Finite Model Reasoning in Horn Description Logics

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Abstract

We study finite model reasoning in the expressive Horn description logics (DLs) Horn-$\mathcal{ALCFI}$ and Horn-$\mathcal{ALCQI}$, starting with a reduction of finite ABox consistency to unrestricted ABox consistency. The reduction relies on reversing certain cycles in the TBox, an approach that originated in database theory, was later adapted to the inexpressive DL *DL-Lite*$_\mathcal{F}$, and is shown here to extend all the way to Horn-$\mathcal{ALCQI}$. The model construction used to establish correctness makes the structure of finite models more explicit than existing approaches to finite model reasoning in expressive DLs that rely on solving systems of inequations over the integers. Since the reduction incurs an exponential blow-up, we then develop a dedicated consequence-based algorithm for finite ABox consistency in Horn-$\mathcal{ALCQI}$ that implements the reduction on-the-fly rather than executing it up-front. The algorithm has optimal worst-case complexity and provides a promising foundation for implementations. Finally, we show that our approach can be adapted to finite (positive existential) query answering relative to Horn-$\mathcal{ALCFI}$ TBoxes, proving that this problem is ExpTime-complete in combined complexity and PTime-complete in data complexity.

1 Introduction

Many popular expressive description logics (DLs) include both inverse roles and some form of counting such as functionality restrictions. This combination is well-known to result in a loss of the finite model property (FMP) and, consequently, reasoning w.r.t. the class of finite models (*finite model reasoning*) does not coincide with reasoning w.r.t. the class of all models (*unrestricted reasoning*). On the one hand, this distinction is gaining importannce because DLs are nowadays regularly used in database applications, where models are generally assumed to be finite. On the other hand, finite model reasoning is rarely used in practice, mainly because for many DLs that lack the FMP, no algorithmic approaches to finite model reasoning are known that lend themselves towards efficient implementation.

Among the most widely-known DLs that include both inverse roles and counting are $\mathcal{ALCFI}$, $\mathcal{ALCQI}$, $\mathcal{SHIF}$, and $\mathcal{SHIQ}$, which are prominent fragments of the OWL2 DL ontology language. While finite model reasoning in these DLs are known to have the same complexity as unrestricted reasoning, namely ExpTime-complete (Lutz, Sattler, and Tendera 2005), the algorithmic approaches are rather different when only finite models are admitted. For unrestricted reasoning, there is a wide range of applicable algorithms such as tableau and resolution calculi, which often perform rather well in practical implementations. For finite model reasoning, all known approaches rely on the construction of some system of inequalities (Calvanese 1996; Lutz, Sattler, and Tendera 2005) and then solve this system over the integers; the crux is that the system of inequalities is of exponential size in the best case, and consequently it is far from obvious how to come up with efficient implementations. This is also true for the two-variable fragment of first-order logic with counting quantifiers (C2), into which the mentioned DLs can be embedded (Pacholski, Szwast, and Tendera 2000; Pratt-Hartmann 2005), that is, all known approaches to finite model reasoning in C2 rely on solving (at least) exponentially large systems of inequalities.

Interestingly, the situation is quite different on the other end of the expressive power spectrum. While $\mathcal{SHIQ}$ et al. belong to the family of expressive DLs, *DL-Lite*$_\mathcal{F}$ is a comparably inexpressive DL that emerged from database applications, but also lacks the FMP because it includes both inverse roles and functionality restrictions. Building on a technique that was developed in database theory by Cosmadakis, Kanellakis, and Vardi to decide the implication of inclusion dependencies and functional dependencies in the finite (1990), Rosati has shown that finite model reasoning in *DL-Lite*$_\mathcal{F}$ can be reduced (in polynomial time) to unrestricted reasoning in *DL-Lite*$_\mathcal{F}$ (2008). In fact, the reduction is conceptually simple and relies on completing the TBox by finding certain cyclic inclusions and 'reversing' them. For example, the cycle

$$\exists r^- \sqsubseteq \exists s \qquad \exists s^- \sqsubseteq \exists r \qquad (\mathsf{funct}\ r^-) \qquad (\mathsf{funct}\ s^-)$$

that consists of existential restrictions in the 'forward direction' and functionality statements in the 'backwards direction' would lead to the addition of the reversed cycle

$$\exists s \sqsubseteq \exists r^- \qquad \exists r \sqsubseteq \exists s^- \qquad (\mathsf{funct}\ r) \qquad (\mathsf{funct}\ s).$$

As a consequence, finite model reasoning in DL-Lite$_\mathcal{F}$ does not require any new algorithmic techniques and can be implemented as efficiently as unrestricted reasoning. Given that DL-Lite$_\mathcal{F}$ is a very small fragment of $\mathcal{ALCFI}$ and $\mathcal{SHIQ}$, these observations raise the question whether the cycle reversion technique extends also to larger fragments of these DLs. In particular, *DL-Lite*$_\mathcal{F}$ is a 'Horn DL', and such logics are well-known to be algorithmically much more well-behaved than non-Horn DLs such as $\mathcal{ALCFI}$ (Baader, Brandt, and Lutz 2005; Calvanese et al. 2007). Maybe, then, this is the reason why cycle reversion works for *DL-Lite*$_\mathcal{F}$?

In this paper, we show that the cycle reversion technique of Cosmadakis et al. extends all the way to the expressive DLs Horn-$\mathcal{ALCFI}$ and Horn-$\mathcal{ALCQI}$. These logics, as well as their extensions Horn-$\mathcal{SHIF}$ and Horn-$\mathcal{SHIQ}$, are popular in ontology-based data access (Hustadt, Motik, and Sattler 2007; Ortiz, Rudolph, and Šimkus 2011; Eiter et al. 2012; Bienvenu, Lutz, and Wolter 2013) and properly extend *DL-Lite*$_\mathcal{F}$ and other relevant Horn DLs such as $\mathcal{ELIF}$ (Krisnadhi and Lutz 2007). We start with showing that finite ABox consistency in Horn-$\mathcal{ALCFI}$ can be reduced to unrestricted ABox consistency in Horn-$\mathcal{ALCFI}$ by cycle reversion; it follows that the same is true for finite satisfiability, finite subsumption, and finite instance checking. While the reduction technique is conceptually similar to that for *DL-Lite*$_\mathcal{F}$, the construction of a finite model in the correctness proof is much more demanding. In comparison to approaches to finite model reasoning that rely on solving systems of inequalities, though, they make the structure of finite models considerably more explicit. Via a reduction, these results extend also to Horn-$\mathcal{ALCQI}$.

Another crucial difference to the *DL-Lite*$_\mathcal{F}$ case is that, when completing Horn-$\mathcal{ALCFI}$ TBoxes, the cycles that have to be considered can be of exponential length, and thus the reduction is not polynomial. Consequently, when used in a naive way it can neither be expected to perform well in practice nor be used to (re)prove tight complexity bounds. To address these shortcomings, we develop a dedicated calculus for finite ABox consistency in Horn-$\mathcal{ALCQI}$ that implements the reduction on-the-fly rather than executing it up-front. The calculus is an extension of a consequence-based procedure for unrestricted satisfiability in Horn-$\mathcal{SHIQ}$ that was introduced by Kazakov in (2009) and implemented in the highly performant reasoner CB, first to classify the notorious Galen ontology. Many other state-of-the art reasoners for Horn-DLs are also based on consequence-based procedures, including ELK (Kazakov, Krötzsch, and Simančík 2011a) and CEL (Baader, Lutz, and Suntisrivaraporn 2006). Our algorithm shares the main feature of other consequence-based procedures to carefully avoid considering 'types' (conjunctions of concept names) that are irrelevant for deciding the problem at hand. We therefore believe that it is a very promising basis for efficient implementations of finite model reasoning in Horn-$\mathcal{ALCQI}$. It also (re)proves the optimal upper EXPTIME complexity bound for finite ABox consistency in this DL.

We then consider the paradigm of ontology-based data access (OBDA), extending our results from finite ABox consistency to answering positive existential queries (PEQs), relative to Horn-$\mathcal{ALCFI}$ TBoxes over finite models. In particular, we show that the reduction based on cycle reversion developed for ABox consistency also works in the case of PEQ answering. The construction of (counter)models in the correctness proofs, however, becomes yet more difficult and technical, and proceeds in two stages. First, we carefully modify the models constructed for finite ABox consistency so that there are no unintended matches of acyclic conjunctive queries (CQs). And second, we take a product with a finite group of high girth to eliminate unintended matches of cyclic CQs. Based on this result, we then prove that finite PEQ entailment (the Boolean version of PEQ answering) in Horn-$\mathcal{ALCFI}$ is EXPTIME-complete regarding combined complexity and PTIME-complete regarding data complexity. Previously, it was only known that finite CQ answering in (non-Horn) $\mathcal{ALCQI}$ is decidable and in CONP regarding data complexity (Pratt-Hartmann 2009).

This paper is a significantly extended version of the workshop paper (Ibáñez-García, Lutz, and Schneider 2013). Some proof details are deferred to the appendix in the long version: http://www.fb3.uni-bremen.de/%7Ets/publ/kr14fmr.pdf

2 Preliminaries

We introduce the DLs Horn-$\mathcal{ALCFI}$ and Horn-$\mathcal{ALCQI}$, as well as the reasoning tasks studied in this paper. The original definition of these DLs is based on a notion of polarity and somewhat unwieldy (Hustadt, Motik, and Sattler 2007); alternative and more direct definitions have been proposed later, see for example (Lutz and Wolter 2012). For brevity, we directly introduce Horn-$\mathcal{ALCQI}$ TBoxes in a normal form that is convenient for our purposes and disallows syntactic nesting of operators. It is a minor variation of the normal form proposed in (Kazakov 2009).

Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be countably infinite and disjoint sets of concept names, role names, and individual names. A *role* is either a role name r or an *inverse role* r^-. A *Horn-$\mathcal{ALCQI}$ TBox* $\mathcal{T}$ is a set of *concept inclusions (CIs)* that can take the following forms:

$$K \sqsubseteq A \qquad K \sqsubseteq \bot \qquad K \sqsubseteq \exists r.K'$$
$$K \sqsubseteq \forall r.K' \qquad K \sqsubseteq (\leqslant 1\ r\ K') \qquad K \sqsubseteq (\geqslant n\ r\ K')$$

where K and K' denote a (possibly empty) conjunction of concept names, A a concept name, r a (potentially inverse) role, and $n \geq 2$. Throughout the paper, we will deliberately confuse conjunctions of concept names and sets of concept names. The empty conjunction is abbreviated by $\top$. As usual, we allow to easily switch between role names and their inverse by identifying $(r^-)^-$ and r. A *Horn-$\mathcal{ALCFI}$ TBox* is a Horn-$\mathcal{ALCQI}$ TBox that does not include CIs of the form $K \sqsubseteq (\geqslant n\ r\ K')$.

The semantics of Horn-$\mathcal{ALCQI}$ is based on interpretations as usual, see (Baader et al. 2003) for details. We write $\mathcal{T} \models C \sqsubseteq D$ if the concept inclusion $C \sqsubseteq D$ is satisfied in all models of the TBox $\mathcal{T}$, and $\mathcal{T} \models_{\mathsf{fin}} C \sqsubseteq D$ if the same holds for all finite models. A concept name A is *(finitely) satisfiable* w.r.t. a TBox $\mathcal{T}$ if $\mathcal{T}$ has a (finite) model $\mathcal{I}$ with $A^\mathcal{I} \neq \emptyset$. If $\mathcal{T} \models A \sqsubseteq B$ (resp. $\mathcal{T} \models_{\mathsf{fin}} A \sqsubseteq B$) with A and B concept names, then we say that B is *(finitely) subsumed* by A.

An *ABox* is a finite set of *concept assertions* $A(a)$ and *role assertions* $r(a, b)$ where A is a concept name, r a role name, and a, b are individual names. For simplicity, we make the *standard names assumption*, that is, every interpretation $\mathcal{I}$

interpretes all individuals as themselves; for example $\mathcal{I}$ satisfies $A(a)$ if $a \in A^{\mathcal{I}}$. The standard names assumption implies the unique name assumption (UNA). The results in this paper, however, do not depend on any of these assumptions. Throughout the paper, we sometimes write $r^-(a,b) \in \mathcal{A}$ for $r(b,a) \in \mathcal{A}$ and use $\mathsf{Ind}(\mathcal{A})$ to denote the set of all individual names that occur in $\mathcal{A}$.

We write $\mathcal{A}, \mathcal{T} \models A(a)$ if the ABox assertion $A(a)$ is satisfied in all common models of the ABox $\mathcal{A}$ and the TBox $\mathcal{T}$, and $\mathcal{A}, \mathcal{T} \models_{\mathsf{fin}} A(a)$ if the same holds for all finite models. We then say that a is a *(finite) instance* of A in $\mathcal{A}$ w.r.t. $\mathcal{T}$. An ABox $\mathcal{A}$ is *(finitely) consistent* w.r.t. $\mathcal{T}$ if there is a (finite) model $\mathcal{I}$ of $\mathcal{T}$ that satisfies all assertions in $\mathcal{A}$.

The above notions give rise to four decision problems studied in this paper, which are *finite satisfiability* (of a concept name w.r.t. a TBox), *finite subsumption* (between two concept names w.r.t. a TBox), *finite ABox consistency* (w.r.t. a TBox) and *finite instance checking* (of an ABox individual and a concept name, w.r.t. an ABox and a TBox). There are easy polynomial time reductions from satisfiability to subsumption to instance checking to ABox consistency, which work both in the finite and in the unrestricted case.

The following examples show that, in Horn-$\mathcal{ALCFI}$, finite and unrestricted reasoning do not coincide.

Example 1 *Let*

$$\mathcal{T} = \{ \quad A \sqsubseteq \exists r.B, \qquad B \sqsubseteq \exists r.B, \quad B \sqsubseteq (\leqslant 1\, r^- \top), \quad A \sqcap B \sqsubseteq \bot \quad \}.$$

Then A is satisfiable w.r.t. $\mathcal{T}$, but not finitely satisfiable. In fact, when $d \in A^{\mathcal{I}}$ in some model $\mathcal{I}$ of $\mathcal{T}$, then there must be an infinite chain $r(d,d_1), r(d_1,d_2), \ldots$ with $d \in A^{\mathcal{I}}$, and $d_2, d_3, \dots \in B^{\mathcal{I}}$. Since d cannot be in $B^{\mathcal{I}}$ and r is inverse functional, no two elements on the chain can be identified.

Let

$$\mathcal{T}' = \{ \; A_1 \sqsubseteq \exists r.A_2, \quad A_2 \sqsubseteq \exists r.(A_1 \sqcap B), \quad \top \sqsubseteq (\leqslant 1\, r^- \top) \quad \}.$$

The reader might want to verify that $\mathcal{T}' \not\models A_1 \sqsubseteq B$, but $\mathcal{T}' \models_{\mathsf{fin}} A_1 \sqsubseteq B$.

It follows form the observations in (Kazakov 2009) that, for the purposes of deciding satisfiability of concepts in unrestricted models, the normal form for TBoxes introduced above can be assumed without loss of generality because every Horn-$\mathcal{ALCQI}$ TBox $\mathcal{T}$ can be converted in polynomial time into a TBox $\mathcal{T}'$ in the above form such that every model of $\mathcal{T}'$ is a model of $\mathcal{T}$ and, conversely, every model of $\mathcal{T}$ can be converted into a model of $\mathcal{T}'$ by interpreting the concept names that were introduced during normalization. It follows that normal form can be assumed w.l.o.g. both for unrestricted reasoning and for finite model reasoning, and for all reasoning problems considered in this paper.

3 From Finite Models to Unrestricted Models

We show that finite ABox consistency can be reduced to unrestricted ABox consistency by reversing certain cycles in the TBox. To ease presentation, we work with Horn-$\mathcal{ALCFI}$ instead of with Horn-$\mathcal{ALCQI}$; see Section 6 for how our results can be lifted to the latter. The reduction exhibited in this section provides a novel decision procedure for finite ABox consistency in Horn-$\mathcal{ALCFI}$ and Horn-$\mathcal{ALCQI}$ (as well as for finite satisfiability, finite subsumption, and finite instance checking) and is the basis for developing a consequence-based procedure in Section 4.

Reversing Cycles

Let $\mathcal{T}$ be a Horn-$\mathcal{ALCFI}$ TBox. A *finmod cycle in $\mathcal{T}$* is a sequence $K_1, r_1, K_2, r_2, \ldots, r_{n-1}, K_n$, with $K_1, \ldots, K_n$ conjunctions of concept names and $r_1, \ldots, r_{n-1}$ (potentially inverse) roles such that $K_n = K_1$ and, for $1 \leq i < n$:

$$\mathcal{T} \models K_i \sqsubseteq \exists r_i.K_{i+1} \text{ and } \mathcal{T} \models K_{i+1} \sqsubseteq (\leqslant 1\, r_i^-\; K_i).$$

By *reversing* a finmod cycle $K_1, r_1, K_2, r_2, \ldots, r_{n-1}, K_n$ in a TBox $\mathcal{T}$, we mean to extend $\mathcal{T}$ with the following concept inclusions, for $1 \leq j < n$:

$$K_{j+1} \sqsubseteq \exists r_j^-.K_j \text{ and } K_j \sqsubseteq (\leqslant 1\, r_j\; K_{j+1}).$$

The *completion* $\mathcal{T}_{\mathsf{f}}$ of a TBox $\mathcal{T}$ is obtained from $\mathcal{T}$ by exhaustively reversing finmod cycles. Note that, although there may be infinitely many finmod cycles, only finitely many CIs can be added by cycle reversion (exponentially many in the size of the original TBox, in the worst case). For finding these finitely many CIs, it clearly suffices to consider finmod cycles in which all triples (r_i, K_{i+1}, r_{i+1}) are distinct. Also note that finding finmod cycles requires deciding unrestricted subsumption, which is decidable and ExpTime-complete.

Example 2 *The TBox $\mathcal{T}'$ from Example 1 entails (in unrestricted models)*

$$A_1 \sqcap B \sqsubseteq \exists r.A_2, \qquad A_2 \sqsubseteq \exists r.(A_1 \sqcap B),$$
$$A_2 \sqsubseteq (\leqslant 1\, r^-\; A_1 \sqcap B), \quad A_2 \sqcap B \sqsubseteq (\leqslant 1\, r^-\; A_1).$$

Thus, $(A_1 \sqcap B), r, A_2, r, (A_1 \sqcap B)$ is a finmod cycle in $\mathcal{T}'$, which is reversed to

$$A_2 \sqsubseteq \exists r^-.(A_1 \sqcap B), \qquad A_1 \sqcap B \sqsubseteq \exists r^-.A_2,$$
$$A_1 \sqcap B \sqsubseteq (\leqslant 1\, r\; A_2), \qquad A_1 \sqsubseteq (\leqslant 1\, r\; A_2 \sqcap B).$$

Another finmod cycle in $\mathcal{T}'$ is A_1, r, A_2, r, A_1, reversed to

$$A_2 \sqsubseteq \exists r^-.A_1, \qquad A_1 \sqsubseteq \exists r^-.A_2,$$
$$A_2 \sqsubseteq (\leqslant 1\, r\; A_1), \qquad A_1 \sqsubseteq (\leqslant 1\, r\; A_2).$$

Note that $\mathcal{T}'_{\mathsf{f}}$ contains $A_1 \sqsubseteq \exists r^-.A_2$, $A_2 \sqsubseteq \exists r.(A_1 \sqcap B)$, and $A_2 \sqsubseteq (\leqslant 1\; r\; A_1)$. Consequently $\mathcal{T}'_{\mathsf{f}} \models A_1 \sqsubseteq B$, in correspondence with $\mathcal{T}' \models_{\mathsf{fin}} A_1 \sqsubseteq B$.

The following result shows that TBox completion provides a reduction from finite ABox consistency to unrestricted ABox consistency.

Theorem 3 *Let $\mathcal{T}$ be a Horn-$\mathcal{ALCFI}$ TBox and $\mathcal{A}$ an ABox. Then $\mathcal{A}$ is finitely consistent w.r.t. $\mathcal{T}$ iff $\mathcal{A}$ is consistent w.r.t. the completion $\mathcal{T}_{\mathsf{f}}$ of $\mathcal{T}$.*

The "only if" direction of Theorem 3 is an immediate consequence of the observation that all CIs added by cycle reversion are entailed by the original TBox in finite models.

Lemma 4 *Let $K_1, r_1, \ldots, r_{n-1}, K_n$ a finmod cycle in $\mathcal{T}$, then $\mathcal{T} \models_{\mathsf{fin}} K_{i+1} \sqsubseteq \exists r_i^-.K_i$ and $\mathcal{T} \models_{\mathsf{fin}} K_i \sqsubseteq (\leqslant 1\; r_i\; K_{i+1})$ for $1 \leq i < n$,.*

Proof. We have to show that, if $K_1, r_1, \ldots, r_{n-1}, K_n$ is a finmod cycle in $\mathcal{T}$ and $\mathcal{I}$ is a finite model of $\mathcal{T}$, then $K_i^\mathcal{I} \subseteq (\leqslant 1\ r_i\ K_{i+1})^\mathcal{I}$ and $K_{i+1}^\mathcal{I} \subseteq (\exists r_i^-.K_i)^\mathcal{I}$ for $1 \leq i < n$. We first note that, by the semantics of Horn-$\mathcal{ALCFI}$, we must have $|K_1^\mathcal{I}| \leq \cdots \leq |K_n^\mathcal{I}|$, thus $K_n = K_1$ yields $|K_1^\mathcal{I}| = \cdots = |K_n^\mathcal{I}|$. Fix some i with $1 \leq i < n$. Using $|K_i^\mathcal{I}| = |K_{i+1}^\mathcal{I}|$, $K_i^\mathcal{I} \subseteq (\exists r_i.K_{i+1})^\mathcal{I}$, and $K_{i+1}^\mathcal{I} \subseteq (\leqslant 1\ r_i^-\ K_i)^\mathcal{I}$, it is now easy to verify that $K_i^\mathcal{I} \subseteq (\leqslant 1\ r_i\ K_{i+1})^\mathcal{I}$ and $K_{i+1}^\mathcal{I} \subseteq (\exists r_i^-.K_i)^\mathcal{I}$, as required. ❏

We now prove the "if" direction of Theorem 3, which is much more demanding as it requires to explicit construct finite models.

Constructing Finite Models

Assume that $\mathcal{A}$ is consistent w.r.t. $\mathcal{T}_\mathsf{f}$. Our aim is to construct a finite model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$ (and thus also of $\mathcal{T}$). Before we give details of the construction, we introduce some relevant preliminaries.

Let $\mathsf{CN}(\mathcal{T})$ denote the set of concept names used in $\mathcal{T}$ (or, equivalently, in $\mathcal{T}_\mathsf{f}$). A *type for* $\mathcal{T}_\mathsf{f}$ is a subset $t \subseteq \mathsf{CN}(\mathcal{T})$ such that there is a (potentially infinite) model $\mathcal{I}$ of $\mathcal{T}_\mathsf{f}$ and a $d \in \Delta^\mathcal{I}$ such that $\mathsf{tp}_\mathcal{I}(d) = t$, where

$$\mathsf{tp}_\mathcal{I}(d) := \{A \in \mathsf{CN}(\mathcal{T}) \mid d \in A^\mathcal{I}\}$$

is the type *realized* at d in $\mathcal{I}$. We use $\mathsf{TP}(\mathcal{T}_\mathsf{f})$ to denote the set of all types for $\mathcal{T}_\mathsf{f}$. For $t, t' \in \mathsf{TP}(\mathcal{T}_\mathsf{f})$ and r a role, we write

- $t \to_r t'$ if $\mathcal{T}_\mathsf{f} \models t \sqsubseteq \exists r.t'$ and t' is maximal with this property;
- $t \to_r^1 t'$ if $t \to_r t'$ and $\mathcal{T}_\mathsf{f} \models t' \sqsubseteq (\leqslant 1\ r^-\ t)$;
- $t\ {}^1\!\!\leftrightarrow_r^1 t'$ if $t \to_r^1 t'$ and $t' \to_{r^-}^1 t$.

Note that when

$$t_1 \to_{r_1}^1 t_2 \to_{r_2}^1 \cdots \to_{r_{n-1}}^1 t_n = t_1 \qquad (*)$$

then $t_1, r_1, \ldots, r_{n-1}, t_n$ is a finmod cycle in $\mathcal{T}_\mathsf{f}$ and the fact that is has been reversed means that all '$\to^1$' in $(*)$ can be replaced with ${}^1\!\!\leftrightarrow^1$.

For all $a \in \mathsf{Ind}(\mathcal{A})$, set

$$\mathsf{tp}_\mathcal{A}(a) := \{A \in \mathsf{CN}(\mathcal{T}) \mid \mathcal{A}, \mathcal{T}_\mathsf{f} \models A(a)\}.$$

The set $\mathsf{TP}(\mathcal{A})$ of *types for* $\mathcal{A}$ *(and* $\mathcal{T}$*)* is the smallest set of types such that $\mathsf{tp}_\mathcal{A}(a) \in \mathsf{TP}(\mathcal{A})$ for all $a \in \mathsf{Ind}(\mathcal{A})$ and whenever $t \in \mathsf{TP}(\mathcal{A})$ and $t \to_r t'$, then $t' \in \mathsf{TP}(\mathcal{A})$. Based on a standard canonical model construction as used for example in (Lutz and Wolter 2012), it is easy to prove that there is a (possibly infinite) model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$ such that the types realized in $\mathcal{I}$ are exactly those in $\mathsf{TP}(\mathcal{A})$. It follows that every element of $\mathsf{TP}(\mathcal{A})$ is a type for $\mathcal{T}_\mathsf{f}$. In the finite model constructed below, we will only realize types from $\mathsf{TP}(\mathcal{A})$.

Types related by ${}^1\!\!\leftrightarrow_r^1$ are connected very tightly by the TBox $\mathcal{T}$ and are best considered together when building finite models. This is formalized by the notion of a *type class*, which is a non-empty set $P \subseteq \mathsf{TP}(\mathcal{A})$ such that $t \in P$ and $t\ {}^1\!\!\leftrightarrow_r^1 t'$ implies $t' \in P$, and is minimal with this condition. Note that the set of all type classes is a partition of $\mathsf{TP}(\mathcal{A})$. We set $P \prec P'$ if there are $t \in P$ and $t' \in P'$ with $t' \subsetneq t$. We will later be referring to the strict partial order that is obtained by taking the transitive closure of $\prec$, denoted by $\prec^+$. A proof of the following observation can be found in the appendix.

Lemma 5 *$\prec^+$ is a strict partial order.*

We construct the desired finite model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$ by starting with an initial interpretation that essentially consists of the ABox $\mathcal{A}$ and then exhaustively applying three *completion rules* denoted with $(\mathbf{c1})$ to $(\mathbf{c3})$, where $(\mathbf{c1})$ is given preference over $(\mathbf{c2})$. Completion repeatedly introduces elements whose existence is required by CIs $K \sqsubseteq \exists r.C$, carefully distinguishing several cases to ensure that no functionality restrictions are violated. We will prove that rule application terminates after finitely many steps, producing a finite model.

During the construction of $\mathcal{I}$, we will make sure that the following invariants are satisfied:

- $(\mathbf{i1})$ $\mathsf{tp}_\mathcal{I}(d) \in \mathsf{TP}(\mathcal{A})$ for all $d \in \Delta^\mathcal{I}$;
- $(\mathbf{i2})$ if $(d, d') \in r^\mathcal{I} \setminus (\mathsf{Ind}(\mathcal{A}) \times \mathsf{Ind}(\mathcal{A}))$, then we have $\mathsf{tp}_\mathcal{I}(d) \to_r \mathsf{tp}_\mathcal{I}(d')$ or $\mathsf{tp}_\mathcal{I}(d') \to_{r^-} \mathsf{tp}_\mathcal{I}(d)$;
- $(\mathbf{i3})$ if $\mathcal{T}_\mathsf{f} \models K \sqsubseteq (\leqslant 1\ r\ K')$, then $\mathcal{I} \models K \sqsubseteq (\leqslant 1\ r\ K')$.

The initial version of $\mathcal{I}$ is defined by introducing an element for every ABox individual, and an element d_t whenever there is some $t' \in \mathsf{TP}(\mathcal{A})$ such that $t' \to_r t$, but not $t' \to_r^1 t$. In detail, we set

$$\begin{aligned}\Delta^\mathcal{I} &= \mathsf{Ind}(\mathcal{A}) \cup \{\, d_t \mid \text{there is a } t' \in \mathsf{TP}(\mathcal{A}) \text{ with} \\ &\qquad\qquad t' \to_r t, \text{ but not } t' \to_r^1 t \,\} \\ A^\mathcal{I} &= \{\, a \in \mathsf{Ind}(\mathcal{A}) \mid A \in \mathsf{tp}_\mathcal{A}(a) \,\} \cup \{d_t \mid A \in t\,\} \\ r^\mathcal{I} &= \{\, (a,b) \mid r(a,b) \in \mathcal{A} \,\}\end{aligned}$$

The completion rules are described in detail below.

$(\mathbf{c1})$ Select a $d \in \Delta^\mathcal{I}$ such that $\mathsf{tp}_\mathcal{I}(d) \to_r^1 t$, $t \not\to_{r^-}^1 \mathsf{tp}_\mathcal{I}(d)$, and $d \notin (\exists r.t)^\mathcal{I}$. Add a fresh domain element e, and modify the extension of concept and role names such that $\mathsf{tp}_\mathcal{I}(e) = t$ and $(d, e) \in r^\mathcal{I}$.

$(\mathbf{c2})$ Select a type class P that is minimal w.r.t. the order $\prec^+$ and such that there is a $\lambda = s\ {}^1\!\!\leftrightarrow_r^1 s'$ with $s \in P$, and select an element $d \in s^\mathcal{I} \setminus (\exists r.s')^\mathcal{I}$.

For each $\lambda = s\ {}^1\!\!\leftrightarrow_r^1 s'$ with $s \in P$, set

$$X_{\lambda,1}^\mathcal{I} = s^\mathcal{I} \setminus (\exists r.s')^\mathcal{I} \qquad X_{\lambda,2}^\mathcal{I} = s'^\mathcal{I} \setminus (\exists r^-.s)^\mathcal{I}.$$

Take (i) a fresh set Δ_s for each $s \in P$ such that $|\biguplus_{s \in P} \Delta_s| \leq 2^{|\mathcal{T}|} \cdot |\Delta^\mathcal{I}|$ and (ii) a bijection π_λ between $X_{\lambda,1}^\mathcal{I} \cup \Delta_s$ and $X_{\lambda,2}^\mathcal{I} \cup \Delta_{s'}$ for each $\lambda = s\ {}^1\!\!\leftrightarrow_r^1 s'$ with $s, s' \in P$ and r a role name (the concrete construction is detailed below). Now extend $\mathcal{I}$ as follows:

- add all domain elements in $\biguplus_{s \in P} \Delta_s$;
- extend $r^\mathcal{I}$ with π_λ, for each $\lambda = s\ {}^1\!\!\leftrightarrow_r^1 s'$ with $s, s' \in P$ and r a role name;
- interpret concept names so that $\mathsf{tp}_\mathcal{I}(d) = s$ for all $d \in \Delta_s$, $s \in P$.

(c3) Select a $d \in \Delta^{\mathcal{I}}$ such that $\mathsf{tp}_{\mathcal{I}}(d) \to_r t$, $\mathsf{tp}_{\mathcal{I}}(d) \not\to_r^1 t$, and $d \notin (\exists r.t)^{\mathcal{I}}$. Add the edge (d, d_t) to $r^{\mathcal{I}}$, where d_t is the element introduced for type t in the initial version of $\mathcal{I}$.

To complete the description of the rules, we have to show that, in (**c2**), the sets Δ_s and bijections π_λ indeed exist. Let $n_{\max} = \max\{|s^{\mathcal{I}}| \mid s \in P\}$. For each $s \in P$, set $\Delta_s := \{d_{s,i} \mid |s^{\mathcal{I}}| < i \leq n_{\max}\}$ and define the set of *s-instances* $I_s := s^{\mathcal{I}} \cup \Delta_s$. For each $\lambda = s \;{}^1\!\!\leftrightarrow_r^1 s'$ with $s, s' \in P$, define

$$R_\lambda := \{(d, e) \in r^{\mathcal{I}} \mid d \in s^{\mathcal{I}} \text{ and } e \in s'^{\mathcal{I}}\}.$$

We first note that it is a consequence of invariant (**i3**) that

($*$) the relation R_λ is functional and inverse functional.

In fact, let $(d, e_1), (d, e_2) \in R_\lambda$. Then $(d, e_1), (d, e_2) \in r^{\mathcal{I}}$, $d \in s^{\mathcal{I}}$, and $e_1, e_2 \in s'^{\mathcal{I}}$. By λ, we have $\mathcal{T}_{\mathsf{f}} \models s \sqsubseteq (\leqslant 1\; r\; s')$. Thus, (**i3**) yields $e_1 = e_2$. Inverse functionality can be shown analogously.

Let R^1_λ be the domain of R_λ, and let R^2_λ be its range. By ($*$), we have $|R^1_\lambda| = |R^2_\lambda|$. By definition of the sets Δ_s, we have $|I_s| = |I_{s'}|$. Moreover, $R^1_\lambda \subseteq I_s$ and $R^2_\lambda \subseteq I_{s'}$. We can thus choose a bijection π_λ between $I_s \setminus R^1_\lambda$ and $I_{s'} \setminus R^2_\lambda$, which is as required since $I_s \setminus R^1_\lambda = X^{\mathcal{I}}_{\lambda,1} \cup \Delta_s$ and $I_{s'} \setminus R^2_\lambda = X^{\mathcal{I}}_{\lambda,2} \cup \Delta_{s'}$.

The following theorem summarizes the statements that remain to be proved in order to show that the construction of $\mathcal{I}$ is well-defined and yields a finite model of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$.

Theorem 6

1. *Applying* (**c1**) *to* (**c3**) *preserves invariants* (**i1**) *to* (**i3**);
2. *Application of* (**c1**) *to* (**c3**) *terminates;*
3. *$\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$.*

Proof. We refer to the appendix for full proofs and only sketch the central idea in the proof of Point 2 here, going back to (Cosmadakis, Kanellakis, and Vardi 1990). The main issue in the termination proof is to show that no infinite role chain $r_0(d_0, d_1), r_1(d_1, d_2), \ldots$ is generated in which all the elements d_i are pairwise distinct. Since every rule application generates only finitely many elements, any such role chain must be generated by infinitely many applications of completion rules. Since there are only finitely many types, we must find elements d_i and d_j with $\mathsf{tp}_{\mathcal{I}}(d_i) = \mathsf{tp}_{\mathcal{I}}(d_j)$ and such that d_i and d_j were generated by different rule applications. It can be shown that, w.l.o.g., we can assume that the elements on the chain are ordered so that if $j > i$, then d_j was not generated by an earlier rule application than d_i. Analysing the completion rules, it is easy to see that this implies $\mathsf{tp}_{\mathcal{I}}(d_i) \to^1_{r_i} \mathsf{tp}_{\mathcal{I}}(d_{i+1}) \to^1_{r_{i+1}} \cdots \to^1_{r_{j-1}} \mathsf{tp}_{\mathcal{I}}(d_j)$. Since $\mathsf{tp}_{\mathcal{I}}(d_i) = \mathsf{tp}_{\mathcal{I}}(d_j)$, this is a finmod cycle, which has been reversed when constructing $\mathcal{T}_{\mathsf{f}}$, and thus all arrows $\to^1_{r_{i+\ell}}$ can be replaced with ${}^1\!\!\leftrightarrow^1_{r_{i+\ell}}$. By definition of the completion rules, this means that all of $d_i, \ldots, d_j$ were introduced in the same application of (**c2**), which is a contradiction to d_i and d_j being generated by different rule applications. ❑

R1 $$\frac{}{K \sqcap A \sqsubseteq A}$$

R2 $$\frac{}{K \sqsubseteq \top}$$

R3 $$\frac{K \sqsubseteq A_i \quad \bigsqcap A_i \sqsubseteq C}{K \sqsubseteq C}$$

R4 $$\frac{K \sqsubseteq \exists r.K_1 \quad K_1 \sqsubseteq \forall r^-.K'}{K \sqsubseteq K'}$$

R5 $$\frac{K \sqsubseteq \exists r.K_1' \quad K \sqsubseteq \forall r.K'}{K \sqsubseteq \exists r.(K_1' \sqcap K')}$$

R6 $$\frac{K \sqsubseteq \exists r.K' \quad K' \sqsubseteq \bot}{K \sqsubseteq \bot}$$

R7 $$\frac{\begin{array}{ll} K \sqsubseteq \exists r.K_1 & K \sqsubseteq \exists r.K_2 \\ K \sqsubseteq (\leqslant 1\; r\; K'') & K'' \subseteq \mathsf{cl}_\vdash(K_1) \cap \mathsf{cl}_\vdash(K_2) \end{array}}{K \sqsubseteq \exists r.(K_1 \sqcap K_2)}$$

R8 $$\frac{\begin{array}{ll} K \sqsubseteq \exists r.K_1 & K_1 \sqsubseteq \exists r^-.K' \\ K_1 \sqsubseteq (\leqslant 1\; r^-\; K'') & K'' \subseteq \mathsf{cl}_\vdash(K) \cap \mathsf{cl}_\vdash(K') \end{array}}{K \sqsubseteq K'}$$

R9 $$\frac{\begin{array}{ll} K_i \sqsubseteq \exists r_i.K_{i \oplus_n 1} & \\ K_{i \oplus_n 1} \sqsubseteq (\leqslant 1\; r_i^-\; K_i') & K_i' \subseteq \mathsf{cl}_\vdash(K_i) \end{array} \quad i < n}{K_1 \sqsubseteq \exists r_0^-.K_0 \qquad K_0 \sqsubseteq (\leqslant 1\; r_0\; K_1')}$$

Figure 1: Inference Rules

4 Consequence-Driven Procedure

While completing TBoxes with reversed cycles yields a reduction of finite model reasoning to infinite model reasoning, it also blows up the TBox exponentially is thus not suited for direct implementation. In this section, we build on the results from the previous section to devise a calculus for ABox consistency in Horn-$\mathcal{ALCFI}$ that does not require TBox completion to be carried out up-front, but instead reverses cycles 'on the fly'. Our calculus belongs to a family of algorithms that are known as consequence-driven procedures and underly modern and highly efficient reasoners for Horn DLs such as CEL, CB, and ELK (Baader, Lutz, and Suntisrivaraporn 2006; Kazakov 2009; Kazakov, Krötzsch, and Simancik 2011b). It thus establishes a promising foundation for actual implementations of finite-model reasoning in Horn-$\mathcal{ALCFI}$ and (via the reduction in Section 6) in Horn-$\mathcal{ALCQI}$. For simplicity, we start with a calculus for finite satisfiability and finite subsumption. An expansion to finite ABox consistency (and thus to finite instance checking) is sketched afterwards.

The calculus starts with a given TBox $\mathcal{T}$ and then exhaustively applies the inference rules displayed in Figure 1 in the sense that, if the concept inclusions in the precondition (above the line) are already present, then those in the postcondition (below the line) are added. Recall that K stands for a conjunction of concept names, which we read here modulo commutativity; in other words, K can be thought of as a set of concept names. Rule **R1** is applied only if $K \sqcap A$ occurs in the current (partially completed) TBox in a CI of the form $K \sqcap A \sqsubseteq C$ or $K' \sqsubseteq \exists r.(K \sqcap A)$. The same is true for rule **R2** with K in place of $K \sqcap A$. In rules **R7** to **R9**, $\mathsf{cl}_\vdash(K)$ denotes the set of concept names A such that $K \sqsubseteq A$ is contained in the current TBox. In rule **R9**, $\oplus_n$ means addition modulo n.

We point out that rules **R1** to **R8** are minor variations of the corresponding rules in the calculus presented by Kazakov (2009), the main difference being that we do not treat role hierarchies because they are not in our language. Rule **R9** is novel and deals with reversing cycles on the fly. Note that only the 'first edge' of each cycle is reversed, and that this is sufficient because the cycle can be rotated to make any edge the 'first' one.

Example 7 *Consider the TBox*

$$\mathcal{T} = \{A \;\sqsubseteq\; \exists r.(A \sqcap A_1 \sqcap \cdots \sqcap A_n), \qquad (1)$$
$$A \;\sqsubseteq\; (\leqslant 1\, r^-\, A)\,\}. \qquad (2)$$

Cycle reversion from Section 3 reverses all of the exponentially many cycles K, r, K with $K \subseteq S := \{A, A_1, \ldots, A_n\}$ and $A \in K$, adding $K \sqsubseteq \exists r^-.K$ and $K \sqsubseteq (\leqslant 1\ r\ K)$ for all such K. In contrast, the calculus avoids introducing 'irrelevant' conjunctions K and instead jointly reverses all these cycles by generating $A \sqsubseteq \exists r^-.S$ and $A \sqsubseteq (\leqslant 1\ r\ A)$:

S	$\sqsubseteq$	A	*from* **R1**	(3)
S	$\sqsubseteq$	$\exists r.S$	*from* $(1), (3)$, **R3**	(4)
S	$\sqsubseteq$	$(\leqslant 1\, r^-\, A)$	*from* $(2), (3)$, **R3**	(5)
S	$\sqsubseteq$	$\exists r^-.S$	*and*	(6)
S	$\sqsubseteq$	$(\leqslant 1\, r\, A)$	*from* $(3), (4), (5)$, **R9**	(7)
A	$\sqsubseteq$	S	*from* $(1), (2), (6)$, **R8**	(8)
A	$\sqsubseteq$	$\exists r^-.S$	*from* $(6), (8)$, **R3**	(9)
A	$\sqsubseteq$	$(\leqslant 1\, r\, A)$	*from* $(7), (8)$, **R3**	(10)

Note that avoiding to introduce 'irrelevant' conjunctions K as illustrated by Example 7 is a main feature of consequence-based procedures which enables the excellent practical performance typically observed for this class of calculi.

The algorithm terminates after at most exponentially many rule applications since there are only exponentially many different concept inclusions over the signature (concept and role names) used in the original TBox. Each rule application can be performed in time polynomial in the size of the partially completed TBox to which it is applied (thus exponential in the size of the input TBox), which is easy to see for the rules **R1**–**R8**. For **R9**, the crucial observation is that it suffices to consider all conjunctions K_0, K_1 and to check whether they are involved in *any* cycle. The latter can easily be done by a variation of graph reachability, where the nodes of the graph are the conjunctions that occur in the current TBox and the edges come from inclusions $K \sqsubseteq \exists r.K'$.

The following theorem, which is the main result of this section, states that the calculus is sound and complete. Note that finite satisfiability (and finite ABox consistency) in Horn-$\mathcal{ALCQI}$ is known to be ExpTime-complete. An upper bound was proved for the non-Horn version of $\mathcal{ALCQI}$ in (Lutz, Sattler, and Tendera 2005) and the ExpTime lower bound for unrestricted satisfiability in (the $\mathcal{ELI}$ fragment of) Horn-$\mathcal{ALCFI}$ given in (Baader, Brandt, and Lutz 2008) can easily be adapted to finite satisfiability. Thus, our algorithm is worst-case optimal.

Theorem 8 *Let $\mathcal{T}$ be a Horn-$\mathcal{ALCFI}$ TBox, $\widehat{\mathcal{T}}$ be obtained by exhaustively applying Rules* **R1**–**R9**, *and let A be a concept name. Then A is finitely satisfiable w.r.t. $\mathcal{T}$ iff $A \sqsubseteq \bot \notin \widehat{\mathcal{T}}$.*

While Theorem 8 is formulated only for finite satisfiability, the algorithm can of course also be used to decide finite subsumption via the usual reduction to finite satisfiability. The following continues Example 7.

Example 9 *Let $\mathcal{T}$ be the TBox from Example 7 and*

$$\mathcal{T}' = \mathcal{T} \cup \{ \qquad A \sqsubseteq \exists r.(A \sqcap X_1), \qquad (11)$$
$$A \sqsubseteq \exists r.(A \sqcap X_2), \qquad (12)$$
$$X_1 \sqcap X_2 \sqsubseteq \bot \quad \} \qquad (13)$$

The calculus derives $A \sqsubseteq \bot$, thus A is finitely unsatisfiable w.r.t. $\mathcal{T}'$:[1]

A	$\sqsubseteq$	$\exists r.(A \sqcap X_1 \sqcap X_2)$	*from* $(10), (11), (12)$, **R7**	(14)
A	$\sqsubseteq$	$\bot$	*from* $(13), (14)$, **R6**	(15)

The rest of this section is devoted to proving Theorem 8. The "only if" direction (soundness) is straightforward by verifying that each rule is sound in finite models. In contrast, the "if" direction (completeness) turns out to be surprisingly subtle to establish. The proof strategy is as follows. Assume that $A \sqsubseteq \bot \notin \widehat{\mathcal{T}}$. We construct an (infinite) model $\widehat{\mathcal{I}}$ of $\widehat{\mathcal{T}}$ with $A^{\widehat{\mathcal{I}}} \neq \emptyset$ and show that $\widehat{\mathcal{I}}$ is actually a model of $\mathcal{T}_\mathsf{f}$. By Theorem 3, it follows that A is finitely satisfiable w.r.t. $\mathcal{T}$. For what follows, assume w.l.o.g. that A actually occurs in $\mathcal{T}$.

To construct $\widehat{\mathcal{I}}$, let $\mathsf{KON}(\widehat{\mathcal{T}})$ denote the set of all conjunctions K such that K occurs on the left-hand side of some concept inclusion in $\widehat{\mathcal{T}}$ and $K \sqsubseteq \bot \notin \widehat{\mathcal{T}}$. The domain $\Delta^{\widehat{\mathcal{I}}}$ consists of finite words $d = K_1 K_2 \cdots K_n \in \mathsf{KON}(\widehat{\mathcal{T}})^*$, and we use $\mathsf{tail}(d)$ to denote K_n. Define $\widehat{\mathcal{I}}$ by starting with

$$\Delta^{\widehat{\mathcal{I}}} = \mathsf{KON}(\widehat{\mathcal{T}})$$
$$A^{\widehat{\mathcal{I}}} = \{K \in \mathsf{KON}(\widehat{\mathcal{T}}) \mid K \sqsubseteq A \in \widehat{\mathcal{T}}\}$$
$$r^{\widehat{\mathcal{I}}} = \emptyset$$

Observe that since A occurs in $\widehat{\mathcal{T}}$ and $A \sqsubseteq \bot \notin \widehat{\mathcal{T}}$, $\Delta^{\widehat{\mathcal{I}}}$ contains the conjunction $K = A$ and thus $A^{\widehat{\mathcal{I}}} \neq \emptyset$. We finish the construction of $\widehat{\mathcal{I}}$ by exhaustively applying the following rule: if there is some $d \in \Delta^{\widehat{\mathcal{I}}}$ with $\mathsf{tail}(d) \sqsubseteq \exists r.K' \in \widehat{\mathcal{T}}$, K' maximal with this property, and $d \notin (\exists r.K')^{\widehat{\mathcal{I}}}$, then add a fresh element $e = dK'$ to $\Delta^{\widehat{\mathcal{I}}}$, add (d, K') to $r^{\widehat{\mathcal{I}}}$, and add dK' to $A^{\widehat{\mathcal{I}}}$ whenever $K' \sqsubseteq A \in \widehat{\mathcal{T}}$.

We first show that $\widehat{\mathcal{I}}$ is a model of $\widehat{\mathcal{T}}$, which amounts to a case distinction over the forms of CIs that can be present in $\widehat{\mathcal{T}}$, in each case relying on the fact that $\widehat{\mathcal{T}}$ is closed under the rules of the calculus. Details are provided in the appendix.

Lemma 10 $\widehat{\mathcal{I}} \models \widehat{\mathcal{T}}$.

It remains to show that $\widehat{\mathcal{I}}$ is a model of $\mathcal{T}_\mathsf{f}$, which is significantly more difficult to prove than Lemma 10 due to the fact that $\mathcal{T}_\mathsf{f}$ is obtained by reversing all cycles in $\mathcal{T}$ whereas the calculus reverses only the relevant ones, as explained above. We start with the observation that, when constructing $\mathcal{T}_\mathsf{f}$, it suffices to close only maximal cycles. More precisely, a

[1] A is obviously satisfiable w.r.t. $\mathcal{T}'$ in unrestricted models.

$$\textbf{R10}\ \frac{K(a) \quad K \sqsubseteq A}{A(a)} \qquad \textbf{R11}\ \frac{K(a) \quad r(a,b) \quad K \sqsubseteq \forall r.K'}{K'(b)}$$

$$\textbf{R12}\ \frac{\begin{array}{c} K_1(a) \quad K_2(a) \quad r(a,b) \quad K(b) \quad K_1 \sqsubseteq (\leqslant 1\ r\ K'') \\ K_2 \sqsubseteq \exists r.K' \quad K'' \subseteq \mathsf{cl}_\vdash(K) \cap \mathsf{cl}_\vdash(K') \end{array}}{K'(b)}$$

Figure 2: Additional Inference Rules

cycle $K_1, r_1, K_2, \ldots, K_n$ in a TBox $\mathcal{T}$ is *maximal* if K_{j+1} is maximal with $\mathcal{T} \models K_j \sqsubseteq \exists r_j.K_{j+1}$, for $1 \leq j \leq n$. Let $\mathcal{T}_\mathsf{f}^{\max}$ be the variation of $\mathcal{T}_\mathsf{f}$ that is obtained by reversing only maximal cycles.

Lemma 11 *$\mathcal{T}_\mathsf{f}$ is equivalent to $\mathcal{T}_\mathsf{f}^{max}$.*

Let $\mathcal{T}_\mathsf{f}^0, \mathcal{T}_\mathsf{f}^1, \ldots$ be the sequence of TBoxes obtained by starting with $\mathcal{T}_\mathsf{f}^0 = \mathcal{T}$ and then exhaustively closing maximal cycles, that is, $\mathcal{T}_\mathsf{f}^{\max}$ is the limit of this sequence. In the appendix, we prove by induction on i that $\widehat{\mathcal{I}}$ is a model of each $\mathcal{T}_\mathsf{f}^i$, thus of $\mathcal{T}_\mathsf{f}$. This finishes the proof of Theorem 8.

Figure 2 displays the rules necessary to extend the calculus to finite ABox consistency. Instead of starting with only a TBox $\mathcal{T}$, the algorithm now begins with a set $\mathcal{T} \cup \mathcal{A}$, where $\mathcal{T}$ is a TBox and $\mathcal{A}$ an ABox, and then exhaustively applies rules **R1** to **R12**. In rules **R10** to **R12**, $K(a)$ is an abbreviation for $A_1(a) \cdots A_k(a)$ when $K = \{A_1, \ldots, A_k\}$. Recall that rules **R1** and **R2** only apply when the conjunction in their precondition occurs in the partially completed TBox. For the extension with ABoxes, an additional way for K to *occur* is that, for some ABox individual a, $K = \{A \mid A(a)$ is in the partial completion$\}$. It is easy to see that rule application still terminates after exponentially many steps. Let Γ be the set of concept inclusions and ABox assertions finally generated. The algorithm is sound and complete in the sense that $\mathcal{A}$ is finitely consistent w.r.t. $\mathcal{T}$ iff there is an ABox individual a and a conjunction K such that Γ contains both $K(a)$ and $K \sqsubseteq \bot$. To prove this, one updates the construction of $\widehat{\mathcal{I}}$ in the obvious way, that is, starts with the initial interpretation defined by setting $\Delta^{\widehat{\mathcal{I}}} = \mathsf{Ind}(\mathcal{A})$, $r^{\widehat{\mathcal{I}}} = \{(a,b) \mid r(a,b) \in \mathcal{A}\}$ for all role names r, and $A^{\widehat{\mathcal{I}}} = \{a \in \mathsf{Ind}(\mathcal{A}) \mid A(a) \in \Gamma\}$ for all concept names A. The rest of the construction of $\widehat{\mathcal{I}}$ is then exactly as before. It is not hard to adapt the proof of Lemma 10 to show that $\widehat{\mathcal{I}}$ satisfies all inclusions and assertions in Γ. As in the case of finite satisfiability, it thus remains to prove that $\widehat{\mathcal{I}}$ is a model of $\mathcal{T}_\mathsf{f}$. Fortunately, the proof of Lemma 11 above goes through without modification.

5 Query Answering in the Finite

In the popular ontology-based data access (OBDA) paradigm, the central reasoning problem is answering database-style queries over ABoxes in the presence of a DL TBox. In this section, we study the finite model version of this problem, assuming that queries are positive existential queries (PEQs) and that TBoxes are formulated in Horn-$\mathcal{ALCFI}$. We show that, as in the case of ABox consistency, finite PEQ answering can be reduced to unrestricted PEQ answering by reversing finmod cycles in the TBox. This result enables the use of algorithms for unrestricted PEQ answering also in the finite case. It also allows us to show that finite PEQ answering w.r.t. Horn-$\mathcal{ALCFI}$ TBoxes is ExpTime-complete regarding combined complexity, and PTime-complete regarding data complexity.

We start with a brief introduction of positive existential queries and of query answering. For simplicity, we concentrate on Boolean queries, that is, queries without answer variables. It is, however, easy to adapt all techniques established in this section to the case of queries with answer variables. A *(Boolean) positive existential query (PEQ)* q takes the form $\exists \mathbf{x}\, \varphi(\mathbf{x})$ where φ is built from atoms of the form $A(x)$ and $r(x,y)$ using conjunction and disjunction, with x,y variables from $\mathbf{x}$, A a concept name, and r a role name. Let $\mathcal{I}$ be an interpretation and $q = \exists \mathbf{x}\, \varphi$ a PEQ. A *match* of q in $\mathcal{I}$ is a mapping $\pi : \mathbf{x} \to \Delta^\mathcal{I}$ such that φ evaluates to true unter the valuation that assigns true to an atom $A(x)$ in φ iff $\pi(x) \in A^\mathcal{I}$ and true to an atom $r(x,y)$ in φ iff $(\pi(x), \pi(y)) \in r^\mathcal{I}$. We write $\mathcal{I} \models q$ if there is a match of q in $\mathcal{I}$. For an ABox $\mathcal{A}$ and a TBox $\mathcal{T}$, we write $\mathcal{A}, \mathcal{T} \models q$ (resp. $\mathcal{A}, \mathcal{T} \models_\mathsf{fin} q$) if $\mathcal{I} \models q$ for all models (resp. finite models) $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{A}$. We then say that $\mathcal{A}, \mathcal{T}$ *entails* (resp. *finitely entails*) q. The problem that we are interested in is *finite query entailment*, that is, given an ABox $\mathcal{A}$, a TBox $\mathcal{T}$, and a query q, to decide whether $\mathcal{A}, \mathcal{T} \models_\mathsf{fin} q$. We will study both the combined complexity and the data complexity of this problem. When studying combined complexity, all of $\mathcal{A}$, $\mathcal{T}$, and q are considered an input. In the case of data complexity, $\mathcal{T}$ and q are assumed to be fixed and q is the only input.

The main result of this section is the following theorem, where $\mathcal{T}_\mathsf{f}$ is the TBox obtained from $\mathcal{T}$ by exhaustively reversing finmod cycles, exactly as in in Section 3.

Theorem 12 *Let $\mathcal{T}$ be a Horn-$\mathcal{ALCFI}$ TBox and $\mathcal{A}$ an ABox that is finitely consistent w.r.t. $\mathcal{T}$. For any PEQ q,*

$$\mathcal{A}, \mathcal{T} \models_\mathsf{fin} q \text{ iff } \mathcal{A}, \mathcal{T}_\mathsf{f} \models q$$

The proof of the "$\Leftarrow$" direction is trivial. Indeed, if $\mathcal{A}, \mathcal{T} \not\models_\mathsf{fin} q$, then there is a finite model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}$ such that $\mathcal{I} \not\models q$. Since every finite model of $\mathcal{T}$ is also a model of $\mathcal{T}_\mathsf{f}$ by Lemma 4, it follows that $\mathcal{A}, \mathcal{T}_\mathsf{f} \not\models q$.

For the proof of the "$\Rightarrow$" direction, we use a well-known (infinite) canonical model $\mathcal{U}$ of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$, constructed by starting with the following initial interpretation

$$\begin{aligned} \Delta^\mathcal{U} &= \mathsf{Ind}(\mathcal{A}) \\ A^\mathcal{U} &= \{a \in \mathsf{Ind}(\mathcal{A}) \mid \mathcal{A}, \mathcal{T}_\mathsf{f} \models A(a)\} \\ r^\mathcal{U} &= \{(a,b) \mid r(a,b) \in \mathcal{A}\} \end{aligned}$$

and then exhaustively applying the following completion rule: for all $d \in \Delta^\mathcal{U}$ such that $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{U}(d) \sqsubseteq \exists r.t'$, where t' is maximal with this property and $d \notin (\exists r.t')^\mathcal{U}$, add a fresh element d' to $\Delta^\mathcal{U}$, the edge (d,d') to $r^\mathcal{U}$, and d' to the interpretation $A^\mathcal{U}$ of all concept names $A \in t'$.

The following properties of $\mathcal{U}$ are well-known and the reason why $\mathcal{U}$ is called canonical (Krisnadhi and Lutz 2007; Eiter et al. 2008; Ortiz, Rudolph, and Šimkus 2011).

Lemma 13

1. $\mathcal{U}$ *is a model of* $\mathcal{A}$ *and of* $\mathcal{T}_{\mathsf{f}}$*;*

2. For any PEQ q*, we have that* $\mathcal{A}, \mathcal{T}_{\mathsf{f}} \models q$ *iff* $\mathcal{U} \models q$*;*

3. for every $t \in \mathsf{TP}(\mathcal{A})$*, there is a* $d \in \Delta^{\mathcal{U}}$ *with* $\mathsf{tp}_{\mathcal{U}}(d) = t$*.*

By Point 2 of Lemma 13, we can establish the "⇒" direction of Theorem 12 by showing that $\mathcal{A}, \mathcal{T} \models_{\mathsf{fin}} q$ implies $\mathcal{U} \models q$. In fact, we shall show the following, stronger property.

Proposition 14 *For every* $n_0 > 0$*, there is a finite model* $\mathcal{J}$ *of* $\mathcal{A}$ *and* $\mathcal{T}$ *such that for any PEQ* q *with at most* n_0 *variables,* $\mathcal{J} \models q$ *implies* $\mathcal{U} \models q$*.*

Thus, the interpretation $\mathcal{J}$ from Proposition 14 is a finite canonical model for PEQs with at most n_0 variables. We will obtain $\mathcal{J}$ by modifying the finite model $\mathcal{I}$ constructed in Section 3. To describe the required modifications in detail, we first introduce some preliminaries.

Let $\mathcal{I}_1, \mathcal{I}_2$ be interpretations. A *homomorphism from* $\mathcal{I}_1$ *to* $\mathcal{I}_2$ is a function $h : \Delta^{\mathcal{I}_1} \to \Delta^{\mathcal{I}_2}$ such that

1. $h(a) = a$ for all $a \in \mathsf{N_I}$;
2. $d \in A^{\mathcal{I}_1}$ implies $h(d) \in A^{\mathcal{I}_2}$ for all concept names A;
3. $(d, e) \in r^{\mathcal{I}_1}$ implies $(h(d), h(e)) \in r^{\mathcal{I}_2}$ for all (possibly inverse) roles r.

A *bounded simulation of* $\mathcal{I}_1$ *in* $\mathcal{I}_2$ is a relation $\rho \subseteq \Delta^{\mathcal{I}_1} \times \mathbb{N} \times \Delta^{\mathcal{I}_2}$ such that for all $(d, i, e) \in \rho$, the following conditions are satisfied:

1. if $d \in A^{\mathcal{I}_1}$, then $e \in A^{\mathcal{I}_2}$;
2. if $i > 0$ and $(d, d') \in r^{\mathcal{I}_1}$ for some (possibly inverse) role r, then there is an $e' \in \Delta^{\mathcal{I}_2}$ with $(e, e') \in r^{\mathcal{I}_2}$ and $(d', i-1, e') \in \rho$.

We write $(\mathcal{I}_1, d) \preceq_k (\mathcal{I}_2, e)$, for $d \in \Delta^{\mathcal{I}_1}$ and $e \in \Delta^{\mathcal{I}_2}$, if there is a bounded simulation of $\mathcal{I}_1$ in $\mathcal{I}_2$ such that $(d, k, e) \in \rho$ and for all $a \in \mathsf{N_I} \cap \Delta^{\mathcal{I}_1}$, we have $(a, k, a) \in \rho$. Then $\mathcal{I}_1 \preceq_k \mathcal{I}_2$ denotes that for every $d \in \Delta^{\mathcal{I}_1}$, there is an $e \in \Delta^{\mathcal{I}_2}$ with $(\mathcal{I}_1, d) \preceq_k (\mathcal{I}_2, e)$. Note that $(\mathcal{I}_1, d) \preceq_k (\mathcal{I}_2, e)$, with $k > 0$, implies that there is a homomorphism from the restriction of $\mathcal{I}_1$ to the elements in $\mathsf{Ind}(\mathcal{A})$ to $\mathcal{I}_2$. We write $(\mathcal{I}_1, d) \sim_k (\mathcal{I}_2, e)$ if $(\mathcal{I}_1, d) \preceq_k (\mathcal{I}_2, e)$ and vice versa.

Returning to Proposition 14, we can rephrase the requirement that for every PEQ q with at most n_0 variables, $\mathcal{J} \models q$ implies $\mathcal{U} \models q$, in the following way:

$(*)$ for all $d \in \Delta^{\mathcal{J}}$, there is a homomorphism from $\mathcal{J}|_d^{n_0}$ to $\mathcal{U}$,

where $\mathcal{J}|_d^{n_0}$ is the restriction of $\mathcal{J}$ to those elements that are reachable in $\mathcal{J}$ from d by traveling at most $2n_0$ role edges.

The finite model $\mathcal{I}$ constructed in Section 3 need not satisfy $(*)$ for all n_0, for several reasons. First and most obviously, $\mathcal{I}$ can contain cycles that do not exclusively consist of ABox elements, and no such cycles are present in $\mathcal{U}$. We will solve this problem by eliminating in $\mathcal{I}$ all non-ABox-cycles of size at most n_0, which can be achieved by taking the product with a suitable finite group of high girth, a technique championed by Otto (2012).

Another reason why $\mathcal{I}$ does not satisfy $(*)$ is that we may not even have $\mathcal{I} \preceq_{n_0} \mathcal{U}$, that is, even if we unravel $\mathcal{I}$ into a tree (thus eliminating all cycles), starting at some $d \in \Delta^{\mathcal{I}}$ and cutting off at depth n_0, we are still not guaranteed to find a homomorphism from the resulting tree into $\mathcal{U}$. There are, in turn, two reasons for why this is the case. First, application of $(\mathbf{c3})$ during the construction of $\mathcal{I}$ can generate role edges $(d_1, d) \in r^{\mathcal{I}}$ and $(d_2, d) \in s^{\mathcal{I}}$ such that $\mathsf{tp}_{\mathcal{I}}(d_1) \to_r \mathsf{tp}_{\mathcal{I}}(d) \ {}_s\!\leftarrow \mathsf{tp}_{\mathcal{I}}(d_2)$ and d is not identified by an ABox element. Such situations are not necessarily reproducible in $\mathcal{U}$. As a concrete example, consider

$$\mathcal{A} = \{ B_1(a), B_2(b) \}$$
$$\mathcal{T} = \{ B_1 \sqsubseteq \exists r.A,\ B_2 \sqsubseteq \exists r.A \}.$$

Then the mentioned edges are $(a, d_t) \in r^{\mathcal{I}}$ and $(b, d_t) \in r^{\mathcal{I}}$ where $t = \{A\}$. Note that we do not even have $(\mathcal{I}, d_t) \preceq_1 \mathcal{U}$.

The second reason for failure of $\mathcal{I} \preceq_{n_0} \mathcal{U}$ is that applications of $(\mathbf{c2})$ can result in similar a situation as above, but where the middle element d is replaced with a sequence of elements $e_0, \ldots, e_k$ such that $(e_i, e_{i+1}) \in r_i^{\mathcal{I}}$ for all $i < k$ (for some roles $r_0, \ldots, r_{k-1}$) and

$$\mathsf{tp}_{\mathcal{I}}(e_0) \ {}^1\!\!\leftrightarrow^1_{r_1} \cdots {}^1\!\!\leftrightarrow^1_{r_{k-1}} \mathsf{tp}_{\mathcal{I}}(e_k). \tag{16}$$

Again, such situations need not be reproducible in $\mathcal{U}$. For a concrete example, take

$$\mathcal{A} = \{ A_1(a),\ B_1(a),\ A_2(b),\ B_2(b) \}$$

and assume that $\mathcal{T}$ is such that $B_1 \ {}^1\!\!\leftrightarrow^1_r B_2$. Then an application of $(\mathbf{c2})$ will simply add $r(a, b)$, an edge that does not exist in $\mathcal{U}$.

Our solution to the problem with $(\mathbf{c3})$ is to modify the construction of $\mathcal{I}$ by replacing the initial elements d_t, $t \in \mathsf{TP}(\mathcal{A})$, used as 'targets' by $(\mathbf{c3})$. Intuitively, the original construction introduces one $(\mathbf{c3})$-target d_t for each type t. In the modified construction, we instead want to introduce one $(\mathbf{c3})$-target for each $2n_0$*-bounded simulation type*, which is an equivalence class of $\sim_{2n_0}$ on the set of all pointed interpretations $(\mathcal{I}_1, d)$.[2] Since simulations need only to consider symbols that occur in the (fixed) ABox $\mathcal{A}$ and (fixed) TBox $\mathcal{T}$, there are only finitely many $2n_0$-simulation types and thus only finitely many $(\mathbf{c3})$-targets.

The problem with $(\mathbf{c2})$ is addressed by modifying the $(\mathbf{c2})$ rule so that the sequences (16) are of length exceeding $2n_0$ and thus the highlighted problem which involves both ends of the sequence is not 'visible' in subinterpretations $\mathcal{I}|_d^{n_0}$. Implementing these modifications yields the following.

Proposition 15 *For every* $n_0 > 0$*, there is a finite model* $\mathcal{I}$ *of* $\mathcal{A}$ *and* $\mathcal{T}$ *such that* $\mathcal{I} \preceq_{2n_0} \mathcal{U}$*.*

To obtain the model stipulated by Proposition 14 from the one delivered by Proposition 15, we then use the product with finite groups of high girth mentioned above. Some technical care needs to be exercised to preserve small cycles in the ABox while eliminating small cycles outside of the ABox. Details for all the constructions sketched above are provided in the appendix. This finishes the proof of Theorem 12.

Apart from enabling the use of algorithms for unrestricted PEQ answering also in the finite case, Theorem 12 allows yields tight complexity bounds for finite PEQ entailment in Horn-$\mathcal{ALCFI}$.

[2]We use $2n_0$ here since interpretations $\mathcal{J}|_d^{n_0}$ can contain simple paths of length $2n_0$.

Theorem 16 *Finite PEQ entailment in Horn-$\mathcal{ALCFI}$ is decidable,* ExpTime*-complete in combined complexity, and* PTime*-complete in data complexity.*

Proof.(sketch) For the unrestricted case, an ExpTime lower bound is in (Baader, Brandt, and Lutz 2008) and a PTime one in (Calvanese et al. 2006). Both results can easily be adapted to the finite case. The upper bounds can be proved using the following straightforward algorithm for PEQ entailment, which resembles existing algorithms such as those presented in (Krisnadhi and Lutz 2007; Eiter et al. 2008; Calì, Gottlob, and Lukasiewicz 2009; Ortiz, Rudolph, and Šimkus 2011). As a consequence of Theorem 3, finite subsumption w.r.t. a TBox $\mathcal{T}$ coincides with unrestricted subsumption w.r.t. $\mathcal{T}_{\mathsf{f}}$. Using our algorithm for computing finite subsumption in Horn-$\mathcal{ALCFI}$ in ExpTime, we can thus compute the set $\mathsf{TP}(\mathcal{A})$ of types as introduced in Section 3 (which is defined with respect to $\mathcal{T}_{\mathsf{f}}$), without computing $\mathcal{T}_{\mathsf{f}}$ or explicitly reasoning w.r.t. this exponentially large TBox. Let $\mathcal{A}'$ be the extension of the input ABox $\mathcal{A}$ with assertions $\{A(a_t) \mid A \in t\}$ for each $t \in \mathsf{TP}(\mathcal{A})$. Now compute an initial piece $\mathcal{U}'$ of the canonical model $\mathcal{U}$ of $\mathcal{A}'$ and $\mathcal{T}_{\mathsf{f}}$, namely its restriction to depth n, where n is the number of variables in the input query q. As before, we can do this by using finite subsumption w.r.t. $\mathcal{T}$ instead of unrestricted subsumption w.r.t. $\mathcal{T}_{\mathsf{f}}$. It is not difficult to prove that $\mathcal{U}' \models q$ iff $\mathcal{U} \models q$. To check whether $\mathcal{U}' \models q$ within the desired time bounds, we can simply enumerate all possible maps of variables in q to elements of $\mathcal{U}'$. ❏

Note that decidability of PEQ entailment in Horn-$\mathcal{ALCFI}$ was expected given a result by Pratt-Hartmann which states that finite CQ answering for the two-variable guarded fragment of first-order logic extended with counting quantifiers is decidable (Pratt-Hartmann 2009). We assume that his proof can be extended to unions of conjunctive queries (UCQs), thus to PEQs. Pratt-Hartmann also analyses the data complexity of finite CQ answering in his logic, but finds it to be coNP-complete. He does not analyse combined complexity.

6 From Horn-$\mathcal{ALCFI}$ to Horn-$\mathcal{ALCQI}$

All results obtained in this paper that concern finite satisfiability and finite subsumption (the reasoning tasks that do not involve ABoxes) extend in a straightforward way from Horn-$\mathcal{ALCFI}$ to Horn-$\mathcal{ALCQI}$. In particular, we can convert a Horn-$\mathcal{ALCQI}$ TBox $\mathcal{T}$ into a Horn-$\mathcal{ALCFI}$ TBox $\mathcal{T}'$ such that finite (un)satisfiability is preserved by replacing each CI $K \sqsubseteq (\geqslant n\ r\ K')$ in $\mathcal{T}$ with the following inclusions, for $1 \leq i < j \leq n$:

$$K \sqsubseteq \exists r.B_i, \quad B_i \sqsubseteq K', \quad B_i \sqcap B_j \sqsubseteq \bot \qquad (*)$$

While an easy unraveling argument can be used to prove that this reduction is correct in the presence of infinite models, more care is required in the finite case. Details can be found in the appendix.

Proposition 17 *$\mathcal{T}$ is finitely satisfiable iff $\mathcal{T}'$ is finitely satisfiable.*

It follows from Proposition 17 and Theorem 3 that a Horn-$\mathcal{ALCQI}$ TBox $\mathcal{T}$ is finitely satisfiable iff $(\mathcal{T}')_{\mathsf{f}}$ is satisfiable. Actually, it is not hard to see that this is the case iff $\mathcal{T}_{\mathsf{f}}$ (the result of applying cycle reversion directly to the Horn-$\mathcal{ALCQI}$ TBox, ignoring all inclusions $A \sqsubseteq (\geqslant n\ r\ C)$) is satisfiable because if any of the existential restrictions in $\mathcal{T}' \setminus \mathcal{T}$ is involved in a finmod cycle, then a simple semantic argument shows that both $\mathcal{T}_{\mathsf{f}}$ and $(\mathcal{T}')_{\mathsf{f}}$ are unsatisfiable. Proposition 17 also enables the use of our consequence-based procedure for deciding finite satisfiability in Horn-$\mathcal{ALCQI}$.

It is not immediately obious how to extend $(*)$ and Proposition 17 to ABox consistency and instance checking. We believe, though, that it is not too hard to modify the proof of Theorem 3 to Horn-$\mathcal{ALCQI}$, to adapt the consequence-based procedure to allow a direct treatment of Horn-$\mathcal{ALCQI}$ TBox without prior reduction to Horn-$\mathcal{ALCFI}$, and to extend all model constructions underlying our results about PEQ entailment to Horn-$\mathcal{ALCQI}$. In particular, such a direct approach should yield ExpTime/ PTime upper bounds for PEQ entailment in Horn-$\mathcal{ALCQI}$ even when the numbers in at least restrictions are coded in binary (node that, in this case, the translation $(*)$ incurs an exponential blowup).

7 Future Work

An interesting direction for future research is to extend the results in this paper to Horn-$\mathcal{SHIF}$, that is, to add role hierarchies and transitive roles. Reducing out role hierarchies does not seem easily possible in the finite, and so they would have to be built directly into all constructions presented in this paper. For query entailment, we expect transitive roles to cause significant additional challenges, see for example (Eiter et al. 2009; Mosurovic et al. 2013). In particular, such an extension results in an additional way in which the finite model property is lost, illustrated by the TBox $\mathcal{T} = \{A \sqsubseteq \exists r.A, \mathsf{trans}(r)\}$ and the conjunctive query $q = \exists x\, r(x,x)$. We have $\{A(a)\}, \mathcal{T} \not\models q$, but $\{A(a)\}, \mathcal{T} \models_{\mathsf{fin}} q$ although neither counting nor inverse roles are present in this example (where the TBox is actually formulated in the DL $\mathcal{EL}_{\mathsf{trans}}$). Finite model reasoning in versions of Datalog$^{\pm}$ that include this and further effects have recently been studied in (Gogacz and Marcinkowski 2013b; 2013a).

In this paper, we have not analyzed the size of finite models. In the case of finite satisfiability it is, however, easy to prove a double exponential lower bound on the size of finite models by enforcing a tree of exponential depth in which no two elements can be identical. A matching upper bound follows from Pratt-Hartmann's result that every finitely satisfiable formula in first-order logic with two variables and counting quantifiers has a model of at most double exponential size (Pratt-Hartmann 2005). Analyzing the size of finite (counter)models for query entailment is left as future work.

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A Proofs for Section 3

Lemma 5. *$\prec^+$ is a strict partial order.*

Proof. Since $\prec^+$ is transitive by definition, it remains to establish irreflexivity and asymmetry. To this end, it suffices to show that $\prec$ is acyclic in the sense that there are no type partitions $P_0, \dots, P_n$, $n \geq 0$, such that $P_0 \prec \cdots \prec P_n = P_0$. Assume to the contrary that there are such $P_0, \dots, P_n$. By reversing the order, we can assume that $P_0 \succ \cdots \succ P_n = P_0$. Then there are, for each $i < n$, types $t_i \in P_i$ and $t'_{i+1} \in P_{i+1}$ such that $t_i \subsetneq t'_{i+1}$. For uniformity, set $t_n = t_0$ and $t'_0 = t'_n$.

Let $i < n$. By definition of type partitions and since $t \;{}^1{\leftrightarrow}^1_r\; t'$ implies $t' \;{}^1{\leftrightarrow}^1_{r^-}\; t$ for all types t, t' and roles r, we can derive from $t_i, t'_i \in P_i$ the existence of types $s_{0,i}, \dots, s_{k_i,i} \in P_i$, $k_i \geq 0$, and roles $r_{0,i}, \dots, r_{k_i-1,i}$ such that

$$t_i = s_{0,i} \;{}^1{\leftrightarrow}^1_{r_{0,i}}\; s_{1,i} \;{}^1{\leftrightarrow}^1_{r_{1,i}}\; \cdots \;{}^1{\leftrightarrow}^1_{r_{k_i-1,i}}\; s_{k_i,i} = t'_i.$$

For each i, we thus find a sequence

$$t_i, r_{0,i}, s_{1,i}, \dots, s_{k_i-1,i}, r_{k_i-1,i}, t'_i \tag{$*$}$$

that satisfies the prerequisites for finmod cycles, namely

$$\mathcal{T} \models s_{j,i} \sqsubseteq \exists r_{j,i}.s_{j+1,i} \tag{17}$$

$$\mathcal{T} \models s_{j+1,i} \sqsubseteq (\leqslant 1\; r_{j,i}\; s_{j,i}) \tag{18}$$

for all $j = 0, \dots, k_i$ (but this sequence need not be a finmod cycle since $t_i = t'_i$ is not guaranteed). Note that we cannot have $k_i = 0$ for all i, since then

$$t_0 \subsetneq t'_1 = t_1 \subsetneq t'_2 = t_2 \subsetneq \cdots \subsetneq t'_n = t_n,$$

in contradiction to $t_n = t_0$. In the following, we can thus assume that $k_i > 0$ for at least one i.

Because of (18), we have $\mathcal{T}_\mathsf{f} \models t_i \sqsubseteq \exists r_{0,i}.s_{1,i}$ and $\mathcal{T}_\mathsf{f} \models s_{1,i} \sqsubseteq (\leqslant 1\; r^-_{0,i}\; t_i)$. Because of $t_i \subsetneq t'_{i+1}$, we thus obtain $\mathcal{T}_\mathsf{f} \models t'_{i+1} \sqsubseteq \exists r_{0,i}.s_{1,i}$ and $\mathcal{T}_\mathsf{f} \models s_{1,i} \sqsubseteq (\leqslant 1\; r^-_{0,i}\; t'_{i+1})$. Consequently, the following sequences also satisfy conditions (17) and (18):

$$t'_n, r_{0,n-1}, s_{1,n-1}, \dots, s_{k_{n-1}-1,n-1}, r_{k_{n-1}-1,n-1}, t'_{n-1}$$
$$t'_{n-1}, r_{0,n-2}, s_{1,n-2}, \dots, s_{k_{n-1}-1,n-2}, r_{k_{n-1}-1,n-2}, t'_{n-2}$$
$$\vdots$$
$$t'_1, r_{0,0}, s_{1,0}, \dots, s_{k_0-1,0}, r_{k_0-1,0}, t'_0.$$

Since $t'_0 = t'_n$, we can concatenate all these sequences to a finmod cycle. As $k_i > 0$ for at least one i, this cycle is non-empty, and the construction of $\mathcal{T}_\mathsf{f}$ ensures that the reversed cycle is also present in $\mathcal{T}_\mathsf{f}$. Let us assume w.l.o.g. that $k_{n-1} > 0$. The presence of the reversed cycle yields $\mathcal{T}_\mathsf{f} \models s_{1,n-1} \sqsubseteq \exists r^-_{0,n-1}.t'_n$. Since $t_n \;{}^1{\leftrightarrow}^1_{r_{0,n-1}}\; s_{1,n-1}$, we have $\mathcal{T}_\mathsf{f} \models s_{1,n-1} \sqsubseteq \exists r^-_{0,n-1}.t_{n-1}$ and t_{n-1} is maximal with this property. This is a contradiction to $t_{n-1} \supsetneq t'_n$. ❑

Proof of Theorem 6

We start with a technical lemma that will be used below to show that applications of $(\mathbf{c2})$ preserve all invariants. The statement by the lemma is meant to refer to a concrete application of $(\mathbf{c2})$.

Lemma 18 *If $\lambda = s \;{}^1{\leftrightarrow}^1_r\; s'$ with $s, s' \in P$ and $(d, d') \in \pi_\lambda$, then $\mathsf{tp}_\mathcal{I}(d) = s$ and $\mathsf{tp}_\mathcal{I}(d') = s'$.*

Proof. Let $\lambda = s \;{}^1{\leftrightarrow}^1_r\; s'$ and $(d, d') \in \pi_\lambda$. We first show that $\mathsf{tp}_\mathcal{I}(d) = s$. Since d is in the domain of π_λ, we have $d \in \Delta_s$ or $d \in X^\mathcal{I}_{\lambda,1}$. In the former case, $\mathsf{tp}_\mathcal{I}(d) = s$ is immediate by construction of $\mathcal{I}$. Thus assume that $d \in X^\mathcal{I}_{\lambda,1}$. Then $s \subseteq \mathsf{tp}_\mathcal{I}(d)$. From λ, we obtain $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{I}(d) \sqsubseteq \exists r.s'$. Let $\hat{s}' \supseteq s'$ be maximal such that $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{I}(d) \sqsubseteq \exists r.\hat{s}'$. Note that, by λ and since $s \subseteq \mathsf{tp}_\mathcal{I}(d)$ and $s' \subseteq \hat{s}'$, we have $\mathcal{T}_\mathsf{f} \models \hat{s}' \sqsubseteq (\leqslant 1\; r\; \mathsf{tp}_\mathcal{I}(d))$. Thus $\mathsf{tp}_\mathcal{I}(d) \to^1_r \hat{s}'$.

Next, observe that $\hat{s}' \to^1_{r^-} \mathsf{tp}_\mathcal{I}(d)$. If this was not the case, then $(\mathbf{c1})$ would be applicable to d and, since its application is preferred over applications of $(\mathbf{c2})$, generate an $e \in \Delta^\mathcal{I}$ such that $(d, e) \in r^\mathcal{I}$ and $e \in (\hat{s}')^\mathcal{I}$ before the $(\mathbf{c2})$ application considered here. This contradicts the fact that $d \in X^\mathcal{I}_{\lambda_1}$, which implies that $d \notin (\exists r.s)^\mathcal{I}$ by the time when $(\mathbf{c2})$ was applied.

In summary, we have established that $\lambda' = \mathsf{tp}_\mathcal{I}(d) \;{}^1{\leftrightarrow}^1_r\; \hat{s}'$ holds. Now assume to the contrary of what we have to show that $s \subsetneq \mathsf{tp}_\mathcal{I}(d)$. Recall that P is the type class that the current $(\mathbf{c2})$ application treats, and that $s, s' \in P$. By λ', there is a type class P' with $\mathsf{tp}_\mathcal{I}(d), \hat{s}' \in P'$. Since $s \subsetneq \mathsf{tp}_\mathcal{I}(d)$, we have $P' \prec P$. Since $d \in X^\mathcal{I}_{\lambda,1}$, we had $d \notin (\exists r.s')^\mathcal{I}$ before the current rule application, thus also $d \notin (\exists r.\hat{s}')^\mathcal{I}$. Summing up, before the current rule application we had $\mathsf{tp}_\mathcal{I}(d), \hat{s}' \in P'$, $\lambda' = \mathsf{tp}_\mathcal{I}(d) \;{}^1{\leftrightarrow}^1_r\; \hat{s}'$, $d \in \mathsf{tp}_\mathcal{I}(d)$, and $d \notin (\exists r.\hat{s}')^\mathcal{I}$. Consequently, rule $(\mathbf{c2})$ was applicable also to type class P'. Since $P' \prec P$ and with the preference order that $(\mathbf{c2})$ imposes on type classes, this contradicts that the current application is treating P.

It remains to show that $\mathsf{tp}_\mathcal{I}(d') = s'$. The argument is exactly the same as above, with r^- playing the role of r, s' playing the role of s and vice versa, λ^- playing the role of λ, and $\mathsf{tp}_\mathcal{I}(d')$ playing the role of $\mathsf{tp}_\mathcal{I}(d)$ and vice versa. ❑

Satisfaction of Invariants It is easy to verify that the initial interpretation $\mathcal{I}$ satisfies all invariants–recall in particular that every element of $\mathsf{TP}(\mathcal{A})$ is from $\mathsf{TP}(\mathcal{T}_\mathsf{f})$. It thus remains to show that each of the rules $(\mathbf{c1})$ to $(\mathbf{c3})$ preserves the invariants.

Application of (c1) preserves all invariants. It is obvious that the invariants $(\mathbf{i1})$ and $(\mathbf{i2})$ are preserved with each single application of $(\mathbf{c1})$. We have to show that the same is true for $(\mathbf{i3})$. Assume that completion processed $d \in \Delta^\mathcal{I}$ with $\mathsf{tp}_\mathcal{I}(d) \to^1_r t$ and $t \not\to^1_{r^-} \mathsf{tp}_\mathcal{I}(d)$, and that after the application $(d, d_1) \in r^\mathcal{I}$, with d_1 the fresh domain element added. Assume to the contrary of what is to be proved that $\mathcal{T}_\mathsf{f} \models K \sqsubseteq (\leqslant 1\; r\; K')$ and there is a $d_2 \in \Delta^\mathcal{I}$ distinct from d_1 such that $d \in K^\mathcal{I}$, $(d, d_2) \in r^\mathcal{I}$, and $d_1, d_2 \in K'^\mathcal{I}$. We aim to show that if such d_2 exists then $t \subseteq \mathsf{tp}_\mathcal{I}(d_2)$, which establishes a contradiction to the fact that $d \notin (\exists r.t)^\mathcal{I}$ was true before the rule application. According to $(\mathbf{i2})$, we can distinguish the following cases:

- $\mathsf{tp}_\mathcal{I}(d) \to_r \mathsf{tp}_\mathcal{I}(d_2)$. Then $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{I}(d) \sqsubseteq \exists r.\mathsf{tp}_\mathcal{I}(d_2)$ and $\mathsf{tp}_\mathcal{I}(d_2)$ is maximal with this property. From $\mathsf{tp}_\mathcal{I}(d) \to_r t$, we additionally get $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{I}(d) \sqsubseteq \exists r.t$. Furthermore, since $K \subseteq \mathsf{tp}_\mathcal{I}(d)$ and $d_1, d_2 \in K'^\mathcal{I}$ implies $K' \subseteq \mathsf{tp}_\mathcal{I}(d_2) \cap t$, a simple semantic argument shows that

$\mathcal{T}_\mathsf{f} \models K \sqsubseteq \exists r.(\mathsf{tp}_\mathcal{I}(d_2) \cup t)$. The maximality of $\mathsf{tp}_\mathcal{I}(d_2)$ thus implies $t \subseteq \mathsf{tp}_\mathcal{I}(d_2)$.

- $\mathsf{tp}_\mathcal{I}(d_2) \rightarrow_{r^-} \mathsf{tp}_\mathcal{I}(d)$. Then $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{I}(d_2) \sqsubseteq \exists r^-.\mathsf{tp}_\mathcal{I}(d)$ and, additionally, we have $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{I}(d) \sqsubseteq \exists r.t$. Since $K \subseteq \mathsf{tp}_\mathcal{I}(d)$ and $K' \subseteq \mathsf{tp}_\mathcal{I}(d_2) \cap t$, a simple semantic argument shows that $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{I}(d_2) \sqsubseteq t$. Since $\mathsf{tp}_\mathcal{I}(d_2)$ is a type for $\mathcal{T}_\mathsf{f}$ by (**i1**), it follows that $t \subseteq \mathsf{tp}_\mathcal{I}(d_2)$.
- $r(d, d_2) \in \mathcal{A}$. Then $\mathsf{tp}_\mathcal{A}(d) = \mathsf{tp}_\mathcal{I}(d)$ by construction of the initial interpretation $\mathcal{I}$. Since $\mathsf{tp}_\mathcal{I}(d) \rightarrow_r t$, we thus have $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{A}(d) \sqsubseteq \exists r.t$. With $r(d, d_2) \in \mathcal{A}$ and by the semantics, $t \subseteq \mathsf{tp}_\mathcal{A}(d_2) = \mathsf{tp}_\mathcal{I}(d_2)$.

Application of (c2) preserves all invariants. Invariant (**i1**) is clearly preserved by each single application of (**c2**). We have to prove that the same is true for (**i2**) and (**i3**).

Assume that (**c2**) is applied to a type class P. To eliminate case distinctions, for each $\lambda = s \ {}^1{\leftrightarrow}_r^1 \ s'$ let λ^- denote $s' \ {}^1{\leftrightarrow}_{r^-}^1 \ s$. Note that λ holds if and only if λ^- does. Then define π_{λ^-} to be the converse of π_λ, for all $\lambda = s \ {}^1{\leftrightarrow}_r^1 \ s'$ with r an inverse role. Note that π_λ is a bijection from $X^\mathcal{I}_{\lambda,1} \cup \Delta_s$ to $X^\mathcal{I}_{\lambda,2} \cup \Delta_{s'}$, just as in the case where r is a role name. Also note that whenever $(d, e) \in r^\mathcal{I}$ is added by the current application of (**c2**) with r a (potentially inverse) role, then there is a $\lambda = s \ {}^1{\leftrightarrow}_r^1 \ s'$ such that $(d, d') \in \pi_\lambda(d)$.

To show that (**i2**) is preserved by (**c2**), consider a (potentially inverse) role r and a pair $(d, d') \in r^\mathcal{I}$ that has been added in a (**c2**) application. Take $\lambda = s \ {}^1{\leftrightarrow}_r^1 \ s'$ such that $(d, d') \in \pi_\lambda(d)$. From Lemma 18, we obtain $\mathsf{tp}_\mathcal{I}(d) = s$ and $\mathsf{tp}_\mathcal{I}(d') = s'$. Consequently, $s \ {}^1{\leftrightarrow}_r^1 \ s'$ yields $\mathsf{tp}_\mathcal{I}(d) \rightarrow_r \mathsf{tp}_\mathcal{I}(d')$ *and* $\mathsf{tp}_\mathcal{I}(d') \rightarrow_{r^-} \mathsf{tp}_\mathcal{I}(d)$.

We now show that (**i3**) is preserved by (**c2**). Let $\mathcal{T}_\mathsf{f} \models K \sqsubseteq (\leqslant 1 \ r \ K')$, and assume to the contrary of what is to be shown that, after some application of (**c2**), there are $(d, d_1), (d, d_2) \in r^\mathcal{I}$ with $d \in K^\mathcal{I}$, $d_1, d_2 \in K'^\mathcal{I}$, and $d_1 \neq d_2$. We distinguish the following cases:

- (d, d_1) was added by an application of (**c2**), (d, d_2) was not. By the former, there is $\lambda = s \ {}^1{\leftrightarrow}_r^1 \ s'$ such that $(d, d_1) \in \pi_\lambda$. By Lemma 18, $\mathsf{tp}_\mathcal{I}(d) = s$ and $\mathsf{tp}_\mathcal{I}(d_1) = s'$.

  We aim to show that $s' \subseteq \mathsf{tp}_\mathcal{I}(d_2)$ because this means that $d \in (\exists r.s')^\mathcal{I}$ was true before the current appliction of (**c2**), in contradiction to d being in the domain of π_λ. Since (d, d_2) was not added by (**c2**), by (**i2**) we can distinguish the following subcases:

  - $\mathsf{tp}_\mathcal{I}(d) \rightarrow_r \mathsf{tp}_\mathcal{I}(d_2)$. Thus $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{I}(d) \sqsubseteq \exists r.\mathsf{tp}_\mathcal{I}(d_2)$ and $\mathsf{tp}_\mathcal{I}(d_2)$ is maximal with this property. Since $\mathsf{tp}_\mathcal{I}(d) = s$ and by λ, $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{I}(d) \sqsubseteq \exists r.s'$. Using the facts that $\mathcal{T}_\mathsf{f} \models K \sqsubseteq (\leqslant 1 \ r \ K')$, $K \subseteq \mathsf{tp}_\mathcal{I}(d) = s$, $K' \subseteq \mathsf{tp}_\mathcal{I}(d_2)$, and $K' \subseteq \mathsf{tp}_\mathcal{I}(d_1) = s'$, an easy semantic argument shows that $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{I}(d) \sqsubseteq \exists r.(\mathsf{tp}_\mathcal{I}(d_2) \cup s')$. The maximality of $\mathsf{tp}_\mathcal{I}(d_2)$ thus yields $s' \subseteq \mathsf{tp}_\mathcal{I}(d_2)$.
  - $\mathsf{tp}_\mathcal{I}(d_2) \rightarrow_{r^-} \mathsf{tp}_\mathcal{I}(d)$. Then $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{I}(d_2) \sqsubseteq \exists r^-.s$. By λ, we have $\mathcal{T}_\mathsf{f} \models s \sqsubseteq \exists r.s'$. Since $K \subseteq s$, $K \subseteq \mathsf{tp}_\mathcal{I}(d_2)$, $K \subseteq \mathsf{tp}_\mathcal{I}(d_1) = s'$, and $\mathcal{T}_\mathsf{f} \models K \sqsubseteq (\leqslant 1 \ r \ K')$, a simple semantic argument shows that $s' \subseteq \mathsf{tp}_\mathcal{I}(d_2)$.
  - $r(d, d_2) \in \mathcal{A}$. Since $d \in K^\mathcal{I}$ and $d_2 \in K'^\mathcal{I}$, we have $K \subseteq \mathsf{tp}_\mathcal{A}(d)$ and $K' \subseteq \mathsf{tp}_\mathcal{A}(d_2)$ by definition of the initial interpretation $\mathcal{I}$. Also, $\mathsf{tp}_\mathcal{A}(d) = s$. By λ, we thus have $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{A}(d) \sqsubseteq \exists r.s'$. With $\mathcal{T}_\mathsf{f} \models K \sqsubseteq (\leqslant 1 \ r \ K')$ and $r(d, d_2) \in \mathcal{A}$, the semantics yields $s' \subseteq \mathsf{tp}_\mathcal{A}(d_2)$, thus $s' \subseteq \mathsf{tp}_\mathcal{I}(d_2)$.
- both (d, d_1) and (d, d_2) were added by an application of (**c2**). Then there are λ_1 and λ_2, such that, for $i \in \{1, 2\}$, we have $\lambda_i = s_i \ {}^1{\leftrightarrow}_r^1 \ s'_i$ and $(d, d_i) \in \pi_{\lambda_i}$. Applying Lemma 18 to λ_i yields $s_i = \mathsf{tp}_\mathcal{I}(d)$, for $i \in \{1, 2\}$. Consequently, $s_1 = s_2$. We next show $s'_1 = s'_2$, thus $\lambda_1 = \lambda_2$.

  For uniformity, we use s to denote s_1 and s_2. From λ_i, we obtain $\mathcal{T}_\mathsf{f} \models s \sqsubseteq \exists r.s'_i$ and s'_i is maximal with this property, for $i \in \{1, 2\}$. Lemma 18 yields $\mathsf{tp}_\mathcal{I}(d_i) = s'_i$. Using the facts that $\mathcal{T}_\mathsf{f} \models s \sqsubseteq \exists r.s'_i$ for $i \in \{1, 2\}$, $K \subseteq \mathsf{tp}_\mathcal{I}(d) = s$, $K' \subseteq \mathsf{tp}_\mathcal{I}(d_i) = s'_i$ for $i \in \{1, 2\}$, and $\mathcal{T}_\mathsf{f} \models K \sqsubseteq (\leqslant 1 \ r \ K')$, an easy semantic argument shows that $\mathcal{T}_\mathsf{f} \models s \sqsubseteq \exists r.(s'_1 \cup s'_2)$. The maximality of s'_1 and s'_2 thus implies $s'_1 = s'_2$ as desired.

  Hence, $\lambda_1 = \lambda_2$ and $(d, d_1), (d, d_2) \in \pi_{\lambda_1}$. Since π_{λ_1} is a bijection, we obtain $d_1 = d_2$, a contradiction.

Application of (c3) preserves all invariants. It is obvious that the invariants (**i1**) and (**i2**) are preserved with each single application of (**c3**). It thus remains to treat (**i3**). Assume that completion processed $d \in \Delta^\mathcal{I}$ with $\mathsf{tp}_\mathcal{I}(d) \rightarrow_r t$ and $\mathsf{tp}_\mathcal{I}(d) \not\rightarrow_r^1 t$, adding the edge (d, d_t) to $r^\mathcal{I}$. Since $\mathsf{tp}_\mathcal{I}(d_t) = t$ and $\mathsf{tp}_\mathcal{I}(d) \not\rightarrow_r^1 t$, there is no $K \sqsubseteq (\leqslant 1 \ r^- \ K') \in \mathcal{T}_\mathsf{f}$ such that $K \subseteq t$ and $K' \subseteq \mathsf{tp}_\mathcal{I}(d)$. Take a $K \sqsubseteq (\leqslant 1 \ r \ K') \in \mathcal{T}_\mathsf{f}$ with $K \subseteq \mathsf{tp}_\mathcal{I}(d)$ and $K' \subseteq t$. We have to prove that there is no $e \in \Delta^\mathcal{I}$ distinct from d_t such that $(d, e) \in r^\mathcal{I}$ and $e \in K'^\mathcal{I}$. This can be done exactly as in the case of the completion rule (**c1**).

Termination of Model Construction

We show that the constructed interpretation $\mathcal{I}$ is indeed finite.

Proposition 19 $\Delta^\mathcal{I}$ *is finite.*

Proof. To analyze the termination of the construction of $\mathcal{I}$, we associate a certain directed tree $T = (V, E)$ with the model $\mathcal{I}$ that makes more explicit the way in which $\mathcal{I}$ was constructed. Note that only the completion rules (**c1**) and (**c2**) introduce new domain elements and that (**c1**) introduces a single new element with each application whereas (**c2**) introduces a whole (finite) set of fresh elements. Also note that each application of a completion rule is triggered by a single domain element d for which some existential restriction is not yet satisfied.[3] Now, the tree T is defined as follows:

- V consists of all subsets of $\Delta^\mathcal{I}$ that were introduced together by a single application of one of the completion rules (**c1**) and (**c2**); additionally, the set of all elements in

[3] In the case of (**c2**), there are potentially many domain elements that trigger the same application. In such a case, we choose one element as the actual trigger; see formulation of (**c2**).

the initial interpretation $\mathcal{I}$ is a node in V (in fact, the root node);

- the edge (v, v') is included in E if the elements in v' were introduced by an application of a completion rule to an element d of v. We call this element the *trigger* of v' and denote it with $d_{v'}$.

To show that $\Delta^{\mathcal{I}}$ is finite, it clearly suffices to show that V is finite. The outdegree of T is finite since every rule application introduces only finitely many elements. By König's Lemma, it thus remains to show that T is of finite depth. We first note that an easy analysis of (**c1**) and (**c2**) reveals the following property:

($*$) if $(v_1, v_2), (v_2, v_3) \in E$, then there are $d_0, \ldots, d_k$ and roles $r_0, \ldots, r_{k-1}$ s.t.

- $d_0 = d_{v_2} \in v_1$ and $d_1, \ldots, d_k = d_{v_3} \in v_2$;
- $\mathsf{tp}_{\mathcal{I}}(d_i) \rightarrow^1_{r_i} \mathsf{tp}_{\mathcal{I}}(d_{i+1})$ for all $i < k$.

Now assume towards a contradiction that the depth of T is larger than $2|\mathsf{TP}(\mathcal{T}_\mathsf{f})|+1$ and choose a concrete path $v_1 \cdots v_n$ with v_1 the root of T and $n > 2|\mathsf{TP}(\mathcal{T}_\mathsf{f})| + 1$. This path gives rise to a corresponding sequence of triggers $d_{v_1}, \ldots, d_{v_n}$. Since the length of this sequence exceeds $2|\mathsf{TP}(\mathcal{T}_\mathsf{f})|$, there must be i, j with $2 \leq i < j \leq n$ and such that $\mathsf{tp}_{\mathcal{I}}(d_{v_i}) = \mathsf{tp}_{\mathcal{I}}(d_{v_j})$ and $j > i + 1$. By applying ($*$) multiple times, we obtain a sequence of domain elements $d_0, \ldots, d_k$ and roles $r_0, \ldots, r_{k-1}$ such that

1. $d_0 = d_{v_i} \in v_{i-1}$, $d_1 \in v_i$, and $d_k = d_{v_j} \in v_{j-1}$;
2. $\mathsf{tp}_{\mathcal{I}}(d_\ell) \rightarrow^1_{r_\ell} \mathsf{tp}_{\mathcal{I}}(d_{\ell+1})$ for $\ell < k$.
3. $d_0, \ldots, d_k$ contains all elements $d_{v_i}, d_{v_{i+1}} \ldots, d_{v_j}$.

Since $\mathsf{tp}_{\mathcal{I}}(d_{v_i}) = \mathsf{tp}_{\mathcal{I}}(d_{v_j})$ and by Point 2, we have that $\mathsf{tp}_{\mathcal{I}}(d_0), r_0, \ldots, r_{k-1}, \mathsf{tp}_{\mathcal{I}}(d_k)$ is a finmod cycle in $\mathcal{T}_\mathsf{f}$. Since all finmod cycles in $\mathcal{T}_\mathsf{f}$ have been reversed, we have

$$\mathsf{tp}_{\mathcal{I}}(d_0) \ {}^1{\leftrightarrow}^1_{r_0} \ \mathsf{tp}_{\mathcal{I}}(d_1) \ {}^1{\leftrightarrow}^1_{r_1} \cdots {}^1{\leftrightarrow}^1_{r_{k-1}} \ \mathsf{tp}_{\mathcal{I}}(d_k). \qquad (\dagger)$$

We prove the following claim:

Claim. If (**c1**) is triggered by $d \in \Delta^{\mathcal{I}}$ generating a new element $e \in \Delta^{\mathcal{I}}$, then there is no role r such that $\mathsf{tp}_{\mathcal{I}}(d) \ {}^1{\leftrightarrow}^1_r \ \mathsf{tp}_{\mathcal{I}}(e)$.

Since $d_{v_i} \in v_{i-1}$ and $d_1 \in v_i$, d_1 was generated by the application of a completion rule triggered by d_0. By ($\dagger$) and the claim, this completion rule must be (**c2**). By definition of (**c2**) and ($\dagger$), all elements $d_1, \ldots, d_k$ have been introduced by exactly this application of (**c2**). This leads to a contradiction: we have $d_1 \in v_i$ and $d_k \in v_{j-1}$, and since $j > i + 1$, $v_i \neq v_{j-1}$. Consequently, d_1 and d_k were introduced by different applications of completion rules. ❑

The following is the remaining ingredient to the termination proof (Claim in the proof of Proposition 19).

Lemma 20 *If* (**c1**) *is triggered by $d \in \Delta^{\mathcal{I}}$ and generates a new element $e \in \Delta^{\mathcal{I}}$, then there is no role r such that $\mathsf{tp}_{\mathcal{I}}(d) \ {}^1{\leftrightarrow}^1_r \ \mathsf{tp}_{\mathcal{I}}(e)$.*

Proof. Observe that if e is introduced by an application of ($\mathbf{c_1}$) to $d \in \Delta^{\mathcal{I}}$, then $\mathcal{T}_\mathsf{f}$ entails that $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow^1_s \mathsf{tp}_{\mathcal{I}}(e)$ for some role s. Assume towards a contradiction that there is a role r such that $\mathsf{tp}_{\mathcal{I}}(d) \ {}^1{\leftrightarrow}^1_r \ \mathsf{tp}_{\mathcal{I}}(e)$. Then, the finmod cycle $\mathsf{tp}_{\mathcal{I}}(d), s, \mathsf{tp}_{\mathcal{I}}(e), r^-, \mathsf{tp}_{\mathcal{I}}(d)$ occurs in $\mathcal{T}_\mathsf{f}$. Since every finmod cycle in $\mathcal{T}_\mathsf{f}$ is reversed, we have $\mathsf{tp}_{\mathcal{I}}(e) \rightarrow^1_{s^-} \mathsf{tp}_{\mathcal{I}}(d)$. This is in contradiction to the assumption that e was introduced by an application of ($\mathbf{c_1}$). ❑

Correctness of Model Construction

To complete the proof of the "if" direction of Theorem 3, it remains to show the following.

Proposition 21 *$\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$.*

Proof. First, we show that for every assertion $\alpha \in \mathcal{A}, \mathcal{I} \models \alpha$. This is a consequence of the definition of $\mathcal{I}$. Indeed, for every individual a, if $\alpha = A(a) \in \mathcal{A}$, then $A \in \mathsf{tp}_{\mathcal{A}}(a)$ which by the definition of $\mathcal{I}$ implies that $a \in A^{\mathcal{I}}$. Further, if $\alpha = r(a, b) \in \mathcal{A}$ then $(a, b) \in r^{\mathcal{I}}$.

Next, we show that for every axiom $K \sqsubseteq C \in \mathcal{T}_\mathsf{f}$, we have that $\mathcal{I} \models K \sqsubseteq C$. We distinguish the following cases:

- $C = A$. Let $d \in K^{\mathcal{I}}$. Then $K \subseteq \mathsf{tp}_{\mathcal{I}}(d)$ and by (**i1**) $\mathsf{tp}_{\mathcal{I}}(d) \in \mathsf{TP}(\mathcal{A}) \subseteq \mathsf{TP}(\mathcal{T}_\mathsf{f})$. Since $\mathcal{T}_\mathsf{f} \models K \sqsubseteq A$, this yields $A \in \mathsf{tp}_{\mathcal{I}}(d)$ and thus $d \in A^{\mathcal{I}}$.

- $C = \bot$. Follows from (**i1**). Indeed since for every $d \in \Delta^{\mathcal{I}}$, $\mathsf{tp}_{\mathcal{I}}(d) \in \mathsf{TP}(\mathcal{A}) \subseteq \mathsf{TP}(\mathcal{T}_\mathsf{f})$, $K^{\mathcal{I}} = \emptyset$.

- $C = \exists r.K'$. Let $d \in K^{\mathcal{I}}$. Then we have that $K \subseteq \mathsf{tp}_{\mathcal{I}}(d)$. Since $\mathcal{T}_\mathsf{f} \models K \sqsubseteq \exists r.K'$, we have that $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r t'$ for some t' with $K' \subseteq t'$. It suffices to show that there is some d' with $(d, d') \in r^{\mathcal{I}}$ and $\mathsf{tp}_{\mathcal{I}}(d') = t'$. The easiest case is that such a d' already exists in the initial $\mathcal{I}$. Assume that this is not the case. Note that one of the following cases must apply: (1) $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow^1_r t'$ and $t' \not\rightarrow^1_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$, (2) $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow^1_r t'$ and $t' \rightarrow^1_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$, and (3) $\mathsf{tp}_{\mathcal{I}}(d) \not\rightarrow^1_r t'$. These cases correspond exactly to the completion rules (**c1**) to (**c3**). Thus, one of these rules will add the required successor.

- $C = \forall r.K'$. Let $d \in K^{\mathcal{I}}$ and $(d, d') \in r^{\mathcal{I}}$, We have that $K \subseteq \mathsf{tp}_{\mathcal{I}}(d)$. Further, by (**i2**), we can distinguish the following cases:
  - $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r \mathsf{tp}_{\mathcal{I}}(d')$. Then $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.\mathsf{tp}_{\mathcal{I}}(d')$ and $\mathsf{tp}_{\mathcal{I}}(d')$ is maximal with this property. Since $\mathcal{T}_\mathsf{f} \models K \sqsubseteq \forall r.K'$, we have that $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.\mathsf{tp}_{\mathcal{I}}(d') \cup K'$, and the maximality of $\mathsf{tp}_{\mathcal{I}}(d')$ yields $K' \subseteq \mathsf{tp}_{\mathcal{I}}(d')$, and thus $d' \in K'^{\mathcal{I}}$.
  - $\mathsf{tp}_{\mathcal{I}}(d') \rightarrow_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$. Then we have $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_{\mathcal{I}}(d') \sqsubseteq \exists r^-.\mathsf{tp}_{\mathcal{I}}(d)$. Together with $\mathcal{T}_\mathsf{f} \models K \sqsubseteq \forall r.K'$, we obtain $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_{\mathcal{I}}(d') \sqsubseteq K'$. Since $\mathsf{tp}_{\mathcal{I}}(d') \in \mathsf{TP}(\mathcal{T}_\mathsf{f})$ by (**i1**), we obtain $K' \subseteq \mathsf{tp}_{\mathcal{I}}(d')$ and thus $d' \in K'^{\mathcal{I}}$.
  - $r(d, d') \in \mathcal{A}$. Then $K \subseteq \mathsf{tp}_{\mathcal{A}}(d)$ by definition of the initial $\mathcal{I}$. By the semantics, we thus have $K' \subseteq \mathsf{tp}_{\mathcal{A}}(d') = \mathsf{tp}_{\mathcal{I}}(d')$, thus $d' \in K'^{\mathcal{I}}$.

- $C = (\leqslant 1 \ r \ K)'$. Follows from (**i3**). ❑

B Proofs for Section 4

Before we can prove $\widehat{\mathcal{I}} \models \mathcal{T}_{\mathsf{f}}$, we will proceed as outlined in Section 4, showing first that $\widehat{\mathcal{I}}$ is a model of $\widehat{\mathcal{T}}$ and that it suffices to close only maximal cycles when constructing $\widehat{\mathcal{T}}$.

We start with an easy auxiliary observation.

Observation 22 *For all $d \in \Delta^{\widehat{\mathcal{I}}}$, that $\mathsf{tail}(d) \sqsubseteq \bot \notin \widehat{\mathcal{T}}$*

Proof. We proceed by induction on the length of d. If $d = K \in \Sigma$, the claim follows from the construction in the initial step. If $|d| > 1$ then d was added due to some element d' with $\mathsf{cl}_\vdash(\mathsf{tail}(d')) \sqsubseteq \exists r.\mathsf{tail}(d) \in \widehat{\mathcal{T}}$. Assume that $\mathsf{tail}(d) \sqsubseteq \bot \in \widehat{\mathcal{T}}$. Then we have $\mathsf{cl}_\vdash(\mathsf{tail}(d')) \sqsubseteq \bot \in \widehat{\mathcal{T}}$ due to **R6**. This implies $\mathsf{tail}(d') \sqsubseteq \bot \in \widehat{\mathcal{T}}$ due to **R3**, which is a contradiction to the inductive hypothesis. ❑

Lemma 10 $\widehat{\mathcal{I}} \models \widehat{\mathcal{T}}$.

Proof. Let $K \sqsubseteq C \in \widehat{\mathcal{T}}$. We distinguish the following cases.

- $C = \bot$.

  Assume towards a contradiction that there is an instance of K, then there is $d \in \Delta^{\mathcal{I}}$ with $K \subseteq \mathsf{tp}_{\mathcal{I}}(d)$. The construction of $\widehat{\mathcal{I}}$ ensures that $\mathsf{tp}_{\widehat{\mathcal{I}}}(d) = \mathsf{cl}_\vdash(\mathsf{tail}(d))$. Hence, from $K \sqsubseteq \bot \in \widehat{\mathcal{T}}$, and **R3** we obtain that $\mathsf{tail}(d) \sqsubseteq \bot \in \widehat{\mathcal{T}}$, contradicting Observation 22.

- $C = A$.

  Let $d \in K^{\widehat{\mathcal{I}}}$. Then $K \subseteq \mathsf{cl}_\vdash(\mathsf{tail}(d))$ by construction of $\widehat{\mathcal{I}}$. Together with $K \sqsubseteq A \in \widehat{\mathcal{T}}$, rules **R1** and **R3** yield $\mathsf{tail}(d) \sqsubseteq A \in \widehat{\mathcal{T}}$, hence $A \in \mathsf{cl}_\vdash(\mathsf{tail}(d))$, that is, $d \in A^{\widehat{\mathcal{I}}}$.

- $C = \exists r.K'$.

  Let $d \in K^{\widehat{\mathcal{I}}}$ and assume $\mathsf{tail}(d) = K''$. By construction of $\widehat{\mathcal{I}}$, we have $K \subseteq \mathsf{cl}_\vdash(K'')$. Further, from $K \sqsubseteq \exists r.K' \in \widehat{\mathcal{T}}$, by **R1** and **R3**, $K'' \sqsubseteq \exists r.K' \in \widehat{\mathcal{T}}$. Then, the the construction ensures that, $d' \in (\exists r.K')^{\widehat{\mathcal{I}}}$ as required.

- $C = \forall r.K'$.

  Let $d \in K^{\widehat{\mathcal{I}}}$ and $(d, d') \in r^{\widehat{\mathcal{I}}}$. Further, let $\mathsf{tail}(d) = K_1$. Since $K \sqsubseteq \forall r.K' \in \widehat{\mathcal{T}}$ and $K \subseteq \mathsf{cl}_\vdash(K_1)$, by rule **R1** and **R3**, $K_1 \sqsubseteq \forall r.K' \in \widehat{\mathcal{T}}$. We distinguish the following cases:

  (i) $d' = dK_2$ i.e., d' was added after d because of some $K_1 \sqsubseteq \exists r.K_2 \in \widehat{\mathcal{T}}$, with K_2 maximal with this property. Since $K_1 \sqsubseteq \forall r.K' \in \widehat{\mathcal{T}}$ and $K_1 \sqsubseteq \exists r.K_2 \in \widehat{\mathcal{T}}$ by **R5** we get that $K_1 \sqsubseteq \exists r.(K_2 \sqcap K') \in \widehat{\mathcal{T}}$. The maximality of K_2 implies that $K' \subseteq K_2$. By **R1**, $K_2 \sqsubseteq A \in \widehat{\mathcal{T}}$ for all $A \in K'$, and thus $K' \in \mathsf{cl}_\vdash(K_2) = \mathsf{tp}_{\widehat{\mathcal{I}}}(d')$. Thus, $d' \in (K')^{\widehat{\mathcal{I}}}$.

  (ii) $d = d'K_1$ i.e., d was added after d' because of some $K_2 \sqsubseteq \exists r^-.K_1 \in \widehat{\mathcal{T}}$ with $K_2 = \mathsf{tail}(d')$. Since $K_1 \sqsubseteq \forall r.K' \in \widehat{\mathcal{T}}$ and $K_2 \sqsubseteq \exists r^-.K_1 \in \widehat{\mathcal{T}}$ by rule **R4** we have $K_2 \sqsubseteq K' \in \widehat{\mathcal{T}}$. Thus, $K' \in \mathsf{cl}_\vdash(K_2) = \mathsf{tp}_{\widehat{\mathcal{I}}}(d')$ and $d' \in K^{\widehat{\mathcal{I}}}$ as required.

- $C = (\leqslant 1\ r\ K')$.

  Let $d \in K^{\widehat{\mathcal{I}}}$ and let $K'' = \mathsf{tail}(d)$. Assume that there are e_1, e_2 with $(d, e_i) \in r^{\widehat{\mathcal{I}}}$ and $e_i \in (K')^{\widehat{\mathcal{I}}}$ for $i = 1, 2$. We have $K \subseteq \mathsf{cl}_\vdash(K'')$, and thus, by **R1** and rule **R3**, $K'' \sqsubseteq (\leqslant 1\ r\ K') \in \widehat{\mathcal{T}}$. Then (d, e_i) is not added by step (2) in the construction of $\widehat{\mathcal{I}}$ for every i.

  Let $K_1 = \mathsf{tail}(e_1)$ and $K_2 = \mathsf{tail}(e_2)$. We then have $K' \in \mathsf{cl}_\vdash(K_1) \cap \mathsf{cl}_\vdash(K_2)$. We distinguish the following cases, according to the construction of $\widehat{\mathcal{I}}$:

  (i) both e_i are added by $K'' \sqsubseteq \exists r.K_1$ and $K'' \sqsubseteq \exists r.K_2 \in \widehat{\mathcal{T}}$ with K_i maximal with this property. That means $e_1 = dK_1$ and $e_2 = dK_2$. Since $K'' \sqsubseteq (\leqslant 1\ r\ K)' \in \widehat{\mathcal{T}}$ and $K' \in \mathsf{cl}_\vdash(K_1) \cap \mathsf{cl}_\vdash(K_2)$ by **R7** we have that $K'' \sqsubseteq \exists r.(K_1 \sqcap K_2) \in \widehat{\mathcal{T}}$. The maximality conditions on both K_i implies $K_1 \subseteq K_2$ and $K_2 \subseteq K_1$. Hence, $e_1 = e_2$, and $d \in (\leqslant 1\ r\ K')^{\widehat{\mathcal{I}}}$ as required.

  (ii) $d = e_1K''$ and $e_2 = dK_2$. That is d is added after e_1 because of some $K_1 \sqsubseteq \exists r^-.K'' \in \widehat{\mathcal{T}}$ with K'' maximal and e_2 is added after d because of some $K'' \sqsubseteq \exists r.K_2 \in \widehat{\mathcal{T}}$ with K_2 maximal. Since $K' \subseteq \mathsf{cl}_\vdash(K_1) \cap \mathsf{cl}_\vdash(K_2)$ and $K'' \sqsubseteq (\leqslant 1\ r\ K') \in \widehat{\mathcal{T}}$, by rule **R8** we have that $K_1 \sqsubseteq K_2 \in \widehat{\mathcal{T}}$. Hence for every $K_2 \sqsubseteq A \in \widehat{\mathcal{T}}$ we have that $K_1 \sqsubseteq A \in \widehat{\mathcal{T}}$ by rule **R3**. This means that $\mathsf{cl}_\vdash(K_2) \subseteq \mathsf{cl}_\vdash(K_1)$. The construction ensures that no e_2 is added as r-successor of d. Thus, $d \in (\leqslant 1\ r\ K')^{\widehat{\mathcal{I}}}$. ❑

Lemma 11. $\mathcal{T}_{\mathsf{f}}$ is equivalent to $\mathcal{T}_{\mathsf{f}}^{\max}$.

Proof. It suffices to show that, for every cycle C in a TBox $\mathcal{S}$, there is a maximal cycle $\widehat{C}$ in $\mathcal{S}$ whose reversal implies the reversal of C. More precisely, let $C = K_1, r_1, K_2, \ldots, K_n$ be a cycle in $\mathcal{S}$. We show that there is a maximal cycle $\widehat{C} = \widehat{K}_1, r_1, \widehat{K}_2, \ldots, \widehat{K}_n$ whose reversal – that is, adding the axioms $\widehat{K}_{j+1} \sqsubseteq \exists r_j^-.\widehat{K}_j$ and $\widehat{K}_j \sqsubseteq (\leqslant 1\ r_j\ \widehat{K}_{j+1})$ to $\mathcal{S}$– will lead to $\mathcal{S}$ implying the reversal of C. We proceed in three steps.

First, we construct $\widehat{C} = \widehat{K}_1, r_1, \widehat{K}_2, \ldots, \widehat{K}_n$ iteratively as follows. Initially, set $\widehat{K}_j = K_j$ for every $j = 1, \ldots, n$. Then exhaustively apply the following step.

> While there is some $\widehat{L}_{j+1} \supsetneq \widehat{K}_{j+1}$ maximal with $\mathcal{S} \models \widehat{K}_j \sqsubseteq \exists r_j.\widehat{L}_{j+1}$ for some $j = 1, \ldots, n-1$, set $\widehat{K}_{j+1} = \widehat{L}_{j+1}$.

The iteration terminates because the supply of conjunctions is bounded and C's length is fixed.

Second, we verify that $\widehat{C}$ is indeed a cycle. It suffices to show that one application of the construction step does not destroy the cycle property, i.e., by replacing $\widehat{K}_{j+1}$ with the larger $\widehat{L}_{j+1}$, the four subsumptions involving $\widehat{K}_{j+1}$ now hold for $\widehat{L}_{j+1}$:

- $\mathcal{S} \models \widehat{K}_j \sqsubseteq \exists r_j.\widehat{L}_{j+1}$ holds due to the step's precondition.

- $\mathcal{S} \models \widehat{L}_{j+1} \sqsubseteq \exists r_j.\widehat{K}_{j+2}$ holds because $\mathcal{S} \models \widehat{L}_{j+1} \sqsubseteq \widehat{K}_{j+1} \sqsubseteq \exists r_j.\widehat{K}_{j+2}$.
- $\mathcal{S} \models \widehat{L}_{j+1} \sqsubseteq (\leqslant 1\ r_j^-\ \widehat{K}_j)$ holds because $\mathcal{S} \models \widehat{L}_{j+1} \sqsubseteq \widehat{K}_{j+1} \sqsubseteq (\leqslant 1\ r_j^-\ \widehat{K}_j)$.
- $\mathcal{S} \models \widehat{K}_{j+2} \sqsubseteq (\leqslant 1\ r_{j+1}^-\ \widehat{L}_{j+1})$ holds because $\mathcal{S} \models \widehat{K}_{j+2} \sqsubseteq (\leqslant 1\ r_{j+1}^-\ \widehat{K}_{j+1})$ and $\mathcal{S} \models \widehat{L}_{j+1} \sqsubseteq \widehat{K}_{j+1}$.

Third, we show that the reversal of $\widehat{C}$ implies the reversal of C. Again, it suffices to show that this is the case when $\widehat{C}$ is obtained from C applying one single construction step. Let $\mathcal{S}^+$ be the TBox obtained from $\mathcal{S}$ after reversing $\widehat{C}$, that is, $\mathcal{S}^+$ equals $\mathcal{S}$ plus the following $2j$ axioms.

$$\begin{array}{lll} \ldots\ \widehat{L}_{j+1} \sqsubseteq \exists r_j^-.\widehat{K}_j & \widehat{K}_{j+2} \sqsubseteq \exists r_{j+1}^-.\widehat{L}_{j+1} \quad (*) & \ldots \\ \ldots\quad \widehat{K}_j \sqsubseteq (\leqslant 1\ r_j\ \widehat{L}_{j+1}) & \widehat{L}_{j+1} \sqsubseteq (\leqslant 1\ r_{j+1}\ \widehat{K}_{j+2}) & \ldots \end{array}$$

To prove that all $2j$ axioms that would be added by reversing C are implied by $\mathcal{S}^+$, it suffices to show that $\mathcal{S}^+ \models \widehat{K}_{j+1} \sqsubseteq \widehat{L}_{j+1}$ (which implies $\mathcal{S}^+ \models \widehat{K}_{j+1} \equiv \widehat{L}_{j+1}$). Consider an arbitrary model $\widehat{\mathcal{I}} \models \mathcal{S}^+$ and an instance d of $\widehat{K}_{j+1}$ in $\widehat{\mathcal{I}}$. Since $\widehat{C}$ is a cycle, there is some e with $(d, e) \in r_{j+1}^{\mathcal{I}}$ and $e \in \widehat{K}_{j+2}^{\mathcal{I}}$. Then, due to the above axiom $(*)$ in $\mathcal{S}^+$, there is some d' with $(d', e) \in r_{j+1}^{\mathcal{I}}$ and $d' \in \widehat{L}_{j+1}^{\mathcal{I}}$. Now, since C is a cycle in $\mathcal{S}$ – i.e., $\widehat{K}_{j+2} \sqsubseteq (\leqslant 1\ r_{j+1}^-\ \widehat{K}_{j+1}) \in \mathcal{S}$ – and because $\widehat{L}_{j+2} \sqsupseteq \widehat{K}_{j+2}$, we obtain that $d' = d$. Hence d is an instance of $\widehat{L}_{j+2}$. ❏

When proving $\widehat{\mathcal{I}} \models \mathcal{T}_{\mathsf{f}}^{\max}$, we have to deal with the complication that $\mathcal{T}_{\mathsf{f}}^{\max} \subseteq \widehat{\mathcal{T}}$ need not hold. As illustrated by Example 7, this is a actually a main feature of our calculus because we are avoiding to introduce conjunctions K that are 'irrelevant' for the reasoning task at hand.

We address this issue by showing that the *relevant consequences* of all concept inclusions in $\mathcal{T}_{\mathsf{f}}^{\max} \setminus \widehat{\mathcal{T}}$ are reflected in $\widehat{\mathcal{T}}$, even if the inclusions themselves are missing. To make this more precise, note that $\mathcal{T}_{\mathsf{f}}^{\max} \setminus \widehat{\mathcal{T}}$ only contains CIs of the form

(i) $K \sqsubseteq \exists r.K'$ and

(ii) $K \sqsubseteq (\leqslant 1\ r\ K')$.

For CIs of the form (i), we show that there is some conjunction $\widehat{K}'$ that satisfies $K' \subseteq \mathsf{cl}_\vdash(\widehat{K}')$ and intuitively replaces K' (since K' itself might be an irrelevant conjunction) such that for all relevant conjunctions $\widehat{K}$ (those that occur in $\widehat{\mathcal{T}}$) with $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$ (note that K itself might also be irrelevant), the inclusion $\widehat{K} \sqsubseteq \exists r.\widehat{K}'$ is contained in $\widehat{\mathcal{T}}$.

For CIs of the form (ii), we show analogously that there is a replacement $\widehat{K}'$ of K' such that for all relevant conjunctions $\widehat{K}$ with $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$, $\widehat{\mathcal{T}}$ contains $\widehat{K} \sqsubseteq (\leqslant 1\ r\ \widehat{K}')$. However, in this case the fact that $\widehat{K}'$ is a replacement of K' has to be formalized even a bit more carefully. Instead of requiring that $K' \subseteq \mathsf{cl}_\vdash(\widehat{K}')$, we need that $K' \subseteq \mathsf{cl}_\vdash(\widetilde{K}')$ implies $\widehat{K}' \subseteq \mathsf{cl}_\vdash(\widetilde{K}')$ for all relevant $\widetilde{K}'$.

What we have just discussed is Lemma 24 below. In order to show that the above concept inclusions $\widehat{K} \sqsubseteq \exists r.\widehat{K}'$ and $\widehat{K} \sqsubseteq (\leqslant 1\ r\ \widehat{K}')$ all are in $\widehat{\mathcal{T}}$, we consider the sequence of TBoxes $\mathcal{T}_{\mathsf{f}}^0, \mathcal{T}_{\mathsf{f}}^1, \ldots$ that are obtained by repeatedly reversing maximal cycles and whose limit is $\mathcal{T}_{\mathsf{f}}$. Note that $\mathcal{T}_{\mathsf{f}}^{i+1}$ is produced from $\mathcal{T}_{\mathsf{f}}^i$ by reversing a cycle, and that cycles are defined in terms of *semantic entailment* of CIs of the form (i) and (ii) by $\mathcal{T}_{\mathsf{f}}^i$, rather than *syntactic containment*. We first establish an auxiliary lemma that helps in bridging this gap. We formulate it here for arbitrary TBoxes $\mathcal{S}$, but will later apply it to ($\widehat{\mathcal{T}}$ and) the TBoxes $\mathcal{T}_{\mathsf{f}}^i$.

Every TBox $\mathcal{S}$ and every conjunction K gives rise to a TBox $(\mathcal{S})_K$ as follows:

1. for all CIs $L \sqsubseteq C \in \mathcal{S}$ with $\mathcal{S} \models K \sqsubseteq L$ and C of one of the forms $\exists r.L'$, $\forall r.L'$, and $(\leqslant 1\ r\ L')$, include $K \sqsubseteq C$;
2. then exhaustively apply rules **R5** and **R7′**, where **R7′** is obtained from **R7** by replacing $\mathsf{cl}_\vdash$ with entailment in $\mathcal{S}$:

$$\textbf{R7}'\quad \frac{\begin{array}{ll} K \sqsubseteq \exists r.K_1 & K \sqsubseteq \exists r.K_2 \\ K \sqsubseteq (\leqslant 1\ r\ K'') & \mathcal{S} \models K_i \sqsubseteq K'',\ i = 1, 2 \end{array}}{K \sqsubseteq \exists r.(K_1 \sqcap K_2)}$$

It is easy to see that $\mathcal{S} \models (\mathcal{S})_K$. Note that Step 1 above addresses the fact that K need not occur syntactically in $\mathcal{S}$.

The proof of the following lemma uses $(\mathcal{S})_K$ to introduce two variants of the canonical model for $\mathcal{S}$ and to extract the required witnesses for entailments.

Lemma 23 *Let $\mathcal{S}$ be a Horn-$\mathcal{ALCFI}$ TBox. Then the following hold.*

1. *If $\mathcal{S} \models K \sqsubseteq \exists r.K'$ and K is satisfiable w.r.t. $\mathcal{S}$ then there is some conjunction L' with*
   - *(a) $\mathcal{S} \models L' \sqsubseteq K'$ and*
   - *(b) $(\mathcal{S})_K \ni K \sqsubseteq \exists r.L'$.*
2. *If $\mathcal{S} \models K \sqsubseteq (\leqslant 1\ r\ K')$ and $\mathcal{S} \models K' \sqsubseteq \exists r^-.K$ such that K is maximal with this property, then there are L, L' with*
   - *(a) $\mathcal{S} \models K \sqsubseteq L$, and*
   - *(b) $\mathcal{S} \models K' \sqsubseteq L'$, and*
   - *(c) $\mathcal{S} \ni L \sqsubseteq (\leqslant 1\ r\ L')$.*

Proof. We begin by constructing a variant of the canonical model for $\mathcal{S}$ that will be used in the proofs of both points of the lemma. Let K be a conjunction satisfiable w.r.t. $\mathcal{S}$. The interpretation $\mathcal{I}_K$ is defined as follows. The domain $\Delta^{\mathcal{I}_K}$ consists of words over the alphabet built up of all conjunctions of concept names that occur in $\mathcal{S}$. Initially, $\Delta^{\mathcal{I}_K}$ is the singleton set $\{d_0\}$ for $d_0 = K$, and the concept and role names are interpreted such that

$$\begin{aligned} \mathsf{tp}_{\mathcal{I}_K}(d_0) &= \{A \mid \mathcal{S} \models K \sqsubseteq A\} \\ r^{\mathcal{I}_K} &= \emptyset \end{aligned}$$

Then we add the required successors to the root node d_0. For every $K \sqsubseteq \exists r.L' \in (\mathcal{S})_K$ such that L' is maximal with this property,

- add a fresh element $e = KL'$ to $\Delta^{\mathcal{I}_K}$;
- add pair (d_0, e) to $r^{\mathcal{I}_K}$;
- interpret concept names such that $\mathsf{tp}_{\mathcal{I}_K}(e) = \{A \mid \mathcal{S} \models L' \sqsubseteq A\}$.

Finally, we exhaustively generate required successors of non-root elements. For every $d = wL \in \Delta^{\mathcal{I}_K}$ with $d \neq d_0$, and every inclusion $L \sqsubseteq \exists r.L'$ such that $\mathcal{S} \models L \sqsubseteq \exists r.L'$, L' is maximal with this property, and $d \notin (\exists r.L')^{\widehat{\mathcal{I}}}$,

- add a fresh element $e = wLL'$ to $\Delta^{\mathcal{I}_K}$;
- add pair (d, e) to $r^{\mathcal{I}_K}$;
- interpret concept names such that $\mathsf{tp}_{\mathcal{I}_K}(e) = \{A \mid \mathcal{S} \models L' \sqsubseteq A\}$.

Note the difference between the treatment of the root node d_0 and all other nodes: for d_0, we consider inclusions $K \sqsubseteq \exists r.L'$ that are syntactically contained in $(\mathcal{S})_K$; for all other nodes, we consider inclusions that semantically follow from $\mathcal{S}$ (equivalently: from $(\mathcal{S})_K$).

Claim. $\mathcal{I}_K \models \mathcal{S}$.

Proof of Claim. Let $L \sqsubseteq C \in \mathcal{S}$. We distinguish the following cases.

- $C = \bot$. Assume that L has an instance $d = w\widehat{L}$ in $\mathcal{I}_K$. Then $\mathcal{S} \models \widehat{L} \sqsubseteq L$ due to the construction of $\mathcal{I}_K$; hence $\mathcal{S} \models \widehat{L} \sqsubseteq \bot$. Observation 22, applied to $\Delta^{\mathcal{I}_K}$ yields that $\mathcal{S} \models K \sqsubseteq \bot$, contradicting the above choice of K.

- $C = A$. Let $d \in L^{\mathcal{I}_K}$ with $d = w\widehat{L}$. Then $\mathcal{S} \models \widehat{L} \sqsubseteq L$ by construction of $\mathcal{I}_K$. Since $L \sqsubseteq A \in \mathcal{S}$, we obtain $\mathcal{S} \models \widehat{L} \sqsubseteq A$; hence $d \in A^{\mathcal{I}_K}$.

- $C = \exists r.L'$. Let $d \in L^{\mathcal{I}_K}$.

  In case $d = d_0 = K$, we have that $\mathcal{S} \models K \sqsubseteq L$. Together with $L \sqsubseteq \exists r.L' \in \mathcal{S}$, this implies that $K \sqsubseteq \exists r.L' \in (\mathcal{S})_K$ by Step 1 of the construction of $(\mathcal{S})_K$. Let K' be maximal with $K \sqsubseteq \exists r.K' \in (\mathcal{S})_K$ and $L' \subseteq K'$. In the construction of $\mathcal{I}_K$, we thus create an r-successor e of d with $\mathsf{tp}_{\mathcal{I}_K}(e) \supseteq L'$. Hence $d \in (\exists r.L')^{\mathcal{I}_K}$.

  In case $d \neq d_0$, let $d = wK'$. Then $\mathcal{S} \models K' \sqsubseteq L$. Together with $L \sqsubseteq \exists r.L' \in \mathcal{S}$, this implies that $\mathcal{S} \models K' \sqsubseteq \exists r.L'$. Then the construction of $\mathcal{I}_K$ ensures that there is an r-successor e of d with $\mathsf{tp}_{\mathcal{I}_K}(e) \supseteq L'$. Hence $d \in (\exists r.L')^{\mathcal{I}_K}$.

- $C = \forall r.L'$. Let $d \in L^{\mathcal{I}_K}$ and $(d, e) \in r^{\mathcal{I}_K}$.

  In case $d = d_0 = K$ and $e = KK'$, we have that $\mathcal{S} \models K \sqsubseteq L$; hence $K \sqsubseteq \forall r.L' \in (\mathcal{S})_K$ as above. Now e was added for some $K \sqsubseteq \exists r.K' \in (\mathcal{S})_K$ with K' maximal. Since $(\mathcal{S})_K$ is closed under application of **R5**, we have that $K \sqsubseteq \exists r.(K' \sqcap L') \in (\mathcal{S})_K$. Maximality of K' implies that $L' \subseteq K'$. The construction of $\mathcal{I}_K$ then implies that $L' \subseteq \mathsf{tp}_{\mathcal{I}_K}(e)$; i.e., $e \in (L')^{\mathcal{I}_K}$.

  In case $e = d_0 = K$ and $d = KK'$, we have that $\mathcal{S} \models K' \sqsubseteq L$; hence $\mathcal{S} \models K' \sqsubseteq \forall r.L'$ (x). Now d was added for some $K \sqsubseteq \exists r^-.K' \in (\mathcal{S})_K$ with K'. Since $\mathcal{S} \models (\mathcal{S})_K$, we obtain $\mathcal{S} \models K \sqsubseteq \exists r^-.K'$. Together with (x), a simple semantic argument implies $\mathcal{S} \models K \sqsubseteq L'$; hence, $e \in (L')^{\mathcal{I}_K}$.

  In case $d = wK_1 \neq d_0$ and $e = wK_1K_2$, we have that $\mathcal{S} \models K_1 \sqsubseteq L$; hence $\mathcal{S} \models K_1 \sqsubseteq \forall r.L'$ as above. Now e was added because $\mathcal{S} \models K_1 \sqsubseteq \exists r.K_2$ with K_2 maximal. A simple semantic argument implies $\mathcal{S} \models K_1 \sqsubseteq \exists r.(K_2 \sqcap L')$, and maximality of K_2 again yields $L' \subseteq K_2$; i.e., $e \in (L')^{\mathcal{I}_K}$.

  In case $e = wK_1 \neq d_0$ and $d = wK_1K_2$, we have that $\mathcal{S} \models K_2 \sqsubseteq L$; hence $\mathcal{S} \models K_2 \sqsubseteq \forall r.L'$. Now d was added because $\mathcal{S} \models K_1 \sqsubseteq \exists r^-.K_2$. A simple semantic argument implies $\mathcal{S} \models K_1 \sqsubseteq L'$, i.e., $e \in (L')^{\mathcal{I}_K}$.

- $C = (\leqslant 1\ r\ L')$. Let $d \in L^{\mathcal{I}_K}$, and let $(d, e_i) \in r^{\mathcal{I}_K}$ and $e_i \in (L')^{\mathcal{I}_K}$ for $i = 1, 2$.

  In case $d = d_0 = K$ and $e_i = KK_i$, we have that (i) $\mathcal{S} \models K \sqsubseteq L$ and (ii) $\mathcal{S} \models K_i \sqsubseteq L'$ for $i = 1, 2$. By construction of $(\mathcal{S})_K$, (i) and the assumption imply (iii) $K \sqsubseteq (\leqslant 1\ r\ L') \in (\mathcal{S})_K$. Now each e_i was added for some (iv) $K \sqsubseteq \exists r.K_i \in (\mathcal{S})_K$ with K_i maximal. Applying **R7′** to (iv), (iii), (ii) yields $K \sqsubseteq \exists r.(K_1 \sqcap K_2) \in (\mathcal{S})_K$. Maximality of the K_i implies that $K_1 = K_2$; hence $e_1 = e_2$.

  In case $e_1 = d_0 = K$, $d = KK_1$, and $e_2 = KK_1K_2$, we have that (i) $\mathcal{S} \models K_1 \sqsubseteq L$ plus (ii) $\mathcal{S} \models K \sqsubseteq L'$ and (iii) $\mathcal{S} \models K_2 \sqsubseteq L'$. By construction of $(\mathcal{S})_K$, (i) and the assumption imply (iv) $\mathcal{S} \models K \sqsubseteq (\leqslant 1\ r\ L')$. Now d was added for some (v) $K \sqsubseteq \exists r^-.K_1 \in (\mathcal{S})_K$, and e_2 was added for some (vi) $K_1 \sqsubseteq \exists r.K_2 \in (\mathcal{S})_K$. A simple semantic argument applied to (v), (vi), (iv), (ii) and (iii) yields $\mathcal{S} \models K \sqsubseteq K_2$. This contradicts the assumption that e_2 was added for (vi).

  In case $d = wK' \neq d_0$ and $e_i = wK'K_i$, we argue as in the first case, but purely on a semantic basis, i.e., referring to entailment by $\mathcal{S}$ instead of containment in $(\mathcal{S})_K$.

  In case $e_1 wK' \neq d_0$, $d = wK'K_1$, and $e_2 = wK'K_1K_2$, we argue "semantically" as in the second case.

◆

To prove claim (1), assume $\mathcal{S} \models K \sqsubseteq \exists r.K'$ with K satisfiable w.r.t. $\mathcal{S}$. Since $\mathcal{I}_K \models \mathcal{S}$ and due to Step 1 of the construction of $\mathcal{I}_K$, there is some L' with $K \sqsubseteq \exists r.L' \in (\mathcal{S})_K$ and $\mathcal{S} \models L' \sqsubseteq K'$.

To prove claim (2), we construct an interpretation $\mathcal{J}$ from the models $\mathcal{I}_K$ and $\mathcal{I}_{K'}$ as follows. Start with two copies of $\mathcal{I}_{K'}$ and one of $\mathcal{I}_K$, pairwise disjoint. Since $\mathcal{S} \models K' \sqsubseteq \exists r^-.K$ with K maximal, the root d_i of each of the two copies of $\mathcal{I}_{K'}$ has an r^--successor e_i of type K. Delete the subtrees starting at e_i and replace the r-edges (e_i, d_i) with (d_0, d_i), where d_0 is the root of the copy of $\mathcal{I}_K$.

Now the proof of the previous claim that $\mathcal{I}_K$ and $\mathcal{I}_{K'}$ are models of $\mathcal{S}$ can be easily refined to yield that

- all axioms $L \sqsubseteq C \in \mathcal{S}$ where C is of the form $A, \bot, \exists s.L', \forall s.L'$ are satisfied by $\mathcal{J}$;

- if an axiom $L \sqsubseteq (\leqslant 1\ s\ L') \in \mathcal{S}$ is violated by $\mathcal{J}$, then it is violated by the root of the copy of $\mathcal{I}_K$, i.e., $s = r$ and $d_0 \in L^{\mathcal{J}}$ but $d_0 \notin (\leqslant 1\ r\ L')^{\mathcal{J}}$.

Since $\mathcal{S} \models K \sqsubseteq (\leqslant 1\ r\ K')$ but obviously $\mathcal{J} \not\models K \sqsubseteq (\leqslant 1\ r\ K')$, we have that $\mathcal{J} \not\models \mathcal{S}$. Consequently, there is an axiom $L \sqsubseteq (\leqslant 1\ r\ L') \in \mathcal{S}$ which is violated by d_0; that is, $d_0 \in L^{\mathcal{J}} \setminus (\leqslant 1\ r\ L')^{\mathcal{J}}$. This has two consequences, which prove (b) and (c): first, by construction of the $\mathcal{I}_{K'}$, we get $\mathcal{S} \models K' \sqsubseteq L'$, which is (b). Second, with $L \sqsubseteq (\leqslant 1\ r\ L') \in \mathcal{S}$ and (b), we obtain $\mathcal{S} \models L \sqsubseteq (\leqslant 1\ r\ K')$. Since K is maximal with this property, we have $L \subseteq K$, which implies (c). ❑

A conjunction K of concept names is *active in* $\widehat{\mathcal{T}}$ if there is a $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_{\vdash}(\widehat{K})$. Point (1) of the following lemma implies the required statement $\widehat{\mathcal{I}} \models \mathcal{T}_{\mathsf{f}}^{\max}$.

Lemma 24 *For every $i \geq 0$, the following hold.*

1. $\widehat{\mathcal{I}} \models \mathcal{T}_{\mathsf{f}}^{i}$
2. *If $\mathcal{T}_{\mathsf{f}}^{i} \models K \sqsubseteq \exists r.K'$ and K is active in $\widehat{\mathcal{T}}$, then there is a $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that*
   (a) $K' \subseteq \mathsf{cl}_{\vdash}(\widehat{K}')$;
   (b) *for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_{\vdash}(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r.\widehat{K}'$.*
3. *If $\mathcal{T}_{\mathsf{f}}^{i} \models K \sqsubseteq (\leqslant 1\ r\ K')$, $\mathcal{T}_{\mathsf{f}}^{i} \models K' \sqsubseteq \exists r^-.K$, K is maximal with this property, and K' is active in $\widehat{\mathcal{T}}$, then there is a $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that*
   (a) *for all $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $K' \subseteq \mathsf{cl}_{\vdash}(\widetilde{K}')$, then $\widehat{K}' \subseteq \mathsf{cl}_{\vdash}(\widetilde{K}')$.*
   (b) *for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_{\vdash}(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ \widehat{K}')$.*

Proof. We simultaneously prove 1–3 by induction on i:

Point 1 of the induction start follows from $\widehat{\mathcal{I}} \models \widehat{\mathcal{T}}$ (Lemma 10) and $\widehat{\mathcal{T}} \supseteq \mathcal{T} = \mathcal{T}_{\mathsf{f}}^{0}$.

For Point 2 of the induction start, assume that $\mathcal{T}_{\mathsf{f}}^{0} = \mathcal{T} \models K \sqsubseteq \exists r.K'$ with K active in $\widehat{\mathcal{T}}$. Consider the TBox $(\widehat{\mathcal{T}})_K$. By Lemma 23 (1), there is some $\widehat{K}'$ such that

(a′) $\widehat{\mathcal{T}} \models \widehat{K}' \sqsubseteq K'$ and

(b′) $(\widehat{\mathcal{T}})_K \ni K \sqsubseteq \exists r.\widehat{K}'$.

From (b′), we conclude that there is some L with

(a″) $\widehat{\mathcal{T}} \models K \sqsubseteq L$ and

(b″) $\widehat{\mathcal{T}} \ni L \sqsubseteq \exists r.\widehat{K}'$.

This is due to a straightforward induction on the construction of $(\widehat{\mathcal{T}})_K$. Hence, $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$, and our claim (a) follows from (a′) by inspection of $\widehat{\mathcal{I}}$: the root element of $\widehat{\mathcal{I}}$ created for $\widehat{K}'$ has to be an instance of K' because of (a′) and $\widehat{\mathcal{I}} \models \widehat{\mathcal{T}}$ (Lemma 10); hence, the construction of $\widehat{\mathcal{I}}$ yields $K' \subseteq \mathsf{cl}_{\vdash}(\widehat{K}')$. For claim (b), take some $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_{\vdash}(\widehat{K})$. From the latter and (a″), we obtain $L \subseteq \mathsf{cl}_{\vdash}(\widehat{K})$ by inspecting $\widehat{\mathcal{I}}$ again: from the latter and the construction of $\widehat{\mathcal{I}}$, we conclude that the root element $\widehat{K}$ in $\widehat{\mathcal{I}}$ is an instance of K; due to (a″) and because $\mathcal{I} \models \widehat{\mathcal{T}}$ (Lemma 10), that element is also an instance of L; by construction of $\widehat{\mathcal{I}}$, we get $L \subseteq \mathsf{cl}_{\vdash}(\widehat{K})$. We can thus apply **R3** to (b′) to conclude $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r.\widehat{K}'$ as required.

For Point 3 of the induction start, assume that $\mathcal{T}_{\mathsf{f}}^{0} = \mathcal{T} \models K \sqsubseteq (\leqslant 1\ r\ K')$, $\mathcal{T} \models K' \sqsubseteq \exists r^-.K$, K is maximal with this property, and K' is active in $\widehat{\mathcal{T}}$. We have to show that there is a $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that

(a) for all $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $K' \subseteq \mathsf{cl}_{\vdash}(\widetilde{K}')$, then $\widehat{K}' \subseteq \mathsf{cl}_{\vdash}(\widetilde{K}')$.

(b) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_{\vdash}(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ \widehat{K}')$.

By Lemma 23 (2), there is some $L \sqsubseteq (\leqslant 1\ r\ L') \in \mathcal{T}$ with

(i) $\mathcal{T} \models \widehat{K} \sqsubseteq L$ and

(ii) $\mathcal{T} \models \widehat{K}' \sqsubseteq L'$.

It remains to show that L' is the required conjunction. For (a), take some $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ and assume that $K' \subseteq \mathsf{cl}_{\vdash}(\widetilde{K}')$. Then (ii) together with $\widehat{\mathcal{I}} \models \widehat{\mathcal{T}}$ (Lemma 10) and $\widehat{\mathcal{T}} \supseteq \mathcal{T}$ implies that $L' \subseteq \mathsf{cl}_{\vdash}(\widetilde{K}')$.

For (b), take some $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_{\vdash}(\widehat{K})$. Then (i) together with $\widehat{\mathcal{I}} \models \widehat{\mathcal{T}} \supseteq \mathcal{T}$ implies that $L \subseteq \mathsf{cl}_{\vdash}(\widehat{K})$. Via Rule **R3**, we obtain from $L \sqsubseteq (\leqslant 1\ r\ L') \in \mathcal{T} \subseteq \widehat{\mathcal{T}}$ that $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ \widehat{L}')$.

Now for the induction step. We start with proving the following:

2′. If $\mathcal{T}_{\mathsf{f}}^{i} \ni K \sqsubseteq \exists r.K'$ with K being active in $\widehat{\mathcal{T}}$, then there is a $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that
   (a) $K' \subseteq \mathsf{cl}_{\vdash}(\widehat{K}')$;
   (b) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_{\vdash}(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r.\widehat{K}'$.

3′. If $\mathcal{T}_{\mathsf{f}}^{i} \ni K \sqsubseteq (\leqslant 1\ r\ K')$ with K being active in $\widehat{\mathcal{T}}$, then there is a $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that
   (a) for all $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $K' \subseteq \mathsf{cl}_{\vdash}(\widetilde{K}')$, then $\widehat{K}' \subseteq \mathsf{cl}_{\vdash}(\widetilde{K}')$.
   (b) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_{\vdash}(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ \widehat{K}')$.

We prove both points simultaneously. If any of the CIs $K \sqsubseteq \exists r.K'$ and $K \sqsubseteq (\leqslant 1\ r\ K')$ is in $\mathcal{T}_{\mathsf{f}}^{i-1}$, then we can use the induction hypothesis for it. Otherwise, the respective CI has been introduced by closing a cycle $K_1, r_1, K_2, \ldots, r_{n-1}, K_n$ in $\mathcal{T}_{\mathsf{f}}^{i-1}$ with $K = K_j$ for some $j \in \{1, \ldots, n-1\}$, and thus K_j is active in $\widehat{\mathcal{T}}$. Take some $\widetilde{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K_j \subseteq \mathsf{cl}_{\vdash}(\widetilde{K})$. Applying Point 2 of the induction hypothesis to $\mathcal{T}_{\mathsf{f}}^{i-1} \models K_j \sqsubseteq \exists r_j.K_{j+1}$, we find a $\widehat{K}_{j+1} \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that

(i) $K_{j+1} \subseteq \mathsf{cl}_\vdash(\widehat{K}_{j+1})$, and

(ii) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K_j \subseteq \mathsf{cl}_\vdash(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r.\widehat{K}_{j+1}$.

By Point (i), K_{j+1} is active in $\widehat{\mathcal{T}}$ and thus we can iterate the argument to find

$$\widehat{K}_{j+2}, \ldots, \widehat{K}_n = \widehat{K}_1, \widehat{K}_2, \ldots, \widehat{K}_j$$

with the following properties, for $1 \leq j < n$.

(iii) $K_j \subseteq \mathsf{cl}_\vdash(\widehat{K}_j)$ and $\widehat{K}_j \in \mathsf{KON}(\widehat{\mathcal{T}})$;

(iv) $\widehat{\mathcal{T}} \ni \widehat{K}_j \sqsubseteq \exists r_j.\widehat{K}_{j+1}$.

Let $1 < j \leq n$. Applying Point 3 of the induction hypothesis to $\mathcal{T}_\mathsf{f}^{i-1} \models K_{j+1} \sqsubseteq (\leqslant 1\ r_j^-\ K_j)$, we find a $\widehat{K}'_j \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that

(v) for all $\widetilde{K}_j \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $K_j \subseteq \mathsf{cl}_\vdash(\widetilde{K}_j)$, then $\widehat{K}'_j \subseteq \mathsf{cl}_\vdash(\widetilde{K}_j)$;

(vi) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K_{j+1} \subseteq \mathsf{cl}_\vdash(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r_j^-\ \widehat{K}'_j)$.

From (vi) and (i), in particular we obtain:

(vii) $\widehat{\mathcal{T}} \ni \widehat{K}_{j+1} \sqsubseteq (\leqslant 1\ r_j^-\ \widehat{K}'_j)$.

From (iii) and (v), we obtain:

(viii) $\widehat{K}'_j \subseteq \mathsf{cl}_\vdash(\widehat{K}_j)$.

We can now apply the cycle rule **R9** to the CIs in (iv) and (vii), obtaining

(ix) $\widehat{\mathcal{T}} \ni \widehat{K}_{j+1} \sqsubseteq \exists r_j^-.\widehat{K}_j$

(x) $\widehat{\mathcal{T}} \ni \widehat{K}_j \sqsubseteq (\leqslant 1\ r_j\ \widehat{K}'_{j+1})$

To establish both Points $2'$ and $3'$, it suffices to show:

(xi) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K_{j+1} \subseteq \mathsf{cl}_\vdash(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r_j^-.\widehat{K}_j$.

(xii) for all $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K_j \subseteq \mathsf{cl}_\vdash(\widehat{K}')$, we have $\widehat{\mathcal{T}} \ni \widehat{K}' \sqsubseteq (\leqslant 1\ r_j\ \widehat{K}'_{j+1})$.

Before we show (xi) and (xii), we make the following observation for every $j \geq 1$. Let $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K_j \subseteq \mathsf{cl}_\vdash(\widehat{K})$. From (ii), we get that

(xiii) $\widehat{K} \sqsubseteq \exists r_j.\widehat{K}_{j+1} \in \widehat{\mathcal{T}}$.

From (viii) we have that $\widehat{K}'_j \subseteq \mathsf{cl}_\vdash(\widehat{K}_j)$. Similarly, from (v) and since $K_j \subseteq \mathsf{cl}_\vdash(\widehat{K})$, we get that $\widehat{K}'_j \subseteq \mathsf{cl}_\vdash(\widehat{K})$. Thus we obtain:

(xiv) $\widehat{K}'_j \in \mathsf{cl}_\vdash(\widehat{K}_j) \cap \mathsf{cl}_\vdash(\widehat{K})$.

From (xiii), (ix), (xiv), (vi), and since $\widehat{\mathcal{T}}$ is closed under **R8** it follows that $\widehat{K} \sqsubseteq A \in \widehat{\mathcal{T}}$ for every $A \in \widehat{K}_i$.

For showing (xi), take a $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K_{j+1} \subseteq \mathsf{cl}_\vdash(\widehat{K})$. Using the argument above we get $\widehat{K} \sqsubseteq A \in \widehat{\mathcal{T}}$ for every $A \in \widehat{K}_{j+1}$. Hence, **R3** and (ix) give us that $\widehat{K} \sqsubseteq \exists.r_i^-.\widehat{K}_i \in \widehat{\mathcal{T}}$. Furthermore, for showing (xii), take $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K_j \subseteq \mathsf{cl}_\vdash(\widehat{K}')$. Using the argument above, once again, we get that $\widehat{K}' \sqsubseteq A \in \widehat{\mathcal{T}}$ for every $A \in \widehat{K}_j$ which by **R3** and (x) yields $\widehat{K}' \sqsubseteq (\leqslant 1\ r_j\ \widehat{K}'_{j+1}) \in \widehat{\mathcal{T}}$ as required.

Now for the induction step of Point 1. For every $K \sqsubseteq C \in \mathcal{T}_\mathsf{f}^i$, if K is realized in $\widehat{\mathcal{I}}$, then in the form of a supertype from $\mathsf{KON}(\widehat{\mathcal{T}})$. Thus we are done.

Now for the induction step of Point 2. Assume that $\mathcal{T}_\mathsf{f}^i \models K \sqsubseteq \exists r.K'$ with K active in $\widehat{\mathcal{T}}$. We also have that K is satisfiable w.r.t. $\mathcal{T}_\mathsf{f}^i$ since $\widehat{\mathcal{I}} \models \mathcal{T}_\mathsf{f}^i$ by Point 1. Consider the TBox $(\mathcal{T}_\mathsf{f}^i)_K$. By Lemma 23 (1), there is some L' such that

(a$'$) $\mathcal{T}_\mathsf{f}^i \models L' \sqsubseteq K'$ and

(b$'$) $(\mathcal{T}_\mathsf{f}^i)_K \ni K \sqsubseteq \exists r.L'$.

We shall show below that, for every $K \sqsubseteq \exists r.L'$ in $(\mathcal{T}_\mathsf{f}^i)_K$, there is some $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ with

(a$''$) $L' \subseteq \mathsf{cl}_\vdash(\widehat{K}')$

(b$''$) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r.\widehat{K}'$.

Note that this implies (a) since $K' \subseteq \mathsf{cl}_\vdash(\widehat{K}')$ follows from (a$'$) and (a$''$) since $\widehat{K}' \in \Delta^{\widehat{\mathcal{I}}}$ must make true K', and (b$'''$) is precisely the required (b).

To prove the above, we use induction on the number of rule applications used to construct $(\mathcal{T}_\mathsf{f}^i)_K$. The base case is that $K \sqsubseteq \exists r.L'$ enters $(\mathcal{T}_\mathsf{f}^i)_K$ in Step 1 of the construction. Then there is some $L \sqsubseteq \exists r.L' \in \mathcal{T}_\mathsf{f}^i$ with $\mathcal{T}_\mathsf{f}^i \models K \sqsubseteq L$. Since K is active in $\widehat{\mathcal{T}}$, so is L: for some $\widetilde{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_\vdash(\widetilde{K})$, the root element $\widetilde{K}$ of $\Delta^{\widehat{\mathcal{I}}}$ must make L true. By Point $2'$ above, there is a $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ with (a$''$) and (b$''$) as required.

In the induction step, $K \sqsubseteq \exists r.L'$ enters $(\mathcal{T}_\mathsf{f}^i)_K$ in Step 2 of the construction. In case this happens via an application of **R5**, we have that $L' = L'_1 \sqcap L'_2$ and

(i) $K \sqsubseteq \exists r.L'_1 \in (\mathcal{T}_\mathsf{f}^i)_K$

(ii) $K \sqsubseteq \forall r.L'_2 \in (\mathcal{T}_\mathsf{f}^i)_K$

Applying the induction hypothesis to (i), we obtain

(i$'$) there is some $\widehat{K}'_1 \in \mathsf{KON}(\widehat{\mathcal{T}})$ with

(a$'''$) $L'_1 \subseteq \mathsf{cl}_\vdash(\widehat{K}'_1)$

(b$'''$) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r.\widehat{K}'_1$.

From (ii), we obtain $L \sqsubseteq \forall r.L'_2 \in \mathcal{T}_\mathsf{f}^i$ for some L with $\mathcal{T}_\mathsf{f}^i \models K \sqsubseteq L$ because axioms with $\forall$-restrictions never enter $(\mathcal{T}_\mathsf{f}^i)_K$ in Step 2 of the construction. Since such axioms are not generated by closing cycles either, we even have $L \sqsubseteq \forall r.L'_2 \in \mathcal{T}$; hence

(ii$'$) $L \sqsubseteq \forall r.L'_2 \in \widehat{\mathcal{T}}$ with $\mathcal{T}_\mathsf{f}^i \models K \sqsubseteq L$.

We now observe that $\mathcal{T}_\mathsf{f}^i \models K \sqsubseteq L$ and $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$ imply $L \subseteq \mathsf{cl}_\vdash(\widehat{K})$ (again by consulting the domain element of $\widehat{\mathcal{I}}$ created for $\widehat{K}$). Hence, application of **R3** to (ii$'$) yields (ii$''$) $\widehat{K} \sqsubseteq \forall r.L'_2 \in \widehat{\mathcal{T}}$.

Now $\widehat{K}' := \widehat{K}'_1 \sqcap L'_2$ is as required:

- $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$.

  Since K is active in $\widehat{\mathcal{T}}$, there is some $\widetilde{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_\vdash(\widetilde{K})$. By (b$'''$), we thus have $\widetilde{K} \sqsubseteq \exists r.\widehat{K}'_1 \in \widehat{\mathcal{T}}$. Applying **R5** to this and (ii$'$) yields $\widehat{K} \sqsubseteq \exists r.(\widehat{K}'_1 \sqcap L'_2) \in \widehat{\mathcal{T}}$. By **R1**, $\widehat{K}'_1 \sqcap L'_2 \in \mathsf{KON}(\widehat{\mathcal{T}})$.

- (a$''$) is satisfied, that is, $L'_1 \sqcap L'_2 \subseteq \mathsf{cl}_\vdash(\widehat{K}'_1 \sqcap L'_2)$.

  Since $L'_1 \subseteq \mathsf{cl}_\vdash(\widehat{K}'_1)$ by (a$'''$), **R3** yields that $L'_1 \sqcap L'_2 \subseteq \mathsf{cl}_\vdash(\widehat{K}'_1 \sqcap L'_2)$.

- (b$''$) is satisfied.

  Let $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$. Then **R5** applied to (b$'''$) and (ii$''$) implies $\widehat{K} \sqsubseteq \exists r.(\widehat{K}'_1 \sqcap L'_2) \in \widehat{\mathcal{T}}$.

In case $K \sqsubseteq \exists r.L'$ enters $(\mathcal{T}^i_{\mathsf{f}})_K$ via an application of **R7$'$**, we have that $L' = L'_1 \sqcap L'_2$ and

(i) $K \sqsubseteq \exists r.L'_j \in (\mathcal{T}^i_{\mathsf{f}})_K$, $j = 1, 2$

(ii) $K \sqsubseteq (\leqslant 1\ r\ L'_3) \in (\mathcal{T}^i_{\mathsf{f}})_K$

(iii) $\mathcal{T}^i_{\mathsf{f}} \models L'_i \sqsubseteq L'_3$, $i = 1, 2$

Applying the induction hypothesis to (i), we obtain

(i$'$) there is some $\widehat{K}'_j$, $j = 1, 2$, with

(a$'''$) $L'_j \subseteq \mathsf{cl}_\vdash(\widehat{K}'_j)$

(b$'''$) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r.\widehat{K}'_j$.

Regarding (ii), we observe that axioms with functionality restrictions never enter $(\mathcal{T}^i_{\mathsf{f}})_K$ in Step 2. Hence, there is some $L \sqsubseteq (\leqslant 1\ r\ L'_3) \in \mathcal{T}^i_{\mathsf{f}}$ with $\mathcal{T}^i_{\mathsf{f}} \models K \sqsubseteq L$, and the same observation as in the previous case yields $L \subseteq \mathsf{cl}_\vdash(\widehat{K})$ whenever $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$. We can thus apply Point 3$'$ above and obtain

(ii$'$) there is some $\widehat{K}'_3$ with

(a^4) for all $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $L'_3 \subseteq \mathsf{cl}_\vdash(\widetilde{K}')$, then $\widehat{K}'_3 \subseteq \mathsf{cl}_\vdash(\widetilde{K}')$.

(b^4) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ \widehat{K}'_3)$.

Furthermore (iii) and (a$'''$) yield

(iii$'$) $L'_3 \subseteq \mathsf{cl}_\vdash(\widehat{K}'_1) \cap \mathsf{cl}_\vdash(\widehat{K}'_2)$,

and with (a^4) we get

(iii$''$) $\widehat{K}'_3 \subseteq \mathsf{cl}_\vdash(\widehat{K}'_1) \cap \mathsf{cl}_\vdash(\widehat{K}'_2)$.

Now $\widehat{K}' = \widehat{K}'_1 \sqcap \widehat{K}'_2$ is as required:

- $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$.

  Since K is active in $\widehat{\mathcal{T}}$, there is some $\widetilde{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_\vdash(\widetilde{K})$. By (b$'''$) and (b^4), we thus have $\widetilde{K} \sqsubseteq \exists r.\widehat{K}'_j \in \widehat{\mathcal{T}}$ and $\widetilde{K} \sqsubseteq (\leqslant 1\ r\ \widehat{K}'_j) \in \widehat{\mathcal{T}}$. Applying **R7** to these and (iii$''$) yields $\widehat{K} \sqsubseteq \exists r.(\widehat{K}'_1 \sqcap \widehat{K}'_2) \in \widehat{\mathcal{T}}$. By **R1**, $\widehat{K}'_1 \sqcap \widehat{K}'_2 \in \mathsf{KON}(\widehat{\mathcal{T}})$.

- (a$''$) is satisfied, that is, $L'_1 \sqcap L'_2 \subseteq \mathsf{cl}_\vdash(\widehat{K}'_1 \sqcap \widehat{K}'_2)$.

  Since $L'_1 \subseteq \mathsf{cl}_\vdash(\widehat{K}'_1)$ by (a$'''$), **R3** yields that $L'_1 \sqcap L'_2 \subseteq \mathsf{cl}_\vdash(\widehat{K}'_1 \sqcap 1\widehat{K}'_2)$.

- (b$''$) is satisfied.

  Let $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$. Then **R7** applied to (b$'''$), (b^4), (iii$''$) implies $\widehat{K} \sqsubseteq \exists r.(\widehat{K}'_1 \sqcap \widehat{K}'_2) \in \widehat{\mathcal{T}}$.

Now for the induction step of Point 3. Assume that $\mathcal{T}^i_{\mathsf{f}} \models K \sqsubseteq (\leqslant 1\ r\ K')$, $\mathcal{T}^i_{\mathsf{f}} \models K' \sqsubseteq \exists r^-.K$, K is maximal with this property, and K' is active in $\widehat{\mathcal{T}}$. We have to show that there is a $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that

(a) for all $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $K' \subseteq \mathsf{cl}_\vdash(\widetilde{K}')$, then $\widehat{K}' \subseteq \mathsf{cl}_\vdash(\widetilde{K}')$.

(b) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ \widehat{K}')$.

By Lemma 23 (2), there is some $L \sqsubseteq (\leqslant 1\ r\ L') \in \mathcal{T}^i_{\mathsf{f}}$ with

(i) $\mathcal{T}^i_{\mathsf{f}} \models K \sqsubseteq L$ and

(ii) $\mathcal{T}^i_{\mathsf{f}} \models K' \sqsubseteq L'$.

Now we apply Point 3$'$ and conclude that there is some $\widehat{L}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ with

(a$'$) $\widehat{\mathcal{T}} \models L' \sqsubseteq \widehat{L}'$ and

(b$'$) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $L \subseteq \mathsf{cl}_\vdash(\widehat{K})$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ \widehat{L}')$.

It remains to show that $\widehat{L}'$ is the required conjunction. For (a), take some $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ and assume that $K' \subseteq \mathsf{cl}_\vdash(\widetilde{K}')$. Then (ii) and Point 1 ($\widehat{\mathcal{I}} \models \mathcal{T}^i_{\mathsf{f}}$) imply that $L' \subseteq \mathsf{cl}_\vdash(\widetilde{K}')$. Now (a$'$) implies that $\widehat{L}' \subseteq \mathsf{cl}_\vdash(\widetilde{K}')$.

For (b), take some $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $K \subseteq \mathsf{cl}_\vdash(\widehat{K})$. Then (i) and $\widehat{\mathcal{I}} \models \mathcal{T}^i_{\mathsf{f}}$ imply that $L \subseteq \mathsf{cl}_\vdash(\widehat{K})$. Now (b$'$) implies that $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ \widehat{L}')$. ❏

C Proofs for Section 5

Recall that, to establish Proposition 14 we have to show the following.

Proposition 25 *For every $n_0 > 0$, there is a finite model $\mathcal{J}$ of $\mathcal{A}$ and $\mathcal{T}$ such that for all $d \in \Delta^{\mathcal{J}}$, there is a homomorphism from $\mathcal{J}|^{n_0}_d$ to $\mathcal{U}$.*

As explained in the main paper, the first step towards proving Proposition 25 is to show Proposition 15, repeated here for convenience

Proposition 15 For every $n_0 > 0$, there is a finite model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}$ such that $\mathcal{I} \preceq_{2n_0} \mathcal{U}$.

As explained in the paper, the central step in proving Proposition 25 from Proposition 15 is to take the product with a finite group of high girth. For technical reasons, we need to interpose another construction that removes cycles of length at most two beforehand; such small cycles are not removed by the product with the finite group because the groups that we use have involutive generators and thus, intuitively, correspond to an undirected graph.

For the rest of Section C, fix a concrete n_0 (and, as in the main paper, an ABox $\mathcal{A}$ and TBox $\mathcal{T}$). To prove Proposition 15, we modify the finite model construction of Section 3. In particular, we modify the initial interpretation, use modified versions of the completion rules $(\mathbf{c2})$ and $(\mathbf{c3})$, and change the strategy of rule application. Here is a more detailed summary of the implemented changes:

- The elements d_t introduced in the initial version of $\mathcal{I}$ in the original construction and used as targets for applications of $(\mathbf{c3})$ are not present.
- Instead of including only $\mathcal{A}$ in the initial version of $\mathcal{I}$, we include an initial piece of the canonical model $\mathcal{U}$ of $\mathcal{A}$ and $\mathcal{T}$, truncated to depth $2n_0$. This does not seem strictly necessary, but simplifies some arguments since it ensures that all paths in $\mathcal{I}$ that connect two ABox elements and not purely inside the ABox have length exceeding $2n_0$.
- We then apply only $(\mathbf{c1})$ and $(\mathbf{c2})$, where the latter is modified in a way so that all new paths introduced between elements that existed already before a rule application exceed length $2n_0$.
- To provide targets for applications of $(\mathbf{c3})$, we determine all the $2n_0$-simulation types needed as targets and introduce these simulation types by adding them in a disjoint way to the model constructed thus far.
- The previous two steps are iterated until no new $2n_0$-simulation types are needed.
- We then apply a modified version of $(\mathbf{c3})$ that respects $2n_0$-simulation types.

Details are provided in what follows.

We introduce some missing bits of notation. An (unbounded) *simulation of* $\mathcal{I}_1$ *in* $\mathcal{I}_2$ is a relation $\rho \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ such that for all $(d, e) \in \rho$, Condition 1 of bounded simulations is satisfied, as well as the following variation of Condition 2:

2′. if $(d, d') \in r^{\mathcal{I}_1}$ for some (possibly inverse) role r with $\mathsf{sig}(r) \subseteq \mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{T})$, then there is an $e' \in \Delta^{\mathcal{I}_2}$ with $(e, e') \in r^{\mathcal{I}_2}$ and $(d', e') \in \rho$.

We write $(\mathcal{I}_1, d) \preceq (\mathcal{I}_2, d)$ if there is a bounded simulation ρ of $\mathcal{I}_1$ in $\mathcal{I}_2$ such that $(d, e) \in \rho$ and for all $a \in \mathsf{Ind}(\mathcal{A}) \cap \Delta^{\mathcal{I}}$, we have $(a, a) \in \rho$.

Proof of Proposition 15: Applying (c1) and (c2)

To define the finite interpretation $\mathcal{I}_0$, we start with the subinterpretation $\mathcal{U}_0$ of $\mathcal{U}$ to those elements that can be reached from an ABox individual by traveling at most $2n_0$ role edges. Note that the ABox $\mathcal{A}$ is a substructure of $\mathcal{U}_0$, but that the elements d_t from the initial interpretation $\mathcal{I}$ in Section 3 are not present (an issue that we will address later). Next, we exhaustively apply the two completion rules $(\mathbf{c1})$ and $(\mathbf{c2'})$ to the initial $\mathcal{I}_0$ just defined, where $(\mathbf{c1})$ is as in Section 3 and $(\mathbf{c2'})$ is a modified version of $(\mathbf{c2})$. Note that, also in the original construction of the finite model $\mathcal{I}$ in Section 3, it is safe to apply $(\mathbf{c3})$ only after no further applications of $(\mathbf{c1})$ and $(\mathbf{c2})$ are possible since $(\mathbf{c3})$ cannot trigger an application of $(\mathbf{c1})$ or $(\mathbf{c2})$.

$(\mathbf{c2'})$ Select a type class P that is minimal w.r.t. the order $\prec^+$ and such that there is a $\lambda = s \; {}^1\!\!\leftrightarrow^1_r \; s'$ with $s \in P$, and an element $d \in \Delta^{\mathcal{I}_0}$ with $d \in s^{\mathcal{I}_0} \setminus (\exists r.s')^{\mathcal{I}_0}$.

For each $\lambda = s \; {}^1\!\!\leftrightarrow^1_r \; s'$ with $s \in P$, set

$$X^{\mathcal{I}_0}_{\lambda,1} = s^{\mathcal{I}_0} \setminus (\exists r.s')^{\mathcal{I}_0} \qquad X^{\mathcal{I}_0}_{\lambda,2} = s'^{\mathcal{I}_0} \setminus (\exists r^-.s)^{\mathcal{I}_0}.$$

Take (i) a fresh set Δ_s for each $s \in P$ and (ii) a bijection π_λ between $X^{\mathcal{I}_0}_{\lambda,1} \cup \Delta_s$ and $X^{\mathcal{I}_0}_{\lambda,2} \cup \Delta_{s'}$ for each $\lambda = s \; {}^1\!\!\leftrightarrow^1_r \; s'$ with $s, s' \in P$ and r a role name, and extend $\mathcal{I}_0$ as follows:

- add all domain elements in $\biguplus_{s \in P} \Delta_s$;
- extend $r^{\mathcal{I}_0}$ with π_λ, for each $\lambda = s \; {}^1\!\!\leftrightarrow^1_r \; s'$ with $s, s' \in P$ and r a role name;
- interpret concept names so that $\mathsf{tp}_{\mathcal{I}_0}(d) = s$ for all $d \in \Delta_s$, $s \in P$.

An element d in the extended $\mathcal{I}_0$ is called *old* if it existed already before the extension (that is, $d \notin \biguplus_s \Delta_s$) and *new* otherwise. A path is a sequence $d_1 r_1 d_2 \cdots d_k r_k d_{k+1}$, with $d_1, \ldots, d_{k+1} \in \Delta^{\mathcal{I}_0}$, $r_1, \ldots, r_k$ (potentially inverse) roles and $(d_i, d_{i+1}) \in r_i^{\mathcal{I}_0}$ for $1 \leq i \leq k$: We will show below that we can choose the sets Δ_s and bijections π_s such that:

1. $|\biguplus_{s \in P} \Delta_s| \leq \mathcal{O}(2^{|\mathcal{T}|} \cdot |\mathcal{T}|^{2n_0} \cdot |\Delta^{\mathcal{I}_0}|)$;
2. no edge $(d_1, d_2) \in r^{\mathcal{I}_0}$ is introduced with both d_1, d_2 old;
3. for each new element $d_1 \in \biguplus_s \Delta_s$, there is at most one simple path $d_1 r_1 d_2 \cdots d_k r_k d_{k+1}$ of length at most $2n_0$ such that $d_1, \ldots, d_k$ are new and d_{k+1} is old.

Rules are applied in the same preference order as in Section 3.

We now argue that, in rule $(\mathbf{c2'})$ above, the sets Δ_s and π_λ indeed exist also with these modified conditions. We first extend $\mathcal{I}_0$ to a new interpretation $\mathcal{I}'_0$ by a bounded unraveling operation. While unraveling, we assign to each newly generated element a level and a type. In detail:

- all elements of $\mathcal{I}_0$ are assigned *level 0*;
- whenever there is an element d on level $\ell < 2n_0$ and a $\lambda = s \; {}^1\!\!\leftrightarrow^1_r \; s'$ with $s, s' \in P$ and r a (potentially inverse) role such that $\mathsf{tp}_{\mathcal{I}'_0}(d) = s$ and there is no $(d, d') \in r^{\mathcal{I}'_0}$ with $d' \in s'^{\mathcal{I}'_0}$, then add a new element d', put $\mathsf{tp}_{\mathcal{I}'_0}(d') = s'$, add (d, d') to $r^{\mathcal{I}'_0}$, and assign to d' level $\ell + 1$.

We can now apply the original $(\mathbf{c2})$ operation to $\mathcal{I}'_0$ instead of $\mathcal{I}_0$. This gives us a set Δ'_s for each $s \in P$ and a bijection π'_λ from $X^{\mathcal{I}'_0}_{\lambda,1} \cup \Delta'_s$ to $X^{\mathcal{I}'_0}_{\lambda,2} \cup \Delta'_{s'}$ for each $\lambda = s \; {}^1\!\!\leftrightarrow^1_r \; s'$ with $s, s' \in P$ and r a role name.

Set $\Delta_s = \Delta'_s \cup \{d \in \Delta^{\mathcal{I}'_0} \setminus \Delta^{\mathcal{I}_0} \mid \mathsf{tp}_{\mathcal{I}'_0}(d) = s\}$, and define π_λ as the extension of π'_λ by all pairs $(d, d') \in r^{\mathcal{I}'_0}$ such that $d, d' \in \Delta^{\mathcal{I}'_0} \setminus \Delta^{\mathcal{I}_0}$. It can be verified that π_λ is a bijection from $X^{\mathcal{I}_0}_{\lambda,1} \cup \Delta_s$ to $X^{\mathcal{I}_0}_{\lambda,2} \cup \Delta_{s'}$. Moreover, we have $|\Delta^{\mathcal{I}'_0}| \leq |\Delta^{\mathcal{I}_0}| + (|\Delta^{\mathcal{I}_0}| \cdot |\mathcal{T}|^{2n_0})$, and thus the size bound in Point 1 above is a consequence of the fact that $|\biguplus_{s \in P} \Delta'_s| \leq 2^{|\mathcal{T}|} \cdot |\Delta^{\mathcal{I}'_0}|$.

The invariants $(\mathbf{i1})$–$(\mathbf{i3})$ from Section 3 are satisfied also with the modified initial interpretation and the modified version of $(\mathbf{c2'})$:

- The modified initial interpretation satisfies the invariants: $(\mathbf{i1})$, $(\mathbf{i2})$ are satisfied by construction of $\mathcal{U}$, and $(\mathbf{i3})$ holds because $\mathcal{U} \models \mathcal{T}_\mathsf{f}$ (Lemma 13).
- To show that the invariants are preserved by applications of $(\mathbf{c1})$ and the *old* $(\mathbf{c2})$ starting from the modified initial interpretation, the same arguments as for Theorem 3 (1) go through.
- The invariants are preserved by the new $(\mathbf{c2'})$, too, because the model extension described in $(\mathbf{c2'})$ is identical to that in $(\mathbf{c2})$ except for the potentially larger number of elements added, and the arguments for why the old $(\mathbf{c2})$ preserves all invariants do not depend on that exact number.

Furthermore, termination can be proved in exactly the same way as before.

Lemma 26 $\mathcal{I}_0 \preceq_{2n_0} \mathcal{U}$.

Proof. Let $d^* \in \Delta^{\mathcal{I}_0}$. We show that $(\mathcal{I}_0|^{2n_0}_{d^*}, d^*) \preceq (\mathcal{U}, e)$ for some $e \in \Delta^{\mathcal{U}}$. For brevity, we use Δ to denote the domain of $\mathcal{I}_0|^{2n_0}_{d^*}$. An element $d \in \Delta$ is an *initial element* if it is present in the initial version of (the modified) $\mathcal{I}$. A *forward path* is a sequence $d_0 r_0 d_1 \cdots d_{k-1} r_{k-1} d_k$ such that

(a) $(d_i, d_{i+1}) \in r_i^{\mathcal{I}_0|^{2n_0}_{d^*}}$ for all $i < k$.

(b) $\mathsf{tp}_{\mathcal{I}_0}(d_i) \rightarrow_{r_i} \mathsf{tp}_{\mathcal{I}_0}(d_{i+1})$ for all $i < k$.

(c) $(r_i^-, d_{i+1}) \neq (r_{i-1}, d_{i-1})$ for $0 < i < k$;

(d) $d_1, \ldots, d_k$ are not initial.

An element $d \in \Delta$ is a *root* if the following two conditions are satisfied:

(i) if $(d, e) \in r^{\mathcal{I}_0|^{2n_0}_{d^*}}$, then both d and e are initial or $\mathsf{tp}_{\mathcal{I}_0}(d) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(e)$;

(ii) for every forward path $d = d_0 r_0 d_1 \cdots r_{k-1} d_k$ and each $(d_k, e) \in r^{\mathcal{I}_0|^{2n_0}_{d^*}}$, we have that $\mathsf{tp}_{\mathcal{I}_0}(d_k) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(e)$ or $e = d_{k-1}$ and $r = r_{k-1}^-$.

We first show the following:

(1) every initial element is a root;

(2) Δ contains at least one root d_r such that d^* is reachable from d_r on a forward path.

To show (1) and (2), we write $d \prec e$ if the rule application that created d happened before the one that created e. We write $d \preceq e$ if $d \preceq e$ or d and e are initial elements or d and e have been created in the same application (thus of $(\mathbf{c2'})$).

Take some element $d_r \in \Delta$ that is minimal w.r.t. $\preceq$, i.e., whenever $d \preceq d_r$, we also have $d_r \preceq d$. We will now show that

(α) d_r is a root, and

(β) d^* is reachable from d_r on a forward path.

Since every initial element is $\preceq$-minimal, (α) establishes (1). Furthermore, (α) and (β) establish (2).

For (α), we have to show Conditions (i) and (ii) of roots.
For (i), take some element e with $(d_r, e) \in r^{\mathcal{I}_0|^{2n_0}_{d^*}}$. If both d_r, e are initial, we are done. If d_r is initial and e is not, we have $\mathsf{tp}_{\mathcal{I}_0}(d_r) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(e)$ due to the construction of $\mathcal{I}_0$. If e is initial and d_r is not, then this contradicts d_r being minimal. Finally, if none of d_r, e is initial, then we consider the rule application that created the edge $(d_r, e) \in r^{\mathcal{I}_0|^{2n_0}_{d^*}}$: if it was $(\mathbf{c1})$, then $\preceq$-minimality of d_r ensures $\mathsf{tp}_{\mathcal{I}_0}(d_r) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(e)$. If it was $(\mathbf{c2'})$, we have $\mathsf{tp}_{\mathcal{I}_0}(d_r) \;{}^1{\leftrightarrow}^1_r\; \mathsf{tp}_{\mathcal{I}_0}(e)$.

To show (ii), let $d_r = d_0 r_0 d_1 \cdots r_{k-1} d_k$ be a forward path and $(d_k, e) \in r^{\mathcal{I}_0|^{2n_0}_{d^*}}$. In case the edge $(d_k, e) \in r^{\mathcal{I}_0|^{2n_0}_{d^*}}$ was created in an application of $(\mathbf{c2'})$, we have $\mathsf{tp}_{\mathcal{I}_0}(d_k) \;{}^1{\leftrightarrow}^1_r\; \mathsf{tp}_{\mathcal{I}_0}(e)$ and we are done. Otherwise, this edge was created in an application of $(\mathbf{c1})$, and either $\mathsf{tp}_{\mathcal{I}_0}(d_k) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(e)$ (in which case we are done) or $\mathsf{tp}_{\mathcal{I}_0}(e) \rightarrow_{r^-} \mathsf{tp}_{\mathcal{I}_0}(d_k)$. Then the edge $(d_{k-1}, d_k) \in r_{k-1}^{\mathcal{I}_0|^{2n_0}_{d^*}}$ was created in an application of $(\mathbf{c1})$ (in which case we must have $e = d_{k-1}$ and $r = r_{k-1}^-$ and are done) or of $(\mathbf{c2'})$.

In the latter case, we trace that $(\mathbf{c2'})$ application backwards on the path. If all elements $d_r = d_0, \ldots, d_k$ have been created by the same application, we have that $e \prec d_0$, which contradicts d_r being $\preceq$-minimal. Otherwise, d_0 is an old element of that application, and we consider a simple path $d_0 = d'_0 r'_0 d'_1 \cdots r'_{\ell-1} d'_\ell = d_k$ of length $\leq 4n_0$ from $d_r = d_0 = d'_0$ to $d_k = d'_\ell$. Due to its construction, $\mathcal{I}_0|^{2n_0}_{d^*}$ has to contain such a path. Then some edge on this new path must have been created in an application of $(\mathbf{c2'})$: otherwise, we would have $d'_0 \prec d'_1 \prec \cdots \prec d'_j \succ \cdots \succ d'_{\ell-1} \succ d'_\ell$, that is, d'_j would have been created in two different $(\mathbf{c1})$ rule applications.

Consider the latest such $(\mathbf{c2'})$ application in the construction of $\mathcal{I}_0$ and observe that $d'_\ell = d_k$ is old for it (we reuse the argument from above). If $d'_0 = d_0$ is old for it as well, then we get a contradiction: since $\ell > 1$ due to Condition 2 of $(\mathbf{c2'})$, there is a middle element $d'_{j'}$ with $j' = \lfloor \frac{\ell}{2} \rfloor$, which has simple paths of length $\leq 2n_0$ to both d'_0 and d'_ℓ. These paths coincide due to Condition 3 of $(\mathbf{c2'})$, which contradicts the assumption that $d'_0 r'_0 d'_1 \cdots r'_{\ell-1} d'_\ell$ is simple.

Otherwise, if d'_0 had been created in the same $(\mathbf{c2'})$ application, then we would get $d_k \prec d'_0$ which contradicts d_r being $\preceq$-minimal. Finally, if d'_0 had been created after that $(\mathbf{c2'})$ application, we would get $d'_1 \prec d'_0$, again contradicting $\preceq$-minimality of d_r.

For (β), we observe that, by construction of $\mathcal{I}_0|^{2n_0}_{d^*}$, there is a simple path $d_r = d_0 r_0 d_1 \ldots r_{k-1} d_k = d^*$ in $\mathcal{I}_0|^{2n_0}_{d^*}$ with $k \leq 2n_0$ such that no element other than possibly d_0 is initial. We pick such a path, and our choice ensures Conditions (a), (c) and (d) of forward paths. This leaves us with showing Condition (b).

Since $(d_i, d_{i+1}) \in r_i^{\mathcal{I}_0|^{2n_0}_{d^*}}$ for all $i < k$, we have that either $\mathsf{tp}_{\mathcal{I}_0}(d_i) \rightarrow_{r_i} \mathsf{tp}_{\mathcal{I}_0}(d_{i+1})$ or $\mathsf{tp}_{\mathcal{I}_0}(d_{i+1}) \rightarrow_{r_i^-}$

$\mathsf{tp}_{\mathcal{I}_0}(d_i)$ for all $i < k$. Assume that there is some i with $\mathsf{tp}_{\mathcal{I}_0}(d_{i+1}) \rightarrow_{r_i^-} \mathsf{tp}_{\mathcal{I}_0}(d_i)$ but $\mathsf{tp}_{\mathcal{I}_0}(d_i) \not\rightarrow_{r_i} \mathsf{tp}_{\mathcal{I}_0}(d_{i+1})$, and take the smallest such i. Then the edge $(d_i, d_{i+1}) \in r_i^{\mathcal{I}_0}$ has been introduced in some application of $(\mathbf{c1})$, and the previous edge $(d_{i-1}, d_i) \in r_{i-1}^{\mathcal{I}_0}$ has been introduced in some application of $(\mathbf{c2'})$ (otherwise, we would have $d_{i+1} = d_{i-1}$ and $r_i = r_{i-1}^-$, contradicting Condition (d) of forward paths). We can now reuse the above argument, tracing back that application of $(\mathbf{c2'})$, and derive a contradiction.

We say that a relation $\rho \subseteq \Delta \times \Delta^{\mathcal{U}}$ is *rooted* if for every $(d, e) \in \rho$, there is a forward path $d_0 r_0 d_1 \cdots r_{k-1} d_k$ and elements $e_0, \ldots, e_k \in \Delta^{\mathcal{U}}$ such that d_0 is a root, $d_k = d$, $e_k = e$, $(d_i, e_i) \in \rho$ for all $i \leq k$ and $(e_i, e_{i+1}) \in r_i^{\mathcal{U}}$ for all $i < k$. We define a sequence of relations

$$\rho_0 \subseteq \rho_1 \subseteq \cdots \subseteq \Delta \times \Delta^{\mathcal{U}}$$

such that

(†) if $(d, e) \in \rho_i$, then $\mathsf{tp}_{\mathcal{I}_0}(d) = \mathsf{tp}_{\mathcal{U}}(e)$;

(‡) ρ_i is rooted.

Choose a root d_r such that d^* is reachable from d_r by a forward path, whose existence is guaranteed by Point 2 above. Also choose an $e_r \in \Delta^{\mathcal{U}}$ with $\mathsf{tp}_{\mathcal{U}}(e_r) = \mathsf{tp}_{\mathcal{I}_0}(d_r)$, which exists by invariant $(\mathbf{i1})$. Then

- set
$$\rho_0 = \{(d, d) \mid d \in \Delta \cap \Delta^{\mathcal{U}_0}\} \cup \{(d_r, e_r)\}$$
Note that (†) and (‡) are trivially satisfied.
- ρ_{i+1} is obained from ρ_i by doing the following for each $(d, e) \in \rho_i$ and $(d, d') \in r^{\mathcal{I}_0|_{d^*}^{2n_0}}$. By (‡), there is a forward path $d_0 r_0 d_1 \cdots r_{k-1} d_k$ and elements $e_0, \ldots, e_k \in \Delta^{\mathcal{U}}$ such that d_0 is a root, $d_k = d$, $e_k = e$, $(d_i, e_i) \in \rho$ for all $i \leq k$ and $(e_i, e_{i+1}) \in r_i^{\mathcal{U}}$ for all $i < k$. By Point 2 above, we can distinguish two cases:
  - $\mathsf{tp}_{\mathcal{I}_0}(d_k) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(d')$ and it is not true that $d' = d_{k-1}$ and $r = r_{k-1}^-$.
  Then $\mathsf{tp}_{\mathcal{I}_0}(d) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(d')$. By (†), we have $\mathsf{tp}_{\mathcal{U}}(e) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(d')$ and thus we find an $e' \in \Delta^{\mathcal{U}}$ with $(e, e') \in r^{\mathcal{U}}$ and $\mathsf{tp}_{\mathcal{U}}(e') = \mathsf{tp}_{\mathcal{I}_0}(d')$. Include (d', e') in ρ_{i+1}. Clearly, (†) is still satisfied. Using that it is not true that $d' = d_{k-1}$ and $r = r_{k-1}^-$, it is also straightforward to show that (‡) is still satisfied.
  - $d' = d_{k-1}$ and $r = r_{k-1}^-$.
  Then we do not need to add an extra tuple to ρ_i since there already is a $(d', e') \in \rho_i$ such that $(e, e') \in r^{\mathcal{U}}$. To see this, recall that e_{k-1} and e_k are such that $(d_{k-1}, e_{k-1}) \in \rho_i$, $(d_k, e_k) \in \rho_i$, and $(e_{k-1}, e_k) \in r_{k-1}^{\mathcal{U}}$. Since $d' = d_{k-1}$ and $r = r_{k-1}^-$, e_{k-1} can serve as the required e'.

Set $\rho = \bigcup_{i \geq 0} \rho_i$. By construction, ρ is a simulation. Since there is a forward path from d_r to d^*, by construction of ρ there must be some $(d^*, e) \in \rho$. Thus we have shown $(\mathcal{I}_0|_{d^*}^{2n_0}, d^*) \preceq (\mathcal{U}, e)$. Note that $\mathsf{tp}_{\mathcal{I}_0}(d^*) = \mathsf{tp}_{\mathcal{U}}(e)$ as required. ❑

Proof of Proposition 15: Generating Witnesses for (c3)

To prepare for the application of (a modified version) of the completion rule $(\mathbf{c3})$, we still need to generate elements that can be used as 'targets' for edges in place of the elements d_t from the original finite interpretation $\mathcal{I}$. To prepare for this, we first extend finite model $\mathcal{I}_0$ constructed so far to an infinite interpretation $\mathcal{I}_0^+$. While $\mathcal{I}_0^+$ will of course not be part of the finite model that we aim to construct, it will guide the further construction.

We obtain $\mathcal{I}_0^+$ from $\mathcal{I}$ by starting with $\mathcal{I}_0^+ = \mathcal{I}$ and then exhaustively applying the completion rule from the construction of the canonical model $\mathcal{U}$, repeated here for convenience: for all $d \in \Delta^{\mathcal{I}_0^+}$ such that $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}_0^+}(d) \sqsubseteq \exists r.t'$, where t' is maximal with this property and $d \notin (\exists r.t')^{\mathcal{I}_0^+}$, add a fresh element d' to $\Delta^{\mathcal{I}_0^+}$, the edge (d, d') to $r^{\mathcal{I}_0^+}$, and d' to the interpretation $A^{\mathcal{I}_0^+}$ of all concept names $A \in t'$.

Lemma 27 $\mathcal{I}_0^+ \preceq_{2n_0} \mathcal{U}$.

Proof. (sketch) Let $d^* \in \Delta^{\mathcal{I}_0^+}$, and let Δ denote the elements reachable from d^* by travelling along at most $2n_0$ role edges. To show that $(\mathcal{I}_0^+, d^*) \preceq_{2n_0} (\mathcal{U}, e)$ for some e, we clearly need to consider only elements from Δ. We distinguish three cases:

1. d^* is from $\Delta^{\mathcal{I}_0}$.
Then Lemma 26 gives us an $2n_0$-bounded simulation ρ of $(\mathcal{I}_0, d^*)$ in $(\mathcal{U}, e)$ for some e. We can extend ρ to the desired $2n_0$-bounded simulation of $(\mathcal{I}_0^+, d^*)$ in $(\mathcal{U}, e)$ by following the applications of the completion rule applied to construct $\mathcal{I}_0^+$ from $\mathcal{I}_0$, and exploiting that $\mathcal{U}$ is constructed by applying the same rule.

2. d^* is not from $\Delta^{\mathcal{I}_0}$ and Δ contains elements from $\Delta^{\mathcal{I}_0}$.
Let d_0 be the unique element from Δ that is in $\Delta^{\mathcal{I}_0}$ and can be reached from d^* in $\mathcal{I}_0^+|_{d^*}^{2n_0}$ on a path of minimal length.[4] Start with a $2n_0$-bounded simulation ρ of $(\mathcal{I}_0, d_0)$ in $(\mathcal{U}, e)$ for some e (given by Lemma 26), restricted to the elements of Δ. Then proceed as in Case 1.

3. Δ contains no elements from $\Delta^{\mathcal{I}_0}$.
Exploiting Invariant $(\mathbf{i1})$, it is easy to show by induction on the number of rule applications used to construct $\mathcal{I}_0^+$ that for every $d \in \Delta^{\mathcal{I}_0^+}$, there is an $e \in \Delta^{\mathcal{U}}$ with $\mathsf{tp}_{\mathcal{I}_0^+}(d) = \mathsf{tp}_{\mathcal{U}}(e)$. For $d, d' \in \Delta$, we write $d \prec d'$ if d' was created by a later rule apliction than d during the construction of $\mathcal{I}_0^+$ from $\mathcal{I}_0$. Let d_0 be the unique element of Δ that is minimal w.r.t. $\prec$. We start with the initial bounded simulation $\rho = \{(d_0, 2n_0, e)\}$ for some e with $\mathsf{tp}_{\mathcal{I}_0^+}(d_0) = \mathsf{tp}_{\mathcal{U}}(e)$ and then proceed as in Case 1 above.

❑

[4]This element d_0 is unique since $\mathcal{I}_0^+$ extends $\mathcal{I}_0$ by attaching tree-shaped structures to existing elements.

We now choose one representative $(\mathcal{J}, d) \in S$ of each $2n_0$-simulation type S realized in $\mathcal{I}_0^+$, i.e., such that there is some $d \in \Delta^{\mathcal{I}_0^+}$ with $(\mathcal{I}_0^+, d) \in S$. Then extend $\mathcal{I}_0$ with pairwise disjoint copies of all the chosen representatives. By Lemma 27, the resulting interpretation $\mathcal{I}_1$ satisfies $\mathcal{I}_1 \preceq_{2n_0} \mathcal{U}$. We treat $\mathcal{I}_1$ as an initial interpretation in the same way as we have treated $\mathcal{U}_0$ as an initial intepretation for constructing $\mathcal{I}_0$ and repeat the application of $(\mathbf{c1})$ and $(\mathbf{c2'})$ as described above, which results in a completed version of the interpretation $\mathcal{I}_1$. Lemmas 26 and 27 apply also to $\mathcal{I}_1$ in place of $\mathcal{I}_0$, with the proofs clearly going through without modification. The same is true for the invariants $(\mathbf{i1})$ to $(\mathbf{i3})$. Since $\mathcal{I}_1^+$ might realize $2n_0$-simulation types that are not realized in $\mathcal{I}_0^+$, we then disjointly add copies of the new simulation types. Repeating this process leads to a sequence of finite interpretations $\mathcal{I}_0, \mathcal{I}_1, \ldots$. Since there are only finitely many $2n_0$-simulation types and since the simulation type of added representatives does not change by applying the rules $(\mathbf{c1})$ and $(\mathbf{c2'})$, this process eventually stabilizes. Call the resulting finite interpretation $\mathcal{I}_\omega$. By what was said above, we have the following.

Lemma 28 *$\mathcal{I}_\omega \prec_{2n_0} \mathcal{U}$ and $\mathcal{I}_\omega^+ \prec_{2n_0} \mathcal{U}$.*

The disjoint copies just added will serve as the desired 'targets' for applying (a modified version) of the completion rule $(\mathbf{c3})$, described in the next section.

Proof of Proposition 15: Applying (c3)

To construct the desired finite interpretation $\mathcal{I}$, it remains to start with $\mathcal{I} = \mathcal{I}_\omega$ and apply a modified version $(\mathbf{c3})'$ of the completion rule $(\mathbf{c3})$.

$(\mathbf{c3'})$ For every $d \in \Delta^{\mathcal{I}}$ such that $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r t$ and $d \notin (\exists r.t)^{\mathcal{I}}$, do the following. By construction of $\mathcal{I}^+$, we find an element $e \in \Delta^{\mathcal{I}^+}$ such that $(d, e) \in r^{\mathcal{I}^+}$ and $\mathsf{tp}_{\mathcal{I}^+}(e) = t$. By construction of $\mathcal{I} = \mathcal{I}_\omega$, there is an element $e' \in \Delta^{\mathcal{I}}$ such that $(\mathcal{I}^+, e)$ and $(\mathcal{I}, e')$ have the same $2n_0$-simulation type. Include in $r^{\mathcal{I}}$ the edge (d, e').

This modification of $(\mathbf{c3})$ preserves all invariants because the same arguments as for the old $(\mathbf{c3})$ go through.

Lemma 29 *$\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$.*

Proof. It is easy to see that the proof of Proposition 21 goes through also for the modified version of $\mathcal{I}$: the essential ingredients of that proof are the invariants $(\mathbf{i1})$–$(\mathbf{i3})$, which hold for $\mathcal{I}$ as argued above, plus the argument after the case distinction (1)–(3) for axioms of the form $K \sqsubseteq \exists r.K'$, which is unaffected by our modification of the rules. ❑

We can now establish the main property that is satisfied by the finite model $\mathcal{I}$ just constructed, but not by the finite modal $\mathcal{I}$ built in Section 3.

Lemma 30 *$\mathcal{I} \preceq_{2n_0} \mathcal{U}$.*

Proof. By Lemma 28, it suffices to show that $\mathcal{I} \preceq_{2n_0} \mathcal{I}_\omega^+$. We call an edge $(d, e) \in r^{\mathcal{I}}$ *special* if it was added in the construction of $\mathcal{I}$ from $\mathcal{I}_\omega$, that is, by applying $(\mathbf{c3'})$. The *source* of the special edge $(d, e) \in r^{\mathcal{I}}$ is the element from $\{d, e\}$ that plays the role of d in the formulation of $(\mathbf{c3'})$.

Let $d^* \in \Delta^{\mathcal{I}}$. In the following, we construct a $2n_0$-bounded simulation ρ of $(\mathcal{I}, d^*)$ in $(\mathcal{I}_\omega^+, d^*)$. To assist with the construction of ρ, we associate with every tuple (d, i, e) in the partially constructed ρ an i-bounded simulation $\rho_{d,i,e}$ of $(\mathcal{I}_\omega^+, d)$ in $(\mathcal{I}_\omega^+, e)$ whose purpose is to guide the further construction.

We start with setting $\rho = \{(d^*, 2n_0, d^*)\}$. As the required $2n_0$-bounded simulation $\rho_{d^*,2n_0,d^*}$ of $(\mathcal{I}_\omega^+, d^*)$ in $(\mathcal{I}_\omega^+, d^*)$, we use the identity, that is, the set of all triples (d, i, e) with $d \in \Delta^{\mathcal{I}_\omega^+}$ and $i \leq 2n_0$. To extend the initial ρ just defined, we distingush three cases.

Assume that $(d, i, e) \in \rho$ with $i > 0$ and $(d, d') \in r^{\mathcal{I}}$ is non-special. Then $(d, d') \in r^{\mathcal{I}_\omega} \subseteq r^{\mathcal{I}_\omega^+}$ and thus we find a triple $(d', i-1, e') \in \rho_{d,i,e}$ with $(e, e') \in r^{\mathcal{I}_\omega^+}$. Add $(e, i-1, e')$ to ρ and set $\rho_{d',i-1,e'} = \rho_{d,i,e}$.

Now assume that $(d, i, e) \in \rho$ with $i > 0$, $(d, d') \in r^{\mathcal{I}}$ is special, and d is the source of this edge. Then there is a $d'' \in \Delta^{\mathcal{I}_\omega^+}$ such that $(d, d'') \in r^{\mathcal{I}_\omega^+}$ and $(\mathcal{I}_\omega^+, d'')$ has the same $2n_0$-simulation type as $(\mathcal{I}_\omega, d')$. Then $(\mathcal{I}_\omega^+, d'')$ must also have the same $2n_0$-simulation type as $(\mathcal{I}_\omega^+, d')$. We can thus find an $i-1$-bounded simulation ν of $(\mathcal{I}_\omega^+, d')$ in $(\mathcal{I}_\omega^+, d'')$. Since $(d, i, e) \in \rho_{d,i,e}$, there must be an $e'' \in \Delta^{\mathcal{I}_\omega^+}$ with $(d'', i-1, e'') \in \rho_{d,i,e}$ and $(e, e'') \in r^{\mathcal{I}_\omega^+}$. We add $(d', i-1, e'')$ to ρ. The required $i-1$-bounded simulation $\rho_{d',i-1,e''}$ of $(\mathcal{I}_\omega^+, d')$ in $(\mathcal{I}_\omega^+, e'')$ is obtained by composing ν with $\rho_{d,i,e}$.

Finally assume that $(d, i, e) \in \rho$ with $i > 0$, $(d, d') \in r^{\mathcal{I}}$ is special, and e is the source of this edge. Then there is a $\widehat{d} \in \Delta^{\mathcal{I}_\omega^+}$ such that $(\widehat{d}, d') \in r^{\mathcal{I}_\omega^+}$ and $(\mathcal{I}_\omega^+, \widehat{d})$ has the same $2n_0$-simulation type as $(\mathcal{I}_\omega, d)$. Then $(\mathcal{I}_\omega^+, \widehat{d})$ must also have the same $2n_0$-simulation type as $(\mathcal{I}_\omega^+, d)$. We can thus find an i-bounded simulation ν of $(\mathcal{I}_\omega^+, \widehat{d})$ in $(\mathcal{I}_\omega^+, d)$. Composing ν with $\rho_{d,i,e}$, we find an i-bounded simulation η of $(\mathcal{I}_\omega^+, \widehat{d})$ in $(\mathcal{I}_\omega^+, e)$. Since $(\widehat{d}, d') \in r^{\mathcal{I}_\omega^+}$, there must be some $\widehat{e} \in \Delta^{\mathcal{I}_\omega^+}$ such that $(d', i-1, \widehat{e}) \in \eta$ and $(e, \widehat{e}) \in r^{\mathcal{I}_\omega^+}$. Add $(d', i-1, \widehat{e})$ to ρ. The required $i-1$-bounded simulation $\rho_{d',i-1,\widehat{e}}$ of $(\mathcal{I}_\omega^+, d')$ in $(\mathcal{I}_\omega^+, \widehat{e})$ is provided by η. ❑

Proof of Proposition 25: Products

Recall that we are looking for a finite model of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$ such that every n_0-neighborhood of this model homomorphically embeds into $\mathcal{U}$. The model $\mathcal{I}$ from the previous section is still not as required since it may contain cycles whose existence is not entailed by $\mathcal{A}, \mathcal{T}_{\mathsf{f}}$. While such cycles cannot be completely avoided, they can be made large enough so as to avoid spurious matches of q. To achieve this, we take the product of $\mathcal{I}$ with a finite group of high girth, see (Otto 2004). We start with recalling some basic notions of group theory.

Let $(G, \circ)$ be a finite group generated by a (finite) set $\{g_i \mid i \geq 0\}$ of *involutive* generators, i.e., $g_i = g_i^{-1}$. The *Cayley graph* of G is the undirected graph that has as vertices the group elements $h \in G$ and where $\{h, h'\}$ is an edge if $h' = h \circ g_i$.

We are interested in groups whose Cayley graph satisfies two properties. First, it should have high girth, where the *girth* of a graph is the length of a shortest cycle contained in that graph. Second, for all group elements h and generators g_1, g_2, we want to have $h \circ g_1 \neq h \circ g_2$; in other words, the outdegree of every node in the Cayley graph should be exactly k, with k the number of generators. Such a graph is called *k-regular*.

Explicit constructions of k-regular graphs with girth greater than m, for any k and m, have been studied in the literature. The following is a known result, see e.g. (Alon 1995) for a full discussion of the construction.

Theorem 31 (Margulis 1982; Imrich 1984)
For every $k, m > 0$ there exists a finite group G which is generated by a set of k involutive generators, and whose Cayley graph has regular degree *k and girth at least m.*

Let $\mathcal{J}$ be an interpretation. We use $E_{\mathcal{J}}$ to denote the set of all edges of $\mathcal{J}$, that is, all sets $\{d, e\}$ such that $(d, e) \in r^{\mathcal{I}}$ for some role r. Let $(G, \circ)$ be a finite group with involutive generators g_S, $S \in E_{\mathcal{J}}$: that is, the set of edges $E_{\mathcal{J}}$ can be embedded via an injection into a set generating G. The existence of such a group G is granted by Theorem 31. We use $\mathcal{J} \otimes G$ to denote the interpretation with domain $\Delta^{\mathcal{J}} \times G$ defined as follows:

$$A^{\mathcal{J}\otimes G} = \{\langle d, h\rangle \in \Delta^{\mathcal{J}} \times G \mid d \in A^{\mathcal{J}}\}$$
$$r^{\mathcal{J}\otimes G} = \{(\langle d, h\rangle, \langle d', h \circ g_{\{d,d'\}}\rangle) \mid (d, d') \in r^{\mathcal{J}}\}.$$

Reduction to simple models. As a preliminary, we transform $\mathcal{I}$ into a simple model, in order to rule out cycles of length 1 (reflexive loops) or 2. More precisely, an interpretation $\mathcal{J}$ is called *simple* if it satisfies the following three conditions for all individuals d, e and (possibly inverse) roles r, s:

1. If $d \notin \mathsf{Ind}(\mathcal{A})$, then $(d, d) \notin r^{\mathcal{J}}$.
2. If $(d, e) \notin \mathsf{Ind}(\mathcal{A}) \times \mathsf{Ind}(\mathcal{A})$ and $(d, e) \in r^{\mathcal{J}} \cap s^{\mathcal{I}}$, then $r = s$.

The following construction shows how to transform $\mathcal{I}$ into a simple model $\mathcal{I}'$. Let $r_1, \ldots, r_R$ be the role names occurring in $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$. We take $2R + 2$ disjoint copies of $\Delta^{\mathcal{I}}$, and interpret ABox elements in the last copy and concept names in every copy the same way as in $\Delta^{\mathcal{I}}$. In the model to be constructed, r_k-edges between non-ABox elements jump over k copies (modulo $2R + 2$), and r_k-edges between ABox elements remain in the last copy if they originate there, or otherwise jump over k copies (modulo $2R+1$ this time). This way, we leave the ABox structure intact in the last copy, and break up all other cycles of length 1 and 2. More precisely,

$$\Delta^{\mathcal{I}'} = \Delta^{\mathcal{I}} \times \{0, \ldots, 2R + 1\}$$
$$A^{\mathcal{I}'} = \{\langle d, i\rangle \mid d \in A^{\mathcal{I}},\ 0 \le i \le 2R + 1\}$$
$$\begin{aligned} r_k^{\mathcal{I}'} = {} & \{(\langle d, i\rangle, \langle e, i \oplus_{2R+2} k\rangle) \mid (d, e) \in r_k^{\mathcal{I}} \setminus \mathsf{Ind}(\mathcal{A})^2\} \\ & \cup \{(\langle d, i\rangle, \langle e, i \oplus_{2R+1} k\rangle) \mid (d, e) \in r_k^{\mathcal{I}} \cap \mathsf{Ind}(\mathcal{A})^2, \\ & \qquad i \le 2R\} \\ & \cup \{(\langle d, 2R + 1\rangle, \langle e, 2R + 1\rangle) \mid (d, e) \in r_k^{\mathcal{I}} \cap \mathsf{Ind}(\mathcal{A})^2\} \end{aligned}$$

ABox individuals are interpreted in the first copy, that is, we identify a with $\langle a, 2R + 1\rangle$. Note that the last line preserves the structure of the ABox, and the preceding lines ensure simplicity (but do not generally rule out cycles of length 3). Indeed, $\mathcal{I}'$ is simple:

1. Whenever $(\langle d, i\rangle, \langle d, i\rangle) \in r_k^{\mathcal{I}'}$, the construction ensures that $i = 2R + 1$ and d is an ABox element.
2. Let $(\langle d, i\rangle, \langle e, j\rangle) \in (r^{\mathcal{I}'} \cap s^{\mathcal{I}'}) \setminus \mathsf{Ind}(\mathcal{A})^2$. We distinguish three cases.

   Both r, s are role names: $r = r_k$, $s = r_\ell$. Then the above pair has been added in the first or second line of the constructions of both $r_k^{\mathcal{I}'}$ and $r_\ell^{\mathcal{I}'}$. If it was added in the second line, then we have $j = i \oplus_{2R+1} k = i \oplus_{2R+1} \ell$ which, due to $0 < k, \ell \le R$, implies $k = \ell$ and hence $r_k = r_\ell$. The case for the first line is analogous.

   One of r, s is a role name; the other is not: $r = r_k$, $s = r_\ell^-$. As in the previous case, the above pair must have been added in the first or second line of the constructions of both role interpretations. If it was added in the second line, then we have $j = i \oplus_{2R+1} k$ and $i = j \oplus_{2R+1} \ell$. Inserting the first equation into the second, we get $i = i \oplus_{2R+1} k \oplus_{2R+1} \ell$, which is impossible because $0 < k + \ell \le 2R$. The case for the first line is analogous.

   None of r, s are role names: $r = r_k^-$, $s = r_\ell^-$. This case reduces to the first case if we swap $\langle d, i\rangle$ and $\langle e, j\rangle$.

It is an easy exercise to show that $\mathcal{I}'$ is a model of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$, and that $\mathcal{I}' \preceq \mathcal{I}$. From Lemma 30, we thus get $\mathcal{I}' \preceq_{2n_0} \mathcal{U}$.

Products of simple models. Now consider the finite model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$ constructed in the previous section. By the reduction just shown, we may assume that $\mathcal{I}$ is simple. By Theorem 31, we can take a finite group G with $|E_{\mathcal{I}}|$ involutive generators $\{g_S \mid S \in E_{\mathcal{I}}\}$, such that the Cayley graph of G has girth higher than $2n_0 + 1$ and is $|E_{\mathcal{I}}|$-regular. We then form the product $\mathcal{I} \otimes G$. This interpretation is almost as required, but does not necessarily satisfy the ABox. To fix this, we consider the interpretation $\widehat{\mathcal{J}}$ that can be obtained as follows:

- start with $\mathcal{I}^- \otimes G$, where $\mathcal{I}^-$ is obtained from $\mathcal{I}$ by removing, for each $r(a, b) \in \mathcal{A}$, the pair (a, b) from $r^{\mathcal{I}}$;
- then take an arbitrary but fixed $h_{\mathcal{A}} \in G$, for every $a \in \mathsf{Ind}(\mathcal{A})$ and identify each ABox element a with $(a, h_{\mathcal{A}})$;
- finally, for each $r(a, b) \in \mathcal{A}$, add $(\langle a, h\rangle, \langle b, h\rangle)$ to $r^{\widehat{\mathcal{J}}}$, for every pair $\langle a, h\rangle, \langle b, h\rangle \in \Delta^{\mathcal{J}}$.

Note that all copies of the ABox in $\widehat{\mathcal{J}}$, not just the 'main' one identified by $h_{\mathcal{A}}$, inherit the relational structure of the ABox. We first observe that $\widehat{\mathcal{J}}$ is still a (finite!) model of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$. This essentially follows from the observations in (Otto 2004).

Lemma 32 *$\widehat{\mathcal{J}}$ is a model of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$.*

Proof. To show that $\widehat{\mathcal{J}}$ is a model of $\mathcal{T}_{\mathsf{f}}$, we use the fact that $\mathcal{I}$ is a model of $\mathcal{T}_{\mathsf{f}}$ and the construction of $\widehat{\mathcal{J}}$. CIs of the form $K \sqsubseteq A$, $K \sqsubseteq \bot$, $K \sqsubseteq \exists r.K'$, and $K \sqsubseteq \forall r.K'$ are easy to deal with. We thus concentrate on CIs $K \sqsubseteq (\leqslant 1\ r\ K')$. Assume that $\langle d, h\rangle \in K^{\widehat{\mathcal{I}}}$. By construction of $\widehat{\mathcal{J}}$, this means $d \in K^{\mathcal{I}}$. Assume to the contrary of what is to be shown that there are $(\langle d, h\rangle, \langle e_i, h_i\rangle) \in r^{\widehat{\mathcal{J}}}$ for $i = 1, 2$ such that $\langle e_i, h_i\rangle \in K'^{\widehat{\mathcal{J}}}$ and $\langle e_1, h_1\rangle \neq \langle e_2, h_2\rangle$. Then $(d, e_i) \in r^{\mathcal{I}}$ and, since $\mathcal{I}$ is a model of $\mathcal{T}_{\mathsf{f}}$, we obtain $e_1 = e_2 =: e$. Now the construction of $\widehat{\mathcal{J}}$ yields that, if both d, e interpret ABox elements in $\mathcal{I}$, then $h_1 = h_2 = h$, and that otherwise $h_i = h \circ g_{\{d,e\}}$ for both $i = 1, 2$. Finally, using the construction of $\widehat{\mathcal{J}}$, it is easy to observe that $\widehat{\mathcal{J}}$ is a model of $\mathcal{A}$. ❑

A *cycle in* $\widehat{\mathcal{J}}$ (of length n) is a path $p_1, r_1, \ldots, r_n, p_{n+1}$, where $p_i \in \Delta^{\widehat{\mathcal{J}}}$, and each r_i is a (possibly inverse) role such that $(p_i, p_{i+1}) \in r_i^{\widehat{\mathcal{J}}}$, and $p_1 = p_{n+1}$. Further, a cycle is *non-degenerate* if, for every $1 \leq i < n$, whenever $r_{i+1} = r_i^-$, then $p_i \neq p_{i+2}$. An element $p = \langle d, h\rangle \in \Delta^{\widehat{\mathcal{J}}}$ is an *ABox element* if there is an $a \in \mathsf{Ind}(\mathcal{A})$ with $d = a$: this definition deliberately includes all "copies" of ABox elements from $\mathcal{I}$, not just those that interpret ABox individuals. We say that $\widehat{\mathcal{J}}$ is *k-acyclic relative to $\mathcal{A}$* if every non-degenerate cycle in $\widehat{\mathcal{J}}$ of length at most k contains exclusively ABox elements.

Lemma 33 *$\widehat{\mathcal{J}}$ is $(2n_0+1)$-acyclic relative to $\mathcal{A}$.*

Proof. Let $\alpha = p_1, r_1, \ldots, r_n, p_{n+1}$ be a non-degenerate cycle in $\widehat{\mathcal{J}}$, with $p_i = \langle d_i, h_i\rangle$ for $1 \leq i \leq n$, such that α does not contain only ABox elements. Assume to the contrary of what is to be shown that $n \leq 2n_0 + 1$. From the construction of $\widehat{\mathcal{J}}$, it follows that if at least one of p_i and p_{i+1} is not an ABox element, then $h_{i+1} = h_i \circ g_i$, where $g_i = g_{\{d_i, d_{i+1}\}}$. In fact, this is immediate if r_i is a role name. If $r_i = s^-$, then $h_i = h_{i+1} \circ g_{\{d_i,d_{i+1}\}}$, which by multiplication with $g_{\{d_i,d_{i+1}\}}$ and due to the generators being involutive yields $h_i \circ g_{\{d_i,d_{i+1}\}} = h_{i+1}$ as stated above.

First assume that $n = 1$. Then p_1 is the only element on α, and thus p_1 cannot be an ABox element. We thus have $h_1 = h_1 \circ g_1$ by what was said above, but this is not possible in a group with k generators whose Cayley graph is $2n_0$-regular.

Now assume that $n = 2$. Then $(d_1, d_2) \in r_1^{\mathcal{I}}$ and $(d_2, d_1) \in r_2^{\mathcal{I}}$. Since $\mathcal{I}$ is simple, by Condition 1 of simplicity, and since at least one of d_1, d_2 is not from $\mathsf{Ind}(\mathcal{A})$, $(d_1, d_2) \in r_1^{\mathcal{I}}$ implies $d_1 \neq d_2$. By Condition 2 of simplicity, we must further have $r_1 = r_2^-$. Because α is non-degenerate, we get $p_1 \neq p_3$, in contradiction to $n = 2$.

Finally, let $n > 2$. Since α does not consist exclusively of ABox elements, there must be some p_i that is not an ABox element. We show that h_{i-1}, h_i, and h_{i+1} are all different. Consequently, α gives rise to a cycle of length between three and $2n_0 + 1$ in the Cayley graph of G (even if some of the other elements on α should coincide), which contradicts the non-existence of such cycles. We have $h_{i-1} \neq h_i$ since otherwise $h_{i-1} = h_{i-1} \circ g_{i-1}$, which is not possible due to $2n_0$-regularity of the Cayley graph of G; for the same reason, $h_i \neq h_{i+1}$. Finally, assume to the contrary of what we want to show that $h_{i-1} = h_{i+1}$. Then $h_{i-1} \circ g_{i-1} \circ g_i = h_{i-1}$. Multiplying with g_i yields $h_{i-1} \circ g_{i-1} = h_{i-1} \circ g_i$, which gives $g_{i-1} = g_i$ by k-regularity of G. Since $g_{i-1} = g_{\{d_{i-1},d_i\}}$ and $g_i = g_{\{d_i,d_{i+1}\}}$, this yields $d_{i-1} = d_{i+1}$, thus $(d_{i-1}, d_i) \in r_{i-1}^{\mathcal{I}}$ and $(d_i, d_{i-1}) \in r_i^{\mathcal{I}}$. Since $d_i \notin \mathsf{Ind}(\mathcal{A})$, this contradicts $\mathcal{I}$ being simple. ❑

Lemma 34 *For every $p_0 \in \Delta^{\widehat{\mathcal{J}}}$, $\widehat{\mathcal{J}}|_{p_0}^{n_0}$ homomorphically embeds into $\mathcal{U}$, the canonical model of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$.*

Proof. We start with making two useful observations.

Claim 1. For each $p_1 \in \Delta^{\widehat{\mathcal{J}}}$ that is not an ABox element, there is at most one simple path $p_1 r_1 p_2 \cdots p_k r_k p_{k+1}$ in $\widehat{\mathcal{J}}|_{p_0}^{n_0}$ such that $p_1, \ldots, p_k$ are not ABox elements and p_{k+1} is an ABox element.

Proof. Assume there is a $p_1 \in \Delta^{\widehat{\mathcal{J}}}$ that is not an ABox element and such that there are two simple paths in $\widehat{\mathcal{J}}|_{p_0}^{n_0}$ of the described form. Each such path $p_1 r_1 p_2 \cdots p_k r_k p_{k+1}$ gives rise to a corresponding path $d_1 r_1 d_2 \cdots d_k r_k d_{k+1}$ in $\mathcal{I}$ such that $d_1, \ldots, d_k$ are not ABox individuals, but d_{k+1} is. Note that the initial version of the modified finite interpretation $\mathcal{I}$ contains $\mathcal{U}_0$, which takes the form of the ABox $\mathcal{A}$ extended with a tree of depth $2n_0$ below each ABox individual, and that later steps in the construction of $\mathcal{I}$ only add successors to leaves in these trees. Therefore and since the length of all mentioned paths is clearly bounded by $2n_0$, both paths in $\mathcal{I}$ must be inside the same tree of $\mathcal{U}_0$. But then, since they start and end at the same element and are simple, they must be identical.[5]

Claim 2. $\widehat{\mathcal{J}}|_{p_0}^{n_0}$ is k-acyclic relative to $\mathcal{A}$, for any $k \geq 0$.

Proof. Assume that $\widehat{\mathcal{J}}|_{p_0}^{n_0}$ contains a non-degenerate cycle $p_1, r_1, \ldots, r_n, p_{n+1}$. We have to show that all p_i are ABox elements. Let $\mathcal{P}_1$ be the shortest path from p_0 to p_i in $\widehat{\mathcal{J}}|_{p_0}^{n_0}$, and $\mathcal{P}_2$ the shortest path from p_0 to p_{i+1}. By definition of $\widehat{\mathcal{J}}|_{p_0}^{n_0}$, the length of both $\mathcal{P}_1$ and $\mathcal{P}_2$ is bounded by n_0. Let p' be the element on the path $\mathcal{P}_1$ that also occurs on $\mathcal{P}_2$ and is furthest away from p_0 (such an element occurs because p_0 is on both paths). The following cycle is of length $2n_0 + 1$:

- from p' to p_i along $\mathcal{P}_1$;
- from p_i to p_{i+1} along r;
- from p_{i+1} to p' backwards along $\mathcal{P}_2$.

The cycle is non-degenerate because no element occurs twice on it. When $p_i = p_{i+1}$, then $p' = p_i = p_{i+1}$ and the above cycle degenerates to a single reflexive r-edge. Otherwise, by choice the two travelled subpaths of $\mathcal{P}_1$ and $\mathcal{P}_2$ do not share any elements (including p_i and p_{i+1}). Since $\widehat{\mathcal{J}}$ is $2n_0 + 1$-acyclic relative to $\mathcal{A}$, p_i must thus be an ABox element as desired.

Fix a $p_0 \in \Delta^{\widehat{\mathcal{J}}}$. By Lemma 30, we know that $\mathcal{I}|_{d_0}^{n_0} \preceq_{2n_0} \mathcal{U}$, witnessed by a bounded simulation ρ. We use ρ to construct the desired homomorphism η from $\widehat{\mathcal{J}}|_{p_0}^{n_0}$ to $\mathcal{U}$.

In what follows, let π be the projection on the first component of elements in $\Delta^{\widehat{\mathcal{J}}}$. Further, let Δ be the domain of $\widehat{\mathcal{J}}|_{p_0}^{n_0}$. We define a sequence of partial homomorphisms η_i, $i \geq 0$, that is, partial functions $\eta_i : \Delta \times \Delta^{\mathcal{U}}$ that satisfy Conditions 1 to 3 of homomorphisms. The desired homomorphism η is then obtained in the limit. We will make sure that all η_i satisfy the following property:

$$(\pi(p), \eta_i(p)) \in \rho \text{ for all } p \in \Delta. \qquad (*)$$

We start by defining η_0 as follows.

[5] The whole reason for inclusing $\mathcal{U}_0$ (instead of only $\mathcal{A}$) in the initial version of the finite interpretation $\mathcal{I}$ is to admit an easy proof of this claim.

- If Δ contains ABox elements, then set $\eta_0(p) = a$ for all $p = \langle a, h\rangle \in \Delta$ with $a \in \mathsf{Ind}(\mathcal{A})$.
- If Δ does not contain ABox elements, then there is some $e \in \Delta^{\mathcal{U}}$ with $(\pi(p_0), e) \in \rho$ by Lemma 30. Choose one such element e and set $\eta_0(p_0) = e$.

Clearly, η_0 satisfies $(*)$ and Condition 1 of homomorphisms. Satisfaction of Condition 2 follows from $(*)$. Finally, satisfaction of Condition 3 follows from the existence of ρ and the fact that, by definition of bounded simulations, ρ preserves all edges between ABox elements.

In the induction step, η_{i+1} is obtained from η_i by defining a value for all $p_2 \in \Delta$ such that there is some edge $(p_1, p_2) \in r^{\widehat{\mathcal{J}}}$ with $\eta_i(p_1)$ defined and $\eta_i(p_2)$ undefined. To define $\eta_{i+1}(p_2)$, we observe that $(\pi(p_1), \pi(p_2)) \in r^{\mathcal{J}}$ follows from $(p_1, p_2) \in r^{\widehat{\mathcal{J}}}$ and $(\pi(p_1), \eta_i(p_1)) \in \rho$ holds by $(*)$. Since $\pi(p_2) \in \Delta^{\mathcal{I}|_d^{n_0}}$, there must be some e such that $(\pi(p_2), e) \in \rho$ and $(\eta_i(p_1), e) \in r^{\mathcal{U}}$. Set $\eta_{i+1}(p_2) = e$.

We next show that η_{i+1} is well-defined, that is, if $(p_1, p) \in r^{\widehat{\mathcal{J}}}$ and $(p_2, p) \in s^{\widehat{\mathcal{J}}}$ with $\eta_i(p_1)$ and $\eta_i(p_2)$ defined and $\eta_i(p)$ undefined, then $(p_1, r) = (p_2, s)$. Assume to the contrary that this is not the case. We distinguish two cases:

- $p_1 = p_2$ and $r \neq s$. Then $\widehat{\mathcal{J}}$ contains the non-degenerate cycle p_1, r, p, s^-, p_1 (from the edges $r(p_1, p), s(p_2, p)$). It follows from Claim 2 that p is an ABox element, in contradiction to $\eta_i(p)$ being undefined.
- $p_1 \neq p_2$. Let an element $\widehat{p} \in \Delta$ be *initial* if $\eta_0(\widehat{p})$ is defined. Since $\eta_i(p_1)$ and $\eta_i(p_2)$ are defined and $\eta_{i+1}(p)$ is not, p_j is reachable from some initial element $\widehat{p}_j$ on a path $\mathcal{P}_j$ of length i and this is the shortest path from any initial element to p_j, for $j \in \{1, 2\}$. If Δ contains no ABox elements, then $\widehat{p}_1 = \widehat{p}_2 = p_0$. Otherwise, $\widehat{p}_1$ and $\widehat{p}_2$ are ABox elements and we obtain from Claim 1 and the fact that both $\widehat{p}_1$ and $\widehat{p}_2$ are reachable from p that $\widehat{p}_1 = \widehat{p}_2$. For readability, we from now on use $\widehat{p}$ to denote $\widehat{p}_1 (= \widehat{p}_2)$.

  Let p' be the element on the path $\mathcal{P}_1$ that also occurs on the path $\mathcal{P}_2$ and is furthest away from $\widehat{p}$ (such an element always exists since $\widehat{p}$ is on both paths). Consider the following cycle in $\widehat{\mathcal{J}}$:

  1. from p' to p_1 along $\mathcal{P}_1$;
  2. from p_1 to p along r;
  3. from p to p_2 along s^-;
  4. from p_2 to p' backwards along $\mathcal{P}_2$.

  We note that the cycle is non-degenerate because no element occurs twice on it. By choice, the two travelled subpaths of $\mathcal{P}_1$ and $\mathcal{P}_2$ do not share any elements, including p_1 and p_2. Moreover, p does not occur on these subpaths because η_i must be defined for all elements on the subpaths whereas it is not defined for p. Thus, p must be an ABox element, which again yields a contradiction.

To finish the proof, we note that it is clear that η_{i+1} satisfies $(*)$ and all three conditions of homomorphisms. ❑

D Proofs for Section 6

Proposition 17 *$\mathcal{T}$ is finitely satisfiable iff $\mathcal{T}'$ is finitely satisfiable.*

Proof. The "if" direction is trivial since every model of $\mathcal{T}'$ is also a model of $\mathcal{T}$. For the "only if" direction, let $\mathcal{I}$ be a finite model of $\mathcal{T}$. We construct a finite model $\mathcal{J}$ of $\mathcal{T}'$ by taking n copies of $\mathcal{I}$ and 'rewiring' all role edges across the concept names B_i can be interpreted in a non-conflicting way.

Specifically, since $\mathcal{I}$ satisfies $K \sqsubseteq (\geqslant n\ r\ K')$ we can choose a function $\mathsf{succ} : K^{\mathcal{I}} \times \{0, \ldots, n-1\} \to \Delta^{\mathcal{I}}$ such that the following conditions are satisfied:

- for all $d \in K^{\mathcal{I}}$ and $i < n$: $(d, \mathsf{succ}(d, i)) \in r^{\mathcal{I}}$ and $\mathsf{succ}(d, i) \in (K')^{\mathcal{I}}$;
- for all $d \in K^{\mathcal{I}}$ and $i < j < n$: $\mathsf{succ}(d, i) \neq \mathsf{succ}(d, j)$.

Then define the desired interpretation $\mathcal{J}$ by setting

$$\begin{aligned}
\Delta^{\mathcal{J}} &= \{d_i \mid d \in \Delta^{\mathcal{I}} \text{ and } i < n\}\\
E^{\mathcal{J}} &= \{d_i \mid d \in E^{\mathcal{I}} \text{ and } i < n\}\\
&\quad \text{for all } E \in \mathsf{N_C} \setminus \{B_0, \ldots, B_{n-1}\}\\
B_i^{\mathcal{J}} &= \{d_i \mid d \in \Delta^{\mathcal{I}}\} \text{ for all } i < n\\
s^{\mathcal{J}} &= \{(d_i, e_i) \mid (d, e) \in s^{\mathcal{I}} \text{ and } i < n\}\\
&\quad \text{for all } s \in \mathsf{N_R} \setminus \{r\}\\
r^{\mathcal{J}} &= \{(d_i, e_i) \mid (d, e) \in r^{\mathcal{I}},\ i < n,\\
&\qquad \text{and } d \notin K^{\mathcal{I}} \text{ or } e \neq \mathsf{succ}(d, j) \text{ for any } j\}\\
&\quad \cup \{(d_i, e_{(i+j) \bmod n}) \mid (d, e) \in r^{\mathcal{I}},\ i, j < n,\\
&\qquad \text{and } e = \mathsf{succ}(d, j)\}
\end{aligned}$$

It remains to verify that $\mathcal{J}$ is indeed a model of $\mathcal{T}'$. Clearly, the CIs in $(*)$ on page 9 are satisfied. To verify that all concept inclusions in $\mathcal{T}$ are satisfied by $\mathcal{J}$, we observe that the construction ensures that the number of r-successors (and -predecessors) in any $A \in \mathsf{CN}$ of every (x, i) is the same as that for x.

We first claim that, for every $d \in \Delta^{\mathcal{I}}$ and every s-successor e of d in $\mathcal{I}$, the i-th copy of d in $\mathcal{J}$ has exactly one copy of e as an s-successor:

Claim. *Let s be a role, $d_i \in \Delta^{\mathcal{J}}$, and let $\{e \in \Delta^{\mathcal{I}} \mid (d, e) \in s^{\mathcal{I}}\} = \{e_1, \ldots, e_\ell\}$ for some $\ell \geq 0$. Then $\{e^j \in \Delta^{\mathcal{I}} \mid (d^i, e^j) \in s^{\mathcal{I}}\} = \{e_1^{j_1}, \ldots, e_\ell^{j_\ell}\}$, for some $j_1, \ldots, j_\ell \in \{0, \ldots, n-1\}$.*

This claim is implied by the construction of $s^{\mathcal{J}}$: consider a given $d_i \in \Delta^{\mathcal{J}}$ and (possibly inverse) role s. If s is neither r nor r^-, then every e_k contributes exactly one s-successor e_k^i of d^i. The same holds if $s = r$ and $d \notin K^{\mathcal{I}}$. If $s = r$ and $d \in K^{\mathcal{I}}$, then each $e_k = \mathsf{succ}(d, j)$ for some j contributes exactly one s-successor $e_k^{(i+j) \bmod n}$ of d^i, and every other e_k contributes e_k^i. For $s = r^-$, then every $e_k \in K^{\mathcal{I}}$ with $d = \mathsf{succ}(e_k, j)$ for some j contributes $e_k^{(i-j) \bmod n}$, and every other e_k contributes e_k^i.

As an immediate consequence, we obtain that all qualified and unqualified number restrictions in $d \in \Delta^{\mathcal{I}}$ are preserved in every $d^i \in \Delta^{\mathcal{J}}$:

Fact. *Let $d^i \in \Delta^{\mathcal{J}}$ and $D = (\bowtie s\ n\ C)$ where $\bowtie \in \{\leqslant, \geqslant\}$, s is a role or inverse role, and C is either a conjunction of concept names, or the negation of such a conjunction, or $\top$, or $\bot$. Then $d \in D^{\mathcal{I}}$ iff $d^i \in D^{\mathcal{J}}$.*

This can be concluded from the previous claim and the observation that e and e^{j_i} satisfy the same concept names. The fact includes the cases $s = r$ and $s = r^-$, and it implies that existential, and universal restrictions are preserved – for the latter it is necessary to allow that C is a negated conjunction.

We are now ready to prove that $\mathcal{J}$ is a model of $\mathcal{T}'$, proceeding by type of CI. We distinguish the following cases.

- $L \sqsubseteq A$ and $L \sqsubseteq \bot$, both in $\mathcal{T}$. These are satisfied because they are satisfied by $\mathcal{I}$ and due to the construction: every d in $\mathcal{I}$ and every d^i in $\mathcal{J}$ are instances of the same non-B_i concept names.
- $L \sqsubseteq \exists s.L'$ in $\mathcal{T}$. Let $d^i \in L^{\mathcal{J}}$. Then $d \in L^{\mathcal{I}}$ due to the construction. Since $\mathcal{I}$ satisfies the axiom, $d \in (\geqslant 1\ s\ L')^{\mathcal{I}}$. With the previous fact, we conclude $d^i \in (\geqslant 1\ s\ L')^{\mathcal{J}}$, hence $d^i \in (\exists s.L')^{\mathcal{J}}$. This argument includes the cases $s = r$ and $s = r^-$.
- $L \sqsubseteq \forall s.L'$ in $\mathcal{T}$. In the argument above, replace "$\in (\geqslant 1\ s\ L')^{\cdots}$" with "$\notin (\geqslant 1\ s\ \neg L')^{\cdots}$".
- $L \sqsubseteq (\leqslant 1\ s\ L')$ in $\mathcal{T}$. Then $d^i \in L^{\mathcal{J}}$ implies $d \in L^{\mathcal{I}}$, hence $d \in (\leqslant 1\ s\ L')^{\mathcal{I}}$ and, due to the previous fact, $d^i \in (\leqslant 1\ s\ L')^{\mathcal{I}}$.
- $L \sqsubseteq (\geqslant m\ s\ L')$ in $\mathcal{T}$. Apply the same argument as above.
- $B_i \sqsubseteq K'$ and $B_i \sqcap B_j \sqsubseteq \bot$. Follows from the construction.
- $K \sqsubseteq \exists r.B_i$. Let $d^j \in K^{\mathcal{J}}$, which implies $d \in K^{\mathcal{I}}$.
  Let $e = \mathsf{succ}(d, (i-j) \bmod n)$. Then the construction yields that $(d^j, e^i) \in r^{\mathcal{J}}$ — because $i = (j + (i-j) \bmod n) \bmod n$ — and $e^i \in B_i^{\mathcal{J}}$. Hence $d^j \in (\exists r.B_i)^{\mathcal{J}}$.

❏