

## chapter 5 Performance Analysis

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# Regret Analysis

- A player chooses an action  $\theta^{(t)} \in K$  every  $t$  period, where  $K$  is a feasible set of actions.
- The cost function  $f^{(t)}$  determines the cost  $f^{(t)}(\theta^{(t)})$  for action  $\theta^{(t)}$ .
- The player decides his action based on the strategy.

# Regret Analysis

- How does the player choose an action which minimizes a total cost  $\sum f^{(t)}(\theta^{(t)})$ ?
- Can the cost function be minimized even if it is not unknown?
- We introduce a regret about the strategy.

## definition (Regret)

*The difference between the total cost of an action based on a strategy  $A$  and the total cost of the optimal strategy  $\theta^*$  is defined as the regret  $\text{Regret}(A)$  of strategy  $A$ .*

$$\text{Regret}(A) = \sum_{t=1}^T f^{(t)}(\theta^{(t)}) - \sum_{t=1}^T f^{(t)}(\theta^*)$$

# Regret Analysis

## Regret analysis in online learning

- Let action be the parameter of the online learner  $\boldsymbol{\theta}^{(t)} \in \mathbb{R}^m$  given the training data  $(\boldsymbol{x}^{(t)}, y^{(t)})$ .
- Let the cost function be a loss function  $f^{(t)} = (\boldsymbol{x}^{(t)}, y^{(t)}, \boldsymbol{\theta})$ .
- In this case, the optimal strategy is the strategy that chooses the action that minimizes the cost function for all training data.

# Follow the Leader

- At first, We consider strategy for choosing action that minimizes the total cost to date.

$$\boldsymbol{\theta}^{(t)} = \arg \min_{\boldsymbol{\theta} \in K} \sum_{i=1}^{t-1} f^{(i)}(\boldsymbol{\theta})$$

- This strategy is called Follow the Leader (FTL).

# Follow the Leader

- However, there are cases where FTL doesn't work.
- Consider action  $\theta \in [-1, 1]$  and cost function  $f^{(t)}(\theta) = (1/2)(-1)^t\theta$ .
- In this case, the action goes back and forth between  $-1$  and  $1$  except for the first, as  $\theta^{(1)} = 0, \theta^{(2)} = -1, \theta^{(3)} = 1, \dots$
- The cost function is  $1/2$  except for the first, as  $f^{(1)}(\theta^{(1)}) = 0, f^{(2)}(\theta^{(2)}) = 1/2, f^{(3)}(\theta^{(3)}) = 1/2, \dots$

# Follow the Leader

- On the other hand, The optimal strategy is  $\theta = 0$  and the total cost of it is 0.
- Therefore, FTL regret in this case doesn't approach 0.
- We have to expand FTL.

# Regularized Follow the Leader

- Regularized Follow the Leader (RFTL)

$$\boldsymbol{\theta}^{(t)} = \arg \min_{\boldsymbol{\theta} \in K} \eta \sum_{i=1}^{t-1} f^{(i)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

- Let  $R(\boldsymbol{\theta})$  be convex regularization function. Let  $\eta \geq 0$  be parameter that determines the degree of regularization.
- When choosing the first action, the cost function is not presented, so action is determined only by the regularization term.

$$\boldsymbol{\theta}^{(1)} = \arg \min_{\boldsymbol{\theta} \in K} R(\boldsymbol{\theta})$$



# Regularized Follow the Leader

- We introduce lemma and definitions to derive RFTL regret.

## lemma

*For any vector  $\mathbf{u} \in K$ , the following holds.*

$$\sum_{t=1}^T \mathbf{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \mathbf{u}) \leq \sum_{t=1}^T \mathbf{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{t+1}) + \frac{1}{\eta} \left\{ R(\mathbf{u}) - R(\boldsymbol{\theta}^{(1)}) \right\} \quad (1)$$

# Regularized Follow the Leader

Proof.

For simplicity, let us assume that  $\mathbf{f}^{(0)} = \frac{1}{\eta}R(\boldsymbol{\theta})$  and the algorithm starts at  $t = 0$ .

$$\sum_{t=0}^T \mathbf{f}^{(t)}(\boldsymbol{\theta}) = \sum_{t=1}^T \mathbf{f}^{(t)}(\boldsymbol{\theta}) + \frac{1}{\eta}R(\boldsymbol{\theta})$$

In this time, the lemma can be expressed as following.

$$\sum_{t=0}^T \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \mathbf{u}) \leq \sum_{t=0}^T \mathbf{f}^{(t)T}(\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})$$

# Regularized Follow the Leader

Proof.

At  $t = 0$ ,  
by definition,  $\boldsymbol{\theta}^{(1)} = \arg \min_{\boldsymbol{\theta}} R(\boldsymbol{\theta})$  and  $\mathbf{f}^{(0)}(\boldsymbol{\theta}^{(1)}) \leq \mathbf{f}^{(0)}(\mathbf{u})$  holds.  
therefore,

$$\mathbf{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \mathbf{f}^{(0)}(\mathbf{u}) \leq \mathbf{f}^{(0)}(\boldsymbol{\theta}^{(0)}) - \mathbf{f}^{(0)}(\boldsymbol{\theta}^{(1)})$$



# Regularized Follow the Leader

## Proof.

At  $t > 0$ ,  
assume that lemma holds for  $t = T$ .  
In this time,

$$\boldsymbol{\theta}^{(T+2)} = \arg \min_{\boldsymbol{\theta}} \sum_{t=0}^{T+1} \mathbf{f}^{(t)}(\boldsymbol{\theta}) \quad (2)$$

$$\boldsymbol{\theta}^{(T+1)} = \arg \min_{\boldsymbol{\theta}} \sum_{t=0}^T \mathbf{f}^{(t)}(\boldsymbol{\theta}) \quad (3)$$



# Regularized Follow the Leader

Proof.

Using equation (2) and (3),

$$\begin{aligned} & \sum_{t=0}^{T+1} \mathbf{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \mathbf{u}) \\ & \leq \sum_{t=0}^{T+1} \mathbf{f}^{(t)} (\boldsymbol{\theta}^{(t)}) - \sum_{t=0}^{T+1} \mathbf{f}^{(t)} (\boldsymbol{\theta}^{(T+2)}) \\ & = \sum_{t=0}^T (\mathbf{f}^{(t)} (\boldsymbol{\theta}^{(t)}) - \mathbf{f}^{(t)} (\boldsymbol{\theta}^{(T+2)})) + \mathbf{f}^{(T+1)} (\boldsymbol{\theta}^{(T+1)}) - \mathbf{f}^{(T+1)} (\boldsymbol{\theta}^{(T+2)}) \\ & \leq \sum_{t=0}^T (\mathbf{f}^{(t)} (\boldsymbol{\theta}^{(t)}) - \mathbf{f}^{(t)} (\boldsymbol{\theta}^{(T+1)})) + \mathbf{f}^{(T+1)} (\boldsymbol{\theta}^{(T+1)}) - \mathbf{f}^{(T+1)} (\boldsymbol{\theta}^{(T+2)}) \\ & = \sum_{t=0}^{T+1} \mathbf{f}^{(t)} (\boldsymbol{\theta}^{(t)}) - \mathbf{f}^{(t)} (\boldsymbol{\theta}^{(t+1)}) \\ & = \sum_{t=0}^{T+1} \mathbf{f}^{(t)} (\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}) \end{aligned}$$



# Regularized Follow the Leader

definition (norm based on positive semi-definite matrix)

We define  $\|\mathbf{x}\|_A = \sqrt{\mathbf{x}^T \mathbf{A} \mathbf{x}}$  as the norm of vector  $\mathbf{x}$  based on positive semi-definite matrix  $\mathbf{A}$ .

We also define  $\|\mathbf{x}\|_{A^{-1}} = \|\mathbf{x}\|_A^*$  as a dual norm.

- In this time, from generalized Cauchy-Schwarz inequality, the following holds.

$$\mathbf{x}^T \mathbf{y} \leq \|\mathbf{x}\|_A \|\mathbf{y}\|_A^*$$

# Regularized Follow the Leader

definition (norm of cost function)

*A norm of cost function measured by the regularization function is defined as*

$$\lambda = \max_{t, \boldsymbol{\theta} \in K} \mathbf{f}^{(t)T} \{\nabla^2 R(\boldsymbol{\theta})\}^{-1} \mathbf{f}^{(t)}$$

# Regularized Follow the Leader

definition (diameter of feasible area)

*A diameter measured by the regularization function is defined as*

$$D = \max_{\boldsymbol{\theta} \in K} R(\boldsymbol{\theta}) - R(\boldsymbol{\theta}^{(1)})$$



# Regularized Follow the Leader

theorem (regret of RFTL)

*RFTL achieves the following regret for any vector  $\mathbf{u} \in K$ .*

$$\text{Regret}(A) = \sum_{t=1}^T \mathbf{f}^{(t)T} (\boldsymbol{\theta}^{(t)} - \mathbf{u}) \leq 2\sqrt{2\lambda DT}$$

# Regularized Follow the Leader

Proof.

At first, we define  $\Phi$  as following.

$$\Phi^{(t)}(\boldsymbol{\theta}) = \eta \sum_{i=1}^t f^{(i)}(\boldsymbol{\theta}) + R(\boldsymbol{\theta})$$

By Taylor expansion of  $\Phi^{(t)}$  around  $\boldsymbol{\theta}^{(t+1)}$  and using intermediate value theorem, we can show following.

$$\begin{aligned}\Phi^{(t)}(\boldsymbol{\theta}^{(t)}) &= \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + (\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)})^T \nabla \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) \\ &\quad + \frac{1}{2} \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\mathbf{z}^{(t)}}^2 \\ &\geq \Phi^{(t)}(\boldsymbol{\theta}^{(t+1)}) + \frac{1}{2} \|\boldsymbol{\theta}^{(t)} - \boldsymbol{\theta}^{(t+1)}\|_{\mathbf{z}^{(t)}}^2\end{aligned}$$



# Regularized Follow the Leader

Proof.

Here, from intermediate value theorem,  $\mathbf{z}^{(t)} \in [\boldsymbol{\theta}^{(t+1)}, \boldsymbol{\theta}^{(t)}]$

This inequality holds because  $\boldsymbol{\theta}^{(t)}$  achieves the minimum value of  $\Phi^{(t)}$ .  $\square$