MATRICES

1. Definition

A rectangular arrangement of numbers in rows and columns, is called a matrix. Such a rectangular arrangement of numbers is enclosed by small ( ) or big [ ] brackets. Generally a matrix is represented by a capital latter A, B, C......... etc. and its element are represented by small letters a, b, c, x, y etc.

Following are some examples of a matrix :

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\quad B = \begin{bmatrix} 1 & 5 & 3 \\ 4 & 0 & 2 \end{bmatrix}$

$C = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\quad D = [1, 5, 6]$, $\quad E = [5]$

2. Order of Matrix

A matrix which has m rows and n columns is called a matrix of order m × n, and its represented by

$$A_{m\times n} \quad \text{or} \quad A = [a_{ij}]_{m\times n}$$

It is obvious to note that a matrix of order m × n contains mn elements. Every row of such a matrix contains n elements and every column contains m elements.

3. Representation of a Matrix

A matrix of order m × n is generally expressed as

$$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{ij} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2j} & .... & a_{2n} \\ .... & .... & .... & .... & .... & .... \\ a_{i1} & a_{i2} & .... & \boxed{a_{ij}} & .... & a_{in} \\ .... & .... & .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mj} & .... & a_{mn} \end{bmatrix}$$

or $\quad A = [a_{ij}]_{m\times n} \quad i = 1, 2, ..., m$

or $\quad A = [a_{ij}] \quad j = 1, 2, ..., n$

From this representation, it is clear that (i, j)th element of a matrix A is written as a_{ij}

It may also be written as $(A)_{ij}$. e.g. $\begin{bmatrix} 1 & 5 & 7 \\ -2 & 6 & 3 \\ 2 & 1 & 9 \end{bmatrix}$

$a_{11} = 1$, $a_{12} = 5$, $a_{13} = 7$; $a_{21} = -2$, $a_{22} = 6$, $a_{23} = 3$; $a_{31} = 2$, $a_{32} = 1$, $a_{33} = 9$

4. Types of Matrices

4.1 Row matrix

If in a matrix, there is only one row, then it is called a Row Matrix.

Thus $A [a_{ij}]_{m\times n}$ is a row matrix if m = 1

eg. [1, 3, 5] is a row matrix of order 1 × 3

4.2 Column Matrix

If in a matrix, there is only one column, then it is called a column matrix.

Thus A = $[a_{ij}]_{m\times n}$ is a column matrix if n = 1.

eg. $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ is column matrix of order 3 × 1.

4.3 Square matrix

If number of rows and number of column in a matrix are equal, then it is called a square matrix.

Thus A = $[a_{ij}]_{m\times n}$ is a square matrix if m = n.

eg. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3 × 3.

Note : (a) If m ≠ n then matrix is called a rectangular matrix.

(b) The elements of a square matrix A for which i = j i.e., a_{11}, a_{22}, a_{33},......a_{nn} are called diagonal elements and the line joining these elements is called the principal diagonal or of leading diagonal of matrix A.

(c) **Trace of a matrix :** The sum of diagonal elements of a square matrix. A is called the trace of matrix A which is denoted by trace A.

trace A = $\sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + ...a_{nn}$

4.4 Singleton matrix

If in a matrix there is only one element then it is called singleton matrix.

Thus

A = $[a_{ij}]_{m\times n}$ is a singleton matrix if m = n = 1.

eg. [4], [2], [b], [–5] are singleton matrices.

4.5 Null or zero matrix

If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by O.

Thus A = $[a_{ij}]_{m\times n}$ is a zero matrix if a_{ij} = 0 for all i and j.

eg. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 3 × 3.

4.6 Diagonal matrix

If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix.

Thus a square matrix

A = $[a_{ij}]$ is a diagonal matrix if a_{ij} = 0, when i ≠ j.

eg. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a diagonal matrix of order 3 × 3, which also can be denoted by diag [5, 6, 7]

Note : (a) No element f principal diagonal in diagonal matrix is zero.

(b) Number of zero is a diagonal matrix is given by $n^2 - n$ where n is a order of the matrix.

4.7 Scalar Matrix

If all the elements of the diagonal of a diagonal matrix are equal, it is called a scalar matrix.

Thus a square matrix A $[a_{ij}]$ is a scalar matrix is

$a_{ij} = \begin{cases} 0 & i \neq j \\ k & i = j \end{cases}$ where k is a constant.

eg. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is a scalar matrix.

4.8 Unit matrix

If all elements of principal diagonal in a **diagonal matrix** are 1, then it is called unit matrix. A unit matrix of order n is denoted by I_n.

Thus a square matrix

A = $[a_{ij}]$ is a unit matrix if

$a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

eg. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note : Every unit matrix is a scalar matrix.

4.9 Triangular matrix

A square matrix $[a_{ij}]$ is said to be triangular if each element above or below the principal diagonal is zero it is of two types -

(a) Upper triangular matrix : A square matrix $[a_{ij}]$ is called the upper triangular matrix, if $a_{ij} = 0$ when i > j.

eg. $\begin{bmatrix} 4 & 2 & 5 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix}$ is a upper triangular matrix of order 3 × 3

(b) Lower triangular matrix : A square matrix $[a_{ij}]$ is called the lower triangular matrix, if $a_{ij} = 0$ when i < j

eg. $\begin{bmatrix} 7 & 0 & 0 \\ 4 & 9 & 0 \\ 2 & 5 & 2 \end{bmatrix}$ is a lower triangular matrix of order 3 × 3.

Note : Minimum number of zero in a triangular matrix is given by $\frac{n(n-1)}{2}$ where n is order of matrix.

4.10 Equal matrix

Two matrix A and B are said to be equal matrix if they are of same order and their corresponding elements are equal

eg. if $A = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ are equal matrix then

$a_1 = 2$, $a_2 = 3$, $a_3 = 4$, $b_1 = 6$, $b_2 = 5$, $b_3 = 1$

4.11 Singular matrix

Matrix A is said to be singular matrix if its determinant |A| = 0, otherwise non-singular matrix i.e.,

If det |A| = 0 $\Rightarrow$ singular

and det |A| ≠ 0 $\Rightarrow$ non-singular

5. Addition and subtraction of Matrices

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of the same order then their sum A + B is a matrix whose each element is the sum of corresponding element.

i.e., $A + B = [a_{ij} + b_{ij}]_{m\times n}$

eg. If $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 2 \\ 5 & 4 \\ 3 & 7 \end{bmatrix}$ $A + B = \begin{bmatrix} 7+6 & 3+2 \\ 4+5 & 2+4 \\ 2+3 & 1+7 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 9 & 6 \\ 5 & 8 \end{bmatrix}$

then $A - B = \begin{bmatrix} 7-6 & 3-2 \\ 4-5 & 2-4 \\ 2-3 & 1-7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \\ -1 & -6 \end{bmatrix}$

Similarly their subtraction A – B is defined as

$A - B\ [a_{ij} - b_{ij}]_{m\times n}$

i.e., in above example

$A - B = \begin{bmatrix} 5-1 & 2-5 \\ 1-2 & 3-2 \\ 4-3 & 1-3 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \\ 1 & -2 \end{bmatrix}$

Note : Matrix addition and subtraction can be possible only when matrices are of same order.

5.1 Properties of matrices addition

If A, B and C are matrices of same order, then-

(i) A + B = B + A (Commutative Law)

(ii) (A + B) + C = A + (B + C) (Associative law)

(iii) A + O = O + A = A, where O is zero matrix which is additive identity of the matrix.

(iv) A + (–A) = 0 = (–A) + A where (–A) is obtained by changing the sign of every element of A which is additive inverse of the matrix

(v) $\left.\begin{matrix} A+B=A+C \\ B+A=C+A \end{matrix}\right\} \Rightarrow$ B = C (Cancellation law)

(vi) trace (A + B) = trace (A) ± trace (B)

6. Scalar Multiplication of Matrices

Let $A = [a_{ij}]_{m\times n}$ be a matrix and k be a number then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it denoted by kA.

Thus

$A = [a_{ij}]_{m\times n} \Rightarrow kA = [ka_{ij}]_{m\times n}$

e.g. if $A = \begin{bmatrix} 4 & 2 \\ 3 & 5 \\ 6 & 7 \end{bmatrix}$ then $5A = \begin{bmatrix} 20 & 10 \\ 15 & 25 \\ 30 & 35 \end{bmatrix}$

6.1 Properties of scalar multiplication

If A, B are matrices of the same order and m, n are any numbers, then the following results can be easily established.

(i) m(A + B) = mA + mB

(ii) (m + n)A = mA + nA

(iii) m(nA) = (mn)A = n(mA)

Ex.1 If $\begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} a & 1 \\ -1 & b \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}$, then value of a, b are-

[1] 1, –2 [2] –1, 2 [3] –1, –2 [4] 1, 2

Sol. Here

$$\begin{bmatrix} 1+a & 0+1 \\ 3-1 & -4+b \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}$$

$\Rightarrow$ 1 + a = 2 and –4 + b = –2 $\Rightarrow$ a = 1, b = 2

Ex.2 If X = $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and 3X – $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ then a is equal to-

[1] 1 [2] 2 [3] 0 [4] –2

Sol. 3X = $\begin{bmatrix} 3 & 3a \\ 0 & 3 \end{bmatrix}$

$\Rightarrow$ L.H.S. = $\begin{bmatrix} 3-2 & 3a-3 \\ 0-0 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 3a-3 \\ 0 & 1 \end{bmatrix}$

No by equality of two matrices, we have 3a – 3 = 3 $\Rightarrow$ a = 2.

Ex.3 If X and Y two matrices are such that X – Y = $\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ and X + Y = $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ then Y matrices is

[1] $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ [2] $\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$ [3] $\begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$ [4] none of these

Sol. Given that X – Y $\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ ....(i)

and X + Y = $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ ....(ii)

Subtracting (2) from (1)

–2 Y = $\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$

(–2) Y = $\begin{bmatrix} 3-1 & 2-(-2) \\ -1-3 & 0-4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -4 & -4 \end{bmatrix}$

$\Rightarrow$ Y = $-\frac{1}{2}\begin{bmatrix} 2 & 4 \\ -4 & -4 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$

Ex.4 A matrix A = $[a_{ij}]$ of order 2 × 3 whose elements are such that $a_{ij} = i + j$ is-

[1] $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ [2] $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 4 & 3 \end{bmatrix}$ [3] $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 5 & 4 \end{bmatrix}$ [4] none of these

Sol. a_{ij} is the element of i^{th} row and j^{th} column of matrix A

$\therefore$ $a_{11} = 1 + 1 = 2$, $a_{12} = 1 + 2 = 3$, $a_{13} = 1 + 3 = 4$

$a_{21} = 2 + 1 = 3$, $a_{22} = 2 + 2 = 4$, $a_{23} = 2 + 3 = 5$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Ex.5 If $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$ and A + B – D = 0 (zero matrix), then D matrix will be-

[1] $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 6 & 5 \end{bmatrix}$ [2] $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$ [3] $\begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$ [4] $\begin{bmatrix} 0 & -2 \\ -3 & -7 \\ -5 & -6 \end{bmatrix}$

Sol. Let $D = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$$\therefore \quad A + B - D = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-1-a & 3-2-b \\ 3+0-c & 2+5-d \\ 2+3-e & 5+1-f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $-a = 0 \Rightarrow a = 0$, $1 - b = 0 \Rightarrow b = 1$,

$3 - c = 0 \Rightarrow c = 3$, $7 - d = 0 \Rightarrow d = 7$,

$5 - e = 0 \Rightarrow e = 5$, $6 - f = 0 \Rightarrow f = 6$

$$\therefore \quad D = \begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$$

Ex.6 If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & k & 5 \\ 4 & 2 & 1 \end{bmatrix}$ is a singular matrix then k is equal to-

[1] –1 [2] 8 [3] 4 [4] –8

Sol. A is singular $\Rightarrow$ $|A| = 0$

$$\Rightarrow \begin{vmatrix} 1 & -3 & 2 \\ 2 & k & 5 \\ 4 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $1(k - 10) + 3(2 - 20) + 2(4 - 4k) = 0$

$\Rightarrow$ $7k + 56 = 0 \Rightarrow k = -8$

7. Multiplication of matrices

If A and B be any two matrices, then their product AB will be defined only when number of column in A is equal to the number of rows in B. If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ then their product $AB = C = [c_{ij}]$, will be matrix of order m × p, where

$$(AB)_{ij} = C_{ij}$$

$$= \sum_{r=1}^{n} a_{ir} b_{rj}$$

eg. If $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

then $AB = \begin{bmatrix} 1.1+4.2+2.1 & 1.2+4.2+2.3 \\ 2.1+3.2+1.1 & 2.2+3.2+1.3 \end{bmatrix}$

$AB = \begin{bmatrix} 11 & 16 \\ 9 & 13 \end{bmatrix}$

7.1 Properties of matrix multiplication

If A, B and C are three matrices such that their product is defined, then

(i) $AB \neq BA$ (Generally not commutative)

(ii) $(AB)C = A(BC)$ (Associative Law)

(iii) $IA = A = AI$ I is identity matrix for matrix multiplication

(iv) $A(B + C) = AB + AC$ (Distributive law)

(v) If $AB = AC \not\Rightarrow B = C$ (cancellation Law is not applicable)

(vi) If $AB = 0$ It does not mean that $A = 0$ or $B = 0$, again product of two non-zero matrix may be zero matrix.

(vii) trance (AB) = trance (BA)

Note : (i) The multiplication of two diagonal matrices is again a diagonal matrix.

(ii) The multiplication of two triangular matrices is again a triangular matrix.

(iii) The multiplication of two scalar matrices is also a scalar matrix.

(iv) If A and B are two matrices of the same order, then

(a) $(A + B)^2 = A^2 + B^2 + AB + BA$

(b) $(A - B)^2 = A^2 + B^2 - AB - BA$

(c) $(A - B)(A + B) = A^2 - B^2 + AB - BA$

(d) $(A + B)(A - B) = A^2 - B^2 - AB + BA$

(e) $A(-B) = (-A)B = -(AB)$

7.2 Positive Integral powers of a matrix

The positive integral powers of a matrix A are defined only when A is a square matrix.

Also then $A^2 = A.A \quad A^3 = A.A.A = A^2A$

Also for any positive integers m, n

(i) $A^m A^n = A^{m+n}$ (ii) $(A^m)^n = A^{mn} = (A^n)^m$

(iii) $I^n = I, I^m = I$ (iv) $A^\circ = I_n$ where A is a square matrices of order n.

Ex.7 If matrix P = $\begin{bmatrix} 0 & -\tan\phi/2 \\ \tan\phi/2 & 0 \end{bmatrix}$ then (I – P) $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ is equal to-

[1] p [2] P + I [3] I [4] none of these

Sol. $I - P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan\phi/2 \\ \tan(\phi/2) & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & +\tan(\phi/2) \\ -\tan(\phi/2) & 1 \end{bmatrix}$

$\therefore$ $(I - P)\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

$= \begin{bmatrix} 1 & \tan(\phi/2) \\ -\tan(\phi/2) & 1 \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

$= \begin{bmatrix} \cos\phi + \tan(\phi/2)\sin\phi & -\sin\phi + \tan(\phi/2)\cos\phi \\ -\tan(\phi/2)\cos\phi + \sin\phi & \tan(\phi/2)\sin\phi + \cos\phi \end{bmatrix}$

$= \begin{bmatrix} 1 - 2\sin^2(\phi/2) + 2\sin^2(\phi/2) & -2\sin(\phi/2)\cos(\phi/2) + \tan(\phi/2)(2\cos^2(\phi/2) - 1) \\ -\tan(\phi/2)(2\cos^2(\phi/2) - 1) + 2\sin(\phi/2)\cos(\phi/2) & \tan(\phi/2)(2\sin(\phi/2)\cos(\phi/2)) + (1 - 2\sin^2(\phi/2)) \end{bmatrix}$

$= \begin{bmatrix} 1 & -\tan(\phi/2) \\ \tan(\phi/2) & 1 \end{bmatrix} = I + P$

Ex.8 If A = $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $A^2 - 4A - nI = 0$, then n is equal to-

[1] 3 [2] –3 [3] 1/3 [4] –1/3

Sol. $A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$, $4A = \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix}$, $nI = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$

$\Rightarrow$ $A^2 - 4A - nI$

$= \begin{bmatrix} 5 - 8 - n & -4 + 4 - 0 \\ -4 + 4 - 0 & 5 - 8 - n \end{bmatrix}$

$= \begin{bmatrix} -3 - n & 0 \\ 0 & -3 - n \end{bmatrix}$

$\therefore$ $A^2 - 4A - nI = 0$

$\Rightarrow$ $\begin{bmatrix} -3 - n & 0 \\ 0 & -3 - n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $-3 - n = 0$

$\Rightarrow$ $n = -3$

Ex.9 If $[1 \times 2]\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ 1 \\ -1 \end{bmatrix} = 0$, then the value of x is-

[1] –1 [2] 0 [3] 1 [4] 2

Sol. The LHS of the equation

$= [2 \;\; 4x + 9 \;\;\; 2x + 5]\begin{bmatrix} x \\ 1 \\ -1 \end{bmatrix}$

$= [2x + 4x + 9 - 2x - 5] = 4x + 4$

Thus $4x + 4 = 0 \;\; \Rightarrow \;\; x = -1$

Ex.10 If $E(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then value of $E(\alpha) \,.\, E(\beta)$ is-

[1] $E(0°)$ [2] $E(90°)$ [3] $E(\alpha + \beta)$ [4] $E(\alpha - \beta)$

Sol. $E(\alpha).\, E(\beta)$

$= \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$

$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$

$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = E(\alpha+\beta)$

Ex.11 If $A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$ then element a_{21} of A^2 is-

[1] 22 [2] –15 [3] –10 [4] 7

Sol. The element a_{21} is product of second row of A to the first column of A

$\therefore \qquad a_{21} = [3 \;\; -4]\begin{bmatrix} -1 \\ 3 \end{bmatrix} = -3 - 12 = -15$

Ex.12 If $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ then

[1] x = 2, y = 1 [2] x = 1, y = 2 [3] x = 3, y = 2 [4] x = 2, y = 3

Sol. The given matrix equation can be written as

$\begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$\Rightarrow \qquad x + 2y = 5$ and $2x + y = 4$

$\Rightarrow \qquad x = 1, y = 2$

8. Transpose of Matrix

If we interchange the rows and column of a matrix A, then the matrix so obtained is called the transpose of A and it is denoted by

A^T or A^t or A'

From this definition it is obvious to note that

(i) Order of A is $m \times n \Rightarrow$ order of A^T is $n \times m$

(ii) $(A^T)_{ij} = (A)_{ji}, \forall$ i, j)

8.1 Properties of Transpose

If A, B are matrices of suitable order then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(A - B)^T = A^T - B^T$

(iv) $(kA)^T = kA^T$

(v) $(AB)^T = B^TA^T$

(vi) $(A_1A_2.....A_n)^T = A_n^T.....A_2^TA_1^T$

(vii) $(A^n)^T = (A^T)^n, n \in N$

Ex.13 If $A = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$ then B^TA^T is equal to-

[1] $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ [2] $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ [3] $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ [4] $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Sol. $B^TA^T = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 8-7 & -28+28 \\ 2-2 & -7+8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ex.14 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$ then $(AB)^T$ equals-

[1] $\begin{bmatrix} 5 & 16 \\ 9 & 16 \end{bmatrix}$ [2] $\begin{bmatrix} 5 & 9 \\ 16 & 12 \end{bmatrix}$ [3] $\begin{bmatrix} 5 & 9 \\ 4 & 3 \end{bmatrix}$ [4] none of these

Sol. $AB = \begin{bmatrix} 3+2 & 4+12 \\ 9+0 & 12+0 \end{bmatrix}$

$= \begin{bmatrix} 5 & 16 \\ 9 & 12 \end{bmatrix}$

$\therefore$ $(AB)^T = \begin{bmatrix} 5 & 9 \\ 16 & 12 \end{bmatrix}$

9. Symmetric and skew-symmetric Matrix

(a) Symmetric matrix : A square matrix A = $[a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all $i_1 j$ or $A^T = A$.

eg. $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Note : (i) Every unit matrix and square zero matrix are symmetric matrices.

(ii) Maximum number of different element in a symmetric matrix is $\frac{n(n+1)}{2}$

(b) Skew-symmetric matrix : A square matrix A = $[a_{ij}]$ is called skew-symmetric matrix if

$a_{ij} = -a_{ji}$ for all i, j

or $A^T = -A$ eg. $\begin{bmatrix} o & h & g \\ -h & o & f \\ -g & -f & o \end{bmatrix}$

Note : (i) All principal diagonal elements of a skew-symmetric matrix are always zero because for any diagonal element -

$$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$$

(ii) Trace of a skew symmetric matrix is always 0

9.1 Properties of symmetric and skew-symmetric matrices

(i) If A is a square matrix, then $A + A^T$, AA^T, A^TA are symmetric matrices while $A - A^T$ is skew-symmetric matrices.

(ii) If A, B are two symmetric matrices, then-

(a) A ± B, AB + BA are also symmetric matrices.

(b) AB – BA is a skew-symmetric matrix.

(c) AB is a symmetric matrix when AB = BA

(iii) If A, B are two skew-symmetric matrices, then-

(a) A ± B, AB – BA are skew-symmetric matrices.

(b) AB + BA is a symmetric matrix.

(iv) If A is a skew-symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

(v) Every square matrix A can unequally be expressed as sum of a symmetric and skew symmetric matrix i.e.,

$$A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$$

Ex.15 If A = $\begin{bmatrix} -1 & 7 \\ 2 & 3 \end{bmatrix}$, then skew-symmetric part of A is-

[1] $\begin{bmatrix} -1 & 9/2 \\ -9/2 & 3 \end{bmatrix}$ [2] $\begin{bmatrix} 0 & -5/2 \\ 5/2 & 0 \end{bmatrix}$ [3] $\begin{bmatrix} -1 & -9/2 \\ 9/2 & 3 \end{bmatrix}$ [4] $\begin{bmatrix} 0 & 5/2 \\ -5/2 & 0 \end{bmatrix}$

Sol. Let A = B + C, where B = $\frac{1}{2}(A + A^T)$ and C = $\frac{1}{2}(A - A^T)$ are respectively symmetric and skew symmetric part of A.

Now C = $\frac{1}{2}\left\{\begin{bmatrix} -1 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 7 & 3 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5/2 \\ -5/2 & 0 \end{bmatrix}$

10. Determinant of a Matrix

If A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix, then its determinant, denoted by |A| or det. (A) is defined as

$$|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

10.1 Properties of the determinant of a matrix

(i) |A| exist $\Leftrightarrow$ A is a square matrix

(ii) |AB| = |A| |B|

(iii) $|A^T| = |A|$

(iv) $|kA| = k^n |A|$, if A is a square matrix of order n.

(v) If A and B are square matrices of same order then |AB| = |BA|

(vi) If A is skew symmetric matrix of odd order then |A| = 0

(vii) If A = diag $(a_1 a_2, ...a_n)$ then $|A| = a_1 a_2....a_n$

(viii) $|A|^n = |A^n|$, $n \in N$

11. Adjoint of a Matrix

If every element of a square matrix A be replaced by its cofactor in |A|, then the transpose of the matrix so obtained is called the adjoint of A and it is denoted by adj A

Thus if $A = [a_{ij}]$ be a square matrix and F_{ij} be the cofactor of a_{ij} in |A|, then

$$\text{adj } A = [F_{ij}]^T$$

$$\Rightarrow \quad (\text{adj } A)_{ij} = F_{ij}$$

Hence if $A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}$, then

$$\text{adj } A = \begin{bmatrix} F_{11} & F_{12} & .... & F_{1n} \\ F_{21} & F_{22} & .... & F_{2n} \\ .... & .... & .... & .... \\ F_{n1} & F_{n2} & .... & F_{nn} \end{bmatrix}^T = \begin{bmatrix} F_{11} & F_{21} & .... & F_{n1} \\ F_{12} & F_{22} & .... & F_{n2} \\ .... & .... & .... & .... \\ F_{1n} & F_{2n} & .... & F_{nn} \end{bmatrix}$$

11.1 Properties of Adjoint Matrix

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A (\text{adj } A) = |A| I_n = (\text{adj } A) A$

(Thus A (adj A) is always a scalar matrix)

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj } (\text{adj } A) = |A|^{n-2} A$

(iv) $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$

(v) $\text{adj } (A^T) = (\text{adj } A)^T$

(vi) adj (AB) = (adj B) (adj A)

(vii) $\text{adj } (A^m) = (\text{adj } A)^m,\ m \in N$

(viii) $\text{adj } (kA) = k^{n-1} (\text{adj } A),\ k \in R$

(ix) $\text{adj } (I_n) = I_n$

(x) adj 0 = 0

(xi) A is symmetric $\Rightarrow$ adj A is also symmetric.

(xii) A is diagonal $\Rightarrow$ adj A is also diagonal.

(xiii) A is triangular $\Rightarrow$ adj A is also triangular.

(xiv) A is singular $\Rightarrow$ |adj A| = 0

Ex.16 If $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, then |adj (adj A)| is equal -

[1] 8 [2] 16 [3] 2 [4] 0

Sol. $|A| = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = 2$

$\therefore$ $|\text{adj } (\text{adj. } A)| = |A|^{(n-1)^2} = |A|^{2^2}$ [$\because$ Here n = 3]

$= 2^4 = 16$

Ex.17 If $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \\ 5 & 1 & 3 \end{bmatrix}$, then adj. A is equal to-

[1] $\begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$ [2] $\begin{bmatrix} -14 & 4 & 22 \\ 4 & 22 & -14 \\ 22 & -14 & 4 \end{bmatrix}$ [3] $\begin{bmatrix} 14 & 4 & -22 \\ 4 & -22 & -14 \\ -22 & -14 & -4 \end{bmatrix}$ [4] none of these

Sol. $\text{adj. } A = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}^T = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$

Ex.18 If $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$, then adj (adj A) is equal to-

[1] $8\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$ [2] $16\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$ [3] $64\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$ [4] none of these

Sol. $|A| = \begin{vmatrix}2&0&0\\2&2&0\\2&2&2\end{vmatrix} = (2)\,(2)\,(2) = 8$

now adj (adj A) = $|A|^{3-2}$ A

$= 8\begin{vmatrix}2&0&0\\2&2&0\\2&2&2\end{vmatrix} = 16\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$

12. Inverse Matrix

If A and B are two matrices such that

$$AB = I = BA$$

then B is called the inverse of A and it is denoted by A^{-1}. Thus

$$A^{-1} = B \iff AB = I = BA$$

Further we may note from above property (i) of adjoint matrix that if $|A| \neq 0$, then

$$A\frac{\text{adj }(A)}{|A|} = I = \frac{(\text{adj }A)}{|A|}A$$

$$\Rightarrow \qquad A^{-1} = \frac{1}{|A|}\text{ adj }A$$

Thus A^{-1} exists $\iff$ $|A| \neq 0$.

Note :

(i) Matrix A is called invertible if A^{-1} exists.

(ii) Inverse of a matrix is unique.

12.1 Properties of Inverse Matrix

(i) $(A^{-1})^{-1} = A$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(A^n)^{-1} = (A^{-1})^n$, $n \in N$

(v) adj (A^{-1}) = $(\text{adj }A)^{-1}$

(vi) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

(vii) $A = \text{diag }(a_1 a_2 .... a_n) \Rightarrow A^{-1} = \text{diag }(a_1^{-1}\, a_2^{-1} ..... a_n^{-1})$

(viii) A is symmetric $\Rightarrow$ A^{-1} is also symmetric.

(ix) A is diagonal $|A| \neq 0 \Rightarrow A^{-1}$ is also diagonal.

(x) A is scalar matrix $\Rightarrow$ A^{-1} is also scalar matrix.

(xi) A is triangular $|A| \neq 0$ $\Rightarrow$ A^{-1} is also triangular.

Ex.19 Inverse matrix of $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix}$ is-

[1] $-\frac{1}{8}\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$ [2] $-\frac{1}{8}\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ [3] $\frac{1}{8}\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$ [4] $\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$

Sol. Let the given matrix is A, then $|A| = -8$

and adj A = $\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{|A|}$ adj A $= \frac{1}{8}\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$

Ex.20 If A = $\begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and M = AB, then M^{-1} is equal to-

[1] $\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$ [2] $\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 1/6 \end{bmatrix}$ [3] $\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$ [4] $\begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/6 \end{bmatrix}$

Sol. M = $\begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$

$|M| = 6$, adj M = $\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$

$\therefore$ $M^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$

13. Some important cases of Matrices

13.1 Orthogonal Matrix

A square matrix A is called orthogonal if

$AA^T = I = A^TA$; i.e., if $A^{-1} = A^T$

eg. A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a orthogonal matrix because here $A^{-1} = A^T$

13.2 Idempotent matrix

A square matrix A is called an idempotent matrix if

$A^2 = A$

eg. $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is a idempotent matrix because here $A^2 = A$

13.3 Involutory Matrix

A square matrix A is called an involutory matrix if

$A^2 = I$ or $A^{-1} = A$

eg. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a involutory matrix.

13.4 Nilpotent matrix

A square matrix A is called a nilpotent matrix if there exist a $p \in N$ such that

$A^P = 0$

eg. $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ is a nilpotent matrix.

13.5 Hermition matrix

A square matrix A is skew-Hermition matrix if

$A^\theta = A$; i.e., $a_{ij} = -\ \overline{a}_{ji}\ \forall\ i, j$

13.6 Skew hermitian matrix

A square matrix A is skew-hermition is

$A = -A^\theta$

i.e., $aij = -\ \overline{a}_{ji}\ \forall\ i, j$

13.7 Period of a matrix

If for any matrix A

$A^{k+1} = A$

then k is called period of matrix (where k is a least positive integer)

eg. If $A^3 = A$, $A^5 = A$, $A^7 = A$,.......then it is a period matrix and $A^{2+1} = A$ so its period is = 2

13.8 Differentiation of matrix

If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & \ell(x) \end{bmatrix}$

then $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & \ell'(x) \end{bmatrix}$ is a differentiation of matrix A

eg. if $A = \begin{bmatrix} x^2 & \sin x \\ 2x & 2 \end{bmatrix}$ then $\frac{dA}{dx}\begin{bmatrix} 2x & \cos x \\ 2 & 0 \end{bmatrix}$

13.9 Submatrix

Let A be m × n matrix, then a matrix obtained by leaving some rows or columns or both of a is called a sub matrix of A eg. if A' = $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 2 \\ 2 & 5 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}$ are sub matrices of matrix A = $\begin{bmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & 2 & 4 \\ 2 & 5 & 3 & 1 \end{bmatrix}$

13.10 Rank of a matrix

A number r is said to be the rank of a m × n matrix A if

(a) every square sub matrix of order (r + 1) or more is singular and

(b) there exists at least one square submatrix of order r which is non-singular.

Thus, the rank of matrix is the order of the highest order non-singular sub matrix.

eg. The rank of matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 4 & 5 \end{bmatrix}$ is

We have |A| = 0 therefore r (A) is less then 3, we observe that $\begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$ is a non-singular square sub matrix of order 2 hence r (A) 2.

Note :

(i) The rank of the null matrix is not defined and the rank of every non null matrix is greater than or equal to one.

(ii) The rank of matrix is same as the rank of its transpose i.e., r(A) = r(A^T)

(iii) Elementary transformation of not alter the rank of matrix.