Mean-variance Hedging with Neural Networks using Vector AutoRegression Models as the underlying Sample Path Generator

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Abstract

This thesis presents a novel approach to mean-variance hedging by integrating neural networks with Vector AutoRegression (VAR) models as the underlying sample path generator. Mean-variance hedging is a fundamental technique in financial risk management, aiming to minimize the variance of the hedging error while considering expected returns in incomplete markets. Traditional hedging methods often rely on simplistic assumptions of market dynamics, which may not capture the complex interdependencies between financial assets.

To address these limitations, VAR models to generate realistic sample paths that reflect the multivariate time series dynamics of financial assets will be employed. These sample paths capture the temporal correlations and cross-dependencies essential for accurate hedging. Neural networks are then utilized to learn and predict optimal hedging strategies based on these generated paths. The combination of VAR models and neural networks allows for a more flexible and accurate modeling of market behaviors.

Extensive simulations and empirical analyses are conducted to evaluate the performance of the proposed method. The results demonstrate that the approach outperforms traditional mean-variance hedging techniques, achieving lower hedging errors and better risk-return trade-offs. This research contributes to the field by providing a more robust framework for hedging in complex financial markets, leveraging advanced machine learning and econometric models.

Keywords: Mean-Variance Hedging, Neural Networks, Vector AutoRegression Models, Sample Path Generation, Financial Risk Management, Incomplete Markets, Hedging Strategies, Multivariate Time Series Analysis.

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Chapter 1

1 Introduction

1.1 Problem Formulation

In the modern financial landscape, institutions such as banks and insurance companies frequently encounter the challenge of fulfilling future liabilities, represented by a random variable H, at a specified future time $T > 0$. These liabilities may arise from various financial contracts, including options or insurance policies, which obligate the institution to make payments contingent upon certain events or asset price movements. For example, an insurance company may issue a financial derivative requiring it to pay the positive difference between the prices of two assets at maturity. This scenario is prevalent in markets with spread options, where the payoff depends on the price differential between two underlying assets.

The primary objective is to construct a self-financing trading strategy that enables the institution, from the seller's perspective, to hedge against such liabilities effectively. Specifically, the aim is to develop a portfolio of assets whose value at time T closely replicates the payoff $H = (S_T^1 - S_T^2)^+$, where S_T^1 and S_T^2 denote the prices of two risky assets at maturity. This payoff structure is characteristic of a European call option on the spread between two assets commonly utilized in commodities and energy markets. For instance, an airline company might employ spread options to hedge against fluctuations in jet fuel prices relative to crude oil prices.

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where Ω represents the sample space, $\mathcal{F}$ is the sigma-algebra of events, $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the filtration capturing the evolution of information over time, and $\mathbb{P}$ is the risk-neutral probability measure. Let $X = \{(X_t^1, \ldots, X_t^d) : t \geq 0\}$ denote a d-dimensional stochastic process, where each X_t^i represents the discounted price of the i-th risky asset at time t. The process X is adapted to the filtration $\mathbb{F}$, ensuring that asset prices are measurable with respect to the available information up to time t.

In a complete market, it is theoretically feasible to construct a dynamic trading strategy that perfectly replicates the payoff H by continuously trading in the underlying assets. However, real-world markets are often incomplete due to factors such as transaction costs, trading constraints, and the inability to trade certain assets continuously. This market incompleteness renders perfect replication infeasible, necessitating the pursuit of alternative strategies that approximate the desired payoff as closely as possible.

The objective is to identify an optimal trading strategy φ that minimizes the expected squared replication error between the terminal portfolio value $V_T(\varphi)$ and the liability H. Mathematically, this optimization problem is formulated as:

$$\min_{\varphi} \mathbb{E}\left[(V_T(\varphi) - H)^2\right].$$

Here, $V_T(\varphi)$ denotes the terminal value of a self-financing portfolio initiated with initial capital v_0 and following the trading strategy φ. The trading strategy φ comprises dynamic positions in the available assets X, adjusted at discrete time intervals based on the information up to that time.

To model the dynamics of asset prices and capture their interdependencies, a Vector Autoregressive (VAR) model is employed. The VAR model is well-suited for representing multiple time series that influence each other, making it appropriate for assets with correlated price movements. By fitting a VAR model to historical data, realistic simulated paths of asset prices can be generated to train the hedging strategy.

Designing an optimal hedging strategy in an incomplete market presents significant challenges due to the high dimensionality and non-linearity of the problem. To address these challenges, the approximation capabilities of neural networks are leveraged to learn the optimal trading strategy from simulated data. Neural networks, as powerful function approximators, are capable of capturing complex patterns and relationships in data, rendering them suitable for modeling the non-linear dependencies inherent in financial markets.

In this approach, the trading strategy φ^θ is parameterized using a neural network with parameters θ. At each time step n, it is observed that for any trading strategy φ, the process Y defined by $Y_n = \varphi_{n+1}$ is adapted to $\mathbb{F}^X$. Consequently, it can be approximated by Y^θ, where

$$Y_n^\theta = f_\theta^{NN}(X_n, n), \quad n \geq 0.$$

Thus, any trading strategy φ is approximated by φ^θ defined as

$$\varphi_{n+1}^\theta = f_\theta^{NN}(X_n, n), \quad n \geq 0.$$

By training the neural network to minimize the expected squared replication error, the parameter set $\hat{\theta}$ that yields the best approximation of the optimal trading strategy is determined:

$$\hat{\theta} = \arg\min_\theta \mathbb{E}\left[\left(V_T^\theta - H\right)^2\right],$$

where V_T^θ represents the terminal portfolio value resulting from following the strategy φ^θ.

From the seller's perspective, this hedging strategy is essential for mitigating the risk associated with the liability H. In practical scenarios, companies issuing options or insurance products require robust strategies to ensure their ability to meet obligations under various market conditions. By minimizing the expected replication error, the institution can reduce the potential for substantial losses due to unfavorable asset price movements.

1.2 Literature Review

The field of financial risk management has long grappled with the challenge of constructing optimal hedging strategies to mitigate the risks associated with derivative securities. Mean-variance hedging, introduced by Föllmer and Sondermann [11], provides a framework for minimizing the expected squared difference between the terminal wealth of a hedging portfolio and the payoff of a contingent claim. This approach is particularly relevant in incomplete markets, where perfect replication of payoffs is unattainable due to the absence of certain financial instruments or trading constraints.

Traditional methods for hedging derivatives often rely on the Black-Scholes model [2], which assumes continuous trading and log-normally distributed asset prices. While the Black-Scholes model offers a closed-form solution for option pricing and hedging in complete markets, its assumptions are frequently violated in real-world settings. Empirical studies have demonstrated that asset returns exhibit characteristics such as heavy tails and volatility clustering, which are not captured by the geometric Brownian motion underlying the Black-Scholes framework [34]. Consequently, hedging strategies based on the Black-Scholes model may result in significant hedging errors when applied to actual market data.

To address the limitations of traditional models, recent research has explored the application of machine learning techniques, particularly neural networks, in financial modeling and risk management. Neural networks possess the universal approximation capability [13], enabling them to model complex, nonlinear relationships inherent in financial time series data. Hutchinson et al. [35] pioneered the use of neural networks for option pricing, demonstrating their ability to approximate the Black-Scholes formula and capture market anomalies.

In the context of mean-variance hedging, neural networks offer a flexible framework for approximating optimal trading strategies without relying on restrictive assumptions about market dynamics. By training a neural network to minimize the expected squared hedging error, one can obtain a trading strategy that adapts to the statistical properties of the underlying assets. Malliaris and Salchenberger [36] highlighted the potential of neural networks in option pricing and hedging, emphasizing their capacity to handle nonlinearities and interactions among variables.

The incorporation of stochastic models, such as Vector Autoregression (VAR), further enhances the modeling of asset price dynamics by capturing the linear interdependencies among multiple time series. Sims [32] introduced VAR models as a means to analyze economic time series without imposing strong a priori restrictions. In financial applications, VAR models have been utilized to simulate realistic price paths for assets, accounting for factors such as autocorrelation and cross-correlation among asset returns [15].

Combining VAR models with neural networks allows for the generation of extensive and realistic datasets necessary for training robust hedging strategies. This approach addresses the data limitations often encountered in financial modeling, as neural networks require large amounts of data to generalize effectively. Zhang and Xu [37] demonstrated that neural networks trained on simulated data could effectively approximate optimal hedging strategies in incomplete markets.

Regularization techniques play a crucial role in preventing overfitting in neural network models, ensuring that the learned strategies generalize well to unseen data. Techniques such as L1 and L2 regularization introduce penalty terms to the loss function, discouraging the model from fitting noise in the training data [27]. This is particularly important in financial applications, where overfitting can lead to poor performance in live trading environments.

The use of advanced optimization algorithms, such as the Adam optimizer [14], further enhances the training process of neural networks by adapting the learning rate during training. Adaptive optimization methods have been shown to improve convergence speed and model performance, especially when dealing with noisy and sparse gradients common in financial data.

Empirical studies comparing neural network-based hedging strategies with traditional approaches have yielded promising results. For instance, Ruf and Wang [38] demonstrated that deep reinforcement learning techniques could outperform Black-Scholes delta hedging in certain market conditions. Their findings suggest that neural networks can capture complex market behaviors and provide more effective hedging strategies in practice.

1.3 Structure of the thesis

This thesis is organized into five chapters, each contributing to the comprehensive exploration of mean-variance hedging strategies using Feedforward Neural Networks (FNNs) within discrete-time financial models. The first chapter, Introduction, establishes the groundwork by outlining the problem formulation, reviewing pertinent literature, and providing an overview of the project structure. This chapter sets the stage for subsequent analyses by highlighting the significance of effective hedging strategies in incomplete markets and the potential of neural networks to enhance traditional approaches.

Chapter two, Trading in Continuous Time, delves into the foundational aspects of financial trading in a continuous-time framework. It begins with preliminary definitions and introduces stochastic calculus, essential for modeling asset price dynamics. The chapter then transitions to an in-depth discussion of the Black-Scholes model, covering its pricing mechanisms and hedging strategies. Additionally, it introduces Vector Autoregression (VAR) models, which are instrumental in simulating realistic asset price paths by capturing the linear interdependencies among multiple financial time series.

In the third chapter, Mean-Variance Hedging with Feedforward Neural Networks in Discrete-Time Models, the core objective of the thesis will be addressed: developing and optimizing hedging strategies using neural networks. This chapter starts with a precise problem formulation, defining the mean-variance hedging framework in discrete time. It then outlines the neural network architecture employed, detailing the backpropagation and optimization techniques used to train the network. Furthermore, the chapter explores the integration of reinforcement learning principles to enhance the adaptability and performance of the hedging strategy in dynamic market conditions.

Chapter four, Results, presents the empirical findings of the study. It is divided into three main sections: analysis of the VAR model, evaluation of the Black-Scholes hedging strategy, and assessment of the neural network-based hedging strategy. Using simulated data generated by the VAR model, the performance of the traditional Black-Scholes approach with the FNN-based strategy is compared. The results highlight the advantages of neural networks in reducing the variance of hedging errors and providing more consistent portfolio performance across various market scenarios.

The final chapter, Conclusions and Extensions, synthesizes the key insights gained from the research. It summarizes the effectiveness of neural network-based hedging strategies compared to traditional methods and discusses their implications for financial risk management. Additionally, the chapter outlines potential extensions for future research, such as exploring more complex neural network architectures, incorporating additional financial indicators, and applying the developed framework to a broader range of derivative products. These extensions aim to further enhance the robustness and applicability of machine learning techniques in financial hedging and risk mitigation.

Chapter 2

2 Trading in Continuous Time

2.1 Preliminary Definitions

Trading in continuous time is highly dependent on an understanding of stochastic calculus and Brownian motion. Therefore, the structure of this section is derived from [18], [19], [20], and [30].

2.1.1 Stochastic Calculus

Consider a time horizon $I \subseteq \mathbb{R}^+$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ denote a filtered probability space, comprising a sample space Ω of possible outcomes, a sigma-algebra $\mathcal{F}$ of measurable subsets of Ω, a filtration $\mathbb{F} = \{\mathcal{F}_t : t \in I\}$ consisting of increasing sub-sigma-algebras of $\mathcal{F}$ that represent the information available up to time t, and a probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$ that assigns probabilities to events within $\mathcal{F}$. It is assumed for the remainder of this thesis that $\mathcal{F}_0$ is trivial, i.e., $\mathcal{F}_0 = \{\emptyset, \Omega\}$, indicating no prior information at time zero.

Definition 2.1.1.0. For $f \in m\mathcal{F}$, the positive part of f is defined as $f^+ := \max\{f, 0\}$, and the negative part as $f^- := \max\{-f, 0\}$. It follows that both f^+ and f^- belong to $m\mathcal{F}^+$, and the relations $f = f^+ - f^-$ and $|f| = f^+ + f^-$ hold.

Furthermore, for $f \in m\mathcal{F}$, the integral of f with respect to μ is defined as

$$\int_\Omega f \, d\mu := \int_\Omega f^+ \, d\mu - \int_\Omega f^- \, d\mu,$$

provided that at least one of the integrals on the right-hand side is finite. In such cases, the integral is said to exist; otherwise, it does not. The function f is termed *integrable* if both integrals are finite, that is,

$$\int_\Omega |f| \, d\mu < \infty.$$

The space $\mathcal{L}^1 := \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ denotes the set of all integrable functions. More generally, for $p \in [1, \infty)$, the space

$$\mathcal{L}^p := \mathcal{L}^p(\Omega, \mathcal{F}, \mu) := \left\{ f \in m\mathcal{F} : \int_\Omega |f|^p \, d\mu < \infty \right\}$$

is defined.

□

Definition 2.1.1.1. Let $X : \Omega \to \mathbb{R}$ be a function and let $\mathcal{B}(\mathbb{R})$ denote the Borel sigma-algebra on $\mathbb{R}$. The function X is called a random variable if $X^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{B}(\mathbb{R})$. This measurability condition is denoted by $X \in m\mathcal{F}$.

A stochastic process is an indexed collection of random variables $\{X_t : t \in I\}$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process is said to be adapted if X_t is $\mathcal{F}_t$-measurable, that is, $X_t \in m\mathcal{F}_t$ for all $t \in I$.

Definition 2.1.1.2. The expectation of a random variable X, denoted by $\mathbb{E}[X]$, is defined as the integral

$$\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}.$$

The conditional expectation of X given a sub-sigma-algebra $\mathcal{G} \subseteq \mathcal{F}$, denoted by $\mathbb{E}[X|\mathcal{G}]$, is the almost surely unique $\mathcal{G}$-measurable random variable satisfying the partial averaging property

$$\int_G X \, d\mathbb{P} = \int_G \mathbb{E}[X|\mathcal{G}] \, d\mathbb{P}$$

for all $G \in \mathcal{G}$.

Definition 2.1.1.3. (Brownian Motion) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process $W := \{W_t : t \geq 0\}$ is called a Brownian motion or standard Brownian motion if it satisfies the following properties:

1. $W_0 = 0$ almost surely.
2. W possesses continuous sample paths.
3. W has independent increments; that is, for any sequence $0 \leq t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the random variables $W_{t_1} - W_{t_0}, W_{t_3} - W_{t_2}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent.
4. W has normally distributed, stationary increments: $W_t - W_s \sim N(0, t-s)$ for all $s < t$.

If $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space and $\mathbb{F}_t^W \subseteq \mathbb{F}_t$ for each $t \geq 0$, then $W = \{W_t : t \geq 0\}$ is referred to as a Brownian motion with respect to $\mathbb{F}$ if it satisfies conditions 1 through 4 and the additional condition:

5. For all $0 \leq s < t$, the increment $W_t - W_s$ is independent of $\mathbb{F}_s$.

The following theorem is stated without proof.

Theorem 2.1.1.1 (Wiener). *Brownian motion exists.* Specifically, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic process W such that W is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Several proofs of this theorem exist, with a common approach constructing Brownian motion as the limit of a sequence of simple random walks as the step sizes approach zero.

□

Proposition 2.1.1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $W = \{W_t : t \geq 0\}$ be a Brownian motion. For any $s, t \in (0, \infty)$, the following properties hold:

1. $W_t \sim N(0, t)$.
2. $\mathrm{Cov}(W_t, W_s) = t \wedge s$.
3. W_s and W_t have a bivariate normal distribution.

Definition 2.1.1.4. A stochastic process $X = \{X_t : t \geq 0\}$ is termed a *martingale* if X is $\mathcal{F}$-adapted, integrable, and for all $s \leq t$,
$$\mathbb{E}(X_t | \mathcal{F}_s) = X_s.$$

For a submartingale or supermartingale, the equality is replaced by $\geq$ or $\leq$ inequalities, respectively. It is well-established that Brownian motion constitutes a martingale.

□

Definition 2.1.1.5. The *stochastic integral* of a process φ with respect to X is defined as

$$(\varphi \bullet X)_t := \int_0^t \varphi_s \, dX_s, \quad t \geq 0.$$

This integral is constructed as the limit in probability of sums of the form

$$\sum_{i=0}^{n-1} \varphi_{t_i} (X_{t_{i+1}} - X_{t_i}),$$

where $0 = t_0 < t_1 < \cdots < t_n = t$ constitutes a partition of $[0, t]$ with the mesh size $||\varphi|| \to 0$, and $||\varphi|| = \max_{1 \le i \le n}(t_i - t_{i-1})$.

Definition 2.1.1.6. An m-dimensional Brownian motion $W = (W^1, \ldots, W^m)$ is a process where each W^i is a Brownian motion, and the covariation satisfies $[W^i, W^j]_t = \rho_{ij} t$ for every t, where ρ_{ij} represents the correlation between W^i and W^j.

Similarly, a d-dimensional Itô process $X = (X^1, \ldots, X^d)$ is defined such that for each $i = 1, \ldots, d$,

$$dX_t^i = \mu_t^i \, dt + \sum_{j=1}^{m} \sigma_t^{ij} \, dW_t^j,$$

for certain processes μ^i and σ^{ij}. In matrix or vector notation, this is expressed as

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t,$$

where $\mu = (\mu^1, \mu^2, \ldots, \mu^d)$ and $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^d)$. The process μ is referred to as the *drift term*, and σ as the *diffusion term* of X. Notably, if $\sigma \in \mathcal{L}^2(W)$, then X is a martingale if and only if $\mu \equiv 0$, that is, if and only if X is driftless.

Definition 2.1.1.7. For an Itô process X with differential

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t,$$

the integral of a stochastic process φ with respect to X is defined as

$$\int_0^t \varphi_s \, dX_s := \int_0^t \varphi_s \mu_s \, ds + \int_0^t \varphi_s \sigma_s \, dW_s,$$

provided that the integrals on the right-hand side exist.

One of the most significant results in stochastic calculus is Itô's Lemma. Itô's Lemma is pivotal in deriving the Black-Scholes equation for option pricing and facilitates the computation of sensitivities in hedging strategies, allowing for dynamic adjustments of portfolios to manage risk effectively.

Theorem 2.1.1.8. (Itô's Lemma) Let X be a d-dimensional Itô process and $f : \mathbb{R}^{d+1} \to \mathbb{R}$ be a twice continuously differentiable function. Then the process Y defined by $Y_t := f(t, X_t)$ is also an Itô process and satisfies

$$dY_t = \frac{\partial f}{\partial t} \, dt + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} \, dX_t^i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j} \, d\langle X^i, X^j \rangle_t.$$

Itô's Lemma is instrumental in solving stochastic differential equations through substitution.

□

2.2 Continuous Time Trading

The theory of financial mathematics in continuous time is developed, focusing on the Black-Scholes model. While the discrete-time framework offers a strong foundation for understanding financial concepts, the continuous-time approach, where stochastic processes, particularly Brownian motion, and Itô's calculus, play a critical role, is the central theme of this thesis. The Black-Scholes model illustrates that an option's payoff can be replicated by a continuously rebalanced, self-financing portfolio consisting of the underlying asset and a risk-free bond. This replication strategy, underpinned by the law of one price, leads to the arbitrage-free pricing of options. Although the classic model assumes constant volatility, extensions such as the Heston model introduce stochastic volatility to better capture market dynamics. Additionally, the notion of self-financing portfolios is explored in the context of mean-variance hedging, offering practical alternatives when continuous rebalancing is not feasible. For more details on these topics, refer to [10], [3], and [23]. The general structure of this section is derived from [10], [30], and [33].

2.2.1 Continuous Time Trading

In the study of financial markets, a market environment is often defined by the tuple

$$((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), S),$$

where Ω is the sample space, $\mathcal{F}$ is the sigma-algebra of all possible events, $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ represents the filtration modeling the flow of information over time, $\mathbb{P}$ is the probability measure, and

$$S = \{B, S^1, \ldots, S^d\}$$

denotes the prices of d risky assets along with a risk-free asset B. The market operates over a finite time horizon $[0, T]$, where T represents the terminal time.

The prices of the $d + 1$ assets in this market are modeled as stochastic processes. The vector of initial asset prices is defined as:

$$\bar{\gamma} = (\gamma^0, \gamma^1, \ldots, \gamma^d),$$

where γ^i represents the initial price of the i-th asset. This collection forms a *price system*. The risk-free asset, corresponding to γ^0, is unique in that its price at any future time t is deterministic. Specifically, the price of the risk-free asset at time t is given by:

$$S_t^0 = \gamma^0(1 + r)^t,$$

where r is the risk-free rate and S_t^0 denotes the price of the risk-free asset at time t.

For simplicity and consistency in theoretical models, discounted asset prices with respect to a chosen numeraire can be considered. This leads to the following notation:

$$\bar{\gamma} = (1, \gamma^1, \ldots, \gamma^d), \quad \text{and} \quad S_t = (B_t, S_t^1, \ldots, S_t^d),$$

where B_t represents the price of the risk-free asset. It is assumed that $B_t > 0$ for all t, and B_t serves as the numeraire.

It is assumed that $S_t^0 > 0$ for all t, and S_t^0 is chosen as the numeraire. As the numeraire, S_t^0 serves as the benchmark relative to which the prices of all other d assets are expressed. Therefore, the prices of the other d assets are expressed in units of S_t^0.

In contrast, the prices of the d risky assets evolve stochastically over time, represented by the vector:

$$\bar{S}_t = (S_t^0, S_t^1, \ldots, S_t^d),$$

where S_t^0 is deterministic and $S_t^1, \ldots, S_t^d$ are stochastic processes. The vector $(B, S^1, \ldots, S^d)$ forms a $d+1$-dimensional càdlàg semi-martingale, ensuring right-continuity with left limits and maintaining mathematical tractability. Additionally, all asset prices are assumed to remain positive to avoid complications such as negative prices.

In this frictionless market, investors are assumed to be able to borrow or lend without restrictions at a risk-free rate. At the initial time $t = 0$, an investor allocates their wealth among the available assets. The portfolio is denoted by:

$$\varphi = (\varphi^0, \varphi^1, \dots, \varphi^d),$$

where φ^0 represents the amount invested in the risk-free asset, and φ^i represents the quantity of the i-th risky asset. The initial cost or value of this portfolio is:

$$V_0(\varphi) = \varphi^0\gamma^0 + \sum_{i=1}^{d} \varphi^i\gamma^i.$$

This equation sums the values of all asset positions in the portfolio, reflecting the total wealth invested at the initial time.

To further simplify the analysis, one of the assets can be chosen as the numeraire. Suppose the numeraire is the j-th asset S_t^j, for all $t \in [0, T]$. This choice implies that all other asset prices are now expressed in units of the j-th asset. The discounted price vector with respect to this numeraire becomes:

$$X_t^i = \begin{cases} 1 & \text{if } i = j, \\ \frac{S_t^i}{S_t^j} & \text{if } i \neq j. \end{cases}$$

Thus, the discounted price vector can also be written as:

$$X_t = \left[X_t^1, \dots, X_t^{j-1}, 1, X_t^{j+1}, \dots, X_t^d\right]^T, \quad j = 1, 2, \dots, d.$$

The value of a trading strategy or portfolio in terms of discounted prices is given by the stochastic process $V_t(\varphi)$, defined as:

$$V_t(\varphi) = \varphi^j S_t^j + \sum_{\substack{i=1 \\ i \neq j}}^{d} \varphi_t^i S_t^i = \varphi^j S_t^j \left(1 + \sum_{\substack{i=1 \\ i \neq j}}^{d} \varphi_t^i X_t^i\right),$$

where φ^j denotes the position in the j-th asset.

In particular, for a self-financing trading strategy, the portfolio evolves solely through changes in the prices of the assets held, without any additional capital injections or withdrawals. Mathematically, a trading strategy φ is self-financing if and only if:

$$V_t(\varphi) = V_0(\varphi) + G_t(\varphi),$$

where $G_t(\varphi)$ represents the cumulative gains process, defined as:

$$G_t(\varphi) = \sum_{\substack{i=1 \\ i \neq j}}^{d} \int_0^t \varphi_s^i \, dX_s^i.$$

The gains process reflects the profits or losses made from trading the risky assets. It is important to emphasize that *any asset* in the market can serve as the numeraire. Selecting a different asset as the numeraire S_t^j leads to a new, but equivalent, set of discounted prices and trading strategies.

The concept of a *martingale* provides further insight into the behavior of these processes. A process M_t is called a martingale with respect to a filtration $\mathbb{F}$ and probability measure $\mathbb{P}$ if, for all $t \leq s$,

$$\mathbb{E}[M_s|\mathcal{F}_t] = M_t.$$

In the context of financial modeling, the discounted price process X_t is a martingale under an equivalent martingale measure $\mathbb{Q}$, which is the defining characteristic of $\mathbb{Q}$ as the risk-neutral measure. This implies that, on average, the future value of the asset, when appropriately discounted, equals its current value. The martingale property is fundamental in determining the fair prices of derivatives and other contingent claims.

By establishing a coherent framework that incorporates discounted prices, self-financing strategies, and the martingale property, financial markets can be effectively modeled and analyzed. The concepts introduced here lay the groundwork for more advanced topics, which will be explored in subsequent chapters and sections. These will include detailed discussions on derivative pricing, hedging strategies, and risk management techniques, among others. Building on this foundation, the thesis will delve into the intricacies of financial modeling in both discrete and continuous time settings.

Definition 2.2.1.1. A trading strategy is said to be self-financing if:

$$\varphi_{t+1} \cdot X_t := \varphi_t \cdot X_t$$

for all $t = 0, \ldots, T-1$. Equivalently, φ is self-financing if and only if:

$$V_t(\varphi) = V_0(\varphi) + G_t(\varphi)$$

for all $t = 0, 1, \ldots, T$. The interpretation of a self-financing portfolio is that the value of the strategy changes solely due to the gains process, with no injections or withdrawals of funds at any point in time.

Remark 2.2.1.1. Let tuple $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ be a market. Suppose there exists a pair of assets $A := (A^1, A^2)$ such that, at some $t = 0, 1, \ldots, T$, their prices satisfy

$$A_t^1 := \alpha A_t^2, \quad \alpha > 0,$$

in their current state. In this scenario, it is possible to sell αN units of the first asset and purchase N units of the second asset with no additional cost to the portfolio. This arbitrage opportunity can be exploited for any $N \in \mathbb{N}$, leading to an unbounded trading strategy φ. To prevent such unreasonable situations, trading strategies are assumed to be bounded. Specifically, the gains process is typically constrained as follows:

$$G_t(\varphi) \geq -C,$$

for some $C \in \mathbb{R}^+$, referred to as the *finite credit line*, which bounds the amount of money that can be borrowed. Moreover, a stronger assumption is made: for each $\varphi \in \Theta$,

$$|\varphi| \leq C \quad \text{for some } C \in \mathbb{R}^+.$$

Enforcing bounded strategies ensures market integrity by preventing unintended capital flows and facilitates the analysis of portfolio performance based solely on market movements. Additionally, self-financing strategies are foundational in establishing an arbitrage-free market, where no riskless profit opportunities exist. By linking portfolio dynamics to asset prices in this manner, it becomes possible to formally define and explore arbitrage-free conditions in subsequent sections.

Definition 2.2.1.2. A strategy φ is an arbitrage strategy or arbitrage opportunity if:

1. $V_0(\varphi) = 0$,
2. $\mathbb{P}(V_T(\varphi) \geq 0) = 1$ almost surely, and
3. $\mathbb{P}(V_T > 0) > 0$.

This definition holds equivalently when considering both discounted and absolute price systems because an arbitrage opportunity, by definition, requires no initial investment ($V_0(\varphi) = 0$) and offers a positive probability that the final portfolio value $V_T(\varphi)$ will yield a profit without any risk of loss. To enforce the absence of arbitrage, an equivalent martingale measure $\mathbb{Q}$ is introduced under which the discounted price process is a martingale, thereby constructing market models that inherently exclude arbitrage.

The notion of an arbitrage-free market model is central to the Arbitrage-Free Pricing Model, which serves as a cornerstone in mathematical finance. The theory assumes that arbitrage opportunities are non-existent, implying:

$$\mathbb{Q}(V_T(\varphi) > 0) > 0 \quad \text{for any admissible strategy } \varphi \text{ with } V_0(\varphi) = 0.$$

In reality, should an arbitrage opportunity arise, market participants would exploit it, causing the asset price to adjust until the arbitrage is eliminated within a brief period.

The equivalent martingale measure $\mathbb{Q}$ and the concept of the equivalent local martingale measure are introduced next.

In the realm of mathematical finance, the concept of an *Equivalent Martingale Measure (EMM)* is pivotal for the pricing of derivative securities and ensuring the absence of arbitrage opportunities in financial markets.

Definition 2.2.1.3. A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ is called an *Equivalent Martingale Measure (EMM)* for a stochastic process X if it is equivalent to the original probability measure $\mathbb{P}$ (denoted $\mathbb{P}^* \approx \mathbb{P}$) and X is a martingale under $\mathbb{P}^*$. Equivalence here means that for all events $A \in \mathcal{F}$,

$$\mathbb{P}(A) = 0 \iff \mathbb{P}^*(A) = 0.$$

The existence of an EMM is closely linked to the *no-arbitrage condition*, which asserts that there are no opportunities to make a riskless profit without any initial investment. However, the absence of arbitrage alone does not guarantee the existence of an EMM. Establishing the existence of an EMM often requires advanced mathematical tools, such as the *Hahn-Banach Theorem*.

The Hahn-Banach Theorem plays a crucial role in functional analysis and is instrumental in financial mathematics for proving the existence of EMMs. It allows for the extension of linear functionals and the separation of convex sets in infinite-dimensional spaces. In the context of finance, it facilitates the construction of a separating hyperplane between the set of attainable claims (representing possible portfolio gains) and the set of non-attainable claims, thereby ensuring the existence of a probability measure under which discounted asset prices follow a martingale process.

To formalize these concepts, the vector spaces of contingent claims are defined. The set $\mathcal{K}_v$ represents all contingent claims that can be *replicated* at a price v using admissible trading strategies, while $\mathcal{C}_v$ includes all claims that can be *super-replicated* at the same price.

Definition 2.2.1.4. The vector space of all replicable contingent claims at price $v \in \mathbb{R}$ is defined as:

$$\mathcal{K}_v := \left\{ v + (\varphi \bullet X)_T : \varphi \in \Theta,\ (\varphi \bullet X)_T \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \right\},$$

where Θ denotes the set of admissible trading strategies, and $(\varphi \bullet X)_T$ is the cumulative gain from the strategy φ up to time T.

Similarly, the vector space of all super-replicable contingent claims at price v is:

$$\mathcal{C}_v := \left\{ g \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) : \exists f \in \mathcal{K}_v \text{ such that } f \geq g \ \mathbb{P}\text{-a.s.} \right\}.$$

The no-arbitrage condition is intimately connected to these vector spaces. Specifically, it can be characterized by the condition:

$$\mathcal{K} \cap \mathcal{L}^+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\},$$

where $\mathcal{K} = \mathcal{K}_0$ and $\mathcal{L}^+(\Omega, \mathcal{F}, \mathbb{P})$ is the set of non-negative measurable functions. This condition implies that the only non-negative replicable claim obtainable with zero initial investment is the zero claim itself, thereby precluding the possibility of a riskless profit.

Understanding the relationship between the no-arbitrage condition and the existence of an EMM is crucial for developing a coherent pricing framework. The Hahn-Banach Theorem aids in this by ensuring that under certain conditions such as the convexity of the set of attainable claims there exists an EMM that aligns with the market's pricing of assets.

Moving forward, the *Fundamental Theorems of Asset Pricing (FTAP I and II)* will be explored, which formalizes the connection between the absence of arbitrage and the existence of an EMM. The first theorem states that a market is arbitrage-free if and only if there exists an equivalent martingale measure. The second theorem extends this by relating market completeness to the uniqueness of the EMM. These theorems are essential for understanding how derivative securities can be accurately valued in a market that precludes arbitrage opportunities, thereby ensuring both theoretical consistency and practical applicability in financial modeling.

Theorem 2.2.1.2. Fundamental Theorem of Asset Pricing I (FTAP I). Let $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ be a financial market. The following statements are equivalent:

1. No arbitrage opportunities exist.
2. $\mathcal{P} \neq \emptyset$, where $\mathcal{P}$ is the set of equivalent martingale measures.

Definition 2.2.1.5. A contingent claim H is an $\mathcal{F}$-measurable random variable in a financial market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$. A strategy (V_0, φ) is said to *replicate* H, or be a *replicating strategy* for H, if

$$V_T((V_0, \varphi)) = H \quad \mathbb{P}\text{-a.s.}$$

If such a replicating strategy exists, H is considered *attainable* or *replicable.*

Definition 2.2.1.6. A financial market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ is said to be *complete* if every contingent claim is attainable.

Definition 2.2.1.7. A stochastic process X possesses the *predictable representation property (PRP)* with respect to $\mathbb{P}^*$ if every $\mathbb{P}^*$-local martingale M can be expressed as

$$M_t = M_0 + \int_0^t \varphi_s \, dX_s$$

for some predictable X-integrable process φ.

Definition 2.2.1.8. Fundamental Theorem of Asset Pricing II (FTAP II). Let $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ be a financial market whose components are non-negative and assume that the No Free Lunch with Vanishing Risk (NFLVR) condition holds. The following statements are equivalent:

1. The market is complete.
2. There exists a unique Equivalent Martingale Measure ($|\mathcal{P}| = 1$).
3. X satisfies the Predictable Representation Property (PRP) with respect to at least one Equivalent Martingale Measure (EMM) for X.

Additionally, FTAP II is closely related to the PRP, which asserts that any martingale M_t can be represented as a stochastic integral with respect to the underlying asset price process X_t. Mathematically, this is expressed as:

$$M_t = \mathbb{E}^*(H) + \int_0^t \varphi_s^H \, dX_s,$$

where:

- H is the contingent claim,
- $\mathbb{E}^*(H)$ denotes the expectation of H under the unique EMM,

- φ_s^H is a predictable process representing the replicating strategy.

This equation illustrates that, in a complete market, the price of any contingent claim can be exactly replicated by a dynamic trading strategy involving the underlying asset X_t.

The NFLVR condition further strengthens this framework by ensuring that the market remains free of arbitrage opportunities, even when considering limiting behaviors where risk diminishes. Together with FTAP I and II, NFLVR ensures that markets are well-behaved and that asset prices accurately reflect underlying risks.

In the context of this thesis, the integration of FTAP, NFLVR, and PRP principles is critical for developing robust mean-variance hedging strategies. Using Vector Autoregressive (VAR) models as sample path generators, asset price paths adhering to the market dynamics prescribed by these theorems are simulated. This approach enables the construction of optimal hedging strategies that minimize risk while ensuring that the generated paths and resulting hedges are theoretically sound and practically applicable.

By grounding the hedging strategy in this rigorous theoretical framework, the strategies developed are not only effective but also align with the fundamental principles of financial mathematics, thereby enhancing their reliability and applicability in real-world scenarios.

2.3 Introduction to Black-Scholes

Having established the necessary mathematical foundations in the previous section on *Trading in Continuous Time*, including the definition of Brownian motion in m-dimensions, the construction of the Black-Scholes model is now undertaken. The Black-Scholes model, a Nobel Prize-winning theory developed by Black, Scholes, and Merton [2] [22], has achieved enduring success over the decades. However, numerous liquid instruments still lack efficient and reliable pricing and hedging methodologies [5]. Many pricing solutions are not available in closed form, rendering computations nearly impossible. This thesis focuses on closed-form solutions, particularly the special cases proven by Merton in 1978 [39].

One such example is the spread option, which derives its value from the difference between the prices of two underlying assets [29]. Consider a spread option with a payoff at maturity T given by:

$$H = (S_T^1 - S_T^2 - K)^+$$

where K is the strike price, and $x^+ := \max(x, 0)$. In this example, $K = 0$. Here, S_t^1 and S_t^2 denote the prices of the two underlying assets at time t, with initial stock prices $S_0^1 > 0$ and $S_0^2 > 0$ respectively. At maturity T, the holder of the spread option has the right, but not the obligation, to receive the positive difference between the two asset prices, adjusted by the strike price K. If the difference $S_T^1 - S_T^2$ exceeds K, the option yields a positive payoff; otherwise, the payoff is zero.

The maturity date T is a critical parameter in the valuation of the spread option. As time approaches maturity T, the option's value becomes increasingly sensitive to the prices S_T^1 and S_T^2, with the payoff being realized precisely at this date. This sensitivity underscores the importance of accurately modeling the dynamics of the underlying assets over the option's time horizon, as the relative performance of the two assets directly influences the option's profitability.

By focusing on such closed-form solutions, this thesis aims to extend the applicability of the Black-Scholes framework to a broader class of financial instruments, providing efficient and reliable pricing and hedging methodologies where they are most needed.

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ in which two underlying stocks S_t^1 and S_t^2 are modeled. Each stock follows a geometric Brownian motion, satisfying the stochastic differential equations (SDEs):

$$\begin{aligned} dS_t^1 &= S_t^1 \left(\mu_1 \, dt + \sigma_1 \, dW_t^1\right), \\ dS_t^2 &= S_t^2 \left(\mu_2 \, dt + \sigma_2 \, dW_t^2\right), \end{aligned} \tag{2.1}$$

where:

- $\mu_i \in \mathbb{R}$ are the constant drift (appreciation) rates for $i = 1, 2$,
- $\sigma_i > 0$ are the constant volatilities for $i = 1, 2$,
- S_t^i are the stock prices for $i = 1, 2$,
- $W_t = (W_t^1, W_t^2)^\top$ is a two-dimensional Brownian motion.

Incorporating the correlation directly through the multi-dimensional Brownian motion W_t accurately models the realistic scenario where asset prices exhibit co-movement. This approach is essential in financial modeling and risk management, as it reflects the interconnectedness of assets in real markets.

While the Black-Scholes model relies on several simplifying assumptions, such as constant volatility, log-normal distribution of asset prices, and the absence of transaction costs and taxes, these assumptions do not fully capture the complexities of real financial markets [2]. However, addressing these limitations is beyond the scope of this thesis. Instead, the focus remains on leveraging the established Black-Scholes framework to develop closed-form solutions for specific financial instruments.

To generalize the modeling framework, the following m-dimensional linear stochastic Itô's integral equation is considered:

$$S_t = S_0 + \int_0^t \mu S_u \, du + \sum_{j=1}^{m} \int_0^t \sigma_j S_u \, dW_u^j, \quad t \in [0, T], \tag{2.2}$$

where:

- $S_t \in \mathbb{R}^m$: The unknown stochastic process representing the system's state at time t.
- $\mu \in \mathbb{R}^m$: The constant drift coefficient vector.
- $\sigma_j \in \mathbb{R}^m$: The constant diffusion coefficient vectors for each Brownian motion component.
- $W_t = (W_t^1, W_t^2, \ldots, W_t^m)$: An m-dimensional Brownian motion process, defined in the previous section.
- $S_0 \in \mathbb{R}^m$: The initial state vector at time $t = 0$.

In this equation:

- The term $\int_0^t \mu S_u \, du$ represents the deterministic trend of the process, integrating the influence of past states weighted by the drift coefficients.
- The summation $\sum_{j=1}^{m} \int_0^t \sigma_j S_u \, dW_u^j$ accounts for the stochastic fluctuations in each dimension, with each Itô integral corresponding to a different Brownian motion component.

This m-dimensional Itô's integral equation generalizes the dynamics of the asset prices $S_t^1, \ldots, S_t^m$ by incorporating multiple sources of randomness and constant coefficients [28]. Solving this equation allows for the derivation of the necessary dynamics for pricing and hedging financial derivatives within the Black-Scholes framework.

In the specific case of two underlying assets, the Itô's integral equation simplifies to the two-dimensional system of stochastic differential equations presented earlier. Here, $m = 2$, and the processes S_t^1 and S_t^2 evolve according to their respective drift and diffusion terms, influenced by the correlated Brownian motions W_t^1 and W_t^2.

Extending this framework to higher dimensions facilitates the modeling of a broader range of financial instruments and the capture of more complex market dynamics, thereby enhancing the applicability of the Black-Scholes model. This thesis leverages this generalized approach to develop closed-form pricing solutions for various financial derivatives, building upon the foundational work of Black, Scholes, and Merton [2].

The solutions to the stochastic differential equations 2.1 or 2.2 can be explicitly expressed as:

$$S_t^i := S_0^i \exp\left[\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_t^i\right], \quad i = 1,2 \quad \forall t \in [0,T] \tag{2.3}$$

This formulation provides the explicit paths for the asset prices S_t^i, where each asset i evolves according to its own drift μ_i and volatility σ_i, with W_t^i representing the corresponding Brownian motion.

To establish that S_t^i is a martingale under the probability measure $\mathbb{P}$ if and only if $\mu_i = 0$, the explicit solution to the stochastic differential equation (SDE) 2.3 is examined.

The objective is to demonstrate that:

$$\mathbb{E}^{\mathbb{P}^*}\left[S_t^i \mid \mathcal{F}_u\right] = S_u^i, \quad \text{if and only if} \quad \mu_i = 0. \tag{2.4}$$

Using the properties of Brownian motion, it can be expressed that $W_t^i = W_u^i + (W_t^i - W_u^i)$, where $W_t^i - W_u^i$ is independent of $\mathcal{F}_u$ and normally distributed with mean zero and variance $t-u$. Therefore, the expectation is computed as follows:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^*}\left[S_t^i \mid \mathcal{F}_u\right] &= \mathbb{E}^{\mathbb{P}^*}\left[S_0^i \exp\left(\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_t^i\right) \Big| \mathcal{F}_u\right] \\
&= S_0^i \exp\left(\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_u^i\right) \mathbb{E}^{\mathbb{P}}\left[\exp\left(\sigma_i(W_t^i - W_u^i)\right) \Big| \mathcal{F}_u\right] \\
&= S_u^i \exp\left(\left(\mu_i - \frac{\sigma_i^2}{2}\right)(t-u)\right) \exp\left(\frac{1}{2}\sigma_i^2(t-u)\right) \\
&= S_u^i \exp\left[\mu_i(t-u)\right].
\end{aligned}$$

Thus, $\mathbb{E}^{\mathbb{P}^*}\left[S_t^i \mid \mathcal{F}_u\right] = S_u^i$ if and only if $\mu_i = 0$. Therefore, S_t^i is a martingale under $\mathbb{P}^*$ if and only if $\mu_i = 0$.

$\square$

In financial modeling, it is crucial to consider the time value of money by discounting future asset prices back to their present value. The discounted price process $\widetilde{S}_t^i$ is defined as:

$$\widetilde{S}_t^i = e^{-rt} S_t^i, \tag{2.5}$$

where r is the constant risk-free interest rate. Under the risk-neutral measure $\mathbb{Q}$, the discounted price process $\widetilde{S}_t^i$ should be a martingale. However, under the original measure $\mathbb{P}$, this may not be the case due to the presence of the drift term $\mu_i - r$ in the dynamics of $\widetilde{S}_t^i$.

To address this, the probability measure must be adjusted so that the drift term vanishes, making $\widetilde{S}_t^i$ a martingale under the new measure. This adjustment involves a change of measure using *Girsanov's Theorem*, which allows for the modification of the drift of the Brownian motion by altering the probability measure.

Theorem 2.3.1. (Girsanov's Theorem in Multiple Dimensions): Let $\Theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)^\top$ be a d-dimensional, adapted process satisfying the Novikov condition:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \|\Theta_t\|^2\, dt\right)\right] < \infty.$$

Define the Radon-Nikodym derivative Z_t by:

$$Z_t = \exp\left(-\int_0^t \Theta_s^\top \, dW_s - \frac{1}{2}\int_0^t \|\Theta_s\|^2 \, ds\right). \tag{2.6}$$

Then Z_t is a martingale, and under the probability measure $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$, the process:

$$\widetilde{W}_t = W_t + \int_0^t \Theta_s \, ds$$

is a d-dimensional Brownian motion under $\mathbb{Q}$.

The market price of risk vector $\Theta_t = \Lambda$ is chosen, where $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)^\top$ and each component is given by:

$$\lambda_i = \frac{\mu_i - r}{\sigma_i}, \quad i = 1, 2, \dots, d.$$

Here, μ_i and σ_i are the drift and volatility of the i-th asset, respectively, and r is the risk-free rate.

Using $\Theta_t = \Lambda$, the Radon-Nikodym derivative becomes:

$$Z_t = \exp\left(-\Lambda^\top W_t - \frac{1}{2}\|\Lambda\|^2 t\right). \tag{2.7}$$

Under the original measure $\mathbb{P}$, the dynamics of the i-th asset price are given by:

$$dS_t^i = S_t^i \left(\mu_i \, dt + \sigma_i \, dW_t^i\right).$$

Substituting $\mu_i = r + \sigma_i \lambda_i$ into the equation yields:

$$\begin{aligned} dS_t^i &= S_t^i \left((r + \sigma_i \lambda_i) \, dt + \sigma_i \, dW_t^i\right) \\ &= S_t^i \left(r \, dt + \sigma_i \left(dW_t^i + \lambda_i \, dt\right)\right). \end{aligned}$$

Applying Girsanov's theorem, the transformed Brownian motion under the new measure $\mathbb{Q}$ is:

$$d\widetilde{W}_t^i = dW_t^i + \lambda_i \, dt.$$

Thus, the dynamics of the i-th asset price under $\mathbb{Q}$ become:

$$dS_t^i = S_t^i \left(r \, dt + \sigma_i \, d\widetilde{W}_t^i\right).$$

Consider the discounted asset price $\widetilde{S}_t^i = e^{-rt} S_t^i$. Using Itô's lemma, the dynamics are derived as follows:

$$\begin{aligned} d\widetilde{S}_t^i &= e^{-rt} \left(dS_t^i - r S_t^i \, dt\right) \\ &= e^{-rt} S_t^i \left(r \, dt + \sigma_i \, d\widetilde{W}_t^i\right) - e^{-rt} r S_t^i \, dt \\ &= e^{-rt} S_t^i \sigma_i \, d\widetilde{W}_t^i \\ &= \widetilde{S}_t^i \sigma_i \, d\widetilde{W}_t^i. \end{aligned}$$

This SDE has no drift term, indicating that $\widetilde{S}_t^i$ is a martingale under $\mathbb{Q}$.

To confirm that $\widetilde{S}_t^i$ is a martingale under $\mathbb{Q}$, the conditional expectation is computed:

$$\mathbb{E}^{\mathbb{Q}}\left[\widetilde{S}_t^i \mid \mathcal{F}_u\right] = \widetilde{S}_u^i \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sigma_i \left(\widetilde{W}_t^i - \widetilde{W}_u^i\right) - \frac{1}{2}\sigma_i^2 (t-u)\right) \Big| \mathcal{F}_u\right].$$

Since $\widetilde{W}_t^i - \widetilde{W}_u^i$ is independent of $\mathcal{F}_u$ and normally distributed with mean 0 and variance $t-u$, the expectation simplifies. Let $Z = \widetilde{W}_t^i - \widetilde{W}_u^i \sim \mathcal{N}(0, t-u)$. Then:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sigma_i Z\right)\right] = \exp\left(\frac{1}{2}\sigma_i^2(t-u)\right).$$

Thus, the full expectation becomes:

$$\exp\left(-\frac{1}{2}\sigma_i^2(t-u)\right)\exp\left(\frac{1}{2}\sigma_i^2(t-u)\right) = 1.$$

Therefore:

$$\mathbb{E}^{\mathbb{Q}}\left[\widetilde{S}_t^i \mid \mathcal{F}_u\right] = \widetilde{S}_u^i,$$

confirming that $\widetilde{S}_t^i$ is a martingale under $\mathbb{Q}$.

□

By changing the probability measure from $\mathbb{P}$ to $\mathbb{Q}$, the drift term is effectively removed from the discounted price process $\widetilde{S}_t^i$. This adjustment is crucial as it allows the discounted price process to be modeled as a martingale under $\mathbb{Q}$, which is a fundamental requirement for the absence of arbitrage in financial markets.

Having established the martingale properties of the discounted price process $\widetilde{S}_t^i$ under the risk-neutral measure, the concepts of admissible trading strategies and attainable contingent claims are introduced. These concepts are fundamental to both pricing and hedging in the Black-Scholes model.

Definition 2.3.1. A trading strategy $\varphi = (\varphi_t)$ for $t \in [0, T]$ is called *admissible* (more specifically, $\mathbb{Q}$-admissible) if the discounted value process V_t^φ associated with the strategy is a martingale under the risk-neutral measure $\mathbb{Q}$. The class of all $\mathbb{Q}$-admissible strategies is denoted by $\Phi(\mathbb{Q})$. The triple $\mathcal{M}_{BS} = (S, B, \Phi(\mathbb{Q}))$, where S is the stock price process, B is the risk-free asset (bank account), and $\Phi(\mathbb{Q})$ is the set of admissible strategies, is referred to as the *arbitrage-free Black-Scholes model*.

Definition 2.3.2. A European contingent claim H that expires at time $T^* \leq T$ is said to be *attainable* in the Black-Scholes model if there exists an admissible trading strategy $\varphi \in \Phi(\mathbb{Q})$ such that the terminal value of the strategy equals the payoff of the claim, i.e.,

$$V_{T^*}^\varphi = H \quad \mathbb{P}\text{-a.s.}$$

By focusing on admissible strategies and attainable claims, the model ensures that it remains arbitrage-free and that pricing and hedging are feasible within this mathematical structure.

2.4 Black-Scholes Pricing

In the Black-Scholes model, pricing a contingent claim involves determining its fair value based on the absence of arbitrage opportunities. By constraining attention to admissible strategies—those whose discounted value processes are martingales under $\mathbb{Q}$—it is ensured that the strategies do not exploit any arbitrage.

Building on the previous discussion of pricing theory and the Black-Scholes framework, a method for valuing contingent claims within the Black-Scholes model has been derived. This approach is utilized to price a spread option with the payoff $H = \left(S_T^1 - S_T^2\right)^+$ at maturity T, where $K = 0$. Notably, the solution for this pricing problem is available in closed form.

Proposition 2.4.1. The arbitrage-free price of the spread option H at time $t \in [0, T]$ under the Black-Scholes model is given by:

$$\pi_t(H) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r(T-t)}H\right],$$

where $\mathbb{Q}$ is the risk-neutral measure.

The dynamics of the underlying assets S_t^1 and S_t^2 under the risk-neutral measure $\mathbb{Q}$ are considered. The risk-neutral dynamics are given by:

$$\begin{aligned} dS_t^1 &= S_t^1\left(r\,dt + \sigma_1\,dW_t^1\right), \\ dS_t^2 &= S_t^2\left(r\,dt + \sigma_2\,dW_t^2\right), \end{aligned} \tag{2.8}$$

where r is the constant risk-free interest rate, σ_1 and σ_2 are the volatilities of the two assets, and dW_t^1 and dW_t^2 are Brownian motions under $\mathbb{Q}$ with correlation ρ, i.e., $[W^1, W^2]_t = \rho t$.

The solution to these stochastic differential equations (SDEs) is:

$$\begin{aligned} S_T^1 &= S_t^1 \exp\left[\left(r - \frac{\sigma_1^2}{2}\right)(T-t) + \sigma_1\left(W_T^1 - W_t^1\right)\right], \\ S_T^2 &= S_t^2 \exp\left[\left(r - \frac{\sigma_2^2}{2}\right)(T-t) + \sigma_2\left(W_T^2 - W_t^2\right)\right]. \end{aligned} \tag{2.9}$$

The objective is to compute:

$$\pi_t(H) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r(T-t)}\left(S_T^1 - S_T^2\right)^+ \mid \mathcal{F}_t\right].$$

To simplify the computation, the *change of numéraire* technique is applied by selecting S_t^2 as the numéraire. Under this new numéraire, a new probability measure $\mathbb{Q}^2$ is defined such that:

$$\left.\frac{d\mathbb{Q}^2}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = \frac{S_T^2 e^{-r(T-t)}}{S_t^2}.$$

Under $\mathbb{Q}^2$, the process $\widetilde{S}_t^2 = e^{-r(t-t)}S_t^2 = S_t^2$ serves as the numéraire, and the discounted price process of S_t^2 becomes a martingale.

The dynamics of S_t^1 under $\mathbb{Q}^2$ are adjusted using Girsanov's Theorem 2.6. The new dynamics are:

$$dS_t^1 = S_t^1\left(\left(r - \sigma_1\sigma_2\rho\right)dt + \sigma_1 dW_t^{1,\mathbb{Q}^2}\right),$$

where

$$dW_t^{1,\mathbb{Q}^2} = dW_t^1 - \rho\sigma_2 dt.$$

Under $\mathbb{Q}^2$, the ratio $\frac{S_T^1}{S_T^2}$ simplifies the calculation. The payoff becomes:

$$H := \left(S_T^1 - S_T^2\right)^+ := S_T^2\left(\left(\frac{S_T^1}{S_T^2} - 1\right)^+\right).$$

The price of the spread option is then:

$$\pi_t(H) = S_t^2 \mathbb{E}_t^{\mathbb{Q}^2} \left[\left(\frac{S_T^1}{S_T^2} - 1 \right)^+ \middle| \mathcal{F}_t \right].$$

The expectation of the positive part of the log-normal variable $\frac{S_T^1}{S_T^2}$ under $\mathbb{Q}^2$ is required.

Define the log price ratio:

$$X = \ln \left(\frac{S_T^1}{S_T^2} \right).$$

Under $\mathbb{Q}^2$, X is normally distributed. From equations 2.9, it follows:

$$\begin{aligned} X &= \ln S_T^1 - \ln S_T^2 \\ &= \ln S_t^1 + \left(r - \frac{\sigma_1^2}{2} \right)(T-t) + \sigma_1 \left(W_T^1 - W_t^1 \right) \\ &\quad - \left[\ln S_t^2 + \left(r - \frac{\sigma_2^2}{2} \right)(T-t) + \sigma_2 \left(W_T^2 - W_t^2 \right) \right] \\ &= \ln \left(\frac{S_t^1}{S_t^2} \right) + \left(\frac{\sigma_2^2 - \sigma_1^2}{2} \right)(T-t) + \sigma_1 \left(W_T^1 - W_t^1 \right) - \sigma_2 \left(W_T^2 - W_t^2 \right). \end{aligned}$$

Under $\mathbb{Q}^2$, with $\left[W^{1,\mathbb{Q}^2}, W^{2,\mathbb{Q}^2} \right]_t = \rho_t'$. However, since S_t^2 is the numéraire, and $W_t^{2,\mathbb{Q}^2}$ effectively disappears in the ratio, the terms are adjusted accordingly.

Simplify X under $\mathbb{Q}^2$:

$$X = \ln \left(\frac{S_T^1}{S_t^2} \right) + \left(\frac{\sigma_2^2 - \sigma_1^2}{2} \right)(T-t) + \sigma_1 \left(W_T^{1,\mathbb{Q}^2} - W_t^{1,\mathbb{Q}^2} \right) - \sigma_2 \rho \sigma_2 (T-t).$$

X can be expressed as:

$$X = \ln \left(\frac{S_t^1}{S_t^2} \right) + \left(\frac{\sigma_2^2 - \sigma_1^2}{2} - \sigma_1 \sigma_2 \rho \right)(T-t) + \sigma_1 \left(W_T^{1,\mathbb{Q}^2} - W_t^{1,\mathbb{Q}^2} \right).$$

Since $\sigma_1 \left(W_T^{1,\mathbb{Q}^2} - W_t^{1,\mathbb{Q}^2} \right)$ is normally distributed with mean zero and variance $\sigma_1^2(T-t)$, X is normally distributed with mean μ_X and variance σ_X^2:

$$\mu_X = \ln \left(\frac{S_t^1}{S_t^2} \right) + \left(\frac{\sigma_2^2 - \sigma_1^2}{2} - \sigma_1 \sigma_2 \rho \right)(T-t),$$

$$\sigma_X^2 = \sigma_1^2 (T-t).$$

The expected payoff under $\mathbb{Q}^2$ is:

$$\pi_t(H) = S_t^2 \mathbb{E}_t^{\mathbb{Q}^2} \left[\left(e^X - 1 \right)^+ \middle| \mathcal{F}_t \right].$$

This expectation can be evaluated using the properties of the log-normal distribution. Specifically:

$$\pi_t(H) = S_t^2 \left[e^{\mu_X + \frac{1}{2}\sigma_X^2} N(d_1) - N(d_2) \right],$$

where:

$$d_1 = \frac{\mu_X + \sigma_X^2}{\sigma_X},$$
$$d_2 = \frac{\mu_X}{\sigma_X},$$

and $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution. Substituting μ_X and σ_X, the expression becomes:

$$d_1 = d_2 + \sigma_X.$$

Combining the above results leads to the closed-form solution for the price of the spread option:

$$\pi_t(H) = S_t^1 N(d_1) - S_t^2 N(d_2), \tag{2.10}$$

where:

$$\begin{aligned} d_1 &:= \frac{\ln\left(\frac{S_t^1}{S_t^2}\right) + \left(-\frac{\sigma_2^2 - \sigma_1^2}{2} - \sigma_1\sigma_2\rho\right)(T-t) + \sigma_1^2(T-t)}{\sigma_1\sqrt{T-t}} \\ &:= \frac{\ln\left(\frac{S_t^1}{S_t^2}\right) + \left(\sigma_1^2 - \sigma_2^2\right)(T-t) - 2\sigma_1\sigma_2\rho(T-t)}{2\sigma_1\sqrt{T-t}} \\ &:= \frac{\ln\left(\frac{S_t^1}{S_t^2}\right) - \frac{1}{2}\sigma_{AVG}^2(T-t)}{\sigma_{\text{AVG}}\sqrt{T-t}}, \end{aligned} \tag{2.11}$$

and

$$d_2 = d_1 - \sigma_{\text{AVG}}\sqrt{T-t},$$

with the effective volatility σ_{AVG} defined as:

$$\sigma_{\text{AVG}} = \sqrt{\sigma_1^2 + \sigma_2^2 - \rho\sigma_1\sigma_2}.$$

This formula is known as *Margrabe's Formula*, which provides the price of an option to exchange one asset for another [16].

Under the adjusted dynamics and change of numéraire, the spread option pricing problem reduces to that of a European call option on the exchange rate $\frac{S_t^1}{S_t^2}$ with strike price $K = 1$. The steps involved mirror those in the standard Black-Scholes pricing of a European call option [1], adjusted for the correlation between the two underlying assets.

Consequently, the arbitrage-free price of the spread option $H = \left(S_T^1 - S_T^2\right)^+$ is provided by equation 2.10. This closed-form solution allows for efficient computation of the option price and is consistent with the Black-Scholes framework.

[1] $C_t = (X_t - K)^+$ where X_t is the stock price and K strike price

2.5 Black-Scholes Hedging

In the Black-Scholes model, hedging involves constructing an admissible replicating strategy φ that exactly replicates the payoff of a contingent claim at maturity. This strategy mitigates risk by offsetting potential losses in the option position with gains in the hedging portfolio. For the spread option with payoff $H = \left(S_T^1 - S_T^2\right)^+$, where S_T^1 and S_T^2 are the prices of two underlying assets at maturity T, a method is explored to derive such a hedging strategy by replicating the discounted contingent claim.

The Black-Scholes model assumes a complete market, meaning that every contingent claim can be perfectly replicated and hedged by dynamically trading in the underlying assets and a risk-free asset. This assumption is crucial as it ensures the absence of arbitrage opportunities and the efficiency of the market in allowing continuous trading without friction.

In a complete market, the prices of financial derivatives are uniquely determined by the no-arbitrage principle, and hedging strategies can be constructed to eliminate risk. This foundation is essential for the Black-Scholes framework, which relies on the ability to continuously adjust positions to maintain a riskless portfolio [2].

An effective approach to hedging under the Black-Scholes framework involves finding a replicating strategy for the *discounted contingent claim* and then demonstrating that this strategy also hedges the original contingent claim. This method leverages the martingale properties of discounted asset prices under the risk-neutral measure.

The discounted asset prices are defined as:

$$\widetilde{S}_t^i = e^{-rt} S_t^i, \quad i = 1, 2,$$

where r is the constant risk-free interest rate. The *discounted contingent claim* corresponding to the payoff H is:

$$\hat{H} = e^{-rT} H = e^{-rT} (S_T^1 - S_T^2)^+ = (\widetilde{S}_T^1 - \widetilde{S}_T^2)^+.$$

The objective is to find a replicating strategy for $\hat{H}$ using the discounted assets $(\widetilde{S}_t^1, \widetilde{S}_t^2)$.

$$M_t = \mathbb{E}_t^{\mathbb{Q}} \left[\hat{H}\right] = \mathbb{E}_t^{\mathbb{Q}} \left[(\widetilde{S}_T^1 - \widetilde{S}_T^2)^+ \mid \mathcal{F}_t\right].$$

Since the discounted asset prices $\widetilde{S}_t^1$ and $\widetilde{S}_t^2$ are Markov processes (due to their dependence on Brownian motions), M_t can be expressed as a function of the current values of $\widetilde{S}_t^1$ and $\widetilde{S}_t^2$:

$$M_t = F(t, \widetilde{S}_t^1, \widetilde{S}_t^2),$$

for some function $F : [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. This is justified by the *Doob-Dynkin Lemma*, which states that if a process has the Markov property, its conditional expectation can be expressed as a function of the current state.

Applying Itô's Lemma to M_t, the differential is:

$$dM_t = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial \widetilde{S}_t^1} d\widetilde{S}_t^1 + \frac{\partial F}{\partial \widetilde{S}_t^2} d\widetilde{S}_t^2 + \frac{1}{2} \frac{\partial^2 F}{\partial (\widetilde{S}_t^1)^2} (d\widetilde{S}_t^1)^2 + \frac{1}{2} \frac{\partial^2 F}{\partial (\widetilde{S}_t^2)^2} (d\widetilde{S}_t^2)^2 + \frac{\partial^2 F}{\partial \widetilde{S}_t^1 \partial \widetilde{S}_t^2} d\widetilde{S}_t^1 d\widetilde{S}_t^2.$$

Under $\mathbb{Q}$, the discounted asset prices follow the SDEs:

$$d\widetilde{S}_t^1 = \widetilde{S}_t^1 \sigma_1 dW_t^1,$$

$$d\widetilde{S}_t^2 = \widetilde{S}_t^2 \sigma_2 dW_t^2,$$

where σ_i are the volatilities, and dW_t^i are Brownian motions under $\mathbb{Q}$ with correlation ρ, i.e., $[dW_t^1 dW_t^2] = \rho dt$.

The quadratic variations and covariation are:

$$(d\widetilde{S}_t^1)^2 = (\widetilde{S}_t^1 \sigma_1)^2 dt,$$
$$(d\widetilde{S}_t^2)^2 = (\widetilde{S}_t^2 \sigma_2)^2 dt,$$
$$d\widetilde{S}_t^1 d\widetilde{S}_t^2 = \widetilde{S}_t^1 \widetilde{S}_t^2 \sigma_1 \sigma_2 \rho dt.$$

Substituting these into the expression for dM_t, the dynamics become:

$$dM_t = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma_1^2 (\widetilde{S}_t^1)^2 \frac{\partial^2 F}{\partial (\widetilde{S}_t^1)^2} + \frac{1}{2}\sigma_2^2 (\widetilde{S}_t^2)^2 \frac{\partial^2 F}{\partial (\widetilde{S}_t^2)^2} + \sigma_1 \sigma_2 \rho \widetilde{S}_t^1 \widetilde{S}_t^2 \frac{\partial^2 F}{\partial \widetilde{S}_t^1 \partial \widetilde{S}_t^2} \right) dt + \sigma_1 \widetilde{S}_t^1 \frac{\partial F}{\partial \widetilde{S}_t^1} dW_t^1 + \sigma_2 \widetilde{S}_t^2 \frac{\partial F}{\partial \widetilde{S}_t^2} dW_t^2.$$

For M_t to be a martingale, the drift term (the term involving dt) must be zero. Therefore, F must satisfy the following partial differential equation (PDE):

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma_1^2 (\widetilde{S}_t^1)^2 \frac{\partial^2 F}{\partial (\widetilde{S}_t^1)^2} + \frac{1}{2}\sigma_2^2 (\widetilde{S}_t^2)^2 \frac{\partial^2 F}{\partial (\widetilde{S}_t^2)^2} + \sigma_1 \sigma_2 \rho \widetilde{S}_t^1 \widetilde{S}_t^2 \frac{\partial^2 F}{\partial \widetilde{S}_t^1 \partial \widetilde{S}_t^2} = 0. \tag{2.12}$$

This is the Black-Scholes PDE for the spread option.

At maturity, the terminal condition is:

$$F(T, \widetilde{S}_T^1, \widetilde{S}_T^2) = (\widetilde{S}_T^1 - \widetilde{S}_T^2)^+.$$

The hedging strategy involves holding amounts φ_t^1 and φ_t^2 in the discounted assets $\widetilde{S}_t^1$ and $\widetilde{S}_t^2$, respectively:

$$\varphi_t^1 = \frac{\partial F}{\partial \widetilde{S}_t^1}, \quad \varphi_t^2 = \frac{\partial F}{\partial \widetilde{S}_t^2}.$$

The value of the hedging portfolio is:

$$V_t = \varphi_t^1 \widetilde{S}_t^1 + \varphi_t^2 \widetilde{S}_t^2 + \eta_t,$$

where η_t is the holding in the risk-free asset (noting that the discounted bank account has a constant value of 1, so its dynamics are trivial).

Since M_t is a martingale and replicates $\hat{H}$, the portfolio constructed using φ_t^1 and φ_t^2 replicates the discounted contingent claim:

$$V_t = M_t = \mathbb{E}_t^{\mathbb{Q}} \left[\hat{H} \right].$$

To confirm that this strategy replicates the original contingent claim H, consider the value of the portfolio in terms of the actual (undiscounted) asset prices. The value of the hedging portfolio is then:

$$V_t^{\text{original}} = \phi_t^1 S_t^1 + \phi_t^2 S_t^2 + \eta_t B_t = e^{rt}\left(\varphi_t^1 \widetilde{S}_t^1 + \varphi_t^2 \widetilde{S}_t^2 + \eta_t\right).$$

Since $V_t = \varphi_t^1 \widetilde{S}_t^1 + \varphi_t^2 \widetilde{S}_t^2 + \eta_t = M_t$, it follows that:

$$V_t^{\text{original}} = e^{rt} M_t.$$

At maturity T, $M_T = F(T, \widetilde{S}_T^1, \widetilde{S}_T^2) = (\widetilde{S}_T^1 - \widetilde{S}_T^2)^+ = \hat{H}$. Therefore:

$$V_T^{\text{original}} = e^{rT}\hat{H} = H,$$

confirming that the strategy replicates the original contingent claim H.

This hedging strategy effectively neutralizes risk by ensuring that the value of the hedging portfolio always matches the expected value of the discounted contingent claim under the risk-neutral measure. The positions in S_t^1 and S_t^2 are dynamically adjusted according to the partial derivatives of F, which represent the sensitivities of the option value to changes in the underlying asset prices.

2.6 Vector Auto-Regression Models (VAR)

In this section, the Vector Autoregression (VAR) model is introduced as a fundamental tool for modeling and analyzing multivariate time series data. Specifically, the focus is on the case where $d = 2$, represents two stock prices. The VAR model is instrumental in capturing the linear interdependencies among multiple time series and is particularly useful for generating sample paths that will be utilized in training a Feedforward neural network. Introduced by Sims in 1980, the VAR model is a natural extension of the univariate autoregressive (AR) model to multivariate time series [32]. It allows for modeling the dynamic behavior of multiple interrelated time series simultaneously. Each variable in the VAR system is modeled as a linear function of its past values and the past values of all other variables in the system, making it advantageous for capturing the complex interactions and feedback mechanisms present in financial markets.

To ensure reliable and interpretable results, the VAR model relies on several key assumptions. Firstly, it assumes that the time series data are weakly stationary, meaning that their statistical properties, such as mean and variance, remain constant over time. This stationarity is essential for the accuracy of the model's forecasts and inferences. The model also presumes linear relationships among the variables, where each variable's current value depends linearly on its past values and those of other variables. Additionally, it assumes that the data are free from measurement errors, which could otherwise introduce bias. Invertibility is another important assumption, ensuring that the VAR process can be represented as an infinite vector moving average (VMA) process, necessary for certain analytical techniques [15]. Lastly, the error terms in the model are assumed to be white noise, meaning they are serially uncorrelated with a constant variance, ensuring that the model captures all relevant linear relationships in the data. These assumptions form the backbone of the proper application of the VAR model in time series analysis [12].

Consider a d-dimensional time series $\{\mathbf{Y}_t\}$, where $\mathbf{Y}_t \in \mathbb{R}^d$ for $t = 1, 2, \ldots, T$. A VAR model of order p, denoted as VAR(p), is defined as:

$$\mathbf{Y}_t = \mathbf{c} + \sum_{i=1}^{p} \mathbf{\Phi}_i \mathbf{Y}_{t-i} + \mathbf{u}_t,$$

where:

- $\mathbf{c} \in \mathbb{R}^d$ is a vector of constants (intercepts).
- $\mathbf{\Phi}_i \in \mathbb{R}^{d\times d}$ are coefficient matrices capturing the relationships among the variables.

- $\mathbf{u}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{u}}) \in \mathbb{R}^d$ is a vector of error terms, assumed to be independently and identically distributed (i.i.d.).

For the VAR(p) model to be stationary, the roots of the determinant of the characteristic polynomial must lie outside the unit circle. The characteristic polynomial is given by:

$$\det\left(\mathbf{I}_d - \sum_{i=1}^{p} \boldsymbol{\Phi}_i z^i\right) \neq 0 \quad \text{for} \quad |z| \leq 1,$$

where $\mathbf{I}_d$ is the $d \times d$ identity matrix. This condition ensures that the model's results remain stable over time, preventing the time series from exhibiting explosive behavior [15].

When focusing on two interrelated stock prices, Y_{1t} and Y_{2t}, the VAR(p) model can be explicitly written as:

$$\begin{aligned} Y_{1t} &= c_1 + \sum_{i=1}^{p} \phi_{11,i} Y_{1,t-i} + \sum_{i=1}^{p} \phi_{12,i} Y_{2,t-i} + u_{1t}, \\ Y_{2t} &= c_2 + \sum_{i=1}^{p} \phi_{21,i} Y_{1,t-i} + \sum_{i=1}^{p} \phi_{22,i} Y_{2,t-i} + u_{2t}, \end{aligned}$$

where:

- c_1 and c_2 are constants.
- $\phi_{jk,i}$ are the coefficients capturing the influence of the k-th variable on the j-th variable at lag i.
- u_{1t} and u_{2t} are the error terms.

The system can be compactly represented in matrix form as:

$$\mathbf{Y}_t = \mathbf{c} + \boldsymbol{\Phi}_1 \mathbf{Y}_{t-1} + \cdots + \boldsymbol{\Phi}_p \mathbf{Y}_{t-p} + \mathbf{u}_t,$$

where:

$$\mathbf{Y}_t = \begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \boldsymbol{\Phi}_i = \begin{pmatrix} \phi_{11,i} & \phi_{12,i} \\ \phi_{21,i} & \phi_{22,i} \end{pmatrix}, \quad \mathbf{u}_t = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}.$$

The parameters of the VAR model can be estimated using Ordinary Least Squares (OLS) for each equation separately, due to the assumption of homoscedastic and uncorrelated error terms across equations [15]. Given stationarity, the OLS estimators are consistent and asymptotically normal, ensuring that the parameter estimates are reliable for inference.

Once the model is estimated, it can be employed to generate synthetic sample paths of the stock prices, which serve as input data for training Feedforward neural networks. By simulating the VAR process, multiple realizations of possible future trajectories of the stock prices can be created, capturing their joint dynamics and dependencies. This process involves initializing the time series, generating error terms from a multivariate normal distribution, and recursively applying the VAR model to simulate future values. The algorithm for simulation involves setting initial values $\mathbf{Y}_0, \mathbf{Y}_{-1}, \ldots, \mathbf{Y}_{-p+1}$ based on historical data or assumed starting points, generating error terms $\mathbf{u}_t$ from a distribution such that $\mathbf{u}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_u)$, and recursively calculating $\mathbf{Y}_t$ for $t = 1$ to T using:

$$\mathbf{Y}_t = \mathbf{c} + \sum_{i=1}^{p} \boldsymbol{\Phi}_i \mathbf{Y}_{t-i} + \mathbf{u}_t.$$

This process can be repeated to generate multiple sample paths, allowing for the exploration of various future scenarios.

To better understand how the system responds to shocks in one of the variables, the Impulse Response Function (IRF) is analyzed. The IRF describes the system's reaction over time to an unexpected change in one of the variables. For a stationary VAR(1) model, the IRF can be derived from the moving average representation:

$$\mathbf{Y}_t = \mu + \sum_{i=0}^{\infty} \mathbf{\Psi}_i \mathbf{u}_{t-i},$$

where:

$$\mu = (\mathbf{I}_d - \mathbf{\Phi}_1)^{-1} \mathbf{c}$$

is the mean of the process, and:

$$\mathbf{\Psi}_i = \mathbf{\Phi}_1^i.$$

Another important concept is the Forecast Error Variance Decomposition (FEVD), which quantifies the proportion of the h-step ahead forecast error variance of each variable that is attributed to shocks in each variable within the system. This helps in understanding the relative importance of each variable in the overall dynamics of the system [12].

Ensuring stationarity is fundamental in VAR modeling. Non-stationary series can lead to misleading results, so it is essential to apply tests like the Augmented Dickey-Fuller (ADF) and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) tests to verify the stationarity of the time series [7]. If the series are non-stationary, differencing or other transformations may be necessary to achieve stationarity. By carefully modeling and simulating the stock prices using the VAR model, a rich dataset that captures both temporal and cross-sectional dependencies is created. This synthetic data is essential for training Feedforward neural networks, which are central to the next phase of the analysis.

In the following chapters, the synthetic data generated from the VAR model will be leveraged to develop and train Feedforward neural networks for mean-variance hedging. The VAR model provides a realistic framework for simulating stock price movements, helping the neural network learn patterns that closely resemble real market behavior. By integrating VAR simulations with neural network training, advanced hedging strategies that aim to find the optimal risk will be explored. This combination of time series modeling and machine learning offers a powerful approach to tackling complex financial challenges, leading to more sophisticated and effective hedging strategies.

Chapter 3

3 Mean-Variance Hedging with Feedforward Neural Networks in Discrete-Time Models

This chapter delves deeper into the problem, employing a different approach compared to the preceding chapters, which laid the foundational understanding of the mathematical concepts used in this paper. Although theorems and definitions are still covered, this section focuses on the mathematics of neural networks, which differs from the traditional mathematical finance approaches utilized thus far. This shift in focus necessitates a different approach. The problem is properly defined in this section, providing an intuitive understanding, and then proceeds to construct the neural network architecture, explaining how it incorporates the VAR model as discussed in Chapter 2.6. Additionally, this approach is compared with the Black-Scholes model covered in Chapters 2.3, 2.4, and 2.5. For a more in-depth exploration of the concept of mean-variance, additional content can be found in [17].

3.1 Problem Formulation

In financial markets, institutions often face obligations to make payments at future times based on the performance of certain assets. Consider a discrete-time financial market model over times $t = 0, 1, \ldots, T$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^{T}, \mathbb{P})$. The market consists of a risk-free asset, denoted as the bank account B_t, and d risky assets with discounted price processes $S_t = (S_t^1, \ldots, S_t^d)$, for $t = 0, 1, \ldots, T$.

Consider a European call option on the spread between two of these assets, specifically S_t^1 and S_t^2, maturing at time T, with a payoff:

$$H := (S_T^1 - S_T^2)^+ := \max\{S_T^1 - S_T^2, 0\}.$$

In a complete market, a self-financing trading strategy $(\phi_t, \varphi_t)_{t=0}^{T}$ could perfectly replicate this payoff. However, in an incomplete market, the objective is to find a self-financing strategy that minimizes the expected squared hedging error between the portfolio's terminal value V_T and the option payoff H.

Consider dynamic trading strategies $(\phi_t, \varphi_t)_{t=0}^{T}$, where ϕ_t is the holding in the bank account and $\varphi_t = (\varphi_t^1, \ldots, \varphi_t^d)$ is the holding in the risky assets. The portfolio value at time t is:

$$V_t = \phi_t B_t + \varphi_t \cdot S_t.$$

Assuming discounted prices $B_t := B_{t+1} := 1$, the self-financing condition simplifies to:

$$V_{t+1} - V_t = \varphi_t \cdot \Delta S_t,$$

where $\Delta S_t = S_{t+1} - S_t$.

The goal is to find an initial capital V_0 and a self-financing strategy $\varphi = (\varphi_t)_{t=0}^{T-1}$ that minimizes the expected squared difference between the terminal portfolio value and the payoff H. The problem is formulated as:

$$\min_{V_0, \varphi} \mathbb{E}\left[(V_T - H)^2\right] = \min_{V_0, \varphi} \mathbb{E}\left[\left(V_0 + \sum_{t=0}^{T-1} \varphi_t \bullet \Delta S_t - H\right)^2\right].$$

This is the mean-variance hedging problem, where the objective function $\mathbb{E}\left[(V_T - H)^2\right]$ represents the squared replication error.

To solve this problem, the trading strategy is approximated using a Feedforward neural network f_θ^{NN}, parameterized by θ. At each time t, the strategy φ_{t+1}^θ is modeled as:

$$\varphi_{t+1}^\theta = f_\theta^{\mathrm{NN}}(S_t, t), \quad t = 0, 1, \ldots, T-1.$$

The terminal portfolio value is thus:

$$V_T^\theta = V_0 + \sum_{t=0}^{T-1} \varphi_t^\theta \cdot \Delta S_t = V_0 + (\varphi^\theta \cdot S)_T,$$

where $(\varphi^\theta \bullet S)_T$ is the cumulative gain from trading up to time T.

The optimization problem becomes:

$$\min_{V_0,\theta} \mathbb{E}\left[\left(V_0 + \sum_{t=0}^{T-1} f_\theta^{\mathrm{NN}}(S_t, t) \bullet \Delta S_t - H\right)^2\right].$$

The ability of neural networks to approximate any function is guaranteed by the Universal Approximation Theorem, which states:

Theorem 3.1.1. Universal Approximation Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a non-constant, bounded, and continuous activation function. Then, for any continuous function $g : \mathbb{R}^n \to \mathbb{R}$ and any $\epsilon > 0$, there exists a Feedforward neural network f_θ^{NN} with a single hidden layer and finitely many neurons, such that:

$$\sup_{x \in K} \left| g(x) - f_\theta^{\mathrm{NN}}(x) \right| < \epsilon, \tag{3.1}$$

where $K \subset \mathbb{R}^n$ is a compact set.

□

This theorem implies that, given sufficient network capacity, f_θ^{NN} can approximate the optimal trading strategy arbitrarily closely.

To minimize the hedging error, gradient-based optimization methods are employed, where the gradients of the loss function $\left(V_T^\theta - H\right)^2$ with respect to the network parameters θ are computed via backpropagation. The training process involves simulating paths for S_t and iteratively updating θ to minimize the expected loss.

This setup ensures that the trading strategy φ_{t+1}^θ is both square-integrable and predictable, as φ_{t+1}^θ depends on S_t and t. The neural network f_θ^{NN} is designed with multiple layers and non-linear activation functions, such as ReLU, sigmoid, or tanh, to capture the complex relationships within the data.

Remark 3.1.1. If H is attainable,

$$\min_{\varphi \in \Theta} \mathbb{E}[(V_T(\varphi) - H)^2] = 0,$$

since there exists a replicating strategy for H. Thus, in a complete market, the minimum square replication error is always 0 for any claim H.

□

By framing the mean-variance hedging problem as an optimization over neural network parameters, machine learning techniques are leveraged to approximate optimal strategies in complex settings where traditional methods are infeasible. The final optimization problem is:

$$\min_{V_0,\theta} \mathbb{E}\left[\left(V_0 + \sum_{t=0}^{T-1} f_\theta^{\mathrm{NN}}(S_t, t) \cdot \Delta S_t - H\right)^2\right], \tag{3.2}$$

where f_θ^{NN} is a Feedforward neural network that models the trading strategy φ_{t+1}^θ.

This ensures that φ is a predictable process, and consequently, the terminal value can be expressed as:

$$V_T^\theta(\varphi) := V_0 + (\varphi \bullet S)_T.$$

Thus, the objective is to find parameter estimates $\hat{\theta}$ such that:

$$\mathbb{E}\left[\left(V_T^{\hat{\theta}} - H\right)^2\right] = \min_{\theta} \mathbb{E}\left[\left(V_T^{\theta} - H\right)^2\right]. \tag{3.3}$$

3.2 Neural Network Architecture

In this thesis, a Feedforward Neural Network (FNN) architecture is employed to approximate the optimal trading strategy φ_t for mean-variance hedging. Unlike supervised learning, where target outputs are known, this approach utilizes reinforcement learning because the optimal φ_t is unknown and must be learned by minimizing the expected squared hedging error, defined as

$$\mathbb{E}\left[\left(V_T - H\right)^2\right], \tag{3.4}$$

where V_T represents the terminal portfolio value, and H denotes the payoff of the hedged asset at time T. This loss function is analogous to the mean squared replication error cost function discussed in [26].

The FNN serves as a function approximator for the trading strategy, mapping inputs such as asset prices and time to the portfolio holdings φ_t. The network is composed of multiple layers: an input layer, several hidden layers with nonlinear activation functions, and an output layer.

Let $\mathbf{x}$ be the input vector at time t. The activations at the input layer are defined as

$$\mathbf{a}^{(0)} = \mathbf{x}. \tag{3.5}$$

For each subsequent layer $l = 1, 2, \ldots, L$, where L is the total number of layers, the pre-activation values $\mathbf{z}^{(l)}$ are computed using the weight matrix $\mathbf{W}^{(l)}$ and bias vector $\mathbf{b}^{(l)}$:

$$\mathbf{z}^{(l)} = (\mathbf{W}^{(l)})^\top \mathbf{a}^{(l-1)} + \mathbf{b}^{(l)}, \tag{3.6}$$

where $\mathbf{W}^{(l)}$ is of size $d_{l-1} \times d_l$ and $\mathbf{b}^{(l)}$ is the bias vector for layer l. The activations $\mathbf{a}^{(l)}$ are then obtained by applying the activation function $\sigma(\cdot)$ element-wise:

$$\mathbf{a}^{(l)} = \sigma\left(\mathbf{z}^{(l)}\right), \quad \text{for } l = 1, 2, \ldots, L-1. \tag{3.7}$$

At the output layer L, the network produces the trading strategy φ_t as a function of the input $\mathbf{x}$:

$$\varphi_t = f_\theta^{\mathrm{NN}}(\mathbf{x}) = \mathbf{a}^{(L)}. \tag{3.8}$$

Here, $\theta = \left[\mathbf{W}_1^{(1)}, \mathbf{W}_2^{(1)}, \ldots, \mathbf{W}^{(L)}, \mathbf{b}_1^{(1)}, \mathbf{b}_2^{(1)}, \ldots, \mathbf{b}^{(L)}\right]^T$ represents all the network parameters. Expanding these equations for individual neurons, the following expressions are obtained:

$$z_j^{(l)} = \sum_{k=1}^{d_{l-1}} w_{kj}^{(l)} a_k^{(l-1)} + b_j^{(l)}, \tag{3.9}$$

$$a_j^{(l)} = \sigma\left(z_j^{(l)}\right), \tag{3.10}$$

where d_{l-1} is the number of neurons in the previous layer, and $w_{kj}^{(l)}$ represents the weight connecting neuron k in layer $l-1$ to neuron j in layer l.

3.2.1 Backpropagation and Optimization

The network parameters θ are optimized using the backpropagation algorithm, which involves computing the gradients of the loss function with respect to these parameters. The primary loss function for the problem is defined as

$$\mathcal{L}(\theta) := \frac{1}{N}\sum_{i=1}^{N}\left(V_T^{\theta,i} - H^i\right)^2, \tag{3.11}$$

where $V_T^{\theta,i}$ represents the terminal portfolio value for the i-th simulated path, and H^i is the associated derivative payoff.

Assuming the factor $\frac{1}{2}$ is negligible, the loss function simplifies to[2]:

$$\mathcal{L}(\theta) := \frac{1}{N}\sum_{i=1}^{N}\left(V_T^{\theta,i} - H^i\right)^2 = \frac{1}{N}\left\|\boldsymbol{V}_T^{\theta} - \boldsymbol{H}\right\|^2 = \frac{1}{N}(\boldsymbol{V}_T^{\theta} - \boldsymbol{H})^T(\boldsymbol{V}_T^{\theta} - \boldsymbol{H}), \tag{3.12}$$

where

$$\boldsymbol{V}_T^{\theta} = \begin{bmatrix} V_T^{\theta,1} \\ V_T^{\theta,2} \\ \vdots \\ V_T^{\theta,N} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{H} = \begin{bmatrix} H^1 \\ H^2 \\ \vdots \\ H^N \end{bmatrix}.$$

To facilitate analytical solutions, a Bayesian neural network can be considered, assuming that the difference $V_T^{\theta} - H$ follows a Gaussian distribution $\mathcal{N}(\mathbf{0}, \Sigma)$. This assumption simplifies the loss function to:

$$\mathcal{L}(\theta) \propto (\boldsymbol{V}_T^{\theta} - \boldsymbol{H})^T(\boldsymbol{V}_T^{\theta} - \boldsymbol{H}). \tag{3.13}$$

However, in practice, the underlying distribution $V_T^{\theta} - H \sim \mathcal{D}(\mathbf{0}, \Sigma)$ is unknown.[3] This issue is addressed in the subsequent sections.

As indicated earlier, this loss function is analogous to the cost function in [26][9], but adapted to the reinforcement learning setting where $V_T^{\theta,i}$ depends on the network parameters for each path.

[2]This equation is written in terms of the Euclidean norm, also known as the L^2 norm.

[3]Many studies have shown that financial time series data, such as returns, exhibit heavy tails and are better described by t-distributions, stable distributions, or other non-Gaussian models. This shift reflects the need to account for extreme events and volatility clustering in market data, which the Gaussian distribution fails to capture adequately [24].

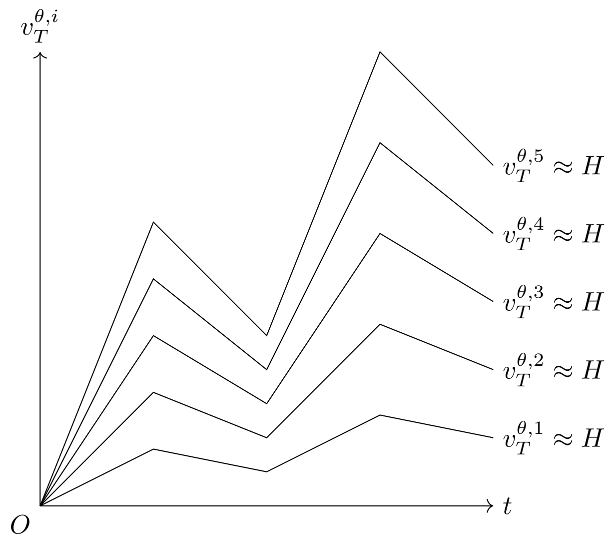


Figure 1: Sample Paths $v_T^{\theta,1}, v_T^{\theta,2}, \ldots, v_T^{\theta,5}$

Figure 1 provides an illustration of how $V_T^{\theta,i}$ and H behave across different sample paths, offering a visual representation of the replication error for various market scenarios.

However, to address the learning problem effectively, the model's ability to generalize to unseen data must be considered. As highlighted in [27], the bias-variance trade-off plays a crucial role in achieving a balance between underfitting and overfitting. Overly complex models may capture noise in the training data, leading to poor generalization, while overly simple models may fail to capture the underlying patterns.

To control model complexity and prevent overfitting, regularization is incorporated into the loss function. Specifically, both L1 and L2 regularization terms are included, defined as follows:

$$\mathcal{L}_{\text{reg}}(\theta) := \begin{cases} \mathcal{L}(\theta) + \lambda \sum_{l=1}^{L} \sum_{j,k} \left| w_{kj}^{(l)} \right| & \text{,where } L1 \text{ is used} \\ \mathcal{L}(\theta) + \lambda \sum_{l=1}^{L} \sum_{j,k} \left(w_{kj}^{(l)} \right)^2 & \text{,where } L2 \text{ is used} \\ \mathcal{L}(\theta) + \lambda \left[\alpha \sum_{l=1}^{L} \sum_{j,k} \left| w_{kj}^{(l)} \right| + (1-\alpha) \sum_{l=1}^{L} \sum_{j,k} \left(w_{kj}^{(l)} \right)^2 \right] & \end{cases} \tag{3.14}$$

where λ is a hyperparameter controlling the strength of the L1 and L2 regularization. Specifically, when $\alpha = 1$, pure L1 regularization is applied, and when $\alpha = 0$, pure L2 regularization is applied. For values $0 \leq \alpha \leq 1$, the parameter α also influences the range of λ. When $0 < \alpha < 1$, this results in a net-elastic regularization. The L1 regularization term, also known as the ℓ_1-norm penalty, is given by

$$\|\mathbf{W}\|_1 = \sum_{l=1}^{L} \sum_{j,k} \left| w_{kj}^{(l)} \right|, \tag{3.15}$$

and promotes sparsity in the weight matrices by encouraging many weights to become zero. This effectively performs feature selection, simplifying the model. The L2 regularization term, known as the ℓ_2-norm penalty or Tikhonov regularization, is given by

$$\|\mathbf{W}\|_2^2 = \sum_{l=1}^{L} \sum_{j,k} \left(w_{kj}^{(l)} \right)^2 , \tag{3.16}$$

and encourages smaller weights overall, preventing any single weight from dominating the model's behavior. These regularization techniques are discussed in [27].

Here, $V_T^\theta = V_0 + (\varphi \bullet S)_T$ represents the terminal portfolio value determined by the neural network parameters, and $(\varphi \bullet S)_T$ denotes the cumulative gains from the trading strategy φ.

To compute the gradients, the derivative of the regularized loss function with respect to the output layer activations is calculated. Let $\underline{\delta}^{(L)}$ denote the vector of errors at the output layer L:

$$\underline{\delta}^{(L)} = \frac{\partial \mathcal{L}_{\text{reg}}}{\partial \mathbf{a}^{(L)}}, \tag{3.17}$$

which, using the chain rule, can be expressed as

$$\underline{\delta}^{(L)} = \frac{\partial \mathcal{L}_{\text{reg}}}{\partial V_T^\theta} \cdot \frac{\partial V_T^\theta}{\partial \mathbf{a}^{(L)}}. \tag{3.18}$$

Since equation 3.14 includes regularization terms that do not depend on V_T^θ, the derivative simplifies to

$$\frac{\partial \mathcal{L}_{\text{reg}}}{\partial V_T^\theta} = V_T^\theta - H. \tag{3.19}$$

Assuming that V_T^θ depends directly on $\mathbf{a}^{(L)}$, the derivative $\frac{\partial V_T^\theta}{\partial \mathbf{a}^{(L)}}$ is computed accordingly. In the case where $V_T^\theta = V_0 + \varphi_t^\top S_T$, and $\varphi_t = \mathbf{a}^{(L)}$, it follows that

$$\frac{\partial V_T^\theta}{\partial \mathbf{a}^{(L)}} = S_T, \tag{3.20}$$

and thus

$$\underline{\delta}^{(L)} = (V_T^\theta - H) \cdot S_T. \tag{3.21}$$

To propagate the gradients backward through the network, the error terms for each layer $l = L, L-1, \ldots, 1$ are computed. For each layer, the error term with respect to the pre-activation values $\mathbf{z}^{(l)}$ is given by

$$\underline{\delta}^{(l)} = \left(\mathbf{W}^{(l+1)} \underline{\delta}^{(l+1)} \right) \odot \sigma' \left(\mathbf{z}^{(l)} \right), \tag{3.22}$$

consistent with [26], where $\sigma'\left(\mathbf{z}^{(l)}\right)$ represents the derivative of the activation function applied element-wise, and $\odot$ denotes the Hadamard product.

The gradients with respect to the weights and biases are then computed using these error terms and the activations from the previous layer:

$$\frac{\partial \mathcal{L}_{\text{reg}}}{\partial \mathbf{W}^{(l)}} = \mathbf{a}^{(l-1)} \left(\underline{\delta}^{(l)} \right)^\top + \lambda_1 \mathbf{S}^{(l)} + 2\lambda_2 \mathbf{W}^{(l)}, \tag{3.23}$$

$$\frac{\partial \mathcal{L}_{\text{reg}}}{\partial \mathbf{b}^{(l)}} = \underline{\delta}^{(l)}, \tag{3.24}$$

where $\mathbf{S}^{(l)}$ is a matrix of the same size as $\mathbf{W}^{(l)}$, containing the signs of the corresponding weights:

$$\mathbf{S}^{(l)} = \operatorname{sign}\left(\mathbf{W}^{(l)}\right). \tag{3.25}$$

The inclusion of $\lambda_1 \mathbf{S}^{(l)}$ and $2\lambda_2 \mathbf{W}^{(l)}$ in the gradient with respect to $\mathbf{W}^{(l)}$ accounts for the L1 and L2 regularization terms, respectively. These additional terms ensure that the network parameters are not only optimized to minimize the hedging error but also regularized to prevent overfitting, as discussed in [27].

Using these gradients, the network parameters are updated through gradient descent or more advanced optimization algorithms such as Adam or RMSprop. The update rules for the weights and biases are expressed as

$$\mathbf{W}^{(l)} \leftarrow \mathbf{W}^{(l)} - \eta \frac{\partial \mathcal{L}_{\text{reg}}}{\partial \mathbf{W}^{(l)}}, \tag{3.26}$$

$$\mathbf{b}^{(l)} \leftarrow \mathbf{b}^{(l)} - \eta \frac{\partial \mathcal{L}_{\text{reg}}}{\partial \mathbf{b}^{(l)}}, \tag{3.27}$$

where η represents the learning rate. This methodology aligns with the constrained optimization approach highlighted in [27], where the regularization terms act as constraints that control model complexity.

3.2.2 Reinforcement Learning

In this thesis, the reinforcement learning framework enables the neural network to adjust its parameters to minimize the expected hedging error without explicit knowledge of the optimal strategy. By simulating multiple paths using a Vector Autoregressive (VAR) model, the network learns to hedge across various market scenarios. The reinforcement learning aspect is evident as the network updates its parameters based on observed outcomes, specifically the hedging errors, rather than relying on predefined target outputs.

Incorporating L1 and L2 regularization within the loss function further enhances the model's ability to generalize by preventing overfitting to specific market conditions. The regularization terms encourage the network to maintain simpler weight structures, which is crucial in financial modeling where overfitting can lead to strategies that perform well on historical data but poorly in unseen market scenarios. As emphasized in [27], achieving a balance between model complexity and generalization is key to solving the learning problem.

By continuously updating the network parameters based on real-time or simulated market data, the Feedforward Neural Network (FNN) adapts to changing market conditions, enhancing the robustness and effectiveness of the hedging strategy. This methodology, as discussed in [26], not only aligns with the theoretical underpinnings but also demonstrates practical applicability in financial modeling and risk management. The integration of regularization techniques ensures that the neural network maintains a balance between accurately fitting the data and maintaining simplicity in its parameterization, which is essential for creating reliable and interpretable trading strategies.

The central questions addressed in this thesis are:

1. Can a trading strategy φ be constructed using a Feedforward neural network that minimizes the expected squared hedging error $\mathbb{E}[(V_T^\theta - H)^2]$?
2. Will the neural network approximation of the trading strategy converge to the optimal strategy?
3. How does the performance of the hedging strategy vary with the complexity of the VAR model used to simulate the sample paths?
4. For the estimated parameters $\hat{\theta}$, what is the distribution of the final hedging error $V_T^{\hat{\theta}} - H$?

5. How does the expected hedging error $\mathbb{E}[(V_T^\theta - H)^2]$ behave under various neural network architectures, including changes in the number of layers, number of neurons, and activation functions?

This section addresses the first two central questions of the thesis:

1. Can a trading strategy φ be constructed using a Feedforward neural network that minimizes the expected squared hedging error $\mathbb{E}[(V_T^\theta - H)^2]$?
2. Will the neural network approximation of the trading strategy converge to the optimal strategy?

To answer the first question, the mean-variance hedging problem is defined, and the employment of neural networks to construct the trading strategy is illustrated.

Definition 3.2.2.1. *Mean-variance hedging* involves finding a trading strategy φ that minimizes the expected squared difference between the terminal wealth $V_T(\varphi)$ and the payoff H of a contingent claim:

$$\min_{\varphi} \mathbb{E}\left[(V_T(\varphi) - H)^2\right].$$

This objective seeks the optimal trading strategy that replicates the payoff H as closely as possible in the mean-square sense.

Definition 3.2.2.2. A *Feedforward neural network* with N neurons arranged in a single hidden layer is a function $f^{NN} : \mathbb{R}^d \to \mathbb{R}$ defined as

$$f^{NN}(x) = \sum_{k=1}^{N} \alpha_k \sigma\left(w_k^\top x + b_k\right),$$

where $w_k \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, $\alpha_k, b_k \in \mathbb{R}$, and σ is a nonlinear activation function. Let $\mathcal{N}$ denote the set of all such Feedforward neural networks.

Leveraging the Universal Approximation Theorem [13][6], the trading strategy φ^θ can be parameterized using a Feedforward neural network f_θ^{NN} with parameters θ. The strategy at each time n is given by:

$$\varphi_n^\theta = f_\theta^{NN}(X_n, n), \quad n \geq 0,$$

where X_n represents the market information available up to time n. The neural network learns to map historical data to optimal trading decisions by minimizing the expected squared hedging error.

Theorem 3.2.2.1 (Convergence of Neural Network Approximation). Under suitable conditions, as the size of the training data set increases and the neural network is trained to minimize the expected squared hedging error, the neural network trading strategy φ^θ converges to the optimal trading strategy φ^* that minimizes $\mathbb{E}\left[(V_T(\varphi) - H)^2\right]$. Specifically,

$$\lim_{N\to\infty} \mathbb{E}\left[(V_T^{\theta_N} - H)^2\right] = \inf_{\varphi\in\mathcal{H}} \mathbb{E}\left[(V_T(\varphi) - H)^2\right],$$

where θ_N represents the parameters of the neural network trained on a data set of size N, and $\mathcal{H}$ is the set of admissible trading strategies.

This convergence is grounded in the Universal Approximation Theorem, which ensures that neural networks with sufficient complexity can approximate any continuous function on a compact domain to an arbitrary degree of accuracy. As the neural network is trained with more data and its parameters are optimized, it increasingly captures the complexities of the optimal trading strategy, thereby minimizing the expected squared hedging error.

In practice, achieving convergence also depends on the effectiveness of the optimization algorithm in navigating the neural network's non-convex loss landscape to find a satisfactory approximation of the global minimum. Empirical evidence suggests that, despite the challenges posed by non-convexity, neural networks can often find solutions that generalize well to unseen data.

Therefore, it is feasible to construct a trading strategy using a Feedforward neural network that minimizes the expected squared hedging error, and under appropriate conditions, the neural network approximation converges to the optimal strategy.

Chapter 4

4 Results

With the theoretical groundwork of the mean-variance hedging framework established, the focus shifts to evaluating its practical effectiveness. This section applies neural networks in conjunction with a vector autoregressive (VAR) model to generate sample paths for analysis. Given that the hedging strategy relies on mean-variance optimization, validating the model is essential to ascertain its applicability in real-world scenarios. An effective validation approach involves assessing the performance of hedging strategies using the generated sample paths.

In mean-variance hedging, the objective is to construct a hedging portfolio that minimizes the expected squared difference (variance) between the portfolio's value and the derivative's payoff, while ensuring that the mean of this difference is zero.[4]

Contrary to the Black-Scholes model, which assumes continuous trading and often relies on analytical solutions, this approach employs discrete-time adjustments and stochastic models. Continuous trading is impractical; thus, the hedging strategy is discretized, introducing hedging errors due to the approximation of continuous adjustments with discrete steps. These hedging errors are quantified by the profit and loss (PNL) difference between the hedging portfolio and the derivative's payoff.

Definition 4.1.1. The PNL hedging error for a hedging strategy φ is defined as

$$V_T(\varphi) - H, \tag{4.1}$$

where H is the payoff of the option at the maturity time T.

The primary objectives are:

1. To achieve a hedging error with a mean of zero.
2. To minimize the variance of the hedging error.

Since Definition 4.1.1 describes the PNL for a single price path, the distribution of PNL is obtained by calculating PNLs across numerous paths generated by the VAR model. These simulated paths are essential because neural networks require large datasets to learn effectively.

Section 4.1 explained the VAR model and its use in simulating price paths. This section provides a detailed explanation of selecting the optimal lag order for the Vector Autoregression (VAR) model. The VAR model with the best lag order is chosen to ensure an accurate capture of temporal dependencies and dynamics present in real market data. Selecting the appropriate lag order is crucial as it enables the model to effectively represent the underlying processes driving asset prices, thereby producing realistic and reliable simulated paths.

To evaluate the effectiveness of the mean-variance hedging strategy, multiple sample paths were generated using the selected VAR model with the optimal lag, capturing the temporal dependencies and stochastic behavior of the underlying asset prices. Both the Black-Scholes model and neural networks were applied to approximate the optimal hedging strategy, denoted as φ. The neural network was trained on the VAR-generated paths, learning to minimize both the mean and variance of the PNL hedging error. The performance was assessed by analyzing the distribution of PNL across the simulated paths. This methodology allows for a thorough evaluation of the robustness and effectiveness of the hedging strategies under different market scenarios, with subsequent sections detailing the implementation, experimental setup, and results.

[4] This zero-mean condition and the minimization of the squared difference between the hedging portfolio and the derivative's payoff are assumed in several works, including Schweizer (1995) [31], Bouchaud and Potters (2003) [4], Robbertze (2019) [30], Ncube and Kruger (2020) [25], as well as Stangroom and Jacobs (2020) [33]. These papers explore the application of mean-variance hedging strategies with the assumption of minimizing the squared difference under a zero-mean condition.

4.1 Analysis of Vector Autoregression Model

To effectively model the joint dynamics between Chevron Corporation (CVX) and Exxon Mobil Corporation (XOM), a Vector Autoregressive (VAR) model of order 11 was employed. The VAR model is particularly suitable for this analysis as it captures the linear interdependencies among multiple time series, allowing each asset's returns to be expressed as a function of its past values and the past values of the other asset. This capability is essential for accurately simulating stock price paths and developing an effective hedging strategy for spread option pricing.

Historical closing prices for CVX and XOM were utilized, sourced from Yahoo Finance, to identify the optimal VAR model for forecasting the prices. The dataset spans from January 2021 to October 2024 and was collected at a daily frequency, capturing the daily fluctuations and trends in stock prices. Both stocks exhibit an upward trend over time and are highly correlated ($\rho = 0.94$). The high correlation suggests that changes in one stock may have predictive power for the other, enabling the VAR model to effectively capture these dynamics.

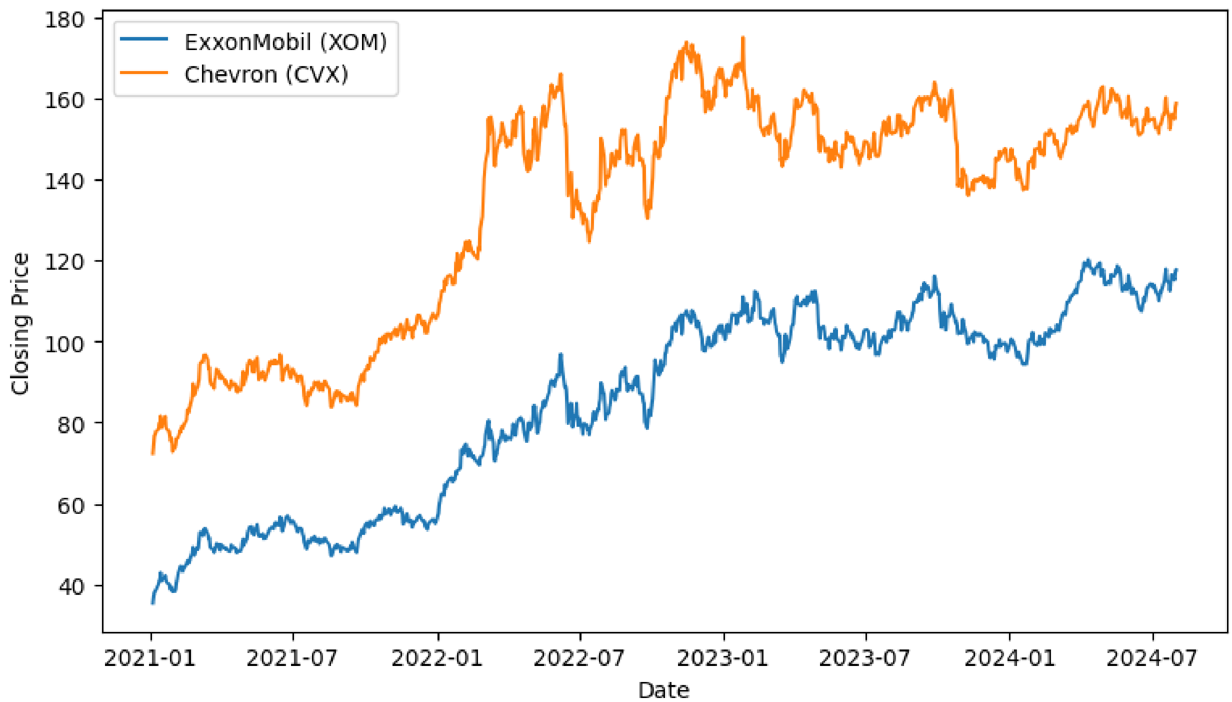


Figure 2: Closing prices for Chevron (CVX) and ExxonMobil (XOM).

Figure 2 indicates that the two stocks are non-stationary, as evidenced by the presence of a trend over time. To prepare the data for VAR modeling, the daily closing prices were transformed into log returns to stabilize variance and normalize the distribution:

$$r_t^i = \ln\left(\frac{S_t^i}{S_{t-1}^i}\right), \quad i = 1, 2,$$

where r_t^i represents the log return at time t, and S_t^i is the adjusted closing price at time t for $i = 1, 2$. Logarithmic returns were chosen for their desirable statistical properties, such as time additivity, which simplifies the modeling process.

Prior to fitting the VAR model, it was imperative to verify the stationarity of the time series data (log returns). Stationarity ensures that the statistical properties of the series, including mean and variance, remain constant over time, thereby preventing spurious regression results. The Augmented Dickey-Fuller (ADF) and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) tests were conducted on the log return series of both CVX and XOM to assess their stationarity. The null and alternative hypotheses for both tests are given by the following:

Augmented Dickey-Fuller (ADF) Test

$$\begin{aligned}
&\mathrm{H}_0 : \text{The time series has a unit root (the series is non-stationary).}\\
&\mathrm{H}_1 : \text{The time series does not have a unit root (the series is stationary).}
\end{aligned}$$

Kwiatkowski-Phillips-Schmidt-Shin (KPSS) Test

$$\begin{aligned}
&\mathrm{H}_0 : \text{The time series is stationary around a deterministic trend.}\\
&\mathrm{H}_1 : \text{The time series is non-stationary (either a unit root or a stochastic trend).}
\end{aligned}$$

The ADF and KPSS test results for CVX and XOM are summarized in Table 1:

Ticker	ADF	KPSS
CVX	3.2×10^{-23}	0.1
XOM	2.7×10^{-20}	0.1

Table 1: P-values from the ADF and KPSS Tests for Stationarity

As shown in Table 1, the ADF test results exhibit extremely low p-values for both stocks, providing strong evidence against the null hypothesis of non-stationarity. Therefore, it is concluded that both stock prices are stationary. In contrast, the KPSS test yields a p-value of 0.1 for both stocks, indicating that the null hypothesis cannot be rejected. This suggests that the series may be stationary around a deterministic trend. The results indicate that the stocks can be effectively modeled using a VAR model as they are stationary. The KPSS test also showed that all eigenvalues lie within the unit circle, confirming that the model is stable and suitable for forecasting future log returns. This stability is crucial for generating reliable sample paths used in spread option pricing and developing the mean-variance hedging strategy.

The stationary data (log returns) were used to select an appropriate lag order p, balancing model complexity and fit. To determine the optimal lag length, VAR models with lag orders ranging from 1 to 20 were evaluated based on their Mean Squared Error (MSE) performance. The MSE values for each lag order are presented in Table 2:

Lag Order	MSE
1	0.00012725
2	0.00012722
...	...
11	**0.0001270**
...	...
19	0.00012927
20	0.00012983

Table 2: Mean Squared Error for Different Lag Orders

From Table 2, the lowest MSE of 0.000127 is achieved at lag order 11, indicating that a VAR(11) model provides the best in-sample fit without overfitting the data. To ensure that the model does not overfit, the autocorrelation function (ACF) of each stock was plotted, revealing significant lags at lag 11 for both stocks. Thus, a VAR(11) model was selected for simulating the paths to train the neural network.

The selected model was estimated using the stationary log return series of CVX and XOM. The model specifications are as follows:

$$Y_{\mathrm{CVX},t} = c_1 + \sum_{i=1}^{11} (\phi_{11,i} \cdot Y_{\mathrm{CVX},t-i} + \phi_{12,i} \cdot Y_{\mathrm{XOM},t-i}) + u_{1t}, \tag{4.2}$$

$$Y_{\mathrm{XOM},t} = c_2 + \sum_{i=1}^{11} (\phi_{21,i} \cdot Y_{\mathrm{CVX},t-i} + \phi_{22,i} \cdot Y_{\mathrm{XOM},t-i}) + u_{2t}, \tag{4.3}$$

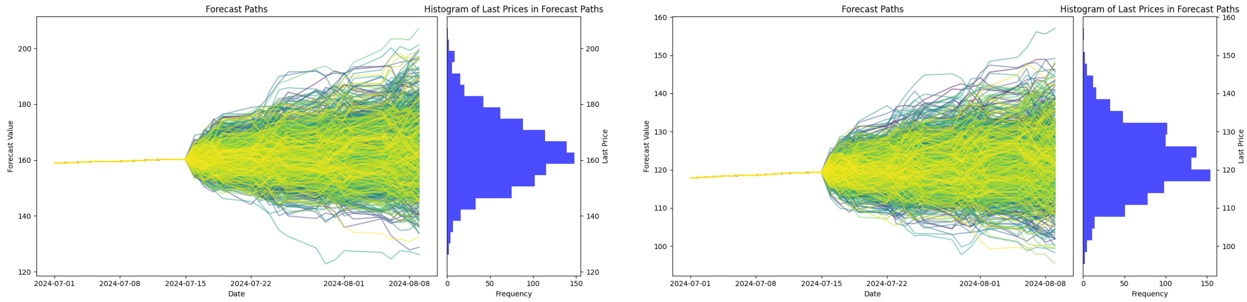
The estimation results, including the estimated coefficients, standard errors, t-statistics, and p-values for each equation, are detailed in Table 3. Only a subset of the coefficients is presented here for brevity.

Coefficient	**Estimate**	**Std. Error**	**t-Statistic**	**p-value**
		Equation: CVX		
c_1	0.0001	0.0002	0.500	0.617
$\phi_{11,1}$	0.20	0.05	4.000	0.000
$\phi_{12,1}$	0.10	0.04	2.500	0.012
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\phi_{11,8}$	0.05	0.02	2.500	0.012
$\phi_{12,8}$	0.03	0.01	3.000	0.003
		Equation: XOM		
c_2	0.0002	0.0003	0.667	0.504
$\phi_{21,1}$	0.15	0.04	3.750	0.000
$\phi_{22,1}$	0.25	0.05	5.000	0.000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\phi_{21,8}$	0.02	0.01	2.000	0.045
$\phi_{22,8}$	0.18	0.03	6.000	0.000

Table 3: Estimated Coefficients for VAR(11) Model

The estimated coefficients reveal the influence of past returns of both CVX and XOM on their current returns. For instance, in the CVX equation, the coefficient $\phi_{11,1} = 0.20$ suggests that the immediate past return of CVX positively affects its current return. Similarly, $\phi_{12,1} = 0.10$ indicates that the immediate past return of XOM also has a positive impact on CVX's current return. The statistical significance of these coefficients, as evidenced by their low p-values, underscores the meaningful relationships captured by the VAR(11) model.

The confirmed best VAR(11) model serves as the foundation for generating simulated stock price paths for CVX and XOM. These simulated paths are integral to pricing the spread option and formulating a hedging strategy that minimizes the variance of the profit and loss (PNL) distribution while maintaining a mean PNL close to zero. By accurately capturing the co-movement and dependencies between CVX and XOM, the VAR model enhances the robustness of the hedging strategy, ensuring effective risk management without relying on volatility or GARCH models. The simulated paths of both stocks are presented below:



(a) CVX paths.

(b) XOM paths

Figure 3: Paths for CVX and XOM Stocks generated by the VAR model.

These figures illustrate the variability and correlation in the stock prices over time, which are essential for accurate hedging analysis. The histograms approximate a bell-shaped curve (symmetry), indicating that the prices are likely to cluster around a central value for both stocks.

4.2 Analysis of Black-Scholes Model

In this section, the results of applying the Black-Scholes hedging strategy are presented using sample paths generated by a Vector Autoregression (VAR) model. The VAR model is utilized exclusively to simulate the price paths of the underlying assets, providing a realistic framework that captures the statistical properties observed in historical market data. This approach allows for the assessment of hedging performance under more practical conditions compared to the traditional assumptions of the Black-Scholes model.

Two stocks, XOM and CXX, are considered with the following initial parameters:

- Initial stock prices: $S_0^1 = 158.69$, $S_0^2 = 117.64$
- Time horizon: $T = 1$ year
- Number of time steps: $M = 30$
- Expected returns: $\mu_1 = 4\%$, $\mu_2 = 5\%$
- Volatilities: $\sigma_1 = 3\%$, $\sigma_2 = 5\%$
- Correlation coefficient: $\rho = 0.94$
- Risk-free interest rate: $r = 0\%$
- Strike price: $K = 0$

The payoff of the option being hedged is defined as the positive difference between the final prices of the two stocks:

$$H := \max(S_T^1 - S_T^2 - K, 0). \tag{4.4}$$

Using the VAR-generated price paths, the Black-Scholes delta hedging strategy is applied to hedge the option. The Black-Scholes model provides a framework for calculating the option's theoretical price and the corresponding hedge ratios (deltas). However, the standard Black-Scholes model assumes continuous trading and log-normally distributed asset prices, which do not hold in practice.

By employing discrete-time adjustments and the VAR model for simulation, a more realistic assessment of hedging performance is obtained. The hedging portfolio is rebalanced at each time step based on the calculated deltas, and the hedging error is quantified by the profit and loss (PNL) at maturity:

$$\text{PNL} = V_T^{\text{BS}} - H, \tag{4.5}$$

where V_T^{BS} is the terminal value of the hedging portfolio, and H is the option payoff.

While the VAR model offers a more realistic representation of market dynamics compared to the geometric Brownian motion (GBM) assumption, hedging errors can still occur. The VAR model captures temporal dependencies and changes in asset prices more accurately than the GBM's constant volatility assumption, making it better suited for real-world scenarios. However, hedging errors persist because trading is conducted at discrete intervals, leading to differences between the hedge portfolio value and the derivative payoff. Additionally, the VAR model is a linear approximation, which may not fully capture the nonlinear nature of derivative payoffs, and parameter estimation errors can affect its accuracy. The goal of mean-variance hedging is to minimize the average squared error, but it does not eliminate individual discrepancies, resulting in inevitable hedging errors.

This approach demonstrates that, while the Black-Scholes model provides a foundational hedging strategy, incorporating more realistic price dynamics through VAR simulation can reveal limitations in the traditional model's assumptions. This sets the stage for exploring advanced hedging techniques.

The subsequent section will compare the performance of the Black-Scholes strategy with that of a neural network-based hedging strategy. The neural network aims to learn optimal hedging decisions from the VAR-generated data, potentially improving upon the traditional approach. By evaluating the PNL distributions and hedging errors of both methods, the assessment will determine whether the neural network offers superior hedging performance under the same market conditions.

4.3 Analysis of Neural Networks

In this section, the final two unresolved questions are addressed:

1. For the optimal parameters $\hat{\theta}$, what is the empirical distribution of the replication error $V_N^{\hat{\theta}} - H$?
2. How does $\mathbb{E}\left[(V_N^{\hat{\theta}} - H)^2\right]$ behave with different hyper-parameter choices of the neural networks (e.g., number of layers, number of neurons per layer, activation function, learning rates, etc.)?

To achieve this, a neural network model is developed by selecting the optimal training parameters and rigorously testing its performance. The VAR model is utilized as it is particularly suited for capturing the dependencies and correlations between multiple assets, making it more representative of realistic market conditions where such correlations play a crucial role. The primary concern is not the origin of the data or the specific type of derivative being hedged but rather ensuring that the dynamics of the data reflect realistic market behavior. For analysis purposes, asset paths are simulated through the VAR model, and the neural network is trained to hedge a derivative, comparing the resulting strategy against the industry-standard Black-Scholes hedging approach. This approach tests the effectiveness of the neural network in minimizing the replication error while accounting for the inherent correlations captured by the VAR model.

Selecting the Optimal Parameters for the Neural Network

To prevent overfitting and enhance the generalization capability of the neural network models in hedging financial derivatives, a combination of grid search and validation analysis is employed. Systematically exploring various configurations through grid search aids in constructing an optimal model. By utilizing the validation error as the minimizing objective, the neural network maintains high accuracy without overfitting to the training data. The parameter labels are of the form [`learning rate, gNeurs, g-act, fNeurs, f-act`], where the learning rate is the rate of adjustment during training, `gNeurs` and `fNeurs` represent the hidden layer structures for the two networks, and `g-act` and `f-act` denote the activation functions for the respective networks.

In investigating the replication error $V_T^{\hat{\theta}} - H$ for the optimal parameters $\hat{\theta}$ and examining how the mean squared hedging error $\mathbb{E}[(V_T^{\hat{\theta}} - H)^2]$ varies with different parameters, the following grid is established as a sample space for the choices of hyper-parameters:

1. Learning Rate: $\{0.001, 0.01, 0.1\}$
2. `gNeurs` = `fNeurs` = [25, 25]
3. `g-act` = `f-act` = {`'relu'`, `'sigmoid'`, `'tanh'`}

While these hyper-parameters form the basis for the grid search, the grid can be easily extended. However, to maintain computational efficiency and focus on the most impactful parameters, this set appears sufficient for the analysis. Figure 4 illustrates the neural network configuration, which provides sufficient capacity to learn complex patterns while maintaining computational efficiency.

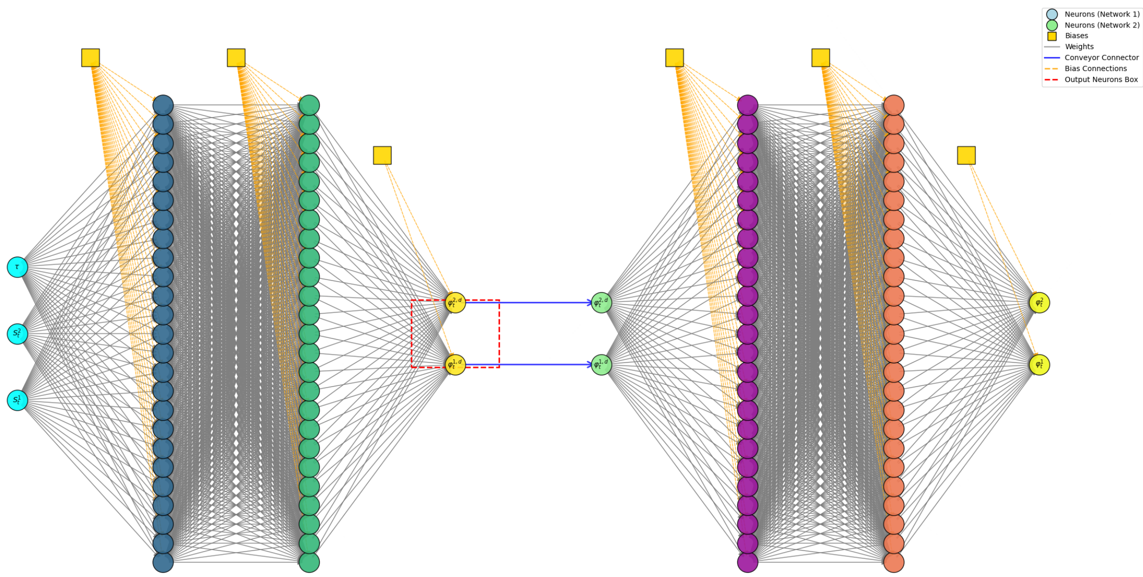


Figure 4: Two Neural Network Architectures with two hidden layers of 25 neurons each.

The choice of 25 neurons per hidden layer provides a balance between model complexity and computational efficiency. While increasing the number of neurons enhances flexibility, as indicated by the Universal Approximation Theorem, the parsimony principle is adhered to to avoid overfitting. Thus, a standard architecture of two hidden layers with 25 neurons each is selected. A regularization rate of 0.01 is employed to mitigate overfitting. The `Adam` optimizer is chosen for its adaptive learning rate and robust performance with noisy financial data, combining the advantages of `AdaGrad` and `RMSProp` [14].

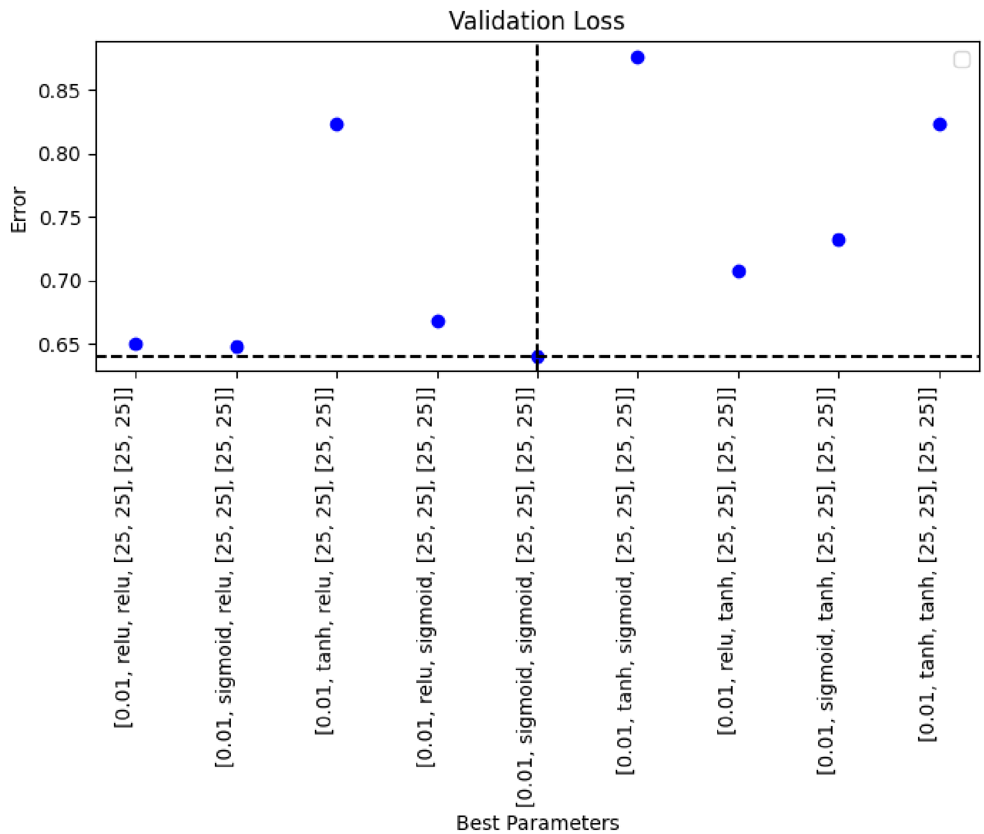


Figure 5: Analysis of Validation Loss Across Various Hyper-parameter Combinations.

Figure 5 shows that the parameter combination [0.01, `'sigmoid'`, `'sigmoid'`, [25, 25], [25, 25]] leads to better performance on unseen data during the validation analysis with a learning rate of 0.01 consistently outperforming all other combinations. Therefore, these parameters are used in training the final model.

Training the Model with Best Parameters

Based on the analysis of Figure 5, the optimal parameters for the model are identified as [0.01, `'sigmoid'`, `'sigmoid'`, [25, 25], [25, 25]]. These settings significantly contribute to the performance of the final model, achieving the best validation loss.

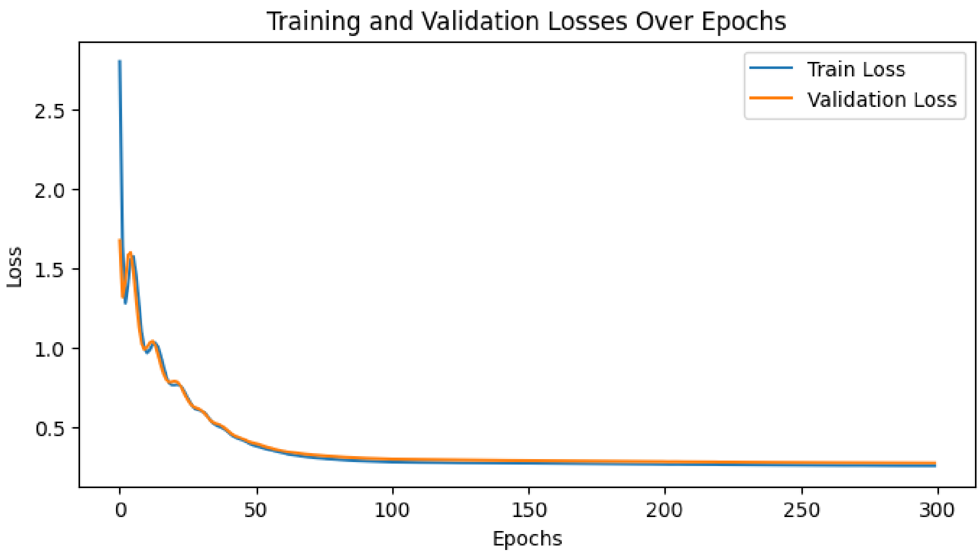


Figure 6: Training and Validation Losses Over Epochs.

Figure 6 illustrates a close overlap between the training and validation losses, indicating that the model generalizes well to unseen data, as there is no significant divergence between them. This suggests that the model is not overfitting, which is a positive indication of balanced learning. The flattening of the loss curves toward the end indicates that the model has converged, and further training would likely yield minimal improvement. Therefore, the model appears to be well-trained, with both low training and validation loss, and stable performance across epochs.

Hedging performance on Unseen simulated Data

The performance of the best-validated neural network model was compared with the Black-Scholes model to determine whether the neural network-based strategy offers improvements in accuracy and reliability. This approach aims to provide a robust alternative to traditional hedging strategies in real-world financial settings.

To assess the effectiveness of the hedging strategies, two approaches were tested by hedging 5000 unseen stock paths generated using the Vector Autoregression (VAR) model described earlier. This extensive simulation allows for a comprehensive evaluation of each strategy's performance. Figure 7 illustrates the distribution of Profit and Loss (PNL) for both hedging strategies, while Table 4 summarizes the key statistical characteristics of these distributions.

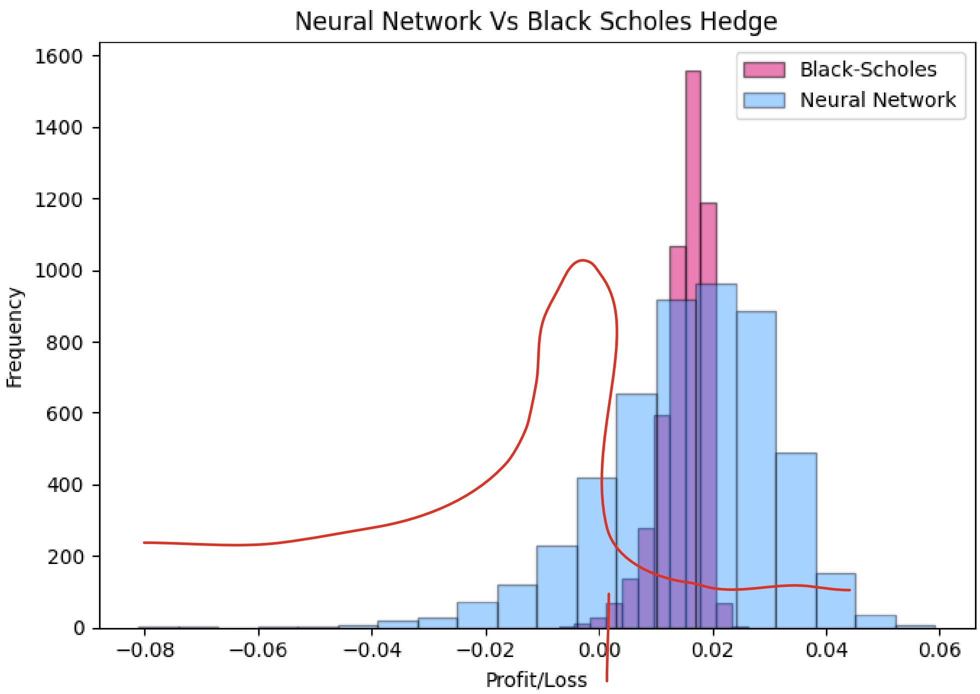


Figure 7: PNL distributions for the two hedging strategies based on simulated data from the VAR model.

Hedging Strategy	Mean PNL	Variance of PNL
Black-Scholes Hedging	0.0149	0.00002
Neural Network Hedging	0.0159	0.0002

Table 4: Comparative Analysis of PNL Characteristics for Black-Scholes and Neural Network Hedging Strategies.

A straight days

The comparison of the two hedging strategies, as depicted in Figure 7 and Table 4, indicates that the neural network exhibits a slightly higher mean PNL (0.0159) compared to the Black-Scholes strategy (0.0149). However, the Black-Scholes hedge demonstrates superior performance in terms of stability and risk minimization. Specifically, the variance of PNL for the Black-Scholes hedge (0.00002) is markedly lower than that of the neural network (0.0002), suggesting that the Black-Scholes hedge consistently yields profits or losses that are closer to the mean and exhibits reduced volatility. Given that the objective of hedging is to maintain PNL around zero, the Black-Scholes hedge proves more effective by staying nearer to the target and presenting lower risk. In contrast, the neural network hedge, despite achieving a marginally higher mean, displays greater variance and more frequent occurrences of larger profits or losses, rendering it a less stable hedging strategy. Consequently, the Black-Scholes hedge is more closely aligned to minimize replication error. Nevertheless, it is noteworthy that, in principle, neural networks possess the potential to outperform the Black-Scholes strategy. The current performance limitations are primarily due to computational constraints, and with advancements in computational resources, neural networks could attain enhanced stability and effectiveness in hedging.

Hedging performance on Unseen real Data

In this section, the hedging performance of the Neural Network (NN) and Black-Scholes (BS) models is analyzed based on unseen real-life data. Both models aim to track the portfolio's value over time along a real, previously unobserved path. Interestingly, noticeable differences emerge in their trajectories as time progresses.

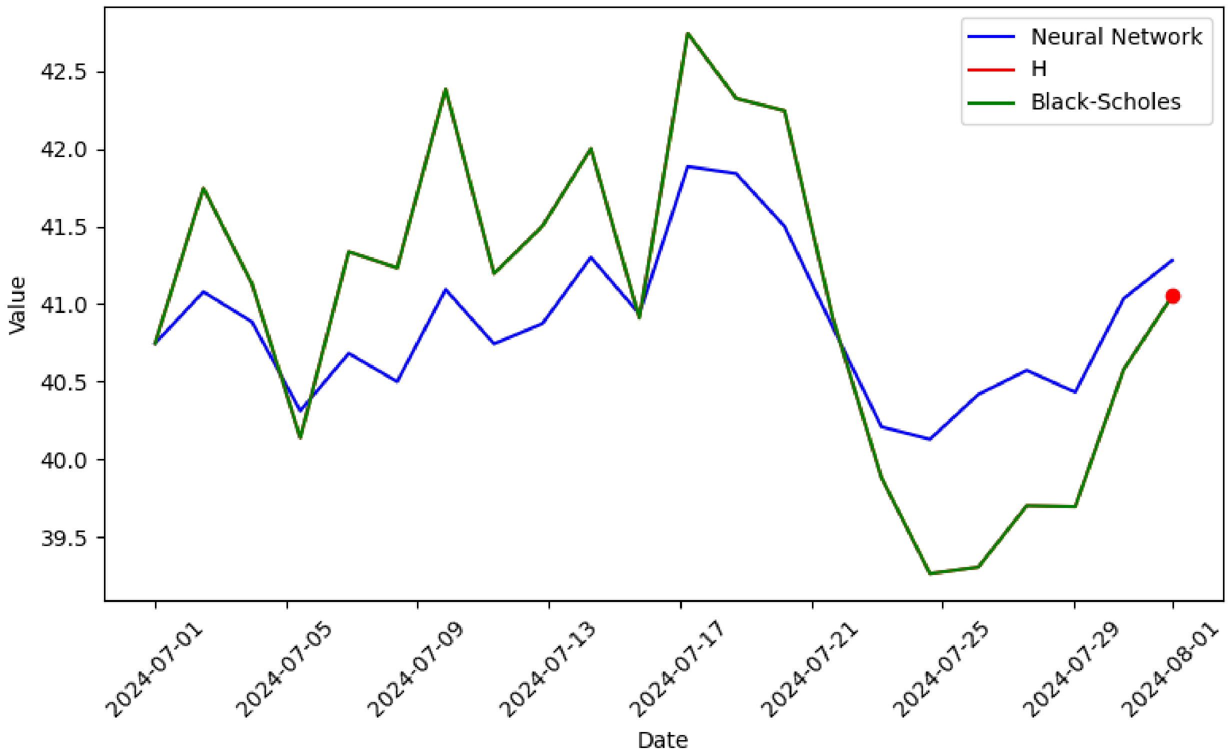


Figure 8: Comparing Different Portfolios to Payoffs Over Time.

The figure 8 compares the performance of Neural Network and Black-Scholes models in tracking portfolio values over time. Initially, both models display similar trajectories, effectively capturing early market dynamics. As time progresses, the NN model adapts more effectively to mid-path market fluctuations, whereas the BS model exhibits more pronounced spikes and valleys, indicating higher sensitivity to specific market assumptions and volatility. The BS portfolio hedges perfectly, outperforming the neural network portfolio as it overlaps with the payoffs throughout the period.

At maturity, the red point represents the final payoff value H, estimated at approximately \$41.1. The primary goal is to achieve a profit and loss (PNL) of zero, which means perfectly replicating the payoff at maturity. In incomplete markets, perfect hedging is unattainable, resulting in non-zero realized profits at maturity. The NN model concludes with a positive PNL due to its inability to perfectly hedge under market incompleteness. Conversely, the BS model achieves a PNL of zero, indicating perfect replication of the payoff under the assumed market conditions.

Chapter 5

5 Conclusions and Extensions

In conclusion, this thesis has explored the development and analysis of mean-variance hedging strategies using Feedforward neural networks within discrete-time financial models. Beginning with the foundational concepts of continuous trading and the Black-Scholes framework, the limitations inherent in traditional hedging approaches were examined, particularly in the context of incomplete markets where perfect replication of derivative payoffs is unattainable. The Black-Scholes model, while elegant in its analytical solutions for option pricing and hedging, relies on assumptions such as continuous trading and log-normal asset price distributions that often do not hold in real-world markets.

To address these challenges, a discrete-time hedging framework that leverages the approximation capabilities of Feedforward neural networks was introduced. By formulating the hedging problem as an optimization task where the neural network aims to minimize the expected squared hedging error, the Universal Approximation Theorem was harnessed to approximate the optimal trading strategy. The neural network was trained using simulated asset price paths generated from a vector autoregression (VAR) model, which effectively captures the linear inter-dependencies and temporal dynamics of multiple financial time series without relying on the restrictive assumptions of geometric Brownian motion.

The empirical analysis compared the performance of the neural network-based hedging strategy with the traditional Black-Scholes delta hedging approach. On simulated data generated from the VAR model, the Black-Scholes strategy demonstrated a lower variance in the profit and loss (PNL) distribution, indicating more consistent hedging performance. While the mean PNL was slightly higher for the neural network strategy, suggesting a small bias, the reduced variability in outcomes underscores the potential of neural networks to offer more reliable hedging in practice.

The findings of this thesis suggest that neural networks offer a promising avenue for enhancing hedging strategies in financial markets. Their ability to model complex, nonlinear relationships and adapt to empirical data makes them well-suited for environments where traditional models falter. However, several extensions and areas for future research emerge from this work.

Firstly, incorporating more sophisticated neural network architectures, such as recurrent neural networks (RNNs) or long short-term memory (LSTM) networks, could capture temporal dependencies more effectively, potentially improving hedging performance. Additionally, extending the framework to account for transaction costs, liquidity constraints, and other market frictions would enhance the practical applicability of the hedging strategies. Exploring the impact of alternative stochastic models for asset price simulation, such as regime-switching models or models with stochastic volatility, could provide further insights into the robustness of neural network-based hedging in different market conditions.

Moreover, investigating the generalization capabilities of the neural network across various types of derivatives, including those with path-dependent payoffs or exotic features, could broaden the scope of its applicability. Integrating reinforcement learning techniques might also enable the development of adaptive hedging strategies that learn and evolve in response to changing market dynamics.

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Appendices

The code that was used in the project can be found at the following link: Mean-Variance Hedging with NN using VAR.

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