MATHEMATICAL FINANCE IN DISCRETE TIME

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Contents

1. Introduction and first example

We start by introducing the first important notions of option pricing and first examples.

1.1. **Notions for option pricing theory.** An **option** is a contract beween two parties (a buyer and a seller). The buyer pays an option price *today* (i.e. at $t = 0$) to the seller and in return obtains the right/*option* but not the obligation to buy a stock for conditions that are fixed *today* at a fixed point in time (*the maturity of the option*). The option to buy is thereby crucial: the buyer can just refrain from buying the stock if te market conditions are disadvantageous.

We only consider options whose *underlyings (german: Basistitel)* are stocks.

Date: Winter Semester 2025/26.

We distinguish the following:

Call option

Buyer has the right to buy the stock for the *strike price* K.

Put option

Buyer has the right to sell the stock for the *strike price* K.

European option

Can only be exercised at maturity T.

American option

Can be exercised at t s.t. $0 \leq t \leq T$.

We call the above **standard** or **vanilla** options. Other options are called **exotic**. Such include *Asian options* (which operate on the mean–value of an asset), *barrier options*, *options on voatility* and so forth. For this course we assume that options have a price at any point in time $0 \leq t \leq T$. Advantages of options include

- The loss is limited: worst case is the loss of the option price,
- leverage effect (in a positive market development),
- can be used to *hedge* against decreasing stock prices via put options (can be sold for a fixed pre–determined price).

1.2. **A first example.** Let us consider a call option in a two–period market model. We set

- $t = 0$: the current time or today,
- $T > 0$: the maturity or exercise date of the option,

- S_T: random price of the underlying stock at time T,
- S_0: known price of the underlying stock at time $t = 0$,
- K: the strike price.

Remark. *No physical transaction of the stock at time T happens. Only the profit is paid out accordingly.*

Clearly, the **payoff** for a call option at time T is

$$H = \max\{S_T - K, 0\} = (S_T - K)^+ .$$

H itself is a random variable since S_T is assumed to be random.

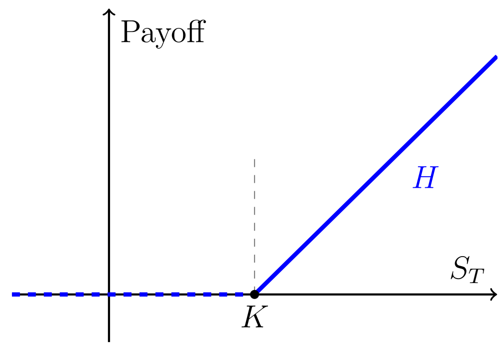

Figure: Visualisation of the payoff H dependent on the strike K.

Take $S_0 = 100$ and assume the stock can attain two prices at T. Either it attains 150 in the case of ω_1 or it attains 90 in the case of ω_2, i.e. $S_T(\omega_1) = 150$ and $S_T(\omega_2) = 90$ with probabilities p and $1 - p$ respectively. We let 130 be the strike K. The payoff is obviously 20 and 0 respectively. We assume that there is an additional, riskless investment opportunity with interest rate $r = 0$.

Question. What is the fair price $\pi(H)$ of the above option at $t = 0$?

Idea: No–arbitrage–principle. There is no arbitrage (i.e. no riskless profit) in the financial market.

The payoff H is replicated by other assets (in this case stocks and riskless investments).

The initial capital that is needed in order to replicate H corresponds to the price of the option. Otherwise, there is arbitrage.

We now want to find a **trading strategy** $(\alpha, \beta) \in \mathbb{R}^2$ with

- α, the number of stocks that we buy at $t = 0$,
- β, the investment in the riskless asset (RA).

The value of the portfolio at $t = 0$ is $V_0(\alpha, \beta) = \beta \cdot 1 + \alpha S_0$ (the RA is normalised) and the value of the portfolio at $t = T$ is $V_T(\alpha, \beta) = \beta \cdot 1 + \alpha S_T$ (recall $r = 0$). We want to *hedge* or *replicate* the payoff, i.e. choose (α, β) such that $V_T(\alpha, \beta) = H$. This means $\beta + \alpha S_T(\omega) = H(\omega)$ for any $\omega \in \{\omega_1, \omega_2\}$. Hence, we get a system of linear equations.

$$\beta + 150\alpha = 20$$
$$\beta + 90\alpha = -0.$$

Solving this for α, β yields $\alpha = 1/3$ and $\beta = -30$. Hence

$$V_0(\alpha, \beta) = -30 + 100/3 = 10/3 = \pi(H).$$

The hedging strategy that we get is to borrow 30 today and buy 1/3 stocks. The overall investment is 10/3. Two scenarios can happen at $t = T$:

(1) $S_T = 150$. Selling the stock yields $1/3 \cdot 150 = 50$. We repay our debt of 30 and yield 20 as profit.
(2) $S_T = 90$. Selling the stock yields $1/3 \cdot 90 = 30$. We repay the debt and nothing happens.

As we see, we hedged H perfectly. Now assume that we sell the call option at a price of 10/3 and invest. Then, the following can happen.

(1) $S_T = 150$. The buyer will exercise the call, he or she will buy the stock for 130 from us,
(2) we have to buy the stock for 150, we lose 20,
(3) but we obtain 20 as by hedging in (1).

(1) $S_T = 90$. Goes analogously.

Using the no–arbitrage–principle, the fair price of the option is $\pi(H) = 10/3$. For any other price, a riskless profit would be possible. Assume $\pi(H) > 10/3$. Then we get the following table:

Action at $t = 0$	**Payoff at** $t = T$
Sell $\pi(H)$	$-(S_T - K)^+$
Borrow 30 RA	-30
Buy $S_0/3$	$S_T/3$
Profit: $\pi(H) + 30 - S_0/3 = \pi(H) - 10/3 > 0$	$\Sigma = 0$

Table 1. Unfair pricing shows the possibility of riskless profit.

Assume now $\pi(H) < 10/3$. The table goes analogous to the above.

Remark. *Notice that the fair price $\pi(H)$ in this example is **independent** of the subjective probability p! In more sophisticated models, we will also adapt to this.*

1.3. Market assumptions.
In this section, we used a perfect market model.

- The markets are frictionless, i.e. there are no taxes on profits and no transaction costs for reallocating portfolios,
- short–selling is allowed at all times and arbitrary shares can be bought and sold,
- interest rates for borrowing and lending are te same as well as investments in RA,
- there are no dividend payments,
- all market participants are rational and maximise their utility.

In a perfect market there is no arbitrage. Supply and demand offset each other perfectly and there are unique prices.

2. Financial markets and the finite state space

Our goal here is to introduce general financial markets. Finite means that there is a finite number of market states and a finite number of trading times $\{0, \dots, T\}$. We describe trading strategies, arbitrage strategies and options formally.

2.1. Definition of the financial market.
We take $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ to be a filtered probability space. I.e.

- Ω is a finite state space of elementary events,
- $\mathcal{F}$ is the power set of Ω and acts as the σ-algebra,
- $(\mathcal{F}_t)_t$ is a filtration,
- $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure.

Remark. *We remark that for a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an ordered index set I, a family of $\mathcal{F}$-sub-σ-algebras $(\mathcal{F}_i)_{i \in I}$ is a filtration if and only if for any $t, s \in I$: $t \leq s \implies \mathcal{F}_t \subseteq \mathcal{F}_s$. Since our index sets I are always finite with maximal element T, we make the additional presumption that $\mathcal{F} = \mathcal{F}_T$.*

3. The Cox–Ross–Rubinstein model

4. Absence of arbitrage and equivalent martingale measures

5. Completeness and equivalent martingale measures

6. Risk–neutral pricing of contingent claims

7. American options

8. Portfolio optimization