

# MATHEMATICAL FINANCE IN DISCRETE TIME

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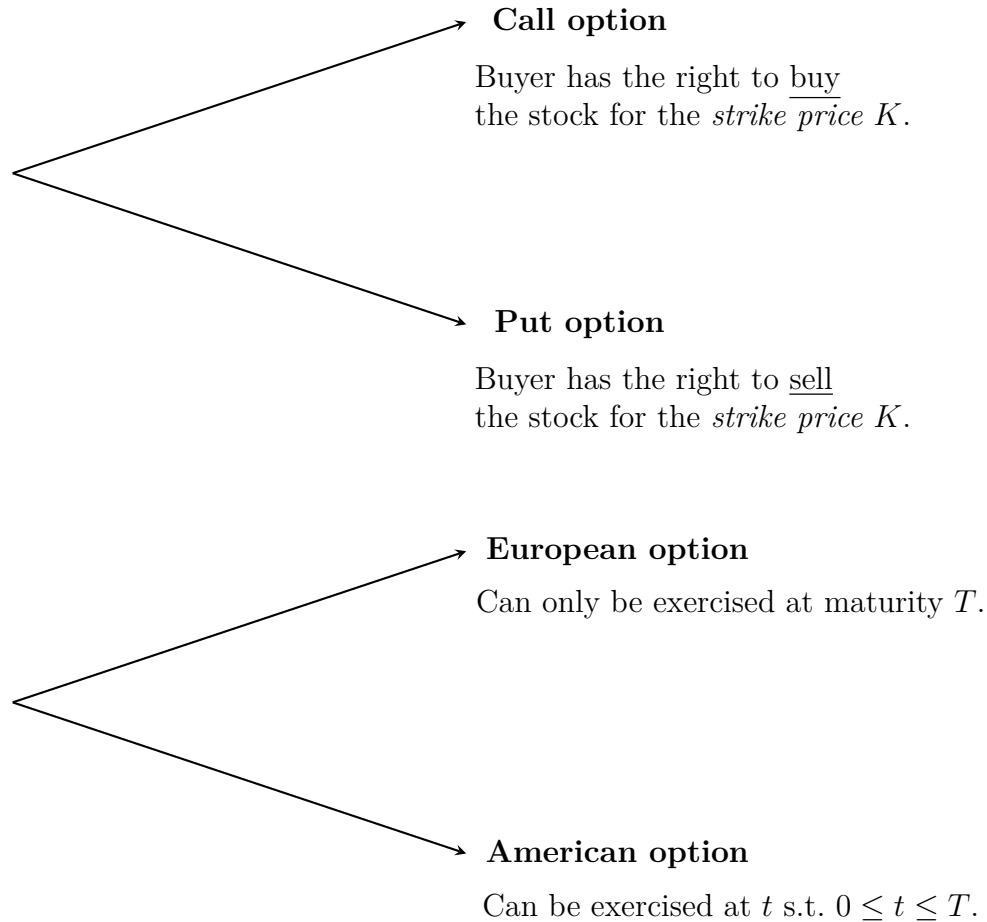
## 1. INTRODUCTION AND FIRST EXAMPLE

We start by introducing the first important notions of option pricing and first examples.

**1.1. Notions for option pricing theory.** An **option** is a contract between two parties (a buyer and a seller). The buyer pays an option price *today* (i.e. at  $t = 0$ ) to the seller and in return obtains the right/*option* but not the obligation to buy a stock for conditions that are fixed *today* at a fixed point in time (*the maturity of the option*). The option to buy is thereby crucial: the buyer can just refrain from buying the stock if the market conditions are disadvantageous.

We only consider options whose *underlyings* (german: *Basistitel*) are stocks.

We distinguish the following:



We call the above **standard** or **vanilla** options. Other options are called **exotic**. Such include *Asian options* (which operate on the mean-value of an asset), *barrier options*, *options on volatility* and so forth. For this course we assume that options have a price at any point in time  $0 \leq t \leq T$ . Advantages of options include

- The loss is limited: worst case is the loss of the option price,
- leverage effect (in a positive market development),
- can be used to *hedge* against decreasing stock prices via put options (can be sold for a fixed pre-determined price).

**1.2. A first example.** Let us consider a call option in a two-period market model. We set

- $t = 0$ : the current time or today,
- $T > 0$ : the maturity or exercise date of the option,

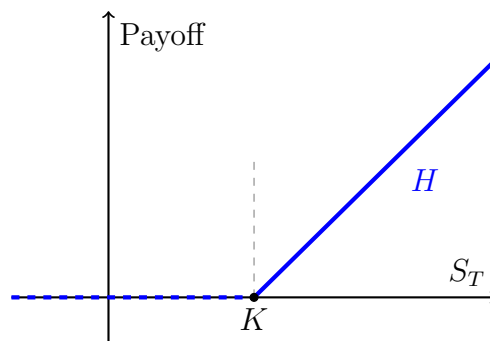
- $S_T$ : random price of the underlying stock at time  $T$ ,
- $S_0$ : known price of the underlying stock at time  $t = 0$ ,
- $K$ : the strike price.

**Remark.** *No physical transaction of the stock at time  $T$  happens. Only the profit is paid out accordingly.*

Clearly, the **payoff** for a call option at time  $T$  is

$$H = \max \{S_T - K, 0\} = (S_T - K)^+.$$

$H$  itself is a random variable since  $S_T$  is assumed to be random.



**Figure:** Visualisation of the payoff  $H$  dependent on the strike  $K$ .

Take  $S_0 = 100$  and assume the stock can attain two prices at  $T$ . Either it attains 150 in the case of  $\omega_1$  or it attains 90 in the case of  $\omega_2$ , i.e.  $S_T(\omega_1) = 150$  and  $S_T(\omega_2) = 90$  with probabilities  $p$  and  $1 - p$  respectively. We let 130 be the strike  $K$ . The payoff is obviously 20 and 0 respectively. We assume that there is an additional, riskless investment opportunity with interest rate  $r = 0$ .

**Question.** What is the fair price  $\pi(H)$  of the above option at  $t = 0$ ?

**Idea: No-arbitrage-principle.** There is no arbitrage (i.e. no riskless profit) in the financial market.

The payoff  $H$  is replicated by other assets (in this case stocks and riskless investments).

The initial capital that is needed in order to replicate  $H$  corresponds to the price of the option. Otherwise, there is arbitrage.

We now want to find a **trading strategy**  $(\alpha, \beta) \in \mathbb{R}^2$  with

- $\alpha$ , the number of stocks that we buy at  $t = 0$ ,
- $\beta$ , the investment in the riskless asset (RA).

The value of the portfolio at  $t = 0$  is  $V_0(\alpha, \beta) = \beta \cdot 1 + \alpha S_0$  (the RA is normalised) and the value of the portfolio at  $t = T$  is  $V_T(\alpha, \beta) = \beta \cdot 1 + \alpha S_T$  (recall  $r = 0$ ). We want to *hedge* or *replicate* the payoff, i.e. choose  $(\alpha, \beta)$  such that  $V_T(\alpha, \beta) = H$ . This

means  $\beta + \alpha S_T(\omega) = H(\omega)$  for any  $\omega \in \{\omega_1, \omega_2\}$ . Hence, we get a system of linear equations.

$$\begin{aligned}\beta + 150\alpha &= 20 \\ \beta + 90\alpha &= -0.\end{aligned}$$

Solving this for  $\alpha, \beta$  yields  $\alpha = 1/3$  and  $\beta = -30$ . Hence

$$V_0(\alpha, \beta) = -30 + 100/3 = 10/3 = \pi(H).$$

The hedging strategy that we get is to borrow 30 today and buy  $1/3$  stocks. The overall investment is  $10/3$ . Two scenarios can happen at  $t = T$ :

- (1)  $S_T = 150$ . Selling the stock yields  $1/3 \cdot 150 = 50$ . We repay our debt of 30 and yield 20 as profit.
- (2)  $S_T = 90$ . Selling the stock yields  $1/3 \cdot 90 = 30$ . We repay the debt and nothing happens.

As we see, we hedged  $H$  perfectly. Now assume that we sell the call option at a price of  $10/3$  and invest. Then, the following can happen.

- (1)  $S_T = 150$ . The buyer will exercise the call, he or she will buy the stock for 130 from us,
- (2) we have to buy the stock for 150, we lose 20,
- (3) but we obtain 20 as by hedging in (1).
- (1)  $S_T = 90$ . Goes analogously.

Using the no-arbitrage-principle, the fair price of the option is  $\pi(H) = 10/3$ . For any other price, a riskless profit would be possible. Assume  $\pi(H) > 10/3$ . Then we get the following table:

Action at $t = 0$	Payoff at $t = T$
Sell $\pi(H)$	$-(S_T - K)^+$
Borrow 30 RA	-30
Buy $S_0/3$	$S_T/3$
Profit: $\pi(H) + 30 - S_0/3 = \pi(H) - 10/3 > 0$	$\Sigma = 0$

TABLE 1. Unfair pricing shows the possibility of riskless profit.

Assume now  $\pi(H) < 10/3$ . The table goes analogous to the above.

**Remark.** Notice that the fair price  $\pi(H)$  in this example is *independent* of the subjective probability  $p$ ! In more sophisticated models, we will also adapt to this.

1.3. **Market assumptions.** In this section, we used a perfect market model.

- The markets are frictionless, i.e. there are no taxes on profits and no transaction costs for reallocating portfolios,
- short-selling is allowed at all times and arbitrary shares can be bought and sold,
- interest rates for borrowing and lending are the same as well as investments in RA,
- there are no dividend payments,
- all market participants are rational and maximise their utility.

In a perfect market there is no arbitrage. Supply and demand offset each other perfectly and there are unique prices.

## 2. FINANCIAL MARKETS AND THE FINITE STATE SPACE

Our goal here is to introduce general financial markets. Finite means that there is a finite number of market states and a finite number of trading times  $\{0, \dots, T\}$ . We describe trading strategies, arbitrage strategies and options formally.

2.1. **Financial market.** We take  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$  to be a filtered probability space. I.e.

- $\Omega$  is a finite state space of elementary events,
- $\mathcal{F}$  is the power set of  $\Omega$  and acts as the  $\sigma$ -algebra,
- $(\mathcal{F}_t)_t$  is a filtration,
- $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$  is a probability measure.

**Remark.** We remark that for a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and an ordered index set  $I$ , a family of  $\mathcal{F}$ -sub- $\sigma$ -algebras  $(\mathcal{F}_i)_{i \in I}$  is a filtration if and only if for any  $t, s \in I : t \leq s \implies \mathcal{F}_t \subseteq \mathcal{F}_s$ . Since our index sets  $I$  are always finite with maximal element  $T$ , we make the additional presumption that  $\mathcal{F} = \mathcal{F}_T$ .

## 3. THE COX–ROSS–RUBINSTEIN MODEL

### 4. ABSENCE OF ARBITRAGE AND EQUIVALENT MARTINGALE MEASURES

### 5. COMPLETENESS AND EQUIVALENT MARTINGALE MEASURES

### 6. RISK–NEUTRAL PRICING OF CONTINGENT CLAIMS

### 7. AMERICAN OPTIONS

### 8. PORTFOLIO OPTIMIZATION