

# MATHEMATICAL FINANCE IN DISCRETE TIME

ELIAS MINDLBERGER

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## 1. INTRODUCTION AND FIRST EXAMPLE

We start by introducing the first important notions of option pricing and first examples.

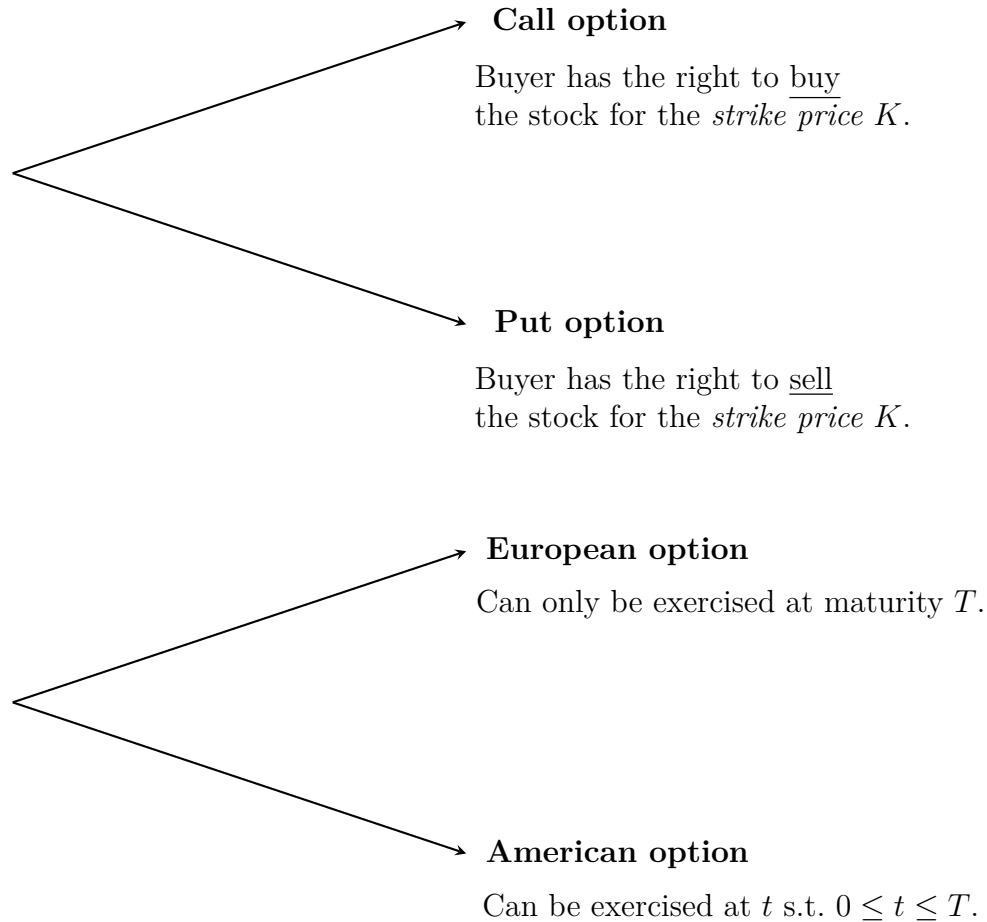
**1.1. Notions for option pricing theory.** An **option** is a contract between two parties (a buyer and a seller). The buyer pays an option price *today* (i.e. at  $t = 0$ ) to the seller and in return obtains the right/*option* but not the obligation to buy a stock for conditions that are fixed *today* at a fixed point in time (*the maturity of the option*). The option to buy is thereby crucial: the buyer can just refrain from buying the stock if the market conditions are disadvantageous.

We only consider options whose *underlyings* (german: *Basistitel*) are stocks.

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We distinguish the following:



We call the above **standard** or **vanilla** options. Other options are called **exotic**. Such include *Asian options* (which operate on the mean-value of an asset), *barrier options*, *options on volatility* and so forth. For this course we assume that options have a price at any point in time  $0 \leq t \leq T$ . Advantages of options include

- The loss is limited: worst case is the loss of the option price,
- leverage effect (in a positive market development),
- can be used to *hedge* against decreasing stock prices via put options (can be sold for a fixed pre-determined price).

**1.2. A first example.** Let us consider a call option in a two-period market model. We set

- $t = 0$ : the current time or today,
- $T > 0$ : the maturity or exercise date of the option,

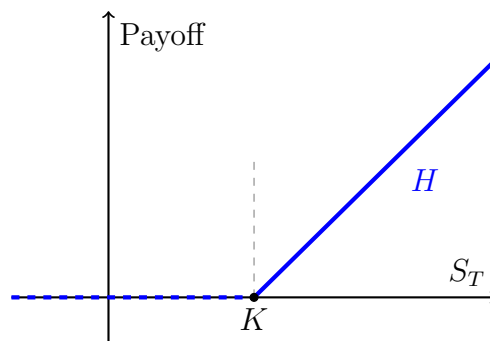
- $S_T$ : random price of the underlying stock at time  $T$ ,
- $S_0$ : known price of the underlying stock at time  $t = 0$ ,
- $K$ : the strike price.

**Remark.** *No physical transaction of the stock at time  $T$  happens. Only the profit is paid out accordingly.*

Clearly, the **payoff** for a call option at time  $T$  is

$$H = \max \{S_T - K, 0\} = (S_T - K)^+.$$

$H$  itself is a random variable since  $S_T$  is assumed to be random.



**Figure:** Visualisation of the payoff  $H$  dependent on the strike  $K$ .

Take  $S_0 = 100$  and assume the stock can attain two prices at  $T$ . Either it attains 150 in the case of  $\omega_1$  or it attains 90 in the case of  $\omega_2$ , i.e.  $S_T(\omega_1) = 150$  and  $S_T(\omega_2) = 90$  with probabilities  $p$  and  $1 - p$  respectively. We let 130 be the strike  $K$ . The payoff is obviously 20 and 0 respectively. We assume that there is an additional, riskless investment opportunity with interest rate  $r = 0$ .

**Question.** What is the fair price  $\pi(H)$  of the above option at  $t = 0$ ?

**Idea: No-arbitrage-principle.** There is no arbitrage (i.e. no riskless profit) in the financial market.

The payoff  $H$  is replicated by other assets (in this case stocks and riskless investments).

The initial capital that is needed in order to replicate  $H$  corresponds to the price of the option. Otherwise, there is arbitrage.

We now want to find a **trading strategy**  $(\alpha, \beta) \in \mathbb{R}^2$  with

- $\alpha$ , the number of stocks that we buy at  $t = 0$ ,
- $\beta$ , the investment in the riskless asset (RA).

The value of the portfolio at  $t = 0$  is  $V_0(\alpha, \beta) = \beta \cdot 1 + \alpha S_0$  (the RA is normalised) and the value of the portfolio at  $t = T$  is  $V_T(\alpha, \beta) = \beta \cdot 1 + \alpha S_T$  (recall  $r = 0$ ). We want to *hedge* or *replicate* the payoff, i.e. choose  $(\alpha, \beta)$  such that  $V_T(\alpha, \beta) = H$ . This

means  $\beta + \alpha S_T(\omega) = H(\omega)$  for any  $\omega \in \{\omega_1, \omega_2\}$ . Hence, we get a system of linear equations.

$$\begin{aligned}\beta + 150\alpha &= 20 \\ \beta + 90\alpha &= -0.\end{aligned}$$

Solving this for  $\alpha, \beta$  yields  $\alpha = 1/3$  and  $\beta = -30$ . Hence

$$V_0(\alpha, \beta) = -30 + 100/3 = 10/3 = \pi(H).$$

The hedging strategy that we get is to borrow 30 today and buy  $1/3$  stocks. The overall investment is  $10/3$ . Two scenarios can happen at  $t = T$ :

- (1)  $S_T = 150$ . Selling the stock yields  $1/3 \cdot 150 = 50$ . We repay our debt of 30 and yield 20 as profit.
- (2)  $S_T = 90$ . Selling the stock yields  $1/3 \cdot 90 = 30$ . We repay the debt and nothing happens.

As we see, we hedged  $H$  perfectly. Now assume that we sell the call option at a price of  $10/3$  and invest. Then, the following can happen.

- (1)  $S_T = 150$ . The buyer will exercise the call, he or she will buy the stock for 130 from us,
- (2) we have to buy the stock for 150, we lose 20,
- (3) but we obtain 20 as by hedging in (1).
- (1)  $S_T = 90$ . Goes analogously.

Using the no-arbitrage-principle, the fair price of the option is  $\pi(H) = 10/3$ . For any other price, a riskless profit would be possible. Assume  $\pi(H) > 10/3$ . Then we get the following table:

Action at $t = 0$	Payoff at $t = T$
Sell $\pi(H)$	$-(S_T - K)^+$
Borrow 30 RA	-30
Buy $S_0/3$	$S_T/3$
Profit: $\pi(H) + 30 - S_0/3 = \pi(H) - 10/3 > 0$	$\Sigma = 0$

TABLE 1. Unfair pricing shows the possibility of riskless profit.

Assume now  $\pi(H) < 10/3$ . The table goes analogous to the above.

**Remark.** Notice that the fair price  $\pi(H)$  in this example is *independent* of the subjective probability  $p$ ! In more sophisticated models, we will also adapt to this.

1.3. **Market assumptions.** In this section, we used a perfect market model.

- The markets are frictionless, i.e. there are no taxes on profits and no transaction costs for reallocating portfolios,
- short-selling is allowed at all times and arbitrary shares can be bought and sold,
- interest rates for borrowing and lending are the same as well as investments in RA,
- there are no dividend payments,
- all market participants are rational and maximise their utility.

In a perfect market there is no arbitrage. Supply and demand offset each other perfectly and there are unique prices.

## 2. FINANCIAL MARKETS AND THE FINITE STATE SPACE

Our goal here is to introduce general financial markets. Finite means that there is a finite number of market states and a finite number of trading times  $\{0, \dots, T\}$ . We describe trading strategies, arbitrage strategies and options formally.

2.1. **Definition of the financial market.** We take  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$  to be a filtered probability space. I.e.

- $\Omega$  is a finite state space of elementary events,
- $\mathcal{F}$  is the power set of  $\Omega$  and acts as the  $\sigma$ -algebra,
- $(\mathcal{F}_t)_t$  is a filtration,
- $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$  is a probability measure.

**Remark.** We remark that for a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and an ordered index set  $I$ , a family of  $\mathcal{F}$ -sub- $\sigma$ -algebras  $(\mathcal{F}_i)_{i \in I}$  is a filtration if and only if for any  $t, s \in I$  :  $t \leq s \implies \mathcal{F}_t \subseteq \mathcal{F}_s$ . Since our index sets  $I$  are always finite with maximal element  $T$ , we make the additional presumption that  $\mathcal{F} = \mathcal{F}_T$ .

A **financial market** consists of  $d+1$  investment opportunities: one **riskless asset** with a deterministic price process  $(B_t) = (B_0, \dots, B_T)$ , i.e. imagine a savings account such that  $B_0 = 1$  and  $B_{t+1} \geq B_t$  for  $t = 0, \dots, T-1$ .  $B_t$  is the value at the time  $t$  of 1 such unit that has been invested at  $t = 0$ , e.g.  $B_t = (1+r)^t$  for interest rate  $r \geq 0$ .

A **risky asset** with stochastic processes  $(S_t^k) = (S_0^k, \dots, S_T^k)$  and  $S_t^k(\omega) > 0$  for  $k = 1, \dots, d$  and  $t = 0, \dots, T$  and all  $\omega \in \Omega$ . We define for  $t = 0, 1, \dots, T$   $S_t$  as

$$S_t \stackrel{\text{def}}{=} (S_t^1, \dots, S_t^d).$$

For example, imagine these to be *stock prices* –  $S_t^k$  is the price of stock  $k$  at time  $t$ . The processes  $(S_t^k)$  are assumed to be adapted ( $k = 1, \dots, d$ ) to the filtration given, i.e.  $(\mathcal{F}_t)$ . Hence

$$\mathcal{F}_t^S \subset \mathcal{F}_t, \quad \text{where} \quad \mathcal{F}_t^S = \sigma(S_0, \dots, S_t)$$

is the  $\sigma$ -algebra generated by  $(S_t)$  up to  $t$ .

**Remark (Repetition).**  $\mathcal{F}_t^X \stackrel{\text{def}}{=} \sigma(\{X_s \mid s \leq t\})$  for all  $t \in I$  is the  $\sigma$ -algebra generated by the stochastic process  $X$  itself.

We can invest in the assets above and trade them, for this we define a trading strategy.

### Definition 2.1: Trading Strategy

A trading strategy (a portfolio) is a  $\mathbb{R}^{d+1}$ -valued,  $(\mathcal{F}_t)$ -adapted stochastic process  $\varphi = (\varphi_0, \dots, \varphi_{T-1})$ , i.e.  $\varphi_t$  is  $\mathcal{F}_t$ -measurable and  $\varphi_t = (\alpha_t, \beta_t)$  for  $t = 0, \dots, T-1$ .

**Remark.** In practice:

- $\beta_t$  is the quantity of the riskless asset that is held in time period  $[t, t+1)$  and
- $\alpha_t = (\alpha_t^1, \dots, \alpha_t^d)$  where  $\alpha_t^k$  is the quantity of the risky asset  $k$  that is held in  $[t, t+1)$ ,
- we denote  $\beta = (\beta_0, \dots, \beta_{T-1})$  and  $\alpha = (\alpha_0, \dots, \alpha_{T-1})$ .

**Remark (Repetition).** A real-valued stochastic process is a mapping  $X : I \times \Omega \rightarrow \mathbb{R}$  such that  $X(t, \cdot) : \Omega \rightarrow \mathbb{R}$  is measurable for every  $t \in I$ , i.e. a sequence of real-valued random variables. For  $\omega \in \Omega$  fixed,  $t \mapsto X_t(\omega)$  is called a path of  $X$ .

**Adapted trading strategy.** If a trading strategy  $\varphi$  is adapted, it means that  $\beta_t = \beta_t(S_0, \dots, S_t)$  and  $\alpha_t^k = \alpha_t^k(S_0, \dots, S_t)$  are functions of the prices  $S_0, \dots, S_t$ . Investors thus observe prices up to time  $t$  and choose – based on this information – a new composition of  $\varphi_t$  of the portfolio at time  $t$  which is then held until  $t+1$ .

The information about the price processes is included in the filtration  $(\mathcal{F}_t^S) \subset (\mathcal{F}_t)$ .

We call the value of a trading strategy also a *wealth process*.

### Definition 2.2: Wealth process

The value of a trading strategy  $\varphi$  at times  $t = 0, \dots, T-1$  is given by

$$V_t^\varphi = \beta_t B_t + \alpha_t \cdot S_t = \beta_t B_t + \sum_{k=1}^d \alpha_t^k S_t^k.$$

Moreover, we let  $V_T^\varphi = \beta_{T-1} B_T + \alpha_{T-1} S_T$ .

**Remark.**  $\beta_t B_t + \alpha_t \cdot S_t$  is the value of the trading strategy immediately after it has been recomposed. Change in the market values due to price changes are then

$$\beta_{t-1}(B_t - B_{t-1}) + \alpha_{t-1} \cdot (S_t - S_{t-1}).$$

**Definition 2.3: Self-financing**

A trading strategy is called self-financing if for  $t = 1, \dots, T - 1$

$$(1) \quad \beta_{t-1}B_t + \alpha_{t-1} \cdot S_t = \beta_t B_t + \alpha_t \cdot S_t.$$

**Remark** (Explanation). *When the investor observes the new prices  $B_t, S_t$ , she adjusts the trading strategy from  $\varphi_{t-1}$  to  $\varphi_t$  without adding or assuming additional wealth. Thus, (1) must hold.*

*We show:* For a self-financing trading strategy  $\varphi = (\alpha, \beta)$ , this is equivalent to knowing  $(V_0^\varphi, \alpha)$  and hence  $\beta$  can be determined from (1).

Denote the set of trading strategies of risky assets as

$$\mathcal{A} \stackrel{\text{def}}{=} \{\alpha = (\alpha_0, \dots, \alpha_{T-1}) : \alpha_t \text{ is } \mathcal{F}_t\text{-measurable for } t = 0, \dots, T-1\}$$

Hence if the initial wealth  $V_0^\varphi$  is known, every  $\alpha \in \mathcal{A}$  can be complemented in a self-financing way.

**Remark** (Notation).  $\Delta X_t = X_t - X_{t-1}$ ,  $\Delta \alpha_t = (\Delta \alpha_t^1, \dots, \Delta \alpha_t^d)$ .

**Lemma 2.1**

Let  $\varphi$  be self-financing. Then we have

$$\beta_t = \beta_0 - \sum_{k=1}^t \Delta \alpha_n \frac{S_n}{B_n} = V_0^\varphi - \sum_{k=0}^t \Delta \alpha_n \frac{S_n}{B_n},$$

for  $t = 0, \dots, T-1$  where  $\Delta \alpha_0^k \stackrel{\text{def}}{=} \alpha_0^k$  for  $k = 1, \dots, d$ .

*Proof.* Let  $t = 0$ , then  $V_0^\varphi = \beta_0 B_0 + \alpha_0 \cdot S_0 = \beta_0 + \alpha_0 \cdot S_0$  by normalising  $B_0 = 1$ . Hence

$$\beta_0 = V_0^\varphi - \alpha_0 \frac{S_0}{B_0} = V_0^\varphi - \Delta \alpha_0 \frac{S_0}{B_0}.$$

Let  $t \in \{1, \dots, T-1\}$ , then, since  $(\alpha, \beta)$  is self-financing, we get

$$\beta_t B_t + \alpha_t \cdot S_t = \beta_{t-1} B_t + \alpha_{t-1} \cdot S_t \iff (\beta_t - \beta_{t-1}) B_t = -(\alpha_t - \alpha_{t-1}) \cdot S_t$$

and hence  $\Delta \beta_t = -\Delta \alpha_t \frac{S_t}{B_t}$ . This implies

$$\beta_t = \beta_0 + \sum_{n=1}^t \Delta \beta_n = \beta_0 - \sum_{n=1}^t \Delta \alpha_n \frac{S_n}{B_n} = V_0^\varphi - \Delta \alpha_0 \frac{S_0}{B_0} - \sum_{n=1}^t \Delta \alpha_n \frac{S_n}{B_n}. \quad \square$$

**Definition 2.4: Discounted stock price**

$\tilde{S}_t^k \stackrel{\text{def}}{=} \frac{S_t^k}{B_t}$  is the discounted stock price of risky asset  $k$  at  $t$ .

**Lemma 2.2**

Let  $\varphi$  be self-financing. Then we have

$$\frac{V_t^\varphi}{B_t} = V_0^\varphi + \sum_{n=1}^t \alpha_{n-1} \Delta \tilde{S}_n.$$

*Proof.* As  $\varphi$  is self-financing, we have

$$\begin{aligned} \frac{V_n^\varphi}{B_n} - \frac{V_{n-1}^\varphi}{B_{n-1}} &= \frac{1}{B_n} (\beta_n B_n + \alpha_n \cdot S_n) - \frac{1}{B_{n-1}} (\beta_{n-1} B_{n-1} + \alpha_{n-1} \cdot S_{n-1}) \\ &\stackrel{\text{SF}}{=} \frac{1}{B_n} (\beta_{n-1} B_n + \alpha_{n-1} S_n) - \frac{1}{B_{n-1}} (\beta_{n-1} B_{n-1} + \alpha_{n-1} \cdot S_{n-1}) \\ &= \underbrace{\beta_{n-1} - \beta_{n-1}}_{=0} + \alpha_{n-1} \left( \frac{S_n}{B_n} - \frac{S_{n-1}}{B_{n-1}} \right) \\ &= \alpha_{n-1} \Delta \tilde{S}_n. \end{aligned}$$

We have that with  $B_0 = 1$ :

$$\frac{V_t^\varphi}{B_t} = \frac{V_0^\varphi}{B_0} + \sum_{n=1}^t \left( \frac{V_n^\varphi}{B_n} - \frac{V_{n-1}^\varphi}{B_{n-1}} \right) = V_0^\varphi + \sum_{n=1}^t \alpha_{n-1} \Delta \tilde{S}_n. \quad \square$$

**Definition 2.5: Gains Process**

For a trading strategy  $\alpha \in \mathcal{A}$ , the process  $(G_t^\alpha)$ , defined by

$$G_0^\alpha = 0 \quad \text{and} \quad G_t^\alpha = \sum_{n=1}^t \alpha_{n-1} \cdot \Delta \tilde{S}_n, \quad t = 1, \dots, T$$

is called the **gains process**.

**No-arbitrage-principle.**



**Definition 2.6: Arbitrage Strategy**

Let  $\varphi$  be self-financing.  $\varphi$  is called arbitrage strategy if

$$V_0^\varphi = 0, \quad \mathbb{P}(V_T^\varphi \geq 0) = 1 \quad \text{and} \quad \mathbb{P}(V_T^\varphi > 0) > 0.$$

**Remark.** We say that there is an arbitrage opportunity if such an arbitrage strategy exists. (NA) means that there is no arbitrage opportunity.

Recall that  $\mathbb{P}(\{\omega\}) > 0$  for any  $\omega \in \Omega$  by definition of our financial market. Thus

$$\mathbb{P}(V_T^\varphi \geq 0) = 1 \iff V_T^\varphi(\omega) \geq 0 \quad \text{for any } \omega \in \Omega.$$

Moreover,

$$\mathbb{P}(V_T^\varphi > 0) > 0 \iff \exists \omega \in \Omega : V_T^\varphi(\omega) > 0.$$

We know from before that

$$\frac{V_T^\varphi}{B_T} = V_0^\varphi + G_T^\alpha, \quad \varphi = (\alpha, \beta)$$

where  $\varphi$  is self-financing. Now, an arbitrage strategy exists if and only if there exists a trading strategy  $\varphi$  with  $V_0^\varphi = 0$  and  $\mathbb{P}(G_T^\alpha \geq 0) = 1$  and  $\mathbb{P}(G_T^\alpha > 0) > 0$ .

In words, the next theorem tells us that a financial market is globally free of arbitrage if and only if it is locally free of arbitrage.

**Theorem 2.1**

The following are equivalent.

- There exists an arbitrage strategy.
- There exists a  $t \in \{1, \dots, T\}$  and a  $\mathcal{F}_{t-1}$ -measurable random vector  $\eta : \Omega \rightarrow \mathbb{R}^d$  such that  $\mathbb{P}(\eta \cdot (\tilde{S}_t - \tilde{S}_{t-1}) \geq 0) = 1$  and  $\mathbb{P}(\eta \cdot (\tilde{S}_t - \tilde{S}_{t-1}) > 0) > 0$ . Here,  $\eta$  depends on  $t$ .

*Proof.* Let  $\varphi = (\alpha, \beta)$  be an arbitrage strategy with wealth process  $(V_t^\varphi)$  and let

$$t = \min \{m \in \mathbb{N} \mid \mathbb{P}(V_m^\varphi \geq 0) = 1 \text{ and } \mathbb{P}(V_m^\varphi > 0) > 0\}.$$

Then  $t \leq T$  and either (a)  $\mathbb{P}(V_{t+1}^\varphi = 0) = 1$  or (b)  $\mathbb{P}(V_{t-1}^\varphi < 0) > 0$ . In case of (a), we have

$$\alpha_{t-1} \cdot (\tilde{S}_t - \tilde{S}_{t-1}) = \frac{V_T^\varphi}{B_t} - \frac{V_{t-1}^\varphi}{B_{t-1}} = \frac{V_t^\varphi}{B_t} \geq 0.$$

Using  $\eta = \alpha_{t-1}$  we obtain  $\mathbb{P}(\eta \cdot (\tilde{S}_t - \tilde{S}_{t-1}) > 0) > 0$  for some development of  $S_t$  and moreover  $\mathbb{P}(\eta \cdot (\tilde{S}_n - \tilde{S}_{n-1}) \geq 0) = 1$ . In case (b) we get by

$$\eta = \alpha_{t-1} \mathbb{1}_{\{V_{t-1}^\varphi < 0\}}$$

that

$$\eta \cdot (\tilde{S}_t - \tilde{S}_{t-1}) = \left( \frac{V_t^\varphi}{B_t} - \frac{V_{t-1}^\varphi}{B_{t-1}} \right) \cdot \mathbb{1}_{\{V_{t-1}^\varphi < 0\}} \geq \frac{V_t^\varphi}{B_t} \mathbb{1}_{\{V_{t-1}^\varphi < 0\}}$$

and hence  $\mathbb{P}(\eta \cdot (\tilde{S}_t - \tilde{S}_{t-1}) \geq 0) = 1$  and  $\mathbb{P}(\eta \cdot (\tilde{S}_t - \tilde{S}_{t-1}) > 0) > 0$ .

Now the backward direction. Define a trading strategy  $\varphi = (\alpha, \beta)$  by

$$\alpha_m = \begin{cases} \eta, & m = t-1 \\ 0, & \text{else.} \end{cases}$$

$\alpha$  can be completed to a self-financing trading strategy with  $V_0^\varphi = 0$ . By the previous lemmas,

$$\frac{V_T^\varphi}{B_T} = V_0^\varphi + \sum_{n=1}^t \alpha_{n-1} \Delta \tilde{S}_n = \eta \cdot (\tilde{S}_t - \tilde{S}_{t-1}).$$

Moreover, using the assertion we get  $\mathbb{P}(\eta \cdot (\tilde{S}_t - \tilde{S}_{t-1}) \geq 0) = 1 = \mathbb{P}(V_T^\varphi / B_T \geq 0)$  and  $\mathbb{P}(\eta \cdot (\tilde{S}_t - \tilde{S}_{t-1}) > 0) = \mathbb{P}(V_T^\varphi / B_T > 0) > 0$ .  $\square$

**Example 2.2.** We assume a financial market with  $T = 2$  periods and a RA with  $B_0 = B_1 = B_2 = 1$ . We assume the following price behaviour.

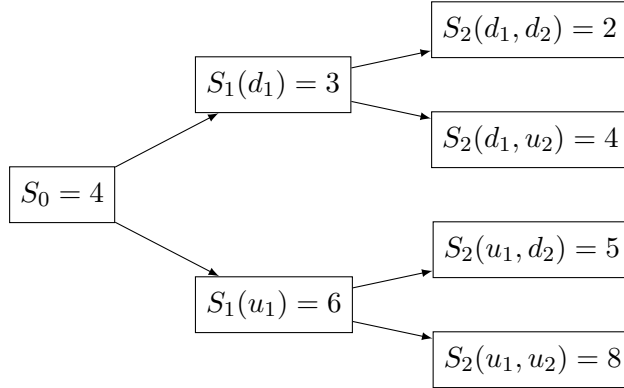


FIGURE 1. Price behaviour in our 2-period market.

We let  $\Omega = \{u_1, d_1\} \times \{u_2, d_2\}$ . We now want to check whether our whole financial market is free of arbitrage by using the previous theorem. We start with  $t = 1$  and obtain

$$\eta \left( \tilde{S}_n(y_1) - S_0 \right) = \begin{cases} \eta(6 - 4) = 2\eta, & y_1 = u_1 \\ \eta(3 - 4) = -\eta, & y_1 = d_1. \end{cases}$$

As the cases have different signs, we cannot choose any common  $\eta \neq 0$  such that  $\eta(\tilde{S}_1(y_1) - S_0) \geq 0$  for all  $y_1$  and hence (NA) holds in period  $t = 1$ . For  $t = 2$ , need to check two cases. Indeed,

$$\eta(u_1) \left( \tilde{S}_2(u_1, y_2) - \tilde{S}_1(u_1) \right) = \begin{cases} \eta(u_1)(8 - 6) = 2\eta(u_1), & y_2 = u_2 \\ \eta(u_1)(5 - 6) = -\eta(u_1), & y_2 = d_2. \end{cases}$$

By the same logic as before, with differing signs, we have (NA). Moreover,

$$\eta(d_1) \left( \tilde{S}_1(d_1, y_2) - \tilde{S}_1(d_1) \right) = \begin{cases} \eta(d_1)(4 - 3) = \eta(d_1), & y_2 = u_2 \\ \eta(d_1)(2 - 3) = -\eta(d_1), & y_2 = d_2 \end{cases}$$

yields differing signs again. Now we have checked all branches of the decision tree and get that (NA) holds for the whole financial market by Theorem 2.1.

**2.2. Options.** Options are characterised by the payoff. For *European options*, the payoff happens at exercise date  $T$ .

#### Definition 2.7: Contingent Claim

A contingent claim is an  $\mathcal{F}_T$ -measurable random variable  $H$  with values in  $\mathbb{R}$ .

**Remark.** If  $H \in \mathcal{F}_T^S$ , then  $H = h(S_0, \dots, S_T)$ .

**Example 2.3** (Examples of Contingent Claims). **European call options** with strike price  $K$ :

$$H = (S_T - K)^+$$

is only exercised when  $S_T > K$ . **European put options** with strike  $K$ :

$$H = (K - S_T)^+$$

is only exercised when  $S_T < K$ . A **Future** is delivered with certainty, thus

$$H = S_T - K.$$

Here,  $K$  is a fixed reference price and  $T$  the delivery date. A **digital call option** with strike  $K$ :

$$H = \mathbb{1}_{\{S_T > K\}}$$

yields payoff of 1 unit if  $S_T > K$ . A **down-and-out-call** with strike  $K$  and barrier  $B$ :

$$H = (S_T - K)^+ \mathbb{1}_{\{\min_{t \in \{0, \dots, T\}} S_t > B\}}$$

depends on the whole path of  $S$ . These are cheaper than classical options. **Asian call options** are characterised by

$$H = \left( S_T - \frac{1}{T} \sum_{t=1}^T S_t \right)^+$$

is only exercised when the stock price is greater than the arithmetic mean. Also these are obviously path-dependent.

### Definition 2.8: Attainability and completeness

- (a) A contingent claim  $H$  is attainable if there is a trading strategy  $\varphi$  with  $V_T^\varphi = H$ . Then  $\pi(H) = V_0^\varphi$  is called a price of  $H$  and  $\varphi$  is called duplication/replication/hedging strategy of  $H$ .
- (b) We say a market is complete if any contingent claim is attainable.

With this definition we get the following lemma.

### Lemma 2.3: Price of attainable claims

Suppose absence of arbitrage (NA). Then, the price  $\pi(H)$  for an attainable contingent claim  $H$  is unique and thus independent of the choice of hedging strategy.

*Proof.* Let  $H$  be an attainable contingent claim. Let  $\varphi = (\alpha, \beta)$  and  $\tilde{\varphi} = (\tilde{\alpha}, \tilde{\beta})$  be hedging strategies for  $H$ . By Lemma 2.1,  $\varphi, \tilde{\varphi}$  can be expressed via  $(V_0^\varphi, \alpha), (V_0^{\tilde{\varphi}}, \tilde{\alpha})$  and by lemma 2.2,

$$V_0^\varphi + G_T^\alpha = \frac{H}{B_T} = V_0^{\tilde{\varphi}} + G_T^{\tilde{\alpha}}$$

and  $H = V_T^\varphi = V_T^{\tilde{\varphi}}$  as  $\varphi$  and  $\tilde{\varphi}$  are hedging strategies. Assume  $d = V_0^{\tilde{\varphi}} - V_0^\varphi > 0$ . Then

$$\begin{aligned} 0 &= V_0^\varphi - V_0^{\tilde{\varphi}} - G_T^{\tilde{\alpha}} + G_T^\alpha \\ &= -d - G_T^{\tilde{\alpha}} + G_T^\alpha \\ &= -d + \sum_{n=1}^T (\alpha_{n-1} - \tilde{\alpha}_{n-1}) \cdot \Delta \tilde{S}_n \\ &= -d + G_T^{\alpha - \tilde{\alpha}} \implies G_T^{\alpha - \tilde{\alpha}} = d > 0. \end{aligned}$$

Now,  $\psi$  is an arbitrage strategy, where  $\psi \stackrel{\text{def}}{=} (\hat{\alpha}, \hat{\beta})$  where  $\hat{\alpha} \stackrel{\text{def}}{=} \alpha - \tilde{\alpha}$  and  $\hat{\beta}$  is determined by lemma 2.1 with  $V_0^\psi = 0$ . This is a contradiction to (NA).  $\square$

**Example 2.4.** We use the tree-based model from the last example. Recall the structure: Consider the digital call with  $H = \mathbb{1}_{\{S_2 \geq 5\}}$ . This will only pay 1 if  $S_2 \geq 5$ . We

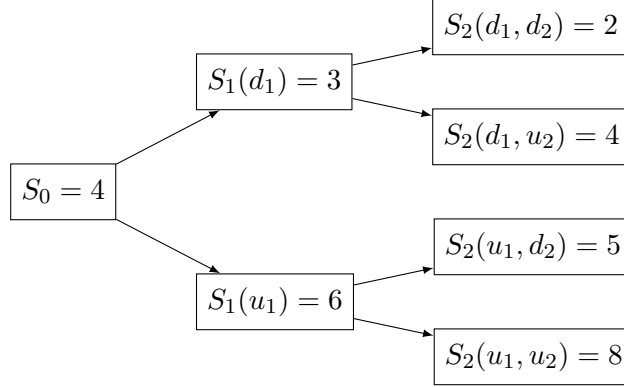


FIGURE 2. Price behaviour in our 2-period market.

get

$$H(u_1, u_2) = H(u_1, d_2) = 1$$

$$H(d_1, u_2) = H(d_1, d_2) = 0.$$

How can we construct a hedging strategy here? We must find  $\varphi$  with  $V_T^\varphi = H$ . For  $t = 2$  and the upper node, we get

$$H(u_1, u_2) = 1 = 8\alpha_1(u_1) + \beta_1(u_1) = V_2^\varphi(u_1, u_2)$$

$$H(u_1, d_2) = 1 = 5\alpha_1(u_1) + \beta_1(u_1) = V_2^\varphi(u_1, d_2).$$

Solving yields  $\alpha_1(u_1) = 0$  and  $\beta_1(u_1) = 1$ . Hence  $V_1^\varphi = 6\alpha_1(u_1) + \beta_1(u_1) = 1$  by the self-financing property. For the lower node, observe

$$H(d_1, u_2) = 0 = 4\alpha_1(d_1) + \beta_1(d_1) = V_2^\varphi(d_1, u_2)$$

$$H(d_1, d_2) = 0 = 2\alpha_1(d_1) + \beta_1(d_1) = V_2^\varphi(d_1, d_2).$$

Solving yields  $\alpha_1(d_1) = 0$  and  $\beta_1(d_1) = 0$  and hence  $V_1^\varphi = 3\alpha_1(d_1) + \beta_1(d_1) = 0$  by self-financing.

We can now look at  $t = 1$ . We have

$$V_1^\varphi(u_1) = 1 = 6\alpha_0 + \beta_0$$

$$V_1^\varphi(d_1) = 0 = 3\alpha_0 + \beta_0,$$

which yields  $\alpha_0 = 1/3$  and  $\beta_0 = -1$  and hence

$$\pi(H) = V_0^\varphi = S_0\alpha_0 + \beta_0B_0 = 4\alpha_0 + \beta_0 = \frac{4}{3} - 1 = \frac{1}{3}.$$

**Remark.** *All considerations in this section are without accounting for transaction costs.*

### 3. THE COX–ROSS–RUBINSTEIN MODEL

This is a simple model in discrete time. We assume 1 riskless asset and 1 risky asset.

**3.1. One-period CRR model.** We assume  $T = 1$  and  $\Omega = \{u, d\}$  and  $\mathcal{F}_T = \mathcal{F} = 2^\Omega$ . For the riskless asset, we set  $B_0 = 1$  and  $B_1 = 1 + r$ ,  $r \geq 0$ . For the risky asset, we assume a random variable  $S_0 > 0$  and

$$S_1(\omega) = \begin{cases} uS_0, & \omega = u \\ dS_0, & \omega = d \end{cases}$$

for functions  $0 < d < u$ . We then call  $u$  *up-factor* and  $d$  the *down-factor*. The case  $u = d$  is not interesting since it reduces to a deterministic case.

The basic question we want to ask is: *When is this model free of arbitrage?*

#### Theorem 3.1

Consider the one-period CRR model. Then, the market is free of arbitrage if and only if  $d < 1 + r < u$ .

*Proof.* Let's assume  $d < 1 + r < u$  or equivalently  $\frac{dS_0}{1+r} - S_0 < 0 < \frac{uS_0}{1+r} - S_0$ . Hence,  $S_1(d)/(1+r) - S_0 < 0 < S_1(u)/(1+r) - S_0$ . Now,  $\tilde{S} = S/B$  with  $B_1 = 1 + r$ . We get  $\tilde{S}_1(d) - S_0 < 0 < \tilde{S}_1(u) - S_0$  and by noting  $\tilde{S}_0 = S_0$  and applying theorem 2.1, we get that there exists an  $\omega$  and some  $\eta \neq 0$  such that  $\eta(\tilde{S}_n(\omega) - \tilde{S}_0) < 0$ , which implies that there is no arbitrage.  $\square$

What can we say about completeness?

#### Theorem 3.2

Suppose there is no arbitrage. Then the CRR model is complete. In particular,

$$\alpha_0 = \frac{H(u) - H(d)}{(u - d)S_0}, \quad \beta_0 = \frac{uH(d) - dH(u)}{(u - d)(1 + r)}.$$

Then,  $\pi(H) = \frac{uH(d) - dH(u)}{(u - d)(1 + r)} + \frac{H(u) - H(d)}{u - d}$  is the unique price.

*Proof.* If  $\varphi$  is a hedging strategy, then at  $T = 1$ , we have

$$V_1^\varphi = \beta_0(1 + r) + \alpha_0 S_1 = H.$$

Hence,

$$\begin{aligned}\beta_0(1+r) + \alpha_0 \underbrace{S_1(u)}_{uS_0} &= H(u) \\ \beta_0(1+r) + \alpha_0 \underbrace{S_1(d)}_{dS_0} &= H(d).\end{aligned}$$

Some calculating yields  $\alpha_0 = \frac{H(u)-H(d)}{(u-d)S_0}$  and  $\beta_0 = \frac{uH(d)-dH(u)}{(u-d)(1+r)}$ . Obviously,  $\pi(H) = V_0^\varphi = \beta_0 B_0 + \alpha_0 S_0$ .  $\square$

**Example 3.3.** Let  $u = 1.1$  and  $d = 0.9$  and  $r = 0.05$ . Due to Theorem 3.1, the market is free of arbitrage. Let  $S_0 = 100$  and  $H(u) = 80$  and  $H(d) = 60$ . The price due to Theorem 3.2 is

$$\pi(H) = \frac{1.1 \cdot 60 - 0.9 \cdot 80}{(1.1 - 0.9)1.05} + \frac{80 - 60}{1.1 - 0.9} = 71.42.$$

The corresponding hedging strategy is

$$\alpha_0 = \frac{80 - 60}{0.2 \cdot 100} = 1$$

and

$$\beta_0 = -28.57.$$

**Remark** (Preparatory remark on Equivalent Martingale Measures (EMMs)). We can rearrange  $\pi(H)$  from before as

$$\pi(H) = \frac{H(u)}{1+r} \cdot \frac{1+r-d}{u-d} + \frac{H(d)}{1+r} \left(1 - \frac{1+r-d}{u-d}\right) =: \star.$$

This can be seen from

$$\begin{aligned}\star &= \frac{H(u)}{1+r} \left(\frac{1+r}{u-d} - \frac{d}{u-d}\right) + \frac{H(d)}{1+r} \left(1 - \frac{1+r}{u-d} + \frac{d}{u-d}\right) \\ &= \frac{H(u)}{u-d} - \frac{dH(u)}{(1+r)(u-d)} + \frac{H(d)}{1+r} - \frac{H(d)}{u-d} + \frac{dH(d)}{(1+r)(u-d)} \\ &= \frac{H(u) - H(d)}{u-d} + \frac{dH(d) - dH(u)}{(1+r)(u-d)} + \frac{H(d)}{1+r} \\ &= \frac{H(u) - H(d)}{u-d} + \frac{dH(d) - dH(u) + uH(d) - dH(d)}{(1+r)(u-d)} \\ &= \frac{H(u) - H(d)}{u-d} + \frac{uH(d) - dH(u)}{(1+r)(u-d)} = \pi(H).\end{aligned}$$

We fix  $q \stackrel{\text{def}}{=} \frac{1+r-d}{u-d}$  and hence  $0 < q < 1$  as by (NA) we know that  $d < 1+r < u$  and thus

$$\pi(H) = \frac{H(u)}{1+r}q + \frac{H(d)}{1+r}(1-q).$$

Define the probability measure  $\mathbb{Q}$  on  $(\Omega, \mathcal{F}_T)$  as

$$\mathbb{Q}(\{u\}) = q, \quad \mathbb{Q}(\{d\}) = 1-q.$$

Hence,  $\pi(H) = \mathbb{E}_{\mathbb{Q}} \left[ \frac{H}{1+r} \right]$  where  $\mathbb{E}_{\mathbb{Q}}$  is the expectation w.r.t.  $\mathbb{Q}$ . We obtain for the discounted price  $\tilde{S}_1 = \frac{S_1}{B_1}$  that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \tilde{S}_1 &= S_0 \left( \frac{u}{1+r}q + \frac{d}{1+r}(1-q) \right) \\ &= S_0 \left( q \left[ \frac{u}{1+r} - \frac{d}{1+r} \right] + \frac{d}{1+r} \right) \\ &= S_0 \left( \frac{1+r-d}{u-d} \frac{u-d}{1+r} + \frac{d}{1+r} \right) \\ &= S_0 \left( \frac{1+r-d+d}{1+r} \right) = S_0 = \tilde{S}_1. \end{aligned}$$

Hence, discounted stock prices are martingales with respect to the risk-neutral measure. One can even show that the measure  $\mathbb{Q}$  is the only (i.e. unique) measure that satisfies the martingale property  $\mathbb{E}_{\mathbb{Q}} \tilde{S}_1 = \tilde{S}_0$ .

#### 4. ABSENCE OF ARBITRAGE AND EQUIVALENT MARTINGALE MEASURES

#### 5. COMPLETENESS AND EQUIVALENT MARTINGALE MEASURES

#### 6. RISK-NEUTRAL PRICING OF CONTINGENT CLAIMS

#### 7. AMERICAN OPTIONS

#### 8. PORTFOLIO OPTIMIZATION