# chapter 5 Performance Analysis

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# Regret Analysis

- A player chooses an action $\theta^{(t)} \in K$ every $t$ period, where $K$ is a feasible set of actions.
- The cost function $f^{(t)}$ determines the cost $f^{(t)}(\theta^{(t)})$ for action $\theta^{(t)}$.
- The player decides his action based on the strategy.

# Regret Analysis

- How does the player choose an action which minimizes a total cost $\Sigma f^{(t)}(\theta^{(t)})$?
- Can the cost function be minimized even if it is not unknown?
- We introduce a regret about the strategy.

**definition (Regret)**

*The difference between the total cost of an action based on a strategy $A$ and the total cost of the optimal strategy $\theta^*$ is defined as the regret $Regret(A)$ of strategy $A$.*

$$Regret(A) = \Sigma_{t=1}^{T} f^{(t)}(\theta^{(t)}) - \Sigma_{t=1}^{T} f^{(t)}(\theta^*)$$

# Regret Analysis

Regret analysis in online learning

- Let action be the parameter of the online learner $\boldsymbol{\theta}^{(t)} \in \mathbb{R}^m$ given the training data $(\boldsymbol{x}^{(t)}, y^{(t)})$.
- Let the cost function be a loss function $f^{(t)} = (\boldsymbol{x}^{(t)}, y^{(t)}, \boldsymbol{\theta})$.
- In this case, the optimal strategy is the strategy that chooses the action that minimizes the cost function for all training data.

# Follow the Leader

- At first, We consider strategy for choosing action that minimizes the total cost to date.

$$\boldsymbol{\theta}^{(t)} = \underset{\boldsymbol{\theta} \in K}{\arg\min} \, \Sigma_{i=1}^{t-1} f^{(t)}(\boldsymbol{\theta})$$

- This strategy is called Follow the Leader (FTL).

# Follow the Leader

- However, there are cases where FTL doesn't work.
- Consider action $\theta \in [-1, 1]$ and cost function $f^{(t)}(\theta) = (1/2)(-1)^t\theta$.
- In this case, the action goes back and forth between $-1$ and $1$ except for the first, as $\theta^{(1)} = 0, \theta^{(2)} = -1, \theta^{(2)} = 1, ....$
- The cost function is $1/2$ except for the first, as $f^{(1)}(\theta^{(1)}) = 0, f^{(2)}(\theta^{(2)}) = 1/2, f^{(3)}(\theta^{(3)}) = 1/2, ....$

# Follow the Leader

- On the other hand, The optimal strategy is $\theta = 0$ and the total cost of it is $0$.
- Therefore, FTL regret in this case doesn't approach $0$.
- We have to expand FTL.

# Regularized Follow the Leader

- Regularized Follow the Leader (RFTL)

$$\boldsymbol{\theta}^{(t)} = \arg\min_{\boldsymbol{\theta}\in K} \eta \Sigma_{i=1}^{t-1} f^{(t)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

- Let $R(\boldsymbol{\theta})$ be convex regularization function. Let $\eta \geq 0$ be parameter that determines the degree of regularization.
- When choosing the first action, the cost function is not presented, so action is determined only by the regularization term.

$$\boldsymbol{\theta}^{(1)} = \arg\min_{\boldsymbol{\theta}\in K} R(\boldsymbol{\theta})$$

# Regularized Follow the Leader

- We introduce lemma and definitions to derive RFTL regret.

**lemma**

*For any vector $\boldsymbol{u} \in K$, the following holds.*

$$\Sigma_{t=1}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq \Sigma_{t=1}^{T} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{t+1}) + \frac{1}{\eta}\left\{R(\boldsymbol{u}) - R(\boldsymbol{\theta}^{(1)})\right\} \tag{1}$$

# Regularized Follow the Leader

**Proof.**

For simplicity, let us assume that $\boldsymbol{f}^{(0)} = \frac{1}{\eta}R(\boldsymbol{\theta})$ and the algorithm starts at $t = 0$.

$$\Sigma_{t=0}^{T}\boldsymbol{f}^{(t)}(\boldsymbol{\theta}) = \Sigma_{t=1}^{T}\boldsymbol{f}^{(t)}(\boldsymbol{\theta}) + \frac{1}{\eta}R(\boldsymbol{\theta})$$

In this time, the lemma can be expressed as following.

$$\Sigma_{t=0}^{T}\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \leq \Sigma_{t=0}^{T}\boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})$$

# Regularized Follow the Leader

**Proof.**

At $t = 0$,

by definition, $\boldsymbol{\theta}^{(1)} = \arg\min_{\boldsymbol{\theta}} R(\boldsymbol{\theta})$ and $\boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(1)}) \leq \boldsymbol{f}^{(0)}(\boldsymbol{u})$ holds.

therefore,

$$\boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \boldsymbol{f}^{(0)}(\boldsymbol{u}) \leq \boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \boldsymbol{f}^{(0)}(\boldsymbol{\theta}^{(1)})$$

$\square$

## Regularized Follow the Leader

**Proof.**

At $t > 0$,
assume that lemma holds for $t = T$.
In this time,

$$\boldsymbol{\theta}^{(T+2)} = \arg\min_{\boldsymbol{\theta}} \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}) \tag{2}$$

$$\boldsymbol{\theta}^{(T+1)} = \arg\min_{\boldsymbol{\theta}} \Sigma_{t=0}^{T} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}) \tag{3}$$

□

# Regularized Follow the Leader

**Proof.**

Using equation $(2)$ and $(3)$,

$$
\begin{aligned}
&\Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{u}) \\
&\leq \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+2)}) \\
&= \Sigma_{t=0}^{T}(\boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+2)})) + \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\
&\leq \Sigma_{t=0}^{T}(\boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(T+1)})) + \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+1)}) - \boldsymbol{f}^{(T+1)}(\boldsymbol{\theta}^{(T+2)}) \\
&= \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)}) - \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t+1)}) \\
&= \Sigma_{t=0}^{T+1} \boldsymbol{f}^{(t)}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})
\end{aligned}
$$

□