Algebraic Properties of Qualitative Spatio-Temporal Calculi

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And now . . .

Qualitative spatio-temporal representation and reasoning

= Symbolic way to

- represent spatio-temporal knowledge
- and draw inferences from it

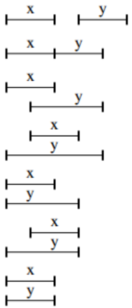
- Common approach:
  define set $\mathcal{R}$ of relations to describe spatial relationships
  - use $\mathcal{R}$ as primitives for representation
  - employ techniques from constraint and qualitative reasoning to reason about the primitives
  - various domains, typically infinite

Applications by domain

Time interval relations

- Medical diagnostics *Simplified Allen*
- Law texts *Allen-13*
- Business and manufacturing: diagnostics *Allen-13*

Relation	Symbol	Inverse	Meaning
x before y	b	bi	
x meets y	m	mi	
x overlaps y	o	oi	
x during y	d	di	
x starts y	s	si	
x finishes y	f	fi	
x equal y	eq	eq	

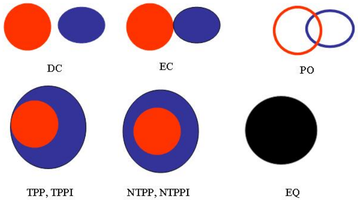


Applications by domain

Positions, regions (topology)

- Planning, robotics, navigation — *RCC, Block Algebra, ROC*
- Natural language processing — *Rectangle Alg., . . .*
- Image understanding — *9-int, CarDir, RCC, Allen*
- GIS, spatial query answering — *9-int, CarDir, RCC*
- Traffic tracking — *9-intersection*
- CAD and manufacturing — *LR, RCC*

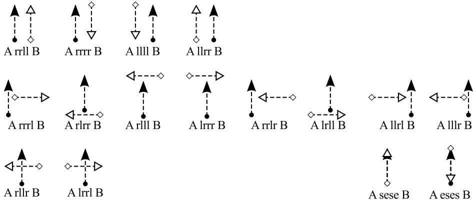
RCC-8

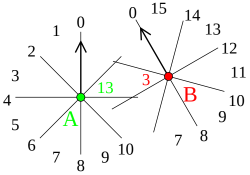


Applications by domain

Moving point objects, directional information

- Robotics, navigation, motion planning — *OPRA, Dipole Flipflop, StarVars*
- GIS, spatial query answering — *QTC*
- Traffic tracking — *Dipole*
- Ambient intelligence, smart environments (scene analysis, task modelling) — *OPRA, RCC*





. . .

Dipole $\mathrm{DRA}_{\mathrm{fp}}$

OPRA_4

Representation and reasoning tasks required

- Knowledge representation
- Data interpretation
- Inference
  - Constraint-based reasoning
    (CSP-SAT, -ENT, -MOD, -MIN)
  - Neighbourhood-based reasoning
    (Relaxing constraints, continuity constraints, dominance space)
  - Logical reasoning
    (Deduction, abduction)
  - Learning
    (Inductive logic programming)

What is a qualitative calculus?

Some answers from the literature

- A weak representation of a non-associative relation algebra
  [Egenhofer & Rodríguez 1999; Ligozat et al. 2003; Ligozat & Renz 2004]
- A system of relations forming a constraint algebra
  [Nebel & Scivos 2002]

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Commonly agreed ingredients

- Set $\mathcal{R}$ of relations
- Operations $\cup, \cap, \bar{\ }, \circ, \breve{\ }$ with certain properties
  (closure, algebraic properties)
- Mapping to a domain with certain properties
  (e.g., JEPD)

And in reality?

Zoo of qualitative calculi

36 implemented in SparQ; many more in the literature

$\updownarrow$

"classical" calculi, usually with strong algebraic properties
(e.g., Allen-13, RCC-8) [Allen 1983, Randell et al. 1992]

more recent calculi, often with weaker algebraic properties
(e.g., Cardinal Direction Relations, Rectangular Card. Relations)
[Skiadopoulos & Koubarakis 2004, Navarrete et al. 2013]

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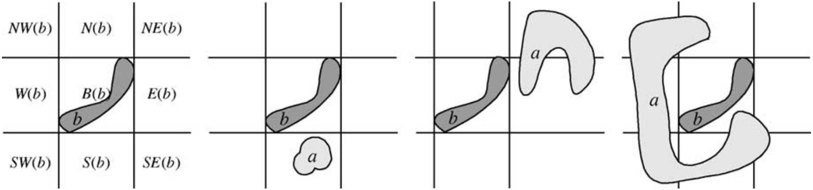

(tiles) $a\ S\ b$ $a\ NE{:}E\ b$ $a\ B{:}E{:}SE{:}\cdots{:}N\ b$

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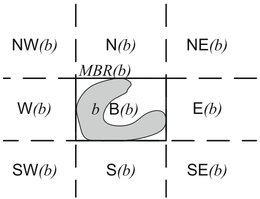

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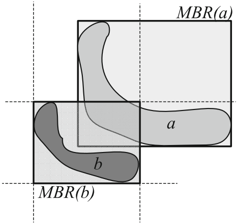
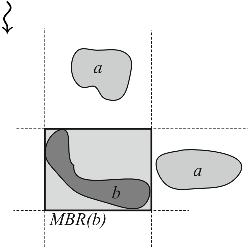


a B:N:E b



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Research programme

Central question

To what extent do existing calculi meet the imposed requirements?

$\rightsquigarrow$ If not, what are minimal requirements?

$\rightsquigarrow$ How can we classify calculi according to their properties?

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On the agenda

- Revisit and generalise definition of a qualitative calculus
- Identify notions of algebras that cover existing calculi
- Discuss relevance of algebraic properties for spatial reasoning
- Evaluate algebraic properties of existing calculi
- Examine information-preservation properties during reasoning

And now . . .

Calculi à la Ligozat & Renz (1)

Basic notions

- **Universe** (domain) $\mathcal{U}$: spatio-temporal entities (points, regions, . . .)
- Set $\mathcal{R}$ of **base relations** over $\mathcal{U}$
  - Restriction to binary relations in this work
  - $\mathcal{R}$ is **JEPD**: jointly exhaustive and pairwise disjoint
    $\rightsquigarrow$ each two entities are in *exactly one* base relation
  - Uncertain information $\rightsquigarrow$ union of base relations

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Partition scheme

- Pair $(\mathcal{U}, \mathcal{R})$ with $\mathcal{R}$ being JEPD
- $\mathcal{R}$ contains the identity relation id and is closed under converse $\breve{}$

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- Plus interpretation φ = mapping to a partition scheme
- Plus symbolic operations $\breve{}, \diamond$ (converse, *weak* composition)

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  - $\breve{}, \diamond$ usually given by tables

These requirements are strong

Some calculi violate them

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$\leadsto$ **Weaken requirements to partition schemes and calculi!**

Our notion of a calculus

Abstract partition scheme

- Pair $(\mathcal{U}, \mathcal{R})$ with $\mathcal{R}$ being JEPD
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Qualitative spatio-temporal reasoning

Qualitative constraint

Formula xRy with x, y variables, R relation from a calculus $\mathcal{C}$

Qualitative spatio-temporal reasoning

Qualitative constraint

Formula xRy with x, y variables, R relation from a calculus $\mathcal{C}$

Qualitative constraint satisfaction problem (QCSP)

- Input: set of constraints
- Question: Is there a mapping from variables to $\mathcal{C}$'s domain that satisfies all constraints?

(Analogous definition for other reasoning problems)

Qualitative spatio-temporal reasoning

Common techniques for solving QCSPs

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- Some taken over from finite-domain CSPs (constraint propagation, *k*-consistency)
- Algebraic closure (a-closure)
  - necessary condition for consistency guaranteed by "$\supseteq$" of abstract composition
  - For some calculi, a-closure known to be sufficient too

Existing qualitative spatio-temporal calculi

Name	Domain	#BR
9-Intersection	simple 2D regions	8
● Allen's interval relations	intervals (order)	13
● Block Algebra	n-dimensional blocks	13^n
● Cardinal Dir. Calculus CDC	directions (point abstr.)	9
Cardinal Dir. Relations CDR	regions	218
CycOrd, binary $\mathrm{CYC_b}$	oriented lines	4
● Dependency Calculus	points (partial order)	5
Dipole Calculus[a] $\mathrm{DRA_f}$	directions from line segm.	72
$\mathrm{DRA_{fp}}$	directions from line segm.	80
DRA-connectivity	connectivity of line segm.	7
Geometric Orientation	relative orientation	4
INDU	intervals (order, rel. dur.n)	25
OPRA_m, $m = 1, \ldots, 8$ (Oriented Point Rel. Algebra)	oriented points	$4m \cdot (4m+1)$
● Point Calculus	points (total order)	3
Qualitat. Traject. Calc. $\mathrm{QTC_{B11}}$	moving point obj.s in 1D	9
$\mathrm{QTC_{B12}}$	"	17
$\mathrm{QTC_{B21}}$	moving point obj.s in 2D	9
$\mathrm{QTC_{B22}}$	"	27
$\mathrm{QTC_{C12}}$	"	81
$\mathrm{QTC_{C22}}$	"	305
● Region Connection Calc. RCC-5	regions	5
● RCC-8	regions	8
● Rectangular Cardinal Rel.s RDR	regions	36
Star Algebra $\mathrm{STAR_4}$	directions from a point	9

a-closure decides consistency

And now . . .

Why relation algebras (RAs)?

Nice computational properties of calculi that are RAs:

e.g., converse being **strong** and **distributive**

$$(r^{\smile})^{\smile} = r \quad \text{and} \quad (r \diamond s)^{\smile} = s^{\smile} \diamond r^{\smile}$$

ensures that $yR'x$ follows from xRy

$$\begin{tikzcd}
x \arrow[r, "r"] \arrow[d, "t"'] & \bullet \arrow[d, "s"] \\
\bullet \arrow[r, "u"'] & y
\end{tikzcd}$$

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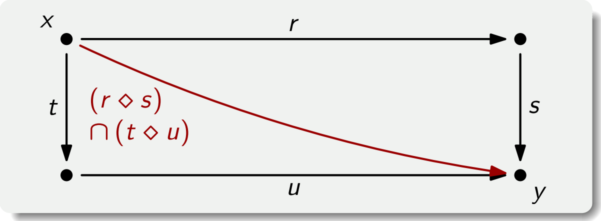

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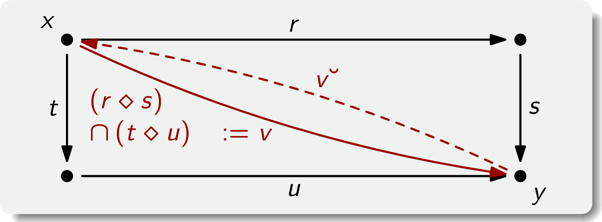

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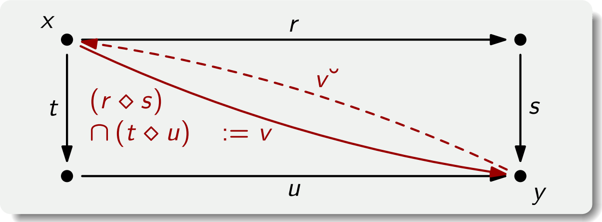

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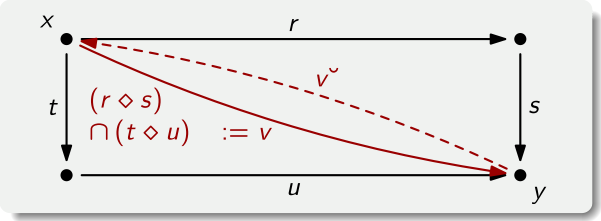

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RAs have been considered for spatio-temporal calculi before.

e.g., [Düntsch et al. 1999, Ligozat & Renz 2004, F. Mossakowski 2007]

What is a relation algebra?

- **RA** = set of relations plus operations $\cup, \cap, \bar{}, \diamond, \breve{}$ satisfying

R_1	$r \cup s$	$=$	$s \cup r$	$\cup$-commutativity
R_2	$r \cup (s \cup t)$	$=$	$(r \cup s) \cup t$	$\cup$-associativity
R_3	$\overline{\bar{r} \cup \bar{s}} \cup \overline{\bar{r} \cup s}$	$=$	r	Huntington's axiom
R_4	$r \diamond (s \diamond t)$	$=$	$(r \diamond s) \diamond t$	$\diamond$-associativity
R_5	$(r \cup s) \diamond t$	$=$	$(r \diamond t) \cup (s \diamond t)$	$\diamond$-distributivity
R_6	$r \diamond \mathsf{id}$	$=$	r	identity law
R_7	$(r^{\smile})^{\smile}$	$=$	r	$^{\smile}$-involution
R_8	$(r \cup s)^{\smile}$	$=$	$r^{\smile} \cup s^{\smile}$	$^{\smile}$-distributivity
R_9	$(r \diamond s)^{\smile}$	$=$	$s^{\smile} \diamond r^{\smile}$	$^{\smile}$-involutive distributivity
R_{10}	$r^{\smile} \diamond \overline{r \diamond s} \cup \bar{s}$	$=$	$\bar{s}$	Tarski/de Morgan axiom

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- **WA, SA** weakly/semi-associative RA weakenings of R_4

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Calculi à la Ligozat & Renz: based on NAs (by definition)

Testing algebraic properties of calculi

Research questions

1. Which calculi correspond to RAs (NAs, WAs, SAs)?
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Experimental setup

- Corpus: 31 calculi listed before
- Used HETS (Heterogeneous Tool Set) to test
  - the SparQ implementation of each calculus
  - against CASL specifications of the RA axioms (+ weakenings)
- Some axioms trivially hold $\leadsto$ no need to test them

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- Parallel tests via SparQ's built-in function `analyze-calculus`

Test results per calculus

Calculus	R_4	SA	WA	R_6	R_{6l}	R_7	R_9	PL	R_{10}
Allen	✓	✓	✓	✓	✓	✓	✓	✓	✓
Block Algebra	✓	✓	✓	✓	✓	✓	✓	✓	✓
Cardinal Direction *Calculus*	✓	✓	✓	✓	✓	✓	✓	✓	✓
CYC_b, Geometric Orientation	✓	✓	✓	✓	✓	✓	✓	✓	✓
DRA_{fp}, DRA-conn.	✓	✓	✓	✓	✓	✓	✓	✓	✓
Point Calculus	✓	✓	✓	✓	✓	✓	✓	✓	✓
RCC-5, Dependency Calc.	✓	✓	✓	✓	✓	✓	✓	✓	✓
RCC-8, 9-Intersection	✓	✓	✓	✓	✓	✓	✓	✓	✓
$STAR_4$	✓	✓	✓	✓	✓	✓	✓	✓	✓
DRA_f	19	✓	✓	✓	✓	✓	✓	✓	✓
INDU	12	✓	✓	✓	✓	✓	✓	✓	✓
$OPRA_n$, $n \leqslant 8$	21–91[b]	✓	✓	✓	✓	✓	✓	✓	✓
QTC_{Bxx}	✓	✓	✓	89–100		✓	✓	✓	✓
QTC_{C21}	55	✓	✓	99	99	✓	2	<1	1
QTC_{C22}	79	✓	✓	99	99	✓	3	<1	1
Rectang. Direction Relations	✓	✓	✓	97	92	89	66	7	52
Cardinal Direction *Relations*	28	17	✓	99	99	98	12	<1	88

Test results per algebra notion

Relation algebra (RA)

9-Intersection, Allen, Block Alg., Card. Dir. *Calculus*, CYC_b, Dependency Calc., DRA_{fp}, DRA-connectivity, Geometric Orientation, Point Calc., RCC-5, -8, $STAR_4$

"RA minus id law"

QTC_{B11}, QTC_{B12}, QTC_{B21}, QTC_{B22}

Semi-associative relation algebra

DRA_f, INDU, $OPRA_n$ ($n = 1, \ldots, 8$)

Associative Boolean alg.

Rectangular Dir. Relations

Semi-associative Boolean alg. with conv-involution

QTC_{C21}, QTC_{C22}

Weakly associative Boolean algebra

Cardinal Direction *Relations*

(Abstract partition scheme yields Boolean algebra with distributivity)

What do we gain from these results?

In the paper: discussion on relevance of axioms to reasoning

- $\leadsto$ Theoretical underpinning of optimisations implemented
- $\leadsto$ Optimisations available for a given calculus
- $\leadsto$ General-purpose reasoning procedure (a-closure) that exploits algebraic properties when applicable

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```
...
iterate
  r(x,y) ← r(x,y) ∩ r(x,z) ◇ r(z,y)


until fixpoint is reached
...
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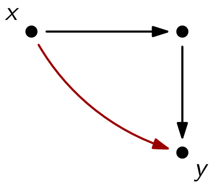

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$$r(y,x) \leftarrow r(y,x) \cap s^{\smile}$$

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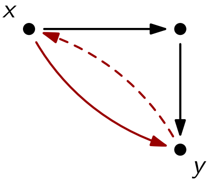

And now . . .

Information preservation by a calculus

Plausible-sounding hypothesis:

Many base relations

$\leadsto$ Finer-grained description of the domain possible

$\leadsto$ More information in a given set of constraints

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Research questions:

- How well do calculi with many relations make use of the potentially higher information content?
- Does the information content differ between the 6 groups of calculi established in the algebraic study?

Measuring information content

To be measured:

- How much additional information is obtained by applying $\diamond$?

  Observe xRy, ySz

  Compute $R \diamond S$ and conclude $x(R \diamond S)z$

  $\rightsquigarrow$ Is it worthwhile to observe xTz too?

- Generalise this to chains $R_1 \diamond \cdots \diamond R_k$

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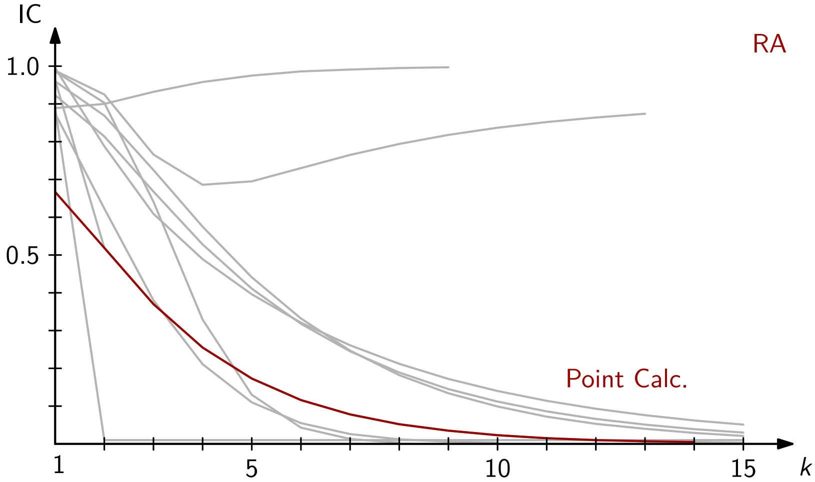
Similar measure: information overlap

measures overlap between entries in composition table

Test results

- Tested 24 calculi, for $k = 1, \ldots, 15$
- Graphical view, selected calculi:



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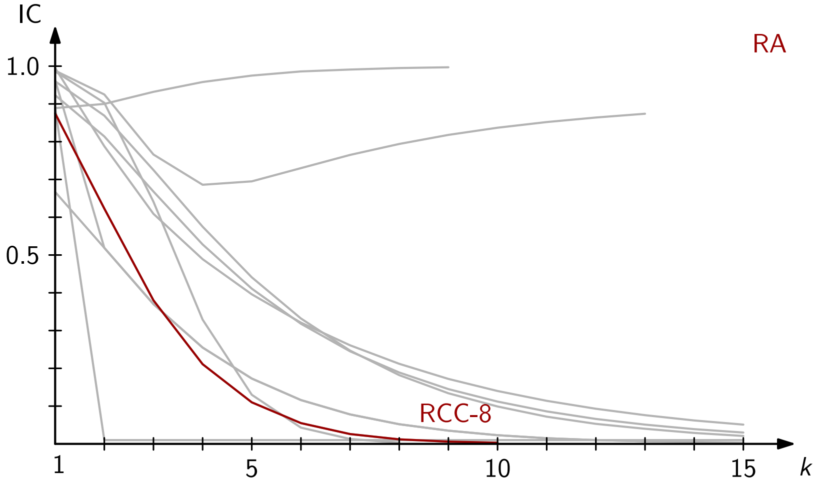

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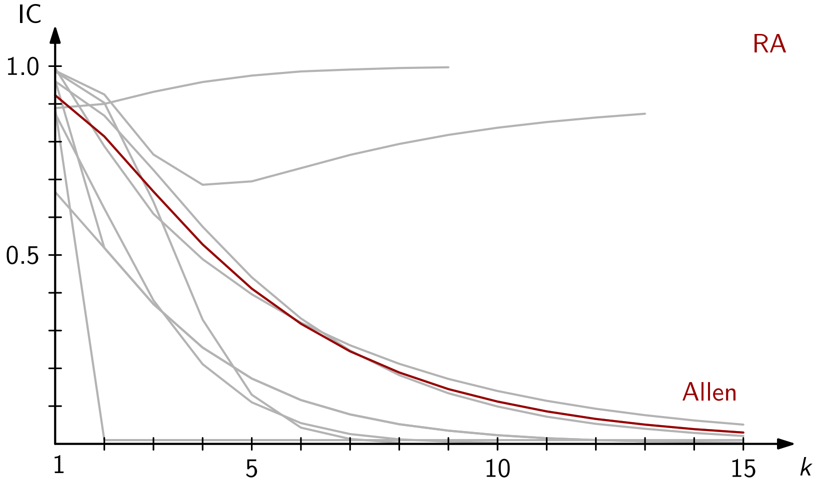

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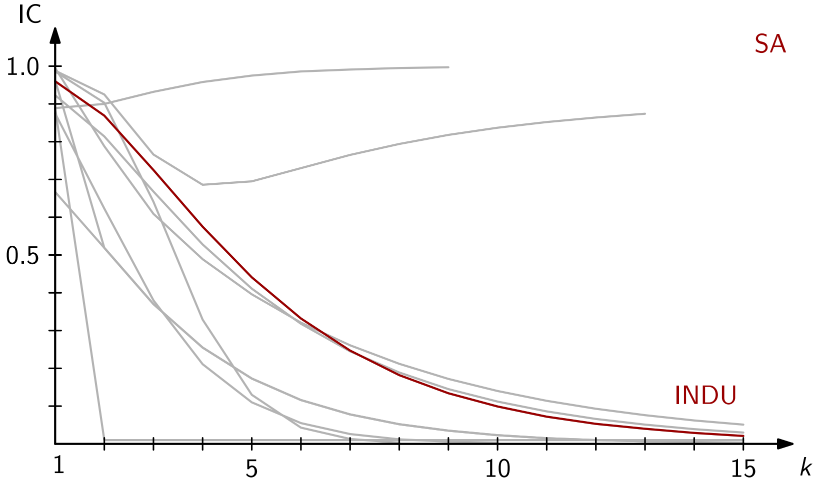

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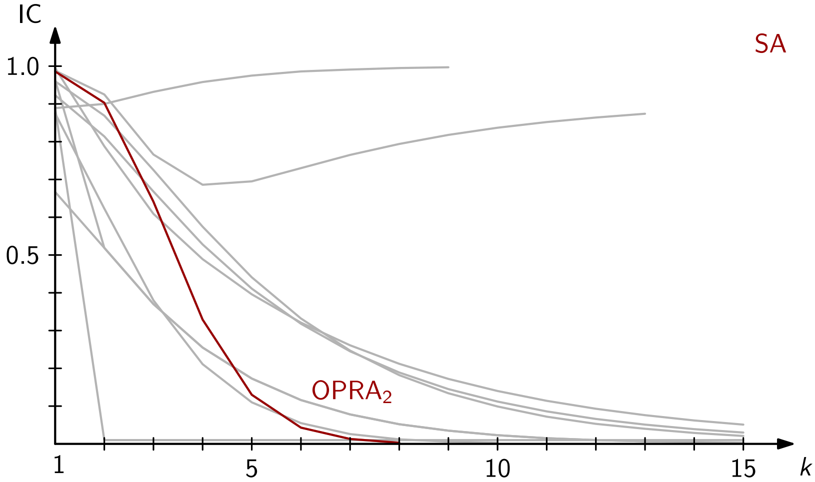

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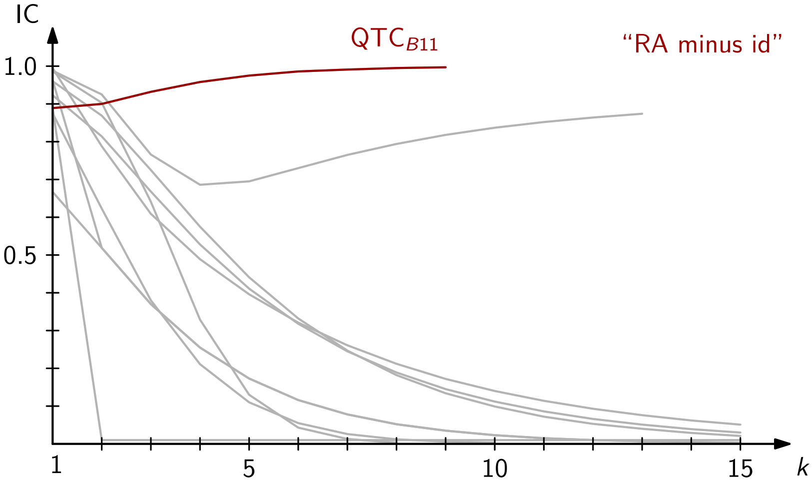

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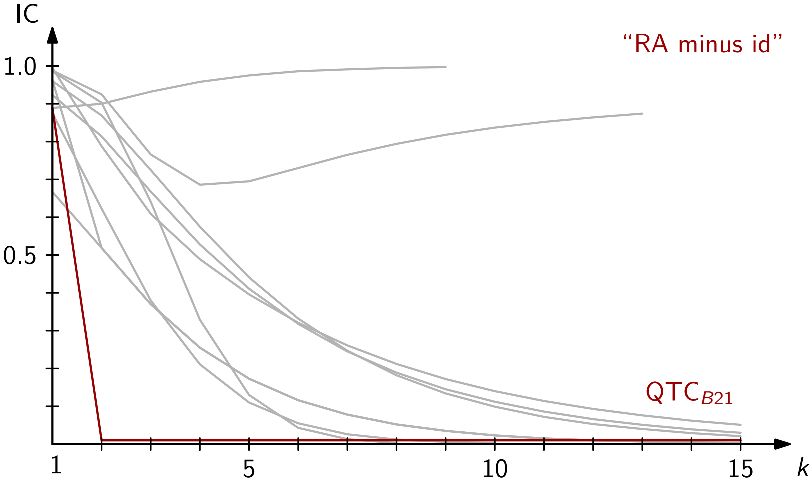

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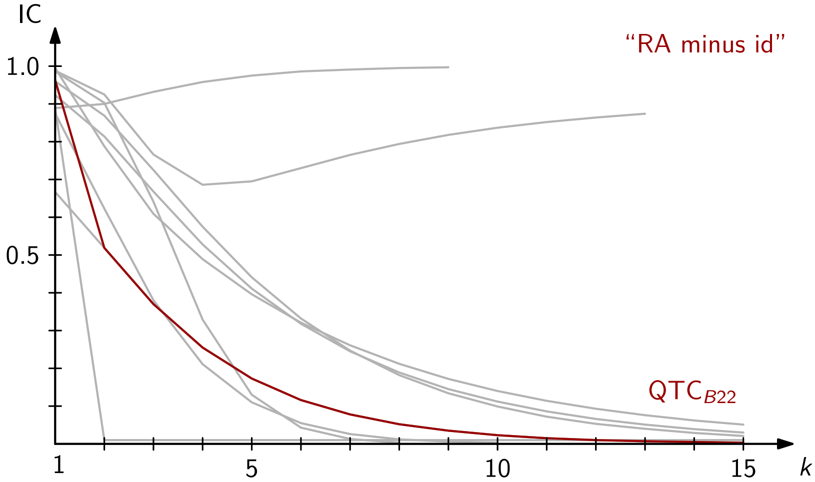

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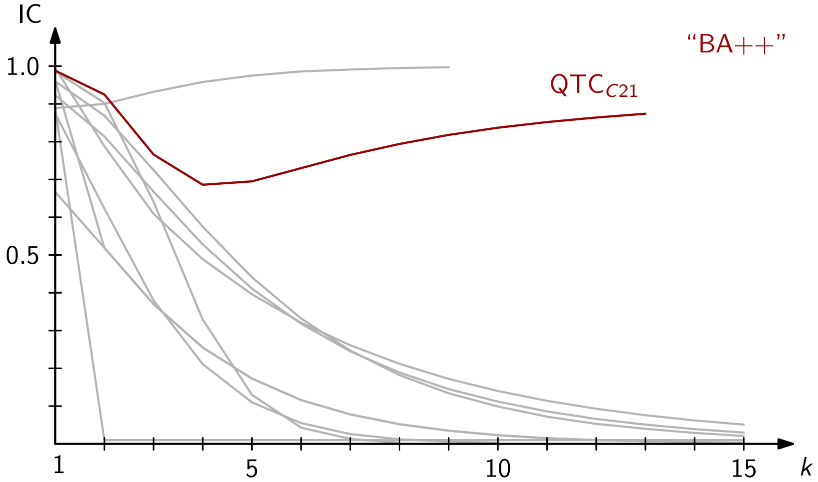

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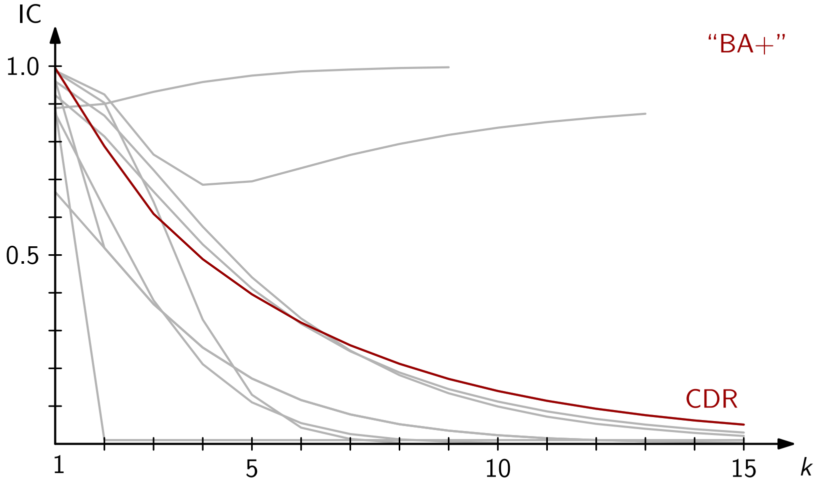

Observations

- **Some** relation between information preservation and number of base relations
- **No** observable relation between information preservation and algebraic properties

And now . . .

Conclusion

We have . . .

- weakened requirements to spatio-temporal calculi to accommodate existing calculi
- discussed algebraic properties of binary calculi
- classified existing calculi according to their algebraic properties
- measured the information preservation by existing calculi

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Thank you.

Why the restriction to binary calculi?

Algebraic properties of ternary relations are

- not as well-understood as those of binary relations
- considerably more complex **(challenges ahead!)**

Work by Isli and Cohn [2000], and Scivos [2000]

Some existing ternary calculi

- Double Cross [Freksa 1992]
- LR calculus [Ligozat 1993]
- CYC_t [Isli & Cohn 2000]
- TPCC [Moratz & Ragni 2008]
- QRPC [Glez-Cabrera et al. 2013]

About the selection of calculi

Our selection includes

- all binary calculi currently implemented in SparQ
- "classical" and more recent calculi (RCC-8 . . . RDR)
- coarse- and fine-grained calculi (3 . . . 1,056 base rel.s)
- and tends towards classical, coarse-grained calculi (more established, easier to implement)

But it misses

- more recent calculi like OPRA* (ongoing implementation)
- . . .

Applications of calculi with weak algebraic properties

QTC (Qualitative Trajectory Calculus) [van de Weghe 2004]

- GIS, spatial query answering
- robotics (analysis of human-robot joint spatial behaviour)

CDR (Cardinal Direction Relations) [Goyal & Egenhofer 2000, Skiadopoulos & Koubarakis 2004, Liu et al. 2010]

- Natural language processing
- potential for GIS

RDR (Rectangular Cardinal Relations) [Navarrete et al. 2013]

- potential for GIS and NLP

$\rightsquigarrow$ these aren't just outliers; they have practical relevance

Quantitative account: IC for all calculi

Calculus	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Allen	92.3	81.4	66.8	52.8	41.1	31.8	24.5	18.9	14.5	11.2	8.6	6.6	5.1	3.9	3.0
Block Algebra	99.4	96.5	89.0	77.7	65.3	53.4	43.0	34.1	27.0	21.1	16.4	12.8	9.9	7.7	5.9
CDC	88.9	76.8	60.4	44.5	31.6	21.9	14.9	10.1	6.8	4.6	3.1	2.0	1.4	0.9	0.6
CYC_b	75.0	62.5	46.9	32.8	21.9	14.1	8.8	5.4	3.2	1.9	1.1	0.6	0.4		
DRA_{fp}	98.8	89.9	69.0	45.0	25.8	13.4	6.5	3.0	1.3	0.6	0.2				
DRA-con	85.7	74.6	59.0	43.4	30.4	20.5	13.5	8.7	5.6	3.5	2.2	1.3	0.8	0.5	0.3
Point Calculus	66.7	51.9	37.0	25.5	17.3	11.6	7.8	5.2	3.5	2.3	1.5	1.0	0.7	0.5	
RCC-5	80.0	56.8	34.9	19.7	10.6	5.5	2.7	1.3	0.6	0.3					
RCC-8	87.5	62.3	38.0	21.1	11.0	5.5	2.6	1.2	0.6	0.3					
STAR_4	88.9	66.9	45.0	28.5	17.4	10.3	6.0	3.5	2.0	1.1	0.6	0.4			
DRA_f	98.6	90.6	70.4	46.3	26.7	13.9	6.7	3.0	1.3	0.6	0.2				
INDU	96.0	86.9	72.5	57.5	44.1	33.2	24.7	18.2	13.4	9.9	7.2	5.3	4.0	2.9	2.1
OPRA_1	95.0	82.0	55.8	30.8	14.5	6.2	2.4	0.9	0.3						
OPRA_2	98.6	90.3	64.1	32.9	13.0	4.3	1.3	0.3							
OPRA_3	99.4	93.1	71.4	40.2	16.7	5.6									
OPRA_4	99.6	94.6	76.7	48.0											
QTC_{B11}	88.9	90.0	93.2	95.8	97.5	98.6	99.1	99.5	99.7						
QTC_{B12}	94.1	91.2	90.5	91.3	92.8	94.2	95.6								
QTC_{B21}	88.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
QTC_{B22}	96.3	51.9	37.0	25.5	17.3	11.6	7.8	5.2	3.5	2.3	1.5	1.0	0.7	0.5	0.3
QTC_{C21}	98.8	92.5	76.6	68.6	69.5	73.0	76.5	79.4	81.8	83.7	85.2	86.4	87.4		
QTC_{C22}	99.5	95.1	78.0	69.3	51.2										
RDR	97.2	82.6	63.2	45.7	32.0	22.0	15.0	10.1	6.8	4.6	3.1	2.0	1.4	0.9	0.6
CDR	99.5	78.8	60.9	48.9	39.6	32.1	26.1	21.2	17.2	14.0	11.4	9.3	7.6	6.2	5.1