Project 1 - Group Phase

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Problems solved

- [x] Newton Method
  - [x] Newton Method with Hessian Modification
  - [x] Standard Newton Comparison
- [x] linear CG
  - [x] Linear CG Method on Hilbert Matrix Problem
  - [x] Comparison of Linear CG with SD on Hilbert Matrix Problem
- [x] nonlinear CG methods
  - [x] FR-Method
  - [x] PR-Methods
- [x] QN methods
  - [x] BFGS
  - [x] SR1
  - [x] SR1 with Trust Region Line Search (Bonus)
- [x] Derivative approximation
  - [x] Gradient Approximation
  - [x] Hessian Approximation
- [x] Outperforming Newton's Method with Quasi-Newton (Bonus)
  - [x] with explicit derivatives

1. Newton Method

1.1 With Hessian Modification

```
Rosenbrock, p=(1.2,1.2)
Minimum point: [1.00000001 1.00000001]
Minimum value: 7.128493336170037e-17
Number of iterations: 12
Gradient: [ 1.94437006e-07 -9.00770569e-08]
Distance to solution: 1.556722369824841e-08

Rosenbrock, p=(-1.2,1)
Minimum value: 5.522678593789617e-17
Number of iterations: 25
Minimum point: [0.99999999 0.99999999]
Gradient: [ 1.45994471e-07 -7.92831134e-08]
Distance to solution: 1.4411304689250775e-08

Rosenbrock, p=(0.2,0.8)
Minimum point: [1.00000003 1.00000006]
Minimum value: 5.522678593789617e-17
Number of iterations: 25
Gradient: [ 1.45994471e-07 -7.92831134e-08]
Distance to solution: 1.4411304689250775e-08

f2, p=(-0.2,1.2)
Minimum point: [-1.48277537e-09  1.00000001e+00]
Minimum value: 4.771095333221083e-16
Number of iterations: 11
Gradient: [-4.32695230e-07  4.85495402e-08]
Distance to solution: 6.607898262226106e-09

f2, p=(3.8,0.1)
Minimum point: [4.00000000e+00 2.84305733e-11]
Minimum value: 3.1940247516021064e-18
Number of iterations: 12
Gradient: [1.11987086e-09 1.40946236e-07]
Distance to solution: 2.126209448519987e-09

f2, p=(1.9,0.6)
Minimum point: [4.0000000e+00 2.9912427e-11]
Minimum value: 3.3168565981356098e-18
Number of iterations: 14
Gradient: [1.08141052e-09 1.47905292e-07]
Distance to solution: 2.0433899159401905e-09
```

1.1 Standard Newton

```
Rosenbrock, p=(1.2,1.2)
Minimum point: [1.00000001 1.00000001]
Minimum value: 7.128493336170037e-17
Number of iterations: 12
Norm of gradient: 2.1428864992956312e-07
Distance to solution: 1.556722369824841e-08

Rosenbrock, p=(-1.2,1)
Minimum value: 6.211761725434095e-16
Number of iterations: 24
Norm of gradient: 5.712667521938612e-07
Distance to solution: 4.791897643096395e-08

Rosenbrock, p=(0.2,0.8)
Minimum point: [0.99999999 0.99999999]
Minimum value: 6.540915293266381e-17
Number of iterations: 18
Norm of gradient: 1.8078557548117722e-07
Distance to solution: 1.5684109617692892e-08

f2, p=(-0.2,1.2)
No convergence after 10001 iterations.
Minimum point: [-0.18727432  0.56730425]
Minimum value: 2.612825050148156
Number of iterations: 10001
Gradient norm: [-19.04040857   2.13277802]
Distance to solution: 0.4714841255137952

f2, p=(3.8,0.1)
No convergence after 10001 iterations.
Minimum value: 2.576003804856032
Number of iterations: 10001
Gradient norm: [-0.10654103 11.32333084]
Distance to solution: 1.2966050653251622

f2, p=(1.9,0.6)
Minimum point: [0.43685073 0.10921268]
Minimum value: 2.784867103719925
Number of iterations: 12
Gradient norm: [1.01302332e-07 2.29186784e-07]
Distance to solution: 0.9921394044367395

Standard Newton seems to have problems with the second function.
```

2. Linear CG

2.1. Linear CG on Hilbert Matrix Problem

```
############################################
Task  1
A = Hilbert, b = (1, 1, 1, 1, 1)
n =  5
Minimum found after 6 iterations: [    4.99999999
  -119.99999999
   630.00000001
 -1119.99999999
   630.00000001]
Minimum value: [1. 1. 1. 1. 1.]
||grad f(x)||: 8.86699613185672e-09
Difference to true minimum:  8.86699613185672e-09
############################################
Task  2
A = Hilbert, b = (1, 1, 1, 1, 1)
n =  8
Minimum found after 19 iterations: [-8.00000407e+00
  5.04000069e+02
 -7.56000003e+03
  4.61999980e+04
 -1.38599992e+05
  2.16215989e+05
 -1.68167993e+05
  5.14799982e+04]
Minimum value: [1. 1. 1. 1. 1. 1. 1. 1.]
||grad f(x)||: 6.1338178330774e-09
Difference to true minimum:  6.1338178330774e-09
############################################
Task  3
A = Hilbert, b = (1, 1, 1, 1, 1)
n =  12
Minimum found after 38 iterations: [-9.60921534e+00
  8.15407288e+02
 -1.64966586e+04
  1.35510701e+05
 -5.36481917e+05
  1.02540008e+06
 -6.42578530e+05
 -6.57591399e+05
  8.04244405e+05
  6.63073081e+05
 -1.24128067e+06
  4.65506943e+05]
Minimum value: [1.00000022 1.00000001 0.99999993 0.99999996 0.99999982
1.00000014
 0.99999956 1.00000016 1.00000011 0.99999962 1.00000014 0.99999983]
||grad f(x)||: 2.510237094806557e-06
Difference to true minimum:  2.510237094806557e-06
############################################
```

```
Task  4
A = Hilbert, b = (1, 1, 1, 1, 1)
n =  20
Minimum found after 75 iterations: [-1.09772145e+01
  1.05100540e+03
 -2.39569833e+04
  2.20428394e+05
 -9.65350725e+05
  1.99010296e+06
 -1.25269794e+06
 -1.34346789e+06
  8.83230941e+05
  1.68795446e+06
  3.88209251e+05
 -1.30552011e+06
 -1.71054295e+06
 -5.28240865e+05
  1.20868164e+06
  2.00288792e+06
  9.44586177e+05
 -1.43405011e+06
 -2.65094980e+06
  1.88785488e+06]
Minimum value: [0.99999962 0.99999975 0.99999962 1.00000003 0.99999993
1.00000016
 0.99999958 0.99999983 1.00000005 0.9999999  0.99999974 0.99999982
 1.00000006 1.0000002  1.00000013 0.99999992 0.9999998  0.99999992
 1.0000001  0.99999969]
||grad f(x)||: 4.303230123701973e-06
Difference to true minimum:  4.303230123701973e-06
############################################
```

```
Task  5
A = Hilbert, b = (1, 1, 1, 1, 1)
n =  30
Minimum found after 157 iterations: [-1.25690103e+01
  1.38807765e+03
 -3.67103190e+04
  3.96178550e+05
 -2.07992111e+06
  5.43183168e+06
 -5.71546372e+06
 -1.60082168e+06
  4.84350708e+06
  3.47462393e+06
 -1.96769217e+06
 -4.97771597e+06
 -3.39492975e+06
  7.57630205e+05
  4.23389237e+06
  4.94829781e+06
  2.77358781e+06
 -9.01660176e+05
 -4.14632007e+06
 -5.41788639e+06
 -4.14123260e+06
 -8.49740334e+05
  3.05420641e+06
  5.79980072e+06
  5.88114162e+06
  2.71283527e+06
 -2.77612654e+06
 -7.65879613e+06
 -6.66512556e+06
  8.02155563e+06]
Minimum value: [0.99999978 0.99999974 0.99999991 1.00000007 0.99999975
0.99999987
 0.99999983 1.00000014 1.00000026 1.00000015 1.00000001 0.99999995
 0.99999993 0.9999999  0.99999984 0.99999978 0.99999976 0.9999998
 0.99999989 1.         1.0000001  1.00000015 1.00000015 1.00000011
 1.00000006 1.00000001 0.99999998 0.99999996 0.99999987 0.9999996 ]
||grad f(x)||: 4.897558151781639e-06
Difference to true minimum:  4.897558151781639e-06
```

2.2. Comparison of Linear CG and SD (exact line search) on Hilbert Matrix Problem

The linear Conjugate Gradient (CG) Method outperformed the Steepest Descent (SD) Method (with exact line search) in every single task. While CG found a viable solution that satisfies the stopping criterion (error < 1e-6) in at most 157 iterations for n=30, SD was not able to find such a solution within the iteration limit (= 1000000) for any problem. Despite this result, the SD method using exact line search performed better than SD using backtracking, finding lower minimum values in the same number of iterations.

```
Task  1
Q = Hilbert, b = (1, 1, 1, 1, 1)
f(x) = (1/2) * x.T * Q * x - b.T * x
n =  5
Minimum found after 1000000 iterations: [[    4.97192388]
 [ -119.46958456]
 [  627.69924445]
 [-1116.51276222]
 [  628.28978481]]
Minimum value: [[-12.49996603]]
||grad f(x)||: 1.7143967334756586e-05
Difference to true minimum:  3.3966622993375495e-05
############################################
Task  2
Q = Hilbert, b = (1, 1, 1, 1, 1)
f(x) = (1/2) * x.T * Q * x - b.T * x
n =  8
Minimum found after 1000000 iterations: [[   -3.69087764]
 [  107.56008761]
 [ -633.82616144]
 [  980.22917269]
 [  407.66746495]
 [-1008.33287566]
 [-1139.3839629 ]
 [ 1331.35421876]]
Minimum value: [[-21.55600514]]
||grad f(x)||: 0.0008437910262655218
Difference to true minimum value:  10.443994902756572
############################################
Task  3
Q = Hilbert, b = (1, 1, 1, 1, 1)
f(x) = (1/2) * x.T * Q * x - b.T * x
n =  12
Minimum found after 1000000 iterations: [[-9.40784457e-01]
 [-5.03178437e+00]
 [ 2.06156614e+02]
 [-6.97318280e+02]
 [ 1.63314105e+02]
 [ 7.93919499e+02]
 [ 5.61899795e+02]
 [-2.08133893e+02]
 [-9.31697146e+02]
 [-1.08890493e+03]
 [-3.18299594e+02]
 [ 1.58612821e+03]]
Minimum value: [[-32.16293912]]
||grad f(x)||: 0.001773012356061572
Difference to true minimum value:  38.591761719860074
############################################
```

```
Task  4
Q = Hilbert, b = (1, 1, 1, 1, 1)
f(x) = (1/2) * x.T * Q * x - b.T * x
n =  20
Minimum found after 1000000 iterations: [[    2.85444964]
 [  -50.20871857]
 [  111.01519004]
 [  240.98486693]
 [ -349.80370273]
 [ -507.99750389]
 [ -189.29975215]
 [  277.16690671]
 [  616.87325736]
 [  704.44474259]
 [  536.8374806 ]
 [  184.08900408]
 [ -251.15328954]
 [ -660.0897437 ]
 [ -942.67611454]
 [-1014.6702833 ]
 [ -809.86687065]
 [ -279.63966939]
 [  608.8581274 ]
 [ 1875.02812606]]
Minimum value: [[-53.36989093]]
||grad f(x)||: 0.0022578964531068963
Difference to true minimum value:  483.5436644318868
############################################
```

```
Task  5
Q = Hilbert, b = (1, 1, 1, 1, 1)
f(x) = (1/2) * x.T * Q * x - b.T * x
n =  30
Minimum found after 1000000 iterations: [[   -3.08783708]
 [   81.39045573]
 [ -412.26412902]
 [  415.50335848]
 [  449.71713684]
 [   -8.94122185]
 [ -433.51999413]
 [ -594.39443178]
 [ -489.48732162]
 [ -214.38955722]
 [  119.50332383]
 [  422.02421308]
 [  635.01042936]
 [  730.64079148]
 [  705.15260252]
 [  571.76553225]
 [  354.42298098]
 [   82.8521705 ]
 [ -211.05762338]
 [ -495.75235607]
 [ -741.62662085]
 [ -921.97812704]
 [-1013.49068336]
 [ -996.39232687]
 [ -854.39873114]
 [ -574.52213403]
 [ -146.80275436]
 [  435.99786136]
 [ 1178.71501107]
 [ 2084.09327436]]
Minimum value: [[-80.5692617]]
||grad f(x)||: 0.0026423621145396114
Difference to true minimum value:  1337.9328586180054
```

3. Nonlinear CG

3.1. FR Method

```
Rosenbrock function with point (1.2, 1.2)
Number of iterations:  55
Solution:  [1.         1.00000001]
Gradient norm:  7.247654104065839e-07
Distance to solution:  8.401592678920707e-09
#############################################
Rosenbrock function with point (-1.2, 1)
Number of iterations:  81
Solution:  [0.99999971 0.99999943]
Gradient norm:  7.755663862962493e-07
Distance to solution:  6.418836722987867e-07
#############################################
Rosenbrock function with point (0.2, 0.8)
Number of iterations:  60
Solution:  [0.99999951 0.99999902]
Gradient norm:  4.383115345597357e-07
Distance to solution:  1.0956967102349871e-06
#############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (-0.2,
1.2)
Number of iterations:  69
Solution:  [-1.86622072e-09  1.00000003e+00]
Gradient norm:  5.551829844773979e-07
Distance to solution:  3.093070567403244e-08
#############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (3.8,
0.1)
Number of iterations:  54
Solution:  [ 4.00000139e+00 -6.32857415e-10]
Gradient norm:  7.413876391102972e-07
Distance to solution:  1.3919047105872326e-06
#############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (1.9,
0.6)
Number of iterations:  54
Solution:  [ 4.00000139e+00 -6.32857415e-10]
Gradient norm:  7.413876391102972e-07
Distance to solution:  1.3919047105872326e-06
#############################################
```

```
USING APPROXIMATED GRADIENT
#############################################
Rosenbrock function with point (1.2, 1.2)
Number of iterations:  125
Solution:  [0.99999932 0.99999863]
Gradient norm:  7.317133571918452e-07
Distance to solution:  1.5333275791379874e-06
#############################################
Rosenbrock function with point (0.2, 0.8)
Number of iterations:  73
Solution:  [1.00000027 1.00000054]
Gradient norm:  6.35768839301528e-07
Distance to solution:  6.00902420318811e-07
#############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (-0.2,
1.2)
Number of iterations:  69
Solution:  [-1.86608762e-09  1.00000003e+00]
Gradient norm:  5.551372698180946e-07
Distance to solution:  3.092492259036113e-08
#############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (3.8,
0.1)
Number of iterations:  60
Solution:  [ 4.00000062e+00 -3.40565480e-10]
Gradient norm:  5.09214033778555e-07
Distance to solution:  6.156406664975145e-07
#############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (1.9,
0.6)
Number of iterations:  96
Solution:  [3.99999859e+00 6.95565649e-10]
Gradient norm:  8.754173566703669e-07
Distance to solution:  1.4125871826125498e-06
#############################################
```

3.2. PR Method

```
Rosenbrock function with point (1.2, 1.2)
Number of iterations:  43
Solution:  [1.00000089 1.00000178]
Gradient norm:  7.962424681971362e-07
Distance to solution:  1.990730248813831e-06
#############################################
Rosenbrock function with point (-1.2, 1)
Number of iterations:  99
Solution:  [0.99999934 0.99999868]
Gradient norm:  9.194286008418815e-07
Distance to solution:  1.471766841120434e-06
#############################################
Rosenbrock function with point (0.2, 0.8)
Number of iterations:  34
Solution:  [1.00000057 1.00000115]
Gradient norm:  5.160249338583101e-07
Distance to solution:  1.2805534504152312e-06
#############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (-0.2,
1.2)
Number of iterations:  34
Solution:  [-1.83774290e-09  1.00000005e+00]
Gradient norm:  6.02509491241137e-07
Distance to solution:  5.034582541958129e-08
#############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (3.8,
0.1)
Number of iterations:  55
Solution:  [3.99999999e+00 1.31446928e-10]
Gradient norm:  6.181319202996985e-07
Distance to solution:  6.937855283616512e-09
#############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (1.9,
0.6)
Number of iterations:  67
Solution:  [ 3.99999984e+00 -2.72239578e-11]
Gradient norm:  4.4799919712483173e-07
Distance to solution:  1.5516404933263727e-07
#############################################
```

USING APPROXIMATED GRADIENT

```
############################################
Rosenbrock function with point (1.2, 1.2)
Number of iterations:  35
Solution:  [1.00000047 1.00000094]
Gradient norm:  4.292772878050737e-07
Distance to solution:  1.0461858903387703e-06
############################################
Rosenbrock function with point (0.2, 0.8)
Number of iterations:  45
Solution:  [1.00000082 1.00000164]
Gradient norm:  7.361712124829501e-07
Distance to solution:  1.831176573977433e-06
############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (-0.2,
1.2)
Number of iterations:  34
Solution:  [-1.83774290e-09  1.00000005e+00]
Gradient norm:  6.025094945900538e-07
Distance to solution:  5.034582608542679e-08
############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (3.8,
0.1)
Number of iterations:  63
Solution:  [3.99999957e+00 1.41326081e-11]
Gradient norm:  8.178183271552984e-07
Distance to solution:  4.2859946370862456e-07
############################################
f(x) = 150 * (x1*x2) ** 2 + (0.5 * x1 + 2 * 2* x2 - 2) ** 2 with point (1.9,
0.6)
Number of iterations:  62
Solution:  [3.99999918e+00 2.90934519e-10]
Gradient norm:  4.6920735309974764e-07
Distance to solution:  8.158430090658931e-07
############################################
```

4. QN Methods

4.1 BFGS Implementation

```
Rosenbrock, p=(1.2,1.2)
Minimum point: [0.99999999 0.99999999]
Minimum value: 6.52104404048973e-17
Number of iterations: 21
Gradient: [ 1.99706922e-07 -1.05948361e-07]
Distance to solution: 1.4104415287228555e-08

Rosenbrock, p=(-1.2,1)
Minimum point: [1.         1.00000001]
Minimum value: 3.272100358852412e-16
Number of iterations: 41
Gradient: [-6.92159081e-07  3.50551099e-07]
Distance to solution: 1.1592955835943192e-08

Rosenbrock, p=(0.2,0.8)
Minimum value: 4.513782224198393e-16
Number of iterations: 26
Gradient: [ 5.66220911e-07 -2.98243386e-07]
Distance to solution: 3.5178356428346117e-08

f2, p=(-0.2,1.2)
Minimum point: [-3.14603913e-09  9.99999999e-01]
Minimum value: 1.4948922651354254e-15
Number of iterations: 15
Gradient: [-9.47014538e-07 -1.28112051e-08]
Distance to solution: 3.2498630668176033e-09

f2, p=(3.8,0.1)
Minimum point: [4.00000000e+00 5.46399999e-11]
Minimum value: 1.0914566549156722e-17
Number of iterations: 12
Gradient: [-1.93630978e-09  2.54526760e-07]
Distance to solution: 4.091544252942489e-09

f2, p=(1.9,0.6)
Minimum point: [ 4.00000009e+00 -2.11925343e-10]
Minimum value: 2.109680589551455e-15
Number of iterations: 26
Gradient: [ 4.47424960e-08 -8.38271709e-07]
Distance to solution: 9.033294191924726e-08
```

4.2 SR1 (Standard Line Search) Implementation

```
Rosenbrock, p=(1.2,1.2)
No convergence after 10001 iterations.
Minimum point: [1.00005422 1.00010893]
Minimum value: 2.9636956892425516e-09
Number of iterations: 10001
Gradient: [ 1.32482596e-04 -1.17606729e-05]
Distance to solution: 0.00012167391422476956
```

Approximate solution is sufficiently close

```
Rosenbrock, p=(-1.2,1)
No convergence after 10001 iterations.
Minimum point: [1.00019406 1.00038996]
Minimum value: 3.798314950619329e-08
Number of iterations: 10001
Gradient: [ 4.85721956e-04 -4.78188522e-05]
Distance to solution: 0.00043557921732176395
```

Approximate solution is sufficiently close

```
Rosenbrock, p=(0.2,0.8)
No convergence after 10001 iterations.
Minimum point: [0.99994404 0.99988752]
Minimum value: 3.163097336858441e-09
Number of iterations: 10001
Gradient: [-1.56715457e-04  2.20835948e-05]
Distance to solution: 0.00012563372146944026
```

Approximate solution is sufficiently close

```
f2, p=(-0.2,1.2)
Minimum point: [-3.22502332e-09  1.00000004e+00]
Minimum value: 8.812968323364963e-15
Number of iterations: 262
Gradient: [-8.82343402e-07  3.40654711e-07]
Distance to solution: 4.350778661540301e-08
```

```
f2, p=(3.8,0.1)
Minimum point: [3.97585717e+00 4.06187129e-06]
Minimum value: 0.00014556215708295145
Number of iterations: 10001
Gradient: [-0.01204041  0.03043719]
Distance to solution: 0.02414283370365608
```

Approximate solution is sufficiently close

```
f2, p=(1.9,0.6)
No convergence after 10001 iterations.
Minimum point: [3.88083739e+00 2.16242984e-05]
Minimum value: 0.0035458366626300373
Number of iterations: 10001
Gradient: [-0.05940689  0.16323327]
Distance to solution: 0.11916261377073409
```

Approximate solution is sufficiently close

4.2 SR1 (Trust Region) Implementation

```
Rosenbrock, p=(1.2,1.2)
Minimum point: [0.99999905 0.9999981 ]
Minimum value: 8.969903467325634e-13
Number of iterations: 8913
Gradient norm: 9.99701924401977e-07
Distance to solution: 2.1192969901940435e-06

Rosenbrock, p=(-1.2,1)
Minimum point: [0.99999905 0.9999981 ]
Minimum value: 8.983410640802916e-13
Number of iterations: 9612
Gradient norm: 9.984625292112995e-07
Distance to solution: 2.1208943892216378e-06

Rosenbrock, p=(0.2,0.8)
Minimum point: [0.99999924 0.99999848]
Minimum value: 5.729736394101963e-13
Number of iterations: 2050
Gradient norm: 6.772409923059122e-07
Distance to solution: 1.693944970289191e-06

f2, p=(-0.2,1.2)
Minimum point: [-1.44547318e-09  1.00000010e+00]
Minimum value: 4.330613839908077e-14
Number of iterations: 183
Gradient norm: 8.59705282055386e-07
Distance to solution: 1.0404485052346189e-07

f2, p=(3.8,0.1)
Minimum point: [3.99999432e+00 1.75182364e-09]
Minimum value: 8.052412526024316e-12
Number of iterations: 49999
Gradient norm: 4.082862640663144e-06
Distance to solution: 5.679766054520579e-06

f2, p=(1.9,0.6)
Minimum point: [3.99997520e+00 1.29903258e-08]
Minimum value: 1.5354038001770454e-10
Number of iterations: 49999
Gradient norm: 1.784240370768631e-05
Distance to solution: 2.4801541548745696e-05
```

5. Derivative Approximation

5.1. Rosenbrock Function

```
========================================================================
Point: 1.2 1.2
Exact Calculation:
f grad(x) [110.80003  -48.000015]
f hessian(x)
 [[1250.00011826 -480.00001907]
 [-480.00001907  200.        ]]
Numerical approximation
f grad approximated(x) [115.04401  -47.769135]
f hessian approximated(x)
 [[1238.0177  -475.39447]
 [-475.39447  198.08067]]
Error gradient: 4.2502484
Error hessian: 13.772593018295234
========================================================================
Point: -1.2 1.0
Exact Calculation:
f grad(x) [-210.80006  -88.00002]
f hessian(x)
 [[1330.00013733  480.00001907]
 [ 480.00001907  200.        ]]
Numerical approximation
f grad approximated(x) [-214.56303   -87.576744]
f hessian approximated(x)
 [[1317.2229   475.39276]
 [ 475.39276  198.0808 ]]
Error gradient: 3.7867014
Error hessian: 14.470479611000002
========================================================================
Point: 0.2 0.8
Exact Calculation:
f grad(x) [-63.2 152. ]
f hessian(x)
 [[-270.00000334  -80.00000119]
 [ -80.00000119  200.        ]]
Numerical approximation
f grad approximated(x) [-62.487984 151.26889 ]
f hessian approximated(x)
 [[-270.75983   -79.728386]
 [ -79.728386  198.08095 ]]
Error gradient: 1.0205339
Error hessian: 2.099438554028578
========================================================================
```

5.2. f (x) = 150(x1x2)^2+(0.5x1+2x2−2)^2

```
======================================================================
Point: -0.2 1.2
Exact Calculation:
f grad(x) [-86.100006  15.600001]
f hessian(x)
 [[ 432.50003433 -142.00000787]
 [-142.00000787   20.00000036]]
Numerical approximation
f grad approximated(x) [-86.2438    15.524965]
f hessian approximated(x)
 [[-4.6298023e+04 -6.9687241e+01]
 [-6.9687241e+01  1.9808064e+01]]
Error gradient: 0.16219223
Error hessian: 46730.63537189417
======================================================================
Point: 3.8 0.1
Exact Calculation:
f grad(x) [ 11.5 433.6]
f hessian(x)
 [[3.50000009e+00 2.30000001e+02]
 [2.30000001e+02 4.33999989e+03]]
Numerical approximation
f grad approximated(x) [ 11.444685 433.64346 ]
f hessian approximated(x)
 [[3.46641111e+00 1.15453156e+02]
 [1.15453156e+02 4.34086865e+03]]
Error gradient: 0.070343904
Error hessian: 161.99603477108425
======================================================================
Point: 1.9 0.6
Exact Calculation:
f grad(x) [205.35002 650.4    ]
f hessian(x)
 [[ 108.50000858  686.0000186 ]
 [ 686.0000186  1090.99997282]]
Numerical approximation
f grad approximated(x) [204.36227 647.2716 ]
f hessian approximated(x)
 [[ 107.45875  340.6987 ]
 [ 340.6987  1080.5299 ]]
Error gradient: 3.2806468
Error hessian: 488.44314696188826
======================================================================
```

6. Outperforming the NM with a QN

Choosing the starting point for f2 different is enough. Starting from p=(2.3,0.1) yields

```
Newton method: 13 iterations
Newton method: [ 4.00000001e+00 -8.23255193e-11]
Newton method: 2.452726727631941e-17
BFGS: 12 iterations
BFGS: [4.00000000e+00 1.06653815e-10]
BFGS: 2.7361209018016954e-17
```

and for p=(3.2, 0.1)

```
Newton method: 12 iterations
Newton method: [ 4.00000001e+00 -4.90935734e-11]
Newton method: 1.237713334780343e-17
BFGS: 11 iterations
BFGS: [ 4.00000001e+00 -1.62146738e-10]
BFGS: 1.0459004863303137e-16
```