a}).$$

## Regret with strong convex loss function

### Proof.

By definition of generalized projections, the point  $\mathbf{z}$  minimizes the function  $f(\boldsymbol{\theta}) = (\boldsymbol{\theta} - \mathbf{y})^\top \mathbf{A}(\boldsymbol{\theta} - \mathbf{y})$  over the convex set. It is a well known fact in optimization that for the optimum  $\mathbf{z}$  the following holds.

$$\forall \mathbf{a} \in K : \nabla f(\mathbf{z})^\top (\mathbf{a} - \mathbf{z}) \geq 0$$

which implies

$$2(\mathbf{z} - \mathbf{y})^\top \mathbf{A}(\mathbf{a} - \mathbf{z}) \geq 0 \Rightarrow 2\mathbf{a}^\top \mathbf{A}(\mathbf{z} - \mathbf{y}) \geq 2\mathbf{z}^\top \mathbf{A}(\mathbf{z} - \mathbf{y}).$$



# Regret with strong convex loss function

Proof.

Now by simple calculation:

$$\begin{aligned}(\mathbf{y} - \mathbf{a})^\top \mathbf{A}(\mathbf{y} - \mathbf{a}) - (\mathbf{z} - \mathbf{a})^\top \mathbf{A}(\mathbf{z} - \mathbf{a}) &= \mathbf{y}^\top \mathbf{A} \mathbf{y} - \mathbf{z}^\top \mathbf{A} \mathbf{z} + 2\mathbf{a}^\top \mathbf{A}(\mathbf{z} - \mathbf{y}) \\&\geq \mathbf{y}^\top \mathbf{A} \mathbf{y} - \mathbf{z}^\top \mathbf{A} \mathbf{z} + 2\mathbf{z}^\top \mathbf{A}(\mathbf{z} - \mathbf{y}) \\&= \mathbf{y}^\top \mathbf{A} \mathbf{y} - 2\mathbf{z}^\top \mathbf{A} \mathbf{y} + \mathbf{z}^\top \mathbf{A} \mathbf{z} \\&= (\mathbf{z} - \mathbf{y})^\top \mathbf{A}(\mathbf{z} - \mathbf{y}) \geq 0.\end{aligned}$$





# Regret with strong convex loss function

## theorem

*Online Gradient Descent with step sizes  $\eta_t = \frac{1}{\alpha_t}$  achieves the following guarantee, for all  $T \geq 1$*

$$\text{Regret}_T(\text{OGD}) \leq \frac{G^2}{2\alpha}(1 + \log T).$$

## Regret with strong convex loss function

Proof.

Let  $\theta^* = \arg \min \sum_{t=1}^T f_t(\theta)$ . Recall the definition of regret

$$\text{Regret}_T(\text{OGD}) = \sum_{t=1}^T f_t(\theta_t) - \sum_{t=1}^T f_t(\theta^*).$$

Define  $\nabla_t = \nabla f_t(\theta_t)$ . By using the Taylor series approximation, we have, for some point  $\xi_t$  on the line segment joining  $\theta_t$  to  $\theta^*$ ,

$$\begin{aligned} f_t(\theta^*) &= f_t(\theta_t) + \nabla_t^\top (\theta^* - \theta_t) + \frac{1}{2} (\theta^* - \theta_t)^\top \nabla^2 f_t(\xi_t) (\theta^* - \theta_t) \\ &\geq f_t(\theta_t) + \nabla_t^\top (\theta^* - \theta_t) + \frac{\alpha}{2} \|\theta^* - \theta_t\|^2. \end{aligned}$$



## Regret with strong convex loss function

Proof.

The inequality follows from  $\alpha$ -strong convexity. Thus, we have

$$2\{f_t(\boldsymbol{\theta}_t - \boldsymbol{\theta}^*)\} \leq 2\nabla^\top(\boldsymbol{\theta}_t - \boldsymbol{\theta}^*) - \alpha\|\boldsymbol{\theta}^* - \boldsymbol{\theta}_t\|^2. \quad (5)$$

Using the update rule for  $\boldsymbol{\theta}_{t+1}$  and lemma, we get

$$\begin{aligned} \|\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}^*\|^2 &= \|\pi(\boldsymbol{\theta}_t - \eta_{t+1} \nabla_t) - \boldsymbol{\theta}^*\|^2 \\ &\leq \|\boldsymbol{\theta}_t - \eta_{t+1} \nabla_t - \boldsymbol{\theta}^*\|^2. \end{aligned}$$



# Regret with strong convex loss function

Proof.

Hence,

$$\begin{aligned} \|\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}^*\|^2 &\leq \|\boldsymbol{\theta}_t - \boldsymbol{\theta}^*\|^2 + \eta_{t+1}^2 \|\nabla_t\|^2 - 2\eta_{t+1} \nabla_t^\top (\boldsymbol{\theta}_t - \boldsymbol{\theta}^*), \\ 2\nabla_t^\top (\boldsymbol{\theta}_t - \boldsymbol{\theta}^*) &\leq \end{aligned} \tag{6}$$

