some homework assignment...

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Problem 1. Consider the flat Minkowski space, initially described in Cartesian coordinates:

$$ds^2 = -dt^2 + (d\mathbf{x})^2.$$

Now consider the transformation to spherical coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta \end{aligned}$$

Define $(x')^1 = r$, $(x')^2 = \theta$, and $(x')^3 = \phi$ (t is unchanged).

(a) Compute the matrix of the derivatives $\partial x^\mu / \partial x'^\nu$, where x^μ are the old coordinates, and x'^ν are the new ones.

(b) Using the transformation law for the metric tensor $g_{\mu\nu}$, find $g'_{\mu\nu}$ (the metric in the new coordinates) and write down the new expression for the interval.

Solution. We want to compute the Jacobian

$$\frac{\partial(x^0, \ldots, x^3)}{\partial(x'^0, \ldots, x'^3)} = \frac{\partial(t, x, y, z)}{\partial(t, r, \theta, \phi)} = \begin{pmatrix} \dfrac{\partial t}{\partial t} & \cdots & \dfrac{\partial t}{\partial \phi} \\ \vdots & & \vdots \\ \dfrac{\partial z}{\partial t} & \cdots & \dfrac{\partial z}{\partial \phi} \end{pmatrix},$$

and so we need to compute the partials of t, x, y, and z with respect to t, r, θ, and ϕ; the only nonzero partial of t is equal to 1. The rest are

$$
\begin{gathered}
\frac{\partial x}{\partial t} = 0, \quad \frac{\partial x}{\partial r} = \sin\theta\cos\phi, \quad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \quad \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \\
\frac{\partial y}{\partial t} = 0, \quad \frac{\partial y}{\partial r} = \sin\theta\sin\phi, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \quad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \\
\frac{\partial z}{\partial t} = 0, \quad \frac{\partial z}{\partial r} = \cos\theta, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta, \quad \frac{\partial z}{\partial \phi} = 0,
\end{gathered}
$$

and so the Jacobian is

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\
0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\
0 & \cos\theta & -r\sin\theta & 0
\end{pmatrix},
$$

this is part (a).

The transformation law for the metric tensor is

$$
\begin{aligned}
g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\
&= \frac{\partial x^0}{\partial x'^\mu}\frac{\partial x^0}{\partial x'^\nu} g_{00} + \frac{\partial x^1}{\partial x'^\mu}\frac{\partial x^1}{\partial x'^\nu} g_{11} + \frac{\partial x^2}{\partial x'^\mu}\frac{\partial x^2}{\partial x'^\nu} g_{22} + \frac{\partial x^3}{\partial x'^\mu}\frac{\partial x^3}{\partial x'^\nu} g_{33} \\
&= -\frac{\partial t}{\partial x'^\mu}\frac{\partial t}{\partial x'^\nu} + \frac{\partial x}{\partial x'^\mu}\frac{\partial x}{\partial x'^\nu} + \frac{\partial y}{\partial x'^\mu}\frac{\partial y}{\partial x'^\nu} + \frac{\partial z}{\partial x'^\mu}\frac{\partial z}{\partial x'^\nu},
\end{aligned}
$$

where the second equality is due to $g_{\mu\nu}$ being diagonal. Note that

$$
g'_{00} = \left(\frac{\partial t}{\partial t}\right)^2 g_{00} = -1, \quad \text{and} \quad g'_{\mu 0} = g'_{0\nu} = 0, \quad \text{for } \mu, \nu = 1, 2, 3.
$$

The last of the off-diagonal entries in the new matrix are symmetric as well, i.e., $g'_{12} = g'_{21}$, and so on. So we have

$$
\begin{aligned}
g'_{12} = g'_{21} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \\
&= r\sin\theta\cos\theta\cos^2\phi + r\sin\theta\cos\theta\sin^2\phi - r\sin\theta\cos\theta = 0, \\
g'_{13} = g'_{31} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \phi} \\
&= -r\sin^2\theta\sin\phi\cos\phi + r\sin^2\theta\sin\phi\cos\phi = 0, \\
g'_{23} = g'_{32} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \phi} \\
&= -r^2\sin\theta\cos\theta\sin\phi\cos\phi + r^2\sin\theta\cos\theta\sin\phi\cos\phi = 0.
\end{aligned}
$$

For the diagonal elements of the new metric, we have

$$
\begin{aligned}
g'_{11} &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1, \\
g'_{22} &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta \\
&= r^2, \\
g'_{33} &= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta.
\end{aligned}
$$

Altogether, the transformed metric $g'_{\mu\nu}$ in matrix form is

$$
g'_{\mu\nu} = \mathrm{diag}(-1, 1, r^2, r^2\sin^2\theta),
$$

and the transformed interval is

$$
ds^2 = g'_{\mu\nu}dx'^{\mu}dx'^{\nu} = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.
$$

This is (b). □

Problem 2. Clocks slow down for moving objects (as predicted by special relativity) but also for objects sitting in gravitational wells. In this problem, you will need to take into account both effects. Suppose we describe the gravitational field of the Earth by the Schwarzschild metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where $f(r) = 1 - r_0/r$ and $r_0 = 8.87$ mm. Consider a GPS satellite in a circular orbit at radial coordinate $r = R_1 = 26,600$ km with period of 11 hours and 58 minutes.

(a) Compute the satellite's proper time interval $d\tau_1$ in terms of the coordinate time interval dt.

(b) Same for the interval $d\tau_2$ measured by a clock at the Earth's North Pole ($r = R_2 = 6400$ km). What is the percent difference between $d\tau_1$ and $d\tau_2$?

Solution. The proper time in the Schwarzschild metric is given by

$$d\tau^2 = -ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.1}$$

If we take the square root of (2.1), this is

$$\begin{aligned} d\tau &= \sqrt{f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2)} \\ &= \sqrt{f(r) - r^2\left(\frac{d\theta}{dt}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2}\, dt, \end{aligned}$$

where the term with $(dr/dt)^2$ vanishes if we take r to be fixed. We may orient our coordinate system so that the satellite observer has a fixed θ component, say, $\theta = \pi/2$; in this case, $(d\theta/dt)^2$ vanishes as well. We may find $d\phi/dt$ by taking the length of the interval over which ϕ takes values and dividing it by the orbital period of the satellite:

$$\frac{d\phi}{dt} = \frac{2\pi}{43,080}\frac{\text{radians}}{s} \approx 1.45849 \times 10^{-4}\frac{\text{radians}}{s}.$$

Now we may write down the proper time interval $d\tau_1$ for the satellite[1]

[1] There may be an issue with my value, although I am unsure. Any help would be appreciated. In order to get a positive value under the square root, I divided R_1^2 by c^2, where c is not normalized to 1.

$$d\tau_1 = \sqrt{f(R_1) - R_1^2 \sin^2 \frac{\pi}{2} \left(\frac{d\phi}{dt}\right)^2} dt \approx (1 - 2.5046 \times 10^{-10}) dt.$$

The observer at the North Pole is commoving, since it has a fixed coordinate $(R_2, 0, 0)$, and thus has the proper time

$$d\tau_2 = \sqrt{1 - \frac{r_0}{R_2}} dt \approx (1 - 6.9297 \times 10^{-10}) dt.$$

The ratio of $d\tau_1$ to $d\tau_2$ is then

$$\frac{|d\tau_1 - d\tau_2|}{d\tau_1} \times 100 \approx (docomputation),$$

a percentage difference of $8.3618 \times 10^{-9}\%$. $\square$